Strength Theory

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Contents

Strength Condition for Simple Stress States

- Tensile/compressive stress:

$$\sigma_{\max} = \left(\frac{F_N}{A}\right)_{\max} \le [\sigma]$$

- Torsional shearing stress:

$$\tau_{\max} = \left(\frac{T}{W_P}\right)_{\max} \le [\tau]$$

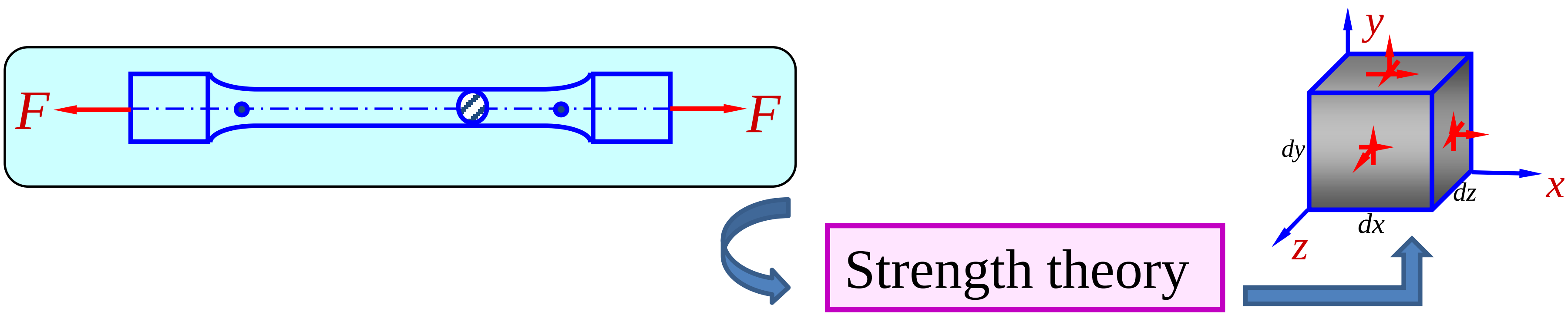
- Bending normal stress:

$$\sigma_{\max} = \left(\frac{M_Z}{W_z}\right)_{\max} \le [\sigma]$$

- Bending shearing stress:

$$\tau_{\max} = \left(\frac{F_S S_Z^*}{I_z b}\right)_{\max} \le [\tau]$$

- Strength condition at locations under complicated stress state:



- Preliminary: find the three principal stress components σ_a, σ_b and σ_c.

Fracture Criteria for Brittle Materials

- Brittle materials fail suddenly in tensile tests. The failure condition is characterized by the ultimate strength σ_U.

1. Maximum tensile stress criteria

Failure criteria: $\sigma_1 = \sigma_U$

Strength condition: $\sigma_1 \leq \sigma_U$

- Accurate for materials primarily under tensile stress state.
- No influences from σ_2, σ_3 are taken into consideration.
- σ_1 must be tensile stress.

2. Maximum tensile strain criteria

Failure criteria: $\varepsilon_1 = \varepsilon_U$

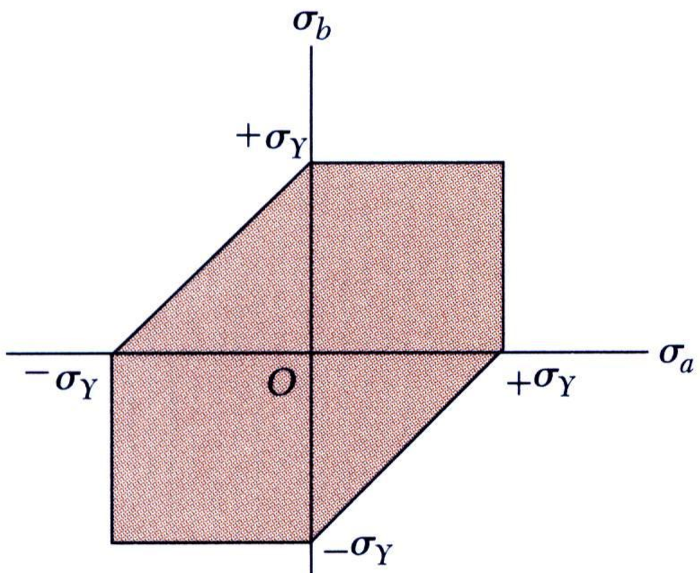
Strength condition:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\nu}{E}\left(\sigma_2 + \sigma_3\right) \leq \varepsilon_U$$

$$\Rightarrow \sigma_1 - \nu\left(\sigma_2 + \sigma_3\right) \leq \sigma_U$$

- Only applicable to linearly elastic brittle materials (until failure)
- Only accurate for few brittle materials

Yield Criteria for Ductile Materials



- Conservative criteria and widely adopted.
- No influences from σ_2 is taken into consideration.

3. Maximum shearing stress (Tresca) criteria

1D at yielding: $\tau_{\max} = \tau_Y = \sigma_Y / 2$

$2D(\sigma_a \neq 0, \sigma_b \neq 0, \sigma_c = 0)$:

For σ_a and σ_b with the same sign,

$$\tau_{\max} = \max\left(\frac{|\sigma_a|}{2}, \frac{|\sigma_b|}{2}\right) = \frac{\sigma_Y}{2}; \ \max\left(|\sigma_a|, |\sigma_b|\right) \leq \sigma_Y$$

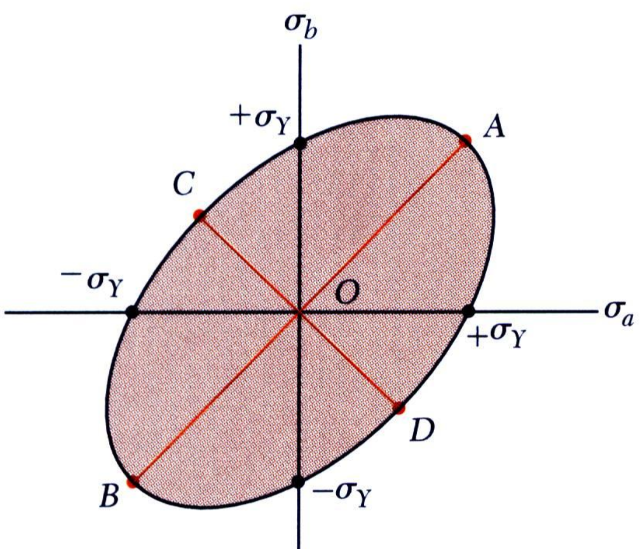
For σ_a and σ_b with opposite signs,

$$\tau_{\max} = \frac{|\sigma_a - \sigma_b|}{2} = \frac{\sigma_Y}{2}; \ |\sigma_a - \sigma_b| \leq \sigma_Y$$

$3D(\sigma_a \neq 0, \sigma_b \neq 0, \sigma_c \neq 0)$:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_Y}{2}; \ \sigma_1 - \sigma_3 \leq \sigma_Y$$

Yield Criteria for Ductile Materials



4. Maximum distortion energy (von Mises) criteria:

$$u_d = \frac{(1+\nu)}{6E}\left[(\sigma_a-\sigma_b)^2+(\sigma_b-\sigma_c)^2+(\sigma_c-\sigma_a)^2\right]$$

1D at yielding $(\sigma_a=\sigma_Y, \sigma_b=\sigma_c=0)$:

$$\Rightarrow u_Y = \frac{(1+\nu)}{3E}\sigma_Y^2$$

$2D(\sigma_a \neq 0, \sigma_b \neq 0, \sigma_c = 0)$:

$$\Rightarrow u_d = \frac{(1+\nu)}{3E}\left[\sigma_a^2-\sigma_a\sigma_b+\sigma_b^2\right]$$

$$u_d = u_Y \quad \Rightarrow \sigma_a^2-\sigma_a\sigma_b+\sigma_b^2=\sigma_Y^2$$

$3D(\sigma_a \neq 0, \sigma_b \neq 0, \sigma_c \neq 0)$: $\quad u_d = u_Y \quad \Rightarrow$

$$\sqrt{\frac{1}{2}\left[(\sigma_a-\sigma_b)^2+(\sigma_b-\sigma_c)^2+(\sigma_c-\sigma_a)^2\right]} \leq \sigma_Y$$

- Economic criteria.
- All three principal stress components are taken into consideration.

Summary of Four Strength Theories

All of the four types of strength theory can be written in a universal form, in terms of an effective stress σ_r:

1. Maximum tensile stress criteria: $\sigma_{r1} = \sigma_1 \leq [\sigma]$

2. Maximum tensile strain criteria: $\sigma_{r2} = \sigma_1 - \nu(\sigma_2 + \sigma_3) \leq [\sigma]$

3. Maximum shearing stress：$\sigma_{r3} = \sigma_1 - \sigma_3 \leq [\sigma]$

4. Maximum distortion energy criteria:

$$\sigma_{r4} = \sqrt{\frac{1}{2}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right]} \leq [\sigma]$$

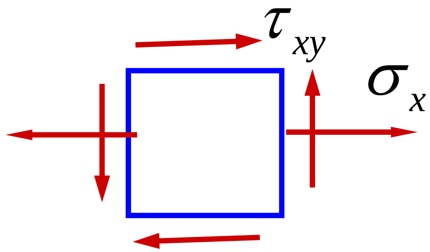
Note: the limit stress has been replaced by the allowable stress.

Remarks on Strength Theory

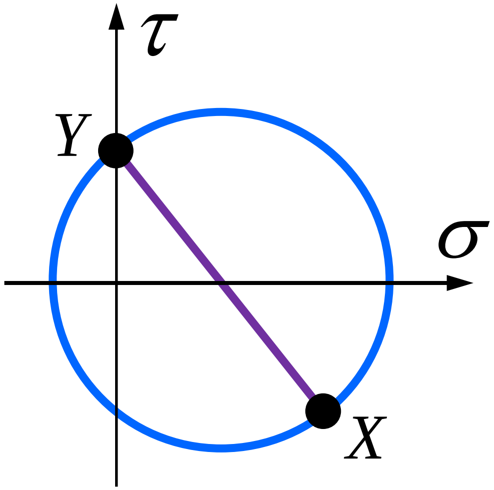
- Failure mechanism depends on not only mechanical behavior but also the stress states.
- Under most stress states, select the maximum tensile stress / strain criteria for brittle materials and the maximum shearing stress / distortion energy criteria for ductile materials.
- Experiments demonstrate: all materials fail by rapture under tri-axial tensile stress state and yielding under tri-axial compressive stress state. Hence, brittle or ductile failure criteria should be selected accordingly.
- For dangerous points under uni-axial stress state, maximum tensile stress criteria is typically selected.
- For dangerous points under pure shearing stress state, maximum shearing stress criteria is typically selected.

Sample Problem

- For the plane stress state shown, analyze the strength condition based on the four types of strength theory.



- Solution



$$\left.\begin{matrix}\sigma_a \\ \sigma_b\end{matrix}\right\} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \left.\begin{matrix}\sigma_a \\ \sigma_b\end{matrix}\right\} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2+4\tau^2}$$

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2+4\tau^2},\ \sigma_2 = 0,\ \sigma_3 = \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2+4\tau^2}$$

1. Maximum tensile stress criteria:

$$\sigma_{r1} = \sigma_1 \le [\sigma] \Rightarrow \sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2+4\tau^2} \le [\sigma]$$

2. Maximum tensile strain criteria:

$$\sigma_{r2} = \sigma_1 - \nu(\sigma_2 + \sigma_3) \le [\sigma]$$

$$\Rightarrow \sigma_{r2} = \frac{(1-\nu)}{2}\sigma + \frac{(1+\nu)}{2}\sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

3. Maximum shearing stress：

$$\sigma_{r3} = \sigma_1 - \sigma_3 \le [\sigma] \quad \Rightarrow \sigma_{r3} = \sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

4. Maximum distortion energy criteria:

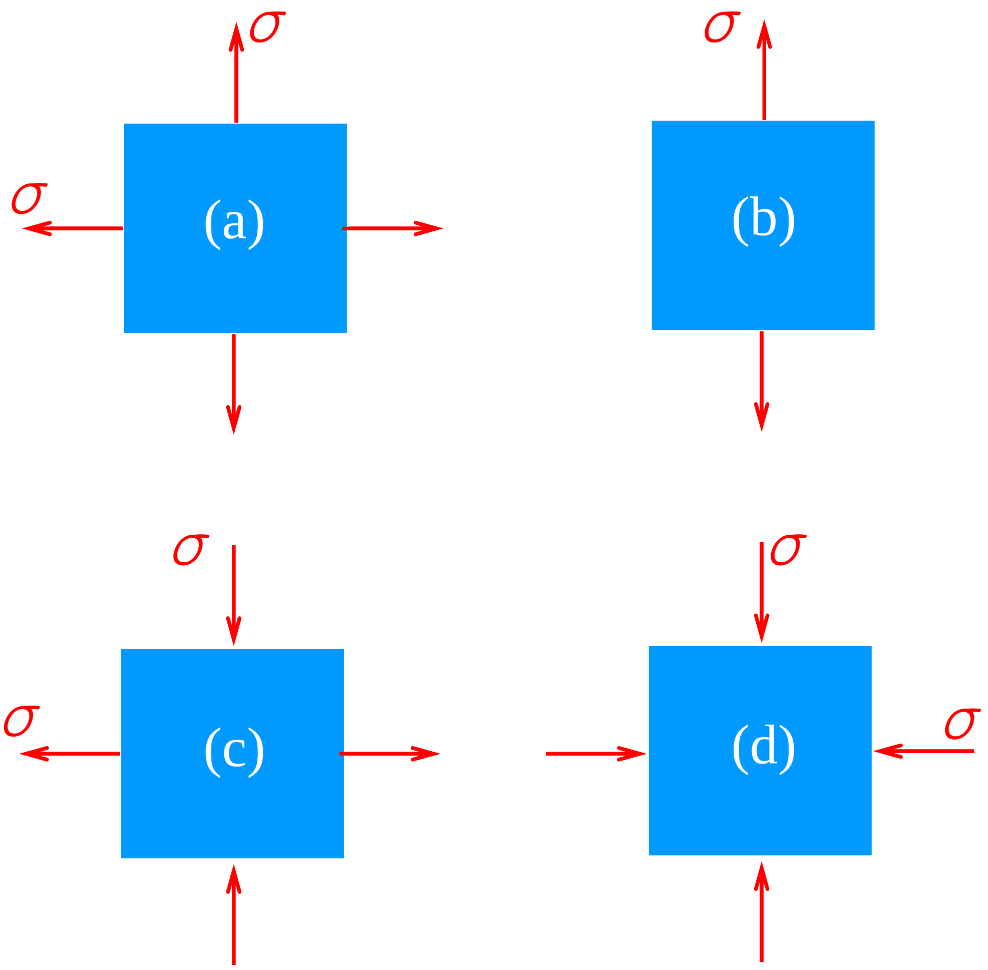
$$\sigma_{r4} = \sqrt{\frac{1}{2}\left[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2\right]} \le [\sigma]$$

$$\Rightarrow \sigma_{r4} = \sqrt{\frac{1}{2}\left[\left(\frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}\right)^2 + \left(\frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}\right)^2 + \left(\sqrt{\sigma^2 + 4\tau^2}\right)^2\right]}$$

$$\Rightarrow \sigma_{r4} = \sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

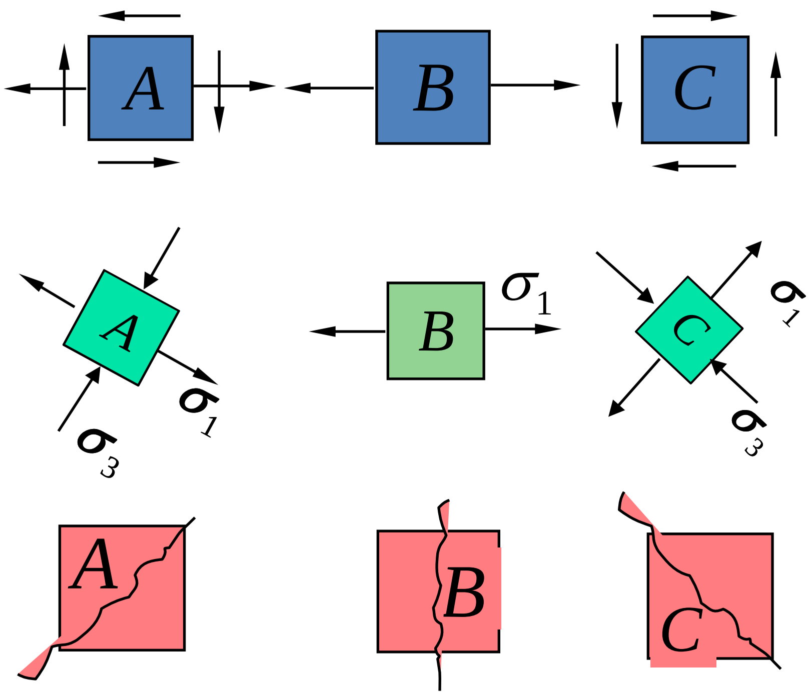
Exercise

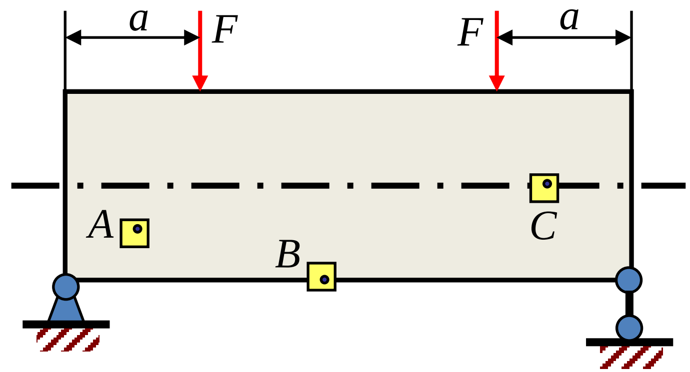
- Based on the maximum shearing stress criteria, which one of the following stress states is most dangerous?

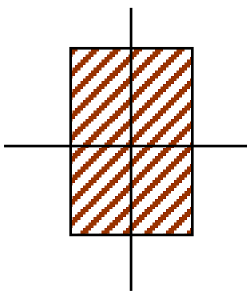


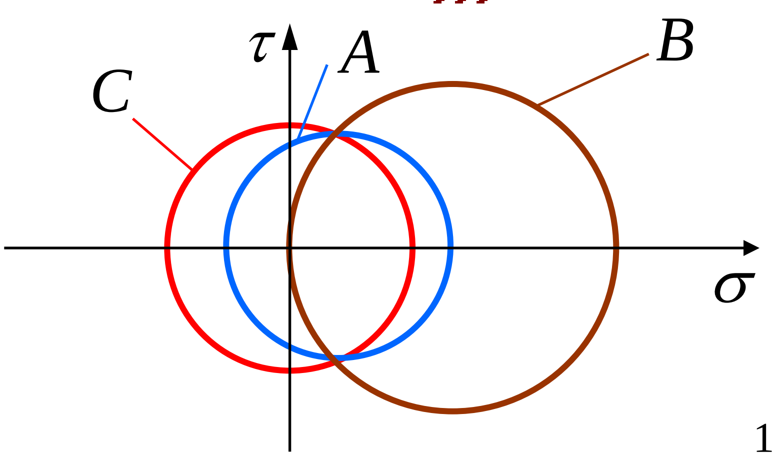
Sample Problem

- For the beam shown below: (a) Sketch the stress state at points *A*, *B* and *C*; (b) Sketch the Mohr's circle for points *A*, *B* and *C*; (c) Sketch the stress state along principle directions for points *A*, *B* and *C*; (d) By satisfying the maximum tensile stress criteria, determine the orientation of the fracture slot at points *A*, *B* and *C*.



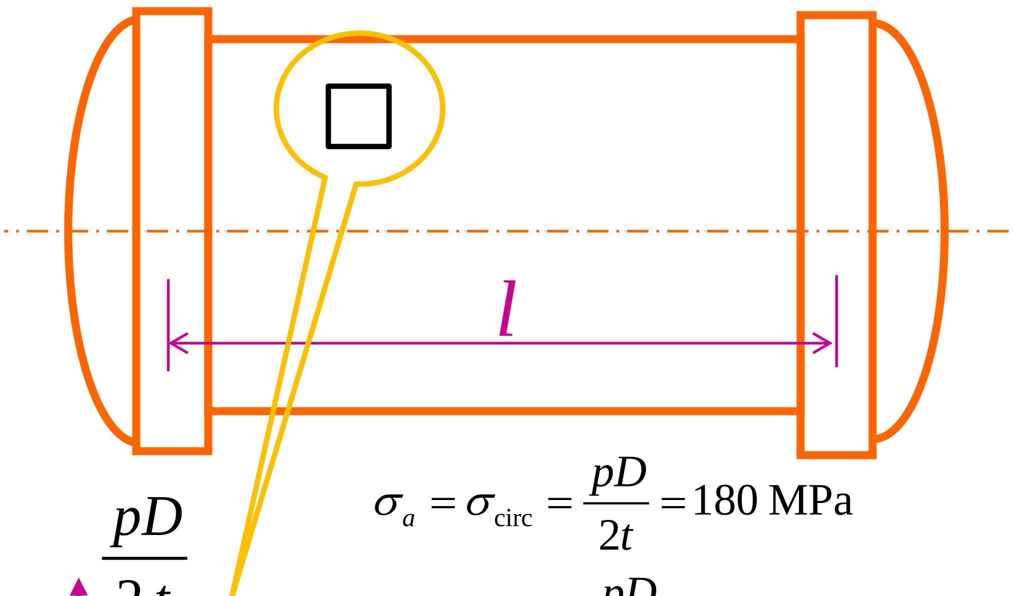


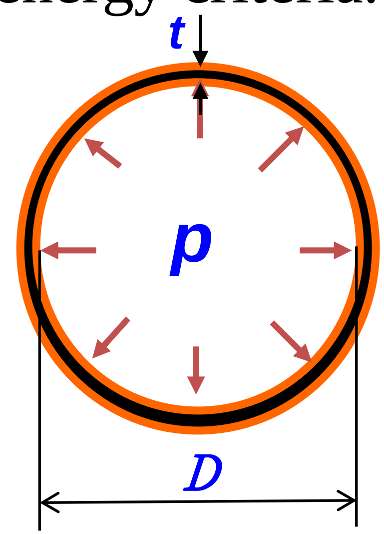


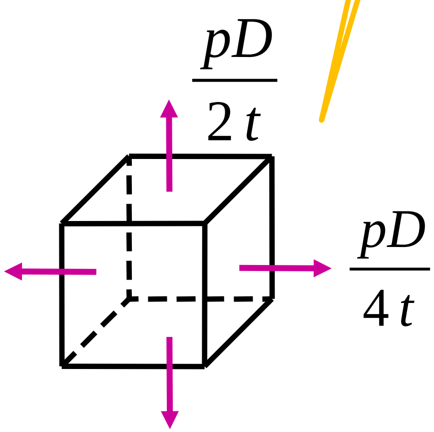


Sample Problem

- For the thin-walled pressure vessel shown below, D = 1000 mm, t = 10 mm, p = 3.6 MPa and [σ] = 170 MPa. denote the inner diameter, wall thickness and pressure respectively. Check the strength condition according to the maximum distortion energy criteria.







$$\sigma_a = \sigma_{\text{circ}} = \frac{pD}{2t} = 180\ \text{MPa}$$

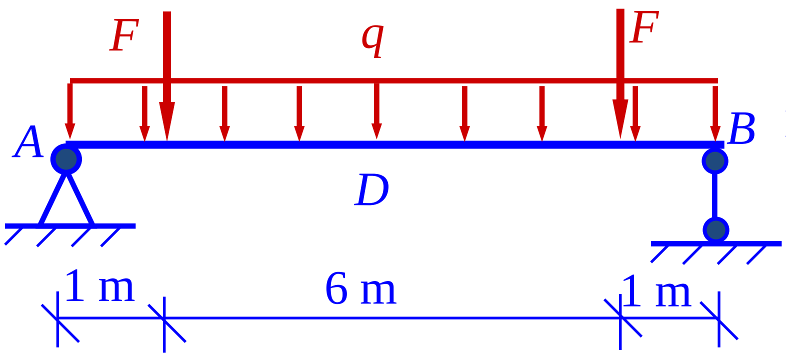
$$\sigma_b = \sigma_{\text{axial}} = \frac{pD}{4t} = 90\ \text{MPa}$$

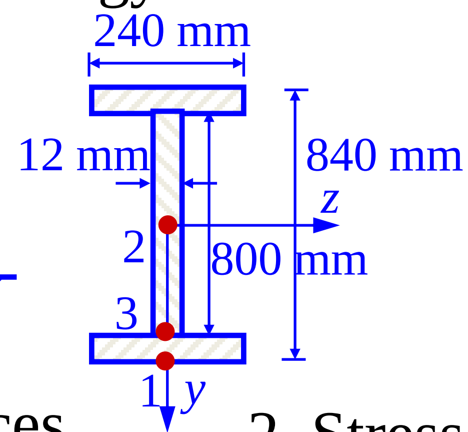
$$\sigma_c = \sigma_{\text{radial}} = -p = -3.6\ \text{MPa} \approx 0$$

$$\sigma_{r4} = \sqrt{\frac{1}{2}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right]} = 156\ \text{MPa} \le 170\ \text{MPa} = [\sigma]$$

Sample Problem

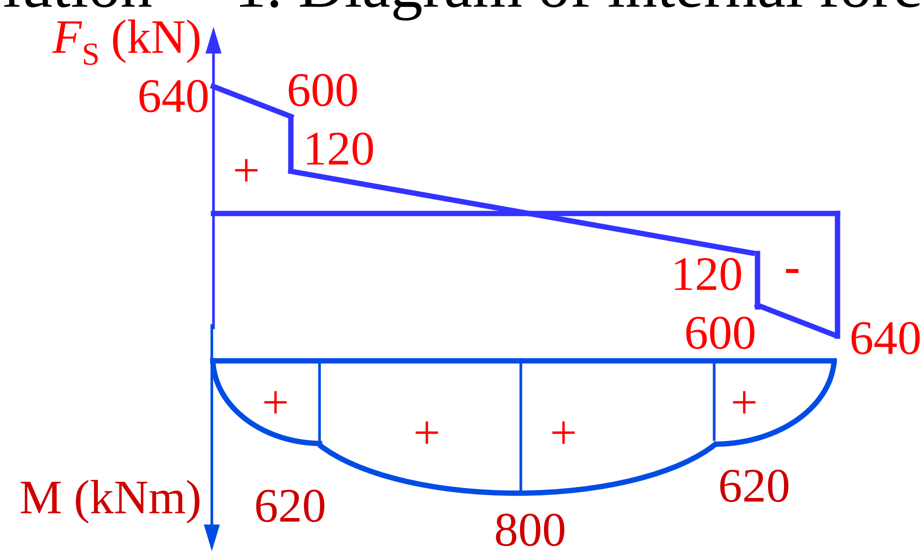
- For the beam shown, q = 40 kN/m, F = 480 kN, and [σ] = 160 MPa. Check the strength condition along longitudinal lines 1, 2 and 3 according to the maximum distortion energy criteria.



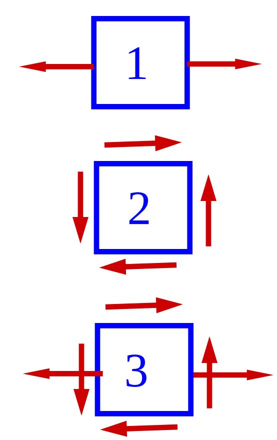


- Solution

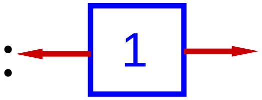
1. Diagram of internal forces.



2. Stress states along Lines 1, 2, and 3.



3. Strength condition at Line 1:

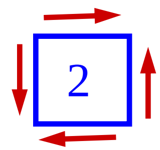


$$\sigma = \left(\frac{M_z y}{I_z}\right)_{max}, \quad M_{max} = M_D = 800\ \textbf{kNm}, \quad y_{max} = 420\ \textbf{mm}$$

$$I_z = 240\times10^{-3}\times840^3\times10^{-9}/12-(240-12)\times10^{-3}\times800^3\times10^{-9}/12 = 2.126\times10^{-3}\ \textbf{m}^4$$

$$\sigma_{r4} = \sigma = \frac{800\times10^3\times420\times10^{-3}}{2.126\times10^{-3}} = 160\ \textbf{MPa} = [\sigma]$$

4. Strength condition at Line 2:



$$\tau_{max} = \left(\frac{F_s S_z}{I_z b}\right)_{max}, \quad F_{s\,max} = F_{sA} = F_{sB} = 640\textbf{kN}, \quad b = 12\textbf{mm}$$

$$S_z = [(240\times10^{-3}\times20\times10^{-3})\times410\times10^{-3}]$$

$$+[(12\times10^{-3}\times400\times10^{-3})\times200\times10^{-3}] = 2.96\times10^{-3}\textbf{m}^3$$

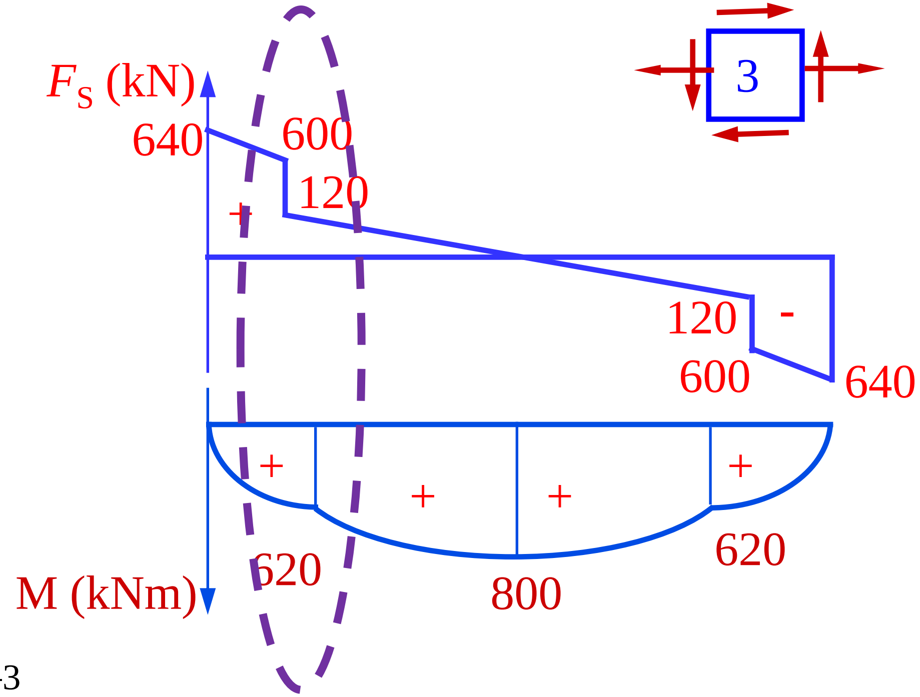
$$\Rightarrow \tau_{max} = \frac{640\times10^3\times2.96\times10^{-3}}{12\times10^{-3}\times2.126\times10^{-3}} = 73.5\textbf{MPa}$$

$$\Rightarrow \sigma_1 = 73.5MPa, \quad \sigma_2 = 0, \quad \sigma_3 = -73.5\textbf{MPa}$$

$$\Rightarrow \sigma_{r4} = \sqrt{\tfrac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} = 127\textbf{MPa} < 160\ \textbf{MPa} = [\sigma]$$

5. Strength condition at Line 3:

- Critical cross-section is identified by both large shearing stress and bending moment (x = 1 m)!



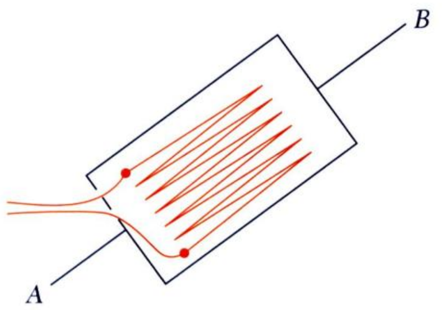
$$\sigma = \frac{My}{I_z} = \frac{620\times10^{3}\times400\times10^{-3}}{2.126\times10^{-3}} = 116.7\ \mathbf{MPa} < [\sigma]$$

$$S_z^{*} = (240\times10^{-3}\times20\times10^{-3})\times410\times10^{-3} = 1.968\times10^{-3}\ \mathbf{m}^3$$

$$\tau = \frac{F_s S_z^{*}}{bI_z} = \frac{600\times10^{3}\times1.968\times10^{-3}}{12\times10^{-3}\times2.126\times10^{-3}} = 46.3\ \mathbf{MPa}$$

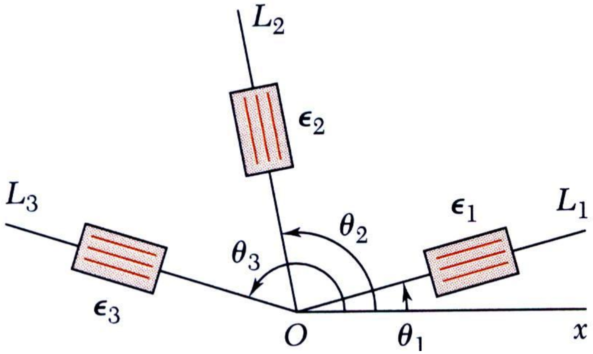
$$\sigma_{r4} = \sqrt{\sigma^2+3\tau^2} = 141.6\ \mathbf{MPa} < 160\ \mathbf{MPa} = [\sigma]$$

Measurement of Strain & Strain Rosettes



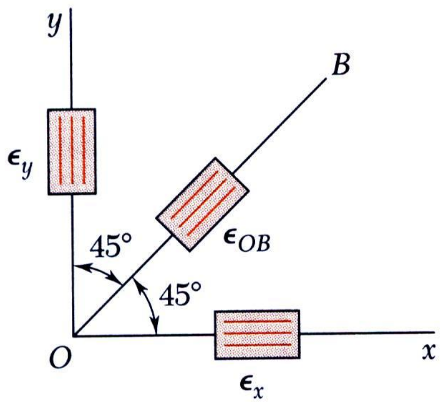
- Strain gages indicate normal strain through changes in resistance.

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$



- Normal and shearing strains may be obtained from normal strains in any three directions,

$$\varepsilon_1 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_1 + \frac{\gamma_{xy}}{2}\sin 2\theta_1$$
$$\varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_2 + \frac{\gamma_{xy}}{2}\sin 2\theta_2$$
$$\varepsilon_3 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_3 + \frac{\gamma_{xy}}{2}\sin 2\theta_3$$



- With a 45^o rosette, ε_x and ε_y are measured directly. γ_{xy} is obtained indirectly with,

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

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