Combined Loading

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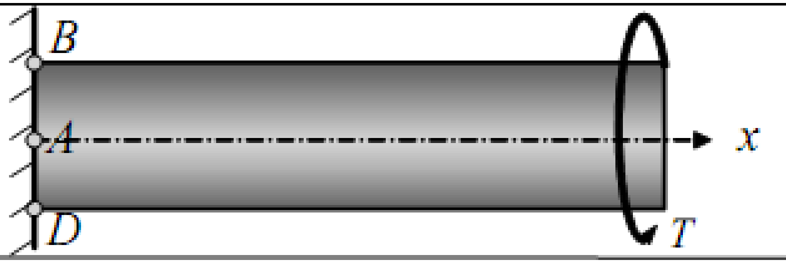
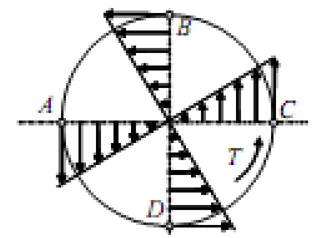
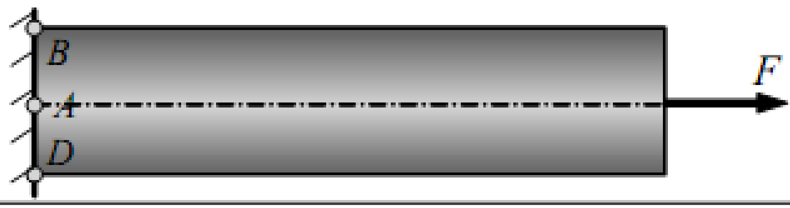
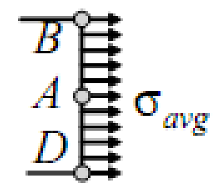
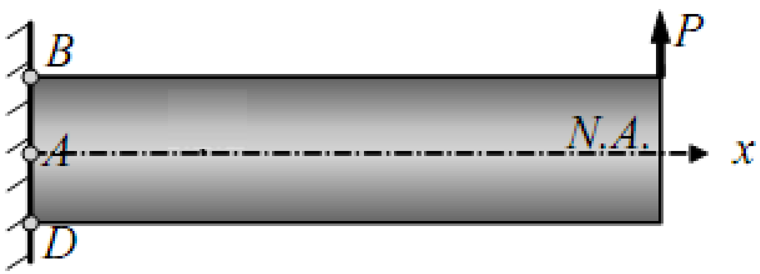
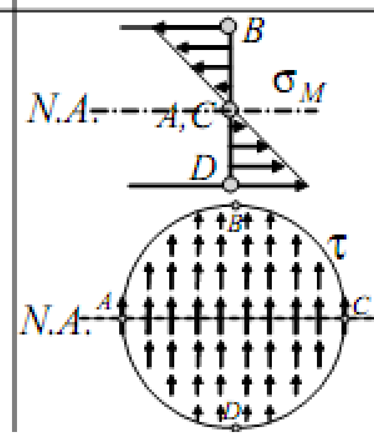
- Bending & Torsion（弯扭组合）
- Tension, Bending & Torsion（拉弯扭组合）

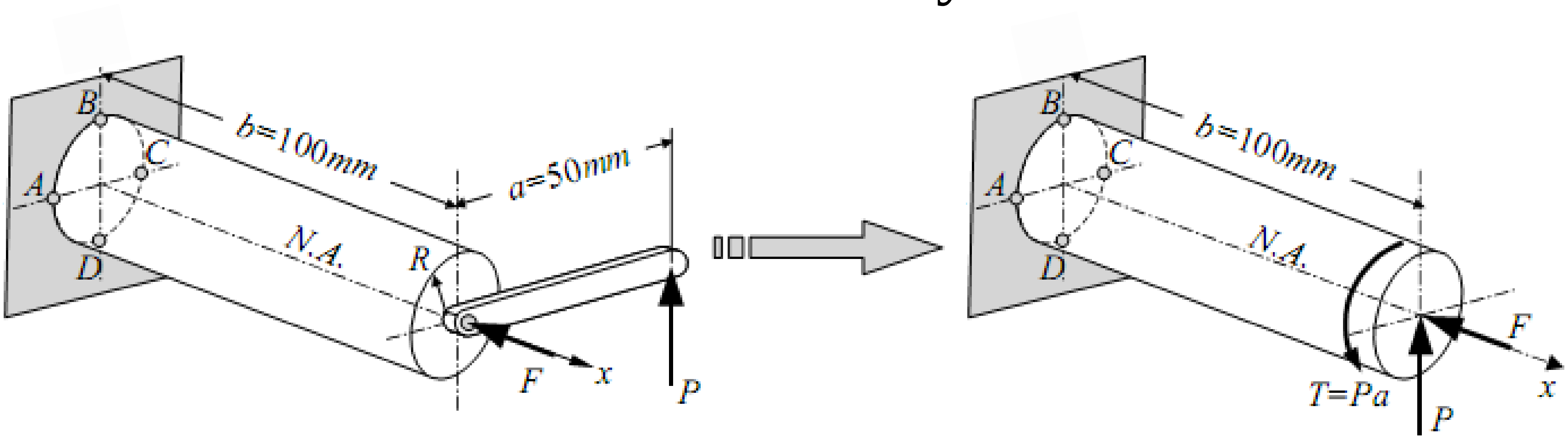
Introduction

- A circular bar subjected to a single type of load

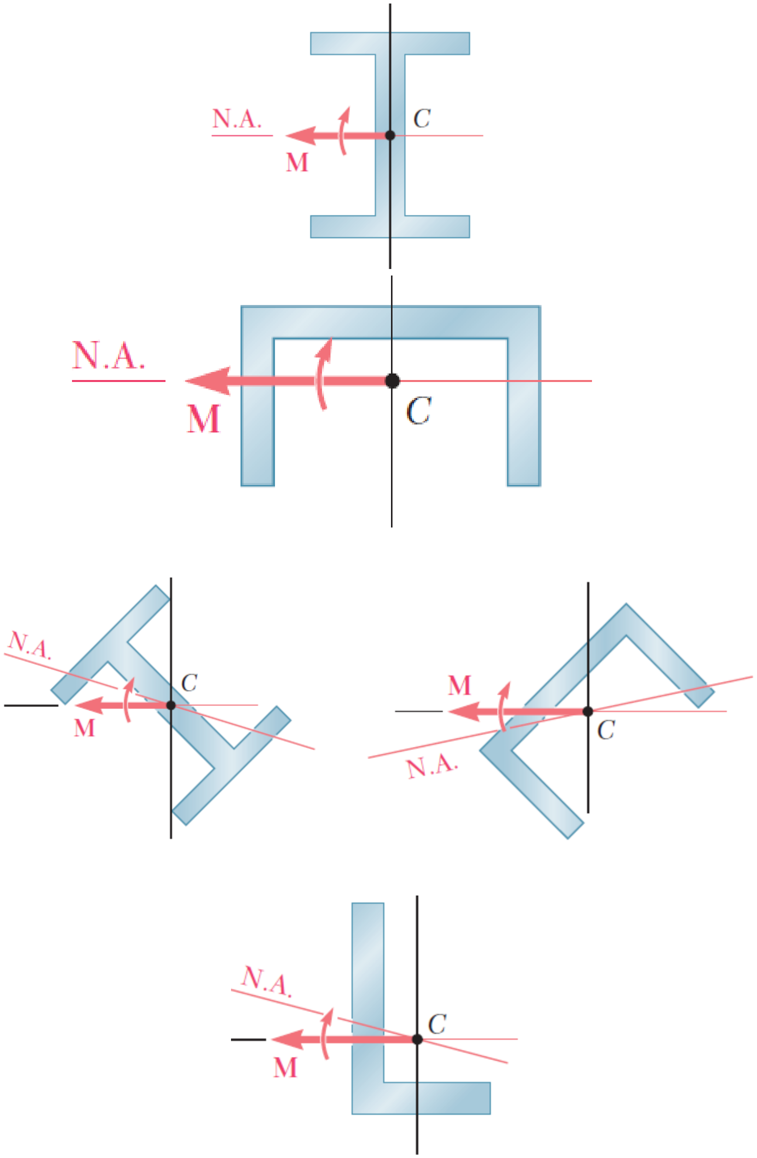
	Stresses Produced by Each Load Individually	Stress Distributions	Stresses
Torsional Load (Torque *T*)			Torsional shear stress $\tau_{x\theta} = T\rho / I_p$
Axial Load (Force *F*)			Tensile average normal stress $\sigma_{avg} = F/A$
Bending Load (Transverse Force *P*)			Bending normal stress $\sigma_x = M_z y / I_z$ Transverse shear stress $\tau_{xy} = F_S S_z^* / I_z b$

Introduction

- Prismatic bars are frequently subjected to several loads simultaneously
- The principle of superposition is used to determine the resultant stress & strain
- Conditions for the principle of superposition
  - Linear elasticity & small deformation
  - No interaction between variously loads

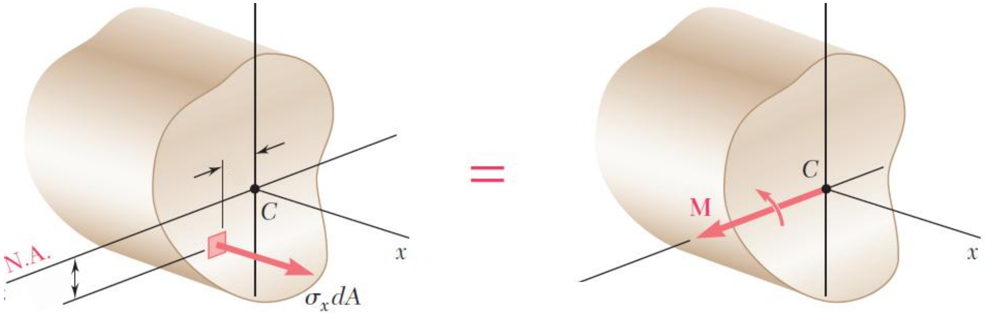


Unsymmetric Bending



- Analysis of pure bending has been limited to members subjected to bending moments acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.

Unsymmetric Bending



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the moment as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

$$F_x = 0 = M_y \quad M_z = M = \text{applied couple}$$

- If neutral axis passes through centroid

$$0 = F_x = \int \sigma_x dA = \int \frac{M_z y}{I_z} dA$$

$$\text{or} \quad 0 = \int y\, dA$$

- Stress distribution

$$M = M_z = \int \frac{M_z y}{I_z}(y)\, dA$$

- **Moment vector must be directed along a principal centroidal axis**

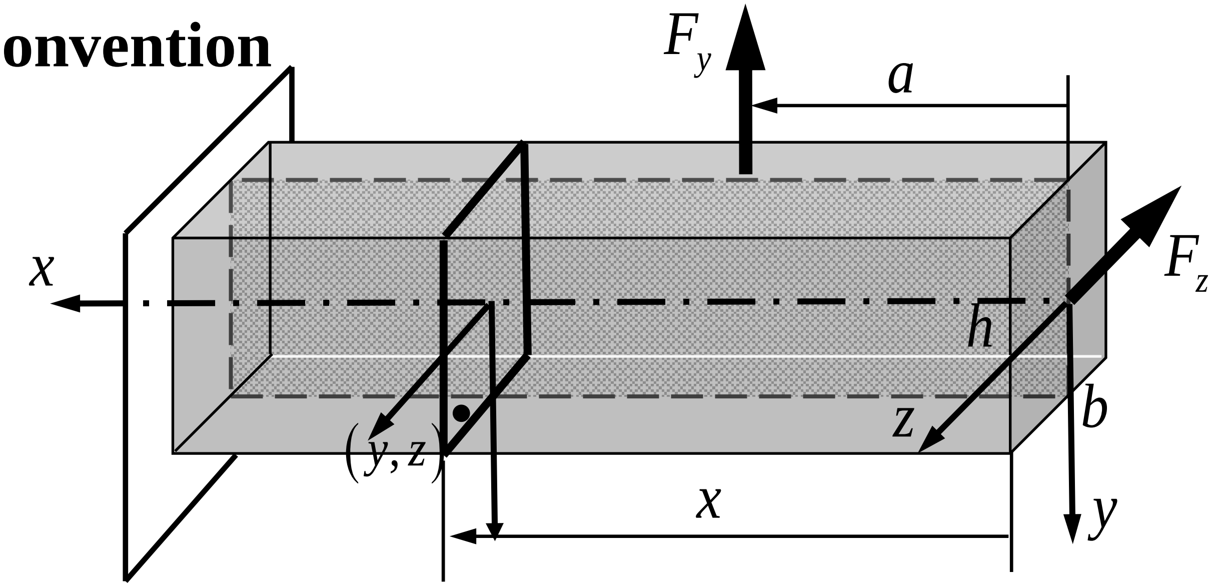
$$0 = M_y = \int \frac{M_z y}{I_z}(z)\, dA$$

$$\text{or} \quad 0 = \int yz\, dA = I_{yz} = \text{product of inertia}$$

- **Superposition is applied to determine stresses in the most general case of unsymmetric bending.**

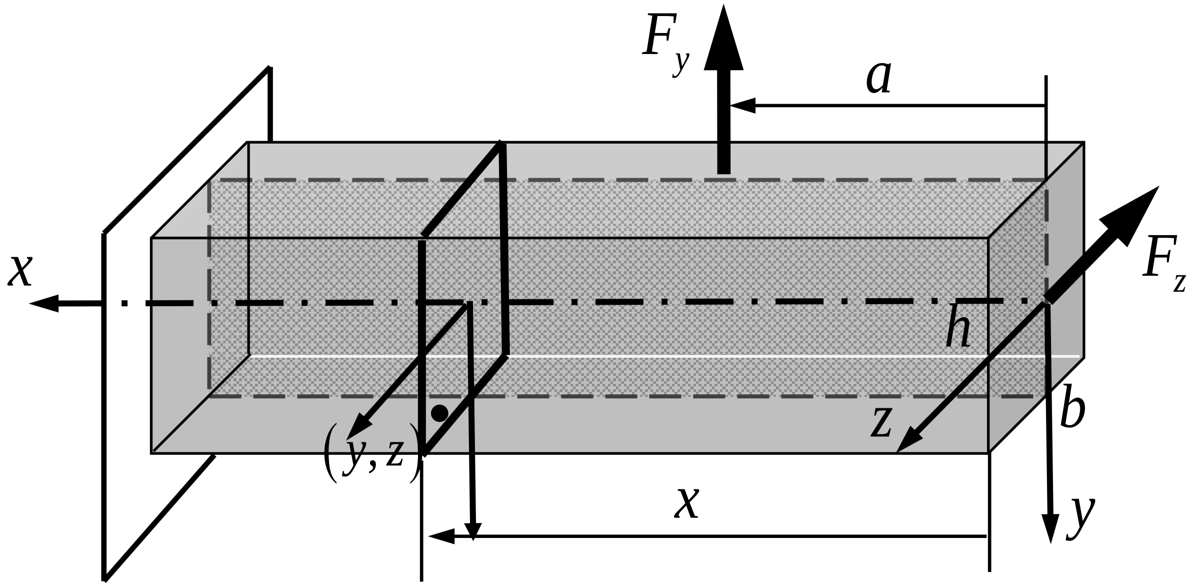
Unsymmetric Bending

- **Sign Convention**



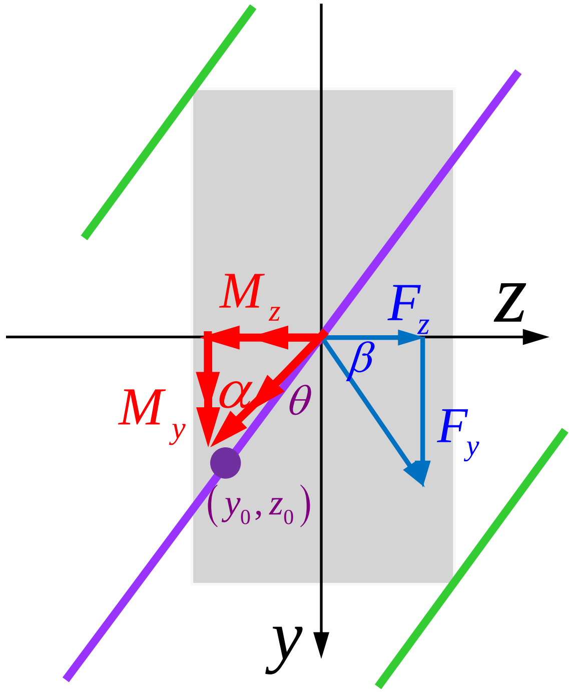
- **Construct a coordinate system.**
- **In x-y plane: positive M_z results in compression for $y < 0$.**
- **Bending stress and deflection:** $\sigma_x = M_z y / I_z, \quad w_y'' = -M_z / EI_z.$
- **The same sign conventions can be used for bending in x-z plane.**
- **Positive M_y results in compression for $z < 0$.**
- **Bending stress and deflection:** $\sigma_x = M_y z / I_y, \quad w_z'' = -M_y / EI_y.$

Unsymmetric Bending



$$\left.\begin{aligned} M_z &= F_y(x-a), \quad \sigma_x' = \frac{M_z y}{I_z}, \quad a \le x \le L \\ M_y &= F_z x, \qquad\qquad \sigma_x'' = \frac{M_y z}{I_y}, \quad 0 \le x \le L \end{aligned}\right\} \Rightarrow \sigma_x = \sigma_x' + \sigma_x'' = \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Unsymmetric Bending



- Equation for Neutral axis:

$$0 = \sigma_x = \frac{M_z y_0}{I_z} + \frac{M_y z_0}{I_y}$$

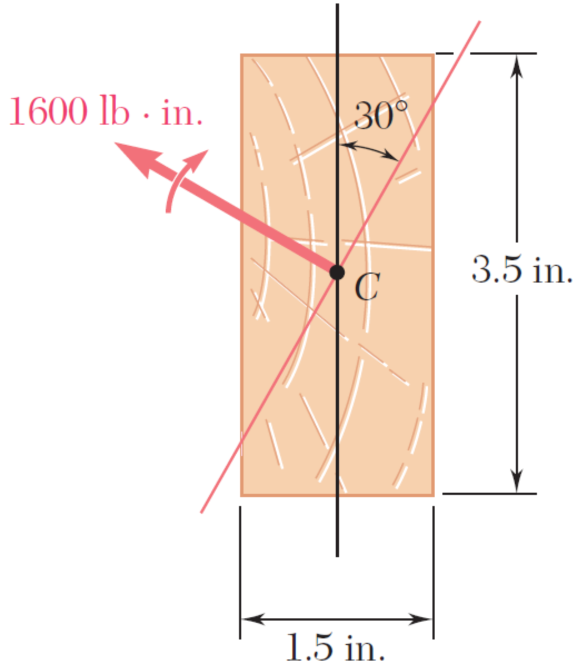
$$\Rightarrow \tan\theta = \frac{-z_0}{y_0} = \frac{M_z}{M_y}\frac{I_y}{I_z} = \frac{F_y}{F_z}\frac{(x-a)}{x}\frac{I_y}{I_z}$$

$$= \tan\alpha \frac{I_y}{I_z} = \tan\beta \frac{(x-a)}{x}\frac{I_y}{I_z}$$

$$\boxed{I_y \neq I_z \quad \Rightarrow \quad \theta \neq \alpha} \qquad \boxed{a \neq 0 \quad \Rightarrow \quad \alpha \neq \beta}$$

- The neutral axis passes through the centroid of the cross-section.
- The maximum stresses occur at the two farthest points from the neutral axis.
- With the exception of $I_y = I_z$, i.e. for circular/square cross-section, the bending stress and deflection cannot be calculated from the resultant moment.
- Bending in a single plane occurs if and only if the orientation of neural axis stays the same for every cross-section, i.e. M_z/M_y=constant for a prismatic beam.
- In general, superposition should be resorted to determine both the bending stress and the deflection.

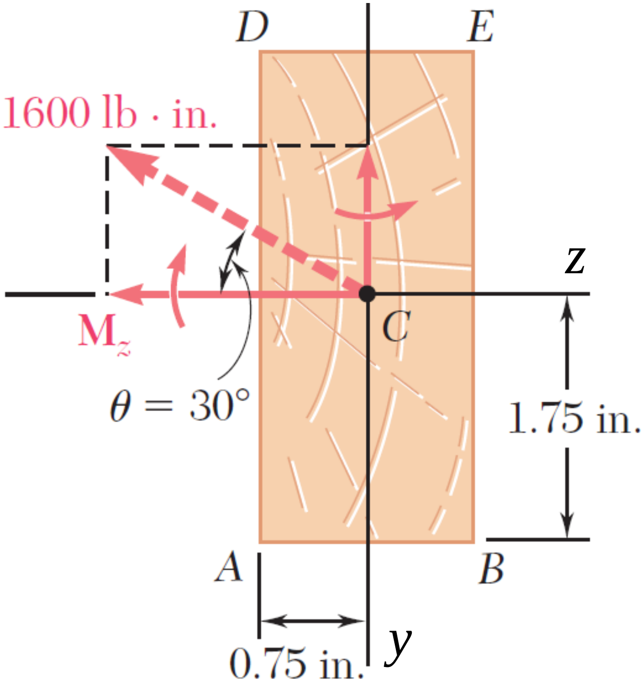
Sample Problem



A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.
- Combine the stresses from the component stress distributions.
- Determine the angle of the neutral axis.



- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600\,\text{lb}\cdot\text{in})\cos 30 = 1386\,\text{lb}\cdot\text{in}$$

$$M_y = -(1600\,\text{lb}\cdot\text{in})\sin 30 = -800\,\text{lb}\cdot\text{in}$$

$$I_z = \tfrac{1}{12}(1.5\,\text{in})(3.5\,\text{in})^3 = 5.359\,\text{in}^4$$

$$I_y = \tfrac{1}{12}(3.5\,\text{in})(1.5\,\text{in})^3 = 0.9844\,\text{in}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386\,\text{lb}\cdot\text{in})(1.75\,\text{in})}{5.359\,\text{in}^4} = 452.6\,\text{psi}$$

The largest tensile stress due to M_z occurs along AD

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(-800\,\text{lb}\cdot\text{in})(-0.75\,\text{in})}{0.9844\,\text{in}^4} = 609.5\,\text{psi}$$

- The largest tensile stress due to the combined loading occurs at *A*.

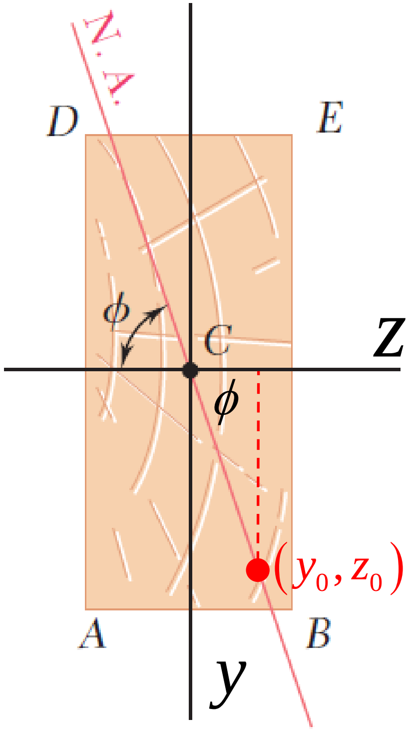
$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5 = 1062\,\text{psi}$$

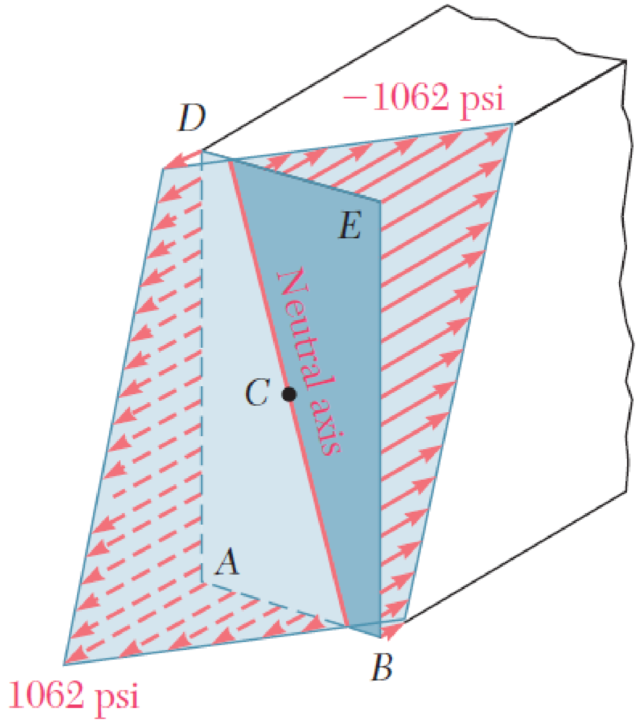
- Determine the angle of the neutral axis.

$$0=\sigma_x=\frac{M_z y_0}{I_z}+\frac{M_y z_0}{I_y}=\frac{(1386\,\text{lb}\cdot\text{in})\,y_0}{5.359\,\text{in}^4}+\frac{(-800\,\text{lb}\cdot\text{in})\,z_0}{0.9844}=258.63y_0-812.68z_0$$

$$\tan\phi=\frac{y_0}{z_0}=\frac{812.68}{258.63}=3.142$$

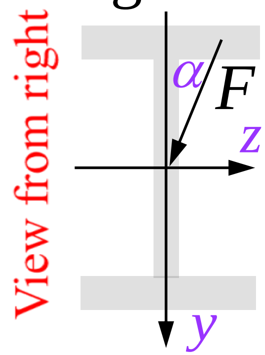
$$\phi=72.4^o$$

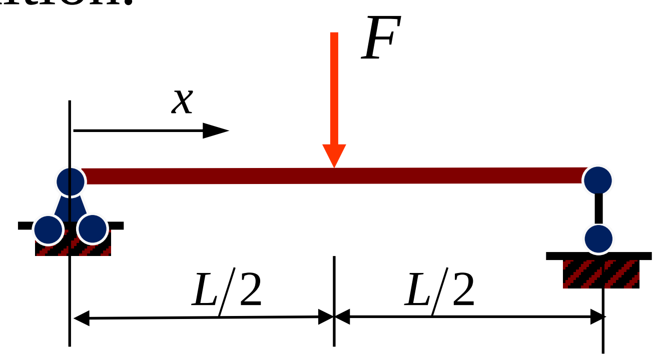




Sample Problem

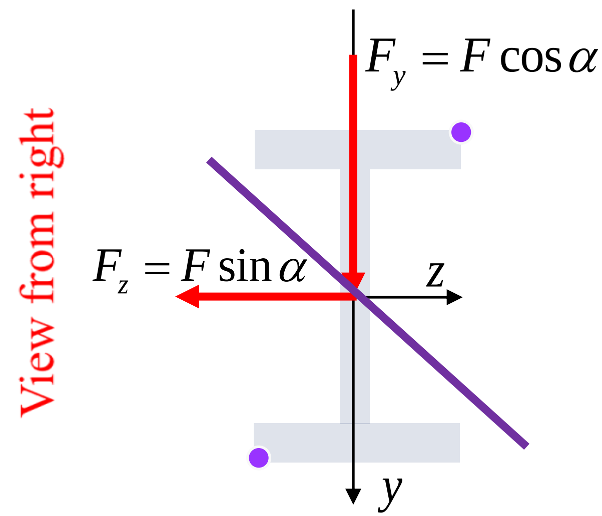
- For the I-32a beam shown, L = 4 m, $[\sigma]$ = 160 MPa, F = 80 kN, α = 5°. Analyze the strength condition.





- Solution

1. Decompose the load



2. Critical cross-section

$$x = \frac{L}{2}; \quad M_z = \frac{F_y L}{4}; \quad M_y = -\frac{F_z L}{4}$$

3. Neutral axis & maximum stresses

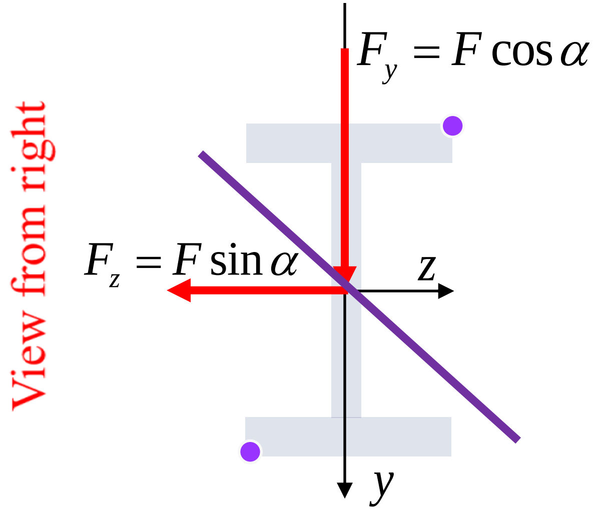
$$\sigma_x = \frac{M_y z}{I_y} + \frac{M_z y}{I_z} = -\frac{F_z L}{4I_y} z + \frac{F_y L}{4I_z} y$$

4. Strength check:

$$\sigma_{\max} = \frac{M_y}{W_y} + \frac{M_z}{W_z} = 217.8\text{MPa} > [\sigma] = 160\text{MPa}$$

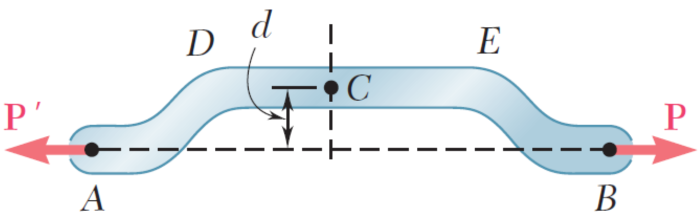
5. Deflections: $w = \sqrt{w_y^2 + w_z^2}$

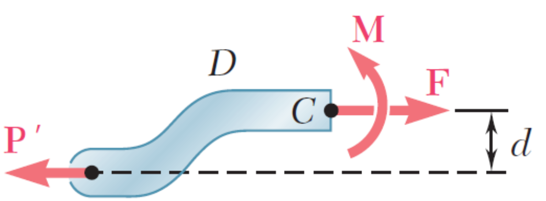
6. If α = 0, $\sigma_{\max} = \frac{M_z}{W_z} = 115.6MPa < [\sigma]$



Remark: The fact that the neutral axis is not necessarily perpendicular to the resultant moment can greatly affect the stresses in a beam, especially if the ratio of the principal moments of inertia is very large. Under these conditions the stresses in the beam are very sensitive to slight changes in the direction of the load and to irregularities in the alignment of the beam itself.

Tension & Bending (Eccentric Tension)





$$F = P, \quad M = Pd$$

- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

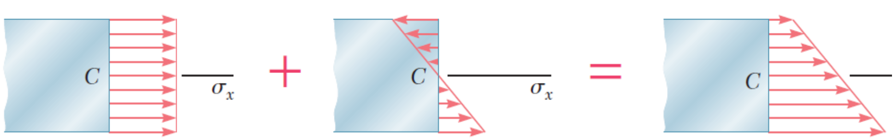
$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

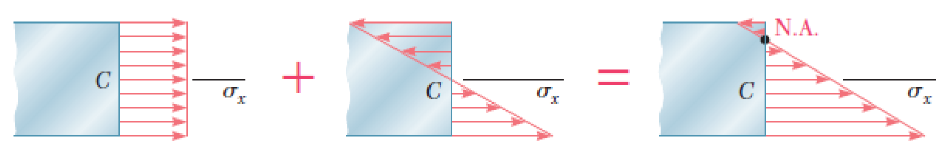
$$= \frac{P}{A} + \frac{M_z y}{I_z}$$

  - Strength condition:

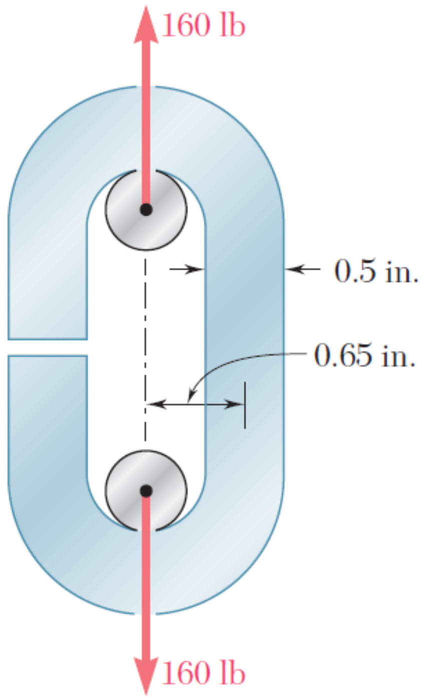
$$\sigma^+_{\max} \le [\sigma^+], \quad |\sigma^-_{\max}| \le [\sigma^-].$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.





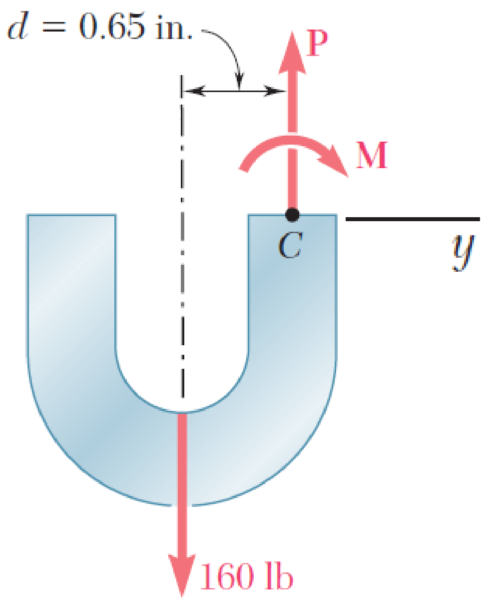
Sample Problem

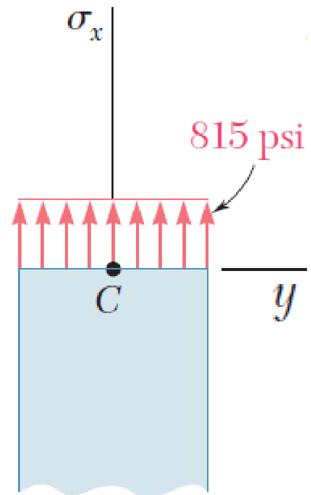


An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

SOLUTION:

- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.



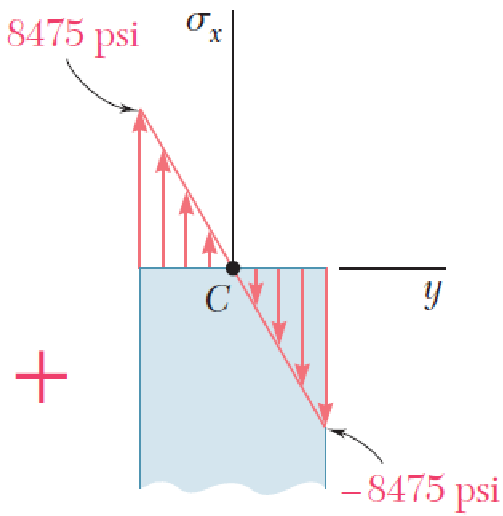


- Normal stress due to a centric load

$$A = \pi c^2 = \pi\left(0.25\,\text{in}\right)^2$$
$$= 0.1963\,\text{in}^2$$
$$\sigma_x = \frac{P}{A} = \frac{160\,\text{lb}}{0.1963\,\text{in}^2}$$
$$= 815\,\text{psi}$$

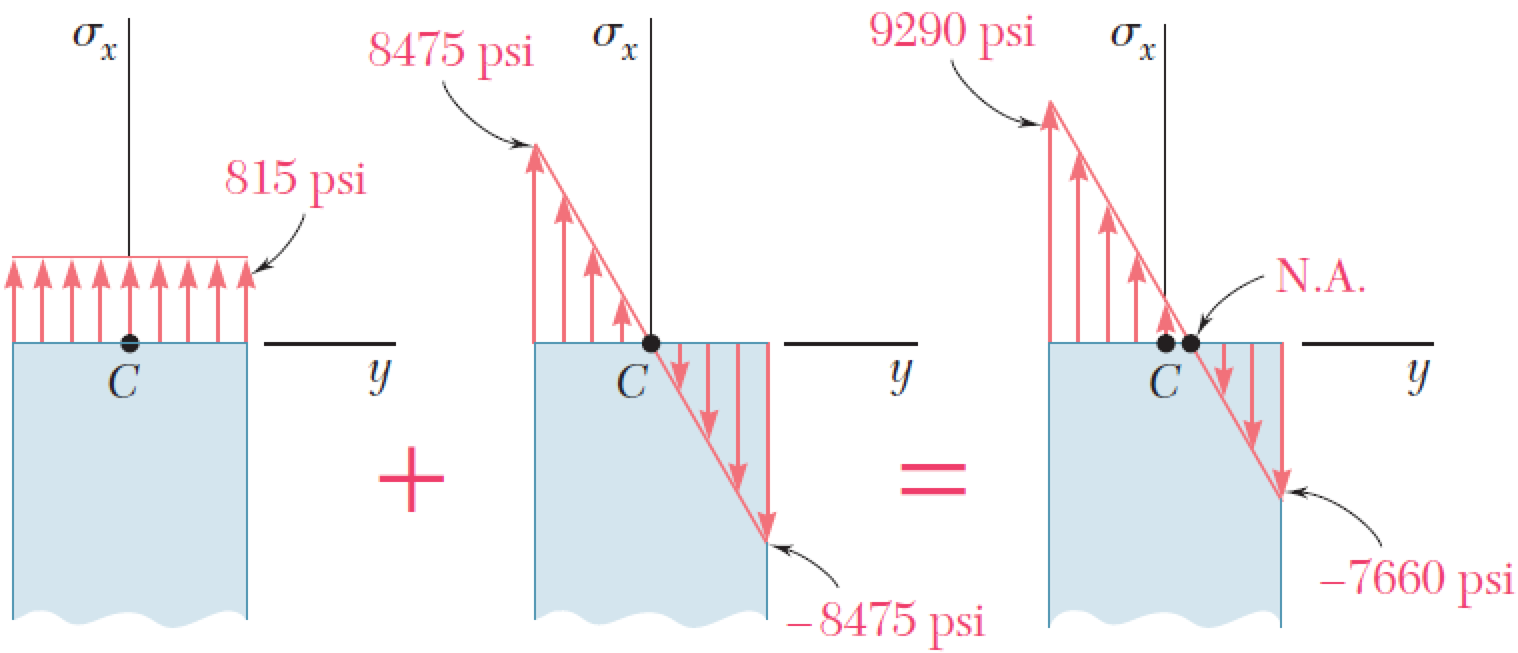
- Equivalent centric load and bending moment

$$P = 160\,\text{lb}$$
$$M_z = -Pd$$
$$= -\left(160\,\text{lb}\right)\left(0.65\,\text{in}\right)$$
$$= -104\,\text{lb}\cdot\text{in}$$



- Normal stress due to bending moment

$$I_z = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi\left(0.25\right)^4$$
$$= 3.068\times10^{-3}\,\text{in}^4$$
$$\sigma_x = \frac{M_z c}{I_z} = \frac{\left(104\,\text{lb}\cdot\text{in}\right)\left(0.25\,\text{in}\right)}{3.068\times10^{-3}\,\text{in}^4}$$
$$= 8475\,\text{psi}$$



- Maximum tensile and compressive stresses

$$\sigma_{\max}^{+} = 815 + 8475 = 9260\,\text{psi}$$

$$\sigma_{\max}^{-} = 815 - 8475 = -7660\,\text{psi}$$

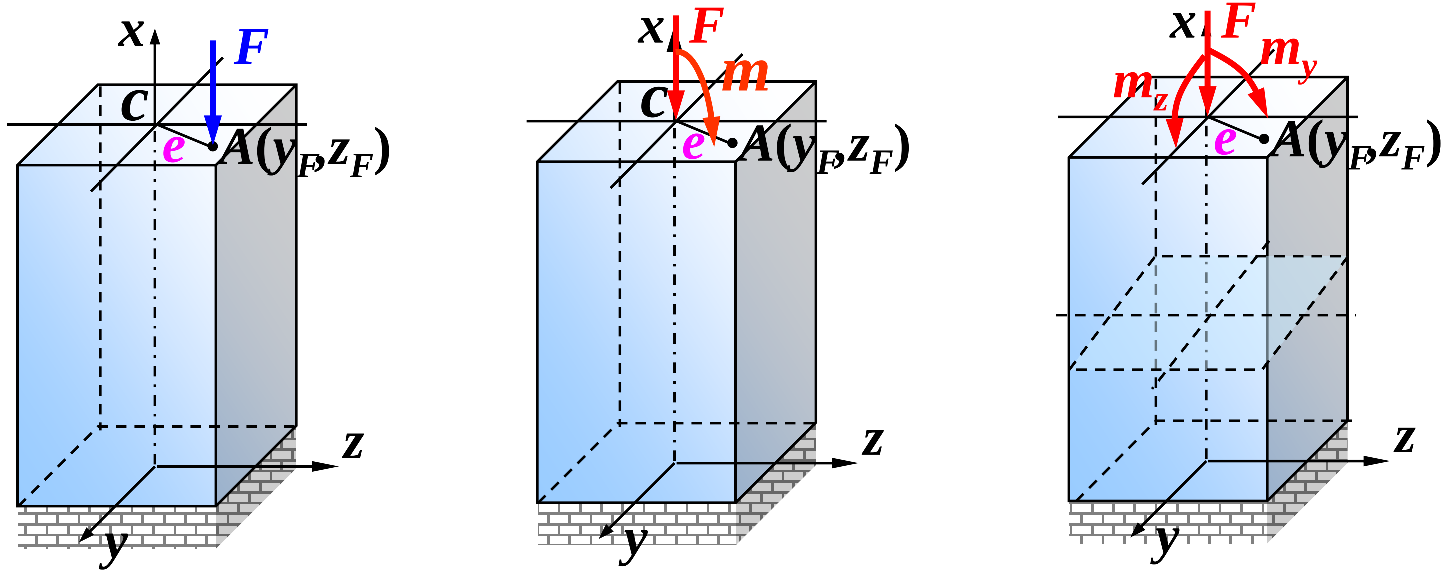
- Neutral axis

$$0 = \frac{P}{A} + \frac{M_z y_0}{I_z}$$

$$y_0 = -\frac{P}{A}\frac{I_z}{M_z} = -\frac{160\,\text{lb}}{0.1963\,\text{in}^2}\frac{3.068\times10^{-3}\,\text{in}^4}{(-104\,\text{lb}\cdot\text{in})}$$

$$= 0.0240\,\text{in}$$

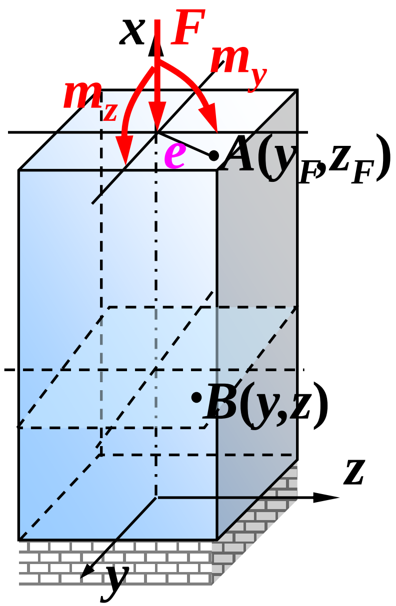
Eccentric Compression



- Bars subjected to axial forces deviated a distance from axis.

1. Find the equivalent system of forces at the centroid.

2. Normal stress on cross-sections.



$$F_{\mathrm{N}} = -F, \qquad \sigma'_x = \frac{F_{\mathrm{N}}}{A} = -\frac{F}{A}$$

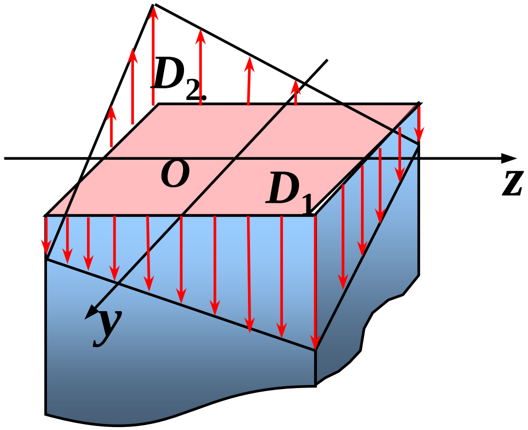
$$M_z = -F \cdot y_F, \qquad \sigma''_x = \frac{M_z \cdot y}{I_z} = -\frac{F \cdot y_F \cdot y}{I_z}$$

$$M_y = -F \cdot z_F, \qquad \sigma'''_x = \frac{M_y \cdot z}{I_y} = -\frac{F \cdot z_F \cdot z}{I_y}$$

$$\Rightarrow \sigma = \sigma' + \sigma'' + \sigma'''$$

$$= -\left(\frac{F}{A} + \frac{F \cdot y_F \cdot y}{I_z} + \frac{F \cdot z_F \cdot z}{I_y} \right)$$

$$I_y = Ai_y^2, \; I_z = Ai_z^2 \quad \Rightarrow \sigma = -\frac{F}{A}\left(1 + \frac{y_F y}{i_z^2} + \frac{z_F z}{i_y^2}\right)$$

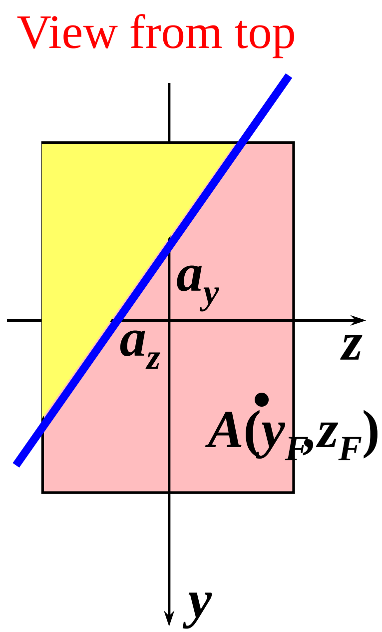


- Note the negative sign in bending stresses.

3. Equation of neutral axis.

$$0=\sigma=-\frac{F}{A}\left(1+\frac{y_F y}{i_z^2}+\frac{z_F z}{i_y^2}\right)$$

$$\Rightarrow 1+\frac{y_F y_0}{i_z^2}+\frac{z_F z_0}{i_y^2}=0$$

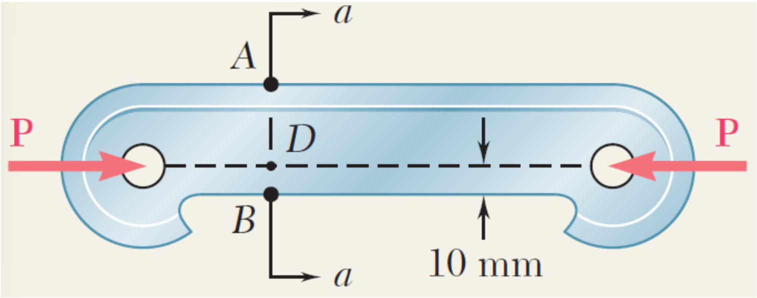


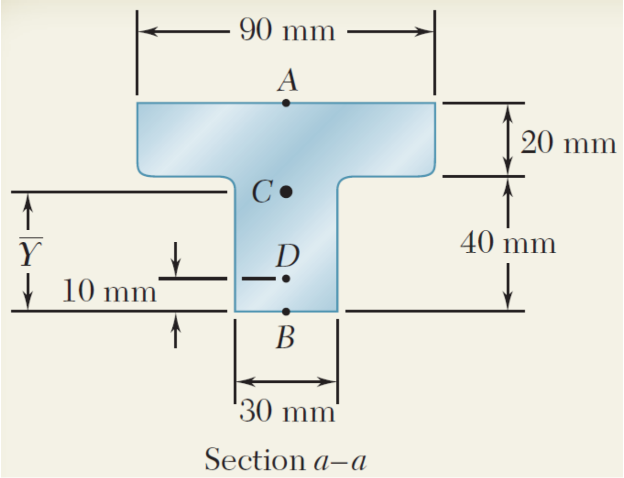
- The intercepts of neutral axis:

$$a_y=y_0\big|_{z_0=0}=-\frac{i_z^2}{y_F} \quad ; \qquad \boldsymbol{a_z=z_0\big|_{y_0=0}=-\frac{i_y^2}{z_F}}$$

4. Strength condition: $\sigma_{\max}^{+}\le\left[\sigma^{+}\right],\quad \left|\sigma_{\max}^{-}\right|\le\left[\sigma^{-}\right].$

Sample Problem



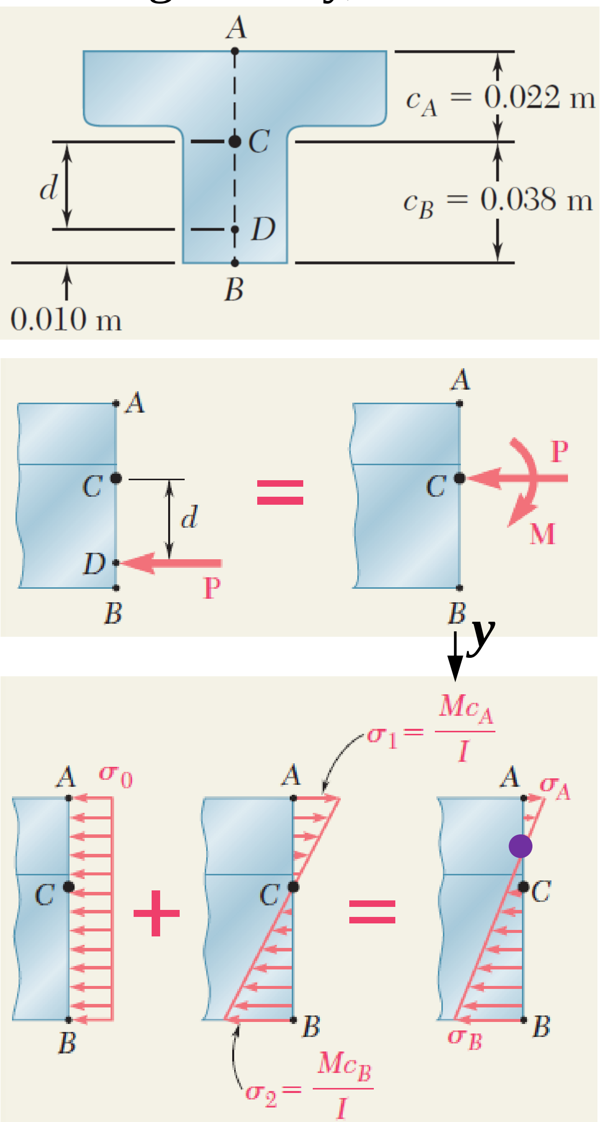


The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force *P* which can be applied to the link and the neutral axis.

SOLUTION:

- Determine an equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

From geometry,



- Determine an equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028\,\text{m}$$

$$M_z = -Pd = -0.028\,P = \text{ bending moment}$$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{M_z c_A}{I_z} = -\frac{P}{3\times10^{-3}} + \frac{(-0.028\,P)(-0.022)}{868\times10^{-9}} = +377\,P$$

$$\sigma_B = -\frac{P}{A} + \frac{M_z c_B}{I_z} = -\frac{P}{3\times10^{-3}} + \frac{(-0.028\,P)(0.038)}{868\times10^{-9}} = -1559\,P$$

- Evaluate critical loads for allowable stresses.

$$\sigma_A = +377\,P = 30\,\text{MPa} \qquad P = 79.6\,\text{kN}$$

$$\sigma_B = -1559\,P = -120\,\text{MPa} \qquad P = 77.0\,\text{kN}$$

- The largest allowable load $\boxed{P = 77.0\ \text{kN}}$

- Neutral axis

$$a_y = y_0\big|_{z_0=0} = -\frac{i_z^2}{y_F} = -\frac{868\times10^{-9}/3\times10^{-3}}{0.028} = -10.33\ \text{mm}$$

$$a_z = z_0\big|_{y_0=0} = -\frac{i_y^2}{z_F} \to \infty$$

Sample Problem

- Find the maximum normal stress for the bar shown.
- Solution:

1. Equivalent System of forces

$$F_N = F = 10\ \text{kN}$$

$$M_z = 10 \times 10^3 \times 5 \times 10^{-2} = 500\ \text{Nm}$$

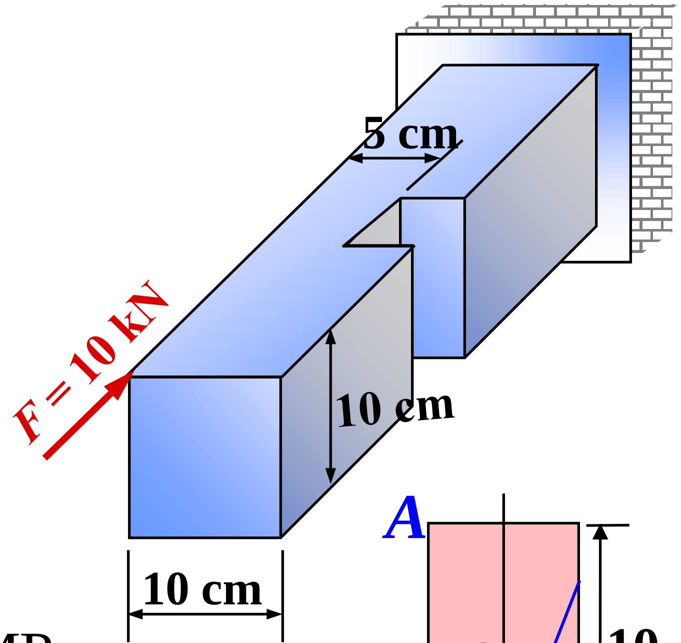
$$M_y = 10 \times 10^3 \times 2.5 \times 10^{-2} = 250\ \text{Nm}$$

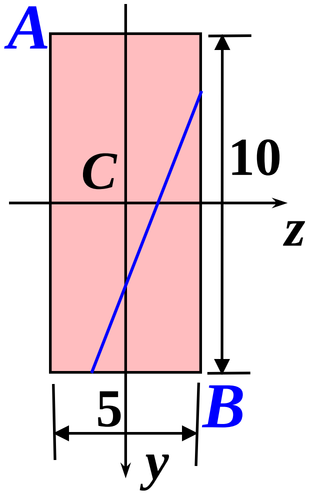
2. Maximum normal stress

$$\sigma_{\max}^{-} = \sigma_A = -\frac{F_N}{A} + \frac{M_z y_A}{I_z} + \frac{M_y z_A}{I_y} = -14\ \text{MPa}$$

$$\boldsymbol{\sigma_{\max}^{+} = \sigma_B = -\frac{F_N}{A} + \frac{M_z y_B}{I_z} + \frac{M_y z_B}{I_y} = 10\ \text{MPa}}$$

Intercepts: $a_y = -\frac{i_z^2}{y_F}$; $a_z = -\frac{i_y^2}{z_F}$





View from left

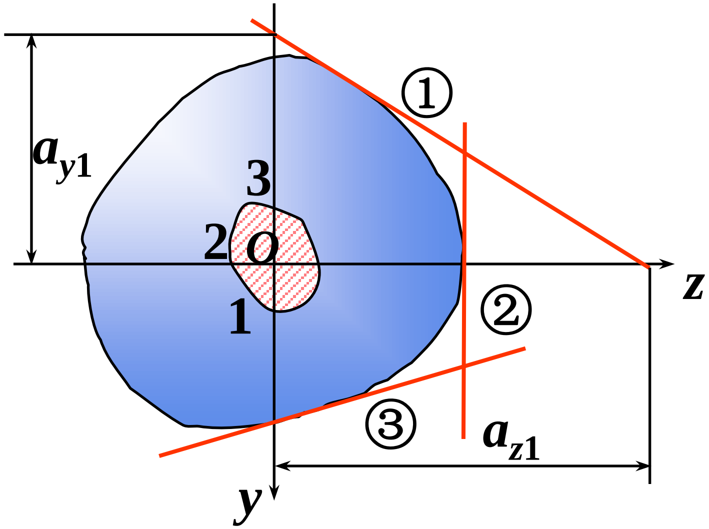
Core of Cross-sections

- Portion of a cross-section within which eccentric loading results in only compressive stresses.

- Intercepts of neutral axis:

$$a_y = -\frac{i_z^2}{y_F};\ \ a_z = -\frac{i_y^2}{z_F}$$

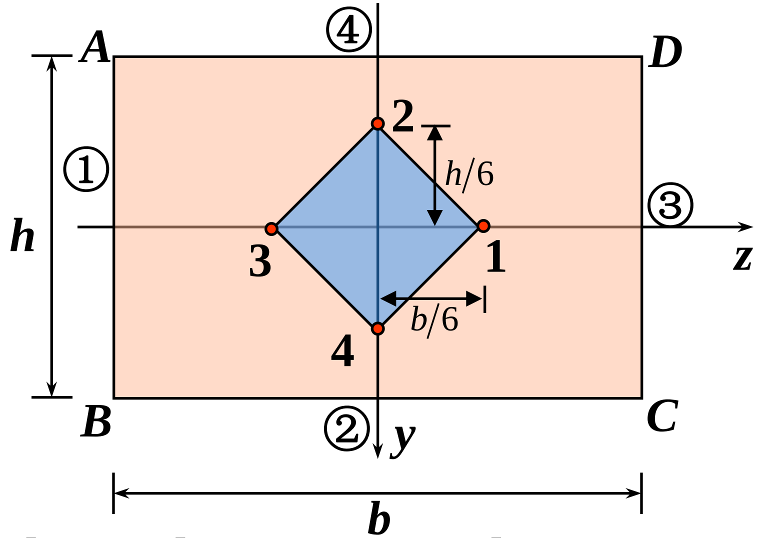
$$\Rightarrow y_F = -\frac{i_z^{\ 2}}{a_y};\ \ z_F = -\frac{i_y^{\ 2}}{a_z}$$



Core of Rectangular Cross-sections

$$y_F = -\frac{i_z^{\,2}}{a_y}, \qquad \boldsymbol{z_F = -\frac{i_y^{\,2}}{a_z}}$$

$$\boldsymbol{i_z^2} = \frac{\boldsymbol{I_z}}{\boldsymbol{A}} = \frac{\boldsymbol{h}^2}{12}, \qquad i_y^2 = \frac{I_y}{A} = \frac{b^2}{12}$$



- Take side AB as neutral axis

$$\boldsymbol{a_{y_1} = \infty, \quad a_{z_1} = -\frac{b}{2}}$$

$$\Rightarrow y_{F_1} = -\frac{i_z^2}{a_{y_1}} = 0, \;\; z_{F_1} = -\frac{i_y^2}{a_{z_1}} = \frac{b}{6}$$

- Take side BC as neutral axis

$$\boldsymbol{y_{F_2} = -\frac{h}{6}, \quad z_{F_2} = 0}$$

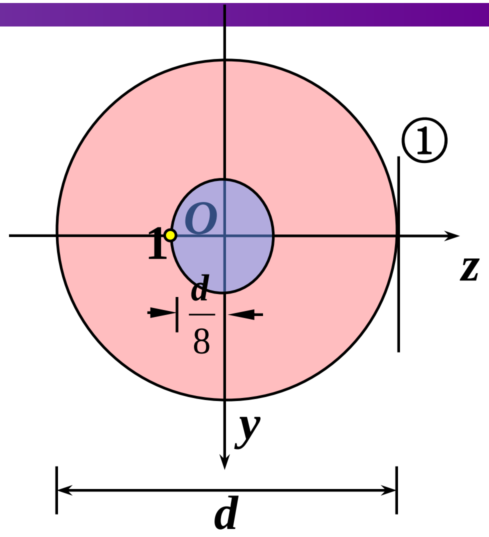
- Through corner B, there exists infinite number of neutral axes we can take.

$$1 + \frac{y_F y_B}{i_z^2} + \frac{z_F z_B}{i_y^2} = 0$$

$$\Rightarrow 1 + \frac{y_F}{h^2/12} h/2 + \frac{z_F}{b^2/12}\left(-b/2\right) = 0$$

$$\Rightarrow 1 + \frac{6y_F}{h} - \frac{6z_F}{b} = 0$$

Core of Circular Cross-sections



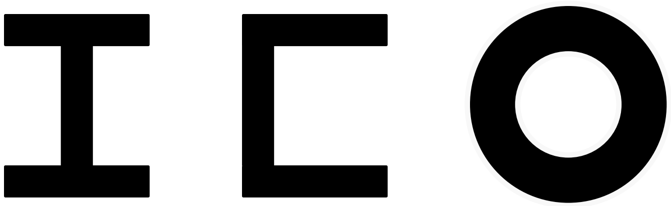
- Due to axial symmetry, only one neutral axis is necessary to determine the radius of the core area:

$$①: a_y = \infty,\ a_z = d/2$$

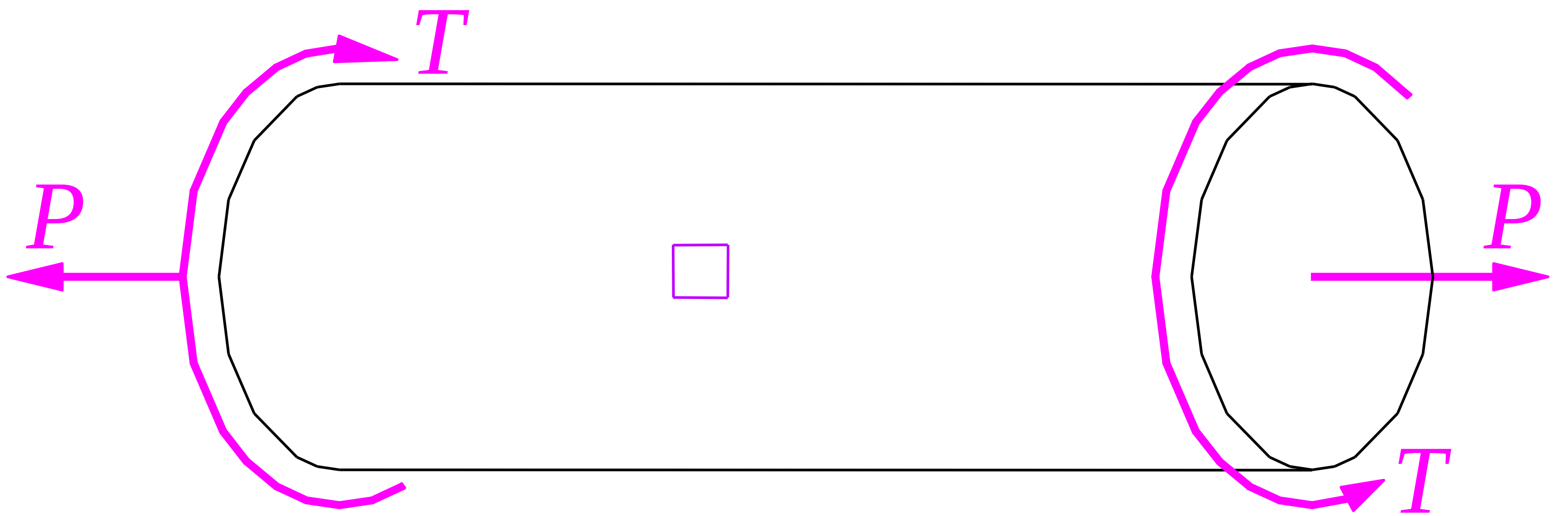
$$i_y = i_z = d/4$$

$$\Rightarrow y_{F1} = -\frac{i_z^2}{a_y} = 0,\ z_{F1} = -\frac{i_y^2}{a_z} = -\frac{\left(d/4\right)^2}{d/2} = -d/8$$

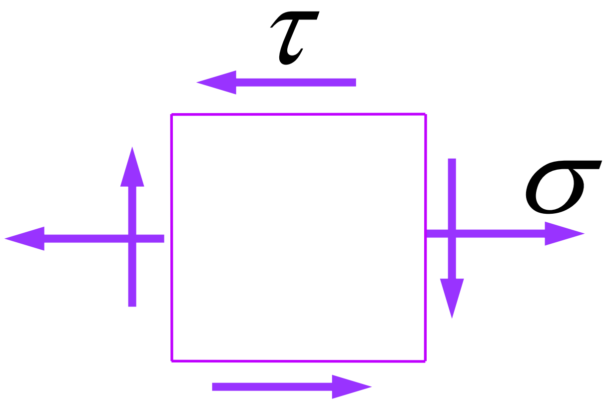
- Note: no matter whatever shape of a cross-section area, the corresponding core area is always solid and non-concave.



Tension & Torsion



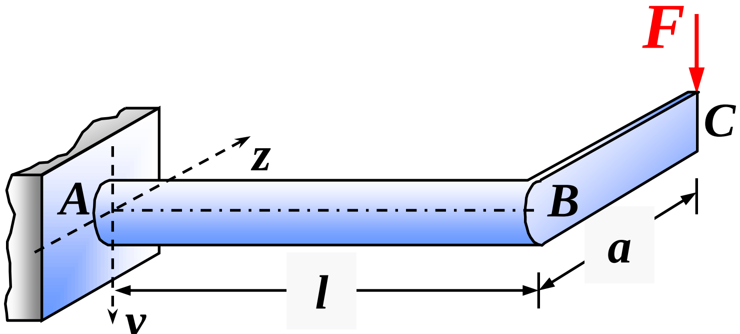
- Solution



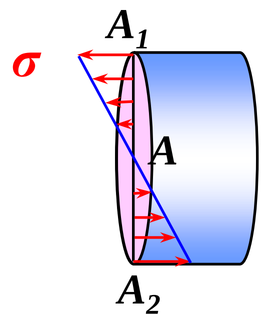
$$\sigma_x = \frac{F_N}{A} = \frac{4P}{\pi d^2}$$

$$\tau_{x\theta} = \frac{T}{W_P} = \frac{T}{\pi d^3/16}$$

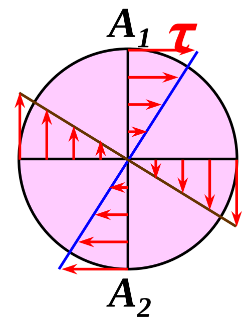
Bending & Torsion



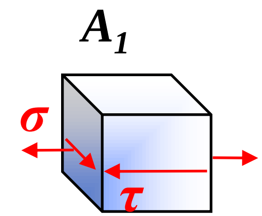
- Critical section: A



View from left

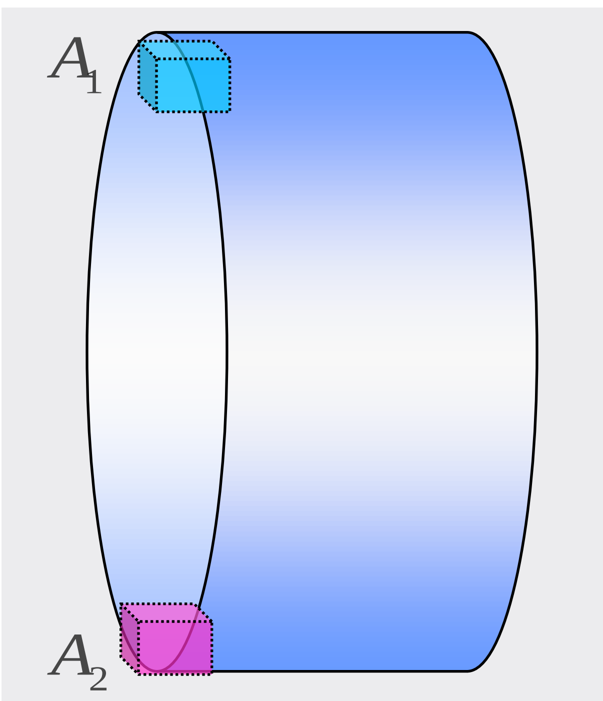


View from right



- Critical points on A: A_1 & A_2

$$\boxed{\sigma = \frac{M_z}{W_z}, \quad \tau = \frac{T}{W_P}}$$



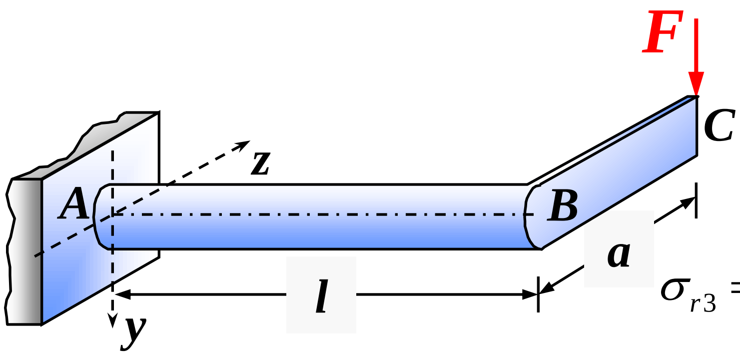
$$\sigma_{r3} \le [\sigma] \qquad \Rightarrow \qquad \sqrt{\sigma^2 + 4\tau^2} \le [\sigma]$$

$$\sigma_{r4} \le [\sigma] \qquad \Rightarrow \qquad \sqrt{\sigma^2 + 3\tau^2} \le [\sigma]$$

$$\Rightarrow \begin{cases} \sigma_{r3} = \sqrt{(\dfrac{M_z}{W_z})^2 + 4(\dfrac{T}{W_P})^2} = \dfrac{1}{W_z}\sqrt{M_z^2 + T^2} \le [\sigma] \\ \sigma_{r4} = \sqrt{(\dfrac{M_z}{W_z})^2 + 3(\dfrac{T}{W_P})^2} = \dfrac{1}{W_z}\sqrt{M_z^2 + 0.75T^2} \le [\sigma] \end{cases}$$

Sample Problem

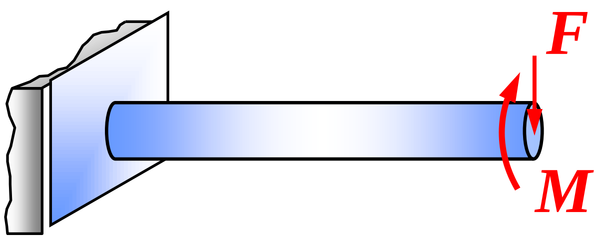
- For the circular shaft shown, L = 50 cm, F = 8 kN, a = 37.5 cm, $[\sigma]$ = 100 MPa. Find the minimum diameter of shaft AB based on maximum shearing stress criteria.



- Solution:

$$T = 3\,\text{kNm}, \quad M_{\max} = FL = 4\,\text{kNm}$$

$$\sigma_{r3} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 4\left(\frac{T}{W_P}\right)^2} = \frac{1}{W_z}\sqrt{M_z^2 + T^2} \le [\sigma]$$



$$\Rightarrow W_z \ge \frac{\sqrt{M_z^2 + T^2}}{[\sigma]} = 5\times 10^{-5} m^3$$

$$\Rightarrow d \ge \sqrt[3]{\frac{32W_z}{\pi}} = 79.8mm$$

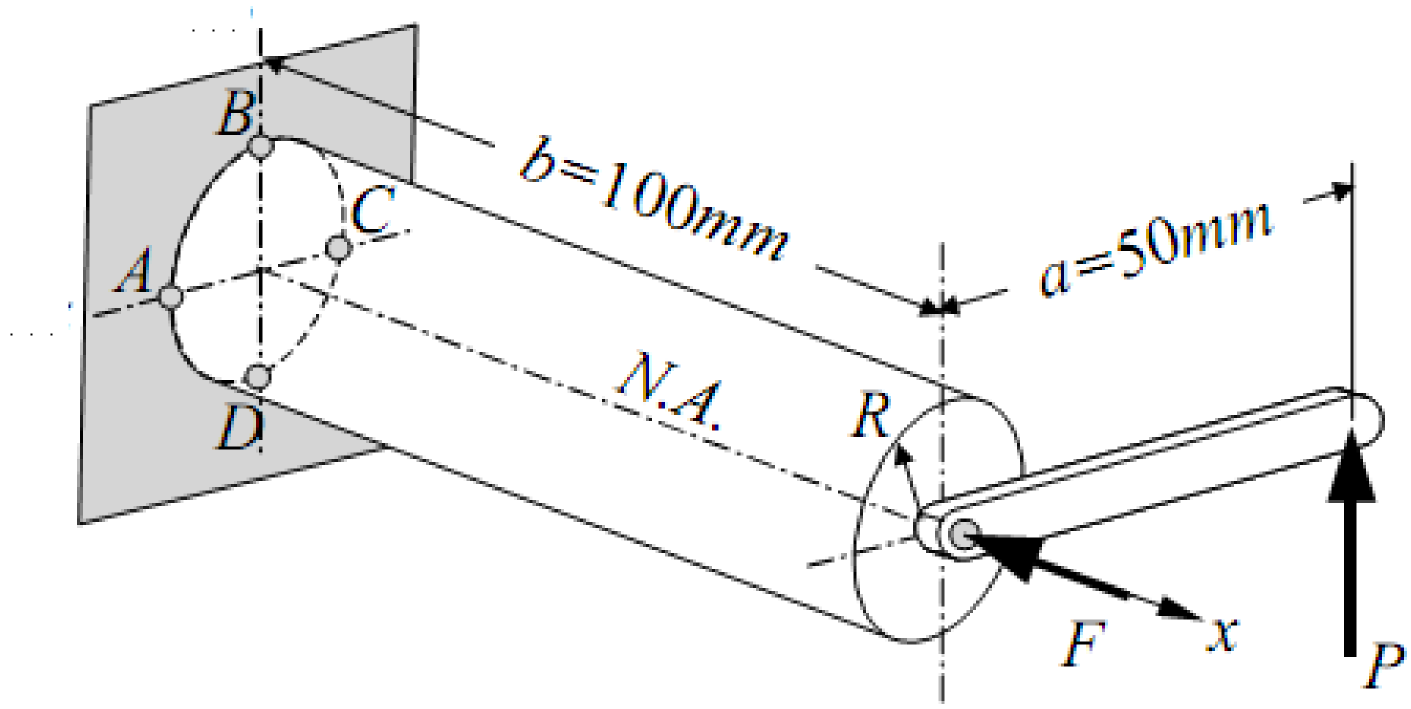
What about the maximum distortion energy criteria?

$$\sigma_{r4} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 3\left(\frac{T}{W_P}\right)^2} = \frac{1}{W_z}\sqrt{M_z^2 + 0.75T^2} \le [\sigma]$$

$$\Rightarrow W_z \ge \frac{\sqrt{M_z^2 + 0.75T^2}}{[\sigma]}$$

Tension, Bending & Torsion – an Exercise

- Two forces P = 18 kN and F = 15 kN are applied to the shaft with a radius of R = 20 mm as shown. Determine the maximum normal and shearing stresses developed in the shaft.



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