Formulation and Solution Strategies

Outline

- Review of Field Equations
- Types of BCs
- BCs on Coordinate Surfaces
- BCs on Oblique Surface
- Line of Symmetry BCs
- Interface BCs
- Problem Classification
- Stress Formulation
- Displacement Formulation
- Principle of Superposition
- Uniqueness of Elastic Solution
- Saint-Venant’s Principle
- Solution Strategies
- Mathematical Techniques

Governing Equations in linear elasticity

- Strain-displacement relations: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (6 eqns)
- Strain compatibility: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$ (6 eqns)
- Equilibrium: $\sigma_{ij,j} + F_i = 0$ (3 eqns)
- Isotropic Hooke's Law:

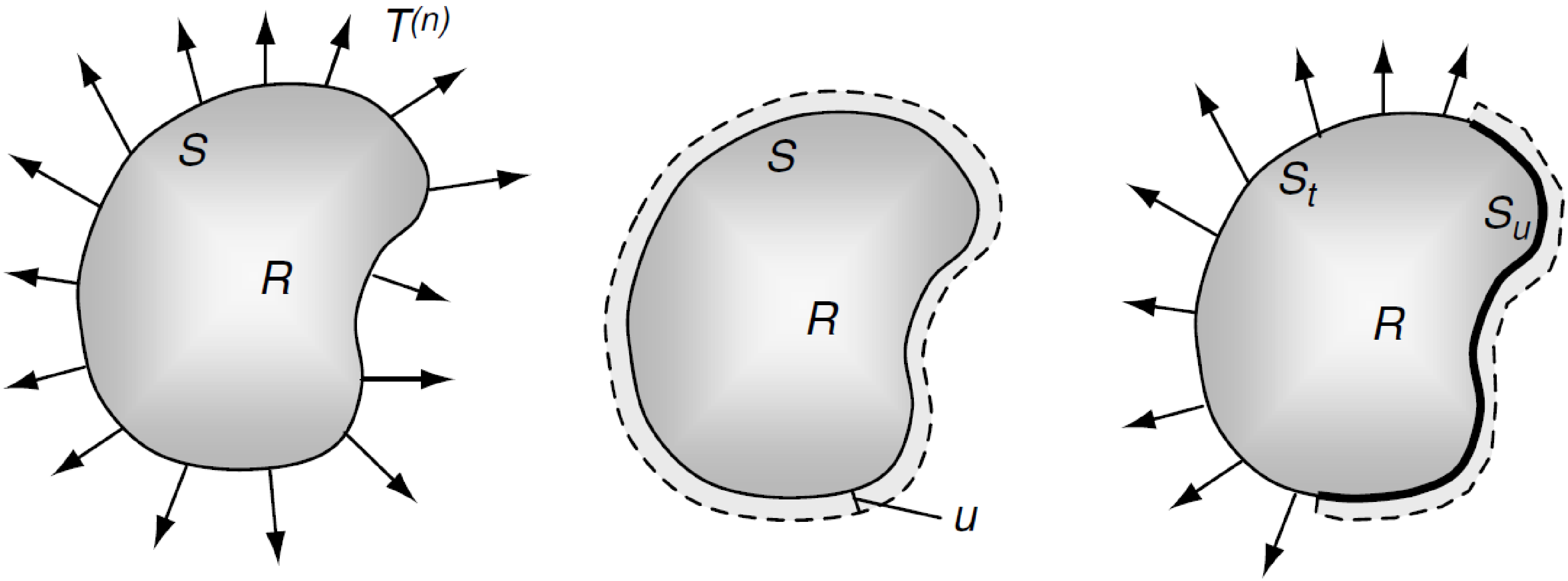
$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}; \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

(6 eqns)

- 15 equations for 15 unknowns (3 displacements, 6 strains, 6 stresses).
- May define the entire system as

$$\boxed{f\{u_i, \varepsilon_{ij}, \sigma_{ij}; \lambda, G, F_i\} = 0}$$

Traction and Displacement Boundary Conditions



Traction Conditions (a) | Displacement Conditions (b) | Mixed Conditions (c)

Boundary Conditions on Coordinate Surfaces

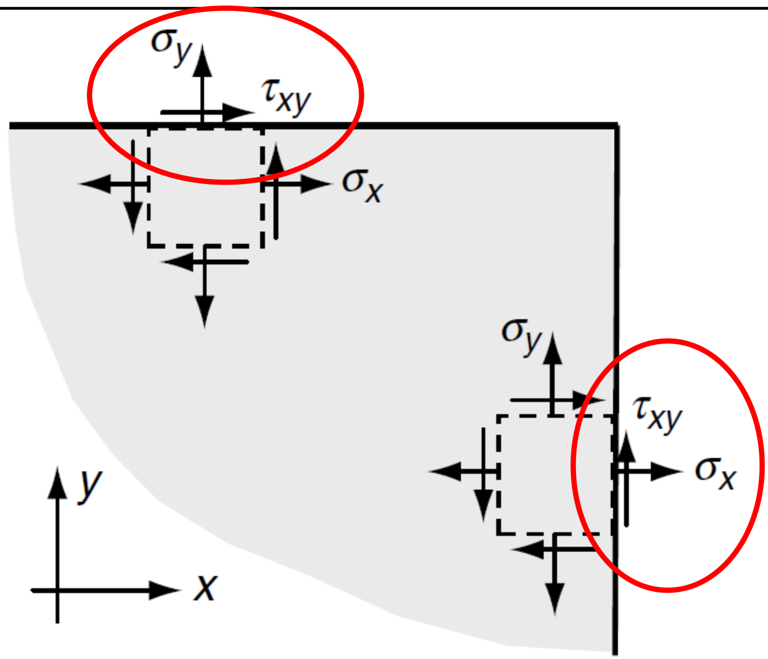
- The traction specification can be reduced to a stress specification.

$$T_x^{(x)} = \sigma_x, T_y^{(x)} = \tau_{xy}. (\sigma_y \text{ is irrelavent.})$$

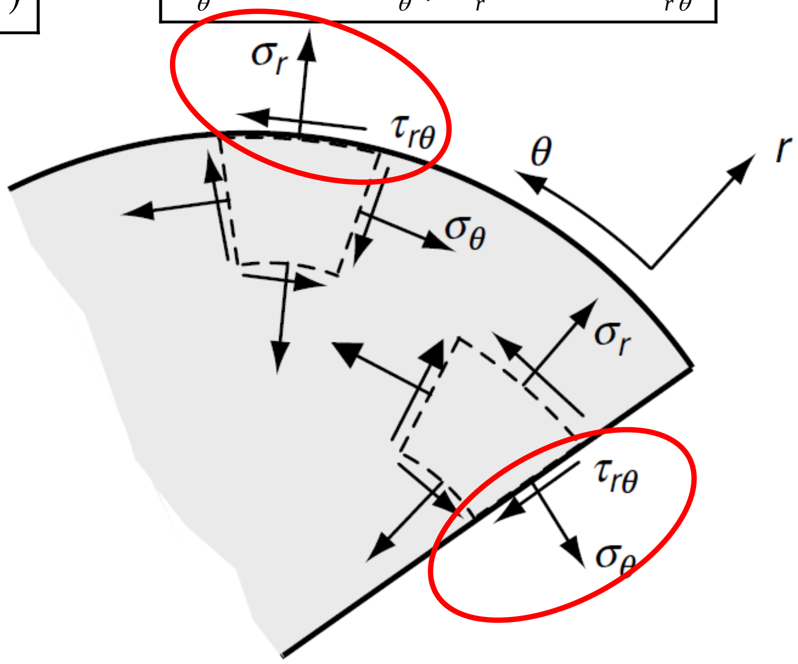
$$T_y^{(y)} = \sigma_y, T_x^{(y)} = \tau_{xy}. (\sigma_x \text{ is irrelavent.})$$

$$T_r^{(r)} = \sigma_r, T_\theta^{(r)} = \tau_{r\theta}$$

$$T_\theta^{(\theta)} = -\sigma_\theta, T_r^{(\theta)} = -\tau_{r\theta}$$



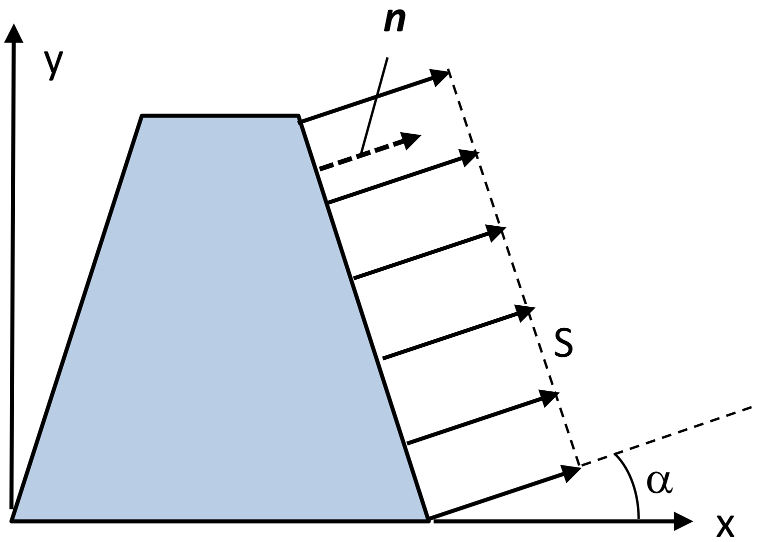
(Cartesian Coordinate Boundaries)

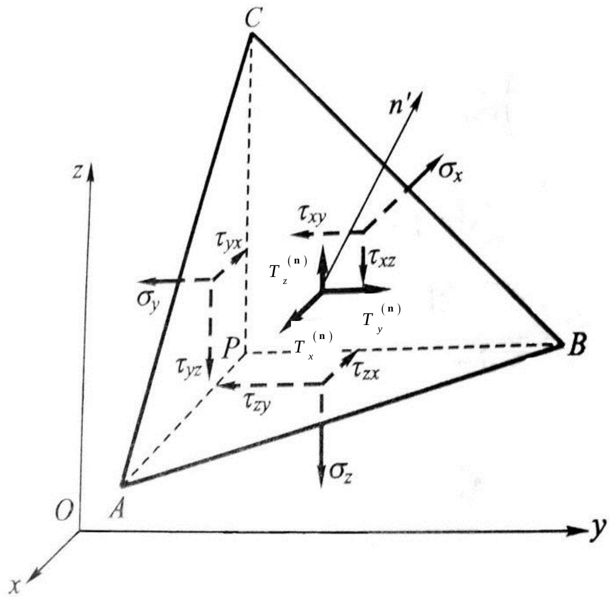


(Polar Coordinate Boundaries)

Boundary Conditions on Oblique Surfaces

- On general non-coordinate surfaces traction vector will not reduce to individual stress components and general form of traction vector must then be used.





$$\sigma_x n_x + \tau_{xy} n_y = T_x^{(\boldsymbol{n})} = S\cos\alpha$$

$$\tau_{xy} n_x + \sigma_y n_y = T_y^{(\boldsymbol{n})} = S\sin\alpha$$

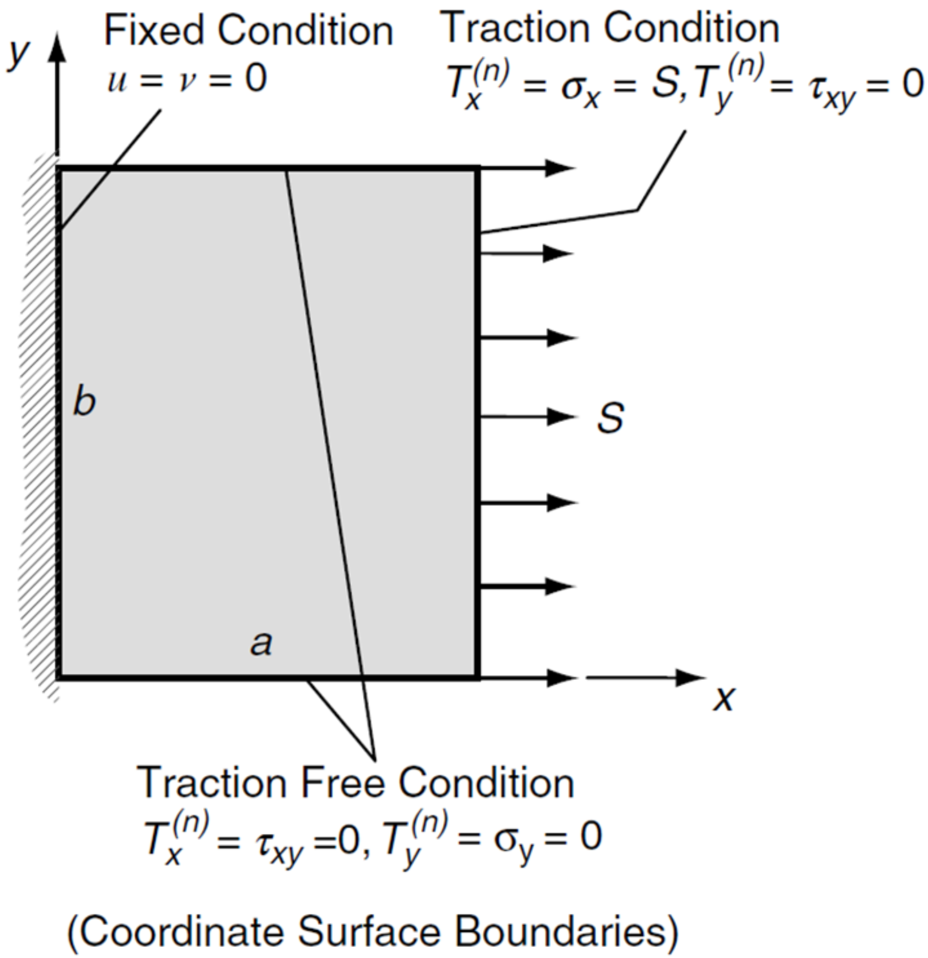
$$\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = T_x^{(\boldsymbol{n})}$$

$$\tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z = T_y^{(\boldsymbol{n})}$$

$$\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = T_z^{(\boldsymbol{n})}$$

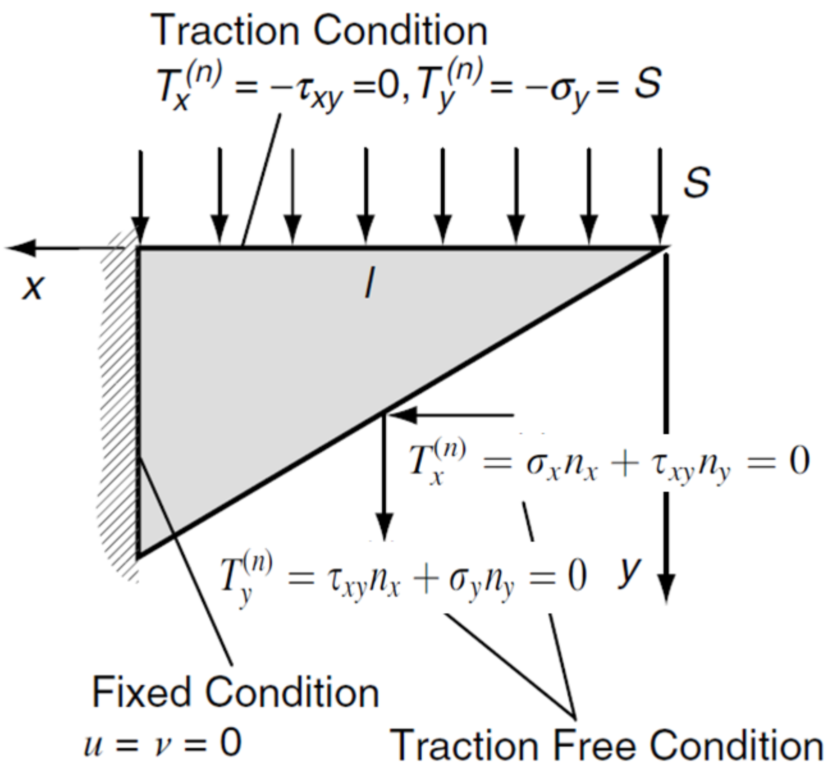
Example of Boundary Conditions

Example (1)



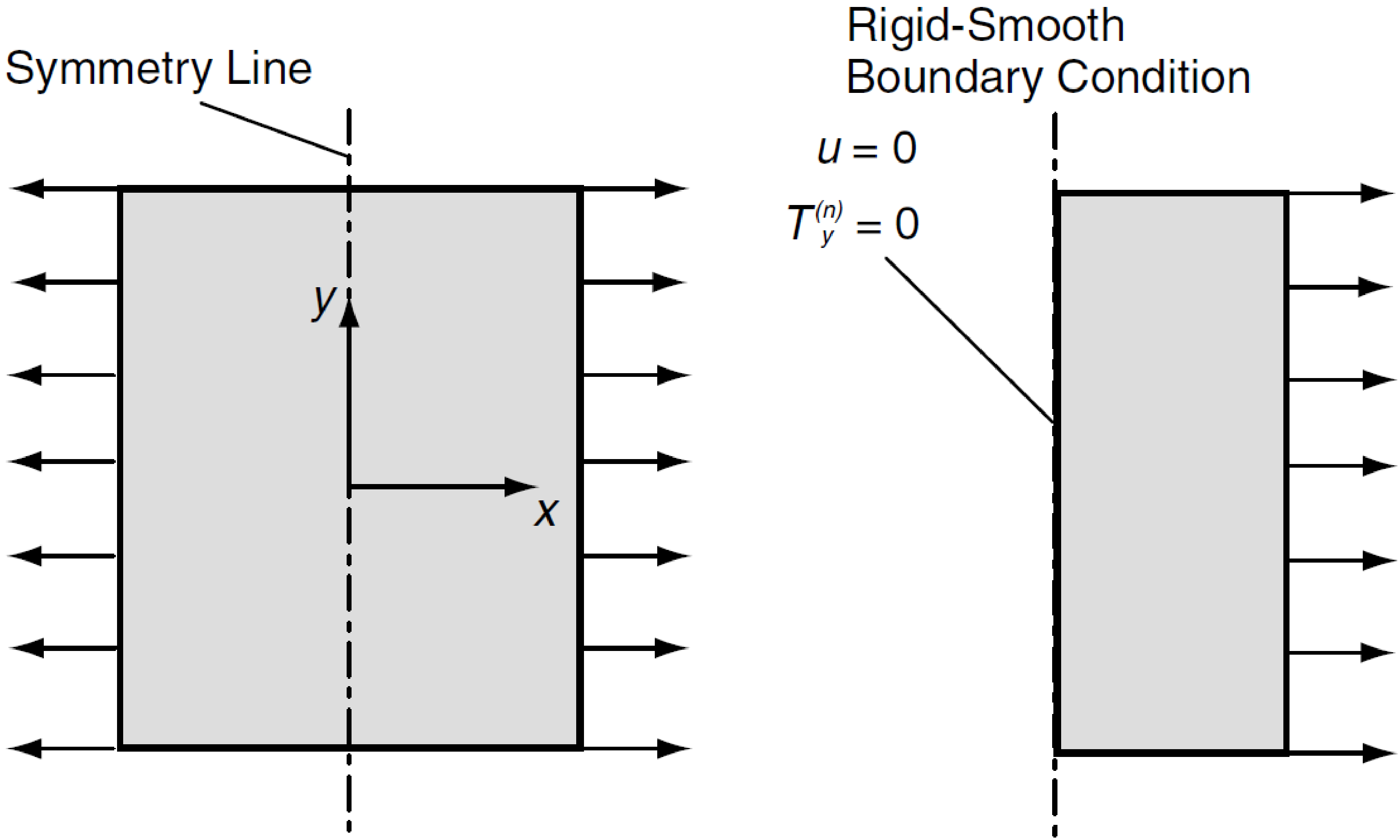
Example of Boundary Conditions

Example (2)



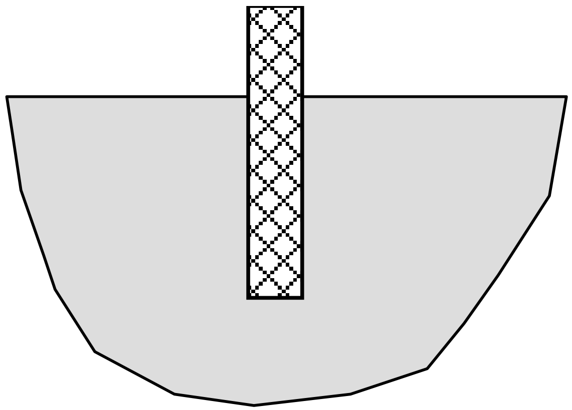
(Non-Coordinate Surface Boundary)

Line of Symmetry Boundary Conditions

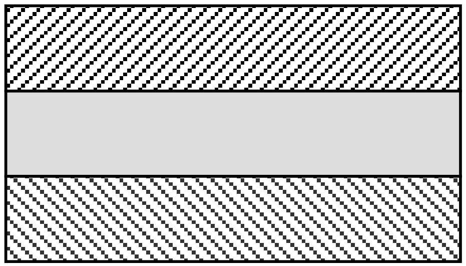


Line of symmetry boundary condition.

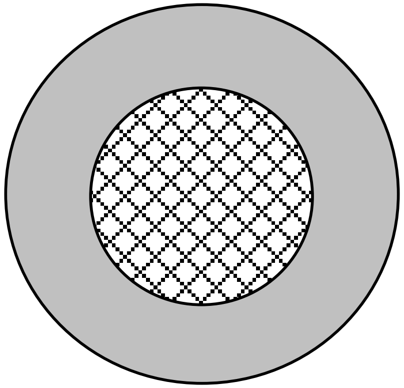
Interface Boundary Conditions



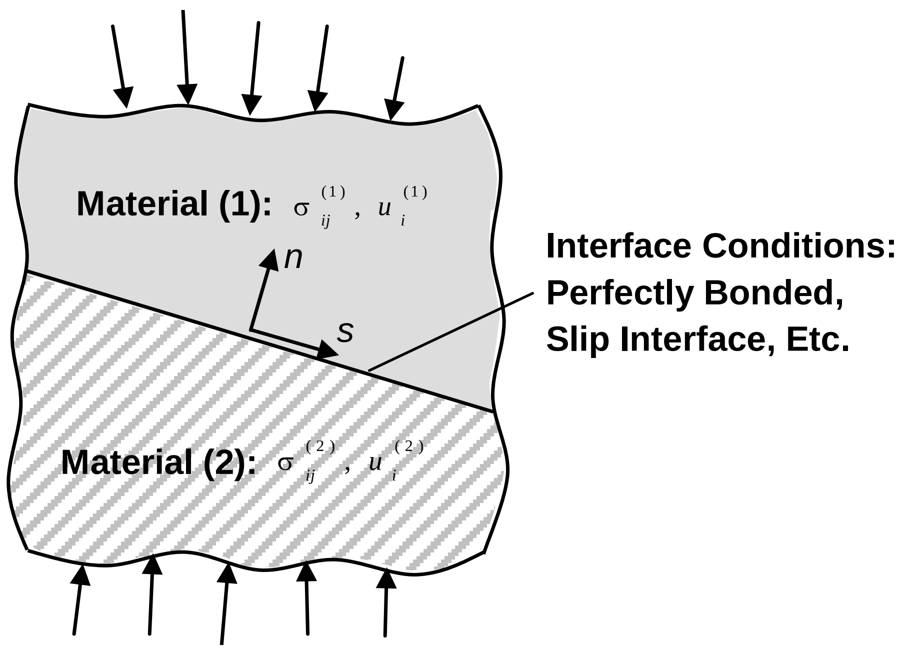
Embedded Fiber or Rod



Layered Composite Plate

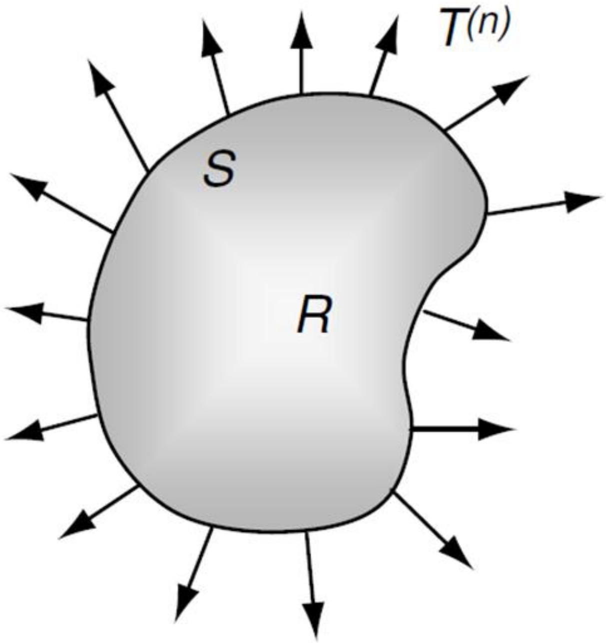


Composite Cylinder or Disk

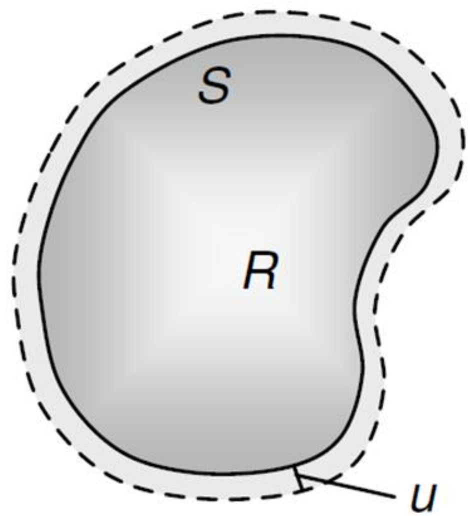


Problem Classification

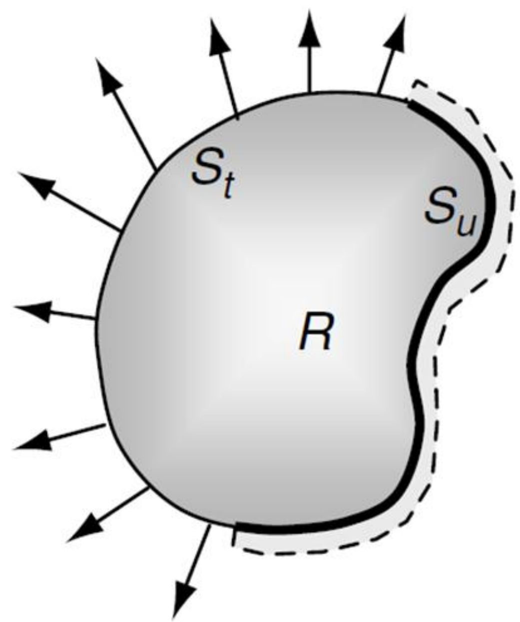
- Goal: determine the distribution of displacements, strains and stresses in the interior of an elastic body under equilibrium when body forces are given.



Traction Problem



Displacement Problem

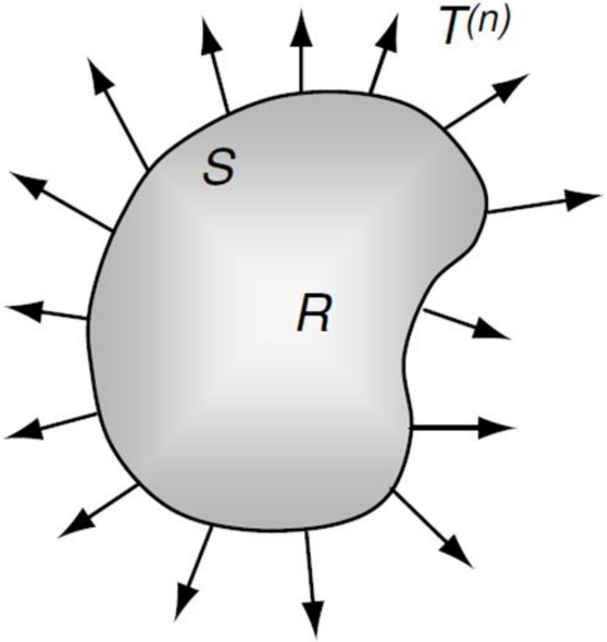


Mixed Problem

$$f\{u_i, \varepsilon_{ij}, \sigma_{ij}; \lambda, G, F_i\} = 0$$ ⟶ Simplification!

Stress Formulation

- Applicable to Traction Problem
- Boundary conditions are given in terms of the tractions or stress components
- Aiming to reformulate the field equations solely in terms of the stresses by eliminating the displacements and strains



Stress Formulation

- Using Hooke's law to rewrite the compatibility in terms of the stresses

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$ (Hooke's Law)

$$\Rightarrow \begin{cases} \varepsilon_{ij,kl} = \dfrac{1+\nu}{E}\sigma_{ij,kl} - \dfrac{\nu}{E}\sigma_{mm,kl}\delta_{ij}, \quad \varepsilon_{kl,ij} = \dfrac{1+\nu}{E}\sigma_{kl,ij} - \dfrac{\nu}{E}\sigma_{mm,ij}\delta_{kl} \\ \varepsilon_{ik,jl} = \dfrac{1+\nu}{E}\sigma_{ik,jl} - \dfrac{\nu}{E}\sigma_{mm,jl}\delta_{ik}, \quad \varepsilon_{jl,ik} = \dfrac{1+\nu}{E}\sigma_{jl,ik} - \dfrac{\nu}{E}\sigma_{mm,ik}\delta_{jl} \end{cases}$$

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$ (Strain compatibility)

$$\Rightarrow \sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} = \frac{\nu}{1+\nu}\left(\sigma_{mm,kl}\delta_{ij} + \sigma_{mm,ij}\delta_{kl} - \sigma_{mm,jl}\delta_{ik} - \sigma_{mm,ik}\delta_{jl}\right)$$

- Recall that, only 6 out of the 81 are meaningful

$$k = l \Rightarrow \boxed{\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} + \sigma_{ik,jk} + \sigma_{jk,ik}}$$

Stress Formulation

- Further simplifying the compatibility using equilibrium equations

$$\sigma_{ik,k} = -F_i \Rightarrow \begin{cases} \sigma_{ik,jk} = -F_{i,j} \\ \sigma_{jk,ik} = -F_{j,i} \end{cases} \Rightarrow \quad \sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} - \left(F_{i,j} + F_{j,i}\right)$$

$$i = j \Rightarrow \sigma_{ii,kk} = -\frac{1+\nu}{1-\nu}F_{i,i}$$

$$\Rightarrow \boxed{\begin{array}{l} \sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}F_{k,k}\delta_{ij} - \left(F_{i,j} + F_{j,i}\right) \\ \nabla^2 \boldsymbol{\sigma} + \frac{1}{1+\nu}\left(\nabla\sigma_{kk}\right)\overleftarrow{\nabla} = -\frac{\nu}{1-\nu}\left(\nabla\cdot\boldsymbol{F}\right)\boldsymbol{I} - \left(\boldsymbol{F}\overleftarrow{\nabla} + \nabla\boldsymbol{F}\right) \end{array}}$$

- Beltrami-Michell Compatibility Equations (Compatibility in terms of stress)

Stress Formulation

- Scalar equations

$$\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2}{\partial x^2}(\sigma_x+\sigma_y+\sigma_z)=-\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)-2\frac{\partial F_x}{\partial x},$$

$$\nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2}{\partial y^2}(\sigma_x+\sigma_y+\sigma_z)=-\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)-2\frac{\partial F_y}{\partial y},$$

$$\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2}{\partial z^2}(\sigma_x+\sigma_y+\sigma_z)=-\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)-2\frac{\partial F_z}{\partial z},$$

$$\nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2}{\partial x\partial y}(\sigma_x+\sigma_y+\sigma_z)=-\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right),$$

$$\nabla^2\tau_{xz}+\frac{1}{1+\nu}\frac{\partial^2}{\partial x\partial z}(\sigma_x+\sigma_y+\sigma_z)=-\left(\frac{\partial F_x}{\partial z}+\frac{\partial F_z}{\partial x}\right),$$

$$\nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2}{\partial y\partial z}(\sigma_x+\sigma_y+\sigma_z)=-\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right).$$

Stress Formulation

- Only 3 are independent, i.e.

$$\frac{\partial^4}{\partial x^2 \partial y^2}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) = (1+\nu)\frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \tau_{xy}}{\partial z} + \frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \tau_{zx}}{\partial y}\right),$$

$$\frac{\partial^4}{\partial y^2 \partial z^2}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) = (1+\nu)\frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \tau_{yz}}{\partial x} + \frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{xy}}{\partial z}\right),$$

$$\frac{\partial^4}{\partial z^2 \partial x^2}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) = (1+\nu)\frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{xy}}{\partial z} + \frac{\partial \tau_{yz}}{\partial x}\right).$$

- 3 equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0.$$

- For 6 stress components

Stress Formulation

- The derivation of strains and displacements

$$\boldsymbol{\sigma} \Longrightarrow \boldsymbol{\varepsilon} \Longrightarrow \boldsymbol{u}$$

Hooke's Law   Strain-displacement relations

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Displacement Formulation

- Rewrite the Hooke's law in terms of displacements.

$$\left.\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right); \varepsilon_{kk} = u_{k,k} \\ \sigma_{ij} &= \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \end{aligned}\right\} \Rightarrow \boxed{\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + G\left(u_{i,j} + u_{j,i}\right)}$$

- Reformulate the equilibrium equations

$$\sigma_{ij,j} + F_i = 0$$

$$\Rightarrow \lambda u_{k,kj}\delta_{ij} + G\left(u_{i,jj} + u_{j,ij}\right) + F_i = 0$$

$$\Rightarrow \boxed{\begin{aligned} & G u_{i,kk} + (\lambda + G) u_{k,ki} + F_i = 0 \\ & G \nabla^2 \boldsymbol{u} + (\lambda + G) \nabla (\nabla \cdot \boldsymbol{u}) + \boldsymbol{F} = 0 \end{aligned}}$$

- Navier's/Lamé's Equations

Displacement Formulation

- Scalar equations

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_x = 0,$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_y = 0,$$

$$G\nabla^2 w + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_z = 0.$$

- 3 equations for 3 displacements

Displacement Formulation

- The derivation of strains and stresses

$$\boldsymbol{\sigma} \Longleftarrow \boldsymbol{\varepsilon} \Longleftarrow \boldsymbol{u}$$

Hooke's Law   Strain-displacement relations

$$\boldsymbol{\sigma} = \lambda \, \mathrm{Tr}(\boldsymbol{\varepsilon}) \boldsymbol{I} + 2G\boldsymbol{\varepsilon}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

$$G = \frac{E}{2(1+\nu)}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} + \nabla \boldsymbol{u}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Summary of Formulations

General Field Equation System
(18 Equations, 15 Unknowns:)

$$f\{u_i, \varepsilon_{ij,} \sigma_{ij}; \lambda, G, F_i\} = 0$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right);$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij};$$

$$\sigma_{ij,j} + F_i = 0;$$

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0.$$

Stress Formulation
(6 Equations, 6 Unknowns)

$$f\{\sigma_{ij}; \nu, F_i\} = 0$$

$$\sigma_{ij,j} + F_i = 0;$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij}$$

$$= -\frac{\nu}{1-\nu} F_{k,k}\delta_{ij} - \left(F_{i,j} + F_{j,i}\right).$$

Displacement Formulation
(3 Equations, 3 Unknowns)

$$f\{u_i, \lambda, G, F_i\} = 0$$

$$G u_{i,kk} + (\lambda + G) u_{k,ki} + F_i = 0$$

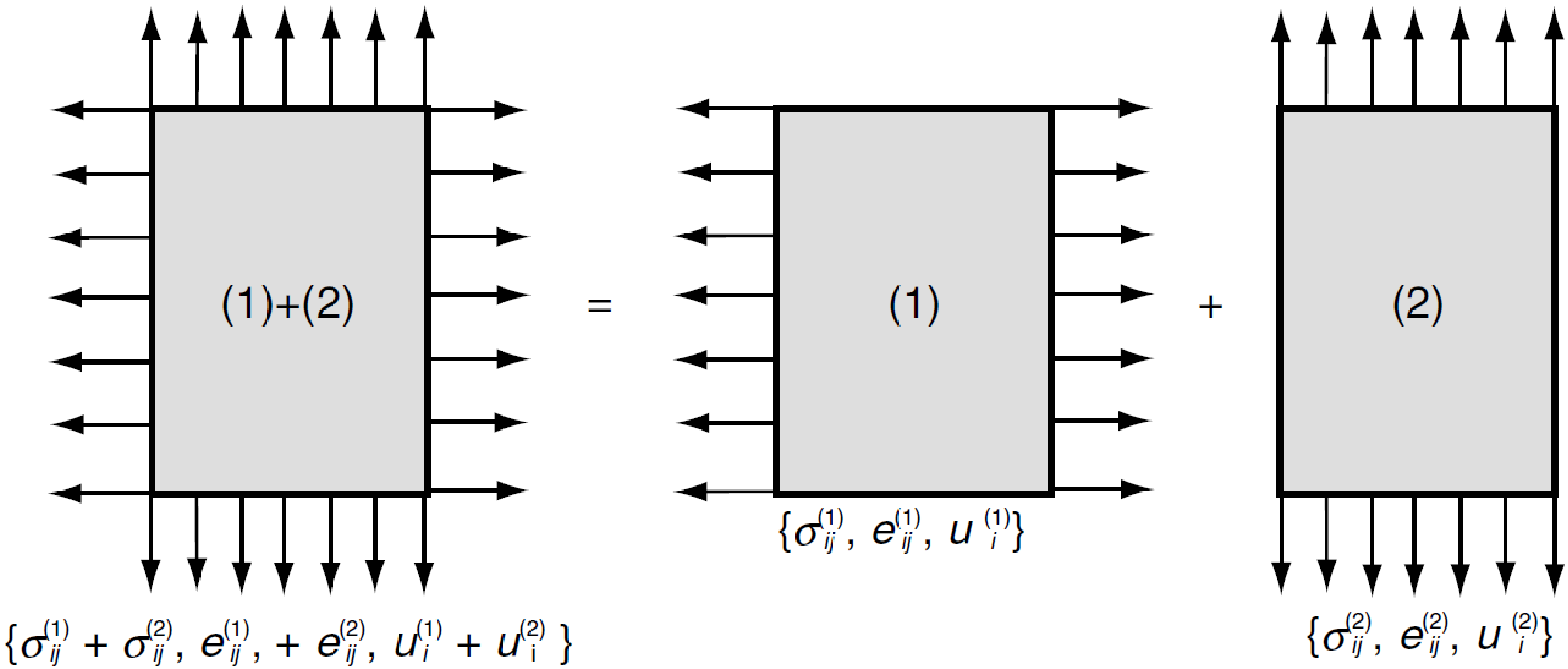
FEM codes!

Principle of Superposition

- Superposition **applies to any problem that is governed by linear equations**.
- Under the assumption of small deformations and linear elastic constitutive behavior, all elasticity field equations are linear.
- The usual displacement and traction boundary conditions are also linear.

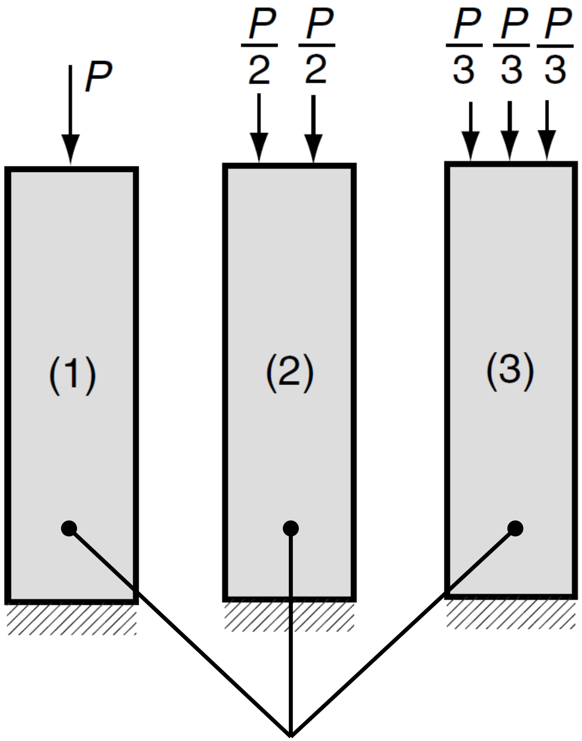
Principle of Superposition: *For a given problem domain, if the state* $\{\sigma_{ij}^{(1)},\ e_{ij}^{(1)},\ u_i^{(1)}\}$ *is a solution to the fundamental elasticity equations with prescribed body forces* $F_i^{(1)}$ *and surface tractions* $T_i^{(1)}$*, and the state* $\{\sigma_{ij}^{(2)},\ e_{ij}^{(2)},\ u_i^{(2)}\}$ *is a solution to the fundamental equations with prescribed body forces* $F_i^{(2)}$ *and surface tractions* $T_i^{(2)}$*, then the state* $\{\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)},\ e_{ij}^{(1)}+e_{ij}^{(2)},\ u_i^{(1)}+u_i^{(2)}\}$ *will be a solution to the problem with body forces* $F_i^{(1)}+F_i^{(2)}$ *and surface tractions* $T_i^{(1)}+T_i^{(2)}$.

Principle of Superposition

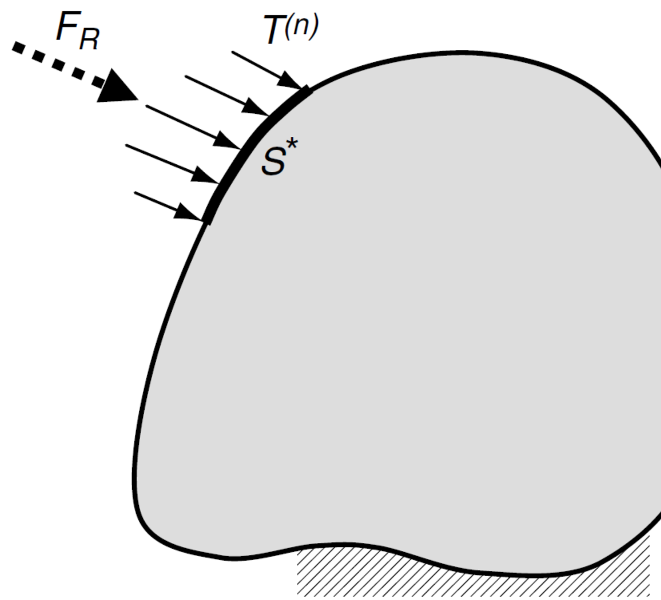


- Must have the same geometry and support type.
- Allow us to solve many more problems by using simpler basic cases whose solutions are already known.

Saint-Venant's Principle



Stresses approximately the same



Boundary loading $\boldsymbol{T}^{(n)}$ would produce detailed and characteristic effects only in the vicinity of S^*. Away from S^* the stresses would generally depend more on the *resultant* $\boldsymbol{F}_R$ of the tractions rather than on the exact distribution

Saint-Venant's Principle

Saint-Venant's Principle: *The stress, strain, and displacement fields caused by two different statically equivalent force distributions on parts of the body far away from the loading points are approximately the same.*

- Allow us to solve a simpler statically equivalent problem with simpler boundary conditions to get a solution (approximate) for the original, more complicated problem.

Solution Strategies – Direct Method

- Seeking the solution of field equations by direct integration.
- Boundary conditions are satisfied exactly.
- Method normally encounters significant mathematical difficulties thus limiting its application to problems with simple geometry.

Solution Strategies – Inverse Method

- Displacements or stresses are selected that satisfy field equations.
- Finding/guessing a solution to the governing equations
- Trying to find a problem whose boundary conditions can be satisfied by the solution
- It is sometimes difficult to construct solutions to a specific problem of practical interest.

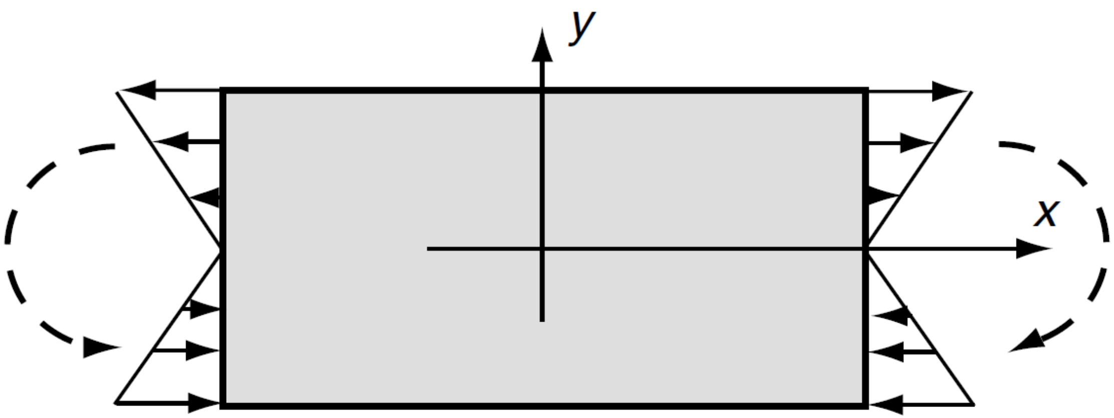
Sample Problem – Pure Bending of a Beam

- Consider the case of an elasticity problem under zero body forces with the following stress field:

$$\sigma_x = Ay, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0.$$

The stress field satisfies the equations of equilibrium and compatibility, and thus the field is a solution to an elasticity problem. What problem would be solved by such a field?

[Hint] Consider some trial domains and investigate the nature of the boundary conditions that would occur using the given stress field.



Pure bending!

Solution Strategies – Semi-Inverse Method

- Part of displacement and/or stress field is specified, while the other remaining portion is determined by the fundamental field equations and the boundary conditions.
- Making an educated guess for part of the solution; then using the governing equations and/or boundary conditions to determine the rest of the solution.
- The usefulness of this approach is greatly enhanced by employing Saint-Venant's principle, whereby a complicated boundary condition can be replaced by a simpler statically equivalent distribution.

Sample Problem – Free Torsion of a Noncircular Shaft

- Based on the torsion problem, We propose the following displacement field:

$$u = -\alpha zy, \quad v = \alpha zx, \quad w = w(x, y).$$

- By using the strain-displacement relations and Hooke’s law, the stress field yields

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0; \quad \tau_{xz} = G\left(\frac{\partial w}{\partial x} - \alpha y\right); \quad \tau_{yz} = G\left(\frac{\partial w}{\partial y} + \alpha x\right).$$

- Substituting these results in equilibrium equations produces

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$ （Navier’s equation）

[Note]

By assuming part of the solution field, the remaining equations to be solved are greatly simplified.

A specific domain in the *x*-*y* plane along with appropriate boundary conditions is needed to complete the solution to a particular problem.

Mathematical Techniques

- Analytical Solution Procedures

- *Power Series Method*
- *Fourier Method*
- *Integral Transform Method*
- *Complex Variable Method*

- Approximate Solution Procedures

- *Ritz Method*
- *Galerkin Method*

- Numerical Solution Procedures

- *Finite Difference Method (FDM)*
- *Finite Element Method (FEM)*
- *Boundary Element Method (BEM)*

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