Three-Dimensional Problems

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Review of Displacement Formulation – RCC

- Navier's equation

$$G\nabla^2 \boldsymbol{u} + \left(\lambda + G\right)\nabla\left(\nabla \cdot \boldsymbol{u}\right) + \boldsymbol{F} = 0$$

$$\boxed{Gu_{i,kk} + \left(\lambda + G\right)u_{k,ki} + F_i = 0}$$

- Displacement-strain relation:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{u}\overleftarrow{\nabla} + \nabla\boldsymbol{u}\right), \qquad \boxed{\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}$$

- Hooke's law:

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}\left(\boldsymbol{\varepsilon}\right)\boldsymbol{I} + 2G\boldsymbol{\varepsilon}, \qquad \boxed{\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}}$$

$$\lambda = \frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}, \quad G = \frac{E}{2\left(1+\nu\right)}$$

Review of Displacement Formulation – Cylindrical

- Navier's equation

$$\boxed{\begin{aligned}
&G\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right) + (\lambda + G)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_r = 0\\
&G\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + (\lambda + G)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_\theta = 0\\
&G\nabla^2 u_z + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right) + F_z = 0
\end{aligned}}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

- Displacement-strain relation:

$$\boxed{\begin{aligned}
&\varepsilon_r = \frac{\partial u_r}{\partial r}, \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \varepsilon_z = \frac{\partial u_z}{\partial z}, \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)\\
&\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)
\end{aligned}}$$

- Hooke's law…

Displacement Formulation – Axi-symmetric

- Navier's equation

$$\boxed{\begin{aligned} &G\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda + G)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + F_r = 0 \\ &G\nabla^2 u_z + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + F_z = 0 \end{aligned}}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

- Displacement-strain relation

$$\boxed{\varepsilon_r = \frac{\partial u_r}{\partial r}, \varepsilon_\theta = \frac{u_r}{r}, \varepsilon_z = \frac{\partial u_z}{\partial z}, \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)}$$

- Hooke's law

$$\boxed{\begin{aligned} &\sigma_r = \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_r = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_r}{\partial r}, \sigma_\theta = \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_\theta = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{u_r}{r} \\ &\sigma_z = \lambda(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_z = \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z}, \tau_{rz} = 2G\varepsilon_{rz} = G\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \end{aligned}}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, G = \frac{E}{2(1+\nu)}$$

Review of Displacement Formulation – Spherical

- Navier's equation

$$G\left(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2\cot\varphi u_\varphi}{R^2} - \frac{2}{R^2}\frac{\partial u_\varphi}{\partial\varphi} - \frac{2}{R^2\sin\varphi}\frac{\partial u_\theta}{\partial\theta}\right) + (\lambda+G)\frac{\partial}{\partial R}\left(\frac{\partial u_R}{\partial R} + \frac{2u_R}{R} + \frac{\cot\varphi u_\varphi}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta}\right) + F_R = 0$$

$$G\left(\nabla^2 u_\varphi + \frac{2}{R^2}\frac{\partial u_R}{\partial\varphi} - \frac{u_\varphi}{R^2\sin^2\varphi} - \frac{2\cot\varphi}{R^2\sin\varphi}\frac{\partial u_\theta}{\partial\theta}\right) + (\lambda+G)\frac{1}{R}\frac{\partial}{\partial\varphi}\left(\frac{\partial u_R}{\partial R} + \frac{2u_R}{R} + \frac{\cot\varphi u_\varphi}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta}\right) + F_\varphi = 0$$

$$G\left(\nabla^2 u_\theta + \frac{2}{R^2\sin\varphi}\frac{\partial u_R}{\partial\theta} + \frac{2\cot\varphi}{R^2\sin\varphi}\frac{\partial u_\varphi}{\partial\theta} - \frac{u_\theta}{R^2\sin^2\varphi}\right) + (\lambda+G)\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\left(\frac{\partial u_R}{\partial R} + \frac{2u_R}{R} + \frac{\cot\varphi u_\varphi}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta}\right) + F_\theta = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{\cot\varphi}{R^2}\frac{\partial}{\partial\varphi} + \frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{R^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

- Displacement-strain relation:

$$\varepsilon_R = \frac{\partial u_R}{\partial R}, \varepsilon_\varphi = \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi}, \varepsilon_\theta = \frac{u_R}{R} + \frac{\cot\varphi u_\varphi}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta}, \varepsilon_{R\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial R} + \frac{1}{R}\frac{\partial u_R}{\partial\varphi} - \frac{u_\varphi}{R}\right)$$

$$\varepsilon_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R} + \frac{\partial u_\theta}{\partial R}\right), \varepsilon_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial\varphi} - \frac{\cot\varphi u_\theta}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial\theta}\right)$$

- Hooke's law…

Displacement Formulation – Centro-symmetric

- Navier's equation

$$G\left(\nabla^2 u_R - \frac{2u_R}{R^2}\right) + (\lambda + G)\frac{d}{dR}\left(\frac{du_R}{dR} + \frac{2u_R}{R}\right) + F_R = 0, \qquad \nabla^2 = \frac{d^2}{\partial R^2} + \frac{2}{R}\frac{d}{\partial R}$$

$$\Rightarrow G\left(\frac{d^2 u_R}{\partial R^2} + \frac{2}{R}\frac{du_R}{\partial R} - \frac{2u_R}{R^2}\right) + (\lambda + G)\frac{d}{dR}\left(\frac{du_R}{dR} + \frac{2u_R}{R}\right) + F_R = 0$$

$$\Rightarrow (\lambda + 2G)\frac{d}{dR}\left(\frac{du_R}{dR} + \frac{2u_R}{R}\right) + F_R = 0$$

$$\Rightarrow \boxed{(\lambda + 2G)\frac{d}{dR}\left(\frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right)\right) + F_R = 0}, \qquad \lambda + 2G = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

- Strain-displacement relation:

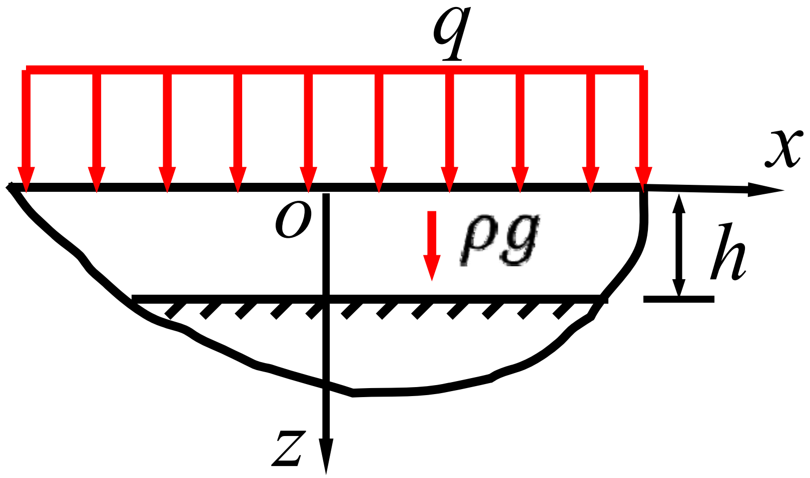
$$\boxed{\varepsilon_R = \frac{\partial u_R}{\partial R}, \quad \varepsilon_\theta = \varepsilon_\varphi = \frac{u_R}{R}}$$

- Hooke's law…

$$\boxed{\sigma_R = \lambda\left(\frac{\partial u_R}{\partial R} + \frac{2u_R}{R}\right) + 2G\frac{\partial u_R}{\partial R}, \quad \sigma_\theta = \sigma_\varphi = \lambda\left(\frac{\partial u_R}{\partial R} + \frac{2u_R}{R}\right) + 2G\frac{u_R}{R}}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}$$

Half-Space under Uniform Pressure and Gravity



- Observations and assumptions

$$F_r = 0, \quad F_z = \rho g$$

$$u_r = 0, u_z = u_z(z)$$

- Navier's equation

$$\boxed{\begin{aligned} &\cancel{G\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + (\lambda + G)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + F_r = 0} \\ &G\left(\cancel{\frac{\partial^2 u_z}{\partial r^2}} + \cancel{\frac{1}{r}\frac{\partial u_z}{\partial r}} + \frac{\partial^2 u_z}{\partial z^2}\right) + (\lambda + G)\frac{\partial}{\partial z}\left(\cancel{\frac{\partial u_r}{\partial r}} + \cancel{\frac{u_r}{r}} + \frac{\partial u_z}{\partial z}\right) + \rho g = 0 \end{aligned}}$$

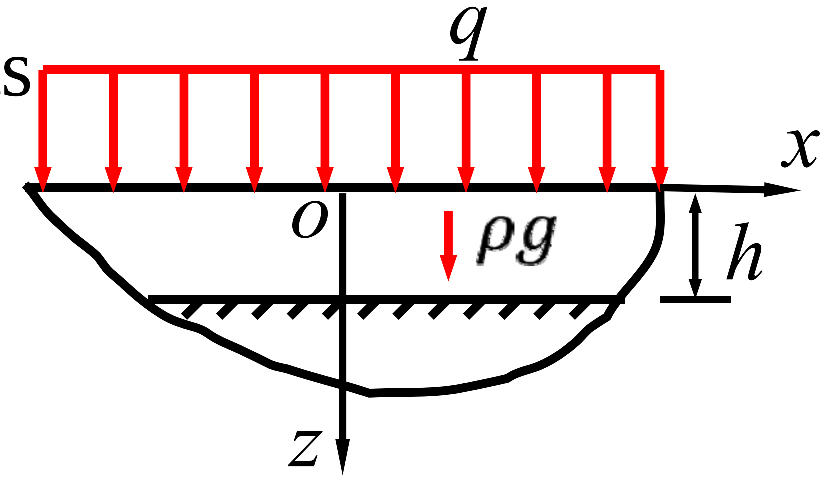
$$\Rightarrow (\lambda + 2G)\frac{d^2 u_z}{dz^2} + \rho g = 0$$

- By direct integration

$$\Rightarrow \frac{d^2 u_z}{dz^2} = -\frac{\rho g}{\lambda + 2G} \quad \Rightarrow \quad \frac{du_z}{dz} = -\frac{\rho g}{\lambda + 2G}(z + A) \quad \Rightarrow \quad u_z = -\frac{\rho g}{2(\lambda + 2G)}(z + A)^2 + B$$

Half-Space under Uniform Pressure and Gravity

- Stresses in terms of displacements



$$
\begin{cases}
\sigma_r = \lambda\left(\cancel{\dfrac{\partial u_r}{\partial r}} + \cancel{\dfrac{u_r}{r}} + \dfrac{\partial u_z}{\partial z}\right) + \cancel{2G\dfrac{\partial u_r}{\partial r}} \\[2ex]
\sigma_\theta = \lambda\left(\cancel{\dfrac{\partial u_r}{\partial r}} + \cancel{\dfrac{u_r}{r}} + \dfrac{\partial u_z}{\partial z}\right) + \cancel{2G\dfrac{u_r}{r}} \\[2ex]
\sigma_z = \lambda\left(\cancel{\dfrac{\partial u_r}{\partial r}} + \cancel{\dfrac{u_r}{r}} + \dfrac{\partial u_z}{\partial z}\right) + 2G\dfrac{\partial u_z}{\partial z}, \quad \tau_{rz} = G\left(\cancel{\dfrac{\partial u_z}{\partial r}} + \cancel{\dfrac{\partial u_r}{\partial z}}\right)
\end{cases}
$$

$$\Rightarrow \sigma_r = \sigma_\theta = \lambda\frac{du_z}{dz} = -\frac{\lambda\rho g}{\lambda+2G}(z+A), \quad \sigma_z = (\lambda+2G)\frac{du_z}{dz} = -\rho g(z+A)$$

- The traction BCs at $z = 0$

$$-q = \sigma_z(z=0) = -\rho g A \quad \Rightarrow \quad A = \frac{q}{\rho g}$$

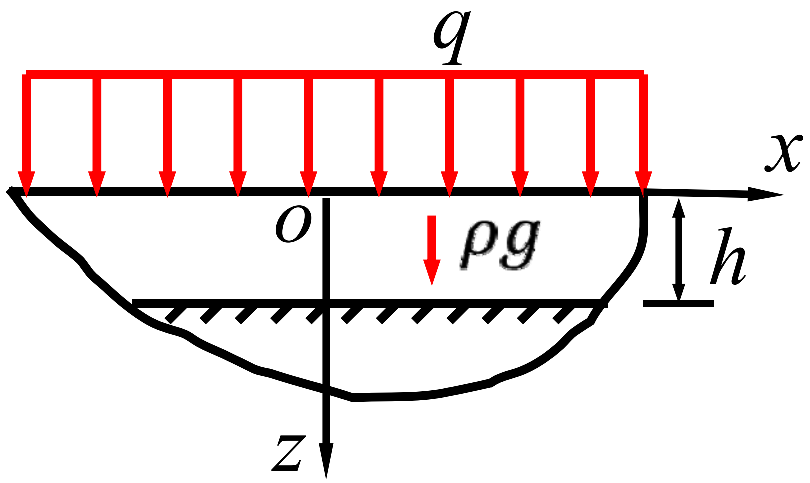
$$\Rightarrow \quad \boxed{\sigma_r = \sigma_\theta = -\frac{\lambda\rho g}{\lambda+2G}\left(z+\frac{q}{\rho g}\right)}, \quad \boxed{\sigma_z = -\rho g\left(z+\frac{q}{\rho g}\right)}$$

Half-Space under Uniform Pressure and Gravity

- Lateral to in-depth stress ratio

$$\frac{\sigma_r}{\sigma_z}=\frac{\sigma_\theta}{\sigma_z}=\frac{\lambda}{\lambda+2G}=\frac{E\nu}{(1+\nu)(1-2\nu)}\Big/\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$=\frac{\nu}{1-\nu}$$



- The displacement BCs at $z = h$

$$\left.\begin{array}{l} u_z=-\dfrac{\rho g}{2(\lambda+2G)}\left(z+\dfrac{q}{\rho g}\right)^2+B \\ 0=u_z(z=h) \end{array}\right\}\Rightarrow B=\frac{\rho g}{2(\lambda+2G)}\left(h+\frac{q}{\rho g}\right)^2$$

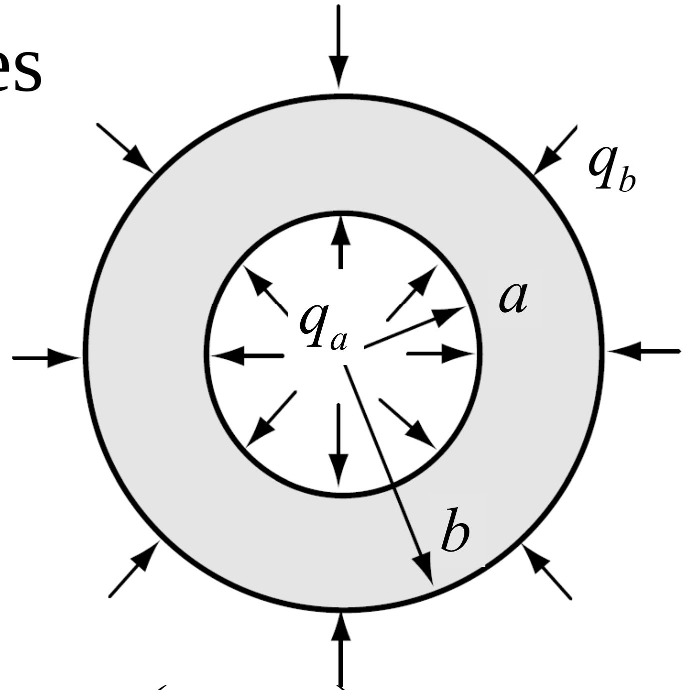
$$\Rightarrow \boxed{u_z=\frac{\rho g}{2(\lambda+2G)}\left[\left(h+\frac{q}{\rho g}\right)^2-\left(z+\frac{q}{\rho g}\right)^2\right]=\frac{\rho g\left(h^2-z^2\right)+2q(h-z)}{2(\lambda+2G)}}$$

$$\Rightarrow (u_z)_{\max}=u_z(z=0)=\frac{\rho g h^2+2qh}{2(\lambda+2G)}$$

The maximum displacement occurs at the top surface.

Spherical Shell under Uniform Pressure

- Centro-symmetry and zero body forces



$$(\lambda + 2G)\frac{d}{dR}\left(\frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right)\right) + \cancel{F_R} = 0$$

$$\Rightarrow \frac{1}{R^2}\frac{d}{dR}\left(R^2 u_R\right) = 3A \quad \Rightarrow \quad \frac{d}{dR}\left(R^2 u_R\right) = 3AR^2$$

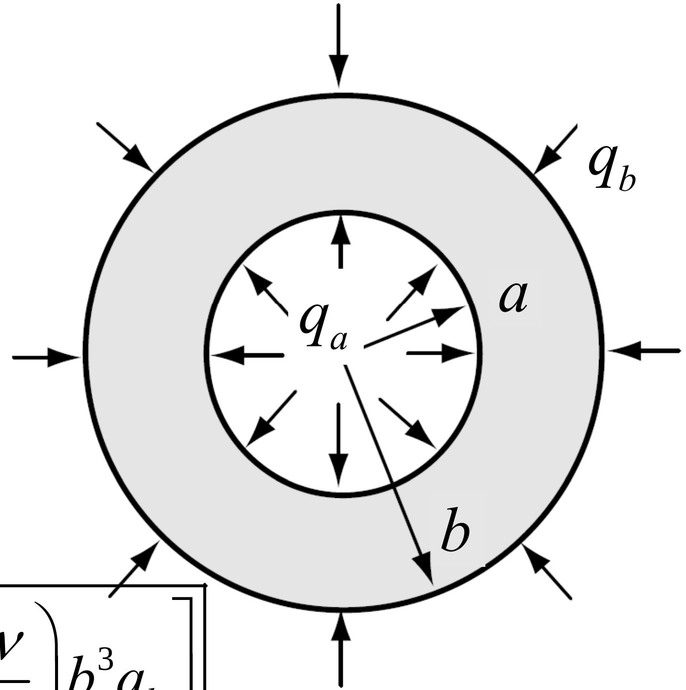
$$\Rightarrow R^2 u_R = AR^3 + B \quad \Rightarrow \quad u_R = AR + \frac{B}{R^2}$$

$$\Rightarrow \begin{cases} \sigma_R = \lambda\left(\dfrac{\partial u_R}{\partial R} + \dfrac{2u_R}{R}\right) + 2G\dfrac{\partial u_R}{\partial R} = \lambda\left(A - \dfrac{2B}{R^3} + 2A + \dfrac{2B}{R^3}\right) + 2G\left(A - \dfrac{2B}{R^3}\right) \\ \sigma_\varphi = \sigma_\theta = \lambda\left(\dfrac{\partial u_R}{\partial R} + \dfrac{2u_R}{R}\right) + 2G\dfrac{u_R}{R} = \lambda\left(A - \dfrac{2B}{R^3} + 2A + 2\dfrac{B}{R^3}\right) + 2G\left(A + \dfrac{B}{R^3}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_R = (3\lambda + 2G)A - \dfrac{4G}{R^3}B = \dfrac{E}{(1-2\nu)}A - \dfrac{2E}{(1+\nu)}\dfrac{1}{R^3}B \\ \sigma_\varphi = \sigma_\theta = (3\lambda + 2G)A + \dfrac{2G}{R^3}B = \dfrac{E}{(1-2\nu)}A + \dfrac{E}{(1+\nu)}\dfrac{1}{R^3}B \end{cases}$$

Spherical Shell under Uniform Pressure

- The traction BCs at $R = a$ and $R = b$



$$\begin{cases} -q_a = (\sigma_R)_{R=a} = \dfrac{E}{1-2\nu}A - \dfrac{2E}{1+\nu}\dfrac{B}{a^3} \\ -q_b = (\sigma_R)_{R=b} = \dfrac{E}{1-2\nu}A + \dfrac{E}{1+\nu}\dfrac{B}{b^3} \end{cases}$$

$$\Rightarrow \quad A = \frac{(1-2\nu)\left(a^3 q_a - b^3 q_b\right)}{E\left(b^3 - a^3\right)}, \quad B = \frac{(1+\nu)(q_a - q_b)a^3 b^3}{2E\left(b^3 - a^3\right)}$$

$$\Rightarrow \boxed{u_R = \frac{(1+\nu)}{E\left(b^3 - a^3\right)} R\left[\left(\frac{b^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right)a^3 q_a - \left(\frac{a^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right)b^3 q_b\right]}$$

$$\boxed{\begin{aligned} \sigma_R &= -\frac{1}{b^3 - a^3}\left(\frac{b^3}{R^3} - 1\right)a^3 q_a - \frac{1}{b^3 - a^3}\left(1 - \frac{a^3}{R^3}\right)b^3 q_b, \\ \sigma_\varphi = \sigma_\theta &= \frac{1}{b^3 - a^3}\left(\frac{b^3}{2R^3} + 1\right)a^3 q_a - \frac{1}{b^3 - a^3}\left(1 + \frac{a^3}{2R^3}\right)b^3 q_b \end{aligned}}$$

- **Here, stress is independent of Poisson's ratio. However, generally in 3-D problems with specified tractions, stress depends on Poisson's ratio.**

General Solution – Displacement Potentials

- **Helmholtz representation:**

$$\boldsymbol{u} = \underbrace{\nabla\phi}_{\text{Irrotational}} + \underbrace{\nabla\times\varphi}_{\text{Solenoidal}}, \qquad \nabla\cdot\varphi \equiv 0.$$

- Dilatation and rotation

$$\varepsilon_{kk} = \nabla\cdot\boldsymbol{u} = \nabla\cdot\left(\nabla\phi + \nabla\times\boldsymbol{\varphi}\right) = \nabla\cdot\nabla\phi + \cancel{\nabla\cdot\nabla\times\boldsymbol{\varphi}} = \nabla^2\phi$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla\times\boldsymbol{u} = \frac{1}{2}\nabla\times\left(\nabla\phi + \nabla\times\boldsymbol{\varphi}\right) = \frac{1}{2}\left(\cancel{\nabla\times\nabla\phi} + \nabla\times\nabla\times\boldsymbol{\varphi}\right) = -\frac{1}{2}\nabla^2\boldsymbol{\varphi}$$

- Navier's equation

$$G\nabla^2\boldsymbol{u} + (\lambda + G)\nabla(\nabla\cdot\boldsymbol{u}) + \boldsymbol{F} = 0$$

$$\Rightarrow G\nabla^2\left(\nabla\phi + \nabla\times\boldsymbol{\varphi}\right) + (\lambda + G)\nabla\left(\nabla^2\phi\right) + \boldsymbol{F} = 0$$

$$\Rightarrow \boxed{G\nabla\times\left(\nabla^2\boldsymbol{\varphi}\right) + (\lambda + 2G)\nabla\left(\nabla^2\phi\right) + \boldsymbol{F} = 0}$$

General Solution – Displacement Potentials

- If divergence and curl is taken of the previous equation

$$\begin{cases} 0 = \nabla \cdot \left[G\nabla \times \left(\nabla^2 \boldsymbol{\varphi} \right) + (\lambda + 2G) \nabla \left(\nabla^2 \phi \right) + \boldsymbol{F} \right] = (\lambda + 2G) \nabla^2 \nabla^2 \phi + \nabla \cdot \boldsymbol{F} \\ 0 = \nabla \times \left[G\nabla \times \left(\nabla^2 \boldsymbol{\varphi} \right) + (\lambda + 2G) \nabla \left(\nabla^2 \phi \right) + \boldsymbol{F} \right] = G\nabla \times \nabla \times \left(\nabla^2 \boldsymbol{\varphi} \right) + \nabla \times \boldsymbol{F} = -G\nabla^2 \nabla^2 \boldsymbol{\varphi} + \nabla \times \boldsymbol{F} \end{cases}$$

$$\Rightarrow \boxed{\nabla^2 \nabla^2 \phi = -\nabla \cdot \boldsymbol{F} / (\lambda + 2G) \qquad \nabla^2 \nabla^2 \boldsymbol{\varphi} = \nabla \times \boldsymbol{F} / G}$$

With zero body forces, both the scalar and vector potential functions are biharmonic.

- These four harmonic functions are **not independent**, since they must satisfy the Navier's equation.

- Summary

$$\boxed{\begin{gathered} G\nabla \times \left(\nabla^2 \boldsymbol{\varphi} \right) + (\lambda + 2G) \nabla \left(\nabla^2 \phi \right) + \boldsymbol{F} = 0 \\ \nabla^2 \nabla^2 \phi = -\nabla \cdot \boldsymbol{F} / (\lambda + 2G), \quad \nabla^2 \nabla^2 \boldsymbol{\varphi} = \nabla \times \boldsymbol{F} / G \end{gathered}}$$

Particular Case – Zero Body Forces

- Consider the special case

$$\boxed{\nabla^2 \phi = \nabla^2 \boldsymbol{\varphi} = 0} \qquad G\nabla \times \left(\nabla^2 \boldsymbol{\varphi}\right) + (\lambda + 2G)\nabla\left(\nabla^2 \phi\right) = 0$$

- Both the scalar and vector potential functions are harmonic.
- This special case may lead to some useful solutions.
- However, there is no guarantee that every elastostatic solution can be represented in terms of these four harmonic functions.

Particular Case – Lamé Strain Potentials

- Consider the further special case with

$$\boxed{\nabla^2\phi = 0, \quad \boldsymbol{\varphi} = 0} \quad \Rightarrow \quad \boldsymbol{u} = \nabla\phi + \cancel{\nabla\times\boldsymbol{\varphi}} = \nabla\phi$$

$$G\nabla\times\left(\nabla^2\boldsymbol{\varphi}\right) + (\lambda+2G)\nabla\left(\nabla^2\phi\right) = 0$$

- The displacement is commonly written as

$$2G\boldsymbol{u} = \nabla\phi, \quad 2Gu_i = \phi_{,i}$$

$$\Rightarrow \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{4G}\left(\phi_{,ij} + \phi_{,ji}\right) = \frac{1}{2G}\phi_{,ij} \quad \Rightarrow \quad \boxed{\varepsilon_{kk} = \phi_{,kk} = \nabla^2\phi = 0}$$

$$\Rightarrow \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} = \lambda\left(\frac{1}{2G}\cancel{\phi_{,kk}}\right)\delta_{ij} + 2G\left(\frac{1}{2G}\phi_{,ij}\right) = \phi_{,ij}$$

- Examples of harmonic functions

$$1, \quad x, y, z, \quad xy, yz, zx, \quad x^2-y^2, y^2-z^2, z^2-x^2, \quad R^2-3x^2, R^2-3y^2, R^2-3z^2, \quad r^n\cos n\theta,$$

$$\ln r, \quad \theta, \quad \frac{1}{R}, \quad \ln(R+z), \quad \ln\frac{\left(\sqrt{r^2+(z-c)^2}+z-c\right)\left(\sqrt{r^2+(z+c)^2}-z-c\right)}{r^2}$$

Particular Case – Lamé Strain Potentials

- In cylindrical coordinates

$$2Gu_r = \frac{\partial \phi}{\partial r},\ 2Gu_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta},\ 2Gw = \frac{\partial \phi}{\partial z},$$

$$\sigma_r = \frac{\partial^2 \phi}{\partial r^2} = -\frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial^2 \phi}{\partial z^2},\ \sigma_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2},\ \sigma_z = \frac{\partial^2 \phi}{\partial z^2},$$

$$\tau_{r\theta} = \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi}{\partial \theta},\ \tau_{\theta z} = \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial z},\ \tau_{rz} = \frac{\partial^2 \phi}{\partial r \partial z}$$

- For axi-symmetric problems

$$2Gu_r = \frac{\partial \phi}{\partial r}, 2Gw = \frac{\partial \phi}{\partial z},$$

$$\sigma_r = \frac{\partial^2 \phi}{\partial r^2} = -\frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial z^2}, \sigma_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r}, \sigma_z = \frac{\partial^2 \phi}{\partial z^2}, \tau_{rz} = \frac{\partial^2 \phi}{\partial r \partial z}$$

Galerkin Vector Potential

- Galerkin vector potential function

$$2G\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{V} - \nabla\left(\nabla\cdot\boldsymbol{V}\right)$$

The three components of Galerkin vector potential are independent.

$$\boxed{\begin{aligned}\phi &= -\frac{1}{2G}\nabla\cdot\boldsymbol{V}\\ \nabla\times\boldsymbol{\varphi} &= \frac{(1-\nu)}{G}\nabla^2\boldsymbol{V}\end{aligned}}$$

- Substitution back into the Navier's equation

$$G\nabla^2\boldsymbol{u} + (\lambda+G)\nabla(\nabla\cdot\boldsymbol{u}) + \boldsymbol{F} = 0$$

$$\Rightarrow \nabla^2\nabla^2\boldsymbol{V} - \frac{1}{2(1-\nu)}\nabla^2\left[\nabla(\nabla\cdot\boldsymbol{V})\right] + \frac{1}{1-2\nu}\nabla\left[\nabla\cdot\left(\nabla^2\boldsymbol{V}\right)\right] - \frac{1}{2(1-\nu)(1-2\nu)}\nabla\left[\nabla\cdot\nabla(\nabla\cdot\boldsymbol{V})\right] = -\frac{\boldsymbol{F}}{1-\nu}$$

$$\Rightarrow \boxed{\nabla^4\boldsymbol{V} = -\frac{\boldsymbol{F}}{1-\nu}}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\quad G = \frac{E}{2(1+\nu)},\quad \lambda+G = \frac{E}{2(1+\nu)(1-2\nu)}$$

Navier's equation has been reduced to a simpler fourth-order vector equation.

Galerkin Vector Potential

- With zero body forces, Galerkin's solution reduces four biharmonic functions in Helmholtz representation to three.

$$\boxed{\nabla^4 \boldsymbol{V} = 0, \qquad V = \xi \boldsymbol{e}_x + \eta \boldsymbol{e}_y + \varsigma \boldsymbol{e}_z}$$

$$\boxed{\nabla^4 \xi = 0} \qquad \boxed{\nabla^4 \eta = 0} \qquad \boxed{\nabla^4 \varsigma = 0}$$

- Displacements

$$2Gu = 2(1-\nu)\nabla^2 \xi - \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right),$$

$$2Gv = 2(1-\nu)\nabla^2 \eta - \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right),$$

$$2Gw = 2(1-\nu)\nabla^2 \varsigma - \frac{\partial}{\partial z}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right).$$

Galerkin Vector Potential

- Stresses

$$\sigma_x = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{\partial u}{\partial x}\right) = 2(1-\nu)\frac{\partial}{\partial x}\nabla^2\xi + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

$$\sigma_y = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{\partial v}{\partial y}\right) = 2(1-\nu)\frac{\partial}{\partial y}\nabla^2\eta + \left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

$$\sigma_z = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{\partial w}{\partial z}\right) = 2(1-\nu)\frac{\partial}{\partial z}\nabla^2\varsigma + \left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = (1-\nu)\left(\frac{\partial}{\partial x}\nabla^2\eta + \frac{\partial}{\partial y}\nabla^2\xi\right) - \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = (1-\nu)\left(\frac{\partial}{\partial z}\nabla^2\xi + \frac{\partial}{\partial x}\nabla^2\varsigma\right) - \frac{\partial^2}{\partial z\partial x}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = (1-\nu)\left(\frac{\partial}{\partial y}\nabla^2\varsigma + \frac{\partial}{\partial z}\nabla^2\eta\right) - \frac{\partial^2}{\partial y\partial z}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \varsigma}{\partial z}\right)$$

Love Strain Potentials – Axi-Symmetry

- **Special case of Galerkin potential:**

$$\boxed{V = \cancel{\xi}\,\boldsymbol{e}_x + \cancel{\eta}\,\boldsymbol{e}_y + \varsigma(r,z)\boldsymbol{e}_z, \quad \nabla^4\varsigma = 0}$$

$$2Gu_r = -\frac{\partial^2\varsigma}{\partial r\partial z}, \qquad 2Gw = 2(1-\nu)\nabla^2\varsigma - \frac{\partial^2\varsigma}{\partial z^2}$$

$$\sigma_r = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{\partial u_r}{\partial r}\right) = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varsigma, \ \sigma_\theta = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{u_r}{r}\right) = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varsigma$$

$$\sigma_z = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\varepsilon_{kk} + \frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varsigma, \tau_{rz} = \frac{E}{2(1+\nu)}\left(\frac{\partial u_r}{\partial z} + \frac{\partial w}{\partial r}\right) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varsigma$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \varepsilon_{kk} = \nabla\cdot\boldsymbol{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

This reduces to the function introduced by Love in 1906 to treat solids of revolution under **axi-symmetric** loading.

$$\boxed{\nabla^2 u_r - \frac{u_r}{r^2} + \frac{1}{1-2\nu}\frac{\partial\varepsilon_{kk}}{\partial r} = 0, \quad \nabla^2 w + \frac{1}{1-2\nu}\frac{\partial\varepsilon_{kk}}{\partial z} = 0}$$

Completeness of Displacement Potentials

- Galerkin vector potential is complete – i.e. that it is capable of describing all possible elastic displacement fields in a three dimensional body.
- Love strain potential is complete for axial symmetric problems.
- Nonetheless, Lamé strain potential is often employed in order to produce a simplified solution form, i.e.

$$2Gu_r = -\frac{\partial^2 \varsigma}{\partial r \partial z}, 2Gw = 2(1-\nu)\nabla^2 \varsigma - \frac{\partial^2 \varsigma}{\partial z^2}$$

$$\sigma_r = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varsigma, \sigma_\theta = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\varsigma$$

$$\sigma_z = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varsigma, \tau_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varsigma$$

$$\boxed{\nabla^4 \varsigma = 0}$$

$$2Gu_r = \frac{\partial \phi}{\partial r}, 2Gw = \frac{\partial \phi}{\partial z},$$

$$\sigma_r = \frac{\partial^2 \phi}{\partial r^2} = -\frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial z^2}, \sigma_\theta = \frac{1}{r}\frac{\partial \phi}{\partial r},$$

$$\sigma_z = \frac{\partial^2 \phi}{\partial z^2}, \tau_{rz} = \frac{\partial^2 \phi}{\partial r \partial z}$$

$$\boxed{\nabla^2 \phi = 0}$$

Harmonic and Bi-harmonic functions

- Consider the identity

$$\nabla^2\left(xf\right)=x\nabla^2 f+2\frac{\partial f}{\partial x} \quad\Rightarrow\quad \nabla^2\nabla^2\left(xf\right)=\nabla^2\left(x\nabla^2 f\right)+2\frac{\partial}{\partial x}\nabla^2 f$$

$$\text{If }\ \nabla^2 f=0 \quad\Rightarrow\quad \nabla^2\nabla^2\left(xf\right)=0.$$

- Similarly: If $\nabla^2 f=0 \quad\Rightarrow\quad \nabla^2\nabla^2\left(R^2 f\right)=0.$
- Generalized representation for bi-harmonic functions

$$\text{If }\ \nabla^2 f_0=\nabla^2 f_1=\nabla^2 f_2=\nabla^2 f_3=\nabla^2 f_4=0$$

$$g=f_0+xf_1+yf_2+zf_3+R^2 f_4, \qquad \nabla^2\nabla^2 g=0$$

- Singular harmonic functions

singular at origin: $\frac{1}{R},\quad \frac{\partial}{\partial x}\frac{1}{R}=-\frac{x}{R^3},\quad \frac{\partial^2}{\partial x\partial y}\frac{1}{R}=\frac{3xy}{R^5},\quad \frac{\partial^2}{\partial x^2}\frac{1}{R}=-\frac{1}{R^3}+\frac{3x^2}{R^5}\quad\cdots$

singular along $-z$: $\ln\left(R+z\right),\quad \frac{\partial}{\partial x}\ln\left(R+z\right)=\frac{x}{R\left(R+z\right)},\quad \frac{\partial^2}{\partial x\partial y}\ln\left(R+z\right)=-\frac{2xy}{R^2\left(R+z\right)^2}-\frac{xyz}{R^3\left(R+z\right)^2},$

$$\frac{\partial^2}{\partial x^2}\ln\left(R+z\right)=\frac{1}{R\left(R+z\right)}-\frac{x^2}{R^3\left(R+z\right)}-\frac{x^2}{R^2\left(R+z\right)^2},\quad \frac{x}{R+z},\quad \frac{y}{R+z},\quad \frac{z}{R+z}\cdots$$

Papkovich-Neuber Displacement Potential

- Define the vector function: $\Phi = -\frac{1}{2}\nabla^2 \boldsymbol{V}, \qquad \nabla^2\Phi = 0$
- The new function must be harmonic since $\boldsymbol{V}$ is biharmonic.

$$\nabla^2(\boldsymbol{r}\cdot\Phi) = \cancel{\nabla^2(\boldsymbol{r})}\cdot\Phi + \boldsymbol{r}\cdot\cancel{\nabla^2\Phi} + 2\nabla\cdot\Phi = -\nabla\cdot\nabla^2\boldsymbol{V} = -\nabla^2(\nabla\cdot\boldsymbol{V})$$

$$\Rightarrow -\nabla\cdot\boldsymbol{V} = \boldsymbol{r}\cdot\Phi + \phi, \qquad \nabla^2\phi = 0$$

- From Galerkin representation of displacements

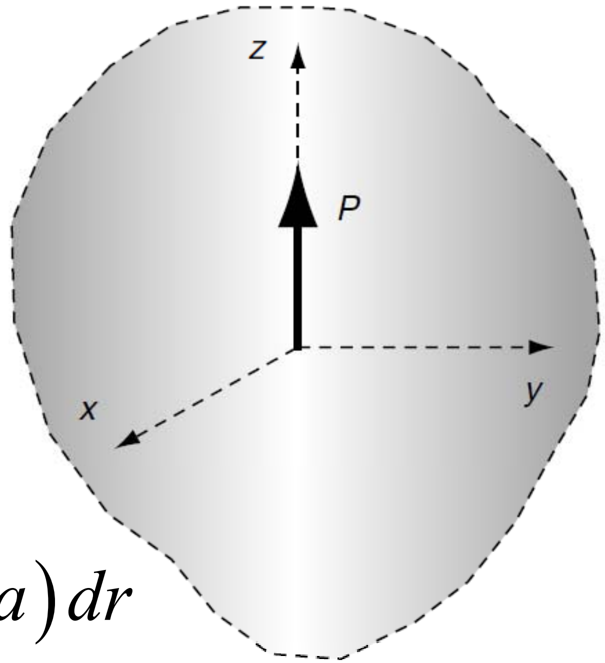
$$2G\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{V} - \nabla(\nabla\cdot\boldsymbol{V})$$

$$\Rightarrow \boxed{2G\boldsymbol{u} = -4(1-\nu)\Phi + \nabla(\boldsymbol{r}\cdot\Phi + \phi), \qquad \nabla^2\Phi = \nabla^2\phi = 0}$$

- The Papkovich-Neuber solution is also complete and it is widely used in modern treatments of elastic problems.
- We note that the scalar function is nothing but the Lamé strain potential introduced previously.

Kelvin's Solution

- A concentrated force in an infinite solid
- The general BCs require that:
- **the stress field vanishes at infinity,**
- **is singular at the origin, and**
- **gives the resultant force $-P\mathbf{e}_z$ on the surface of a cylinder ($r = r'$, $z = \pm a$).**



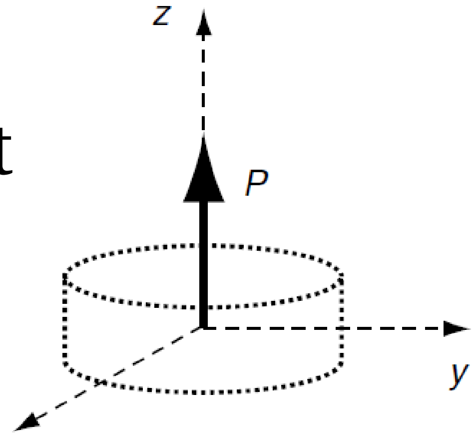
$$\int_0^{r'} 2\pi r\sigma_z(r,a)\,dr - \int_0^{r'} 2\pi r\sigma_z(r,-a)\,dr$$

$$+\int_{-a}^{a} 2\pi r'\tau_{rz}(r',z)\,dz + P = 0.$$

- Geometry and loading configuration suggest that we may wish to try the Love potential

$$\sigma_z = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\varsigma, \tau_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\varsigma \;\Rightarrow\; \boxed{\varsigma \propto [\mathrm{N}\cdot\mathrm{m}]}$$

$$g = \cancel{f_0 + xf_1 + yf_2 + zf_3} + R^2 f_4 \quad\Rightarrow\quad \varsigma = APR^2\frac{1}{R} = APR = AP\sqrt{r^2+z^2}$$



Resultant boundary condition evaluation

Kelvin's Solution

- Displacements and stresses

$$\boxed{\varsigma = APR = AP\sqrt{r^2 + z^2},}$$

$$\Rightarrow \begin{cases} 2Gu_r = -\dfrac{\partial^2 \varsigma}{\partial r \partial z} = AP\dfrac{rz}{R^3}, \\ 2Gw = 2(1-\nu)\nabla^2 \varsigma - \dfrac{\partial^2 \varsigma}{\partial z^2} = AP\left(\dfrac{2(1-2\nu)}{R} + \dfrac{1}{R} + \dfrac{z^2}{R^3}\right) \\ \sigma_r = \dfrac{\partial}{\partial z}\left(\nu\nabla^2 - \dfrac{\partial^2}{\partial r^2}\right)\varsigma = AP\left(\dfrac{(1-2\nu)z}{R^3} - \dfrac{3r^2 z}{R^5}\right), \\ \sigma_\theta = \dfrac{\partial}{\partial z}\left(\nu\nabla^2 - \dfrac{1}{r}\dfrac{\partial}{\partial r}\right)\varsigma = AP\dfrac{(1-2\nu)z}{R^3} \\ \sigma_z = \dfrac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \dfrac{\partial^2}{\partial z^2}\right)\varsigma = -AP\left(\dfrac{(1-2\nu)z}{R^3} + \dfrac{3z^3}{R^5}\right), \\ \tau_{rz} = \dfrac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \dfrac{\partial^2}{\partial z^2}\right]\varsigma = -AP\left(\dfrac{(1-2\nu)r}{R^3} + \dfrac{3rz^2}{R^5}\right) \end{cases}$$

Kelvin's Solution

- Applying the BCs to determine the potential coefficient A

$$\left.\begin{aligned}&\sigma_z=-AP\left(\frac{(1-2\nu)z}{R^3}+\frac{3z^3}{R^5}\right),\tau_{zr}=-AP\left(\frac{(1-2\nu)r}{R^3}+\frac{3rz^2}{R^5}\right)\\&\int_0^{r'}2\pi r\sigma_z(r,a)dr-\int_0^{r'}2\pi r\sigma_z(r,-a)dr+\int_{-a}^{a}2\pi r'\tau_{rz}(r',z)dz+P=0\end{aligned}\right\}$$

$$\Rightarrow 2\int_0^{r'}2\pi r\sigma_z(r,a)dr+\int_{-a}^{a}2\pi r'\tau_{rz}(r',z)dz+P=0$$

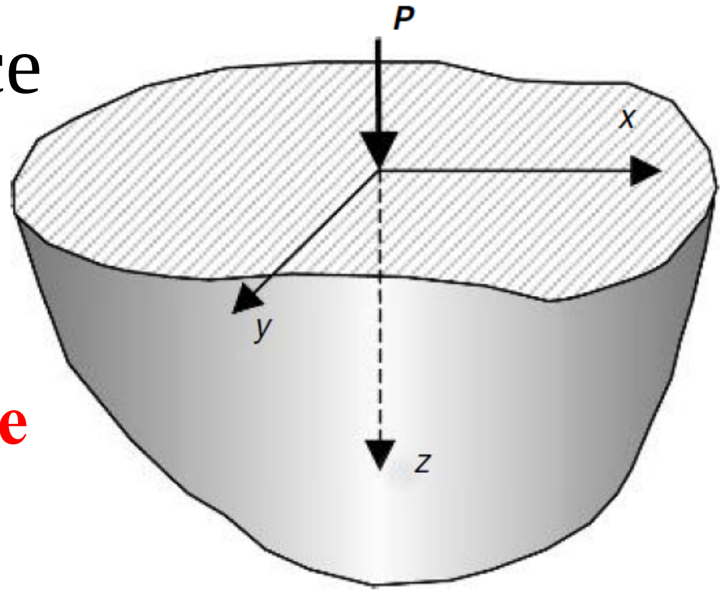
$$\Rightarrow -4\pi AP\int_0^{r'}r\left(\frac{(1-2\nu)a}{\left(r^2+a^2\right)^{3/2}}+\frac{3a^3}{\left(r^2+a^2\right)^{5/2}}\right)dr-2\pi AP\int_{-a}^{a}r'\left(\frac{(1-2\nu)r'}{\left(r'^2+z^2\right)^{3/2}}+\frac{3r'z^2}{\left(r'^2+z^2\right)^{5/2}}\right)dz+P=0$$

- For the limiting case of $r'\to\infty$, the second term of the above vanishes.

$$\Rightarrow\ 8\pi(1-\nu)A=1\ \Rightarrow\ \boxed{A=\frac{1}{8\pi(1-\nu)}}$$

Boussinesq's Solution

- A concentrated force on an half-space
- The general BCs require that:
- **the stress field vanishes at infinity,**
- **is singular at the origin, but**
- **produces zero tractions on the half-space boundary, and**
- **gives the resultant force $-P\mathbf{e}_z$ on any surface parallel to the boundary.**



$$(\sigma_z)_{z=0,r\neq 0}=0,\quad (\tau_{zr})_{z=0,r\neq 0}=0,\quad \int_0^\infty 2\pi r\sigma_z(r,a)\,dr+P=0.$$

- Resort to the Love potential of Kelvin solution

$$(\sigma_z)_{z=0,r\neq 0}=-AP\left(\frac{(1-2\nu)z}{R^3}+\frac{3z^3}{R^5}\right)_{z=0,r\neq 0}=0,\quad (\tau_{zr})_{z=0,r\neq 0}=-AP\left(\frac{(1-2\nu)r}{R^3}+\frac{3rz^2}{R^5}\right)_{z=0,r\neq 0}=-AP\frac{(1-2\nu)}{r^2}$$

The shear traction BC on z = 0 cannot be satisfied.

Boussinesq's Solution

- Amend the solution with the axi-symmetric part of Lamé Strain Potential

$$\tau_{rz} = \frac{\partial^2 \phi}{\partial r \partial z} \Rightarrow \boxed{\phi \propto [\mathrm{N}]}$$

- Singular at origin and zero length scale suggests

$$\phi = BP \ln\left(R + z\right)$$

- Displacements and stresses due to Lamé Potential

$$2Gu'_r = BP\frac{r}{R(R+z)}, 2Gu'_z = BP\frac{1}{R}$$

$$\sigma'_r = BP\left(\frac{z}{R^3} - \frac{1}{R(R+z)}\right), \sigma'_\theta = \frac{BP}{R(R+z)}, \sigma'_z = -BP\frac{z}{R^3}, \tau'_{zr} = -BP\frac{r}{R^3}$$

- The normal and shear stress on the half-space boundary

$$\left(\sigma'_z\right)_{z=0,r\neq 0} = -BP\left(\frac{z}{R^3}\right)_{z=0,r\neq 0} = 0, \quad \left(\tau'_{zr}\right)_{z=0,r\neq 0} = -BP\left(\frac{r}{R^3}\right)_{z=0,r\neq 0} = -BP\frac{1}{r^2}$$

Boussinesq's Solution

- Apply the BCs

$$\cancel{(\sigma_z)_{z=0,r\neq 0} = 0,} \quad (\tau_{zr})_{z=0,r\neq 0} = 0, \quad \int_0^\infty 2\pi r\sigma_z\left(r,a\right)dr + P = 0.$$

$$\Rightarrow \begin{cases} -AP\dfrac{(1-2\nu)}{r^2} - BP\dfrac{1}{r^2} = 0 \\ -2\pi AP\displaystyle\int_0^\infty\left[\frac{(1-2\nu)ar}{\left(r^2+a^2\right)^{3/2}} + \frac{3a^3r}{\left(r^2+a^2\right)^{5/2}}\right]dr - 2\pi BP\int_0^\infty \frac{ar}{\left(r^2+a^2\right)^{3/2}}dr + P = 0 \end{cases}$$

$$\Rightarrow \quad \boxed{A = \frac{1}{2\pi}, \qquad B = -\frac{(1-2\nu)}{2\pi}}$$

- Total displacements and stresses in cylindrical coordinates

$$u_r = \frac{P}{4\pi G}\frac{1}{R}\left[\frac{rz}{R^2} - \frac{(1-2\nu)r}{R+z}\right], u_z = \frac{P}{4\pi G}\frac{1}{R}\left[2(1-\nu) + \frac{z^2}{R^2}\right]$$

$$\sigma_r = \frac{P}{2\pi}\frac{1}{R^2}\left[\frac{(1-2\nu)R}{R+z} - \frac{3r^2z}{R^3}\right], \sigma_\theta = \frac{(1-2\nu)P}{2\pi}\frac{1}{R^2}\left(\frac{z}{R} - \frac{R}{R+z}\right), \sigma_z = \frac{3P}{2\pi}\frac{z^3}{R^5}, \tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{R^5}$$

Boussinesq's Solution

- Total displacements and stresses in RCC

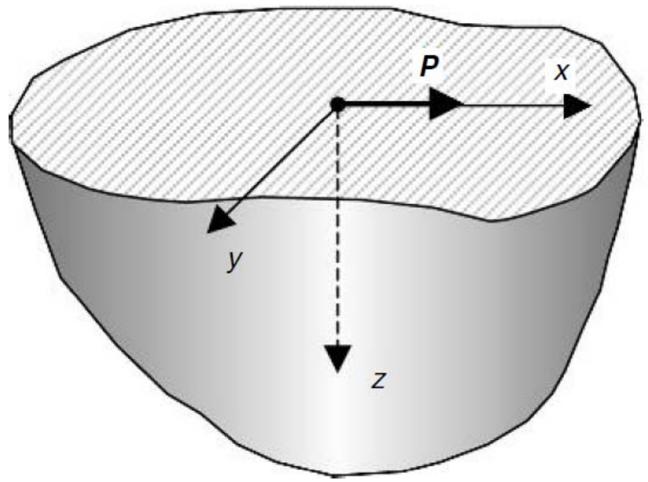
$$u=\frac{P}{4\pi G}\left[\frac{xz}{R^3}-\frac{(1-2\nu)x}{R(R+z)}\right],v=\frac{P}{4\pi G}\left[\frac{yz}{R^3}-\frac{(1-2\nu)y}{R(R+z)}\right],w=\frac{P}{4\pi G}\left[\frac{z^2}{R^3}+\frac{2(1-\nu)}{R}\right]$$

$$\sigma_x=-\frac{P}{2\pi}\left[\frac{3x^2z}{R^5}-(1-2\nu)\left(\frac{Rz-y^2}{R^3(R+z)}-\frac{x^2}{R^2(R+z)^2}\right)\right],$$

$$\sigma_y=-\frac{P}{2\pi}\left[\frac{3y^2z}{R^5}-(1-2\nu)\left(\frac{Rz-x^2}{R^3(R+z)}-\frac{y^2}{R^2(R+z)^2}\right)\right],\sigma_z=\frac{3Pz^3}{2\pi R^5},$$

$$\tau_{xy}=-\frac{P}{2\pi}\left(\frac{3xyz}{R^5}+\frac{(1-2\nu)xy(2R+z)}{R^3(R+z)}\right),\tau_{xz}=-\frac{3P}{2\pi}\frac{xz^2}{R^5},\tau_{yz}=-\frac{3P}{2\pi}\frac{yz^2}{R^5}$$

Cerruti's Solution



- A concentrated tangential force
- The general BCs require that:
- **the stress field vanishes at infinity,**
- **is singular at the origin, but**
- **produces zero tractions on the half-space boundary, and**
- **gives the resultant force $-P\mathbf{e}_x$ on any surface parallel to the half-space boundary**

$$(\sigma_z, \tau_{zx}, \tau_{zy})_{z=0,\, r\neq 0} = 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau_{zx}dxdy + P = 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau_{zy}dxdy = 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_z dxdy = 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y\sigma_z - z\tau_{zy})dxdy = 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x\sigma_z - z\tau_{zx})dxdy = 0, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y\tau_{zx} - x\tau_{zy})dxdy = 0$$

Cerruti's Solution

- Galerkin vector components ~ [N·m], since stress is the third derivative of the potentials
- Lamé strain potential ~ [N], since stress is the second order derivative of the potential
- Generalized representation for bi-harmonic functions

$$g = f_0 + xf_1 + yf_2 + zf_3 + R^2 f_4, \quad \nabla^2\nabla^2 g = 0, \quad \nabla^2 f_0 = \nabla^2 f_1 = \nabla^2 f_2 = \nabla^2 f_3 = \nabla^2 f_4 = 0$$

- Singular harmonic functions

singular at origin: $\frac{1}{R}, \quad \frac{\partial}{\partial x}\frac{1}{R} = -\frac{x}{R^3}, \quad \frac{\partial^2}{\partial x \partial y}\frac{1}{R} = \frac{3xy}{R^5}, \quad \frac{\partial^2}{\partial x^2}\frac{1}{R} = -\frac{1}{R^3} + \frac{3x^2}{R^5} \quad \cdots$

singular along $-z$: $\ln(R+z), \quad \frac{\partial}{\partial x}\ln(R+z) = \frac{x}{R(R+z)}, \quad \frac{\partial^2}{\partial x \partial y}\ln(R+z) = -\frac{2xy}{R^2(R+z)^2} - \frac{xyz}{R^3(R+z)^2},$

$$\frac{\partial^2}{\partial x^2}\ln(R+z) = \frac{1}{R(R+z)} - \frac{x^2}{R^3(R+z)} - \frac{x^2}{R^2(R+z)^2}, \quad \frac{x}{R+z}, \quad \frac{y}{R+z}, \quad \frac{z}{R+z} \cdots$$

$$\boxed{\xi = A_1 R, \quad \eta = 0, \quad \varsigma = A_2 x \ln(R+z)}$$

$$\boxed{\phi = \frac{A_3 x}{R+z}}$$

Cerruti's Solution

$$\boxed{\xi = A_1 R\,,\quad \eta = 0\,,\quad \varsigma = A_2 x\ln(R+z)}$$

$$\boxed{\phi = \frac{A_3 x}{R+z}}$$

- Applying the BCs

$$A_1 = \frac{P}{4\pi(1-\nu)},\quad A_2 = \frac{(1-2\nu)P}{4\pi(1-\nu)},\quad A_3 = \frac{(1-2\nu)P}{2\pi}$$

- Displacements

$$u = \frac{P}{4\pi G}\frac{1}{R}\left[1+\frac{x^2}{R^2}+(1-2\nu)\left(\frac{R}{R+z}-\frac{x^2}{(R+z)^2}\right)\right]$$

$$v = \frac{P}{4\pi G}\frac{1}{R}\left[\frac{xy}{R^2}-\frac{(1-2\nu)xy}{(R+z)^2}\right],\quad w = \frac{P}{4\pi G}\frac{1}{R}\left[\frac{xz}{R^2}+\frac{(1-2\nu)x}{R+z}\right].$$

Cerruti's Solution

- Stresses

$$\sigma_x = \frac{P}{2\pi}\frac{x}{R^3}\left[\frac{1-2\nu}{(R+z)^2}\left(R^2 - y^2 - \frac{2Ry^2}{R+z}\right) - \frac{3x^2}{R^2}\right]$$

$$\sigma_y = \frac{P}{2\pi}\frac{x}{R^3}\left[\frac{1-2\nu}{(R+z)^2}\left(3R^2 - x^2 - \frac{2Rx^2}{R+z}\right) - \frac{3y^2}{R^2}\right]$$

$$\sigma_z = -\frac{3P}{2\pi}\frac{xz^2}{R^5}, \tau_{xy} = \frac{P}{2\pi}\frac{y}{R^3}\left[\frac{1-2\nu}{(R+z)^2}\left(-R^2 + x^2 + \frac{2Rx^2}{R+z}\right) - \frac{3x^2}{R^2}\right]$$

$$\tau_{xz} = -\frac{3P}{2\pi}\frac{x^2 z}{R^5}, \tau_{yz} = -\frac{3P}{2\pi}\frac{xyz}{R^5}$$

Cerruti's Solution in Cylindrical Coordinates

- Displacements

$$\frac{u_r}{\cos\theta}=\frac{P}{4\pi G}\frac{1}{R}\left\{1+\frac{r^2}{R^2}+(1-2\nu)\left(\frac{R}{R+z}-\frac{r^2}{(R+z)^2}\right)\right\},$$

$$\frac{u_\theta}{\sin\theta}=-\frac{P}{4\pi G}\left\{\frac{1}{R}+\frac{(1-2\nu)}{R+z}\right\},\frac{u_z}{\cos\theta}=\frac{P}{4\pi G}\frac{r}{R}\left\{\frac{z}{R^2}+\frac{1-2\nu}{R+z}\right\};$$

- Stresses

$$\frac{\sigma_r}{\cos\theta}=\frac{P}{2\pi}\frac{r}{R}\left\{\frac{(1-2\nu)}{(R+z)^2}-\frac{3r^2}{R^4}\right\},\frac{\sigma_\theta}{\cos\theta}=\frac{P}{2\pi}\frac{(1-2\nu)r}{(R+z)^2R^3}\left(3R^2-r^2-\frac{2Rr^2}{R+z}\right),$$

$$\frac{\sigma_z}{\cos\theta}=-\frac{3P}{2\pi}\frac{rz^2}{R^5},\frac{\tau_{r\theta}}{\sin\theta}=\frac{P}{2\pi}\frac{(1-2\nu)r}{(R+z)^2R},\frac{\tau_{rz}}{\cos\theta}=-\frac{3P}{2\pi}\frac{r^2z}{R^5},\tau_{\theta z}=0.$$

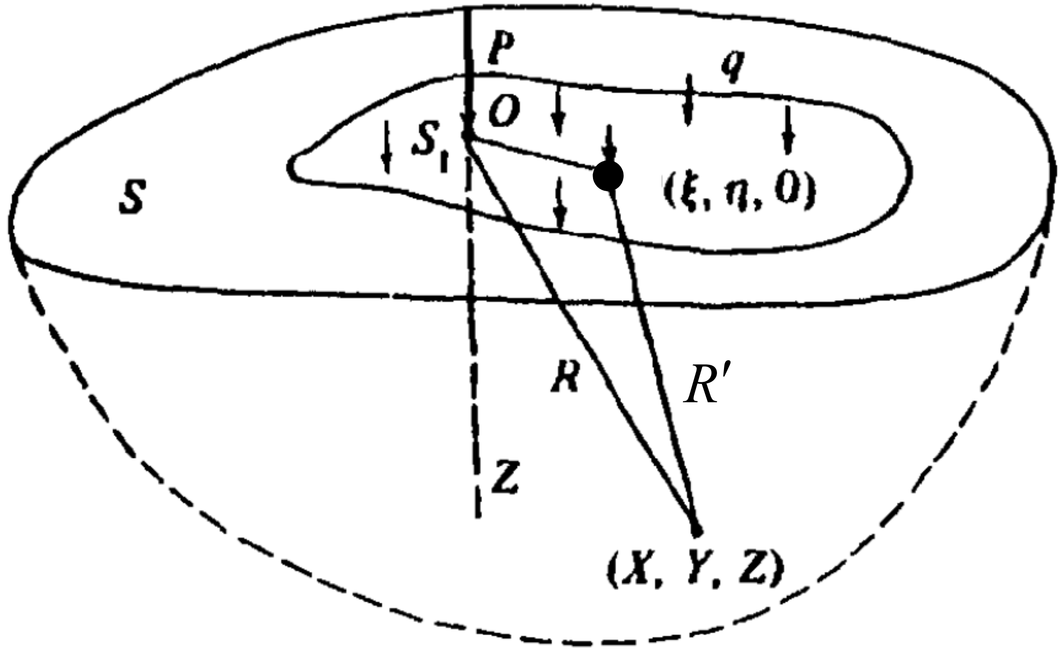
Distributed Pressure on Half-Space Boundary

- Boussinesq's solution

$$u=\frac{P}{4\pi G}\left[\frac{xz}{R^3}-\frac{(1-2\nu)x}{R(R+z)}\right],$$

$$v=\frac{P}{4\pi G}\left[\frac{yz}{R^3}-\frac{(1-2\nu)y}{R(R+z)}\right],$$

$$w=\frac{P}{4\pi G}\left[\frac{z^2}{R^3}+\frac{2(1-\nu)}{R}\right]$$



- Move away from the origin, let $P = qd\xi d\eta$, and integrate

$$u=\frac{1}{4\pi G}\iint_{S_1} q(\xi,\eta)(x-\xi)\left[\frac{z}{R'^3}-\frac{(1-2\nu)}{R'(R'+z)}\right]d\xi\, d\eta$$

$$v=\frac{1}{4\pi G}\iint_{S_1} q(\xi,\eta)(y-\eta)\left[\frac{z}{R'^3}-\frac{(1-2\nu)}{R'(R'+z)}\right]d\xi\, d\eta$$

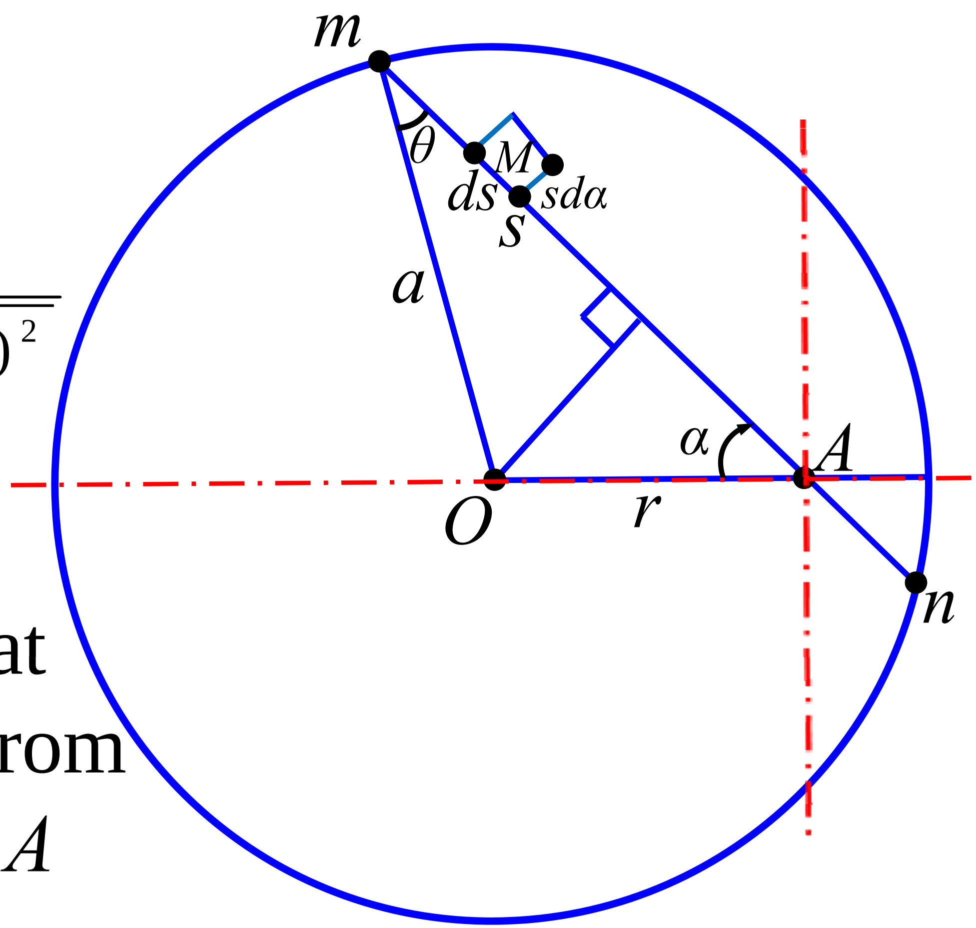
$$w=\frac{1}{4\pi G}\iint_{S_1} q(\xi,\eta)\left[\frac{z^2}{R'^3}+\frac{2(1-\nu)}{R'}\right]d\xi\, d\eta,\quad R'^2=(x-\xi)^2+(y-\eta)^2+z^2$$

Vertical Displacement within the Loading Zone

- Vertical displacement on the half-space boundary

$$(w)_{z=0} = \frac{1-\nu^2}{\pi E}\iint_{S_1} \frac{q(\xi,\eta)\,d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$$

- **For uniformly distributed pressure q in a circle $r \le a$**
- By use of a polar coordinate at **field point A** and measured from AO, the vertical deflection at A due to the differential normal pressure at the **source point M**



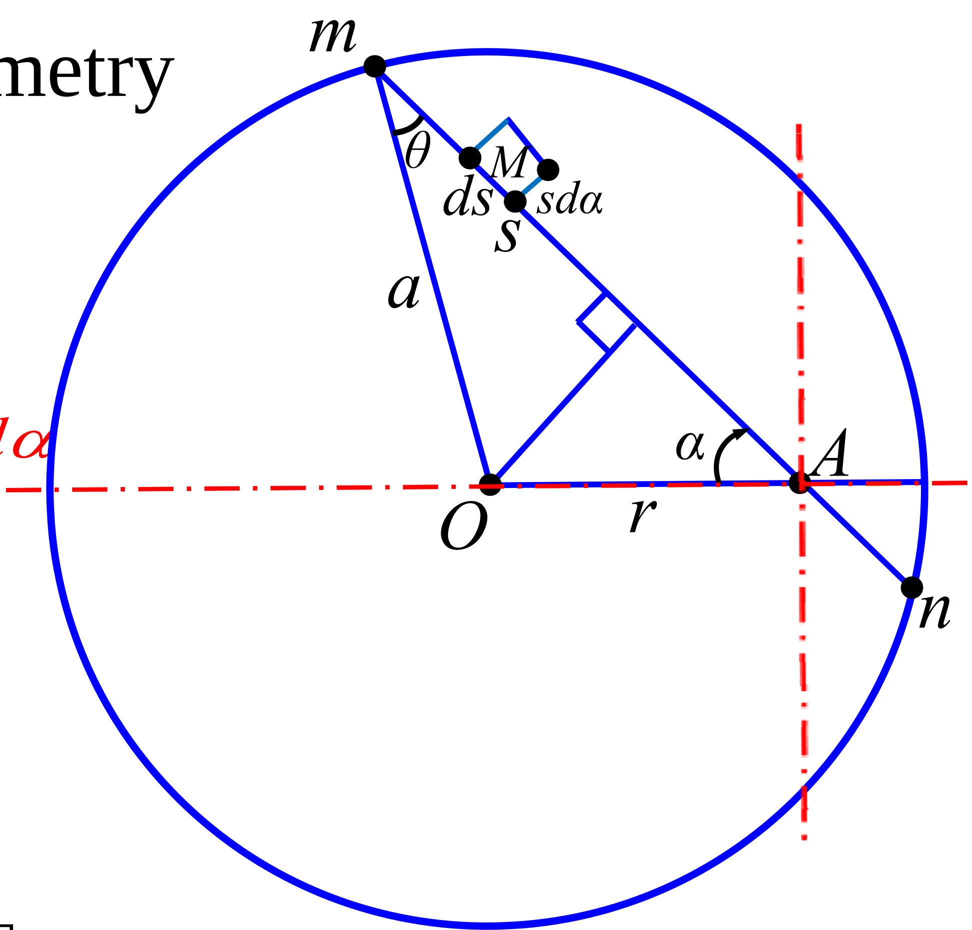
$$(w)_A = \frac{(1-\nu^2)q}{\pi E}\iint_{S_1} \frac{\cancel{s}\,ds\,d\alpha}{\cancel{s}}$$

$$= \frac{(1-\nu^2)q}{\pi E}\left[\int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi}\right]\overline{Am}\,d\alpha = \frac{2(1-\nu^2)q}{\pi E}\left[\int_0^{\pi/2} + \int_{\pi}^{3\pi/2}\right]\overline{Am}\,d\alpha$$

Vertical Displacement within the Loading Zone

- Further inspection of the geometry

$$\overline{mn} = 2a\cos\theta, \qquad a\sin\theta = r\sin\alpha$$
$$\overline{Am}(\alpha) + \overline{Am}(\alpha+\pi) = \overline{mn}$$

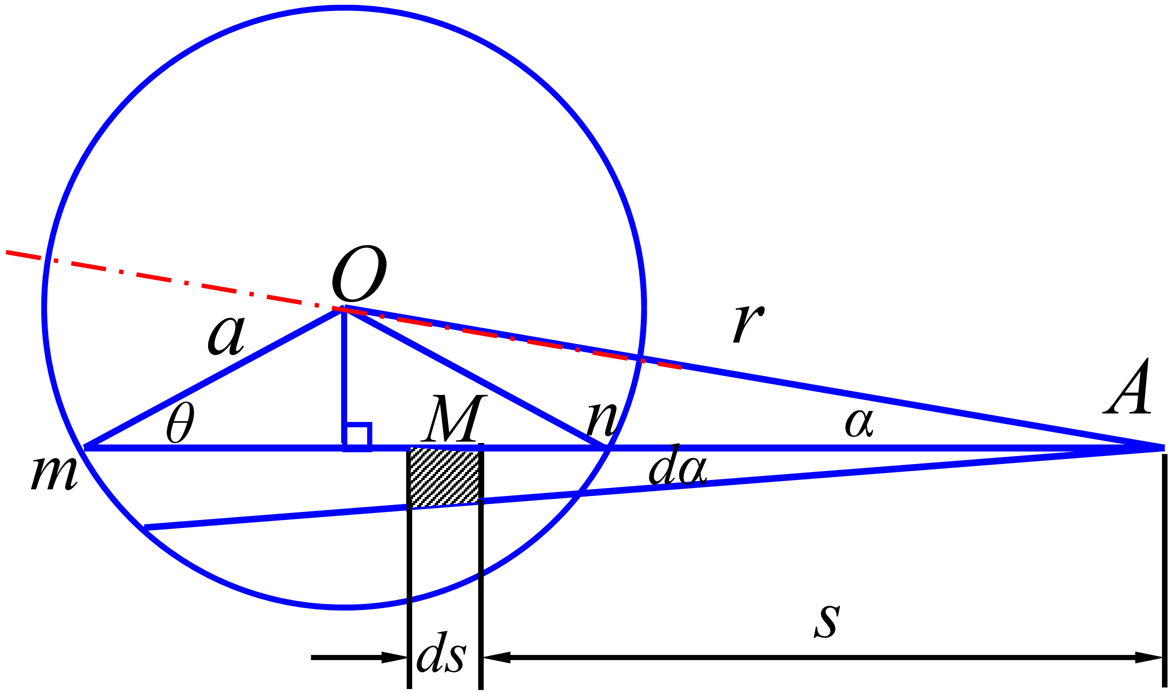


$$\Rightarrow (w)_A = \frac{2(1-\nu^2)q}{\pi E}\left[\int_0^{\pi/2} + \int_{\pi}^{3\pi/2}\right]\overline{Am}d\alpha$$
$$= \frac{2(1-\nu^2)q}{\pi E}\int_0^{\pi/2}\overline{mn}d\alpha$$
$$= \frac{4(1-\nu^2)qa}{\pi E}\int_0^{\pi/2}\left[\cos\theta\right]d\alpha$$
$$= \frac{4(1-\nu^2)qa}{\pi E}\int_0^{\pi/2}\left[\sqrt{1-\tfrac{r^2}{a^2}\sin^2\alpha}\right]d\alpha$$

- The integral in the last equality is elliptic integral.

$$w_{max} = w_{r=0} = \frac{2(1-\nu^2)qa}{E}, \qquad w_{r=a} = \frac{4(1-\nu^2)qa}{\pi E} = \frac{2}{\pi}w_{max}$$

Vertical Displacement outside the Loading Zone



- **For a field point *A* outside of the circle**
- By use of a polar coordinate at A and measured from AO, the vertical deflection at A due to the differential normal pressure at the **source point *M***

- Inspection of the geometry

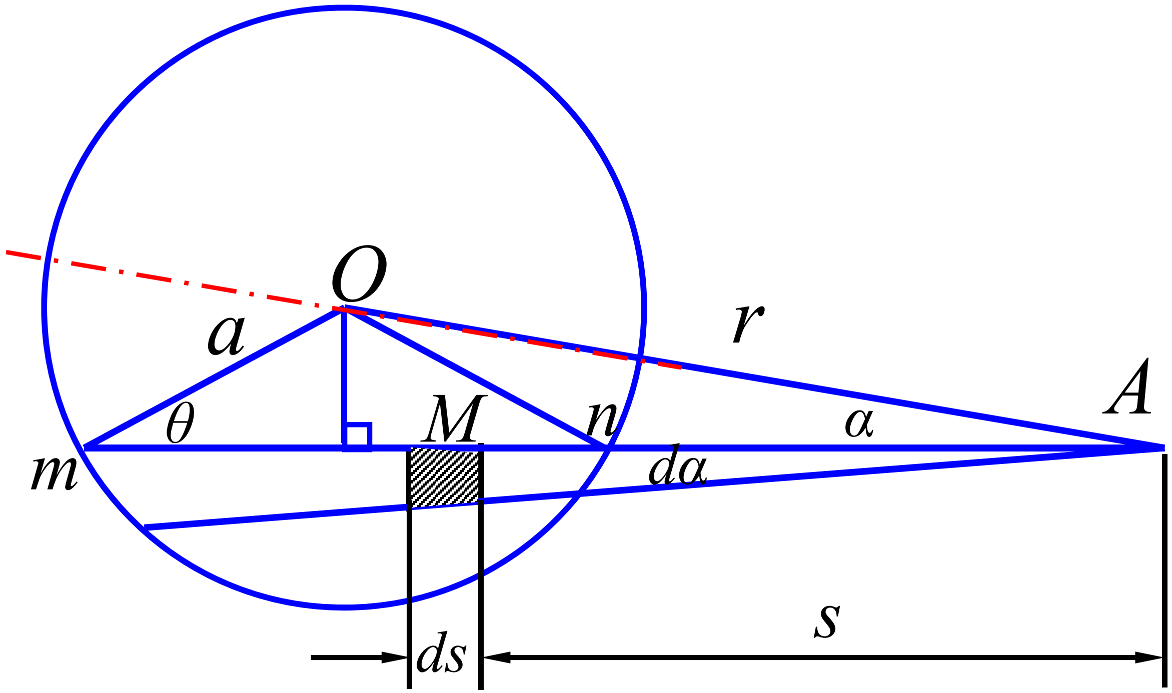
$$\boxed{\overline{mn} = 2\sqrt{a^2 - r^2 \sin\alpha^2}, \quad a\sin\theta = r\sin\alpha}$$

Symmetric about AO

$$\Rightarrow (w)_A = \frac{(1-\nu^2)q}{\pi E}\iint_{S_1} ds\,d\alpha$$

$$= \frac{2(1-\nu^2)q}{\pi E}\int_0^{\alpha_{\max}} \overline{mn}\,d\alpha = \frac{4(1-\nu^2)q}{\pi E}\int_0^{\alpha_{\max}} d\alpha\sqrt{a^2 - r^2\sin\alpha^2}$$

Vertical Displacement outside the Loading Zone



- **Change of integral variable**

$$a\sin\theta = r\sin\alpha$$

$$\Rightarrow d\alpha = \frac{a\cos\theta}{r\sqrt{1-\frac{a^2}{r^2}\sin^2\theta}}d\theta$$

$$0 \le \alpha \le \alpha_{\max} \Rightarrow 0 \le \theta \le \pi/2$$

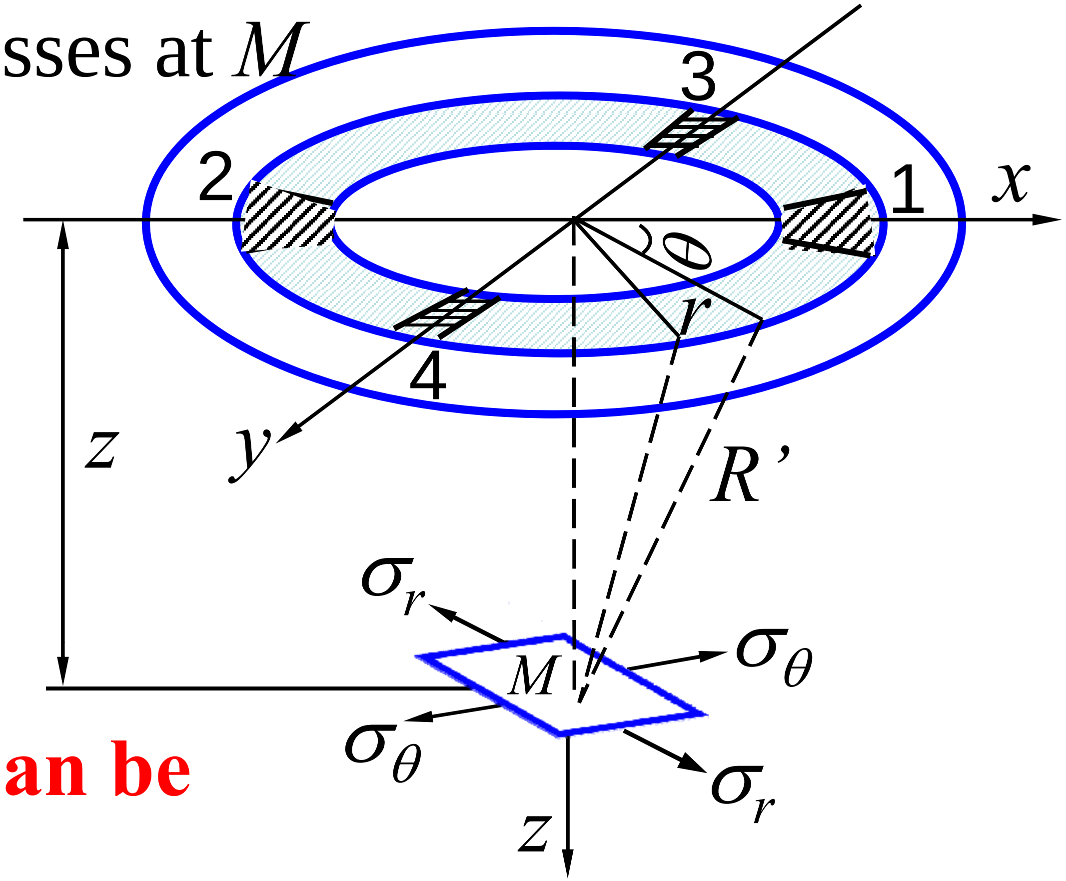
$$\Rightarrow (w)_A = \frac{4(1-\nu^2)q}{\pi E}\int_0^{\alpha_{\max}} d\alpha\sqrt{a^2 - r^2\sin\alpha^2} = \frac{4(1-\nu^2)q}{\pi E}\int_0^{\pi/2}\frac{a^2\cos^2\theta}{r\sqrt{1-\frac{a^2}{r^2}\sin^2\theta}}d\theta$$

$$= \frac{4(1-\nu^2)qr}{\pi E}\left[\int_0^{\pi/2} d\theta\sqrt{1-\frac{a^2}{r^2}\sin^2\theta} - \left(1-\frac{a^2}{r^2}\right)\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{a^2}{r^2}\sin^2\theta}}\right], \quad \Rightarrow \boxed{w_{r=a} = \frac{4(1-\nu^2)qa}{\pi E}}$$

- The integrals in the last equality are elliptic integrals.

Stresses along the Symmetry Axis

- Solve for the non-zero stresses at M
- Boussinesq's solution



$$\sigma_z = \frac{3P}{2\pi}\frac{z^3}{R^5},$$

$$\sigma_r = \frac{P}{2\pi}\frac{1}{R^2}\left[\frac{(1-2\nu)R}{R+z} - \frac{3r^2 z}{R^3}\right],$$

$$\sigma_\theta = \frac{(1-2\nu)P}{2\pi}\frac{1}{R^2}\left(\frac{z}{R} - \frac{R}{R+z}\right)$$

- **Only Cartesian stresses can be directly added. For σ_z:**
- Consider the total pressure acting on a differential annular: $dP = 2\pi rqdr$

$$\sigma_z = \int_{S_1}\frac{3dP}{2\pi}\frac{z^3}{R'^5} = 3qz^3\int_0^a \frac{r}{(r^2+z^2)^{5/2}}dr = q\left[1 - \frac{z^3}{(a^2+z^2)^{3/2}}\right]$$

Stresses along the Symmetry Axis

- **Polar stresses cannot be directly added.**
- Consider element 1 and 2:

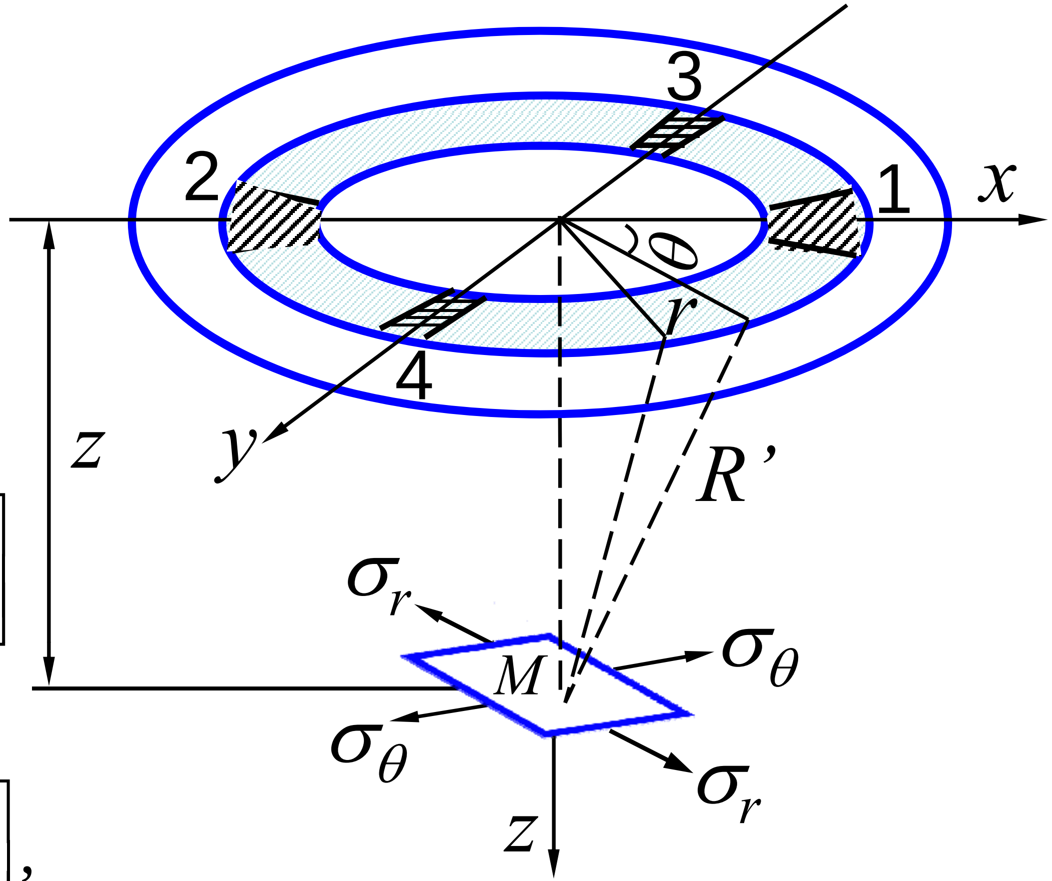
$$d\sigma_r' = 2\frac{qrdrd\theta}{2\pi}\frac{1}{R'^2}\left[\frac{(1-2\nu)R'}{R'+z} - \frac{3r^2 z}{R'^3}\right]$$

$$d\sigma_\theta' = 2\frac{(1-2\nu)qrdrd\theta}{2\pi}\frac{1}{R'^2}\left[\frac{z}{R'} - \frac{R'}{R'+z}\right]$$

- Consider element 3 and 4:

$$d\sigma_r'' = 2\frac{(1-2\nu)qrdrd\theta}{2\pi}\frac{1}{R'^2}\left[\frac{z}{R'} - \frac{R'}{R'+z}\right],$$

$$d\sigma_\theta'' = 2\frac{qrdrd\theta}{2\pi}\frac{1}{R'^2}\left[\frac{(1-2\nu)R'}{R'+z} - \frac{3r^2 z}{R'^3}\right]$$



$$\boxed{\sigma_r' \perp \sigma_r'', \quad \sigma_\theta' \perp \sigma_\theta''}$$

- Together: $d\sigma_r = d\sigma_\theta = \dfrac{qrdrd\theta}{\pi}\left[(1-2\nu)\dfrac{z}{R'^3} - \dfrac{3r^2 z}{R'^5}\right].$

Stresses along the Symmetry Axis

- Integrate over the interval $0 \le \theta \le \pi/2$:

$$\sigma_r = \sigma_\theta = \int_0^{\pi/2} \frac{qrdrd\theta}{\pi}\left[(1-2\nu)\frac{z}{R'^3} - \frac{3r^2 z}{R'^5}\right]$$

$$= -\frac{q}{2}\left[(1+2\nu) + \frac{z^3}{(a^2+z^2)^{3/2}} - \frac{2(1+\nu)z}{(a^2+z^2)^{1/2}}\right]$$

$$\sigma_z = q\left[1 - \frac{z^3}{(a^2+z^2)^{3/2}}\right]$$

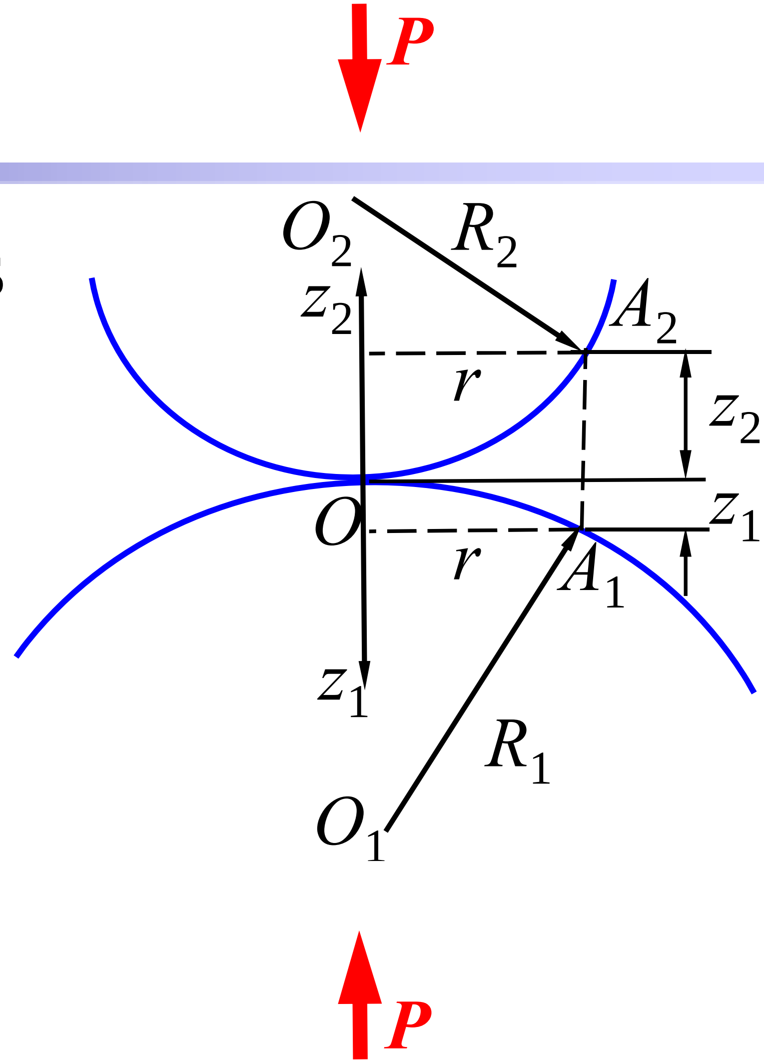
- $\tau_{\max}$ occurs on planes $\pi/4$ from z-axis:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_\theta - \sigma_z}{2} = \frac{q}{2}\left[\frac{1-2\nu}{2} + \frac{(1+\nu)z}{(a^2+z^2)^{1/2}} - \frac{3z^3}{2(a^2+z^2)^{3/2}}\right]$$

Hertz Contact – Geometry

- Contact stress between two spheres under concentrated force *P*
- By geometry:

$$\begin{cases}(R_1 - z_1)^2 + r^2 = R_1^2 \\ (R_2 - z_2)^2 + r^2 = R_2^2\end{cases} \Rightarrow \begin{cases} z_1 = \dfrac{r^2}{2R_1 - z_1} \approx \dfrac{r^2}{2R_1} \\ z_2 = \dfrac{r^2}{2R_2 - z_2} \approx \dfrac{r^2}{2R_2}\end{cases}$$



- Prior to deformation

$$\overline{A_1A_2} = z_1 + z_2 = \frac{r^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = Br^2$$

- Relative rigid-body displacement after deformation

$$\overline{A_1A_2}' = D = z_1 + z_2 + w_1 + w_2$$

$$\Rightarrow w_1 + w_2 = D - \left(z_1 + z_2\right) = D - Br^2, \qquad B = \frac{R_1 + R_2}{2R_1R_2}$$

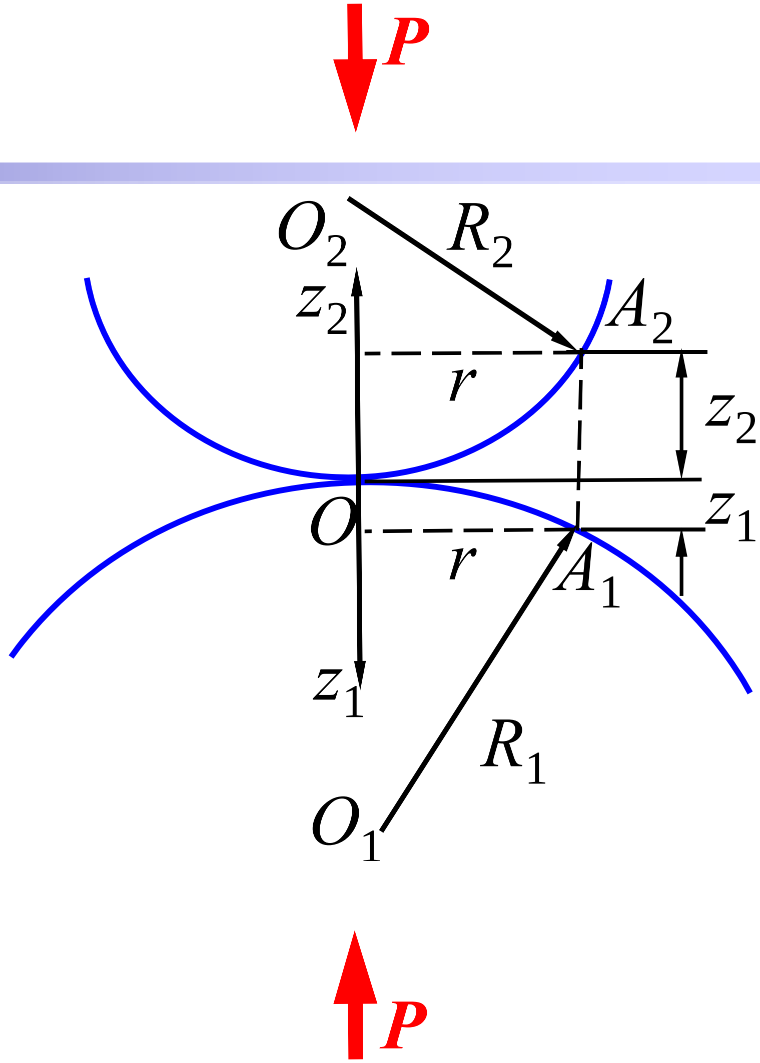
Hertz Contact – Assumptions

- **The contact area is a circle of radius *a***

$$w_1 = \frac{1-\nu_1^2}{\pi E_1}\iint q\,ds\,d\alpha = k_1\iint q\,ds\,d\alpha$$

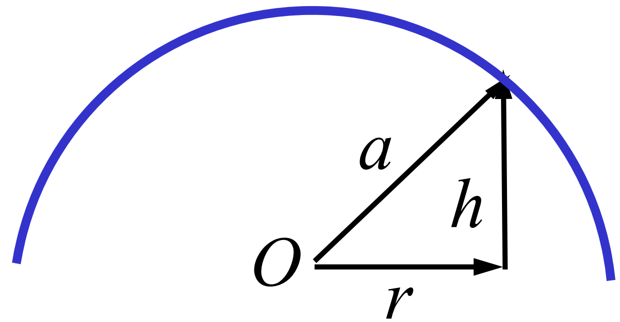
$$w_2 = \frac{1-\nu_2^2}{\pi E_2}\iint q\,ds\,d\alpha = k_2\iint q\,ds\,d\alpha$$

$$\Rightarrow \frac{(w_1+w_2)_r}{(k_1+k_2)} = \iint q\,ds\,d\alpha$$



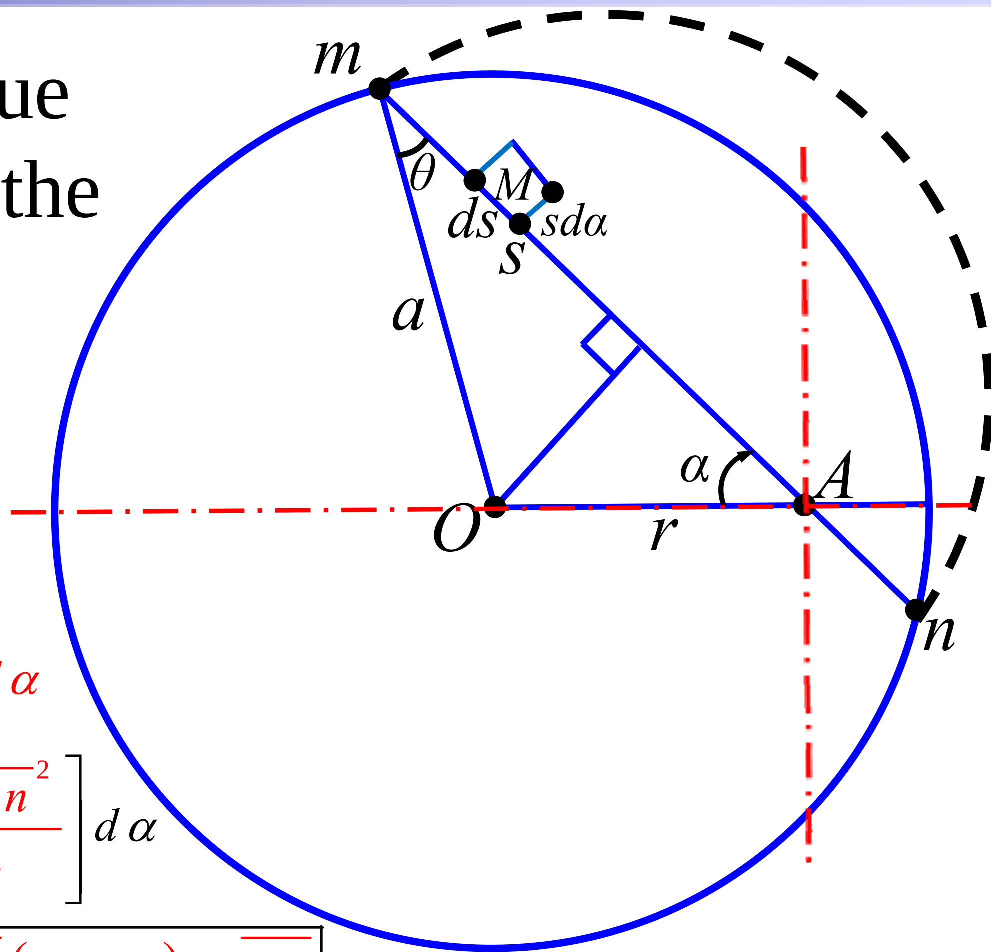
- **The distribution of contact pressure is a hemi-sphere**

$$q(r) = \frac{q_0}{a}\sqrt{a^2-r^2} = \frac{q_0}{a}h(r), \quad q(0) = q_0$$



Hertz Contact – Displacement due to Contact

- The vertical deflection at A due to the differential pressure at the source point M



$$\Rightarrow \frac{(w_1+w_2)_r}{(k_1+k_2)} = \iint q\,ds\,d\alpha$$

$$= \left[\int_0^{\pi/2}+\int_{\pi/2}^{\pi}+\int_{\pi}^{3\pi/2}+\int_{3\pi/2}^{2\pi}\right]\left[\int_{\overline{Am}} q\,ds\right]d\alpha$$

$$= 2\left[\int_0^{\pi/2}+\int_{\pi}^{3\pi/2}\right]\left[\int_{\overline{Am}} q\,ds\right]d\alpha = 2\int_0^{\pi/2}\left[\int_{\overline{mn}} q\,ds\right]d\alpha$$

$$= 2\int_0^{\pi/2}\left[\int_{\overline{mn}} \frac{q_0}{a} h(s,\alpha)\,ds\right]d\alpha = 2\int_0^{\pi/2}\frac{q_0}{a}\left[\frac{1}{2}\frac{\pi\,\overline{mn}^2}{4}\right]d\alpha$$

$$\boxed{\overline{mn} = 2a\cos\theta, \qquad a\sin\theta = r\sin\alpha} \quad \boxed{\overline{Am}(\alpha)+\overline{Am}(\alpha+\pi) = \overline{mn}}$$

$$\Rightarrow \frac{(w_1+w_2)_r}{(k_1+k_2)} = \frac{\pi q_0}{a}\int_0^{\pi/2}\left[a^2 - r^2\sin^2\alpha\right]d\alpha = \frac{\pi q_0}{a}\left[a^2\frac{\pi}{2} - r^2\int_0^{\pi/2}\sin^2\alpha\,d\alpha\right]$$

$$= \frac{\pi q_0}{a}\left[a^2\frac{\pi}{2} - r^2\int_0^{\pi/2}\frac{1-\cos 2\alpha}{2}d\alpha\right] = \frac{\pi^2 q_0}{4a}\left(2a^2 - r^2\right)$$

Hertz Contact – Back to Geometry

$$\left(w_1 + w_2\right)_r = \frac{\left(k_1 + k_2\right)\pi^2 q_0}{4a}\left(2a^2 - r^2\right) = D - Br^2$$

$$\Rightarrow \boxed{1}: D = \frac{\left(k_1 + k_2\right)\pi^2 q_0 a}{2} = \overline{A_1 A_2}', \qquad \boxed{2}: B = \frac{\left(k_1 + k_2\right)\pi^2 q_0}{4a} = \frac{R_1 + R_2}{2R_1R_2}$$

- Force equilibrium

$$P = \int_A q\,dA = \int_A \frac{q_0}{a} h\,dA = \frac{q_0}{a}\frac{2}{3}\pi a^3 = \frac{2\pi a^2}{3} q_0 \quad \Rightarrow \quad \boxed{3}: q_0 = \frac{3P}{2\pi a^2}$$

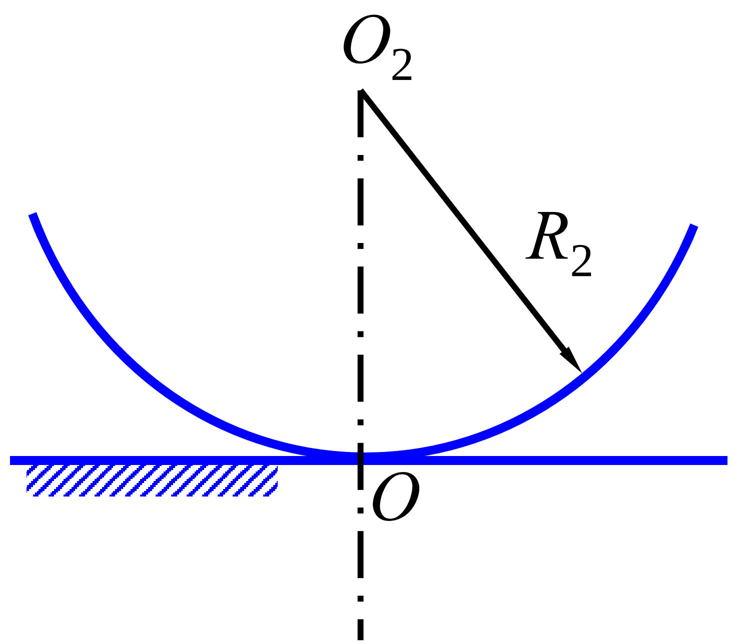
- **Three equations for contact radius (*a*), maximum pressure (q_0), and rigid-body displacement (*D*):**

$$a = \left[\frac{3\pi P\left(k_1 + k_2\right)}{4}\frac{R_1R_2}{\left(R_1 + R_2\right)}\right]^{1/3}, \qquad q_0 = \frac{3P}{2\pi}\left[\frac{4}{3\pi P\left(k_1 + k_2\right)}\frac{\left(R_1 + R_2\right)}{R_1R_2}\right]^{2/3}$$

$$D = \left[\frac{9\pi^2 P^2\left(k_1 + k_2\right)^2}{16}\frac{\left(R_1 + R_2\right)}{R_1R_2}\right]^{1/3}$$

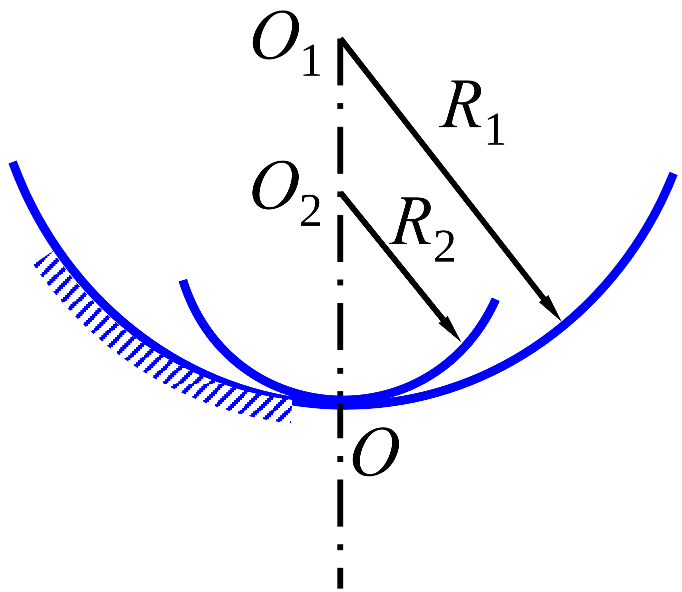
Hertz Contact – Special Cases

- Contact between a sphere and a flat surface



$$R_1 \rightarrow \infty$$

- Contact between a sphere and a spherical cavity



$$R_1 \rightarrow -\left|R_1\right|$$

Outline

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- Half-Space under Uniform Pressure and Gravity
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- Particular Case – Lamé Strain Potential
- Galerkin Vector Potential
- Love Strain Potential – Axi-symmetry
- Completeness of Displacement Potentials
- Harmonic and Bi-harmonic Functions
- Kelvin’s Problem
- Boussinesq’s Problem
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- Distributed Pressure on Half-Space
- Hertz Contact Problem