Moments and Product of Inertia

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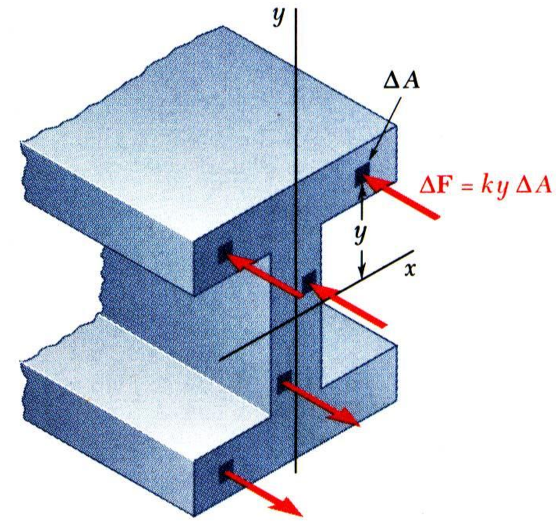
Introduction

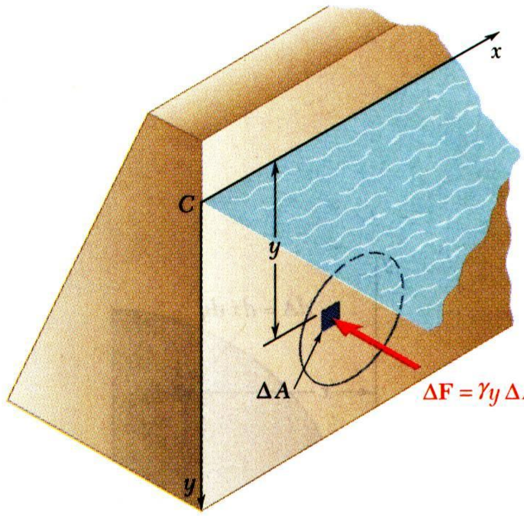
- Previously considered distributed forces which were proportional to the area or volume over which they act.
  - The resultant was obtained by summing or integrating over the areas or volumes.
  - The moment of the resultant about any axis was determined by computing the first moments of the areas or volumes about that axis.

- Will now consider forces which are proportional to the area or volume over which they act but also vary linearly with distance from a given axis.
  - It will be shown that the magnitude of the resultant depends on the first moments of the force distribution with respect to the axis.
  - The point of application of the resultant depends on the second moments of the distribution with respect to the axis.

- Current chapter will present methods for computing the moments and products of inertia for areas.

Moments of Inertia of an Area





- Consider distributed forces $\Delta\vec{F}$ whose magnitudes are proportional to the elemental areas ΔA on which they act and also vary linearly with the distance of ΔA from a given axis.
- Example: Consider a beam subjected to pure bending. Internal forces vary linearly with distance from the neutral axis which passes through the section centroid.

$$\Delta\vec{F} = ky\Delta A$$

$$R = k\int y\,dA = 0 \qquad \int y\,dA = S_x = \text{ first moment}$$

$$M = k\int y^2 dA \qquad \int y^2 dA = \text{ second moment}$$

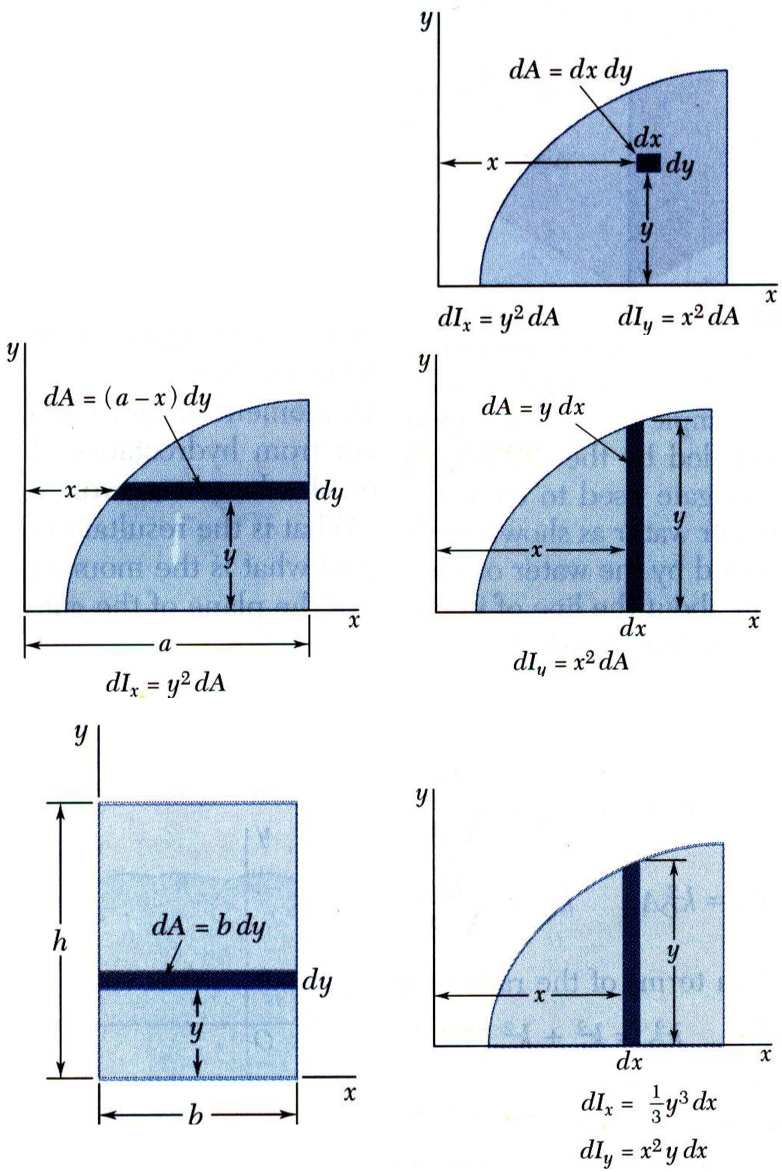
- Example: Consider the net hydrostatic force on a submerged circular gate.

$$\Delta F = p\Delta A = \rho g y \Delta A$$

$$R = \rho g \int y\,dA$$

$$M_x = \rho g \int y^2 dA$$

Moments of Inertia of an Area by Integration



- *Second moments* or *moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

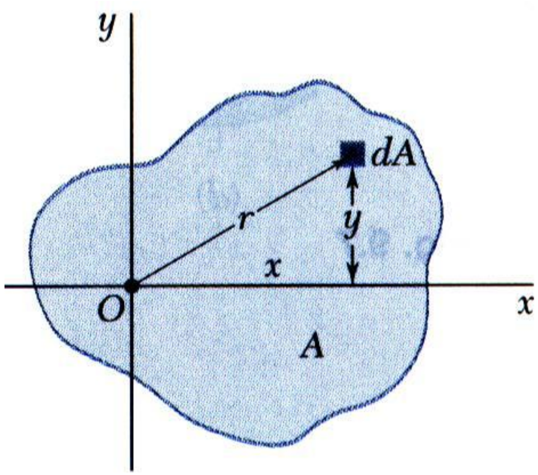
- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

Polar Moments of Inertia



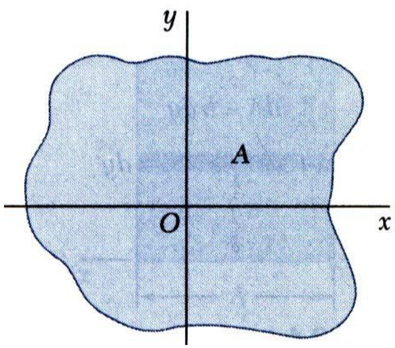
- The *polar moments of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

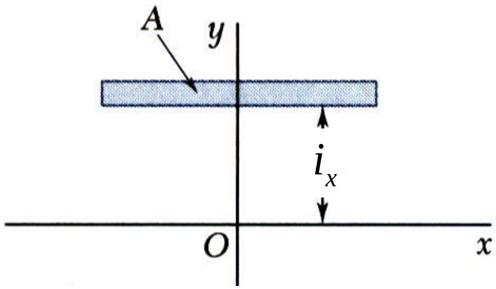
$$I_p = \int r^2 dA$$

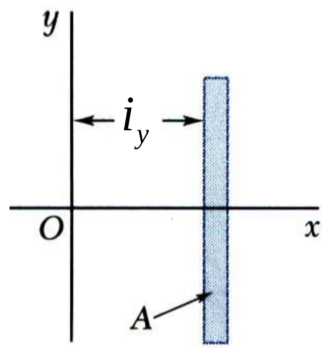
- The polar moments of inertia is related to the rectangular moments of inertia,

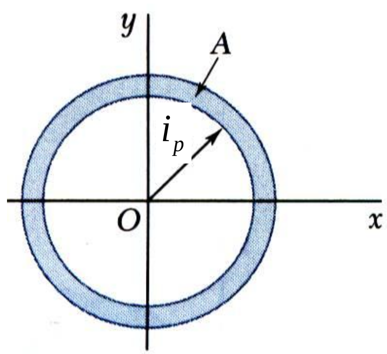
$$I_p = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Radius of Gyration of an Area









- Consider area A with moments of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = i_x^2 A \qquad i_x = \sqrt{\frac{I_x}{A}}$$

$i_x =$ *radius of gyration* with respect to the x axis

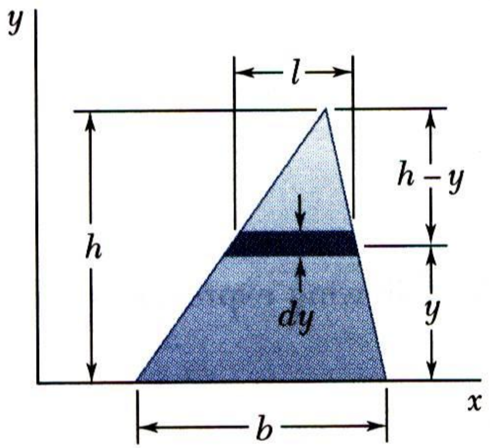
- Similarly,

$$I_y = i_y^2 A \quad i_y = \sqrt{\frac{I_y}{A}}$$

$$I_p = i_p^2 A \quad i_p = \sqrt{\frac{I_p}{A}}$$

$$i_p^2 = i_x^2 + i_y^2$$

Sample Problem



Determine the moments of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

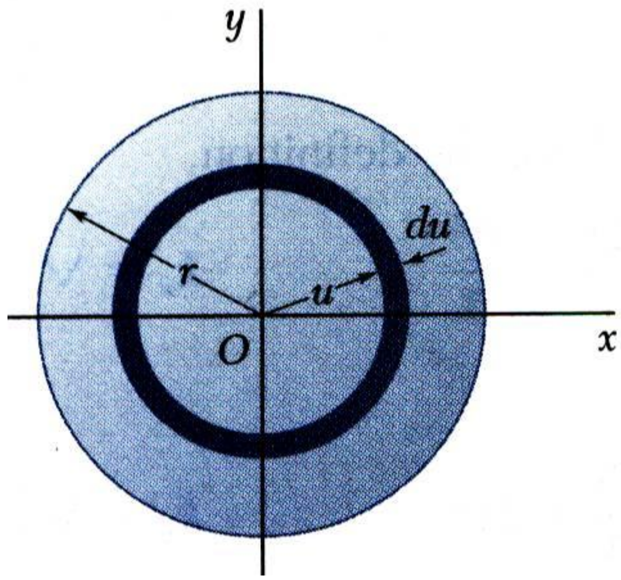
- Integrating dI_x from $y = 0$ to $y = h$,

$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Sample Problem



a) Determine the centroidal polar moments of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moments of inertia of a circular area with respect to a diameter.

SOLUTION:

- An annular differential area element is chosen,

$$dI_p = u^2 dA \qquad dA = 2\pi u\, du$$

$$I_p = \int dI_p = \int_0^r u^2 \left(2\pi u\, du\right) = 2\pi \int_0^r u^3 du$$

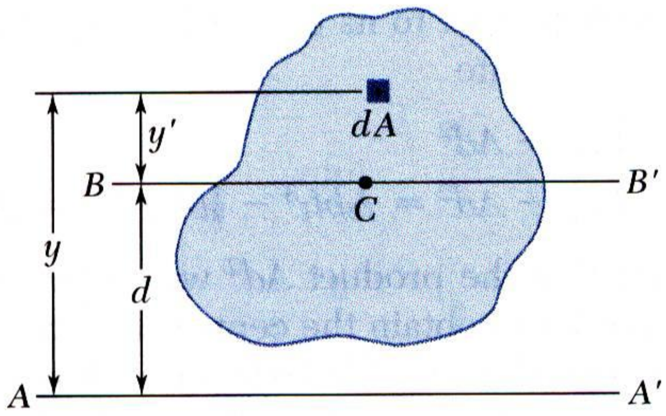
$$\boxed{\Rightarrow I_p = \frac{\pi}{2} r^4}$$

- From symmetry, $I_x = I_y$,

$$I_p = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$\boxed{\Rightarrow I_x = \frac{\pi}{4} r^4 = I_{\text{diameter}}}$$

Parallel Axis Theorem



- Consider moments of inertia I of an area A with respect to the axis AA'

$$I_{AA'} = \int y^2 dA$$

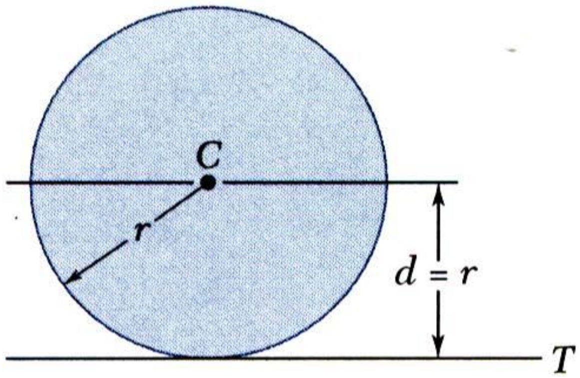
- The axis BB' passes through the area centroid and is called a *centroidal axis.*

$$I_{AA'} = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + \cancel{2d \int y' dA} + d^2 \int dA$$

$$\Rightarrow I_{AA'} = I_{BB'} + Ad^2 = \bar{I} + Ad^2$$

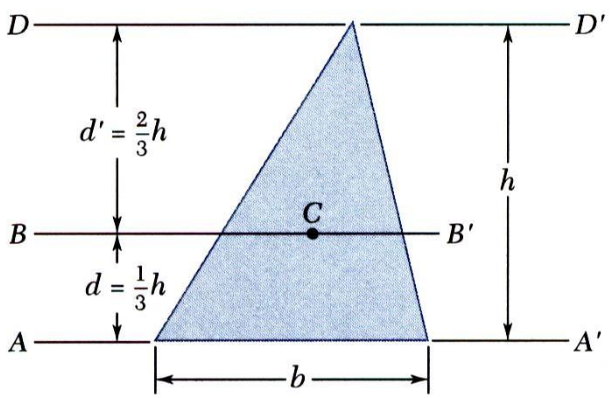
- For a group of parallel axes, the moment of inertia reaches the minimum value when the reference axis is the centroid axis.

Parallel Axis Theorem



- Moments of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$



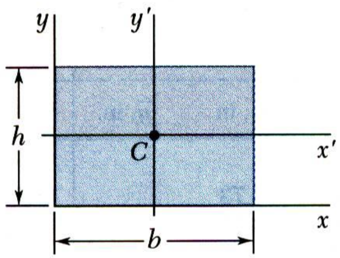
- Moments of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

Moments of Inertia of Common Shapes of Areas

- The moments of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas $A_1, A_2, A_3, \ldots$, with respect to the same axis.

Rectangle



$\bar{I}_{x'} = \frac{1}{12}bh^3$

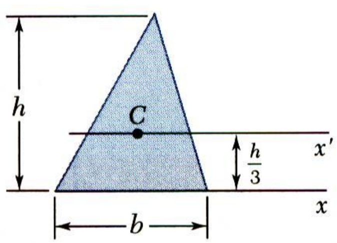
$\bar{I}_{y'} = \frac{1}{12}b^3h$

$I_x = \frac{1}{3}bh^3$

$I_y = \frac{1}{3}b^3h$

$I_p = \frac{1}{12}bh(b^2 + h^2)$

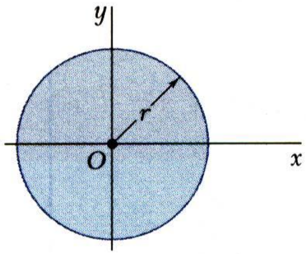
Triangle



$\bar{I}_{x'} = \frac{1}{36}bh^3$

$I_x = \frac{1}{12}bh^3$

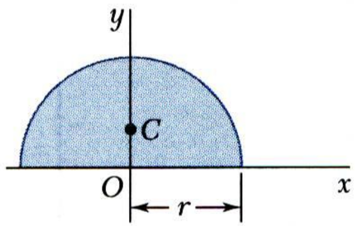
Circle



$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$

$I_p = \frac{1}{2}\pi r^4$

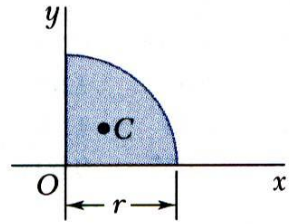
Semicircle



$I_x = I_y = \frac{1}{8}\pi r^4$

$I_p = \frac{1}{4}\pi r^4$

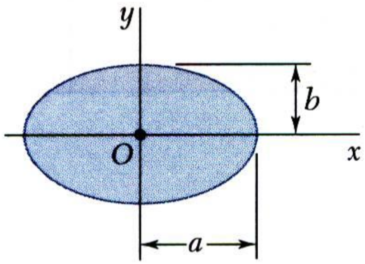
Quarter circle



$I_x = I_y = \frac{1}{16}\pi r^4$

$I_p = \frac{1}{8}\pi r^4$

Ellipse

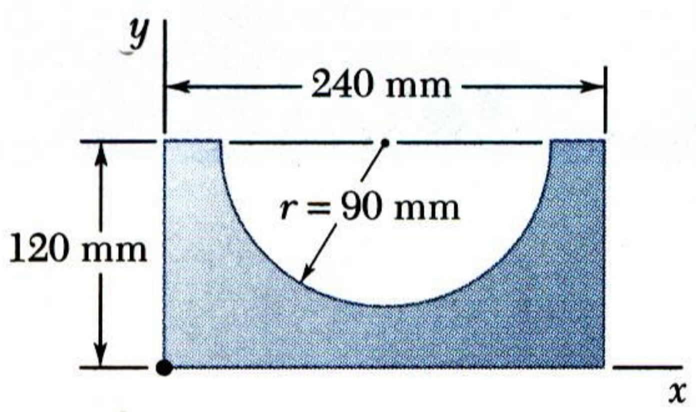


$\bar{I}_x = \frac{1}{4}\pi ab^3$

$\bar{I}_y = \frac{1}{4}\pi a^3b$

$I_p = \frac{1}{4}\pi ab(a^2 + b^2)$

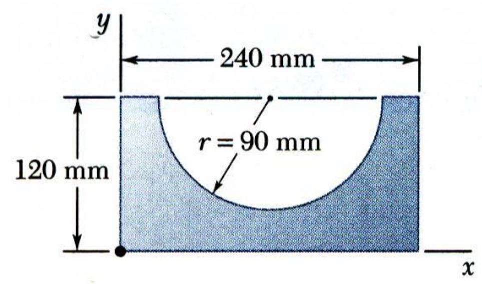
Sample Problem



Determine the moments of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moments of inertia of the shaded area is obtained by subtracting the moments of inertia of the half-circle from the moments of inertia of the rectangle.

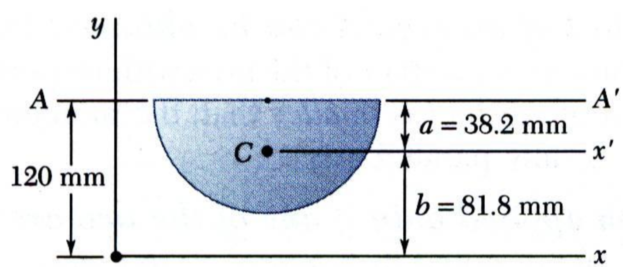


SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2 \times 10^6 \,\mathrm{mm}^4$$



Half-circle:

moments of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76 \times 10^6 \,\mathrm{mm}^4$$

moments of inertia with respect to *x'*,

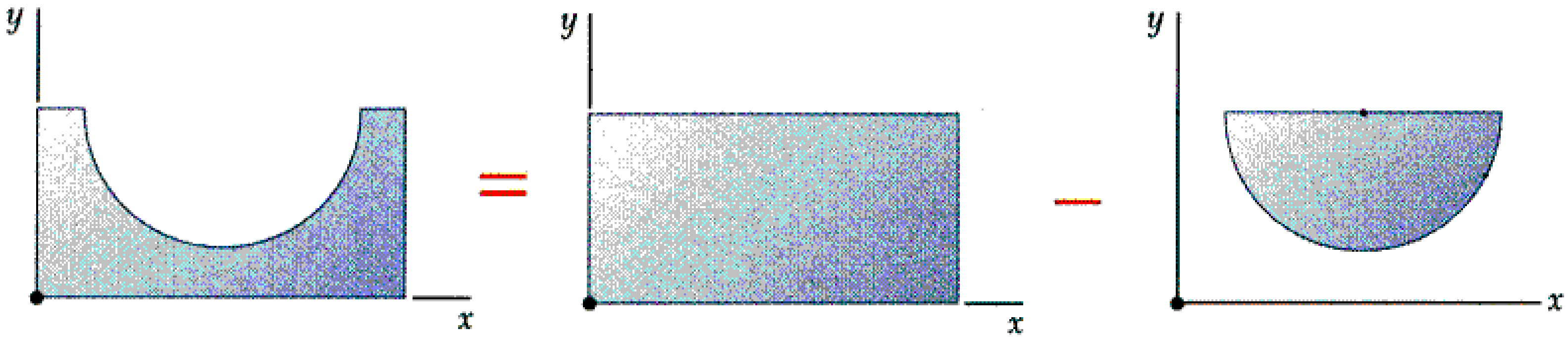
$$\bar{I}_{x'} = I_{AA'} - Aa^2 = 25.76 \times 10^6 - 12.72 \times 10^3 \times 38.2^2$$
$$= 7.20 \times 10^6 \,\mathrm{mm}^4$$

$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2 \,\mathrm{mm}$$
$$\mathrm{b} = 120 - \mathrm{a} = 81.8 \,\mathrm{mm}$$
$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$
$$= 12.72 \times 10^3 \,\mathrm{mm}^2$$

moments of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + \left(12.72 \times 10^3\right)(81.8)^2$$
$$= 92.3 \times 10^6 \,\mathrm{mm}^4$$

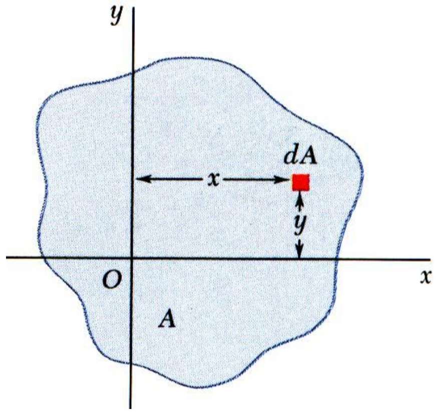
- The moments of inertia of the shaded area is obtained by subtracting the moments of inertia of the half-circle from the moments of inertia of the rectangle.



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

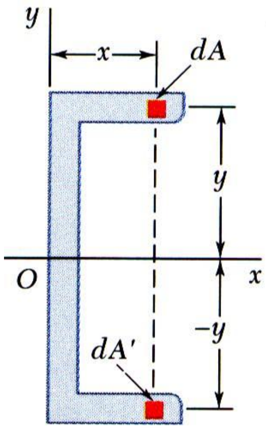
Product of Inertia

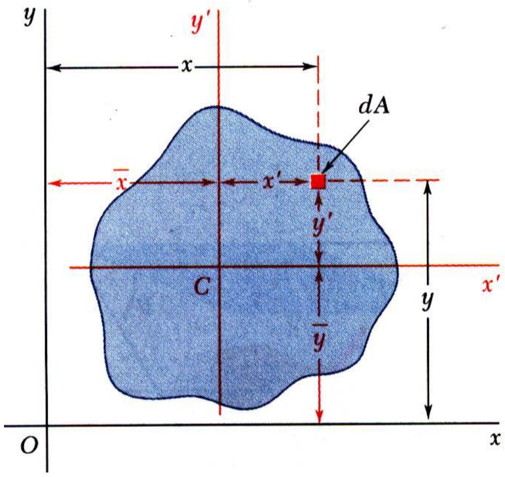


- *Product of Inertia:*

$$I_{xy} = \int xy\, dA$$

- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.

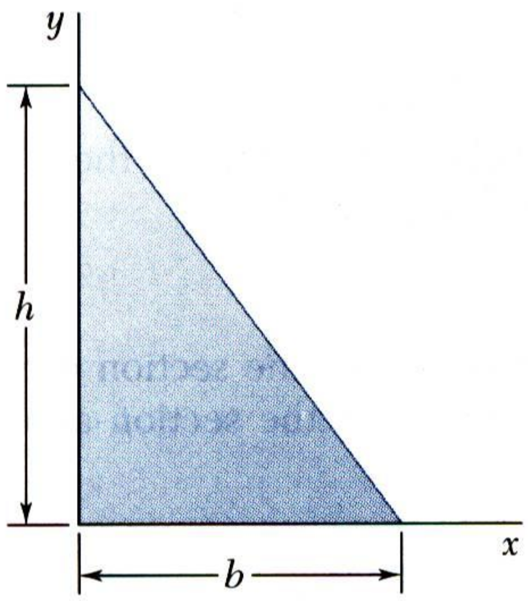




- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$

Sample Problem

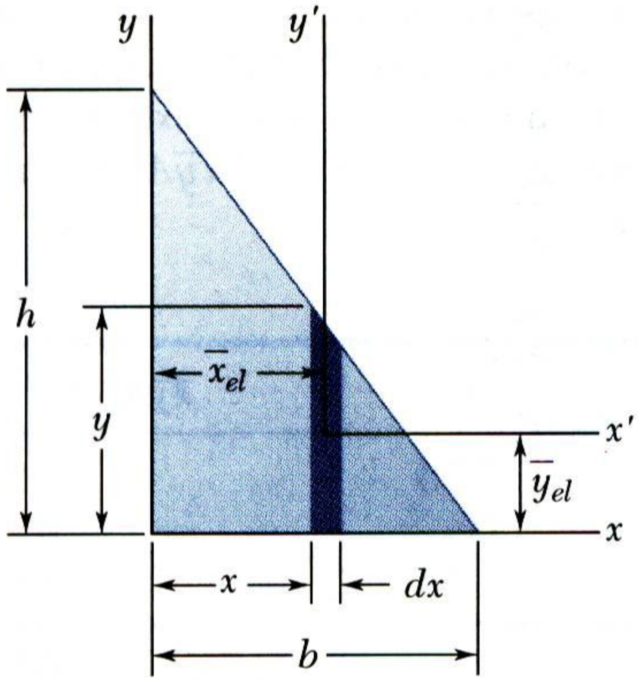


Determine the product of inertia of the right triangle (a) with respect to the *x* and *y* axes and
(b) with respect to centroidal axes parallel to the *x* and *y* axes.

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

SOLUTION:



- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1-\frac{x}{b}\right) \qquad dA = y\,dx = h\left(1-\frac{x}{b}\right)dx$$

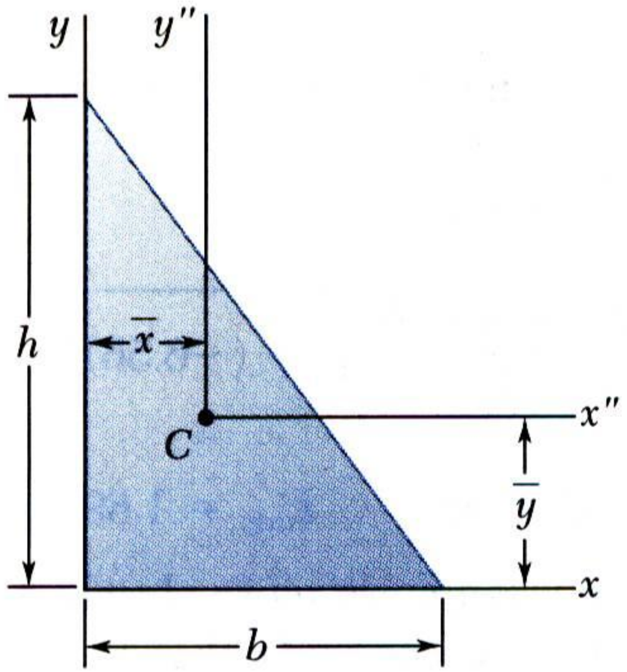
$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1-\frac{x}{b}\right)$$

Integrating dI_{xy} from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}\,dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1-\frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2}-\frac{x^2}{b}+\frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4}-\frac{x^3}{3b}+\frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$



- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \tfrac{1}{3}b \qquad \bar{y} = \tfrac{1}{3}h$$

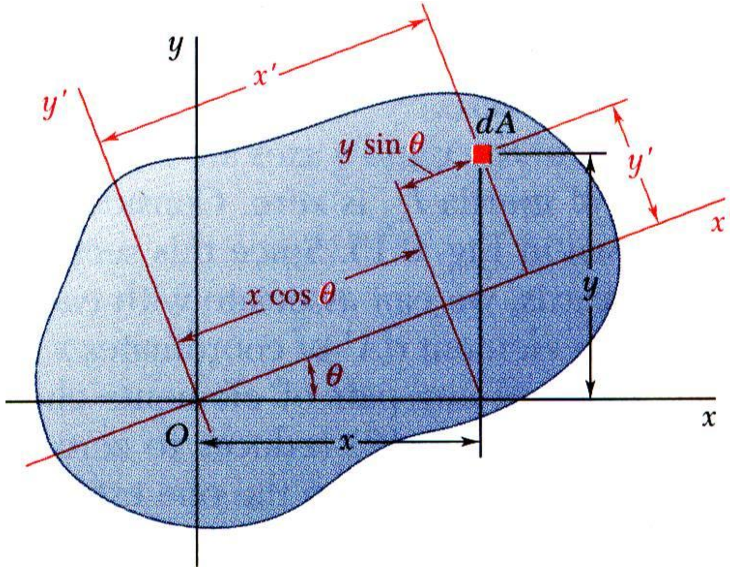
With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

$$\bar{I}_{x''y''} = \tfrac{1}{24}b^2h^2 - \left(\tfrac{1}{3}b\right)\left(\tfrac{1}{3}h\right)\left(\tfrac{1}{2}bh\right)$$

$$\boxed{\bar{I}_{x''y''} = -\tfrac{1}{72}b^2h^2}$$

Principal Axes and Principal Moments of Inertia



Given $I_x = \int y^2 dA \quad I_y = \int x^2 dA$

$$I_{xy} = \int xy\, dA$$

We wish to determine moments and product of inertia with respect to new axes x' and y'.

Note: $x' = x\cos\theta + y\sin\theta$

$$y' = y\cos\theta - x\sin\theta$$

- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

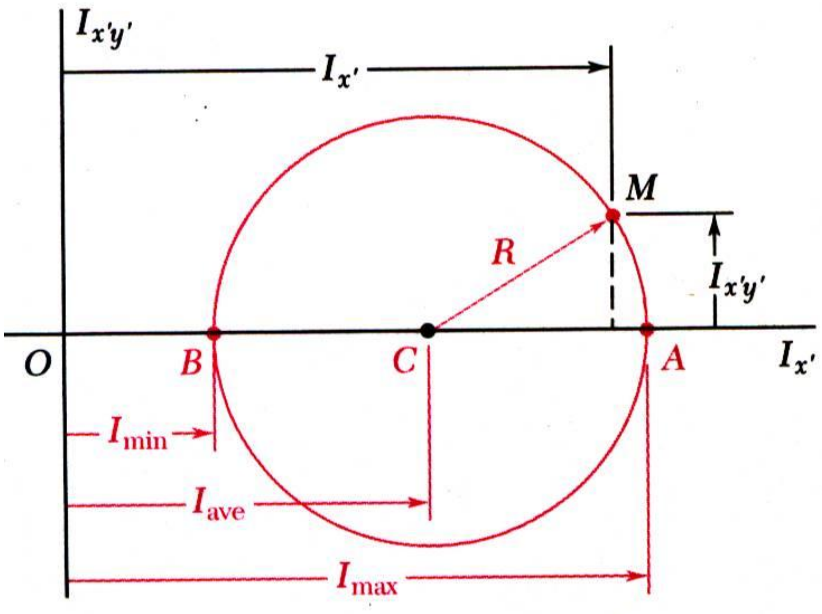
- The equations for $I_{x'}$ and $I_{x'y'}$ are the parametric equations for a circle,

$$\left(I_{x'} - I_{ave}\right)^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- The equations for $I_{y'}$ and $I_{x'y'}$ lead to the same circle.

Principal Axes and Principal Moments of Inertia



$$\left(I_{x'} - I_{ave}\right)^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

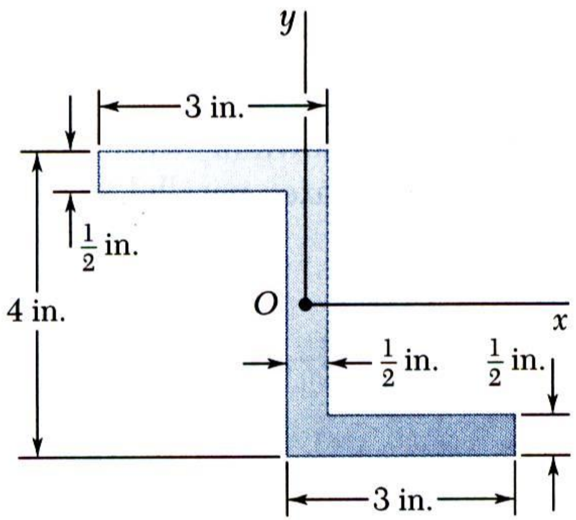
- At the points *A* and *B*, $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

$$\tan 2\theta_m = -2I_{xy} \big/ \left(I_x - I_y\right)$$

- The equation for θ_m defines two angles, 90º apart which correspond to the ***principal axes*** of the area about *O*.
- One method to determine the ***principal moments of inertia*** of the area about O is to substitute these θ_m back into the equation for $I_{x'}$.
- The advantage is that we know which of the two principal angles corresponds to each principal moment of inertia.
- Alternatively, the principal moments of inertia may be determined by

$$I_{\max,\min} = I_{ave} \pm R$$

Sample Problem



For the section shown, the moments of inertia with respect to the *x* and *y* axes are $I_x = 10.38\ \text{in}^4$ and $I_y = 6.97\ \text{in}^4$.

Determine (a) the orientation of the principal axes of the section about *O*, and (b) the values of the principal moments of inertia about *O*.

SOLUTION:

- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles and applying the parallel axis theorem to each.
- Determine the orientation of the principal axes and the principal moments of inertia.

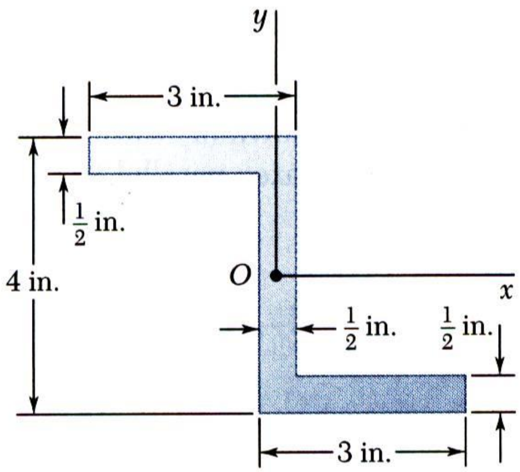
SOLUTION:

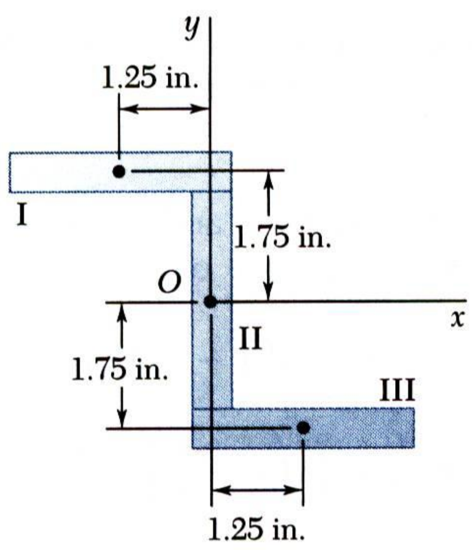
- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles.

  Apply the parallel axis theorem to each rectangle,

  $$I_{xy} = \sum\left(\bar{I}_{x'y'} + \bar{x}\bar{y}A\right)$$

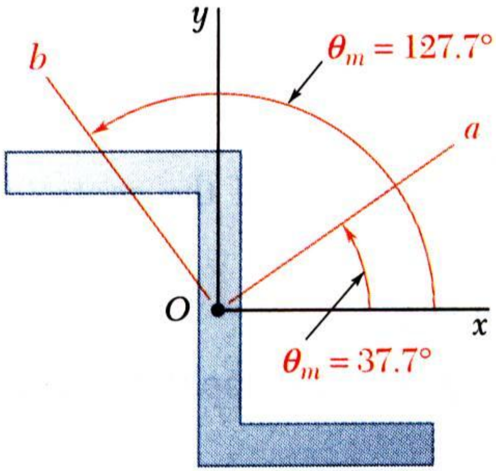
  Note that the product of inertia with respect to centroidal axes parallel to the *xy* axes is zero for each rectangle.





Rectangle	Area, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$, in^4
I	1.5	-1.25	$+1.75$	-3.28
II	1.5	0	0	0
III	1.5	$+1.25$	-1.75	-3.28
				$\sum \bar{x}\bar{y}A = -6.56$

$$\boxed{I_{xy} = \sum \bar{x}\bar{y}A = -6.56\ \text{in}^4}$$



$$I_x = 10.38 \text{ in}^4$$
$$I_y = 6.97 \text{ in}^4$$
$$I_{xy} = -6.56 \text{ in}^4$$

- Determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

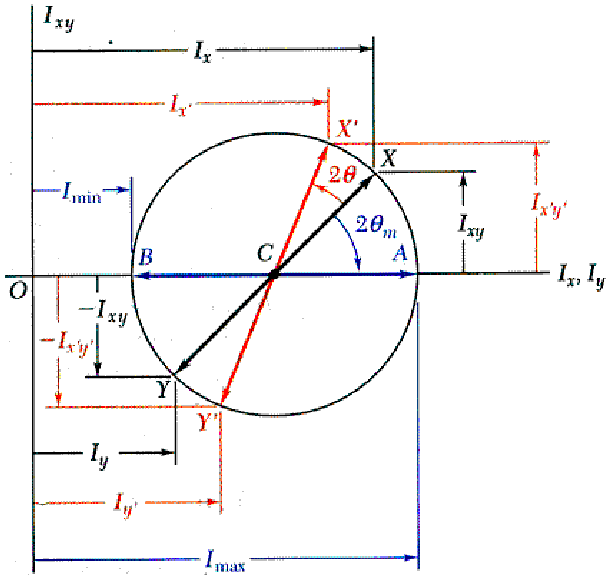
$$2\theta_m = 75.4° \text{ and } 255.4°$$

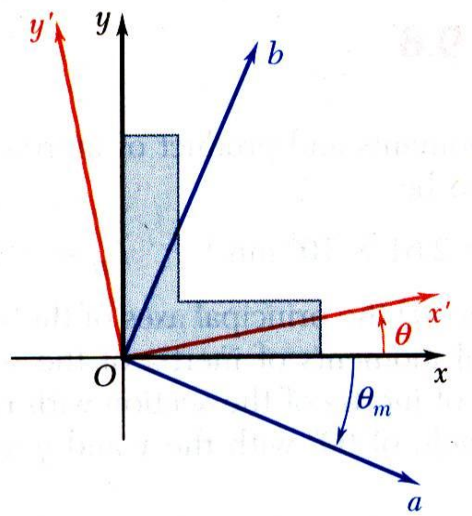
$$\boxed{\theta_m = 37.7° \text{ and } \theta_m = 127.7°}$$

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$
$$= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2}$$

$$\boxed{\begin{aligned} I_a &= I_{\max} = 15.45 \text{ in}^4 \\ I_b &= I_{\min} = 1.897 \text{ in}^4 \end{aligned}}$$

Mohr's Circle for Moments of Inertia



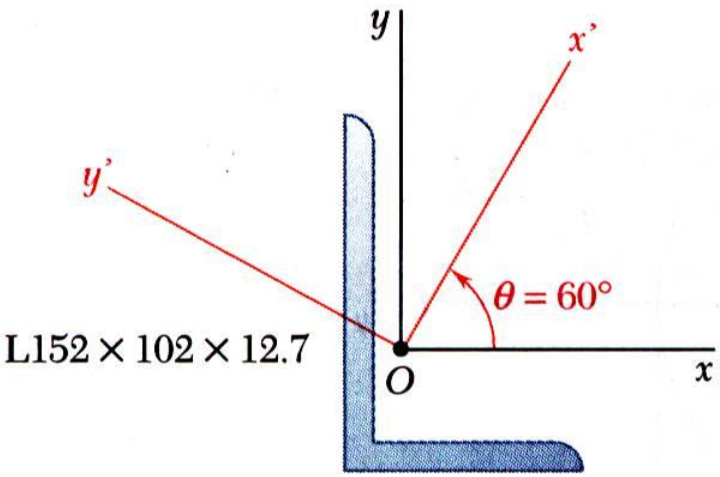


- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle*,

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes.

Sample Problem

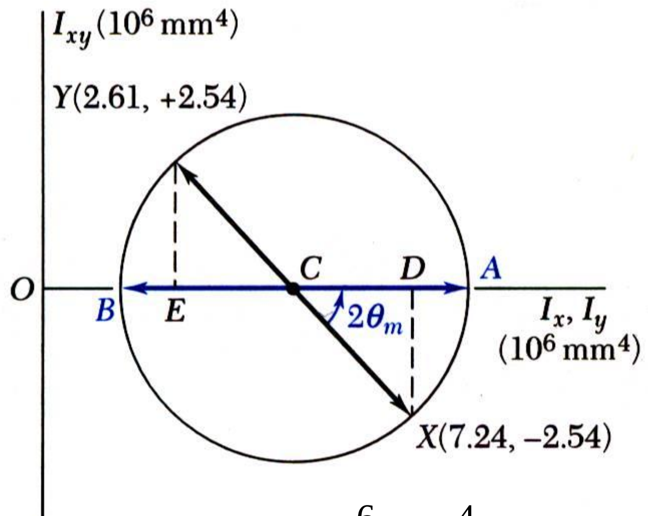


The moments and product of inertia with respect to the x and y axes are I_x = 7.24x106 mm^4, I_y = 2.61x106 mm^4, and I_{xy} = -2.54x10^6 mm^4.

Using Mohr's circle, determine (a) the principal axes about O, (b) the values of the principal moments about O, and (c) the values of the moments and product of inertia about the x' and y' axes

SOLUTION:

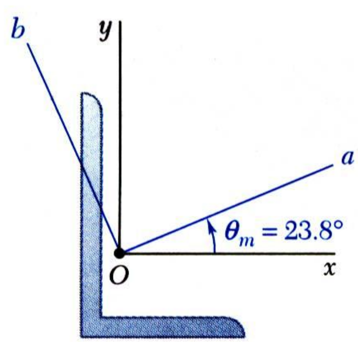
- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.



$$I_x = 7.24 \times 10^6 \text{mm}^4$$

$$I_y = 2.61 \times 10^6 \text{mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \text{mm}^4$$



SOLUTION:

- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \tfrac{1}{2}\left(I_x + I_y\right) = 4.925 \times 10^6 \text{mm}^4$$

$$CD = \tfrac{1}{2}\left(I_x - I_y\right) = 2.315 \times 10^6 \text{mm}^4$$

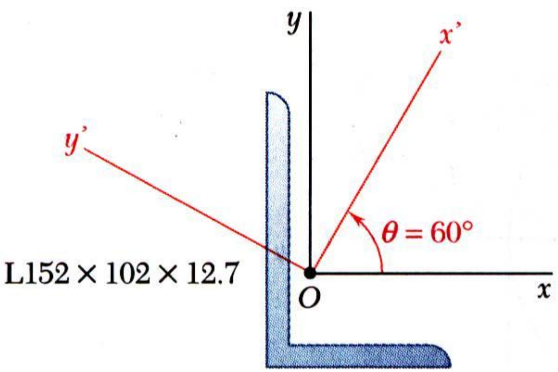
$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{mm}^4$$

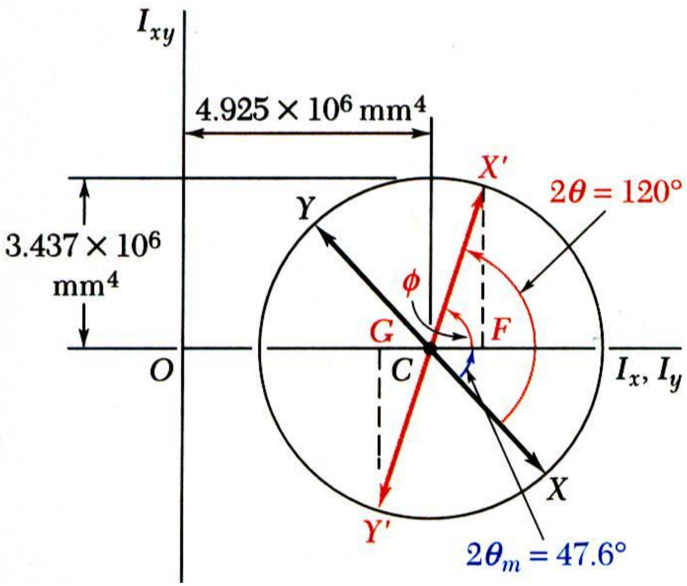
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6° \qquad \boxed{\theta_m = 23.8°}$$

$$I_{max} = OA = I_{ave} + R \qquad \boxed{I_{max} = 8.36 \times 10^6 \text{mm}^4}$$

$$I_{min} = OB = I_{ave} - R \qquad \boxed{I_{min} = 1.49 \times 10^6 \text{mm}^4}$$





$$OC = I_{ave} = 4.925 \times 10^6 \text{mm}^4$$

$$R = 3.437 \times 10^6 \text{mm}^4$$

- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

The points *X'* and *Y'* corresponding to the *x'* and *y'* axes are obtained by rotating *CX* and *CY* counterclockwise through an angle $\theta = 2(60^o) = 120^o$. The angle that *CX'* forms with the *x'* axes is $\phi = 120^o - 47.6^o = 72.4^o$.

$$I_{x'} = OF = OC + CX' \cos\phi = I_{ave} + R\cos 72.4^o$$

$$\boxed{I_{x'} = 5.96 \times 10^6 \text{mm}^4}$$

$$I_{y'} = OG = OC - CY' \cos\phi = I_{ave} - R\cos 72.4^o$$

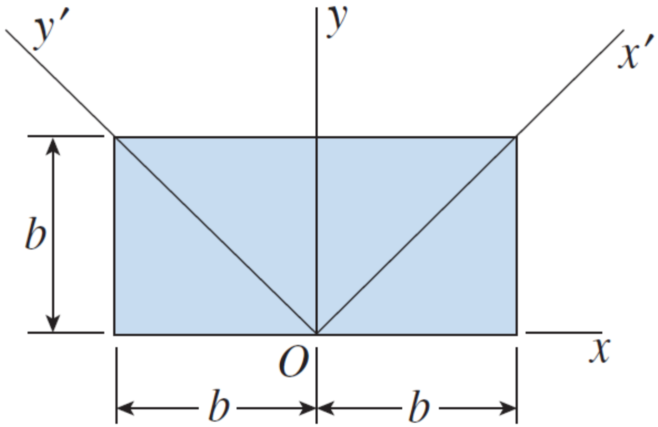
$$\boxed{I_{y'} = 3.89 \times 10^6 \text{mm}^4}$$

$$I_{x'y'} = FX' = CY' \sin\phi = R\sin 72.4^o$$

$$\boxed{I_{x'y'} = 3.28 \times 10^6 \text{mm}^4}$$

Principal Points

- Consider a pair of principal axes with origin at a given point *O*.
- If there exists a *different pair of principal axes* through that same point, then every pair of axes through that point is a set of principal axes and all moments of inertia are the same.



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

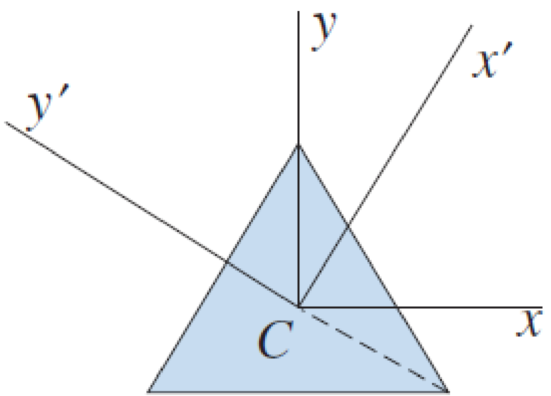
$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

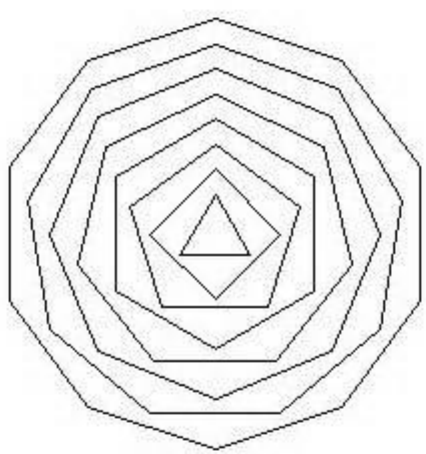
- A point so located that every axis through the point is a principal axis, and hence the moments of inertia are the same for all axes through the point, is called a **principal point.**

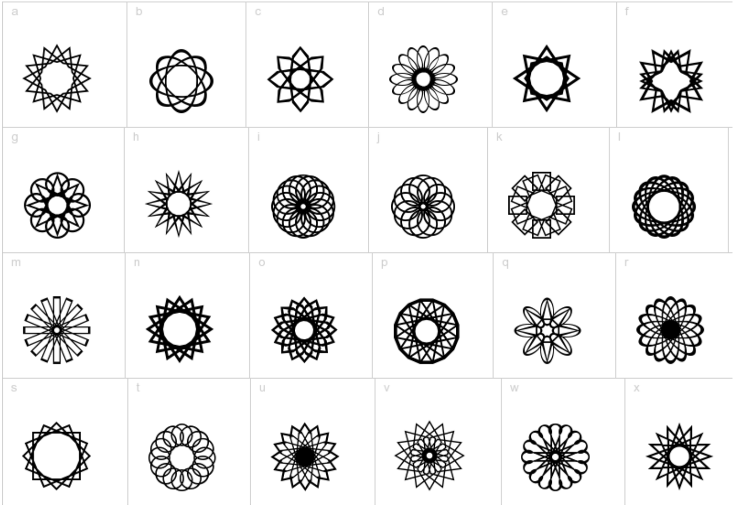
- In general, every plane area has two principal points. These points lie equidistant from the centroid on the principal centroidal axis having the larger principal moment of inertia.

Principal Points

- Apply the concepts described above to axes through the centroid of an area.
- If an area has three or more axes of symmetry, the centroid is a principal point and every axis through the centroid is a principal axis and has the same moment of inertia.
- These conditions are fulfilled for a circle, for all regular polygons (equilateral triangle, square, regular pentagon, regular hexagon, and so on), and for many other symmetric shapes.







- When the two principal centroidal moments of inertia are equal; then the two principal points merge at the centroid, which becomes the sole principal point.