Bending Stress

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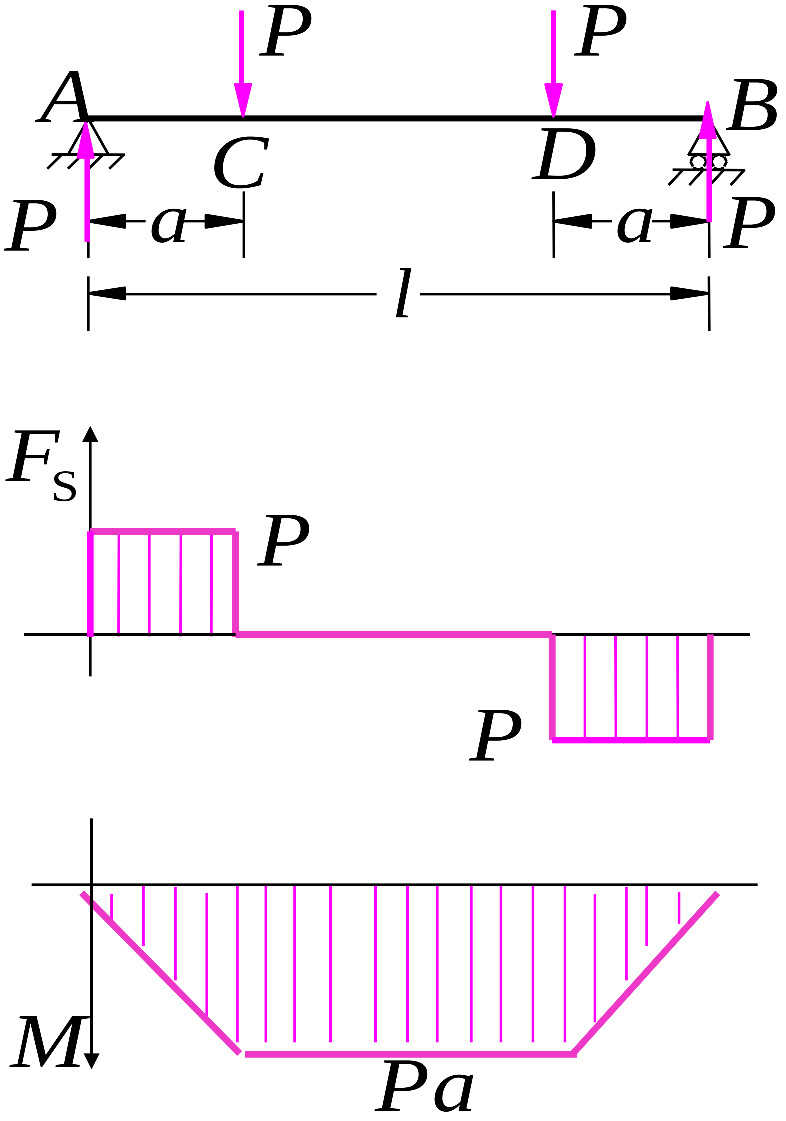
Pure Bending vs. Nonuniform Bending

- Pure bending (*CD*)

$$F_S = 0, \quad M = \text{const}$$

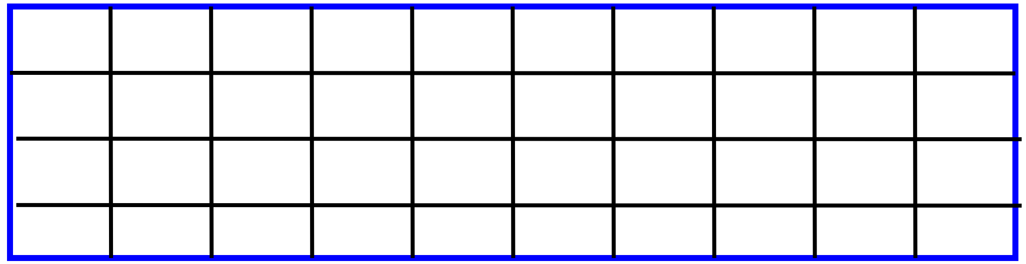
- Nonuniform bending (*AC* & *DB*)

$$F_S \neq 0, \quad M \neq 0$$

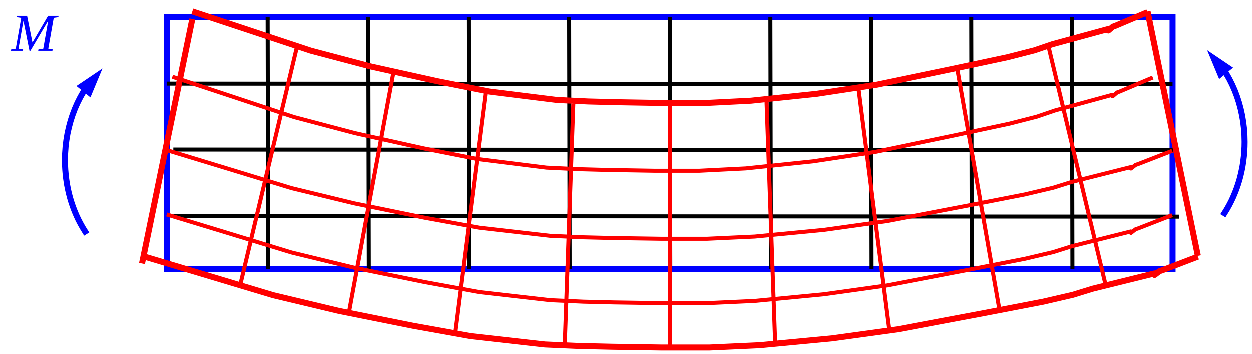


Deformation Characteristics

- Before:



- After:



- Straight longitudinal lines turns into curves.
- Longitudinal lines get shortened under compression and lengthened under tension.
- Cross-section lines remain straight and perpendicular to longitudinal curves.

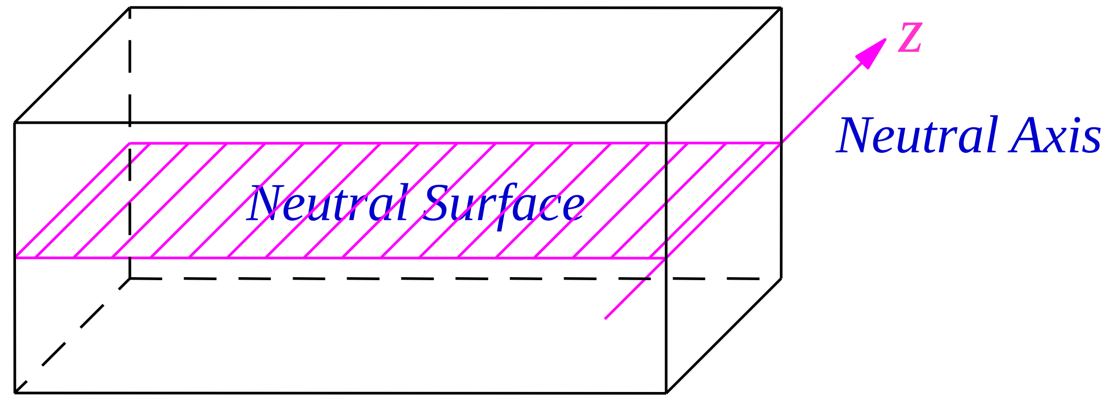
Assumptions for Pure Bending

- **Plane assumption:** under pure bending, cross-sections of beams remain planar and perpendicular to beam axis and only rotate a small angle.

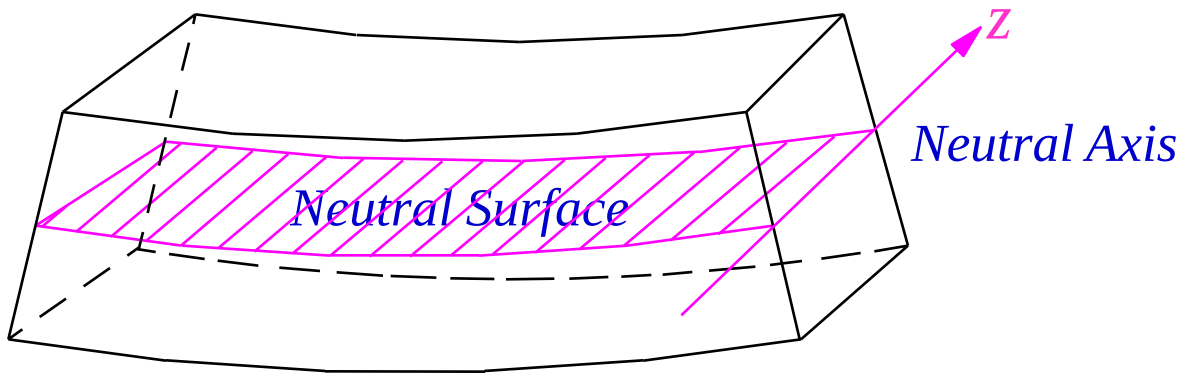
- **Assumption of uniaxial stress state:** individual longitudinal layers are under uniaxial tension/compression along beam axis, without stresses acting in between.

Neutral Surface & Neutral Axes

- Before:

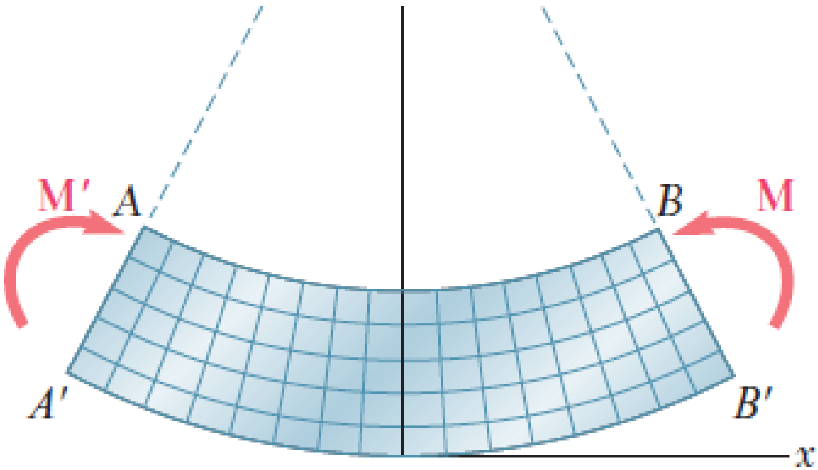


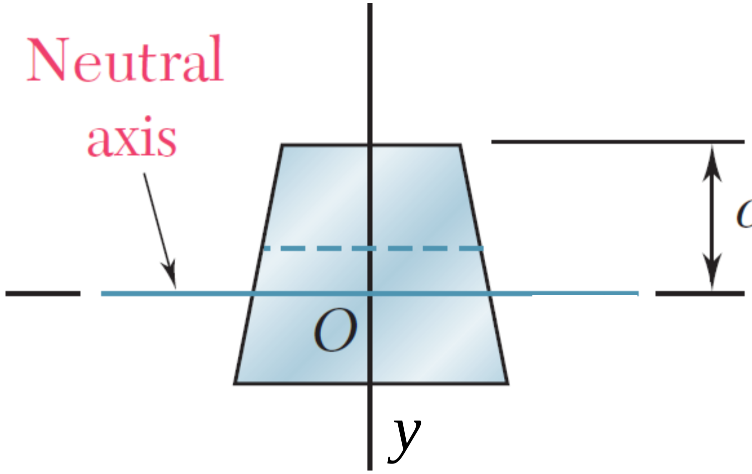
- After:

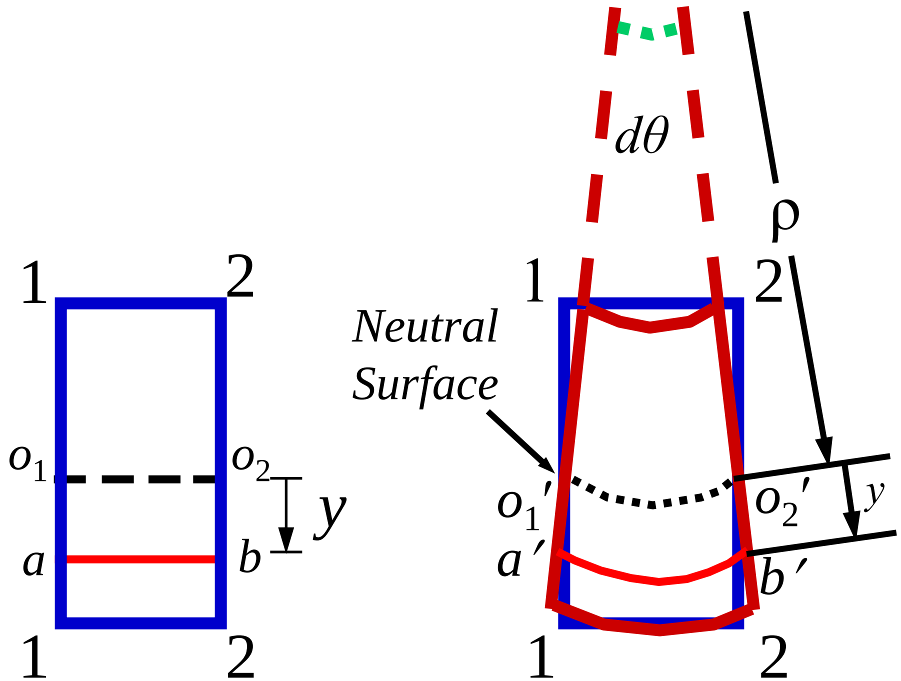


- **Neutral Surface** the longitudinal layer under neither tension nor compression.
- **Neutral Axes:** intersecting lines of the neutral surface & cross sections.

Kinematics







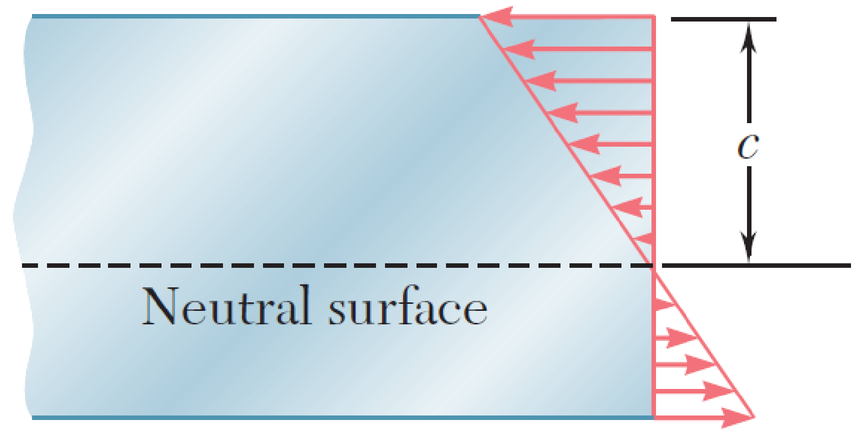
$$\varepsilon(y) = \frac{(\rho + y)d\theta - \rho d\theta}{\rho d\theta}$$

$$= \frac{y}{\rho}$$

- The y-coordinate is measured from the proposed neutral axis.

Hooke's Law

$$\sigma(y) = E\varepsilon(y) = E\frac{y}{\rho}$$



- Normal stress acting on a longitudinal layer is linearly proportional to its distance from the neutral surface, positive for layers under tension / negative for layers under compression.

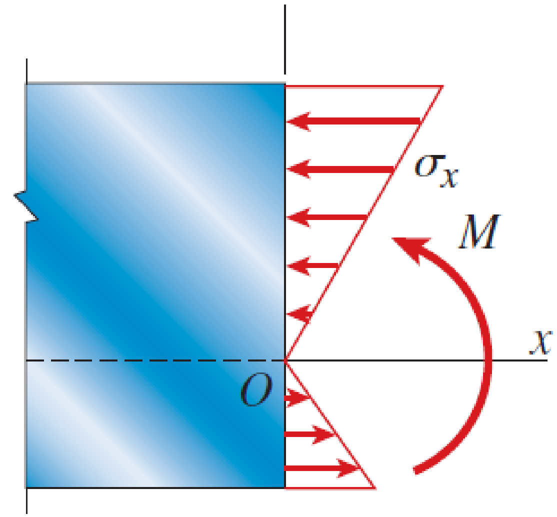
- Remark: the above equation can only be used for qualitative analysis of stresses in bending beams since it is difficult to measure the curvature of radius (ρ) of individual longitudinal layers.

Static Equivalency

$$0 = F_N = \int_A \sigma dA = \int_A \frac{E}{\rho} y dA = \frac{E}{\rho} A\bar{y}$$

- Neutral axis passes through the centroid: $\bar{y} = 0.$
- for an arbitrarily defined y-coordinate:

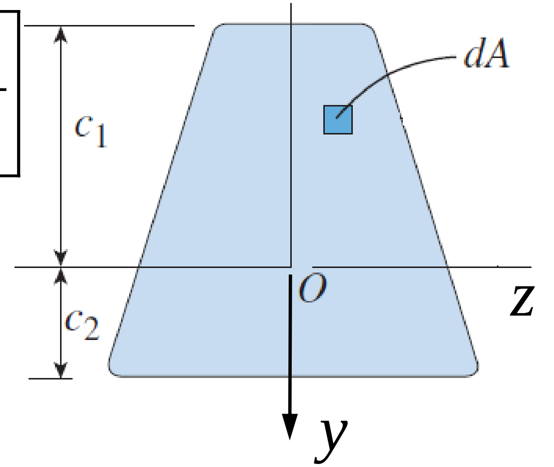
$$\bar{y} = \sum_i A_i \bar{y}_i \Big/ A$$



$$0 = M_y = \int_A z \cdot \sigma dA = \frac{E}{\rho} \int_A yz dA$$

$$M_z = \int_A y \cdot \sigma dA = \int_A \frac{E}{\rho} y^2 dA = \frac{E}{\rho} I_z \quad \Rightarrow \quad \boxed{\frac{1}{\rho} = \frac{M_z}{EI_z}}$$

$$\Rightarrow \sigma = \frac{E}{\rho} y \quad \Rightarrow \quad \boxed{\sigma = \frac{M_z y}{I_z}}$$



EI_z : flexural rigidity

$I_z = \int_A y^2 dA$: second moment of cross-section w.r.t. z.

Pure Bending Normal Stress Formula

- Normal stress on cross-sections: $\sigma = M_z\ y / I_z$.
- Maximum normal stress on cross-sections:

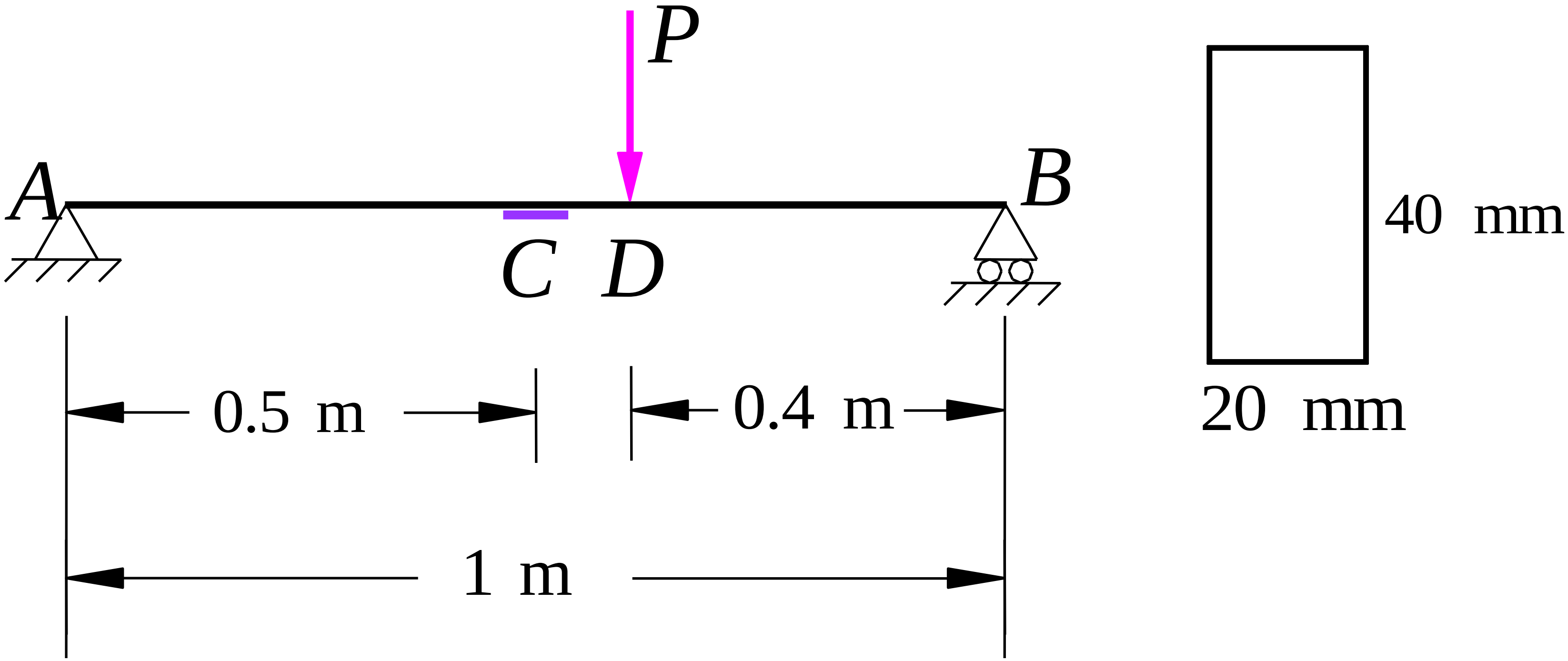
$$\sigma_{\max} = M_z\ y_{\max} / I_z = M_z / W_z$$

$W_z = I_z / y_{\max} = 2I_z / h$: bending section modulus

- Remarks:
- The neutral axis passes through the centroid of the cross-sectional area when the material follows Hooke's law and there is no axial force acting on the cross section.
- Our discussion is limited to beams for which the y axis is an axis of symmetry. Consequently, the origin of coordinates is the centroid.
- Because the *y*-axis is an axis of symmetry, it follows that the *y*-axis is a principal axis. So is the *z*-axis.

Sample Problem

- A strain gauge is placed under cross-section C of a simply supported beam shown. Under the concentrated load P, the strain gauge reads $\varepsilon = 6\times10^{-4}$. Find the magnitude of P for E = 200 GPa.



- Solution:

$$\sigma_C = E\varepsilon = 200\times10^3\times6\times10^{-4} = 120\ \text{MPa}$$

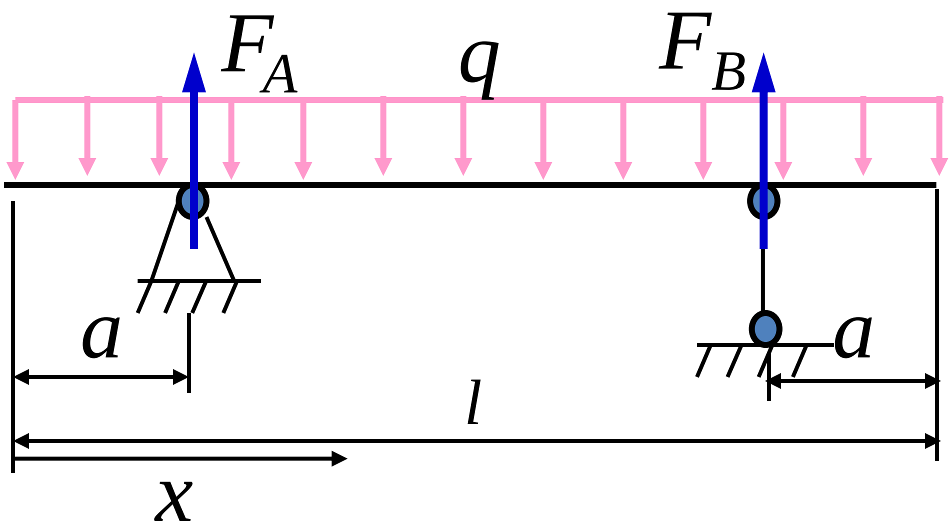
$$M_C = \sigma_C W_z = 640\ \text{N}\cdot\text{m}$$

$$M_C = 0.5R_A = 0.5\times0.4P = 0.2P = 640\ \text{N}\cdot\text{m}$$

$$\Rightarrow P = 3.2\ \text{kN}$$

Sample Problem

- Find the support position (a) at the condition of minimum "maximum normal stress" for the overhanging I-beam shown below, under uniformly distributed load q.



- Solution:

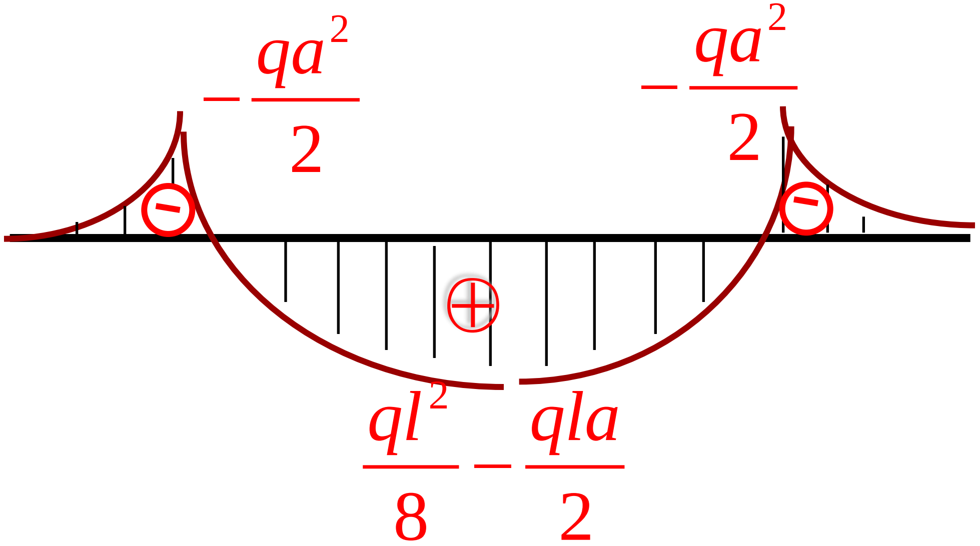
1. reaction force at the supports. Due to symmetry:

$$F_A = F_B = ql/2$$

2. Equation of bending moments:

$$\begin{cases} M = -\dfrac{1}{2}qx^2 & x \in [0, a) \\ M = -\dfrac{1}{2}qx^2 + \dfrac{1}{2}ql(x-a) & x \in [a, l-a) \end{cases}$$

3. Diagram of bending moments:



- Equating the absolute value of the negative and positive moment extremities results in minimum bending moments and hence minimum "maximum normal stress."

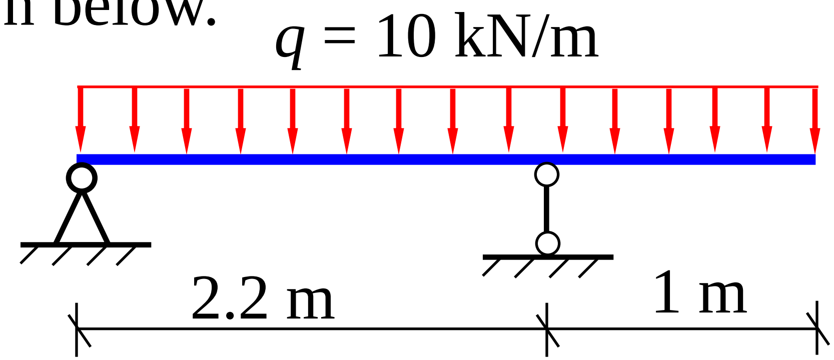
$$\left|-\frac{qa^2}{2}\right| = \left|-\frac{1}{2}q\left(\frac{l}{2}\right)^2 + \frac{1}{2}ql\left(\frac{l}{2}-a\right)\right|$$

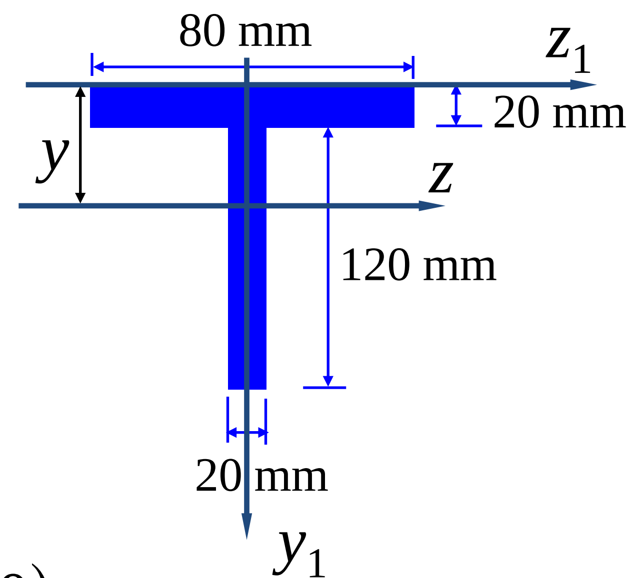
$$\Rightarrow \frac{qa^2}{2} = \frac{ql^2}{8} - \frac{qla}{2} \Rightarrow a^2 + la - \frac{1}{4}l^2 = 0 \Rightarrow a = \frac{1}{2}\left(-l \pm \sqrt{l^2 + l^2}\right)$$

$$\Rightarrow a \approx 0.207 \cdot l$$

Sample Problem

- Find the maximum tensile and compressive stress in the *T*-beam shown below.





- Solution:

1. Centroid (neutral surface, neutral axis):

$$y = \frac{\sum A_i y_i}{\sum A_i} = \frac{80\times20\times10+20\times120\times(60+20)}{80\times20+20\times120} = 52 \text{ mm}$$

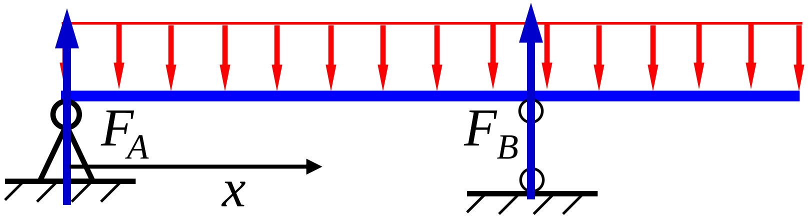
2. Moment of inertia:

By the Parallel Axis Theorem: $I_{z'} = I_z + Ad_{zz'}^2$

$$I_z = \frac{80\times20^3}{12} + 80\times20\times(52-10)^2 + \frac{20\times120^3}{12} + 20\times120\times(80-52)^2$$

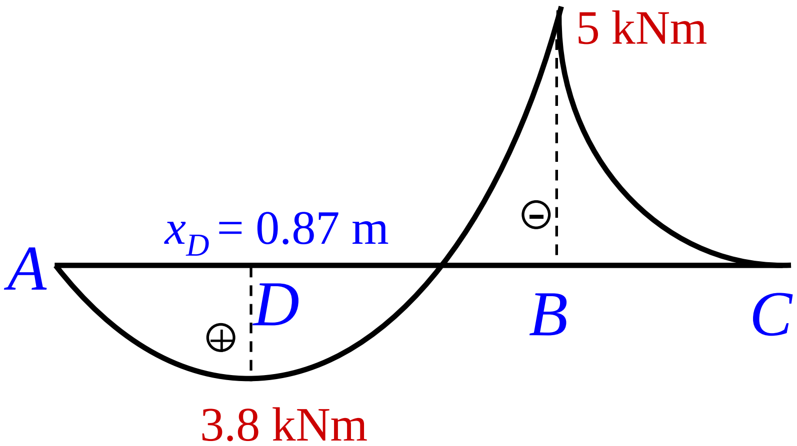
$$= 764\times10^4 \text{ mm}^4 = 7.64\times10^{-6} \text{ m}^4$$

3. Reaction forces and diagram of bending moments



$$0 = \sum M_{Ay} \Rightarrow F_B = 23.27\text{KN}$$

$$\Rightarrow F_A = 10\times 3.2 - F_B = 8.73\text{KN}$$



$$M(x) = F_A x - \frac{qx^2}{2} \quad x \in [0, 2.2\text{ m})$$

$$M(x) = F_A x + F_B(x-2.2) - \frac{qx^2}{2} \quad x \in [2.2\text{ m}, 3.2\text{ m})$$

4. Maximum normal stress (At cross-section *B*)

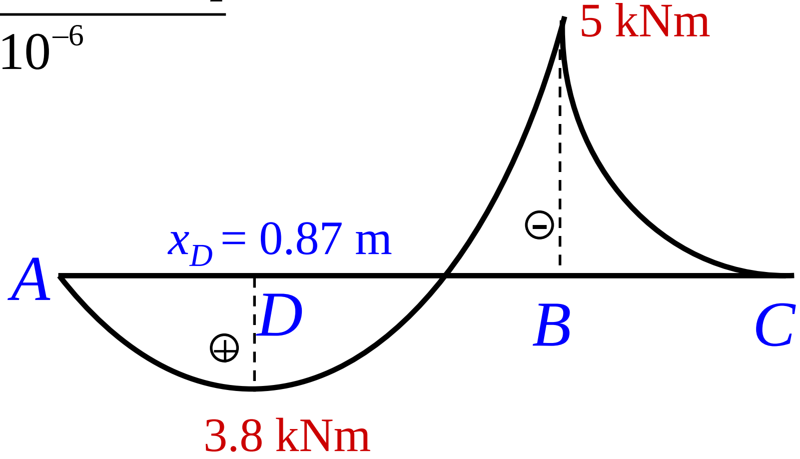
$$\sigma^{+}_{\max} = \frac{(-5\times 10^{3})\times(-52\times 10^{-3})}{7.64\times 10^{-6}} = 34\times 10^{6}\ \text{Pa} = 34\ \text{MPa}$$

$$\sigma^{-}_{\max} = \frac{(-5\times 10^{3})\times[(140-52)\times 10^{-3}]}{7.64\times 10^{-6}} = -57.6\times 10^{6}\ \text{Pa} = -57.6\ \text{MPa}$$

5. Maximum normal stress (At cross-section D)

$$\sigma^{+}_{\max} = \frac{(3.8\times10^{3})\times[(140-52)\times10^{-3}]}{7.64\times10^{-6}} = 43.8\times10^{6}\ \text{Pa} = 43.8\ \text{MPa}$$

$$\sigma^{-}_{\max} = \frac{(3.8\times10^{3})\times[-52\times10^{-3}]}{7.64\times10^{-6}}$$



6. Maximum normal stress

- Maximum tensile stress: lower edge of cross-section D (43.8 MPa).
- Maximum compressive stress: lower edge of cross-section B (-57.6 MPa).

Normal Stress Strength Condition

- For ductile materials

$$\sigma_{\max} = \left(\frac{M}{W_z}\right)_{\max} \leq [\sigma]$$

- For brittle materials

$$\sigma_{\max}^{+} = \left(\frac{M}{W_z}\right)_{\max} \leq \left[\sigma^{+}\right], \qquad \left|\sigma_{\max}^{-}\right| = \left|\frac{M}{W_z}\right|_{\max} \leq \left[\sigma^{-}\right]$$

- The maximum positive and negative bending moments in a beam may occur at the following places: (1) a cross section where a concentrated load is applied and the shear force changes sign, (2) a cross section where the shear force equals zero, (3) a point of support where a vertical reaction is present, and (4) a cross section where a couple is applied.

Remarks on Strength Condition

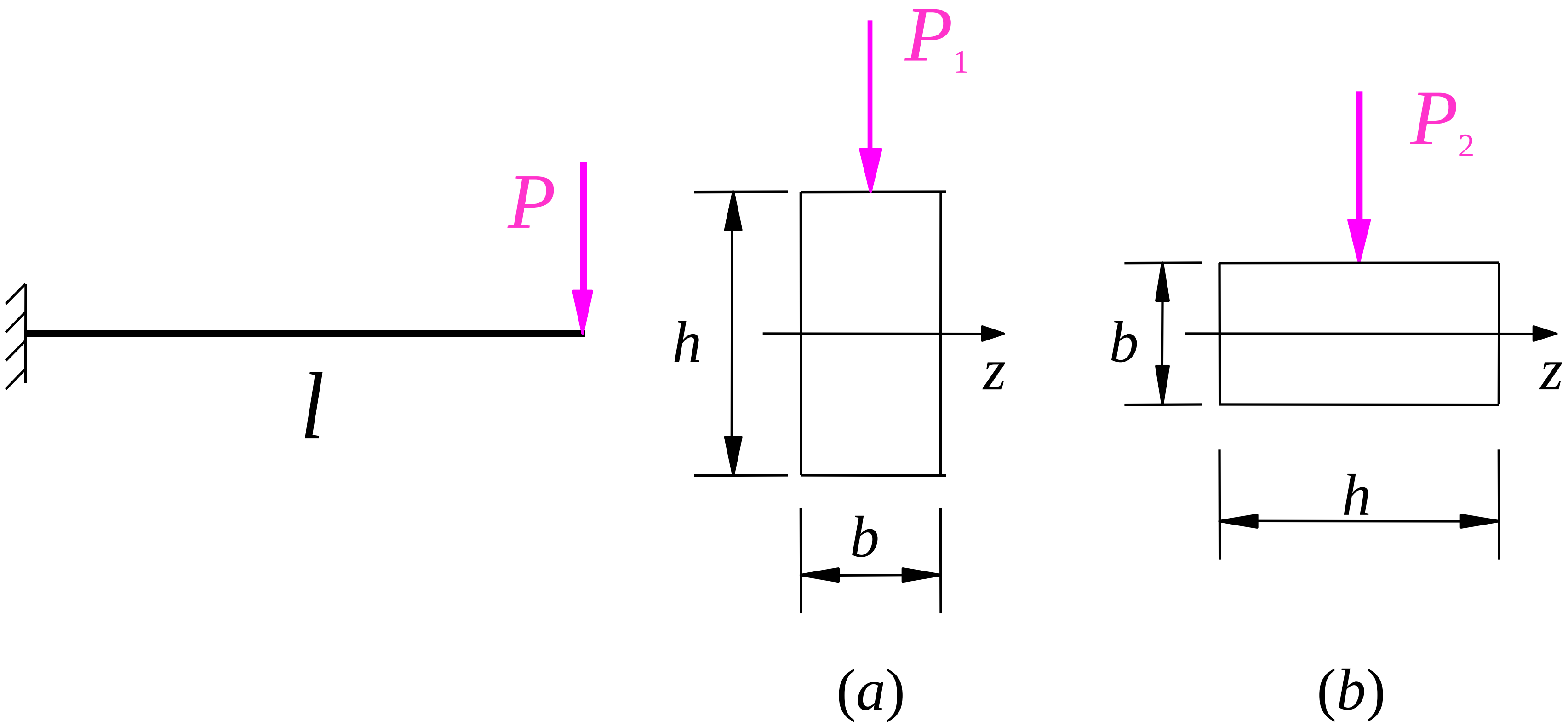
- The maximum tensile stress and the maximum compressive stress sometimes don't occur on the same cross-section.

- Usually, the allowable bending stress is slightly higher than the allowable uniaxial tensile/compressive stress. This is because the bending stress only takes extremities at the upper/lower edges of bending beams while the maximum axial stress is uniformly distributed on bar cross-sections.

- Three types problem that are typically addressed by strength analysis:
  - Strength check
  - Cross-section design
  - Allowable load

Sample Problem

- The dimension and material of the two cantilever beams shown are identical. Find the allowable load ratio of these two beam based on the normal stress strength condition: $P_1/P_2 = ?$



- Solution：

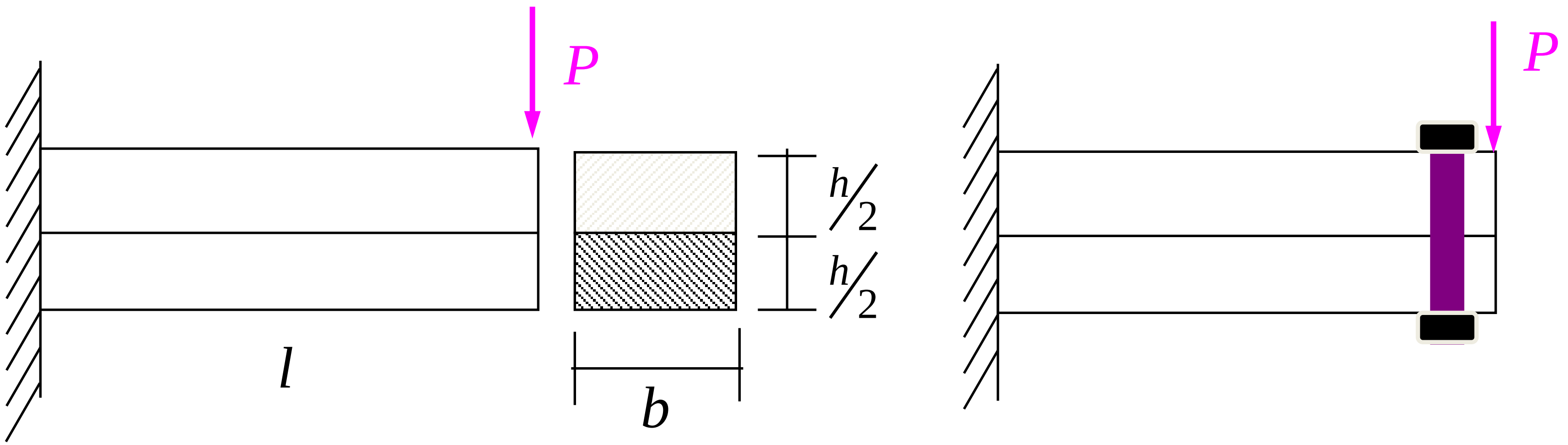
$$\sigma_{\max 1} = \frac{M_{\max 1}}{W_{z1}} = \frac{P_1 l}{\frac{bh^3}{12} \Big/ \frac{h}{2}} = \frac{P_1 l}{\frac{bh^2}{6}}$$

$$\sigma_{\max 2} = \frac{M_{\max 2}}{W_{z2}} = \frac{P_2 l}{\frac{hb^2}{6}}$$

$$\sigma_{\max 1} = \sigma_{\max 2} \Rightarrow \frac{P_1}{P_2} = \frac{h}{b}$$

Sample Problem

- Two identical rectangular beams are placed together and subjected to a concentrated load as shown. Find the allowable load $[P]$ if the allowable normal stress is given as $[\sigma]$. What is $[P]$ if the two beams are pinned together?



- Solution

1. when the beams are *not* pinned together, each beam has its own neutral surface and carries half of the bending moments.

$$W_1 = \frac{b\left(h/2\right)^2}{6} = \frac{bh^2}{24}$$

$$\sigma_{\max} = \frac{M_{\max}/2}{W_1} = \frac{M_{\max}}{2W_1} = \frac{12Pl}{bh^2} \le [\sigma]$$

$$\Rightarrow [P] \le \frac{bh^2[\sigma]}{12l}$$

2. After the beams are pinned, there exists only one neutral surface

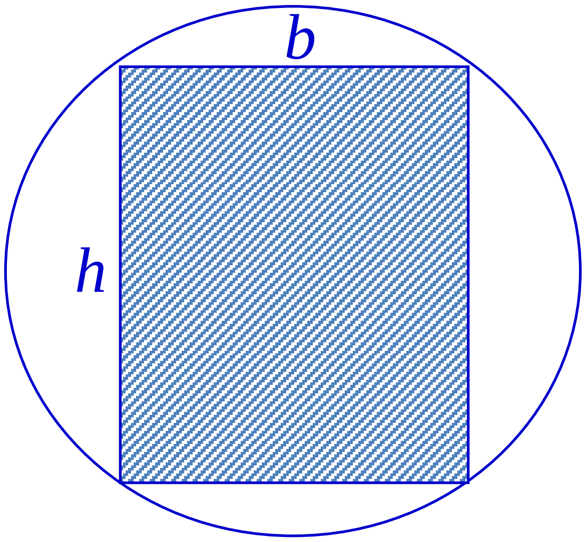
$$\sigma_{\max} = \frac{M_{\max}}{W} = \frac{Pl}{bh^2/6} \leq [\sigma]$$

$$\Rightarrow [P] \leq \frac{bh^2[\sigma]}{6l}$$

- It can be seen that the load carrying ability are doubled after pinning.

Sample Problem

- In ancient China, the typical aspect ratio of the cross-section of rectangular beams is given as *h*:*b* = 3:2. If beams were made from circular trees, employing the strength theory prove that the above ratio is close to the optimal aspect ratio.



- Proof

  Optimal aspect ratio means that W_z achieves the maximum value.

$$b^2 + h^2 = d^2$$

$$W_z = \frac{bh^2}{6} = \frac{b(d^2 - b^2)}{6}$$

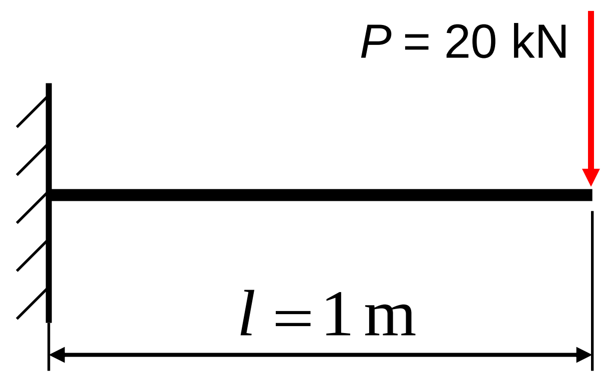
$$\Rightarrow \frac{\partial W_z}{\partial b} = \frac{d^2}{6} - \frac{b^2}{2} = 0 \quad \Rightarrow \quad b = \frac{d}{\sqrt{3}} \quad \Rightarrow \quad \frac{h}{b} = \sqrt{2}$$

Sample Problem

- Given P = 20 kN, [σ] = 140 MPa. Compare the material consumption for the following three types of cross-sections: (1) rectangle with h/b = 2; (2) circle; (3) I-shaped.

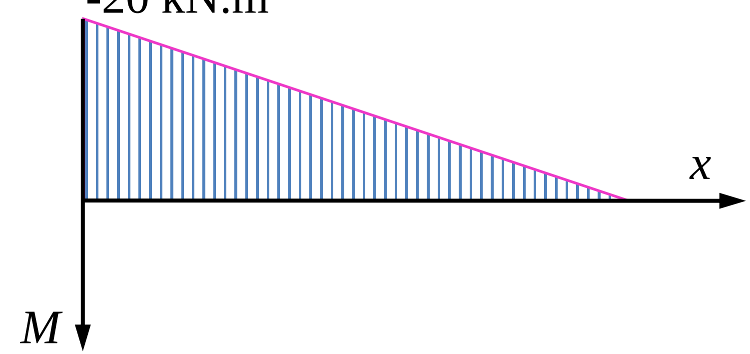
- Solution:

Diagram of bending moment.



$$\sigma_{\max} = \frac{M_{\max}}{W_z} \le [\sigma] \Rightarrow$$

$$W_z \ge \frac{M_{\max}}{[\sigma]} = \frac{20 \times 10^3}{140 \times 10^6}\ \text{m}^3 \approx 143\,\text{cm}^3$$



(1) For rectangular cross-section

$$W_z = \frac{bh^2}{6} \Rightarrow A_1 = 72\,\text{cm}^2$$

(2) For circular cross-section

$$W_z = \frac{\pi d^3}{32} \Rightarrow d = 11.3\,\text{cm} \quad \Rightarrow \quad A_2 \approx 100\,\text{cm}^2$$

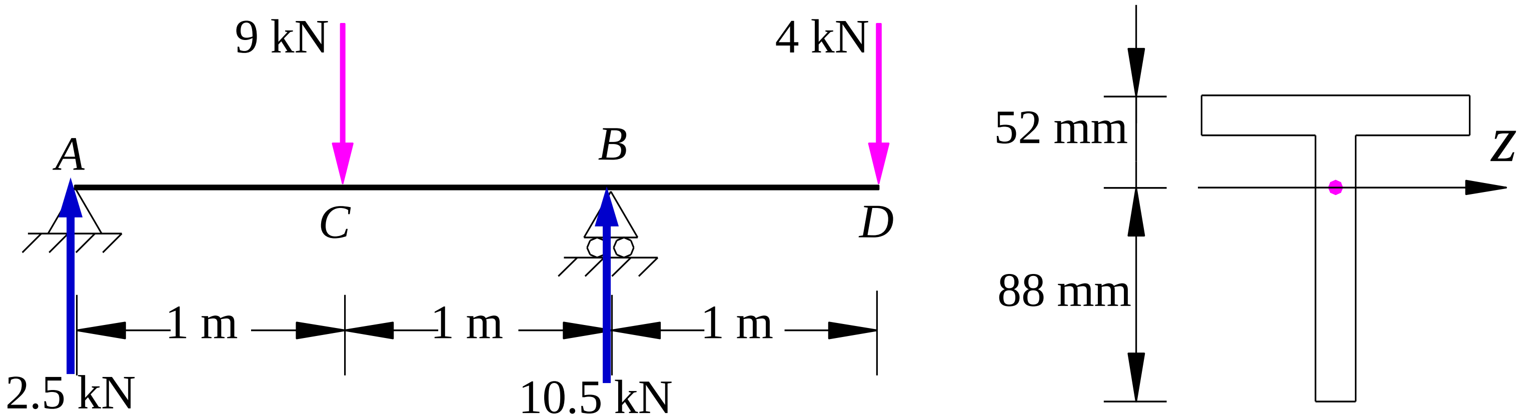
(3) For *I*-shaped cross-section

Check the table for I-beam: $W_z = 141\,\text{cm}^3 \Rightarrow A_3 = 26.1\,\text{cm}^2$

- *I*-beam consumes the least material while circular beam costs the most.
- The maximum stress in the I-beam exceeds the maximum allowable stress less than 5%. This is allowable in engineering practice.

Sample Problem

- For the casting iron T-beam shown, the allowable tensile stress $[\sigma^{+}]$ = 30 MPa, allowable compressive stress $[\sigma^{-}]$ = 60 MPa, moment of inertia $I_z = 7.63\times10^{-6}\ \mathrm{m}^4$. Analyze the strength condition.



- Solution:

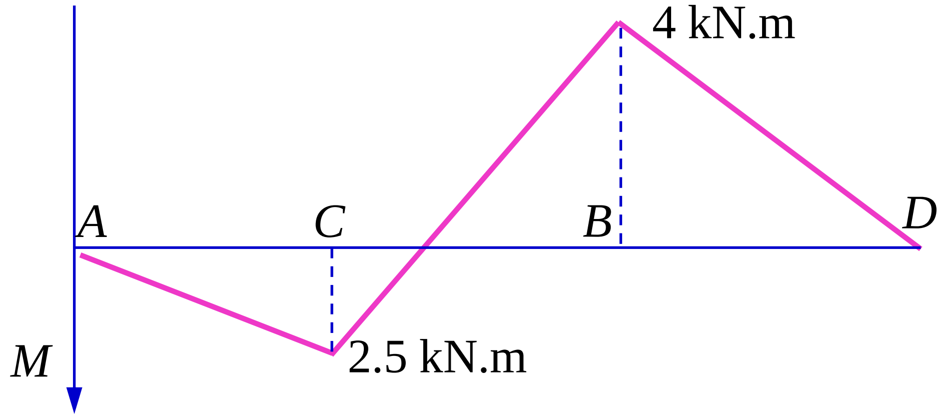


Diagram of bending moment.

- For cross-section *C*:

$$\sigma_{\max}^{+} = \frac{2.5\times 88}{I_z} = 28.8\text{MP}_\text{a} \le [\sigma^{+}]$$

$$\left|\sigma_{\max}^{-}\right| = \left|\frac{2.5\times(-52)}{I_z}\right| \le [\sigma^{-}]$$

- For cross-section *B*:

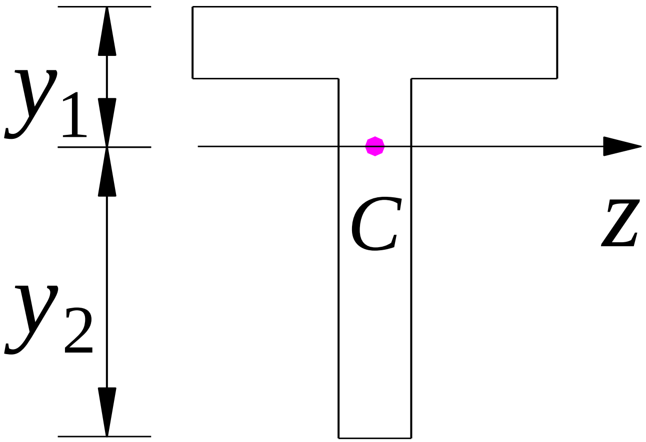
$$\sigma_{\max}^{+} = \frac{(-4)\times(-52)}{I_z} = 27.3\text{MP}_\text{a} \le [\sigma^{+}]$$

$$\left|\sigma_{\max}^{-}\right| = \left|\frac{(-4)\times 88}{I_z}\right| = 46.1\text{MP}_\text{a} \le [\sigma^{-}]$$

- The strength condition of the beam is satisfied.

Sample Problem

- For the *T*-beam shown below, the allowable tensile and compressive stress are given as $[\sigma^{+}]$ and $[\sigma^{-}]$ respectively. Find the optimal ratio for y_1/y_2. (*C* denotes the centroid of the beam cross-section.)



- Solution:
- The maximum bending moment occurs at the fixed end *A*. Make the upper and lower edge of cross-section *A* reach $[\sigma^+]$ and $[\sigma^-]$ simultaneously:

$$\sigma_{\max}^{+} = \frac{M_{\max} y_1}{I_z} = [\sigma^{+}]$$

$$\left|\sigma_{\max}^{-}\right| = \left|\frac{M_{\max} y_2}{I_z}\right| = [\sigma^{-}]$$

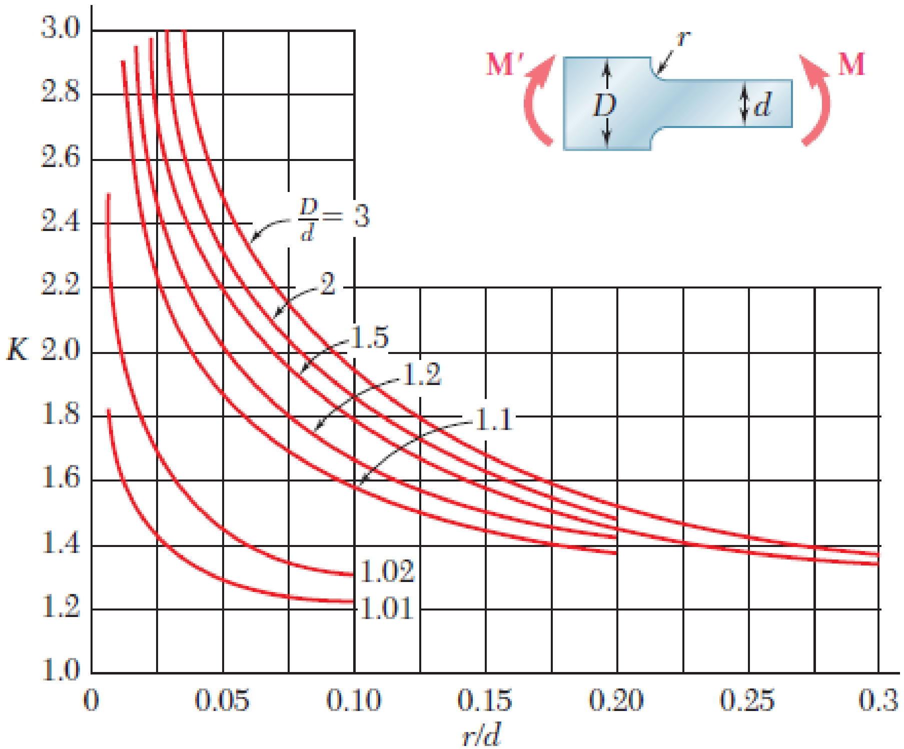
$$\frac{\sigma_{\max}^{+}}{\left|\sigma_{\max}^{-}\right|} \Rightarrow \frac{\left|y_1\right|}{y_2} = \frac{[\sigma^{+}]}{[\sigma^{-}]}$$

Stress Concentrations

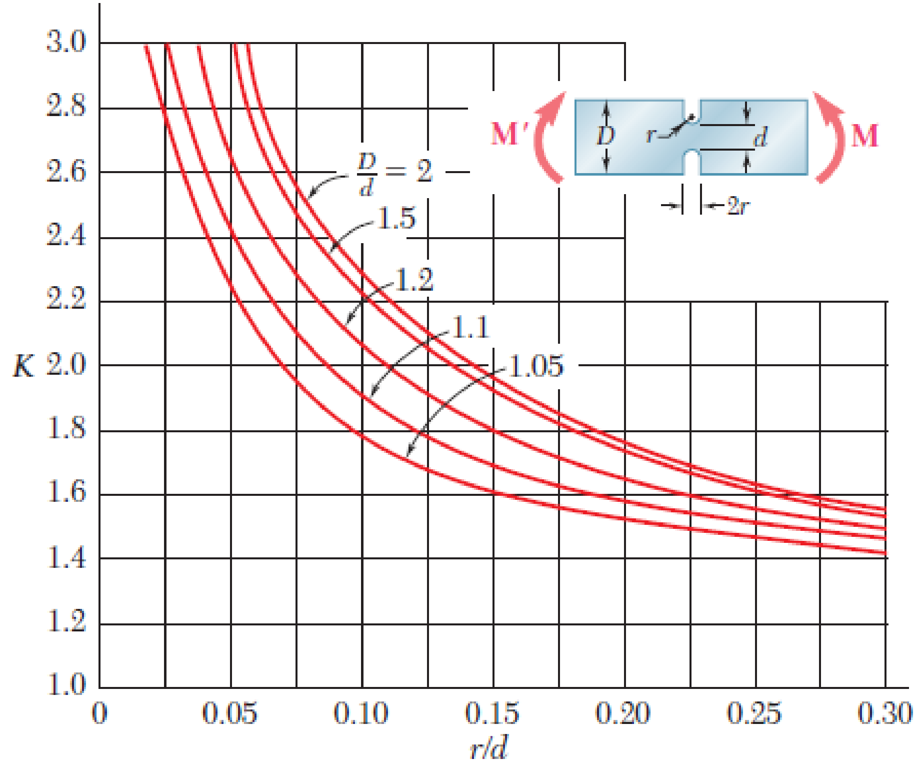
$$\sigma_{max} = K\left(\frac{M_z}{W_z}\right)_{max}$$

Stress concentrations may occur:

- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

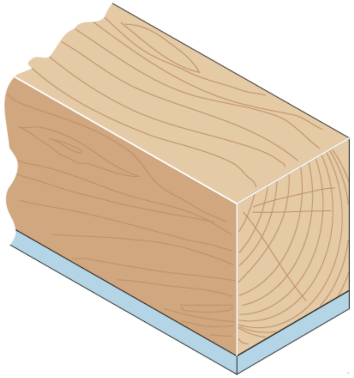


Stress-concentration factors for flat bars with fillets under pure bending

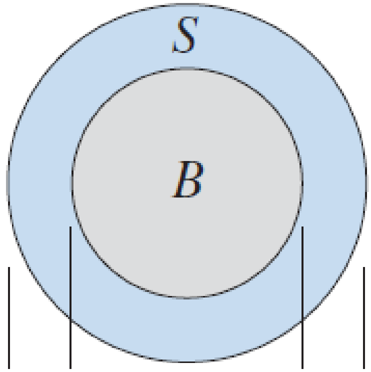


Stress-concentration factors for flat bars with grooves under pure bending

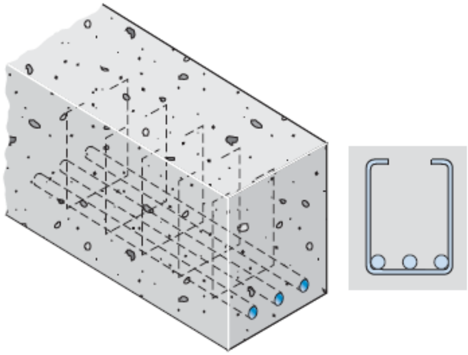
Bending of a Composite Beam



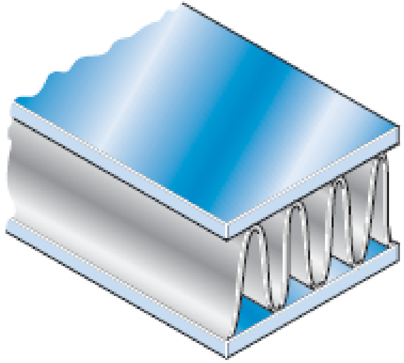
Wood-steel beam



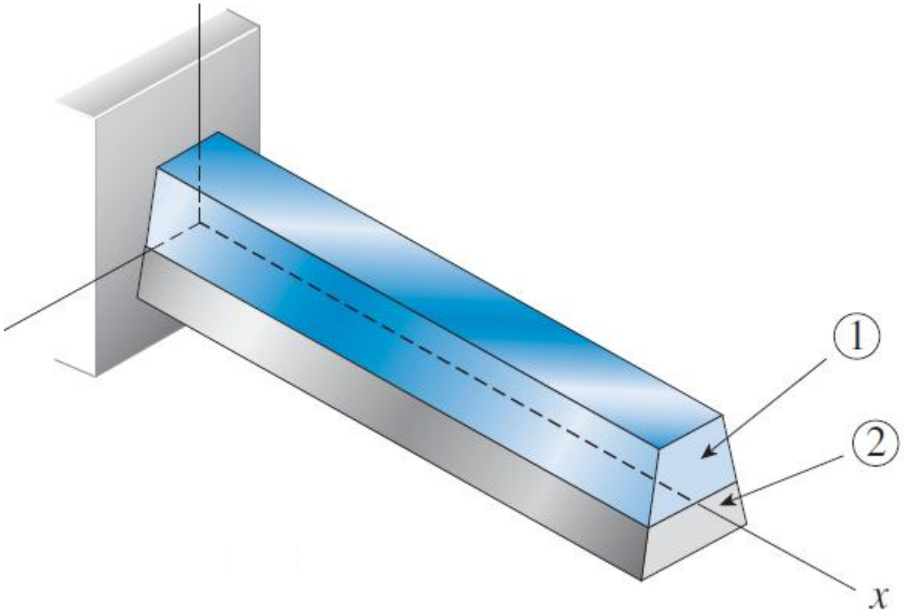
Bimetallic beam



Reinforced concrete Beam



Sandwich beam



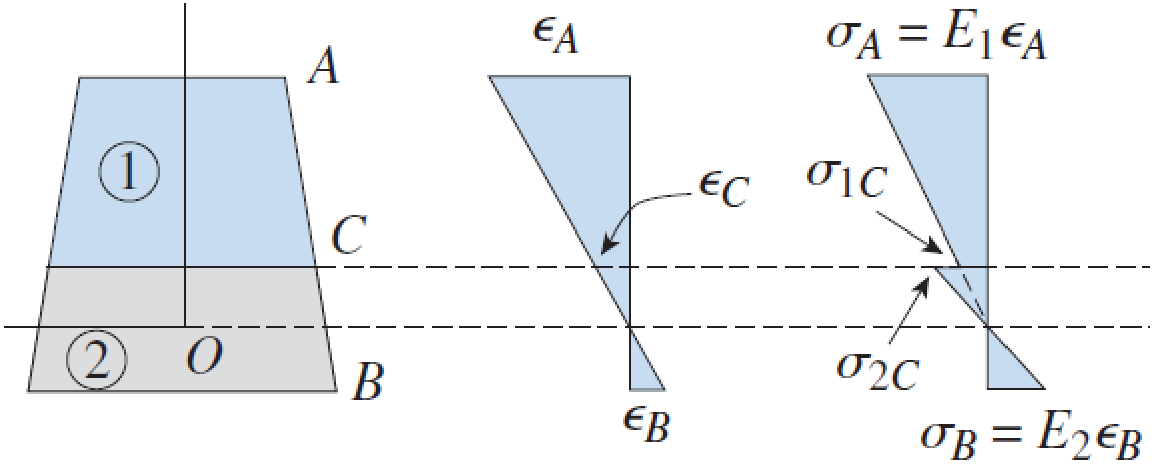
$$\varepsilon = \frac{y}{\rho}; \qquad \sigma = E\varepsilon = E\frac{y}{\rho}$$

- At the contact surface the stresses in the two materials are different.
- The *y*-coordinate is measured from the proposed neutral axis.

Bending of a Composite Beam

$$0 = F_N = \int_{A_1} \sigma_1 dA + \int_{A_2} \sigma_2 dA = \int_{A_1} \frac{E_1}{\rho} y dA + \int_{A_2} \frac{E_2}{\rho} y dA$$

$$\Rightarrow \boxed{0 = \sum_i E_i A_i \bar{y}_i}$$

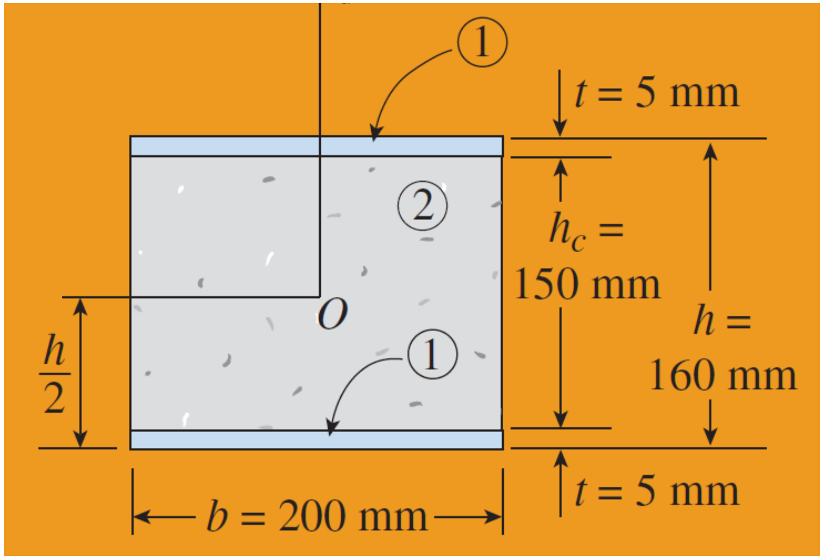


- This equation determines the exact position of neutral axis.
- For an arbitrarily defined y-coordinate: $\bar{y} = \sum_i E_i A_i \bar{y}_i \Big/ \sum_i E_i A_i$.
- Bending stress & Moment-curvature relationship

$$M_Z = \int_A y \cdot \sigma dA = \sum_i \frac{E_i}{\rho} \int_{A_i} y^2 dA = \frac{1}{\rho} \sum_i E_i I_{z \cdot i} \quad \Rightarrow \quad \boxed{\frac{1}{\rho} = \frac{M_z}{\sum_i E_i I_{z \cdot i}}}$$

$$\Rightarrow \sigma_i = E_i \varepsilon = E_i \frac{y}{\rho} \quad \Rightarrow \quad \boxed{\sigma_i = \frac{E_i}{\sum_i E_i I_{z \cdot i}} M_z y}$$

Approximate Theory for a Sandwich Beam



- Provided that: $E_1 >> E_2$

$$I_{z1} = \frac{b}{12}\left(h^3 - h_c^3\right), I_{z2} = \frac{b}{12}h_c^3$$

$$\Rightarrow \frac{I_{z1}}{I_{z2}} = \frac{b\left(h^3 - h_c^3\right)/12}{bh_c^3/12} = \left(1 + \frac{2t}{h_c}\right)^3 - 1 \approx \frac{6t}{h_c}$$

$$\Rightarrow \frac{E_1 I_{z1}}{E_2 I_{z2}} \approx \frac{E_1}{E_2}\frac{6t}{h_c}$$

- If E_1 = 72 Gpa (Al), E_2 = 800 Mpa (Plastic), $2t/h_c$ = 1/15:

$$\Rightarrow \frac{E_1 I_{z1}}{E_2 I_{z2}} \approx \frac{72}{0.8}\frac{3}{15} = 18$$

- Provided that: $E_1 I_{z1} >> E_2 I_{z2}$

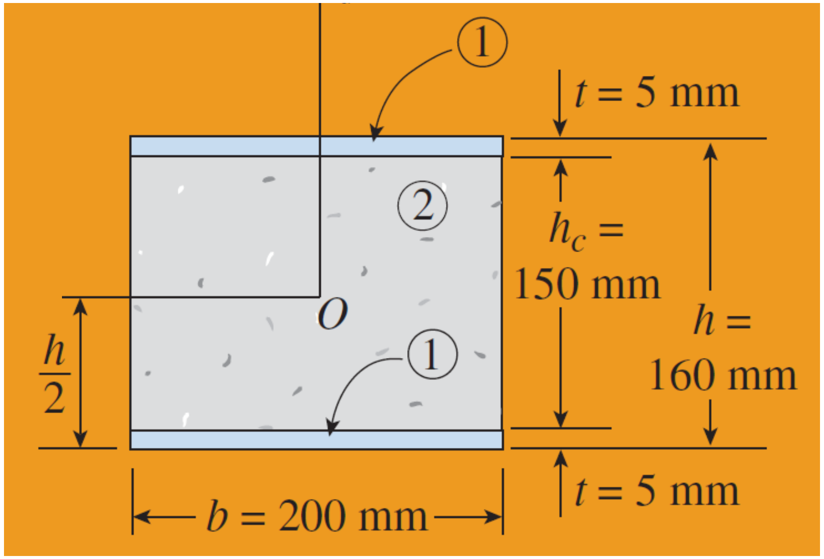
$$\Rightarrow \boxed{\frac{1}{\rho} = \frac{M_z}{E_1 I_{z1} + E_2 I_{z2}} \approx \frac{M_z}{E_1 I_{z1}}}$$

- A conservative theory.

$$\Rightarrow \sigma_i = \frac{E_i}{E_1 I_{z1} + E_2 I_{z2}} M_z y = \frac{E_i}{E_1 I_{z1}} M_z y \quad \Rightarrow \quad \boxed{\sigma_1 \approx \frac{M_z y}{I_{z1}}, \qquad \sigma_2 \approx 0}$$

Sample problem

- Determine the maximum normal stress in the faces (Al, E_1 = 72 Gpa) and the core (E_2 = 800 Mpa) using: (a) the general theory for composite beams, and (b) the approximate theory for sandwich beams. M = 3.0 kN-m.



- Solution:
- (a) the general theory

$$I_{z1} = \frac{b}{12}\left(h^3 - h_c^3\right) = 12.017\times10^{-12}\ \mathrm{m}^4, \quad I_{z2} = 56.25\times10^{-12}\ \mathrm{m}^4$$

$$\Rightarrow E_1 I_{z1} + E_2 I_{z2} = 910.2\times10^3\ \mathrm{kN}\cdot\mathrm{m}^2$$

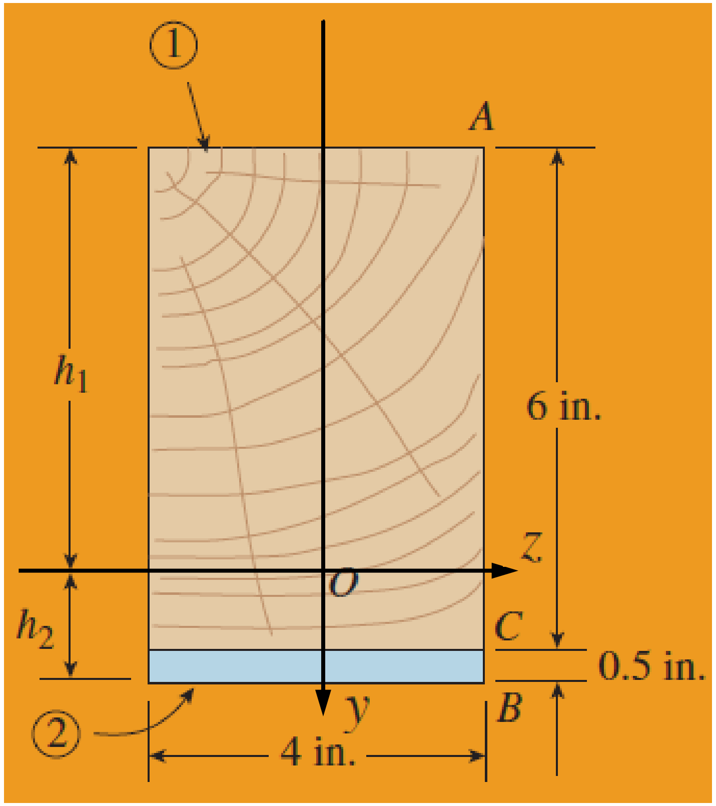
$$\Rightarrow (\sigma_1)_{\max} = \frac{E_1 M_z (h/2)}{E_1 I_{z1} + E_2 I_{z2}} = 19.0\ \mathrm{MPa}, \quad (\sigma_2)_{\max} = \frac{E_2 M_z (h_c/2)}{E_1 I_{z1} + E_2 I_{z2}} = 0.198\ \mathrm{MPa}$$

- (b) the approximate theory

$$(\sigma_1)_{\max} \approx \frac{M_z (h/2)}{I_{z1}} = 20.0\ \mathrm{MPa}, \qquad (\sigma_2)_{\max} \approx 0$$

Sample problem

- Calculate the largest tensile and compressive stresses in the wood (E_1 = 1500 ksi) and the maximum and minimum tensile stresses in the steel (material E_2 = 30,000 ksi) M = 60 kip-in.



- Solution:
- Neutral axis:

$$\left.\begin{aligned} &0 = E_1 A_1 \bar{y}_1 + E_2 A_2 \bar{y}_2 = -1500(4\times 6)(h_1 - 3) + 30000(4\times 0.5)(h_2 - 0.25) \\ &h_1 + h_2 = 6.25 \end{aligned}\right\}$$

$$\Rightarrow h_1 = 5.031 \text{ in}, \quad h_2 = 1.469 \text{ in}$$

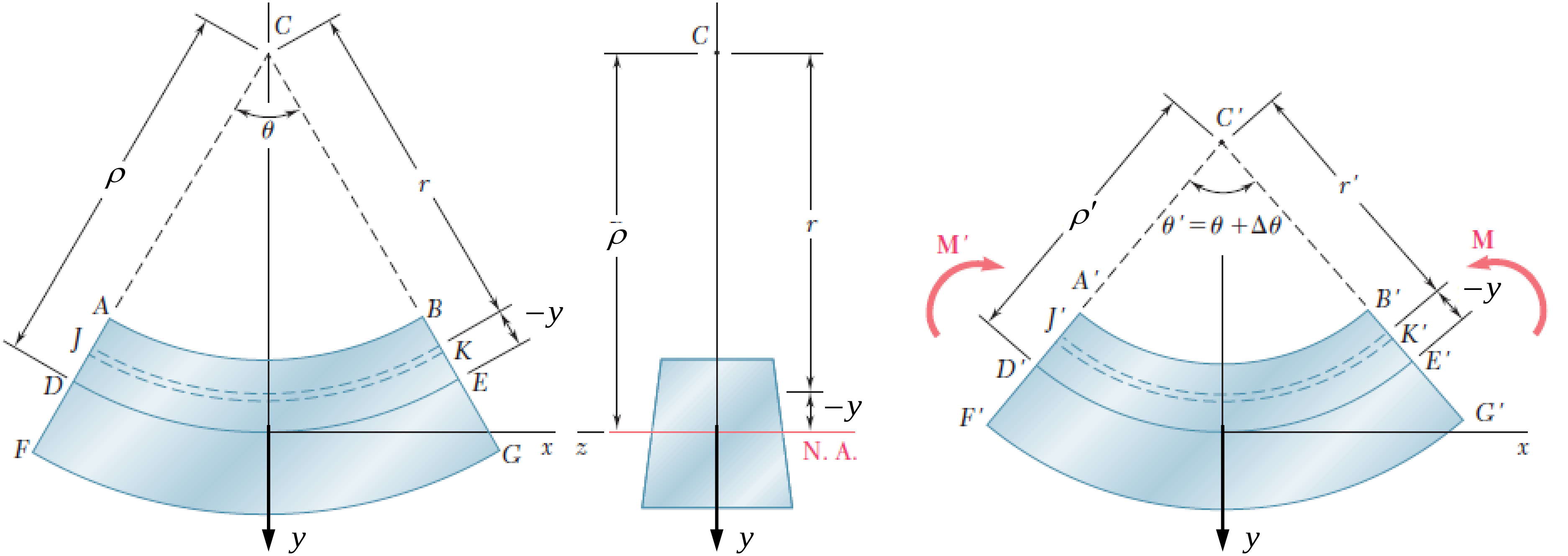
- Stresses along line A, C and B

$$\Rightarrow (\sigma_1)_A = -\frac{E_1 M_z h_1}{E_1 I_{z1} + E_2 I_{z2}} = -1.31 \text{ ksi}, \quad (\sigma_1)_C = \frac{E_1 M_z (h_2 - 0.5)}{E_1 I_{z1} + E_2 I_{z2}} = 0.251 \text{ ksi}$$

$$\Rightarrow (\sigma_2)_C = \frac{E_2 M_z (h_2 - 0.5)}{E_1 I_{z1} + E_2 I_{z2}} = 5.030 \text{ ksi} \quad (\sigma_2)_B = \frac{E_2 M_z h_2}{E_1 I_{z1} + E_2 I_{z2}} = 7.62 \text{ ksi}$$

$$\boxed{\frac{(\sigma_2)_C}{(\sigma_1)_C} = \frac{E_2 \varepsilon_C}{E_1 \varepsilon_C} = 20}$$

Bending of a Curved Beam



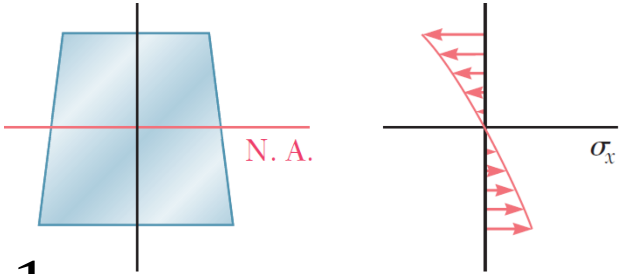
- Geometry
- Length of neutral surface remain unchanged: $\rho\theta = \rho'\theta'$
- Change of arc length: $\Delta JK = r'\theta' - r\theta = (\rho' + y)\theta' - (\rho + y)\theta = y\Delta\theta$
- Longitudinal strain: $\varepsilon = \dfrac{\Delta JK}{JK} = \dfrac{y}{r}\dfrac{\Delta\theta}{\theta} = \dfrac{y}{\rho + y}\dfrac{\Delta\theta}{\theta} = \dfrac{r-\rho}{r}\dfrac{\Delta\theta}{\theta}$
- Hooke's law: $\sigma = E\varepsilon = \dfrac{y}{r}\dfrac{E\Delta\theta}{\theta} = \dfrac{y}{\rho + y}\dfrac{E\Delta\theta}{\theta} = \dfrac{r-\rho}{r}\dfrac{E\Delta\theta}{\theta}$

Bending of a Curved Beam

- Neutral axis:

$$0 = F_N = \int_A \sigma dA = \int_A \frac{r-\rho}{r}\frac{E\Delta\theta}{\theta} dA = \frac{E\Delta\theta}{\theta}\int_A \left(1-\frac{\rho}{r}\right) dA$$

$$\Rightarrow \boxed{\rho = \frac{A}{\int_A \frac{dA}{r}}, \qquad \frac{1}{\rho} = \frac{1}{A}\int_A \frac{dA}{r}}$$



- Distance between C and centroid: $\bar{r} = \frac{1}{A}\int_A r dA \neq \rho$
- Static equilibrium and bending stress

$$M_z = \int_A y\cdot\sigma dA = \int_A \frac{y^2}{r}\frac{E\Delta\theta}{\theta} dA = \int_A \frac{(r-\rho)^2}{r}\frac{E\Delta\theta}{\theta} dA = \frac{E\Delta\theta}{\theta}\int_A \frac{r^2-2\rho r+\rho^2}{r} dA$$

$$= \frac{E\Delta\theta}{\theta}\left(A\bar{r} - 2\rho A + \rho A\right) \qquad \Rightarrow \qquad \boxed{\frac{E\Delta\theta}{\theta} = \frac{M_z}{A(\bar{r}-\rho)}, \qquad \bar{r} > \rho}$$

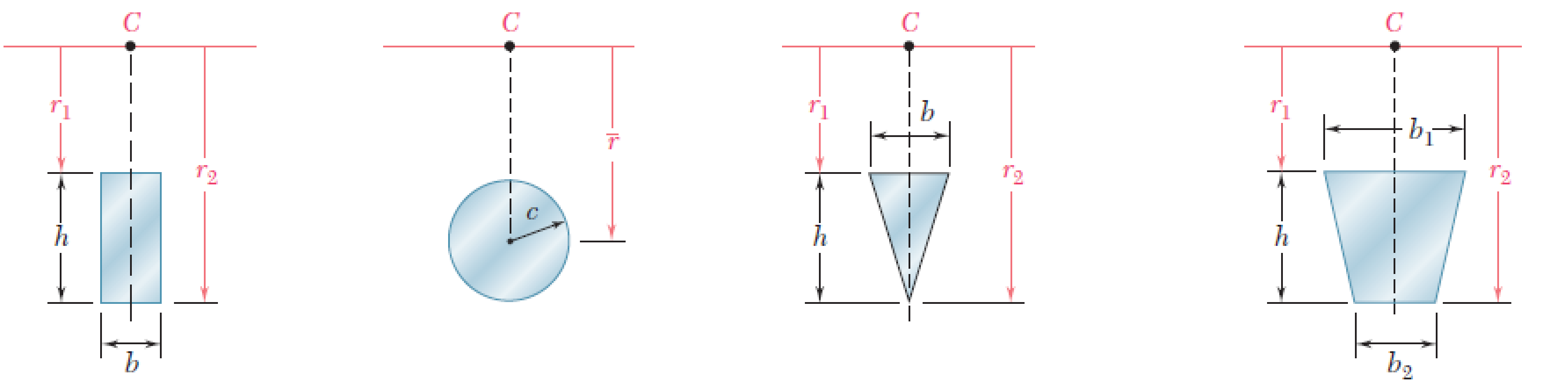
$$\sigma = \frac{y}{\rho+y}\frac{E\Delta\theta}{\theta} = \frac{r-\rho}{r}\frac{E\Delta\theta}{\theta} \qquad \Rightarrow \qquad \boxed{\sigma = \frac{M_z y}{A(\bar{r}-\rho)(\rho+y)} = \frac{M_z(r-\rho)}{A(\bar{r}-\rho)r}}$$

Bending of a Curved Beam

- The change in curvature of the neutral surface :

$$\rho\theta = \rho'\theta', \quad \frac{\Delta\theta}{\theta} = \frac{M_z}{EA(\bar{r}-\rho)} \Rightarrow \boxed{\frac{1}{\rho'} - \frac{1}{\rho} = \frac{\theta'}{\rho\theta} - \frac{1}{\rho} = \frac{1}{\rho}\frac{\Delta\theta}{\theta} = \frac{M_z}{EA(\bar{r}-\rho)\rho}}$$

- Radius of neutral surface for various cross-sectional shapes..



Rectangle

$$\rho = \frac{h}{\ln \frac{r_2}{r_1}}$$

Circle

$$\rho = \tfrac{1}{2}\left(\bar{r} + \sqrt{\bar{r}^2 - c^2}\right)$$

Triangle

$$\rho = \frac{\tfrac{1}{2}h}{\frac{r_2}{h}\ln\frac{r_2}{r_1} - 1}$$

Trapezoid

$$\rho = \frac{\tfrac{1}{2}h^2(b_1 + b_2)}{(b_1 r_2 - b_2 r_1)\ln\frac{r_2}{r_1} - h(b_1 - b_2)}$$

Sample Problem

- Determine the largest tensile and compressive stresses for a curved rectangular bean shown below, knowing that

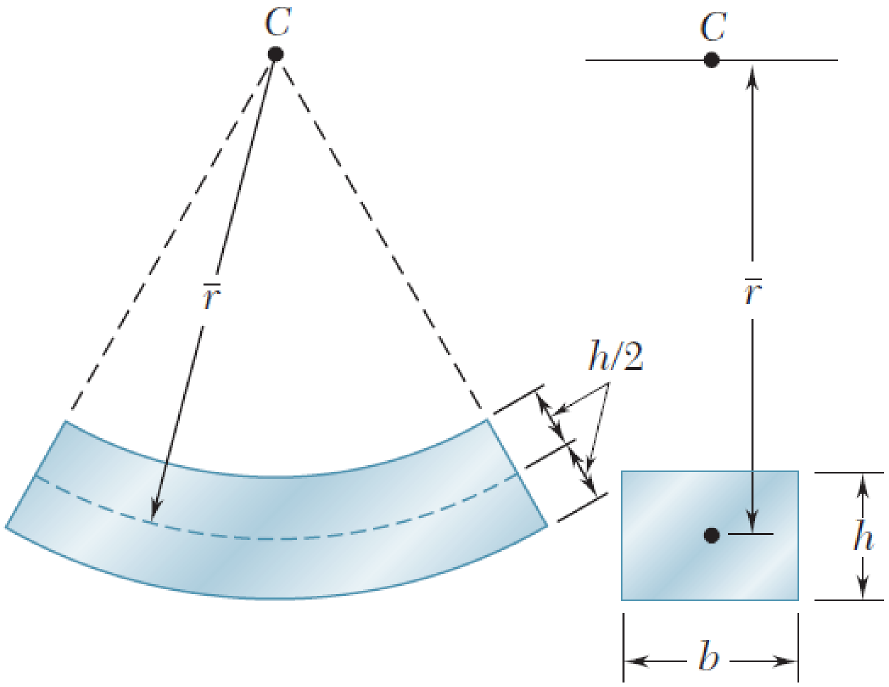
$$b = 2.5 \text{ in., } h = 1.5 \text{ in., } \bar{r} = 6 \text{ in., } M = 8 \text{ kip}\cdot\text{in., } E = 1500 \text{ ksi.}$$

- Solution:

$$\int_A \frac{dA}{r} = b\int_{\bar{r}-h/2}^{\bar{r}+h/2} \frac{dr}{r} = b\ln\frac{\bar{r}+h/2}{\bar{r}-h/2}$$

$$\Rightarrow \rho = \frac{A}{\int_A \frac{dA}{r}} = \frac{h}{\ln\frac{\bar{r}+h/2}{\bar{r}-h/2}} = 5.9686 \text{ in.}$$

$$\Rightarrow \bar{r} - \rho = \bar{r} - \frac{h}{\ln\frac{\bar{r}+h/2}{\bar{r}-h/2}} = 0.0314 \text{ in.}$$



- It is necessary to calculate ρ with enough significant figures in order to obtain the usual degree of accuracy.

- Largest tensile and compressiv

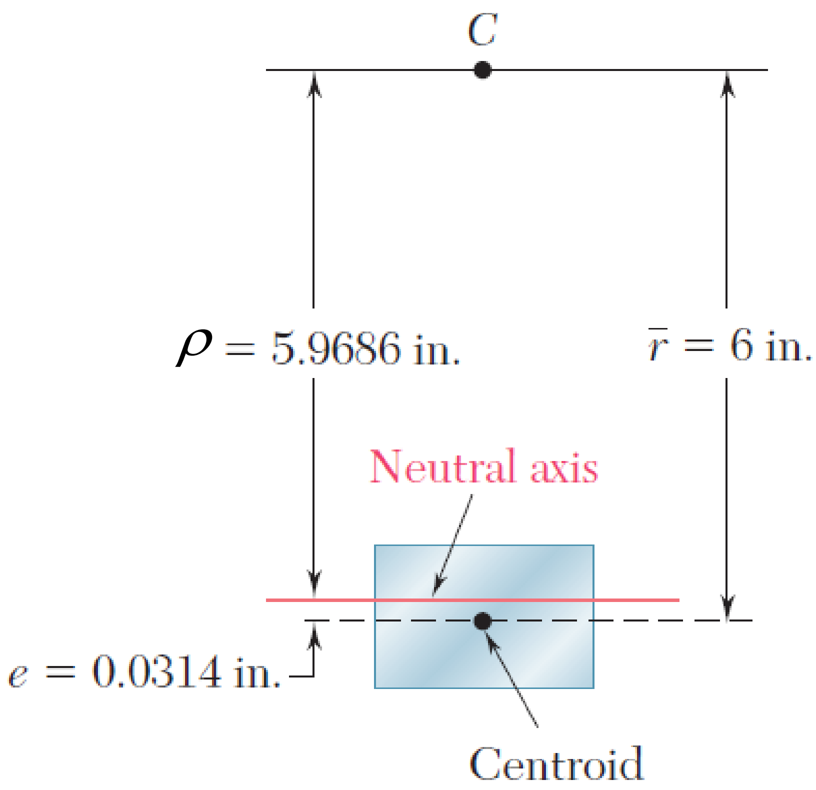
$$\sigma = \frac{M_z y}{A(\bar{r}-\rho)(\rho+y)} = \frac{M_z(r-\rho)}{A(\bar{r}-\rho)r}$$

$$\Rightarrow \sigma_{\max} = \sigma(r=6.75) = 7.86 \text{ ksi}$$

$$\Rightarrow \sigma_{\min} = \sigma(r=5.25) = -9.30 \text{ ksi}$$

- Stresses approximated by the theory for a straight bar:

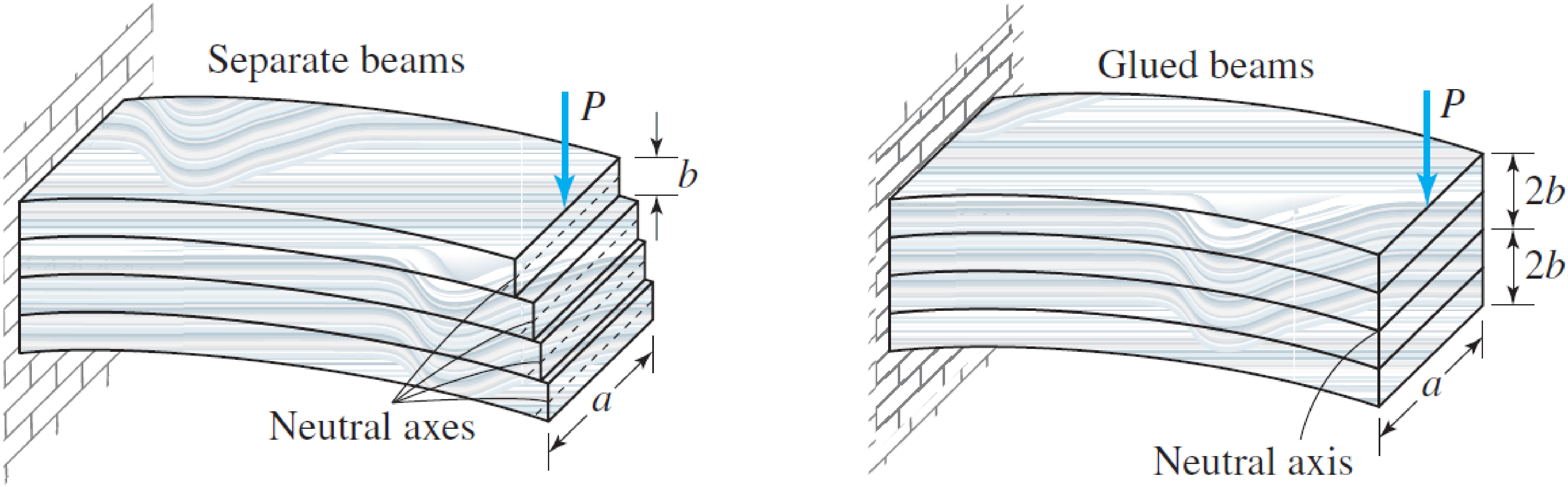
$$\sigma_{\max,\min} = \pm\frac{M_z(h/2)}{I_z} = \pm 8.53 \text{ ksi}$$



- The change in curvature of the neutral surface :

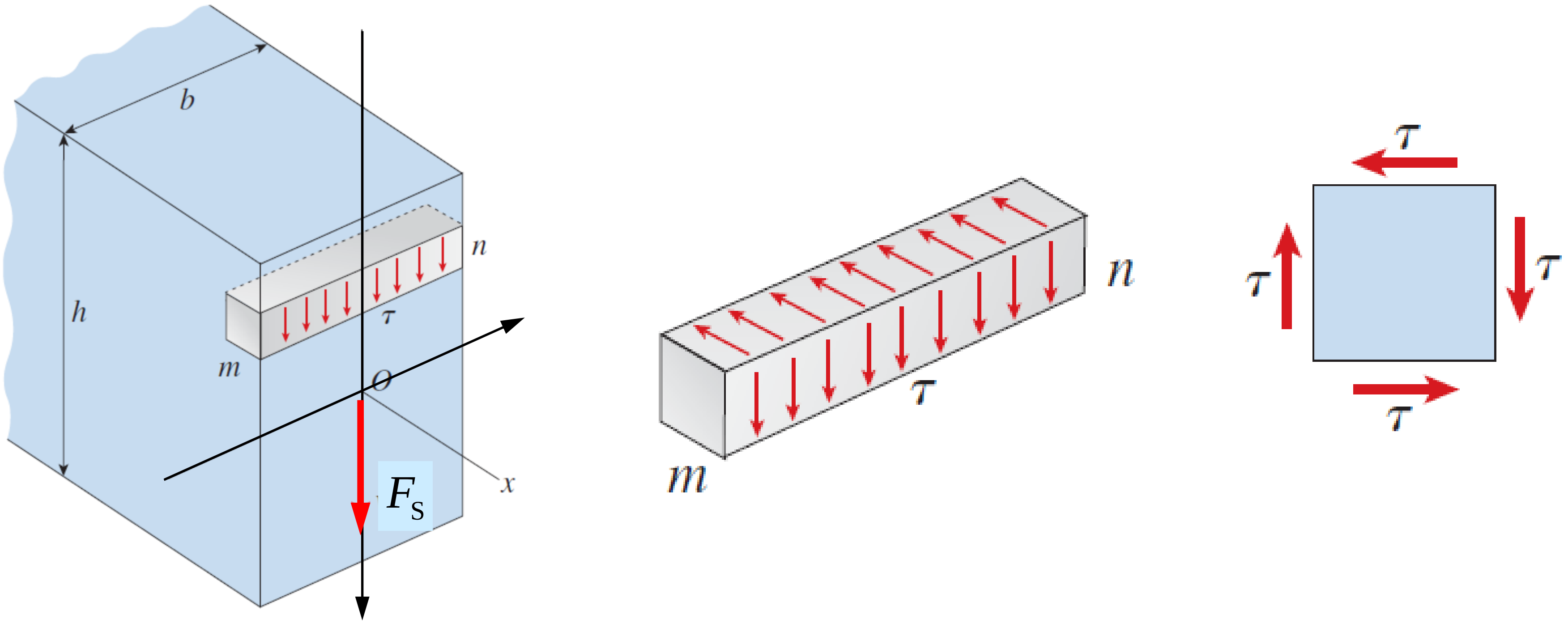
$$\frac{1}{\rho'} - \frac{1}{\rho} = \frac{M_z}{EA(\bar{r}-\rho)\rho} = 0.00758866$$

Shearing Stresses due to Transverse Loads



- If friction among beams is small, they will bend independently.
- The bottom surface of the upper beams will slide with respect to the top surface of the lower beams.
- Horizontal shearing stresses must develop along the glued surfaces in order to prevent the sliding.
- Because of the presence of these shearing stresses, the single solid beam is much stiffer and stronger than the separate beams.

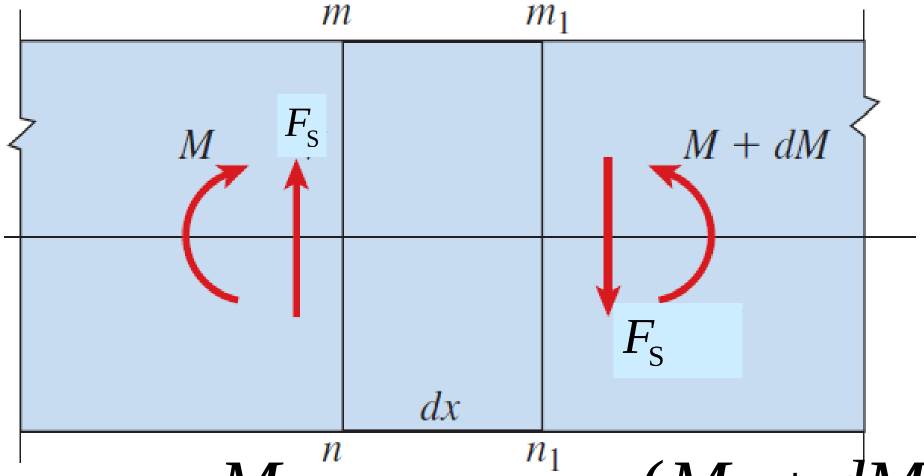
Shearing Stresses in a Rectangular Beam

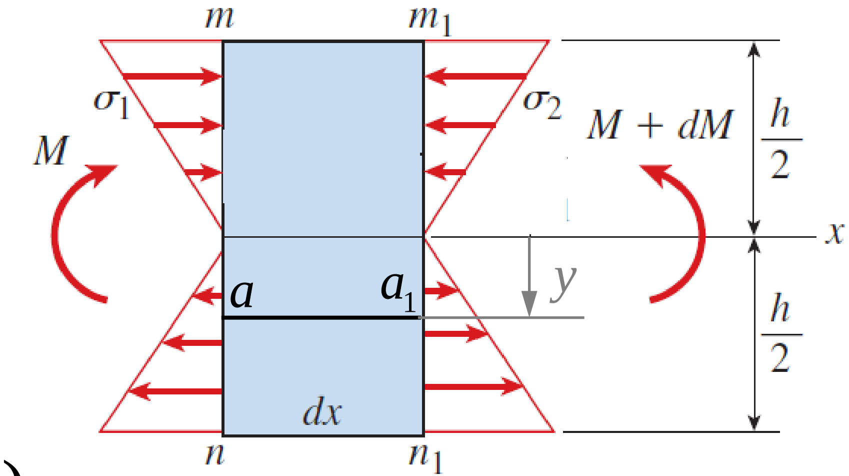


- Two assumptions:

- Shearing stresses acting on the cross section are parallel to shear force.
- Shearing stresses are uniformly distributed across the width of the beam, although they may vary over the height.

Shearing Stresses in a Rectangular Beam





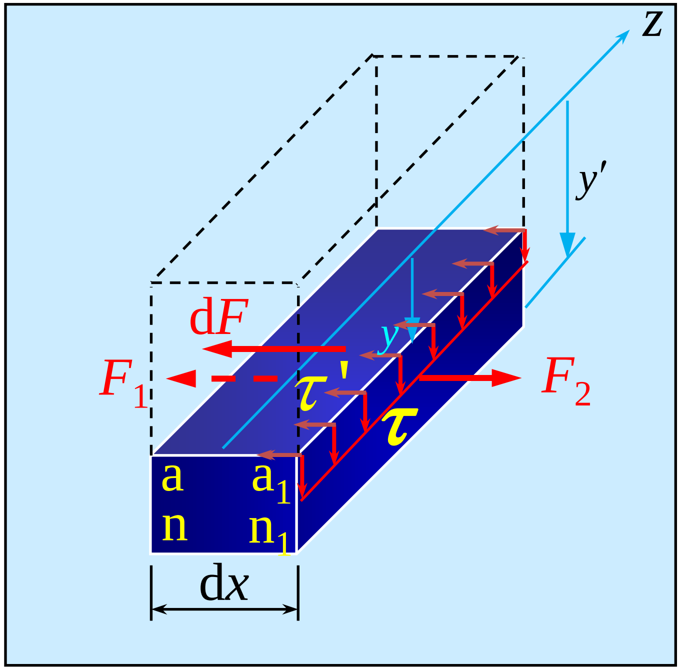
$$\sigma_1 = \frac{M_z y}{I_z}, \qquad \sigma_2 = \frac{(M_z + dM_z)y}{I_z}$$

- Horizontal shear forces/stresses

$$dF = F_2 - F_1 = \frac{dM_z}{I_z}\int_{A_1} y' \mathrm{d}A$$

$$\tau'_{\text{ave}}(y) = \frac{dF}{bdx} = \frac{\mathrm{d}M_z}{\mathrm{d}x}\frac{\int_{A_1} y' \mathrm{d}A}{I_z b} = \frac{F_S S_z^*}{I_z b} = \tau_{\text{ave}}(y)$$

$$\text{shear flow} \equiv f = \frac{dF}{dx} = \tau'_{\text{ave}} b = \frac{F_S S_z^*}{I_z}$$



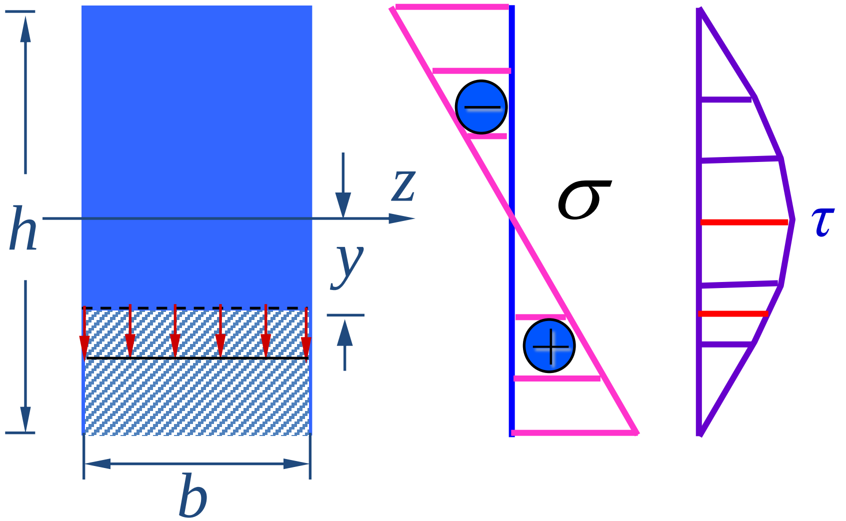
Shearing Stresses in a Rectangular Beam

- Vertical shearing stresses:

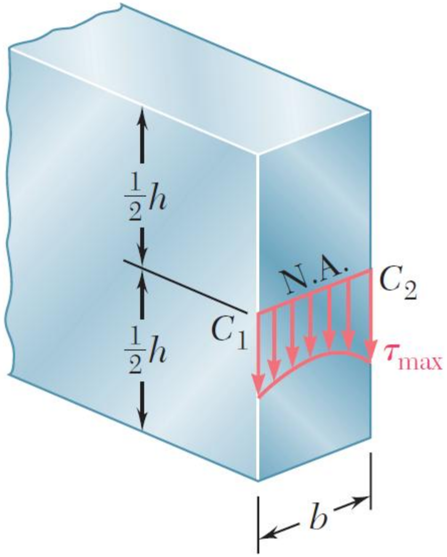
$$S_z^* = \int_{-b/2}^{b/2}\int_{y}^{h/2} y'\mathrm{d}A = b\int_{y}^{h/2} y'\mathrm{d}A = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\Rightarrow \tau_{\text{ave}}(y) = \frac{F_S \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)}{\frac{bh^3}{12} b} = \frac{3F_S}{2A}\left(1 - \frac{y^2}{(h/2)^2}\right)$$

$$\Rightarrow (\tau_{\text{ave}})_{\min} = \tau_{\text{ave}}(\pm h/2) = 0; \quad (\tau_{\text{ave}})_{\max} = \tau_{\text{ave}}(0) = \frac{3F_S}{2A}$$

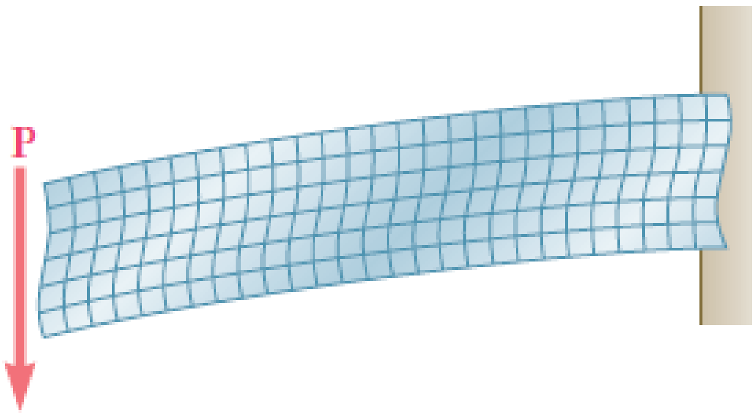


- If the width of the beam is comparable or large relative to its depth, the shearing stresses at C_1 and C_2 are significantly higher than their midpoint.
- Theory of elasticity shows that, for $h \geq 4b$, the maximum shearing stress *does not* exceed by more than 0.8% than the average value.



Effect of Shearing Stress/Strain

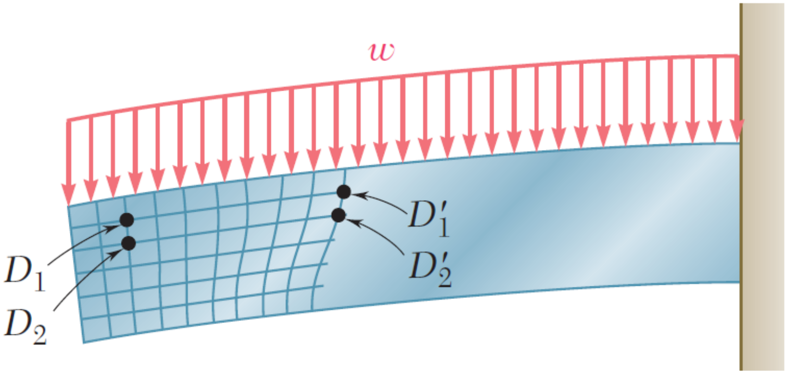
- If the shear force is constant along beam axis, warping is the same at every cross section.

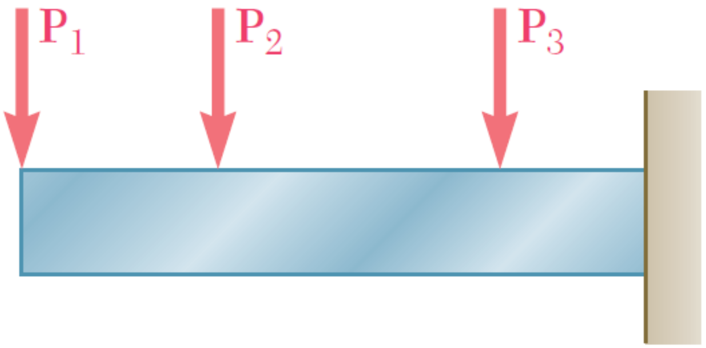


$$\gamma = \tau/G = \frac{3F_S}{2GA}\left(1 - \frac{y^2}{(h/2)^2}\right)$$

$$\Rightarrow \begin{cases} \gamma_{\min} = \gamma(\pm h/2) = 0 \\ \gamma_{\max} = \gamma(0) = \dfrac{3F_S}{2GA} \end{cases}$$

- In portions of the beam located under a distributed or concentrated load, normal stresses will be exerted on the horizontal faces of a cubic element of material, in addition to the stresses.



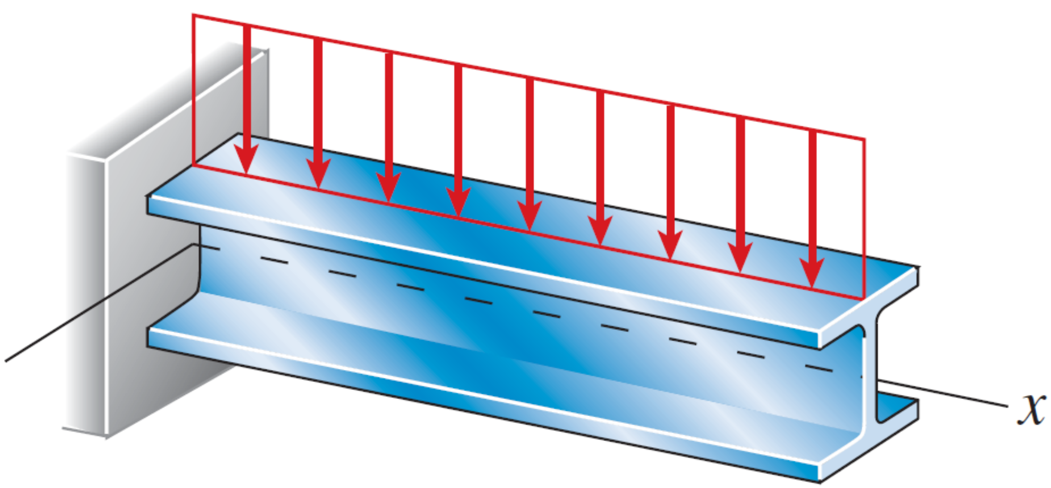


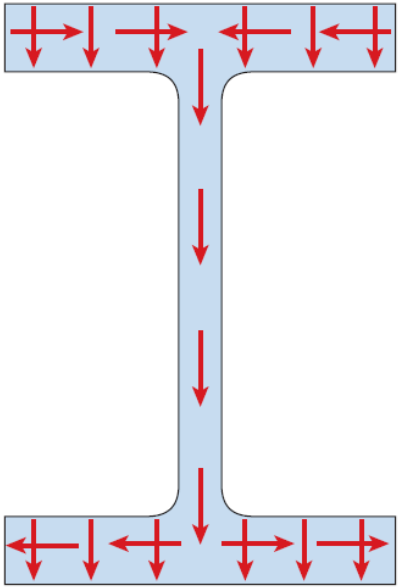
Effect of Shearing Stress/Strain

- Uniaxial stress state is violated due to the existence of shearing stress..
- Plane hypothesis is violated due to the existence of shear strain.
- **The error involved, however, is small for the values of the span-depth ratio encountered in practice.**
- Warping does not substantially affect the longitudinal strains even when the shear force varies continuously along the length.
- Thus, under most conditions it is justifiable to use the flexure formula for nonuniform bending, even though the formula was derived for pure bending.

Shearing Stresses in a Wide-flange Beam

- The shearing stresses in the web of a wide-flange beam act only in the vertical direction and are larger than the stresses in the flanges.
- The shearing stresses in the flanges of the beam act in both vertical and horizontal directions.
- The shear formula cannot be used to determine the vertical shearing stresses in the flanges.
- However, the shear formula does give good results for the shearing stresses acting horizontally in the flanges.





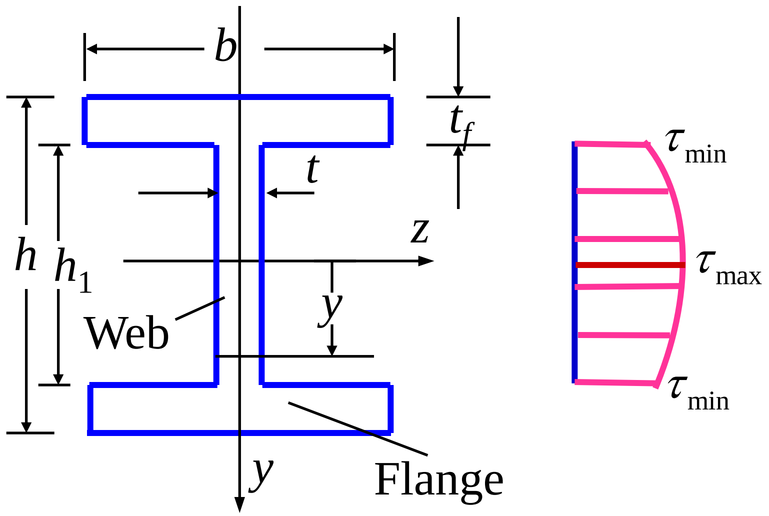
Shearing Stresses in a Wide-flange Beam

$$S_z^* = \int_{A_1} y\,dA = \frac{b}{2}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{t}{2}\left(\frac{h_1^2}{4} - y^2\right)$$

$$\tau = \frac{F_S S_z^*}{I_z t} = \frac{F_S}{8 I_z t}\left\{b\left(h^2 - h_1^2\right) + t\left(h_1^2 - 4y^2\right)\right\}$$

$$I_z = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}\left(bh^3 - bh_1^3 + th_1^3\right)$$

$$\Rightarrow \begin{cases} \tau_{\min} = \tau\left(y = \pm h_1/2\right) = \dfrac{F_S}{8 I_z t}\left(bh^2 - bh_1^2\right) \\ \tau_{\max} = \tau\left(y = 0\right) = \dfrac{F_S}{8 I_z t}\left(bh^2 - bh_1^2 + th_1^2\right) \end{cases}$$



- Shear force in the web

$$F_{S\cdot Web} = \int_{A_1} \tau\,dA = t\int_{-h_1/2}^{h_1/2} \tau\,dy = t\,\frac{F_S}{8 I_z t}\left\{\left(bh^2 - bh_1^2 + th_1^2\right)h_1 - 4t\,\frac{1}{3}\,\frac{h_1^3}{4}\right\}$$

$$= \frac{th_1}{3}\,\frac{F_S}{8 I_z t}\left(3bh^2 - 3bh_1^2 + 2th_1^2\right) = \frac{th_1}{3}\left(2\tau_{\max} + \tau_{\min}\right)$$

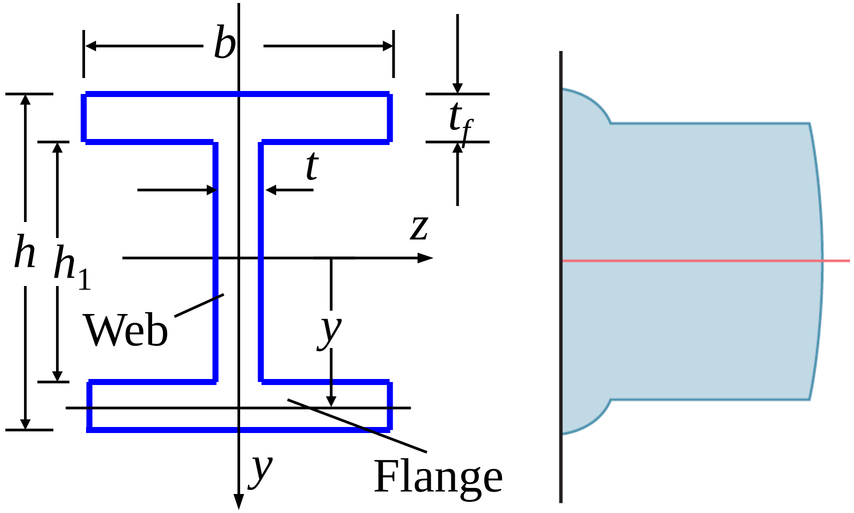
- For beams of typical proportions, shear force in the web is greater than 90% of the total shear force; the remainder is carried by shear in the flanges.

Shearing Stresses in a Wide-flange Beam

$$S_z^* = \int_{A_1} y dA = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$\tau = \frac{F_S S_z^*}{I_z b} = \frac{F_S}{8 I_z b}\left(bh^2 - 4by^2\right)$$

$$\Rightarrow \begin{cases} \tau_{\max} = \tau\left(y = \pm h_1/2\right) = \dfrac{F_S}{8 I_z b}\left(bh^2 - bh_1^2\right) \\ \tau_{\min} = \tau\left(y = \pm h/2\right) = 0 \end{cases}$$



- Discontinuities exist along the web/flange boundaries.
- The ratio between the minimum shearing stress in the web and the maximum stress in the flange is *b*/*t*.
- In practice, one usually assumes that the entire shear load is uniformly carried by the web ($\tau = F_S/A_{web}$).
- **We should note, however, that while the vertical shearing stress in the flanges can be neglected, its horizontal component has a significant value that will be determined as follows.**

Shearing Stresses in a Wide-flange Beam

- Consider a segment of a wide-flange beam subjected to the vertical shear F_S.
- The longitudinal shear force on the vertical cut

$$dF = \frac{(dM_z)S_z^*}{I_z} = \frac{(F_S dx)S_z^*}{I_z}$$

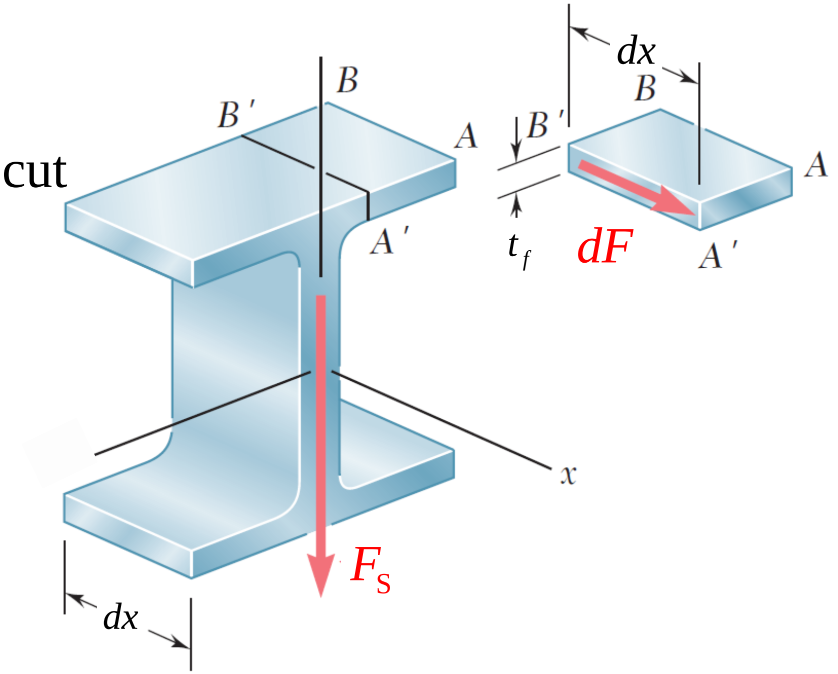
- The corresponding **average** shearing stress

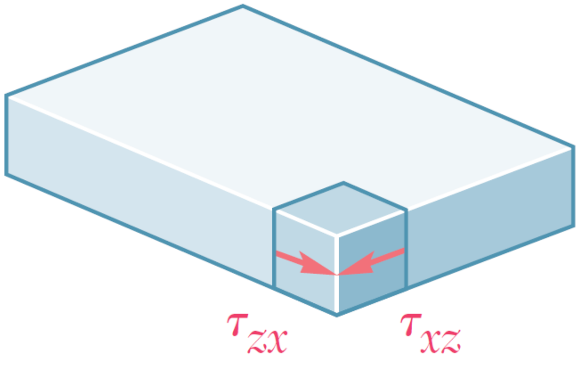
$$(\tau'_{zx})_{ave} = (\tau_{xz})_{ave} \approx \frac{dF}{t_f dx} = \frac{F_S S_z^*}{I_z t_f}$$

- Previously found a similar expression for the shearing stress in the web

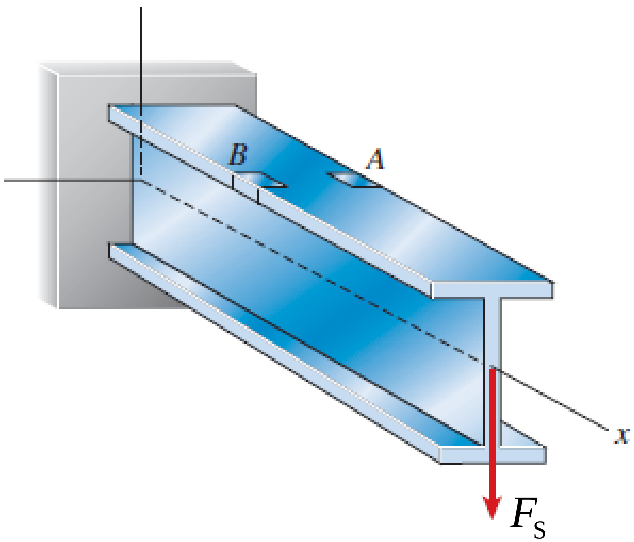
$$(\tau_{xy})_{ave} = \frac{F_S S_z^*}{I_z t}$$

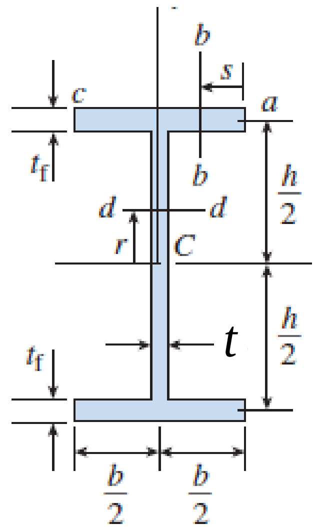
- NOTE: $\tau_{xy} \approx 0$ in the flanges
  $\tau_{xz} \approx 0$ in the web





Shearing Stresses in a Wide-flange Beam



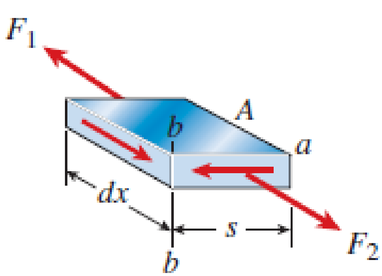


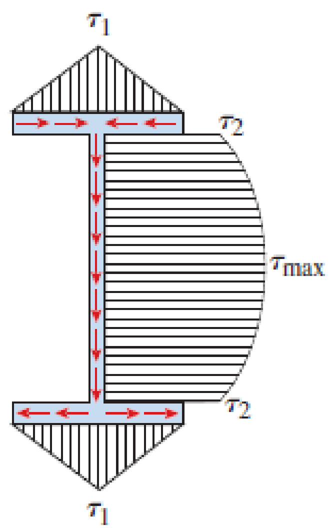
$$S_z^* = \int_s y\mathrm{d}A = st_f\, h/2$$

$$\Rightarrow \tau = \frac{F_S S_z^*}{I_z t_f} = \frac{shF_S}{2I_z}$$

$$\Rightarrow \tau_1 = \tau\left(s = b/2\right) = \frac{bhF_S}{4I_z}$$

$$\Rightarrow f_1 = \tau_1 t_f = \frac{bht_f F_S}{4I_z}$$





$$\tau_2 = \frac{F_S S_z^*}{I_z t} = \frac{F_S\left(bt_f\, h/2\right)}{I_z t} = \frac{bht_f F_S}{2I_z t}$$

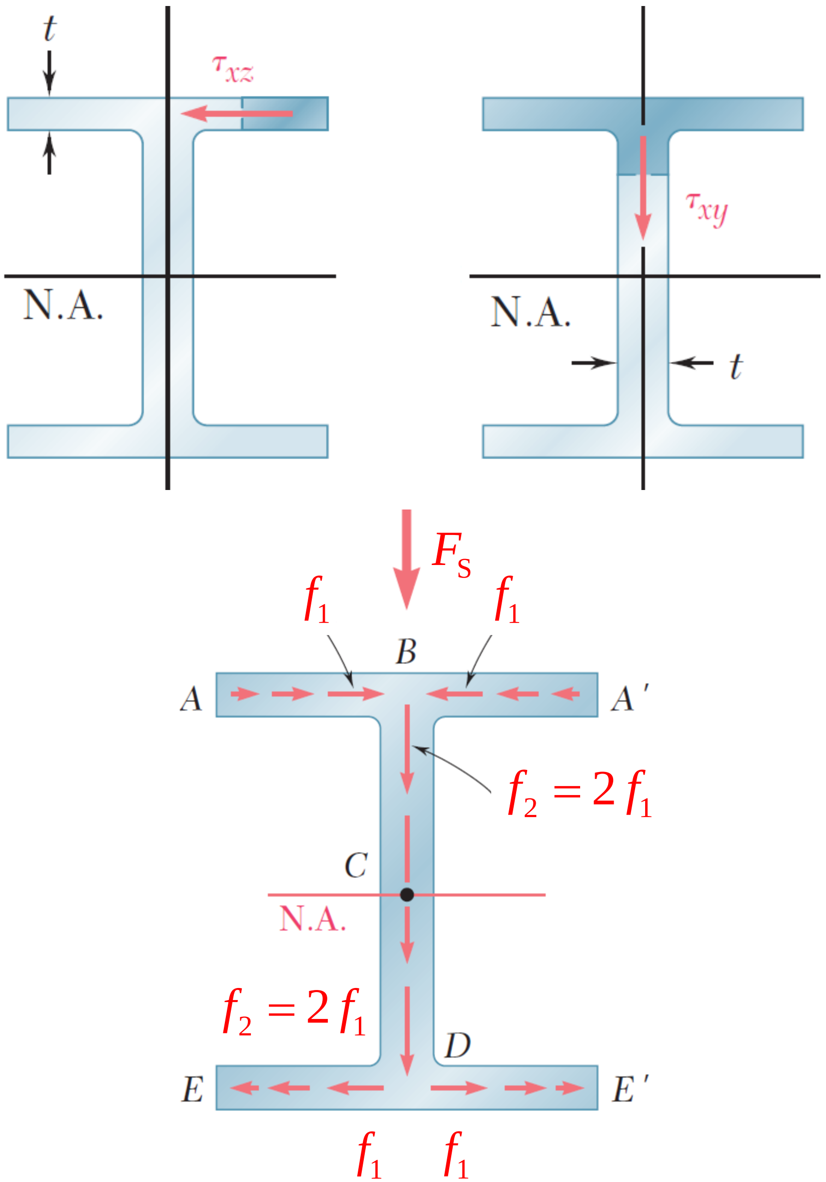
$$\Rightarrow f_2 = \tau_2 t = \frac{bht_f F_S}{2I_z} = 2 f_1$$

$$\tau_{\max} = \frac{F_S S_z^*}{I_z t} = \frac{F_S\left\{bt_f\, h/2 + \left(t\, h/2\right)\left(h/4\right)\right\}}{I_z t}$$

$$\Rightarrow f_{\max} = \frac{htF_S}{2I_z}\left(\frac{bt_f}{t} + \frac{h}{4}\right)$$

$$\frac{\tau_{\max}}{\tau_2} = 1 + \frac{ht}{4bt_f}, \qquad \frac{\tau_2}{\tau_1} = \frac{2t_f}{t}$$

Shear Flow in a Wide-flange Beam



- The variation of shear flow across the section depends only on the variation of the first moment.

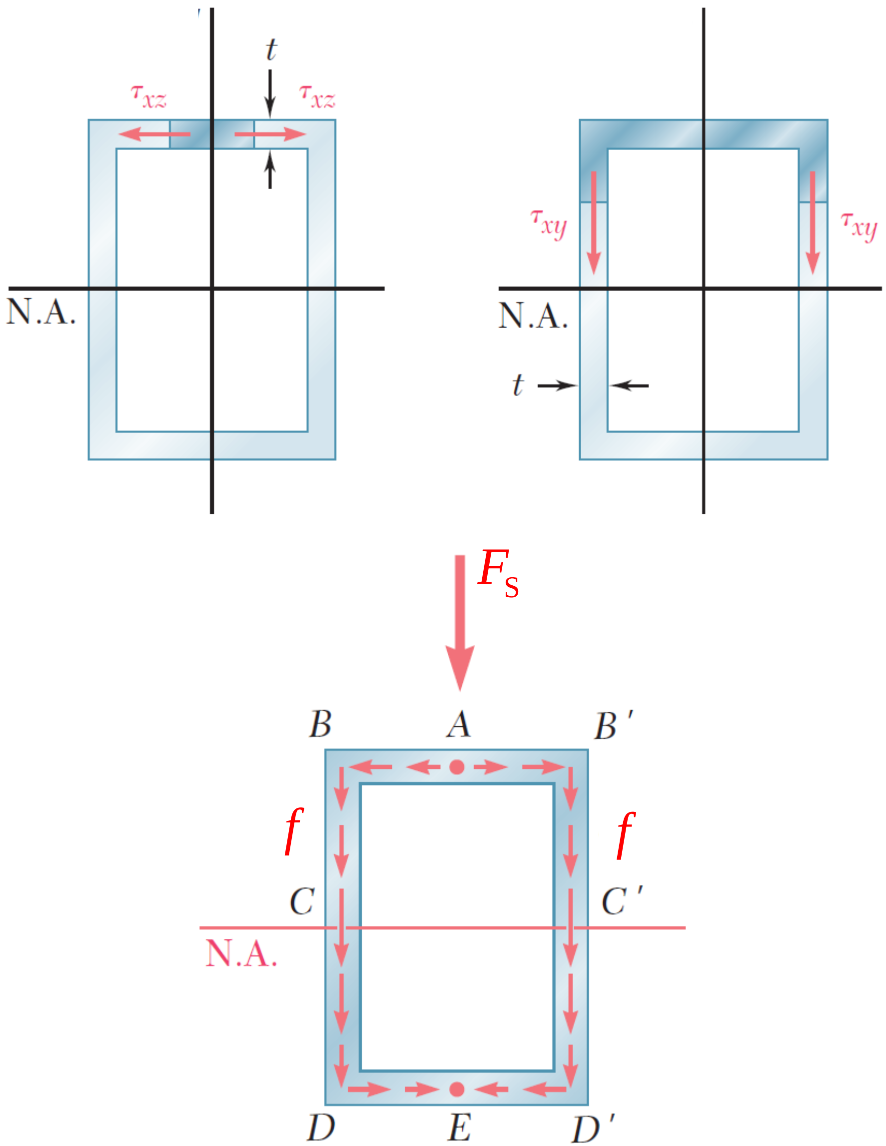
$$f = \tau'_{ave} t = \frac{F_S S_z^*}{I_z}$$

- For a wide-flange beam, the shear flow increases symmetrically from zero at *A* and *A'*, reaches a maximum at *C* and the decreases to zero at *E* and *E'*.

- The sense of f in the horizontal portions of the section may be deduced from the sense in the vertical portions or the sense of the shear F_S.

- The continuity of the variation in f and the merging of f from section branches suggests an analogy to fluid flow.

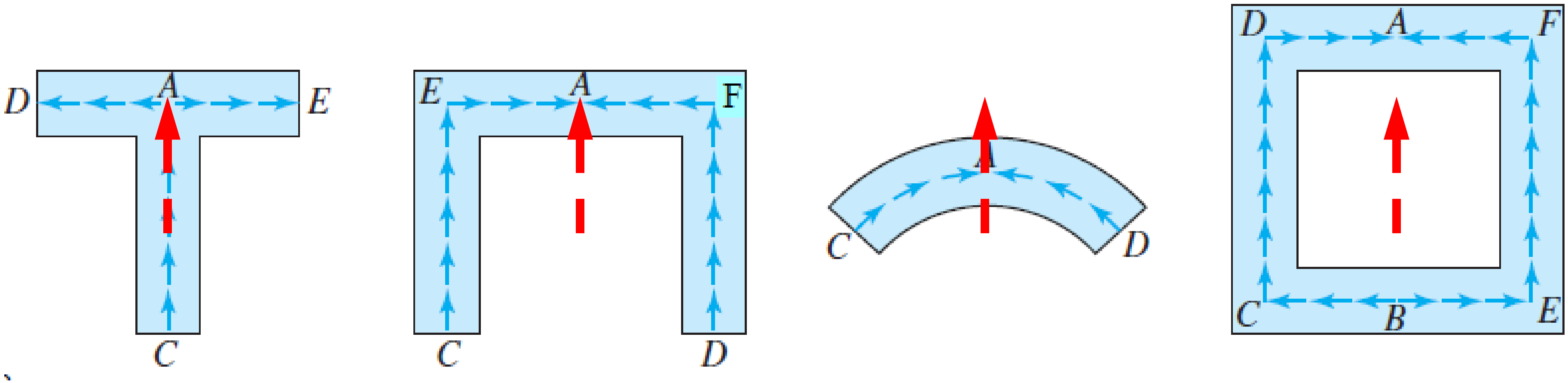
Shear Flow in a Box Beam



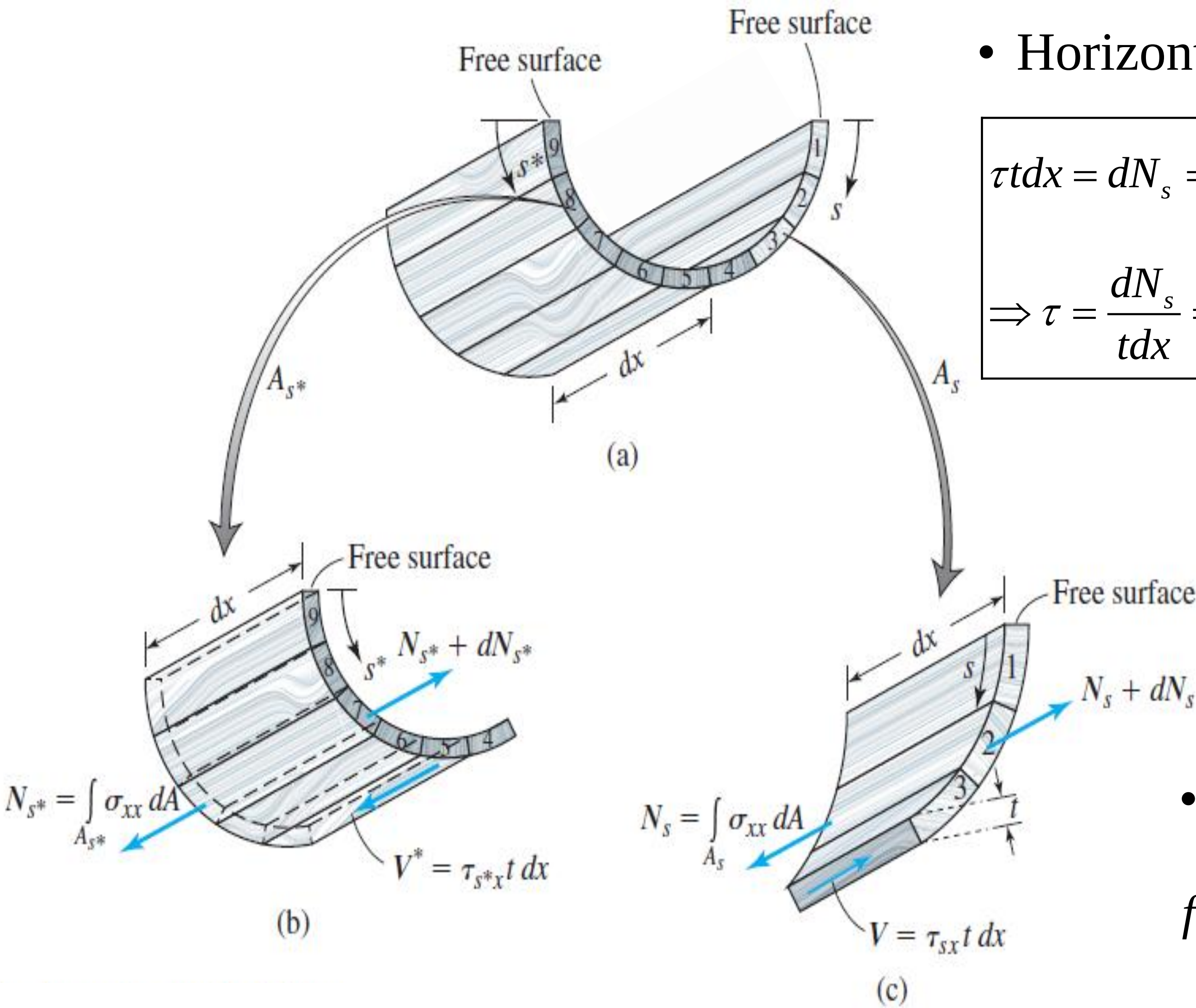
- For a box beam, f grows smoothly from zero at A to a maximum at C and C' and then decreases back to zero at E.

Shear Flow in a Thin-walled Beam

- The shearing stress formulae can be used to determine shearing stresses in thin-walled beams, as long as the loads are applied in a plane of symmetry of the member.
- In each case, the cut must be perpendicular to the surface of the member, and the shearing stress formulae will yield the component of the shearing stress in the direction of the tangent to that surface.
- The other component may be assumed equal to zero, in view of the proximity of the two free surfaces.



Shear Flow in a Thin-walled Beam



- Horizontal shearing stress

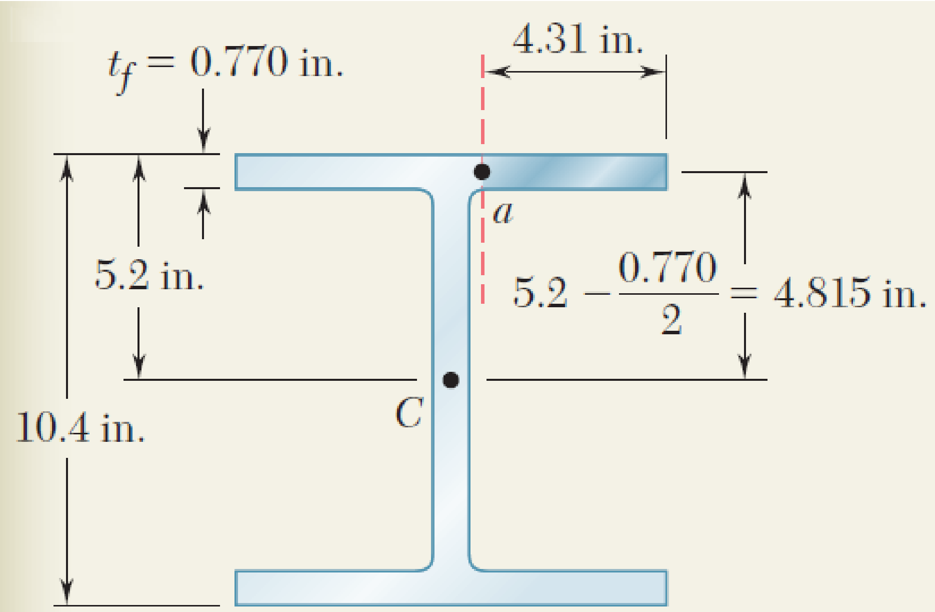
$$\tau t dx = dN_s = \frac{dM_z}{I_z}\int_s y dA$$

$$\Rightarrow \tau = \frac{dN_s}{tdx} = \frac{dM_z}{dx}\frac{\int_s ydA}{I_z t} = \frac{F_S S_z^*}{I_z t}$$

- Shear flow

$$f = \tau t = \frac{dF}{dx} = \frac{F_S S_z^*}{I_z}$$

Sample Problem – Shearing Stresses in Flanges



$I_z = 394\,\text{in}^4$

Knowing that the vertical shear is 50 kips in a rolled-steel beam, determine the horizontal shearing stress in the top flange at the point *a*.

SOLUTION:

- For the shaded area

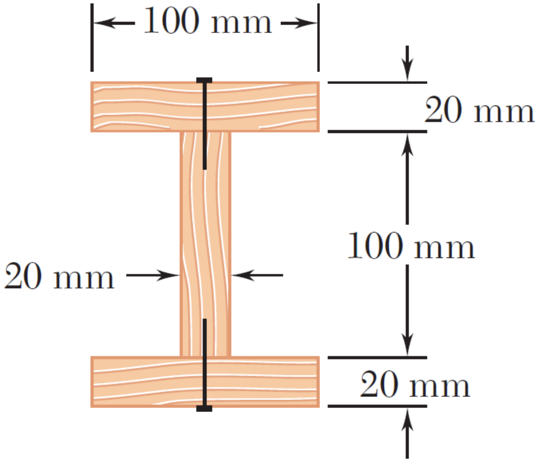
$$S_z^* = (4.31\,\text{in})(0.770\,\text{in})(4.815\,\text{in}) = 15.98\,\text{in}^3$$

- The shearing stress at *a*

$$\tau = \frac{F_S S_z^*}{I_z t} = \frac{(50\,\text{kips})(15.98\,\text{in}^3)}{(394\,\text{in}^4)(0.770\,\text{in})} = 2.63\,\text{ksi}$$

Sample Problem – Shear Force in a Web

- A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is F_S = 500 N, determine the shear force in each nail.



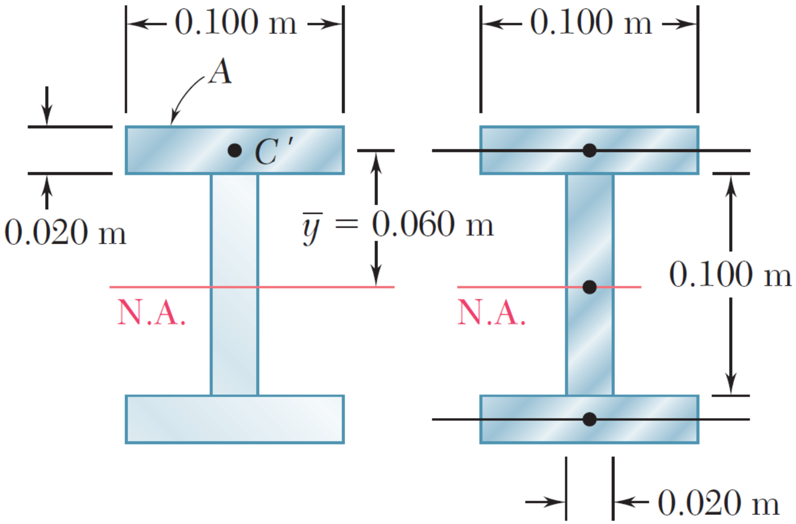
SOLUTION:

- Determine the horizontal force per unit length or shear flow f on the lower surface of the upper plank.

$$S_z^* = A\bar{y} = (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m}) = 120 \times 10^{-6}\ \text{m}^3$$

$$I_z = 16.20 \times 10^{-6}\ \text{m}^4$$

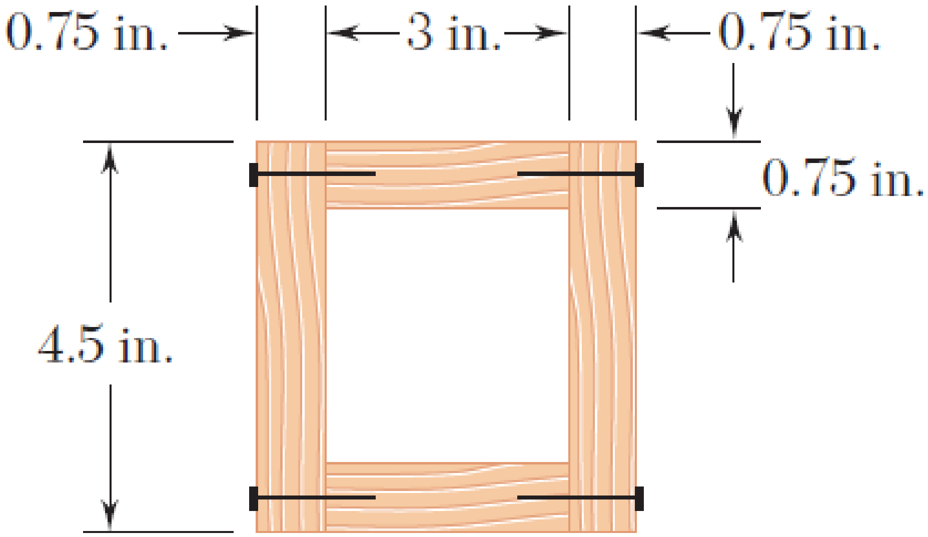
$$f = \frac{F_S S_z^*}{I_z} = \frac{(500\,\text{N})(120 \times 10^{-6}\,\text{m}^3)}{16.20 \times 10^{-6}\,\text{m}^4} = 3704\ \text{N/m}$$



- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

$$F = 0.025 f = 0.025(3704) = 92.6\,\text{N}$$

Sample Problem – Shear Force in Flanges

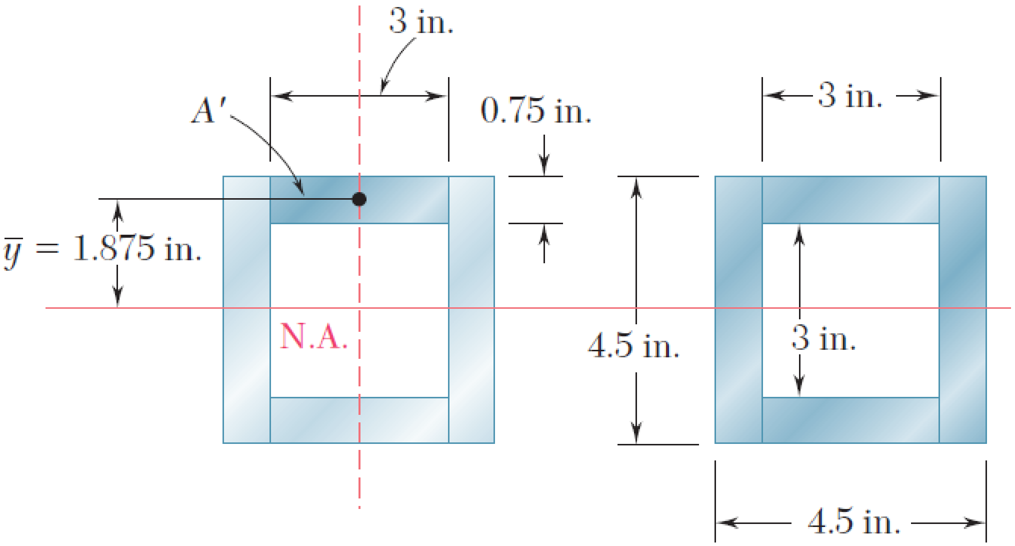


A square box beam is constructed from four planks as shown. Knowing that the spacing between nails is 1.5 in. and the beam is subjected to a vertical shear of magnitude F_S = 600 lb, determine the shearing force in each nail.

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.
- Based on the spacing between nails, determine the shear force in each nail.

Sample Problem – Shear Force in Flanges



- For the upper plank

$$S_z^* = A'y = (0.75\text{in.})(3\text{in.})(1.875\text{in.}) = 4.22\,\text{in}^3$$

- For the overall beam cross-section

$$I_z = \tfrac{1}{12}(4.5\,\text{in})^4 - \tfrac{1}{12}(3\,\text{in})^4 = 27.42\,\text{in}^4$$

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.

$$2f = \frac{F_S S_z^*}{I_z} = \frac{(600\,\text{lb})(4.22\,\text{in}^3)}{27.42\,\text{in}^4} = 92.3\frac{\text{lb}}{\text{in}}$$

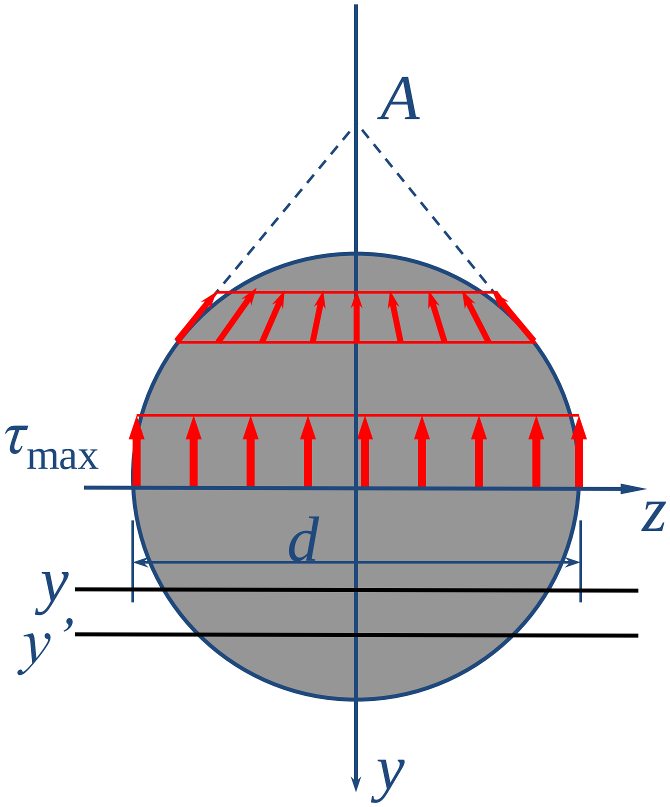
$$\Rightarrow f = 46.15\frac{\text{lb}}{\text{in}}$$

- Based on the spacing between nails, determine the shear force in each nail.

$$F = f\,\ell = \left(46.15\frac{\text{lb}}{\text{in}}\right)(1.5\,\text{in}) = 69.225\,\text{lb}$$

Shearing Stresses in a Circular Beam

- The shearing stresses can no longer be assumed parallel to the *y*-axis.
- On the boundary of the cross section, the shearing stress must act tangent to the boundary.
- Only the neutral axis is an exception.



- However, at a horizontal line we may further assume:

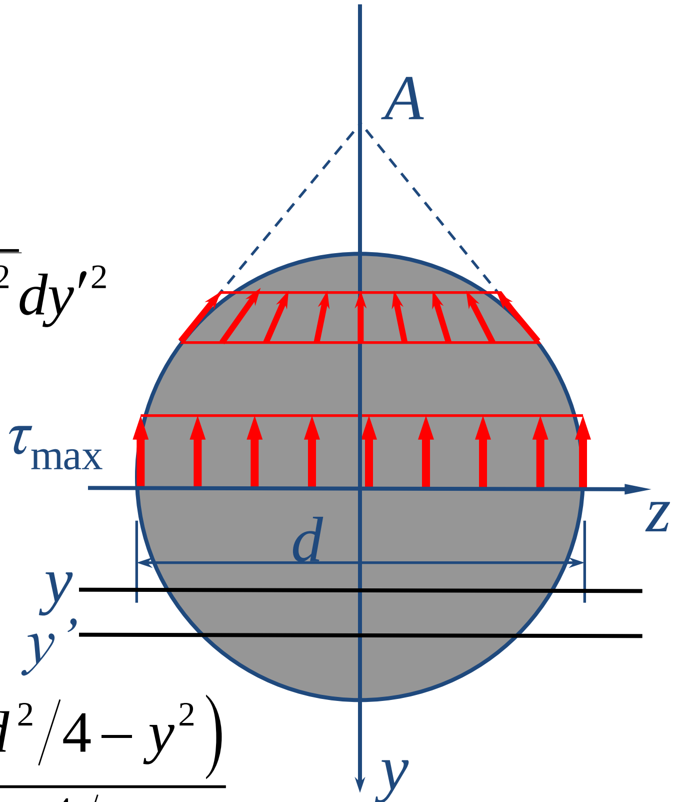
- Shearing stresses are concurrent at the intersection of boundary tangent and *y*-axis.
- The projection of shearing stresses on *y*-axis are uniformly distributed across the width of the beam.

Shearing Stresses in a Circular Beam

$$S_z^* = \int_{A_1} y' dA = \int_{y'=y}^{y'=d/2} \int_{z=-\sqrt{d^2/4-y'^2}}^{z=\sqrt{d^2/4-y'^2}} y' dy' dz$$

$$= \int_{y'=y}^{y'=d/2} 2y'\sqrt{d^2/4-y'^2}\, dy' = \int_{y'=y}^{y'=d/2} \sqrt{d^2/4-y'^2}\, dy'^2$$

$$= -\frac{2}{3}\left(d^2/4-y'^2\right)^{3/2}\Big|_{y'=y}^{y'=d/2} = \frac{2}{3}\left(d^2/4-y^2\right)^{3/2}$$



$$\Rightarrow \tau(y) = \frac{F_S S_z^*}{I_z b} = \frac{F_S \times \frac{2}{3}\left(d^2/4-y^2\right)^{3/2}}{\left(\pi d^4/64\right) \times 2\sqrt{d^2/4-y^2}} = \frac{F_S\left(d^2/4-y^2\right)}{3\pi d^4/64}$$

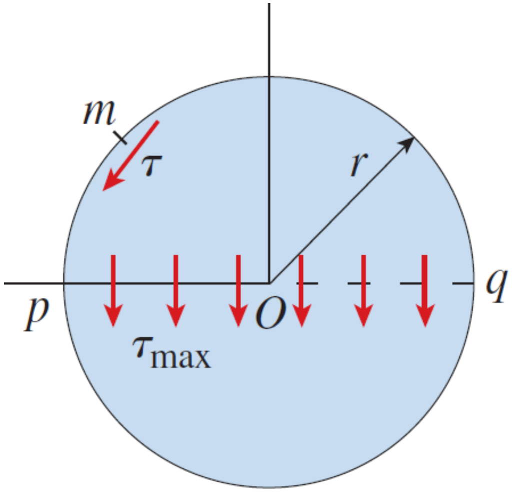
$$\Rightarrow \tau_{\min} = \tau\left(y=\pm d/2\right) = 0, \quad \tau_{\max} = \tau\left(y=0\right) = \frac{F_S \times \frac{2}{3}\frac{d^3}{8}}{\left(\pi d^4/64\right)\times d} = \frac{F_S \times \frac{4}{3}}{\pi d^2/4} = \frac{4}{3}\frac{F_S}{A}$$

Shearing Stresses in a Hollow Circular Beam

- Solid circular cross-sections

$$S_z^*(y=0) = A\bar{y} = (\pi r^2/2)(4r/3\pi) = 2r^3/3$$

$$\Rightarrow \tau_{\max} = \tau(y=0) = \frac{F_S S_z^*}{I_z b} = \frac{F_S(2r^3/3)}{(\pi r^4/4)(2r)} = \frac{4F_S}{3A}$$

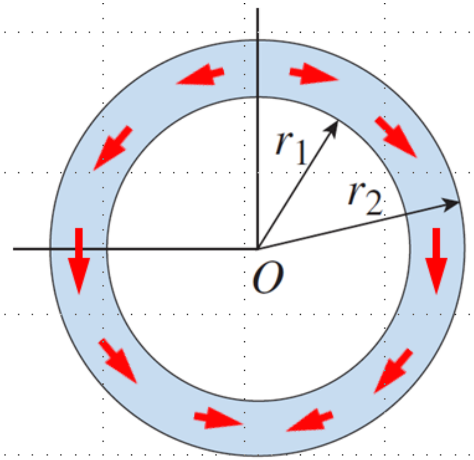


- Hollow circular cross-sections

$$S_z^*(y=0) = A_2\bar{y}_2 - A_1\bar{y}_1 = 2(r_2^3 - r_1^3)/3$$

$$\Rightarrow \tau_{\max} = \tau(y=0) = \frac{F_S\{2(r_2^3 - r_1^3)/3\}}{\{\pi(r_2^4 - r_1^4)/4\}\{2(r_2 - r_1)\}}$$

$$= \frac{4F_S}{3A}\frac{(r_2^2 + r_2 r_1 + r_1^2)}{(r_2^2 + r_1^2)}$$



- For a thin-walled circular beam: $\tau_{\max} = \frac{4F_S}{3A}\frac{(r_2^2 + r_2 r_1 + r_1^2)}{(r_2^2 + r_1^2)} \approx \frac{2F_S}{A}$

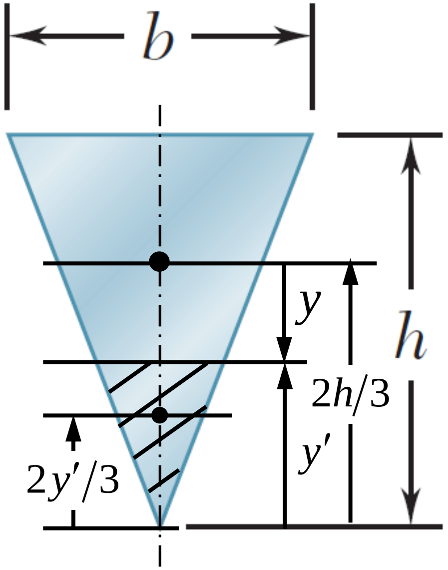
Shearing Stresses in an Equilateral Triangular Beam

$$I_z = \frac{bh^3}{12} - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2 = \frac{bh^3}{36}$$

$$S_z^* = \int_{A_1} y\,dA = A_1\bar{y}_1 = \left\{\frac{1}{2}\left(\frac{y'}{h}b\right)y'\right\}\left\{\frac{2}{3}(h-y')\right\} = \frac{by'^2(h-y')}{3h}$$

$$t(y) = \frac{y'}{h}b$$

$$\Rightarrow \boxed{\begin{aligned}\tau(y) &= \frac{F_S S_z^*}{I_z t} = \frac{F_S\left(by'^2(h-y')/3h\right)}{\left(bh^3/36\right)\left(by'/h\right)} = \frac{12F_S y'(h-y')}{bh^3} \\ \tau_{\max} &= \tau\left(y' = h/2\right) = \frac{3F_S}{bh} = \frac{3F_S}{2A}\end{aligned}}$$



- Although the theory for maximum shearing stresses in beams is approximate, it gives results differing by only a few percent from those obtained using the exact theory of elasticity.

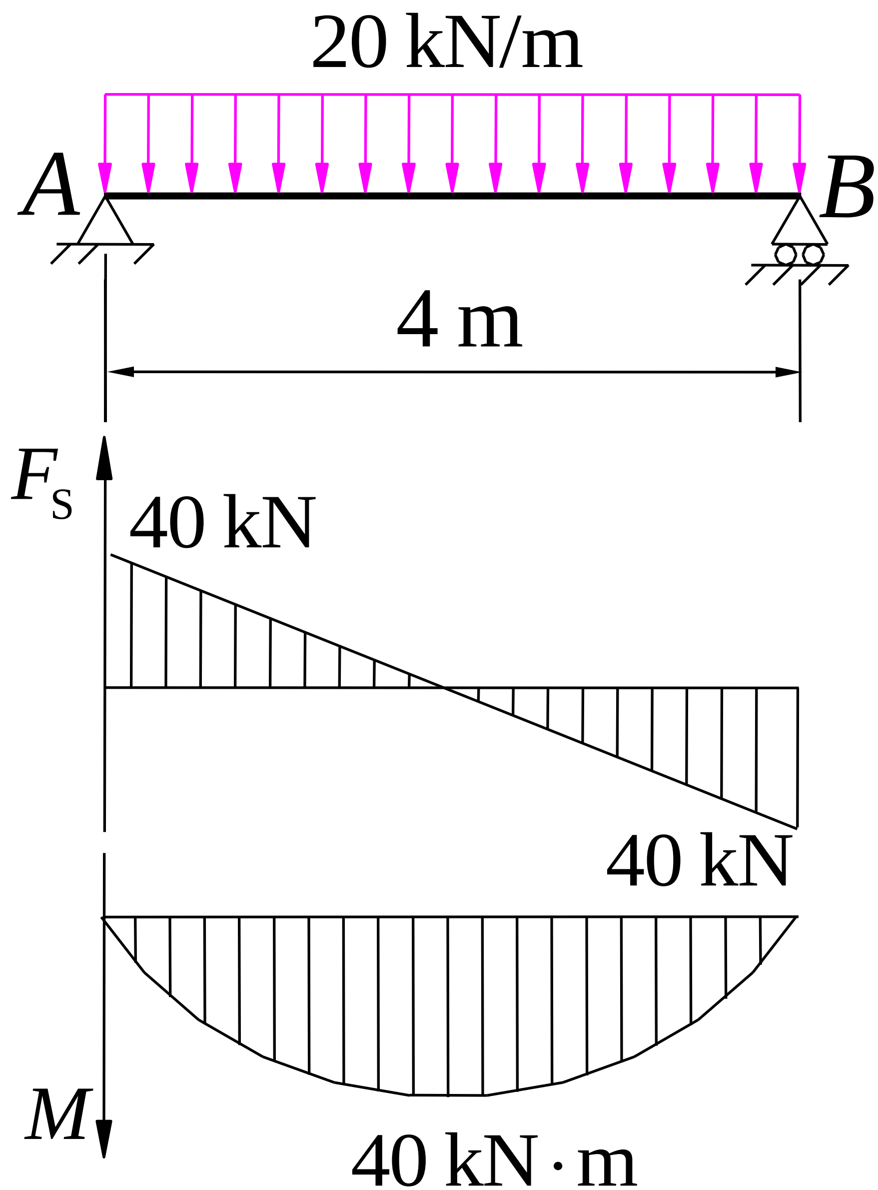
Shearing Stress Strength Condition

$$\tau_{\max} = \left(\frac{F_S S_z^*}{I_z b} \right)_{\max} \leq [\tau]$$

Sample Problem

- A circular beam is subjected to a uniformly distributed load q = 20 kN/m. The allowable normal and shearing stresses are $[\sigma]$ = 160 Mpa, $[\tau]$ = 100 MPa. Find the minimum required beam diameter.

- Solution
- Diagram of shearing forces & bending moments



- For normal stress:

$$\sigma_{\max} = \frac{M_{\max}}{W_z} \leq [\sigma] \Rightarrow \quad \frac{40 \times 10^3}{\frac{\pi d^3}{32}} \leq 160 \times 10^6$$

$$\Rightarrow d \geq 137 \, \text{mm}$$

- For shearing stress:

$$\tau_{\max} = \frac{4}{3} \frac{F_{S\max}}{A} \leq [\tau] \Rightarrow \frac{4}{3} \times \frac{40 \times 10^3}{\frac{\pi d^2}{4}} \leq 100 \times 10^6$$

$$\Rightarrow d \geq 26.1 \text{mm}$$

$$\Rightarrow \quad \boxed{d_{\min} = 137 \, \text{mm}}$$

Rational Design of Beams

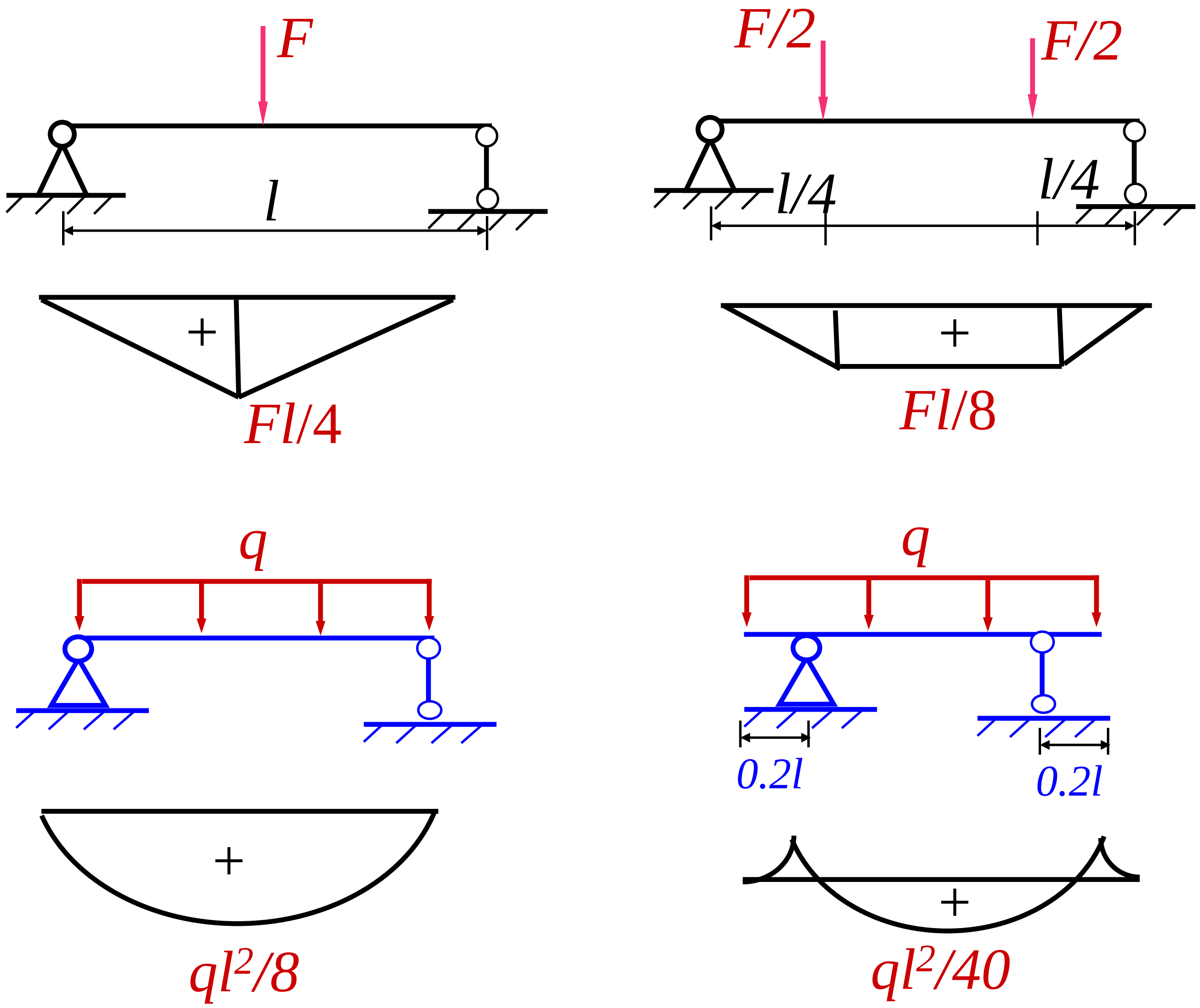
- Normal stress plays the most important role in satisfying the strength condition of beams under bending.

$$\sigma_{\max} = \left(\frac{M_z}{W_z} \right)_{\max} \leq [\sigma]$$

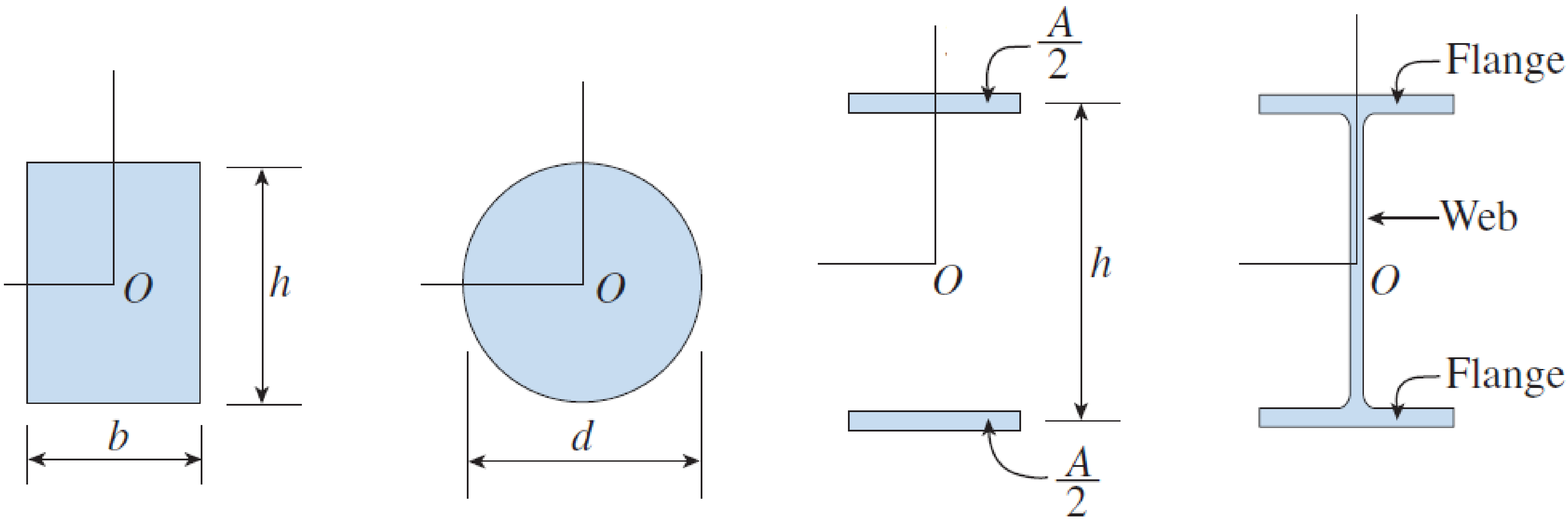
- Minimize the maximum bending moments by proper arrangements of the form and position of loading and constraints.

- Proper design of cross-sections to maximize bending section modulus.

Rational Design of Loads & Constraints



Rational Design of Cross-sections



$$W_z = \frac{bh^2}{6} = 0.167Ah$$

$$W_z = \frac{\pi d^3}{32} = 0.125Ad$$

$$W_z = \frac{2(A/2)(h^2/4)}{h/2} = 0.5Ah$$

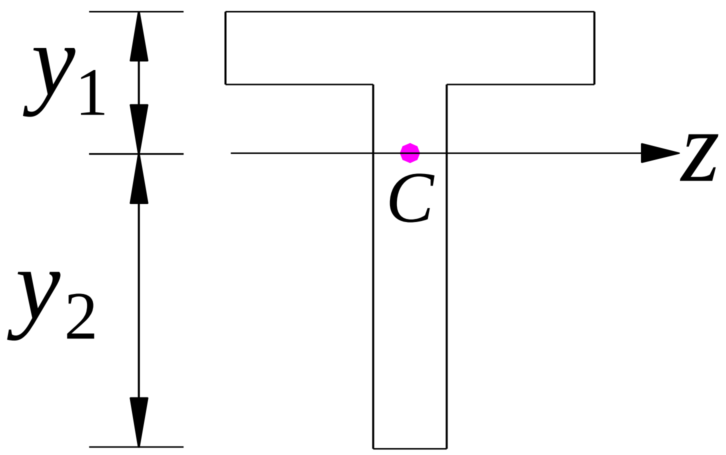
$$W_z \approx 0.35Ah$$

- Compare a square with a circle: $W_z = \dfrac{hh^2}{6} = \dfrac{\sqrt{\pi}}{12}Ad = 0.1477Ad$
- Among beam section choices which have an acceptable section modulus, the one with the smallest weight per unit length or cross sectional area will be the least expensive and the best choice.

Symmetry vs. Asymmetry

- For materials with $[\sigma^+] = [\sigma^-]$, symmetric cross-sections may be used such that the maximum tensile and compressive stress are equal in magnitude at the upper/lower edges.

- For materials with $[\sigma^+] < [\sigma^-]$, i.e. casting irons, cross-sectional neutral axis should deviate toward the tensile side.

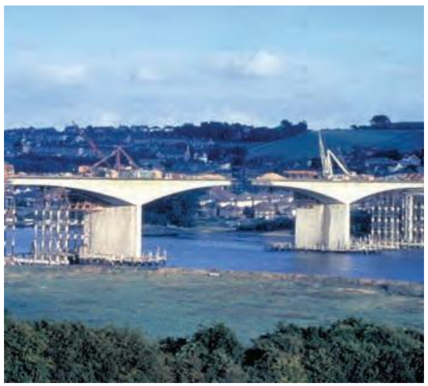


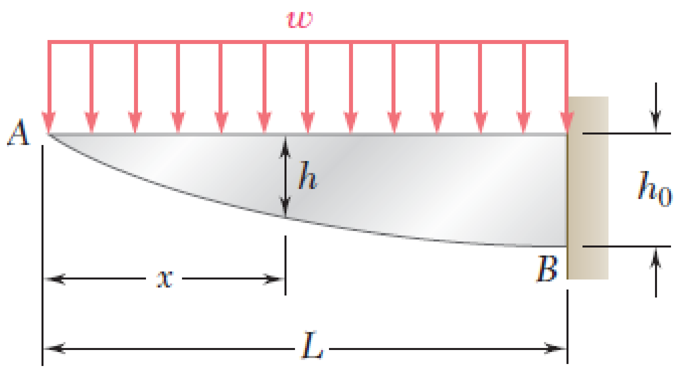
$$\frac{\sigma^{+}_{\max}}{\left|\sigma^{-}_{\max}\right|} \Rightarrow \frac{|y_1|}{y_2} = \frac{[\sigma^{+}]}{[\sigma^{-}]}$$

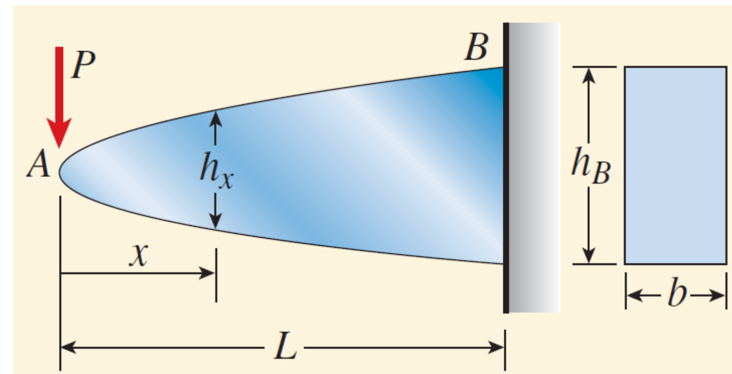
Nonprismatic and Constant-strength Beams

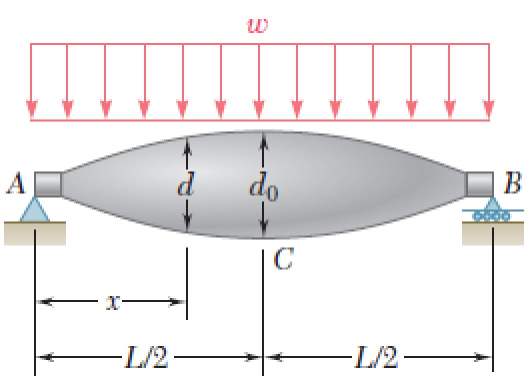
- The maximum normal stress stays the same for every cross-section.

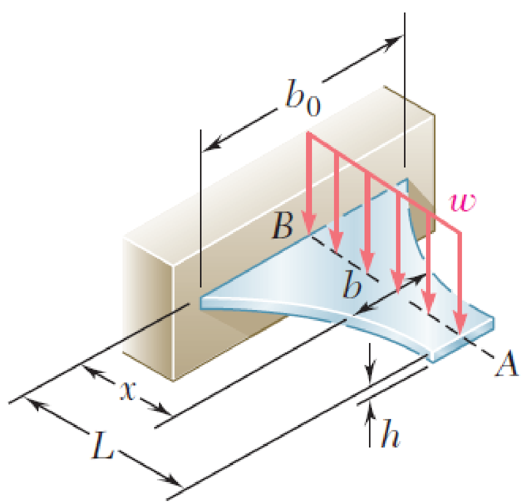
$$\sigma_{\max} = \{M(x)/W(x)\}_{\max} \le [\sigma]$$

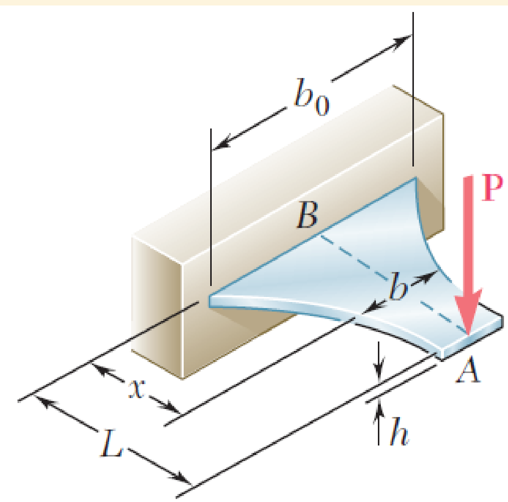




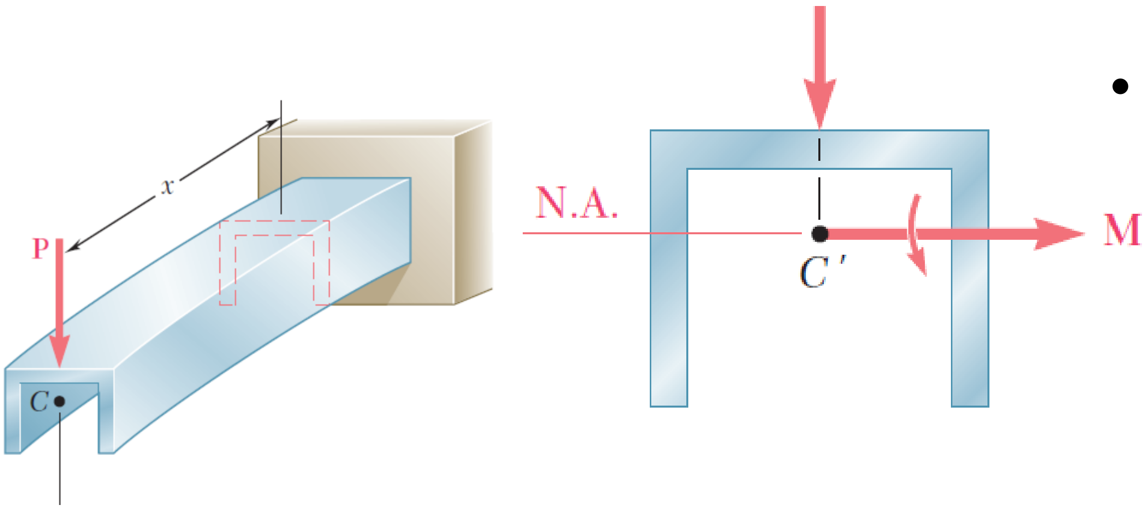






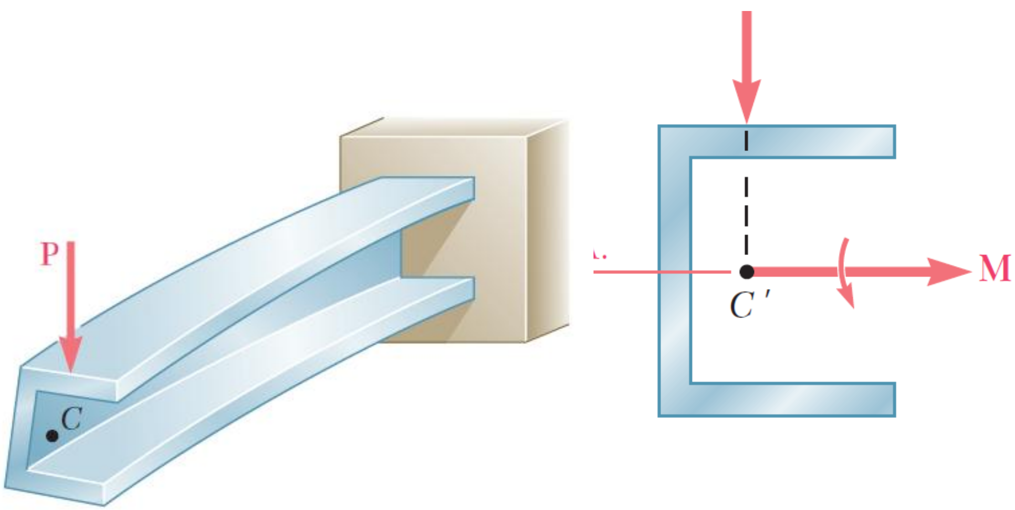


Unsymmetric Loading of Thin-Walled Members



- Beams loaded in a vertical plane of symmetry result in deforms in the symmetry plane without twisting.

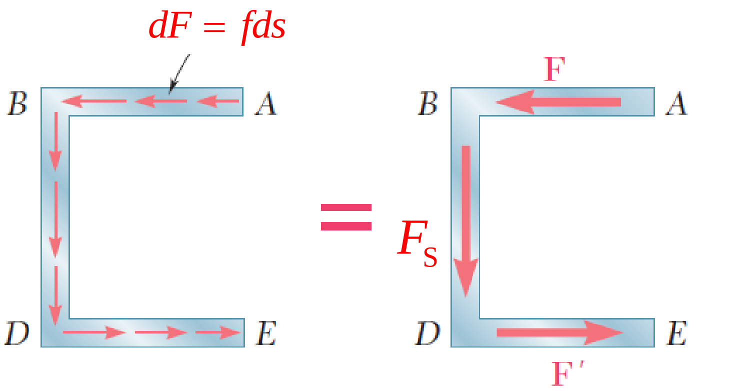
$$\sigma = \frac{M_z y}{I_z}, \qquad \tau_{\text{ave}} = \frac{F_S S_z^*}{I_z t}$$

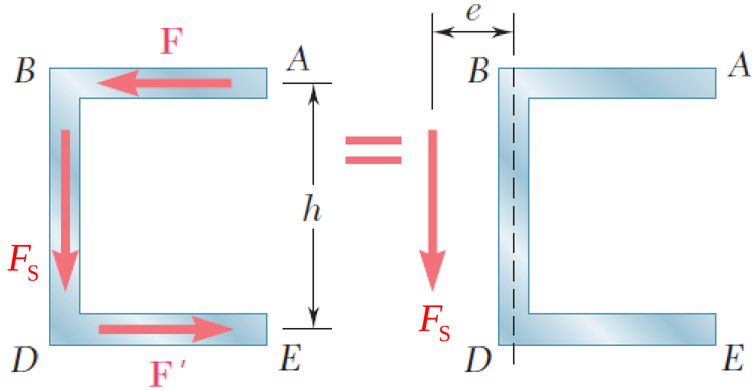


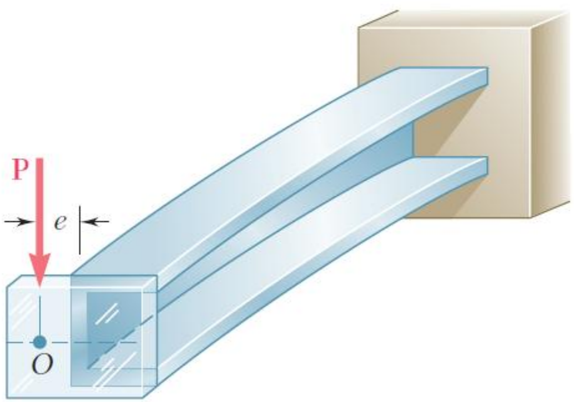
- Beams without a vertical plane of symmetry bend and twist under transverse loading.

$$\sigma = \frac{M_z y}{I_z}, \qquad \tau_{\text{ave}} \neq \frac{F_S S_z^*}{I_z b}$$

Unsymmetric Loading of Thin-Walled Members







- If the shear load is applied such that the beam does not twist, then the shearing stress distribution satisfies

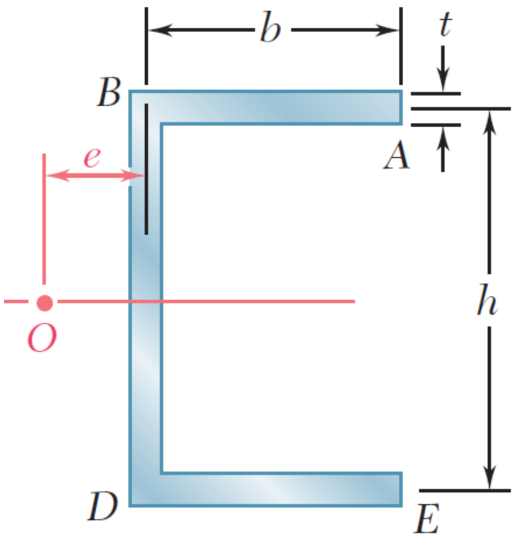
$$\tau_{ave} = \frac{F_S S_z^*}{I_z t}, \quad F_S = \int_B^D f\, ds, \quad F = \int_A^B f\, ds = -\int_D^E f\, ds = -F'$$

- *F* and *F'* indicate a couple *Fh* and the need for the application of a torque as well as the shear load.

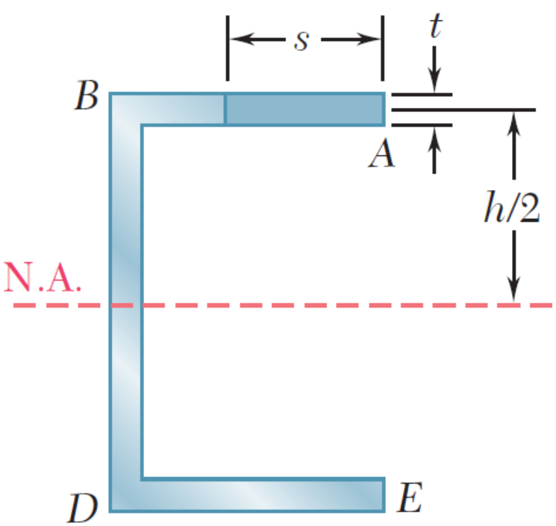
$$Fh = Ve$$

- When the force *P* is applied at a distance e to the left of the web centerline (**shear center**), the member bends in a vertical plane without twisting.

Sample Problem



- Determine the location for the shear center of the channel section with b = 4 in., h = 6 in., and t = 0.15 in.
- Determine the shearing stress distribution for F_S = 2.5 kips applied at the shear center.

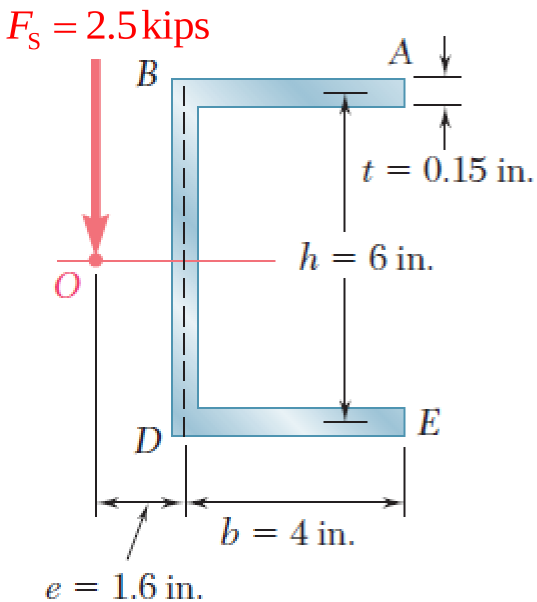


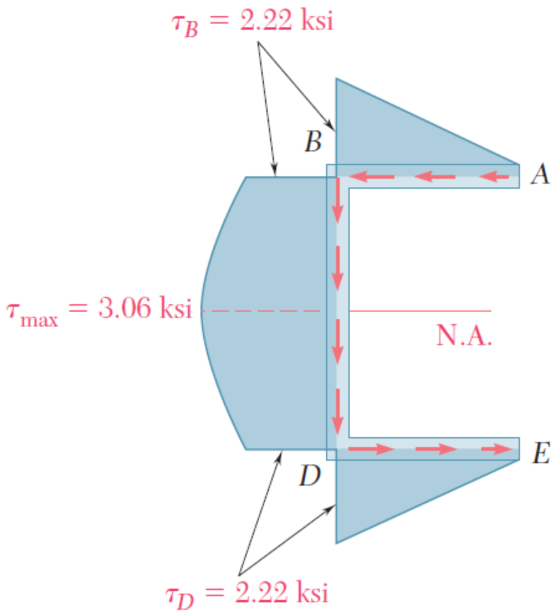
- Solution

$$I_z = I_{\text{web}} + 2I_{\text{flange}} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right] \approx \frac{1}{12}th^2\left(h+6b\right)$$

$$F = \int_0^b f ds = \int_0^b \frac{F_S S_z^*}{I_z} ds = \frac{F_S}{I_z}\int_0^b st\frac{h}{2} ds = \frac{F_S thb^2}{4I_z} = \frac{3F_S b^2}{h\left(h+6b\right)}$$

$$\Rightarrow e = \frac{F h}{F_S} = \frac{3b^2}{\left(h+6b\right)} = \frac{b}{2+\dfrac{h}{3b}} = \frac{4\,\text{in.}}{2+\dfrac{6\,\text{in.}}{3\left(4\,\text{in.}\right)}} = 1.6\,\text{in.}$$





- Determine the shearing stress distribution for $F_S = 2.5$ kips applied at the shear center.

- The maximum shearing stress in the flanges

$$\tau = \frac{F_S S_z^*}{I_z t} = \frac{F_S}{I_z t}(st)\frac{h}{2} = \frac{F_S h}{2I_z} s$$

$$\tau_{max} = \tau_B = \frac{F_S hb}{2\left(\frac{1}{12}th^2\right)(h+6b)} = \frac{6F_S b}{th(h+6b)}$$

$$= \frac{6(2.5\,\text{kips})(4\,\text{in})}{(0.15\,\text{in})(6\,\text{in})(6\times 4\,\text{in}+6\,\text{in})} = 2.22\,\text{ksi}$$

- The maximum shearing stress in the web

$$\left(S_z^*\right)_{max} = bt\left(\tfrac{1}{2}h\right) + \left(\tfrac{1}{2}ht\right)\left(\tfrac{1}{4}h\right) = \tfrac{1}{8}ht(h+4b)$$

$$\tau_{max} = \frac{F_S S_z^*}{I_z t} = \frac{F_S \frac{1}{8}ht(h+4b)}{\frac{1}{12}th^2(h+6b)t} = \frac{3F_S(h+4b)}{2th(h+6b)}$$

$$= \frac{3(2.5\,\text{kips})(4\times 4\,\text{in}+6\,\text{in})}{2(0.15\,\text{in})(6\,\text{in})(6\times 6\,\text{in}+6\,\text{in})} = 3.06\,\text{ksi}$$

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