Constitutive Relations

Outline

- Constitutive Laws
- Strain Energy
- Linear Constitutive Relations
- Anisotropy and Stiffness Tensor
- Isotropic Hooke’s Law
- Physical Meaning of Elastic Moduli
- Simple Engineering Tests
- Relationship among Elastic Constants
- Hooke’s Law in Curvilinear Coordinates
- Thermoelastic Constitutive Relations
- Typical Values of Elastic Constants
- Inhomogeneity

Constitutive Laws

- Relations that characterize the mechanical properties of materials
- Perhaps one of the most challenging fields in mechanics, due to the endless variety of materials and loadings

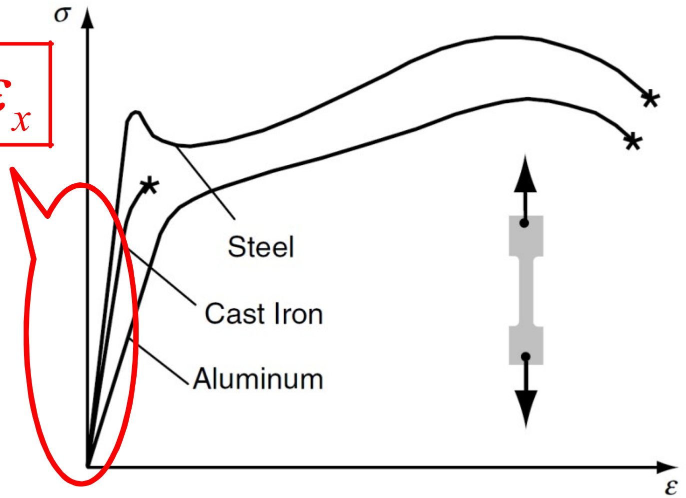
- The mechanical behavior of solids is normally defined by constitutive stress-strain relations
- Generally

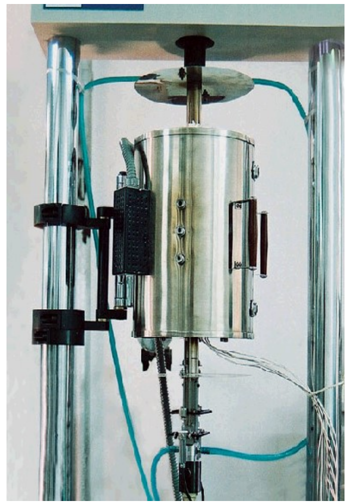
$$\boldsymbol{\sigma} = \boldsymbol{f}\left(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, t, T, \cdots\right)$$

Perfect (Linear) Elastic Solids

- Neglect strain rate, time and loading history dependency
- Set aside thermal, electric, pore-pressure, and other loads
- Include only mechanical loads
- Assume linear stress-strain relationship
- Defined as materials that recover original configuration when mechanical loads are removed
- Agree well with experimental tests

$$\sigma_x = E\varepsilon_x$$





Strain Energy (Stress-Deformation Work)

- In physical terms, stress represents the variation of strain energy with respect to strain change

$$U\left(\varepsilon_{ij}\right) = U\left(0\right) + \frac{\partial U\left(0\right)}{\partial \varepsilon_{kl}} \varepsilon_{kl} + \frac{\partial^2 U\left(0\right)}{\partial \varepsilon_{kl} \partial \varepsilon_{mn}} \varepsilon_{kl} \varepsilon_{mn} + O\left(\varepsilon^3\right)$$

$$U\left(\varepsilon_{ij}\right) = U\left(0\right) + b_{kl}\varepsilon_{kl} + c_{klmn}\varepsilon_{kl}\varepsilon_{mn} + O\left(\varepsilon^3\right)$$

$$\sigma_{ij} = \frac{\partial U\left(\varepsilon_{ij}\right)}{\partial \varepsilon_{ij}} = b_{ij} + \left(c_{ijkl} + c_{klij}\right)\varepsilon_{kl} + O\left(\varepsilon^2\right)$$

Stiffness of Perfect Elastic Solids

- Isothermal and perfect elasticity results in

$$\sigma_{ij} \approx \left(c_{ijkl} + c_{klij}\right)\varepsilon_{kl} \approx C_{ijkl}\varepsilon_{kl}$$

- Symmetry property

$$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl}$$

$$\varepsilon_{kl} = \varepsilon_{lk} \Rightarrow C_{ijkl} = C_{ijlk}$$

$$C_{ijkl} = c_{ijkl} + c_{klij} \Rightarrow C_{ijkl} = C_{klij}$$

- There remain 21 independent elastic components

Generalized Hooke's Law in Matrix Form

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

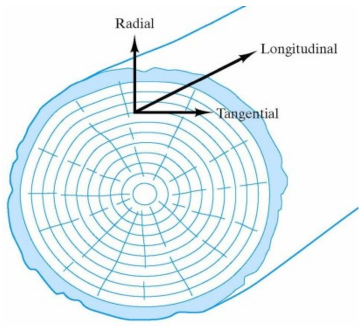
$$= C_{ij11}\varepsilon_{11} + C_{ij22}\varepsilon_{22} + C_{ij33}\varepsilon_{33} + 2C_{ij12}\varepsilon_{12} + 2C_{ij13}\varepsilon_{13} + 2C_{ij23}\varepsilon_{23}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ & & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ & & & C_{1212} & C_{1213} & C_{1223} \\ & \text{Symm.} & & & C_{1313} & C_{1323} \\ & & & & & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

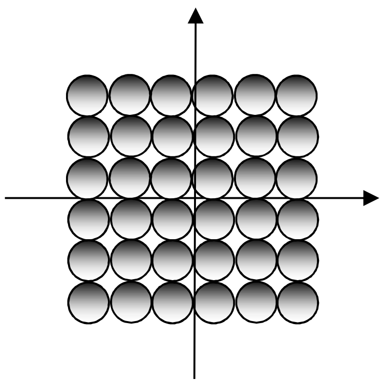
- 21 elastic constants

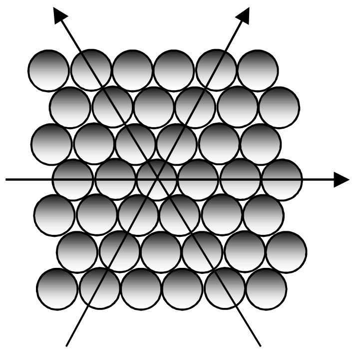
Anisotropy

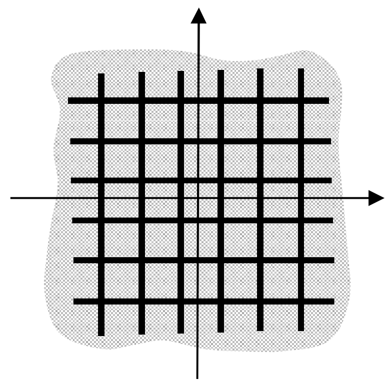
- Differences in material properties along different directions.
- Materials like wood, crystalline minerals, fiber-reinforced composites have such behavior.



Typical Wood Structure


Body-Centered Cubic Crystal

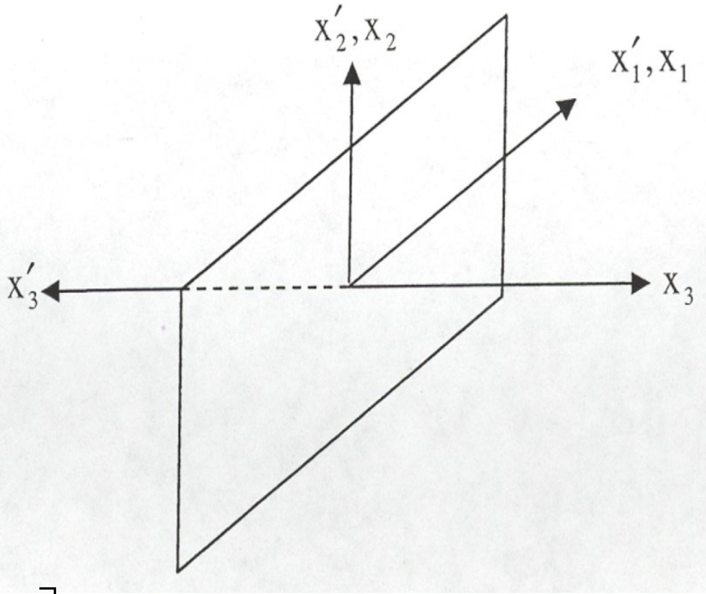

Hexagonal Crystal


Fiber Reinforced Composite

- Note particular material symmetries indicated by arrows.

Monotropic Materials

- Symmetric about one plane



Examine: $C'_{1113}, C'_{1123}, C'_{2213}, C'_{2223}, C'_{3313}, C'_{3323}, C'_{1213}, C'_{1223}$

$$\left.\begin{aligned} &C'_{1113} = Q_{1m}Q_{1n}Q_{1o}Q_{3p}C_{mnop} \qquad (m=1; n=1; o=1; p=3) \\ &\Rightarrow C'_{1113} = Q_{11}Q_{11}Q_{11}Q_{33}C_{1113} = -C_{1113} = C_{1113} \end{aligned}\right\}$$

$$\Rightarrow \boxed{C_{1113}=0} \quad \boxed{C_{1123}=0}\boxed{C_{2213}=0}\boxed{C_{2223}=0}$$

$$\boxed{C_{3313}=0}\boxed{C_{3323}=0}\boxed{C_{2312}=0}\boxed{C_{1312}=0}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & \cancel{C_{1113}} & \cancel{C_{1123}} \\ & C_{2222} & C_{2233} & C_{2212} & \cancel{C_{2213}} & \cancel{C_{2223}} \\ & & C_{3333} & C_{3312} & \cancel{C_{3313}} & \cancel{C_{3323}} \\ & & & C_{1212} & \cancel{C_{1213}} & \cancel{C_{1223}} \\ & \text{Symm.} & & & C_{1313} & C_{1323} \\ & & & & & C_{2323} \end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{bmatrix}$$

$$\left[Q_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- 13 different elastic constants

Orthotropic Materials

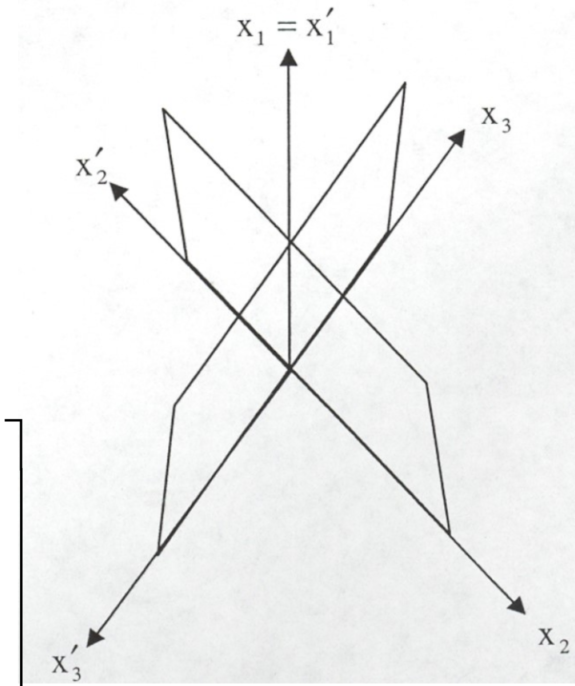
- Symmetric about two orthogonal planes

Examine: $C'_{1112}, C'_{2212}, C'_{3312}, C'_{1323}$

$$\left.\begin{aligned} C'_{1112} &= Q_{1m}Q_{1n}Q_{1o}Q_{2p}C_{mnop} \qquad (m=1; n=1; o=1; p=2) \\ \Rightarrow C'_{1112} &= Q_{11}Q_{11}Q_{11}Q_{22}C_{1112} = -C_{1112} = C_{1112} \end{aligned}\right\}$$

$$\Rightarrow \boxed{C_{1112}=0} \qquad \boxed{C_{2212}=0}\,\boxed{C_{3312}=0}\,\boxed{C_{2313}=0}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \cancel{C_{1112}} & \cancel{C_{1113}} & \cancel{C_{1123}} \\ & C_{2222} & C_{2233} & \cancel{C_{2212}} & \cancel{C_{2213}} & \cancel{C_{2223}} \\ & & C_{3333} & \cancel{C_{3312}} & \cancel{C_{3313}} & \cancel{C_{3323}} \\ & & & C_{1212} & \cancel{C_{1213}} & \cancel{C_{1223}} \\ & \text{Symm.} & & & C_{1313} & \cancel{C_{1323}} \\ & & & & & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

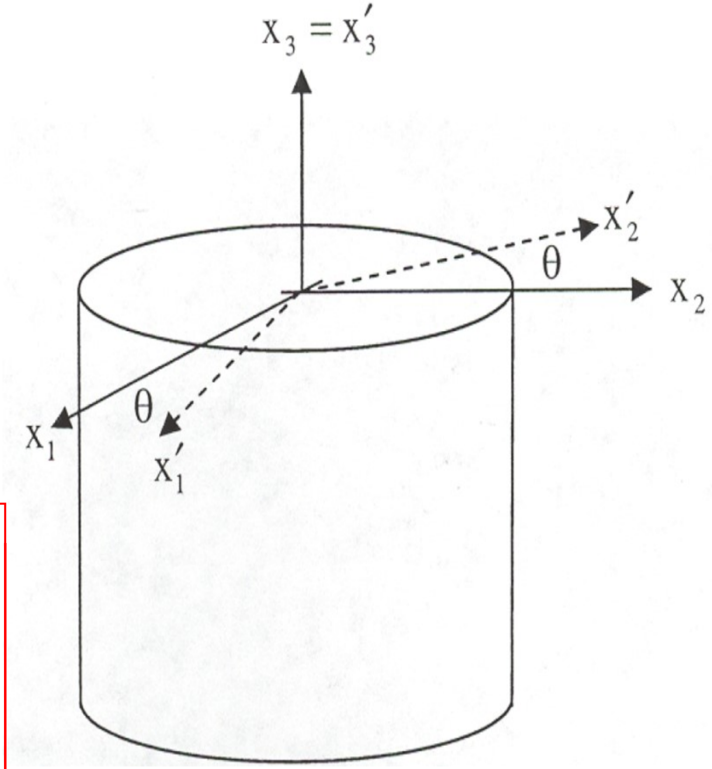


$$[Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ & 0 & -1 \end{bmatrix}$$

- 9 different elastic constants

Transversely Isotropic Materials

- Symmetric about one single axis
- Wood is one example
- Start from the constitutive relation of orthotropic materials



$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & & & \\ & C_{2222} & C_{2233} & & & \\ & & C_{3333} & & & \\ & & & C_{1212} & & \\ & \text{Symm.} & & & C_{1313} & \\ & & & & & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

$$\left[Q_{ij}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transversely Isotropic Materials

Examine: $C'_{1111}, C'_{2222}, C'_{1122}, C'_{1212}$

$$C'_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}C_{mnop} \quad (\text{for } i,j,k,l = 1,2:\ m,n,o,p = 1,2)$$

$$\Rightarrow C'_{ijkl} = \begin{pmatrix} Q_{i1}Q_{j1}Q_{k1}Q_{l1}C_{1111} + Q_{i2}Q_{j2}Q_{k2}Q_{l2}C_{2222} + \left(Q_{i1}Q_{j1}Q_{k2}Q_{l2} + Q_{i2}Q_{j2}Q_{k1}Q_{l1}\right)C_{1122} \\ +\left(Q_{i1}Q_{j2}Q_{k1}Q_{l2} + Q_{i1}Q_{j2}Q_{k2}Q_{l1} + Q_{i2}Q_{j1}Q_{k1}Q_{l2} + Q_{i2}Q_{j1}Q_{k2}Q_{l1}\right)C_{1212} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \boxed{1}: C'_{1111} = C_{1111}\cos^4\theta + C_{2222}\sin^4\theta + \left(2C_{1122} + 4C_{1212}\right)\cos^2\theta\sin^2\theta = C_{1111} \\ \boxed{2}: C'_{2222} = C_{2222}\cos^4\theta + C_{1111}\sin^4\theta + \left(2C_{1122} + 4C_{1212}\right)\cos^2\theta\sin^2\theta = C_{2222} \\ \boxed{3}: C'_{1122} = C_{1122}\left(\cos^4\theta + \sin^4\theta\right) + \left(C_{1111} + C_{2222} - 4C_{1212}\right)\cos^2\theta\sin^2\theta = C_{1122} \\ \boxed{4}: C'_{1212} = C_{1212}\left(\cos^4\theta + \sin^4\theta\right) + \left(C_{1111} + C_{2222} - 2C_{1122} - 2C_{1212}\right)\cos^2\theta\sin^2\theta = C_{1212} \end{cases}$$

Consider symmetry and Eqs. $\boxed{1}$ and $\boxed{2}$ $\Rightarrow$ $\boxed{C_{1111} = C_{2222}}$

$$\begin{cases} \boxed{1} - \boxed{3}: C_{1111} - C_{1122} = \left(C_{1111} - C_{1122}\right)\left(\cos^4\theta + \sin^4\theta\right) + \left(2C_{1122} + 8C_{1212} - 2C_{1111}\right)\cos^2\theta\sin^2\theta \\ \boxed{4}\times 2: 2C_{1212} = 2C_{1212}\left(\cos^4\theta + \sin^4\theta\right) + \left(4C_{1111} - 4C_{1122} - 4C_{1212}\right)\cos^2\theta\sin^2\theta \end{cases}$$

$$\Rightarrow \boxed{C_{1212} = \frac{1}{2}\left(C_{1111} - C_{1122}\right)}$$

Transverselly Isotropic Materials

Examine: $C'_{1133}, C'_{2233}, C'_{1313}, C'_{2323}$

$$\left.\begin{aligned} &C'_{1133} = Q_{1m}Q_{1n}Q_{3o}Q_{3p}C_{mnop} \qquad (o=p=3) \\ &\Rightarrow C'_{1133} = Q_{11}Q_{11}C_{1133} + Q_{11}Q_{12}\cancel{C_{1233}} + Q_{12}Q_{11}\cancel{C_{2133}} + Q_{12}Q_{12}C_{2233} \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} C'_{1133} &= \cos^2\theta C_{1133} + \sin^2\theta C_{2233} = C_{1133} \\ C'_{2233} &= \cos^2\theta C_{2233} + \sin^2\theta C_{1133} = C_{2233} \end{aligned}\right\} \Rightarrow \boxed{C_{1133} = C_{2233}}$$

$$\left.\begin{aligned} C'_{1313} &= \cos^2\theta C_{1313} + \sin^2\theta C_{2323} = C_{1313} \\ C'_{2323} &= \cos^2\theta C_{2323} + \sin^2\theta C_{1313} = C_{2323} \end{aligned}\right\} \Rightarrow \boxed{C_{1313} = C_{2323}}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & & & \\ & C_{1111} & C_{1133} & & & \\ & & C_{3333} & & & \\ & & & (C_{1111}-C_{1122})/2 & & \\ & \text{Symm.} & & & C_{1313} & \\ & & & & & C_{1313} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

- 5 different elastic constants

Isotropic Materials

- Symmetric about all three axes
- Repeat the previous analysis for x_1 & x_2

$$[Q'_{ij}]=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix},[Q''_{ij}]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix},[Q'''_{ij}]=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix}=\begin{bmatrix}C_{1111} & C_{1122} & C_{1122} & & & \\ & C_{1111} & C_{1122} & & & \\ & & C_{1111} & & & \\ & & & (C_{1111}-C_{1122})/2 & & \\ & \text{Symm.} & & & (C_{1111}-C_{1122})/2 & \\ & & & & & (C_{1111}-C_{1122})/2\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{bmatrix}$$

- 2 different elastic constants

Isotropic Hooke's Law

Define Lame constants: $\lambda = C_{1122}, G = \left(C_{1111} - C_{1122}\right)/2.$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & & & \\ & \lambda + 2G & \lambda & & & \\ & & \lambda + 2G & & & \\ & & & G & & \\ & \text{Symm.} & & & G & \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

$$\boxed{C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)}$$

$$\sigma_{ij} = \left(\lambda \delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right)\varepsilon_{kl}$$

$$\Rightarrow \boxed{\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}}$$

Inverse Relation

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$$

$$\Rightarrow \sigma_{kk} = \lambda\varepsilon_{ll}\delta_{kk} + 2G\varepsilon_{kk} = \left(3\lambda + 2G\right)\varepsilon_{kk} \Rightarrow \varepsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2G}$$

$$\Rightarrow \varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \lambda\varepsilon_{kk}\delta_{ij}\right) \Rightarrow \boxed{\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2G}\sigma_{kk}\delta_{ij}\right)}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} \frac{\lambda + G}{G\left(3\lambda + 2G\right)} & -\frac{\lambda}{2G\left(3\lambda + 2G\right)} & -\frac{\lambda}{2G\left(3\lambda + 2G\right)} & & & \\ & \frac{\lambda + G}{G\left(3\lambda + 2G\right)} & -\frac{\lambda}{2G\left(3\lambda + 2G\right)} & & & \\ & & \frac{\lambda + G}{G\left(3\lambda + 2G\right)} & & & \\ & & & 1/G & & \\ & \text{Symm.} & & & 1/G & \\ & & & & & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}$$

Physical Meaning of Elastic Moduli

- Lamé constants were only the consequence of isotropy
- Their physical interpretation requires material testing

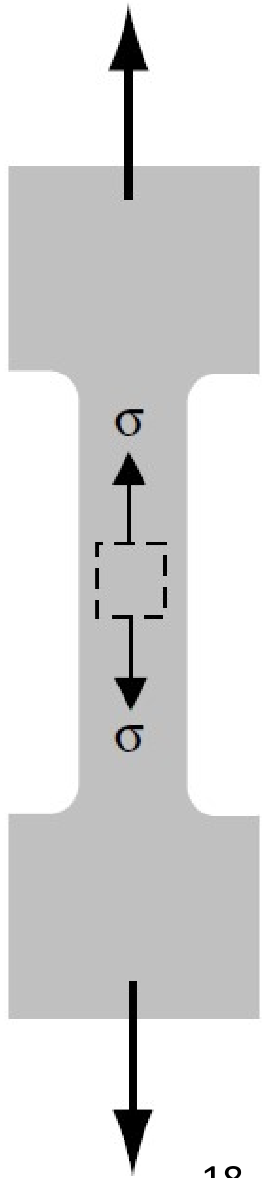
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & & \\ & 1/E & -\nu/E & & & \\ & & 1/E & & & \\ & & & 1/G & & \\ & \text{Symm.} & & & 1/G & \\ & & & & & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}$$

$$\boxed{\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}}$$

$$\begin{cases} \dfrac{\lambda + G}{G(3\lambda + 2G)} = \dfrac{1}{E} \\ \dfrac{\lambda}{2G(3\lambda + 2G)} = \dfrac{\nu}{E} \end{cases} \Rightarrow \boxed{\begin{aligned} \lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)} \\ G &= \frac{E}{2(1+\nu)} \end{aligned}} \boxed{\begin{aligned} E &= \frac{G(3\lambda + 2G)}{\lambda + G} \\ \nu &= \frac{\lambda}{2(\lambda + G)} \end{aligned}}$$

Uniaxial Tension Test

$$
\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}
=
\begin{bmatrix}
1/E & -\nu/E & -\nu/E & & & \\
-\nu/E & 1/E & -\nu/E & & & \\
-\nu/E & -\nu/E & 1/E & & & \\
 & & & 1/G & & \\
 & & & & 1/G & \\
 & & & & & 1/G
\end{bmatrix}
\begin{Bmatrix} \sigma_1 \\ \cancel{\sigma_2} \\ \cancel{\sigma_3} \\ \cancel{\tau_{12}} \\ \cancel{\tau_{13}} \\ \cancel{\tau_{23}} \end{Bmatrix}
$$

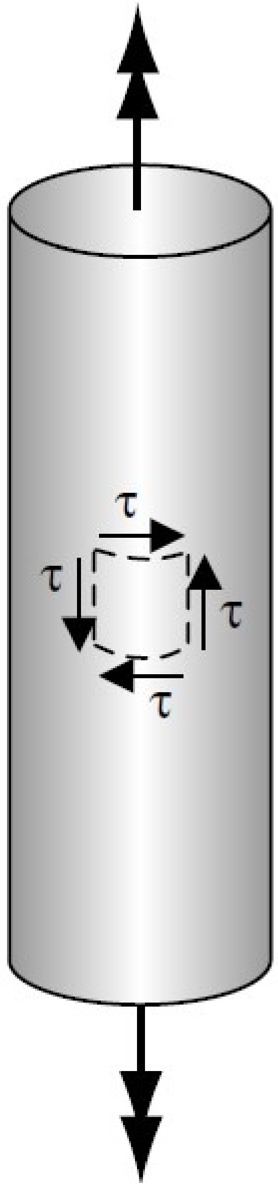


E : Young's modulus

ν : Poisson's ratio $\boxed{\nu > 0}$

Simple Shear (Torsion)

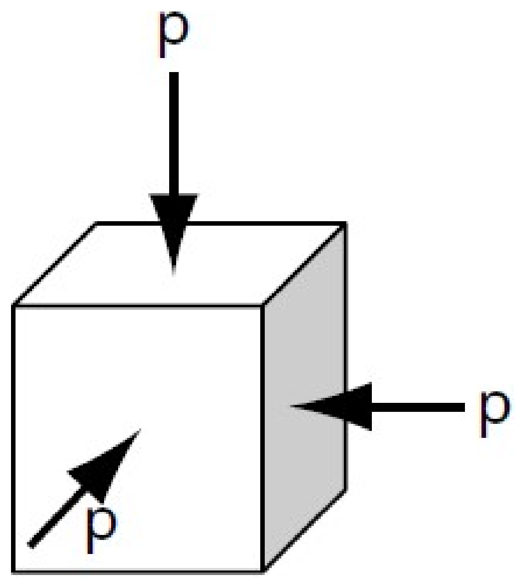
$$\begin{Bmatrix} 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix} = \begin{bmatrix} 1/G & & \\ & 1/G & \\ & & 1/G \end{bmatrix} \begin{Bmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}$$



G : Shear modulus

Hydrostatic Compression

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E \\ -\nu/E & 1/E & -\nu/E \\ -\nu/E & -\nu/E & 1/E \end{bmatrix} \begin{Bmatrix} -p \\ -p \\ -p \end{Bmatrix}$$



$$\varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -p\frac{3(1-2\nu)}{E} = \frac{-3p}{3\lambda + 2G}$$

$$K = \frac{-p}{\varepsilon_{kk}} = \frac{E}{3(1-2\nu)} = \frac{3\lambda + 2G}{3}$$: Bulk modulus $\boxed{K > 0 \Rightarrow \nu < 0.5}$

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + \hat{\sigma}_{ij}$$

$$\sigma_{kk}/3 = K\varepsilon_{kk}$$

$$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - \frac{1}{3}(3\lambda + 2G)\varepsilon_{kk}\delta_{ij} = 2G\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right) = 2G\hat{\varepsilon}_{ij}$$

Relationship among Isotropic Elastic Constants

- The elastic constants we have encountered include

$$\begin{array}{|l|}\hline E: \text{Young's modulus} \\ \nu: \text{Poisson's ratio} \\ G: \text{Shear's modulus} \\ K: \text{Bulk modulus} \\ \lambda, G: \text{Lame constants} \\ \hline\end{array}$$

- It has been proved that **only two out of the five elastic constants are independent** for isotropic materials.
- Given any two, the remaining three can be solved.

Relationship among Isotropic Elastic Constants

$$\left(E,\nu\right): \lambda = \frac{E\nu}{\left(1-2\nu\right)\left(1+\nu\right)}, G = \frac{E}{2\left(1+\nu\right)}, K = \frac{E}{3\left(1-2\nu\right)}$$

$$\left(\lambda,G\right): E = \frac{G\left(3\lambda+2G\right)}{\lambda+G}, \nu = \frac{\lambda}{2\left(\lambda+G\right)}, K = \frac{3\lambda+2G}{3}$$

$$\left(G,\nu\right): E = 2G\left(1+\nu\right), \lambda = \frac{2G\nu}{1-2\nu}, K = \frac{2G\left(1+\nu\right)}{3\left(1-2\nu\right)}$$

$$\left(\lambda,\nu\right): E = \frac{\lambda\left(1+\nu\right)\left(1-2\nu\right)}{\nu}, G = \frac{\lambda\left(1-2\nu\right)}{2\nu}, K = \frac{\lambda\left(1+\nu\right)}{3\nu}$$

$$\left(E,G\right): \nu = \frac{E-2G}{2G}, K = \frac{EG}{3\left(3G-E\right)}, \lambda = \frac{G\left(E-2G\right)}{3G-E}; \quad \left(E,K\right): \nu = \frac{3K-E}{6K}, G = \frac{3KE}{9K-E}, \lambda = \frac{3K\left(3K-E\right)}{9K-E}$$

$$\left(E,\lambda\right): \nu = \frac{2\lambda}{E+\lambda+R}, G = \frac{E-3\lambda+R}{4}, K = \frac{E+3\lambda+R}{6}; \quad \left(K,\nu\right): E = 3K\left(1-2\nu\right), G = \frac{3K\left(1-2\nu\right)}{2\left(1+\nu\right)}, \lambda = \frac{3K\nu}{1+\nu}$$

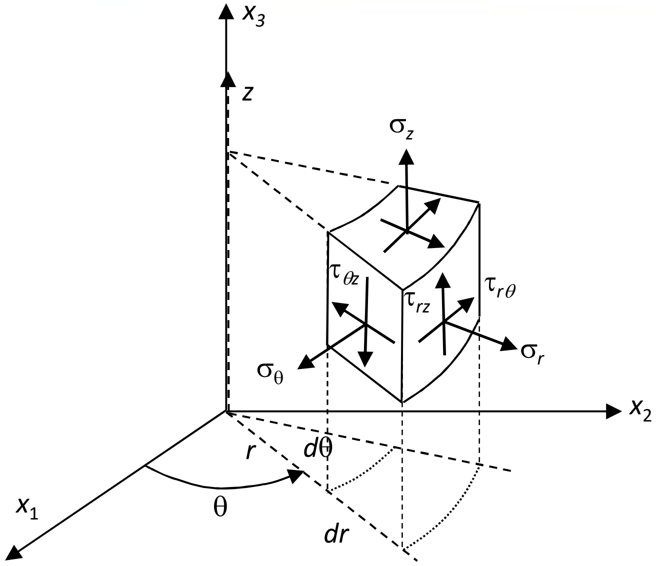
$$\left(G,K\right): E = \frac{9KG}{3K+G}, \nu = \frac{3K-2G}{6K+2G}, \lambda = \frac{3K-2G}{3}; \quad \left(K,\lambda\right): E = \frac{9K\left(K-\lambda\right)}{3K-\lambda}, \nu = \frac{\lambda}{3K-\lambda}, G = \frac{3}{2}\left(K-\lambda\right)$$

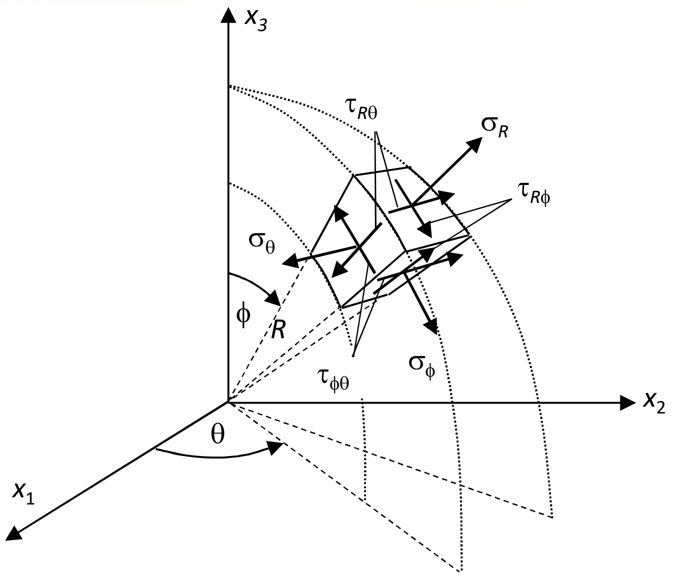
$$R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$$

Isotropic Hooke's Law in Curvilinear Coordinates

- Recall that, **the elastic stiffness tensor *C* is a fourth order isotropic tensor**.
- Its components remain unchanged under any orthogonal coordinate systems.
- The isotropic Hooke's law stays the same.

Isotropic Hooke's Law in Curvilinear Coordinates





$$\sigma_r = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z\right) + 2G\varepsilon_r$$

$$\sigma_\theta = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z\right) + 2G\varepsilon_\theta$$

$$\sigma_z = \lambda\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z\right) + 2G\varepsilon_z$$

$$\tau_{r\theta} = 2G\varepsilon_{r\theta}$$

$$\tau_{rz} = 2G\varepsilon_{rz}$$

$$\tau_{\theta z} = 2G\varepsilon_{\theta z}$$

$$\sigma_R = \lambda\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + 2G\varepsilon_R$$

$$\sigma_\varphi = \lambda\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + 2G\varepsilon_\varphi$$

$$\sigma_\theta = \lambda\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + 2G\varepsilon_\theta$$

$$\tau_{R\varphi} = 2G\varepsilon_{R\varphi}$$

$$\tau_{R\theta} = 2G\varepsilon_{R\theta}$$

$$\tau_{\theta\varphi} = 2G\varepsilon_{\theta\varphi}$$

Thermoelastic Constitutive Relations

- A temperature change in an elastic solid produces deformation.
- The total strain can be decomposed into the sum of mechanical and thermal components.
- It is extremely important to understand that, the elastic stiffness tensor (***C***) correlates mechanical stress and mechanical strain.

$$\varepsilon_{ij}^{\text{Total}} = \varepsilon_{ij}^{M} + \varepsilon_{ij}^{T} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}^{M}\delta_{ij} + 2G\varepsilon_{ij}^{M} = \lambda\left(\varepsilon_{kk}^{\text{Total}} - \varepsilon_{kk}^{T}\right)\delta_{ij} + 2G\left(\varepsilon_{ij}^{\text{Total}} - \varepsilon_{ij}^{T}\right)$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}^{\text{Total}}\delta_{ij} + 2G\varepsilon_{ij}^{\text{Total}} - (3\lambda+2G)\alpha\Delta T\delta_{ij} = \lambda\varepsilon_{kk}^{\text{Total}}\delta_{ij} + 2G\varepsilon_{ij}^{\text{Total}} - 3K\alpha\Delta T\delta_{ij}$$

Typical Values of Elastic and Thermal Moduli

	E (GPa)	ν	G (GPa)	λ (GPa)	K (GPa)	α ($10^{-6}/^{o}C$)
Aluminum	68.9	0.34	25.7	54.6	71.8	25.5
Concrete	27.6	0.20	11.5	7.7	15.3	11
Cooper	89.6	0.34	33.4	71	93.3	18
Glass	68.9	0.25	27.6	27.6	45.9	8.8
Nylon	28.3	0.40	10.1	4.04	47.2	102
Rubber	0.0019	0.499	0.654×10^{-3}	0.326	0.326	200
Steel	207	0.29	80.2	111	164	13.5

Outline

- Constitutive Laws
- Strain Energy
- Linear Constitutive Relations
- Anisotropy and Stiffness Tensor
- Isotropic Hooke’s Law
- Physical Meaning of Elastic Moduli
- Simple Engineering Tests
- Relationship among Elastic Constants
- Hooke’s Law in Curvilinear Coordinates
- Thermoelastic Constitutive Relations
- Typical Values of Elastic Constants
- Inhomogeneity