Bending of Thin Plates

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- Relations between Internal Force and Stress
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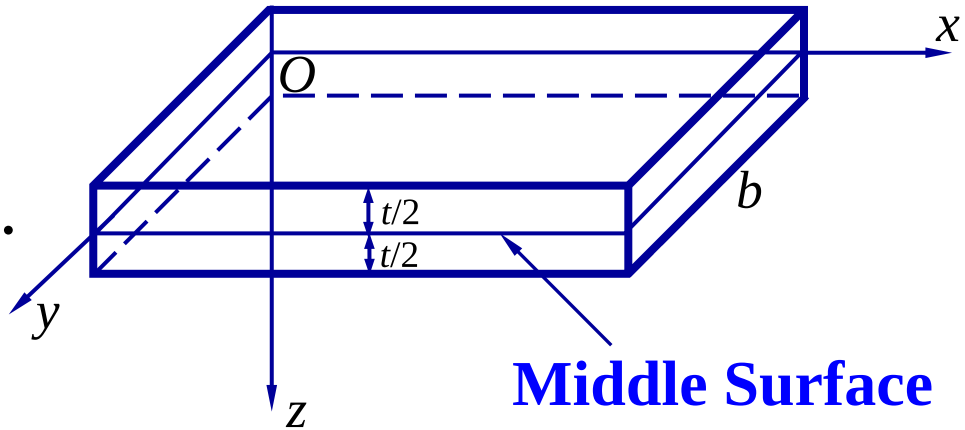
Introduction

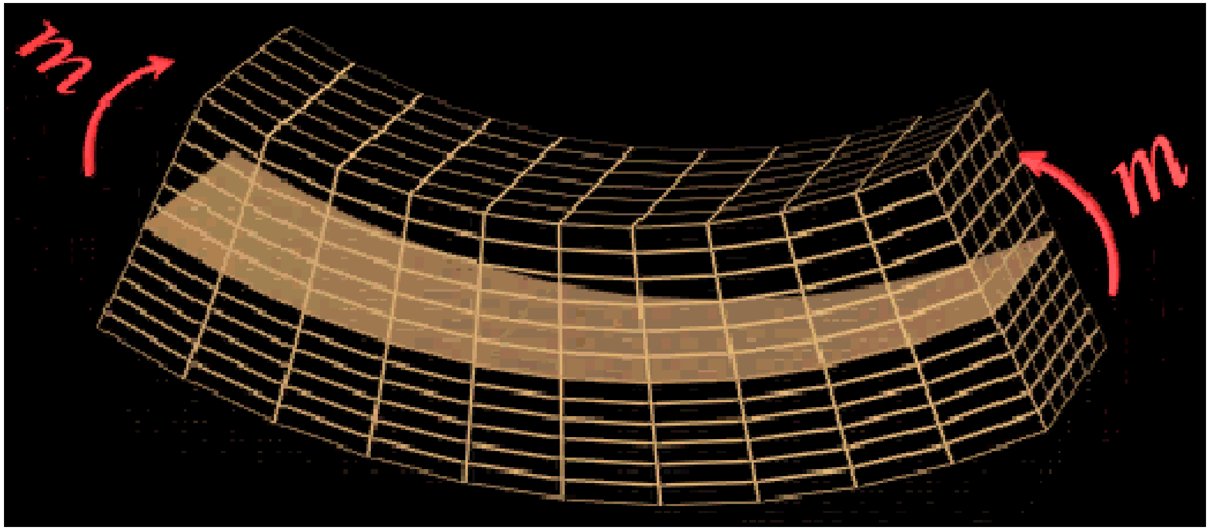


- One dimension (the thickness) is significantly smaller than the other two. (1/8-1/5) > t/b > (1/80-1/100)
- Middle Surface: $z = 0$.
- Only subjected to transvers loads.

- If a plate is only subjected to longitudinal loads, the problem is reduced to plane stress state.
- The bending problem of thin plates is analyzed with strategies similar to those of elastic beams.

Review of the Elementary Beam Theory

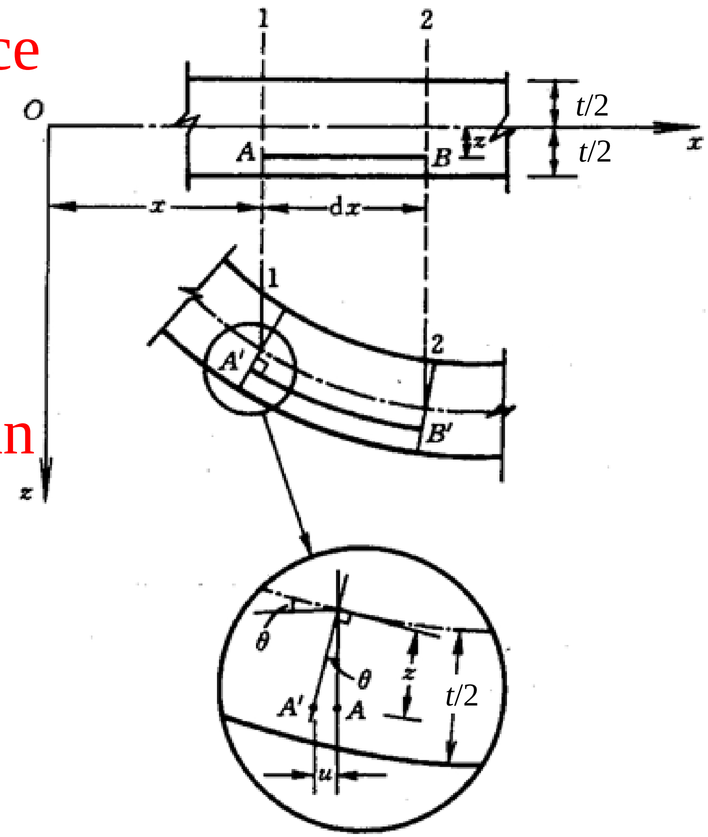
- Plane sections normal to the longitudinal axis of the beam remain planar.
- Only uniaxial longitudinal stress is assumed.



$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = M, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = q$$

Assumptions

- Straight lines normal to the middle surface remain straight and the same length.
- Stress components acting on planes parallel to the middle surface are significantly smaller than other components. The corresponding strain can therefore be neglected.



$$0=\varepsilon_z=\frac{\partial w}{\partial z}\quad\Rightarrow\quad\boxed{w=w(x,y)}$$

$$0=\varepsilon_{zx}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\quad\Rightarrow\quad\boxed{\frac{\partial u}{\partial z}=-\frac{\partial w}{\partial x}}$$

$$0=\varepsilon_{zy}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\quad\Rightarrow\quad\boxed{\frac{\partial v}{\partial z}=-\frac{\partial w}{\partial y}}$$

Discard: $\varepsilon_z=\dfrac{\sigma_z-\nu(\sigma_x+\sigma_y)}{E},\ \varepsilon_{zx}=\dfrac{1}{2G}\tau_{zx},\ \varepsilon_{zy}=\dfrac{1}{2G}\tau_{zy}.$

Assumptions

- Constitutive relations

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \quad \varepsilon_{xy} = \frac{1}{2G}\tau_{xy}.$$

- The middle surface of the plate is not strained during bending.

$$\begin{cases} (u)_{z=0} = 0 \\ (v)_{z=0} = 0 \end{cases} \Rightarrow \begin{cases} (\varepsilon_x)_{z=0} = \left(\dfrac{\partial u}{\partial x}\right)_{z=0} = 0 \\ (\varepsilon_y)_{z=0} = \left(\dfrac{\partial v}{\partial y}\right)_{z=0} = 0 \\ (\varepsilon_{xy})_{z=0} = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)_{z=0} = 0 \end{cases}$$

Governing Equation in terms of Deflection *w*(*x*, *y*)

- Longitudinal displacements formulated in terms of the vertical deflection $w = w(x,y)$

$$\begin{cases} \dfrac{\partial u}{\partial z} = -\dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} = -\dfrac{\partial w}{\partial y} \end{cases} \Rightarrow \begin{cases} u = -\dfrac{\partial w}{\partial x} z \cancel{+ f_1(x,y)} \\ v = -\dfrac{\partial w}{\partial y} z \cancel{+ f_2(x,y)} \end{cases} \Rightarrow \boxed{\begin{aligned} u &= -\dfrac{\partial w}{\partial x} z \\ v &= -\dfrac{\partial w}{\partial y} z \end{aligned}}$$

$$\begin{cases} (u)_{z=0} = 0 \\ (v)_{z=0} = 0 \end{cases} \Rightarrow \begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

- Longitudinal strains in terms of *w*

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\partial^2 w}{\partial x^2} z, \quad \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\partial^2 w}{\partial y^2} z, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{\partial^2 w}{\partial x \partial y} z$$

Governing Equation in terms of Deflection *w*(*x*, *y*)

- Longitudinal stresses in terms of *w*

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\ \sigma_y = \dfrac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\ \tau_{xy} = \dfrac{E}{(1+\nu)}\varepsilon_{xy} \end{cases} \Rightarrow \boxed{\begin{aligned} \sigma_x &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ \sigma_y &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ \tau_{xy} &= -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} \end{aligned}}$$

- Transvers shear stresses in terms of *w*

$$\begin{cases} \dfrac{\partial \tau_{zx}}{\partial z} = -\dfrac{\partial \sigma_x}{\partial x} - \dfrac{\partial \tau_{yx}}{\partial y} \\ \dfrac{\partial \tau_{zy}}{\partial z} = -\dfrac{\partial \sigma_y}{\partial y} - \dfrac{\partial \tau_{xy}}{\partial x} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \tau_{zx}}{\partial z} = \dfrac{Ez}{1-\nu^2}\left(\dfrac{\partial^3 w}{\partial x^3} + \dfrac{\partial^3 w}{\partial x \partial y^2}\right) = \dfrac{Ez}{1-\nu^2}\dfrac{\partial}{\partial x}\nabla^2 w \\ \dfrac{\partial \tau_{zy}}{\partial z} = \dfrac{Ez}{1-\nu^2}\left(\dfrac{\partial^3 w}{\partial y^3} + \dfrac{\partial^3 w}{\partial y \partial x^2}\right) = \dfrac{Ez}{1-\nu^2}\dfrac{\partial}{\partial y}\nabla^2 w \end{cases}$$

$$\text{Integrate w.r.t } z \cdots$$

Governing Equation in terms of Deflection *w*(*x*, *y*)

- Transvers shear stresses in terms of *w*

$$\tau_{zx} = \frac{Ez^2}{2(1-\nu^2)}\frac{\partial}{\partial x}\nabla^2 w + F_1(x,y), \qquad \tau_{zy} = \frac{Ez^2}{2(1-\nu^2)}\frac{\partial}{\partial y}\nabla^2 w + F_2(x,y)$$

- Applying the BCs at the top/bottom surface

$$\begin{cases} (\tau_{zx})_{z=\pm t/2} = 0 \\ (\tau_{zy})_{z=\pm t/2} = 0 \end{cases} \Rightarrow \boxed{\begin{aligned} \tau_{zx} &= \frac{E}{2(1-\nu^2)}\left(z^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial x}\nabla^2 w \\ \tau_{zy} &= \frac{E}{2(1-\nu^2)}\left(z^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial y}\nabla^2 w \end{aligned}}$$

- Transvers normal stress in terms of *w*

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} = \frac{E}{2(1-\nu^2)}\left(\frac{t^2}{4} - z^2\right)\nabla^4 w$$

$$\Rightarrow \sigma_z = \frac{E}{2(1-\nu^2)}\left(\frac{t^2}{4}z - \frac{z^3}{3}\right)\nabla^4 w + F_3(x,y)$$

Governing Equation in terms of Deflection *w*(*x*, *y*)

- Applying the BCs at the bottom surface

$$(\sigma_z)_{z=t/2} = 0$$

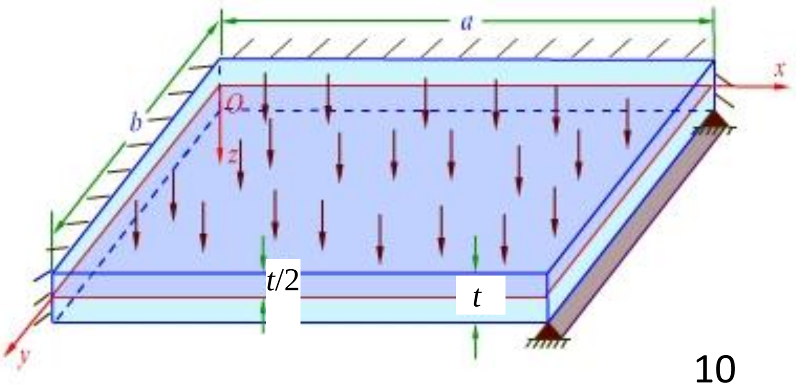
$$\Rightarrow \sigma_z = \frac{E}{2(1-\nu^2)}\left[\frac{t^2}{4}\left(z-\frac{t}{2}\right)-\frac{1}{3}\left(z^3-\frac{t^3}{8}\right)\right]\nabla^4 w$$

$$\Rightarrow \boxed{\sigma_z = -\frac{E}{6(1-\nu^2)}\left(z-\frac{t}{2}\right)^2(z+t)\nabla^4 w}$$

- Further applying the BCs at the top surface

$$(\sigma_z)_{z=-t/2} = -q \quad \Rightarrow \quad \frac{Et^3}{12(1-\nu^2)}\nabla^4 w = q$$

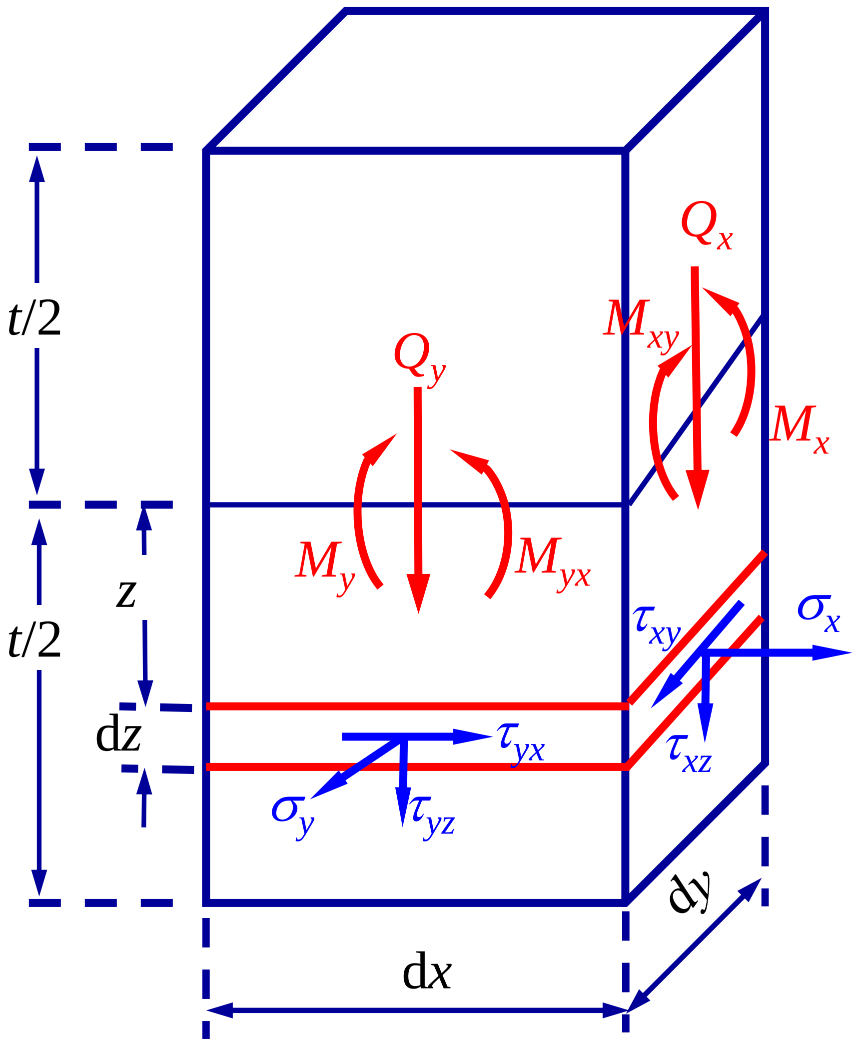
$$\boxed{D\nabla^4 w = q, \quad D = \frac{Et^3}{12(1-\nu^2)}}$$



- ***D*: Flexural Rigidity**

Internal Forces per Unit Length

- **Definition:** It is customary to integrate the stresses over the (constant) plate thickness.
- **Design requirements**
- **Dealing with the Boundary Conditions** (Saint-Venant BCs)



$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_x \mathrm{d}z$$

$$= -\frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2 \mathrm{d}z$$

$$= -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

Internal Forces per Unit Length

$$M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\tau_{xy}\mathrm{d}z$$

$$= -\frac{E}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2\mathrm{d}z$$

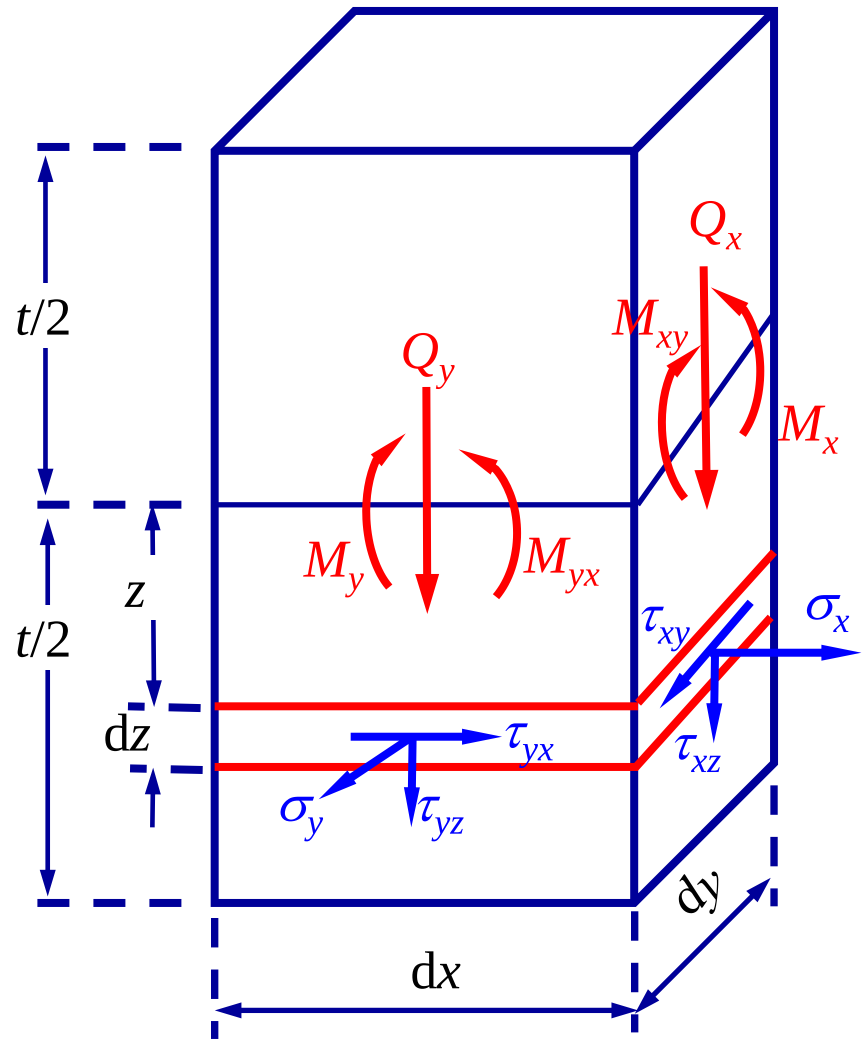
$$= -\frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x\partial y} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\sigma_y\mathrm{d}z$$

$$= -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{yx} = M_{xy}$$



Internal Forces per Unit Length

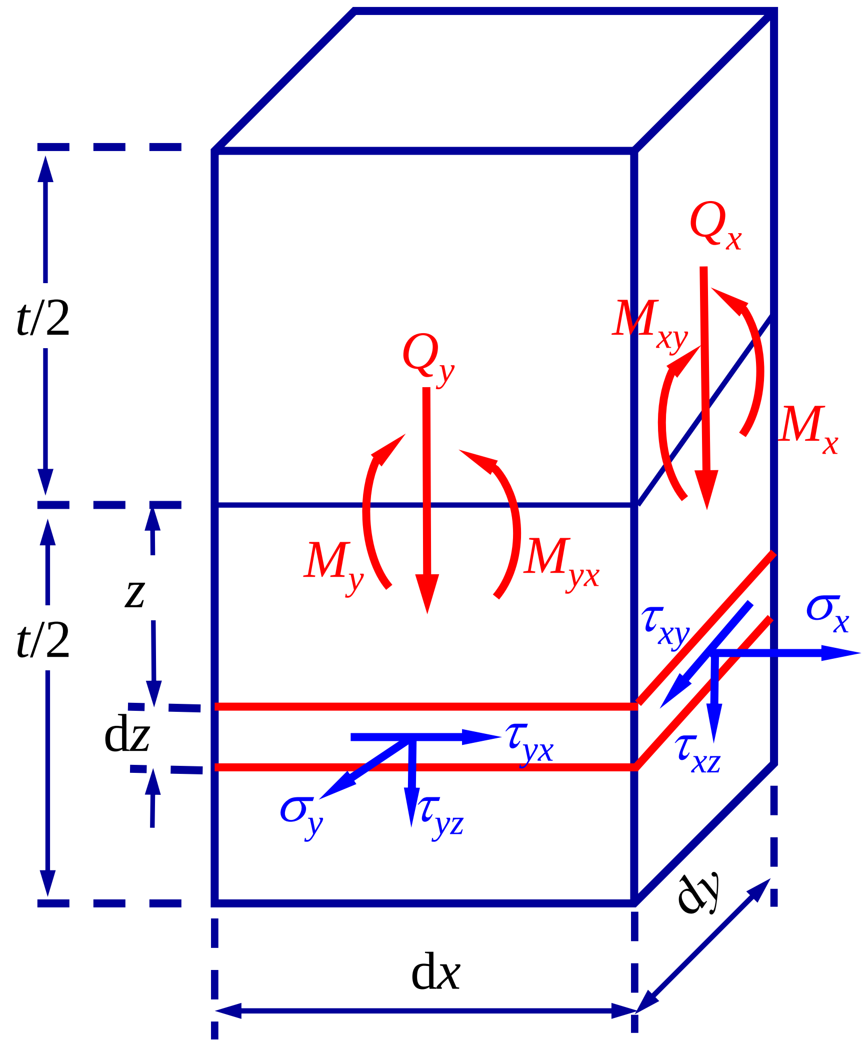
$$Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} \mathrm{d}z$$

$$= \frac{E}{2(1-\nu^2)} \frac{\partial}{\partial x} \nabla^2 w \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(z^2 - \frac{t^2}{4} \right) \mathrm{d}z$$

$$= -\frac{Et^3}{12(1-\nu^2)} \frac{\partial}{\partial x} \nabla^2 w = -D \frac{\partial}{\partial x} \nabla^2 w$$

$$Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} \mathrm{d}z = -\frac{Et^3}{12(1-\nu^2)} \frac{\partial}{\partial y} \nabla^2 w$$

$$= -D \frac{\partial}{\partial y} \nabla^2 w$$



Relations between Internal Forces and Stresses

$$\left.\begin{aligned}\sigma_x &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\\ M_x &= -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\end{aligned}\right\} \Rightarrow \boxed{\sigma_x = \frac{12z}{t^3}M_x}$$

$$\left.\begin{aligned}\sigma_y &= -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ M_y &= -\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\end{aligned}\right\} \Rightarrow \boxed{\sigma_y = \frac{12z}{t^3}M_y}$$

$$\left.\begin{aligned}\tau_{xy} &= -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}\\ M_{xy} &= -\frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x\partial y}\end{aligned}\right\} \Rightarrow \boxed{\tau_{xy} = \frac{12z}{t^3}M_{xy}}$$

Relations between Internal Forces and Stresses

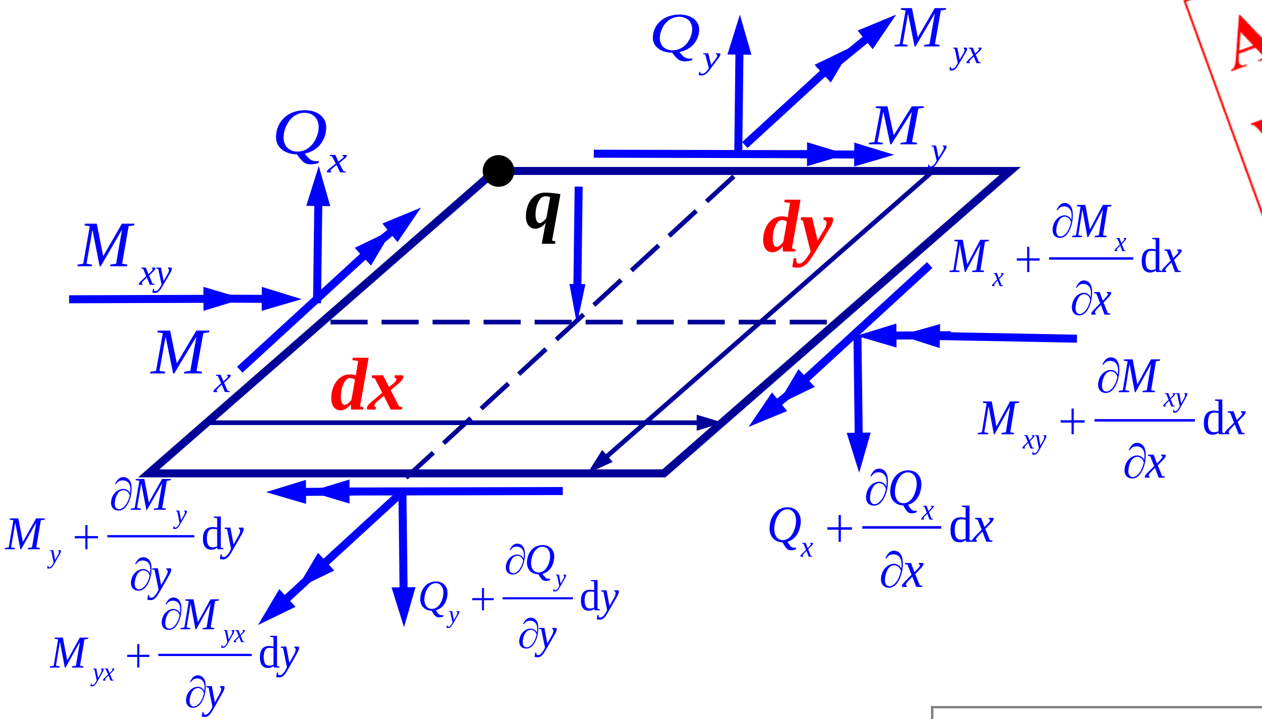
$$\left.\begin{aligned}\tau_{zx} &= \frac{E}{2(1-\nu^2)}\left(z^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial x}\nabla^2 w \\ Q_x &= -\frac{Et^3}{12(1-\nu^2)}\frac{\partial}{\partial x}\nabla^2 w\end{aligned}\right\} \Rightarrow \boxed{\tau_{zx} = \frac{6}{t^3}\left(\frac{t^2}{4} - z^2\right)Q_x}$$

$$\left.\begin{aligned}\tau_{zy} &= \frac{E}{2(1-\nu^2)}\left(z^2 - \frac{t^2}{4}\right)\frac{\partial}{\partial y}\nabla^2 w \\ Q_y &= -\frac{Et^3}{12(1-\nu^2)}\frac{\partial}{\partial y}\nabla^2 w\end{aligned}\right\} \Rightarrow \boxed{\tau_{zy} = \frac{6}{t^3}\left(\frac{t^2}{4} - z^2\right)Q_y}$$

$$\left.\begin{aligned}\sigma_z &= -\frac{E}{6(1-\nu^2)}\left(z - \frac{t}{2}\right)^2 (z+t)\nabla^4 w \\ &\frac{Et^3}{12(1-\nu^2)}\nabla^4 w = q\end{aligned}\right\} \Rightarrow \boxed{\sigma_z = -2q\left(\frac{z}{t} - \frac{1}{2}\right)^2\left(\frac{z}{t} + 1\right)}$$

Differential Element Equilibrium

Alternative way to derive governing equations



$$0=\sum F_z = \frac{\partial Q_x}{\partial x}dxdy + \frac{\partial Q_y}{\partial y}dydx + q\,dxdy \Rightarrow \boxed{\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0}$$

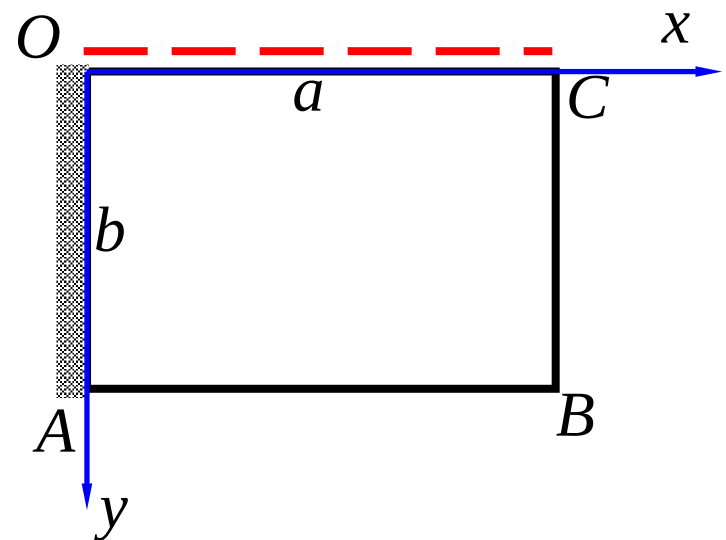
$$0=\sum M_x = \frac{\partial M_{xy}}{\partial x}dxdy + \frac{\partial M_y}{\partial y}dydx - Q_y dxdy - \cancel{\frac{\partial Q_y}{\partial y}dy\frac{1}{2}dx} \Rightarrow \boxed{\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y}$$

$$0=\sum M_y \Rightarrow \boxed{\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}=Q_x} \quad \Rightarrow \quad \boxed{\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0}$$

Boundary Conditions

- Built-in / clamped edge along *OA*

$$(w)_{x=0} = 0, \quad \left(\frac{\partial w}{\partial x}\right)_{x=0} = 0$$



- Simply supported edge along *OC*

$$\boxed{0 = (w)_{y=0},} \quad 0 = (M_y)_{y=0} = -D\left(\frac{\partial^2 w}{\partial y^2} + \cancel{\nu \frac{\partial^2 w}{\partial x^2}}\right)_{y=0} \Rightarrow \boxed{\left(\frac{\partial^2 w}{\partial y^2}\right)_{y=0} = 0}$$

- Completely free edges, i.e. *BC*

$$(M_x)_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0, \quad (M_{xy})_{x=a} = 0, \quad (Q_x)_{x=a} = 0$$

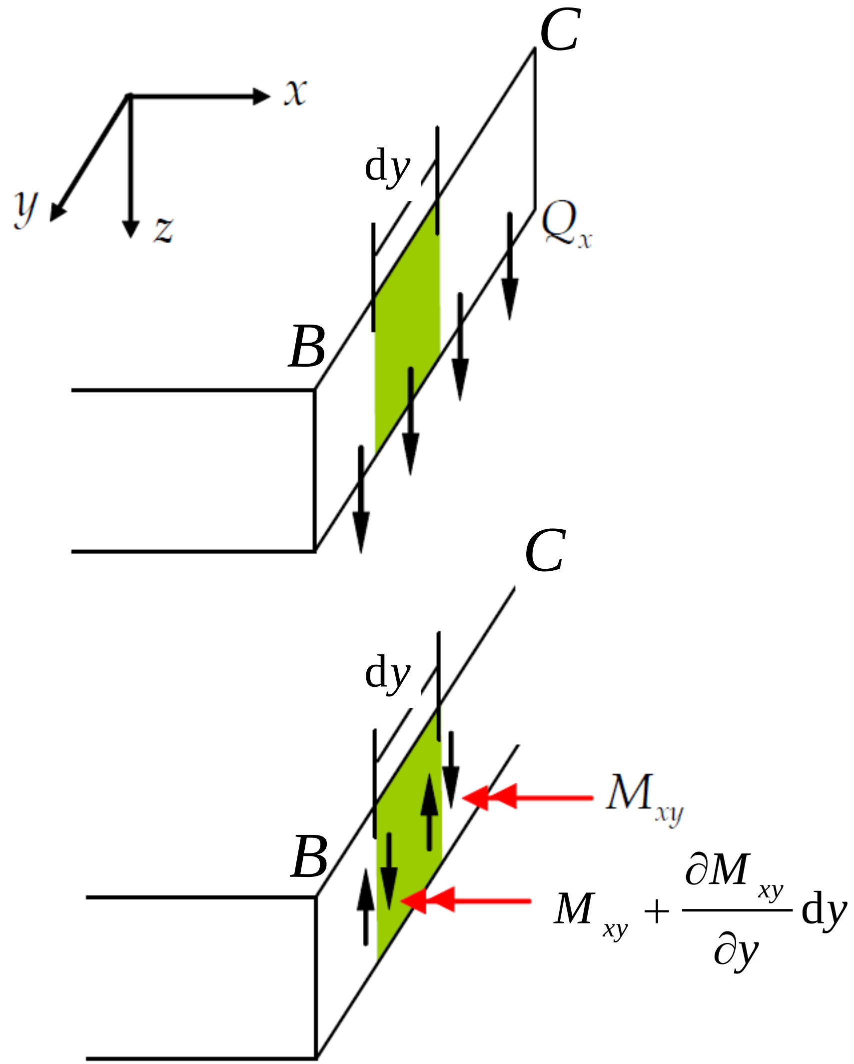
The boundary conditions for a free edge were expressed by Poisson in this form.

Boundary Conditions – Free Edges

- Kirchhoff proved that the two requirements of Poisson dealing with the twisting moment M_{xy} and with the shearing force Q_x must be replaced by one condition.
- Transforming every twisting moment into a force couple (Saint-Venant's principle)

$$\bar{V}_x \mathrm{d}y = Q_x \mathrm{d}y + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y} \mathrm{d}y \right) - M_{x}$$

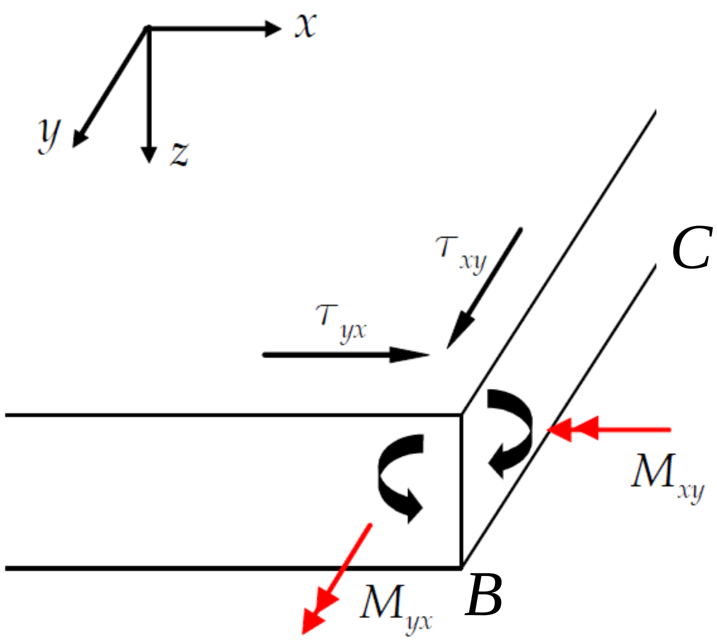
$$\Rightarrow \boxed{\bar{V}_x = Q_x + \frac{\partial M_{xy}}{\partial y}}$$



Boundary Conditions – Free Edges

- We are left with two concentrated forces at the corners *B* and *C*

$$R_B = (M_{xy})_B \uparrow, \qquad R_C = (M_{xy})_C \downarrow$$

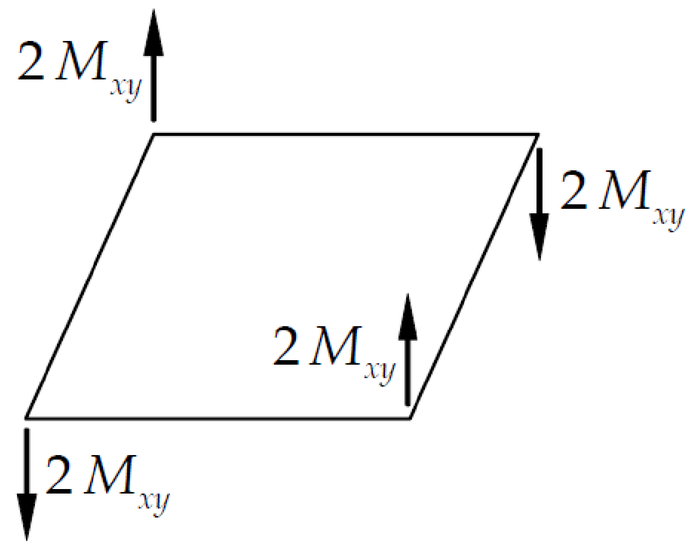


- At the common corner *B* of the Edges *AB* and *BC*

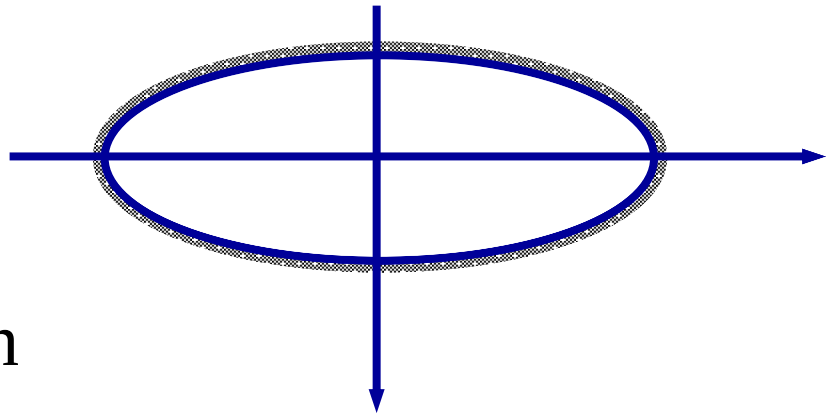
$$R_B = (M_{xy})_B \uparrow + (M_{yx})_B \uparrow = 2(M_{xy})_B \uparrow$$

$$= -2D(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)_B$$

- For all four corners:



Boundary Equation Method – Elliptic Plate

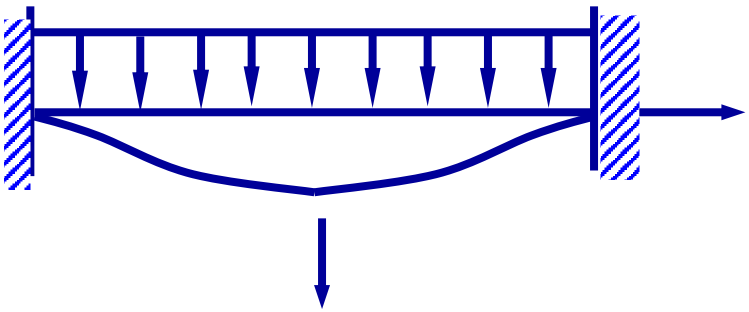


- The boundary equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

- Proposed deflection function

$$w = A\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$$



- On the boundary

$$w=0,\quad \frac{\partial w}{\partial x}=\frac{4Ax}{a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0,\quad \frac{\partial w}{\partial y}=\frac{4Ay}{b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0$$

This solution can only address elliptic plates with fixed boundary.

Boundary Equation Method – Elliptic Plate

- By the governing equation

$$D\left(\frac{24A}{a^4}+\frac{16A}{a^2b^2}+\frac{24A}{b^4}\right)=q$$

This solution can only address elliptic plates under constant pressure.

- The deflection

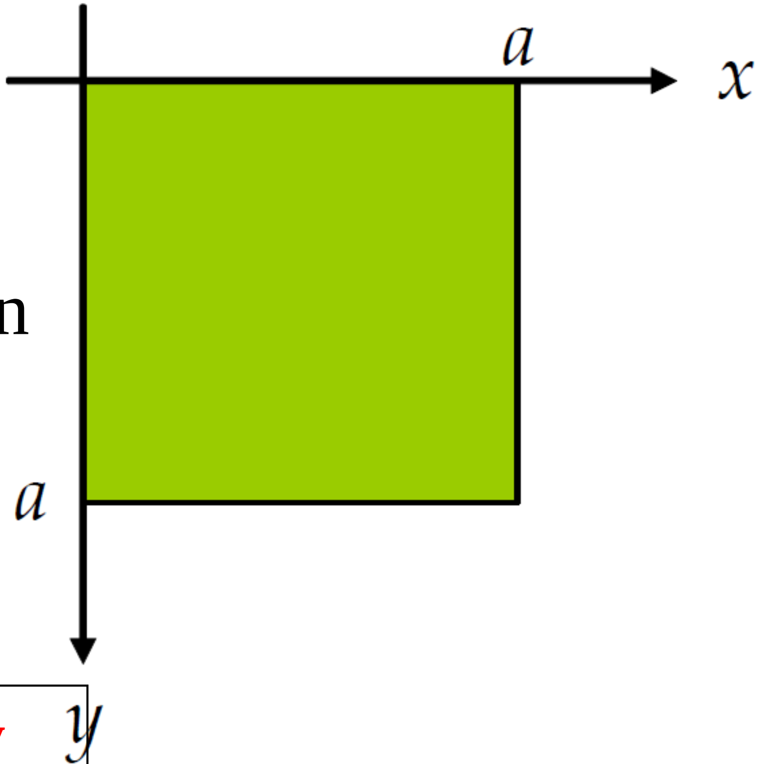
$$A=\frac{qa^4b^4}{8D\left(3a^4+2a^2b^2+3b^4\right)} \quad \Rightarrow \quad \boxed{w=\frac{qa^4b^4}{8D\left(3a^4+2a^2b^2+3b^4\right)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2}$$

The applicability of this method is thus very limited.

Boundary Equation Method – Square Plate

- Consider a simply supported square plate subjected to sinusoidal load distribution

$$q(x,y) = q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$



- The proposed deflection function

$$w(x,y) = w_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$

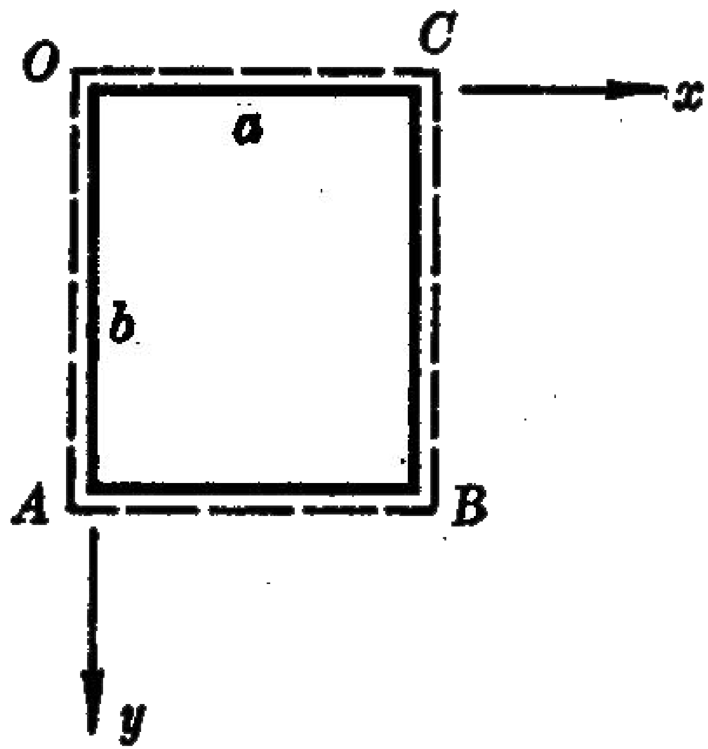
This form satisfies both the BCs and governing equations.

This solution works only for simply supported square plate.

- As an exercise, finish the problem by examining BCs, w_0, bending moments, shear forces, effective shear (reaction) forces at the edges, and corner forces. Check the force balance.

Fourier Method – Rectangular Plate

- Calculate the deflection of a **simply supported** rectangular plate, which is subjected to a distributed lateral load $q(x, y)$.



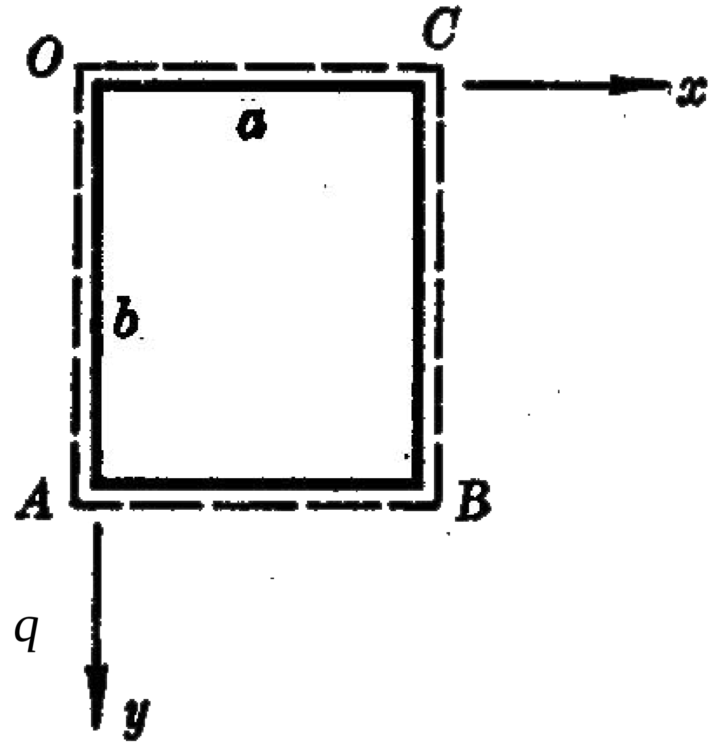
- The governing equation

$$D\nabla^4 w = q, \quad D = \frac{Et^3}{12(1-\nu^2)}$$

- BCs

$$\boxed{\begin{gathered}(w)_{x=0} = (w)_{x=a} = (w)_{y=0} = (w)_{y=b} = 0 \\ (M_x)_{x=0} = (M_x)_{x=a} = (M_y)_{y=0} = (M_y)_{y=b} = 0\end{gathered}}$$

Fourier Method – Rectangular Plate



- Double Fourier Series solution

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

- This solution satisfies all the BCs.
- By the governing equation

$$\Rightarrow \pi^4 D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=q$$

- To derive the coefficients, expand q in Fourier series

$$q=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\int_0^a\int_0^b q\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Fourier Method – Rectangular Plate

- Matching the coefficients A_{mn} and C_{mn}

$$\Rightarrow \boxed{A_{mn} = \frac{4\int_0^a \int_0^b q \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \mathrm{d}x\mathrm{d}y}{\pi^4 abD\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}}$$

- **For constant pressure $q = q_0$**

$$A_{mn} = \frac{16q_0}{\pi^6 Dmn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}, \quad m = 1,3,5,\cdots; \quad n = 1,3,5,\cdots$$

$$\Rightarrow \boxed{w = \frac{16q_0}{\pi^6 D} \sum_{m=1,3,5,\cdots}^{\infty} \sum_{n=1,3,5,\cdots}^{\infty} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}}$$

Fourier Method – Rectangular Plate

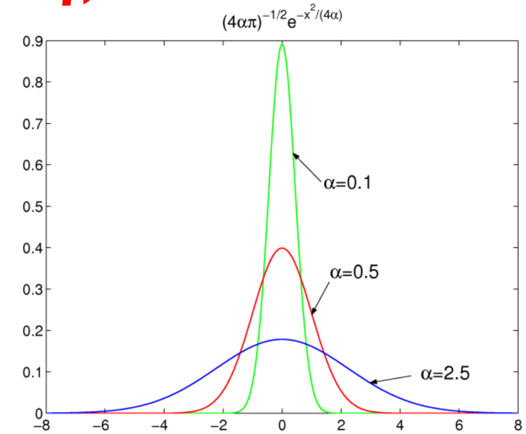
- **For concentrated load *F* applied at (*ξ*, *η*)**
- Dirac Delta Function

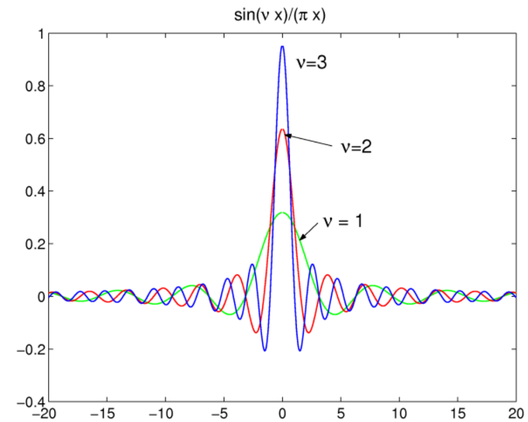
$$q(x,y)=\delta(x,y)=\begin{cases}\infty, & (x,y)=(\xi,\eta)\\ 0, & (x,y)\neq(\xi,\eta)\end{cases}$$

$$\iint \delta(x,y)\,\mathrm{d}x\mathrm{d}y = F$$

$$\iint \delta(x,y)\,f(x,y)\,\mathrm{d}x\mathrm{d}y = Ff(\xi,\eta)$$

where *f*(*x*,*y*) should be sufficiently smooth.





Two examples of Dirac delta functions

- Fourier Series coefficients in *w*

$$A_{mn}=\frac{4\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\mathrm{d}x\mathrm{d}y}{\pi^4 abD\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}=\frac{4F}{\pi^4 abD\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

Summary

- The whole problem is formulated in terms of deflection w.
- The governing equation

$$D\nabla^4 w = q, \quad D = \frac{Et^3}{12(1-\nu^2)}$$

- Boundary conditions: three classical cases
- Built-in / clamped boundary: $w = 0, \quad \dfrac{\partial w}{\partial x} = 0.$
- Simply supported boundary: $w = 0, \qquad M = 0.$
- Free edges: $M = 0, \quad Q + \dfrac{\partial M_{nt}}{\partial t} = 0.$

Summary

- Longitudinal displacements

$$u=-\frac{\partial w}{\partial x}z,\qquad v=-\frac{\partial w}{\partial y}z$$

- Stress field

$$\boxed{\begin{aligned}\sigma_x&=-\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\\ \sigma_y&=-\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\\ \tau_{xy}&=-\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\partial y}\end{aligned}}$$

$$\boxed{\begin{aligned}\tau_{zx}&=\frac{E}{2(1-\nu^2)}\left(z^2-\frac{t^2}{4}\right)\frac{\partial}{\partial x}\nabla^2 w\\ \tau_{zy}&=\frac{E}{2(1-\nu^2)}\left(z^2-\frac{t^2}{4}\right)\frac{\partial}{\partial y}\nabla^2 w\\ \sigma_z&=-\frac{E}{6(1-\nu^2)}\left(z-\frac{t}{2}\right)^2(z+t)\nabla^4 w\end{aligned}}$$

- Internal forces

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\quad M_{yx}=M_{xy}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$Q_x=-D\frac{\partial}{\partial x}\nabla^2 w,\qquad Q_y=-D\frac{\partial}{\partial y}\nabla^2 w$$

Outline

- Introduction
- Elementary Beam Theory
- Assumptions
- Formulation in terms of Deflection
- Internal Force per Unit Length
- Relations between Internal Force and Stress
- Differential Element Equilibrium – Alternative Approach
- Boundary Conditions
- Boundary Equation Method
- Fourier Method
- Summary