Mathematical Preliminary

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Outline

- Scalar, Vector and Matrix（标量、矢量、矩阵）
- Indicial Notation and Summation Convention（指标记法与求和约定）
- Kronecker Delta（克罗内克 δ）
- Levi-Civita symbol（轮换记号）
- Coordinate Transformation（坐标变换）
- Tensor（张量）
- Principal Values and Directions（特征值与特征向量）
- Tensor Algebra（张量代数）
- Tensor Calculus（张量微分）
- Integral Theorems（积分定理）
- Cylindrical and Spherical Coordinates（柱与球坐标）
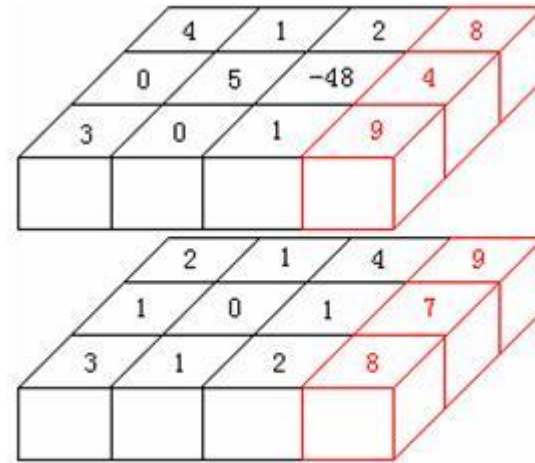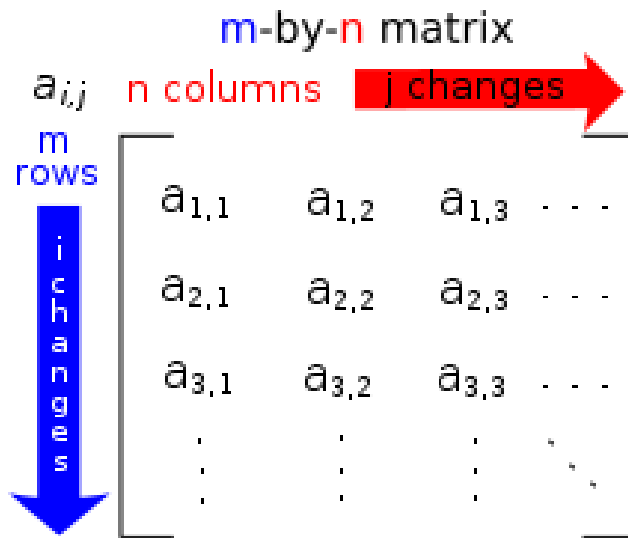
Scalar

- Scalar: representing a single magnitude at each point in space
- Examples:

- Material density
- Young's modulus
- Poisson's ratio
- Shear modulus
- Temperature

Vector

- Vector: representing physical quantities that have both magnitude and direction
- Examples:

- Force
- Displacement of material points
- Rotation of material points

Matrix (Array)

- Matrix: a rectangular array of numbers

- Array: a data structure in which similar elements of data are arranged in a table
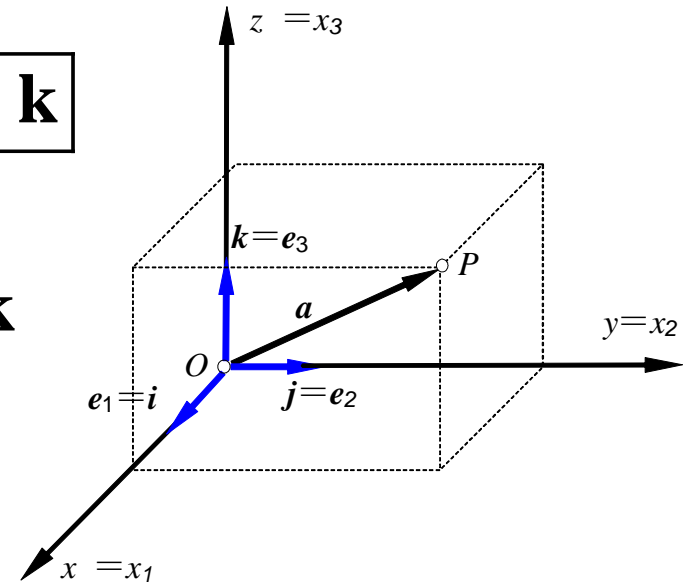
Indicial Notation

- Orthogonal unit vectors: $\boxed{\mathbf{i} \times \mathbf{j} = \mathbf{k}}$
- Vector decomposition:

$$\mathbf{a} = \mathbf{OP} = (\mathbf{a} \cdot \mathbf{i})\mathbf{i} + (\mathbf{a} \cdot \mathbf{j})\mathbf{j} + (\mathbf{a} \cdot \mathbf{k})\mathbf{k}$$

$$= a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} = \sum_{i=1}^{3} a_i\mathbf{e}_i$$

- Indicial notation: a shorthand scheme whereby a whole set of components is represented by a single symbol with subscripts

$$x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{e}_i = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \ a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_3 & a_{32} & a_{33} \end{bmatrix}$$

Indicial Notation

- Addition and subtraction

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \; a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

- Scalar multiplication

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \; \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix}$$

- Outer multiplication

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Indicial Notation

- Commutative, associative and distributive laws

$$a_i + b_i = b_i + a_i$$

$$a_{ij}b_k = b_k a_{ij}$$

$$a_i + (b_i + c_i) = (a_i + b_i) + c_i$$

$$a_i(b_{jk}c_l) = (a_i b_{jk})c_l$$

$$a_{ij}(b_k + c_k) = a_{ij}b_k + a_{ij}c_k$$

- Equality of two symbols

$$a_i = b_i \Rightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}, \qquad a_{ij} = b_{ij} \Rightarrow \begin{bmatrix} a_{11} = b_{11} & a_{12} = b_{12} & a_{13} = b_{13} \\ a_{21} = b_{21} & a_{22} = b_{22} & a_{23} = b_{23} \\ a_{31} = b_{31} & a_{32} = b_{32} & a_{33} = b_{33} \end{bmatrix}$$

- Avoid

$$a_i = b_j, \qquad a_{ij} = b_{kl} \cdots$$

Summation Convention

- Summation convention: if a subscript appears twice in the same term, then summation over that subscript from one to three is implied

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_{ij}x_i x_j = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_i x_j$$

$$a_{ij}b_j = \sum_{j=1}^{3} a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$

$$a_{ii} = a_{jj} = a_{kk} \cdots$$

- In a single term, no index can appear more than twice.
- Dummy (repeated) indices vs. free (distinct) indices
- Among terms, index property must match.

Contraction and Symmetry

- Contraction: for example, a_{ii} is obtained from a_{ij} by contraction on i and j
- Outer multiplication $\rightarrow$ contraction $\rightarrow$ inner product

$$a_i b_j \xrightarrow{\left(\text{contraction on } ij\right)} a_i b_i$$

$$a_{ij} b_{kl} \xrightarrow{\left(\text{contraction on } jk\right)} a_{ik} b_{kl}$$

- Symmetric vs. antisymmetric (skewsymmetric) w.r.t. two indices, i.e. m and n

$$a_{ij...m...n...k} = a_{ij...n...m...k} \qquad a_{ij...m...n...k} = -a_{ij...n...m...k}$$

- Useful identity: $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

Kronecker Delta

- Kronecker delta: a useful special symbol commonly used in index notational schemes
- Symmetric

$$\delta_{ij} = \begin{cases} 1, if\ i = j\ (no\ sum) \\ 0, if\ i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Replacement property

$$\delta_{ij} = \delta_{ji}$$
$$\delta_{ii} = 3, \delta_{i\underline{i}} = 1$$
$$\delta_{ij}a_j = a_i, \delta_{ij}a_i = a_j$$
$$\delta_{ij}a_{jk} = a_{ik}, \delta_{jk}a_{ik} = a_{ij}$$
$$\delta_{ij}a_{ij} = a_{ii}, \delta_{ij}\delta_{ij} = 3$$

Levi-Civita (Permutation) Symbol

- Antisymmetric w.r.t. any pair of its indices

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1, \varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \ldots = 0.$$

- Cross product of two vectors

$$\mathbf{a} \times \mathbf{b} = \left(a_i \mathbf{e}_i\right) \times \left(b_j \mathbf{e}_j\right) = a_i b_j \left(\mathbf{e}_i \times \mathbf{e}_j\right) = a_i b_j \left(\varepsilon_{ijk} \mathbf{e}_k\right) = \varepsilon_{ijk} a_i b_j \mathbf{e}_k$$

- Scalar triple product of three vectors

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \varepsilon_{ijk} a_i b_j \mathbf{e}_k \cdot c_m \mathbf{e}_m = \varepsilon_{ijk} a_i b_j c_m \left(\mathbf{e}_k \cdot \mathbf{e}_m\right) = \varepsilon_{ijk} a_i b_j c_m \delta_{km} = \varepsilon_{ijk} a_i b_j c_k$$

Levi-Civita (Permutation) Symbol

- Determinant of a matrix (easily verifiable)

$$\det\left[a_{ij}\right] = \left|a_{ij}\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_3 & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

- By the definition of Levi-Civita and note the following

$$\varepsilon_{ijk} \varepsilon_{rst} a_{ir} a_{js} a_{kt} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3} - \varepsilon_{ijk} a_{i1} a_{j3} a_{k2} + \varepsilon_{ijk} a_{i2} a_{j3} a_{k1}$$

$$- \varepsilon_{ijk} a_{i2} a_{j1} a_{k3} + \varepsilon_{ijk} a_{i3} a_{j1} a_{k2} - \varepsilon_{ijk} a_{i3} a_{j2} a_{k1}$$

$$\Rightarrow \boxed{\varepsilon_{ijk} \varepsilon_{rst} a_{ir} a_{js} a_{kt} = 6\det\left[a_{ij}\right], \qquad \varepsilon_{ijk} a_{il} a_{jm} a_{kn} = \varepsilon_{lmn} \det\left[a_{ij}\right]}$$

Levi-Civita (Permutation) Symbol

- ε-δ property

$$\varepsilon_{ijk} = \mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = \left(\delta_{ip}\mathbf{e}_p\right) \cdot \left(\delta_{jq}\mathbf{e}_q\right) \times \left(\delta_{kr}\mathbf{e}_r\right) \quad \Rightarrow \quad \varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

$$\varepsilon_{ijk}\varepsilon_{pqr} = \big\| [A] \big\| \big\| [B] \big\| = \big\| [A] \big\| \big\| [B]^T \big| = \big\| [A][B]^T \big|$$

$$= \begin{Vmatrix} \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix} \end{Vmatrix} = \begin{vmatrix} \delta_{im}\delta_{pm} & \delta_{im}\delta_{qm} & \delta_{im}\delta_{rm} \\ \delta_{jm}\delta_{pm} & \delta_{jm}\delta_{qm} & \delta_{jm}\delta_{rm} \\ \delta_{km}\delta_{pm} & \delta_{km}\delta_{qm} & \delta_{km}\delta_{rm} \end{vmatrix}$$

$$\Rightarrow \quad \varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \quad \Rightarrow \quad \varepsilon_{ijk}\varepsilon_{pqk} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{vmatrix} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Sample Problem

- The matrix a_{ij} and vector b_i are specified by

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \ b_i = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

- Determine the following quantities, and indicate whether they are a scalar, vector or matrix.

$$a_{ii}, a_{ij}a_{ij}, a_{ij}a_{jk}, a_{ij}b_j, a_{ij}b_ib_j, b_ib_i, b_ib_j, a_{\text{symm.}}, a_{\text{anti.}}$$

- **Solution**:

$$a_{ii} = a_{11} + a_{22} + a_{33} = 7 \ (\text{scalar})$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$

$$= 1 + 4 + 0 + 0 + 16 + 9 + 4 + 1 + 4 = 39 \ (\text{scalar})$$

Sample Problem - Solution

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = \begin{bmatrix} 1 & 10 & 6 \\ 6 & 19 & 18 \\ 6 & 10 & 7 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 10 \\ 16 \\ 8 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + \cdots = 84 \text{ (scalar)}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 16 + 0 = 20 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} 4 & 8 & 0 \\ 8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$$a_{\text{symm.}} = \frac{1}{2}\left(a_{ij} + a_{ji}\right) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} \text{ (matrix)}$$

$$a_{\text{anti.}} = \frac{1}{2}\left(a_{ij} - a_{ji}\right) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \text{ (matrix)}$$

Coordinate Transformation

- Unit base vectors for each frame

$$\{\mathbf{e_i}\} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}; \qquad \{\mathbf{e_i'}\} = \{\mathbf{e_1'}, \mathbf{e_2'}, \mathbf{e_3'}\}$$
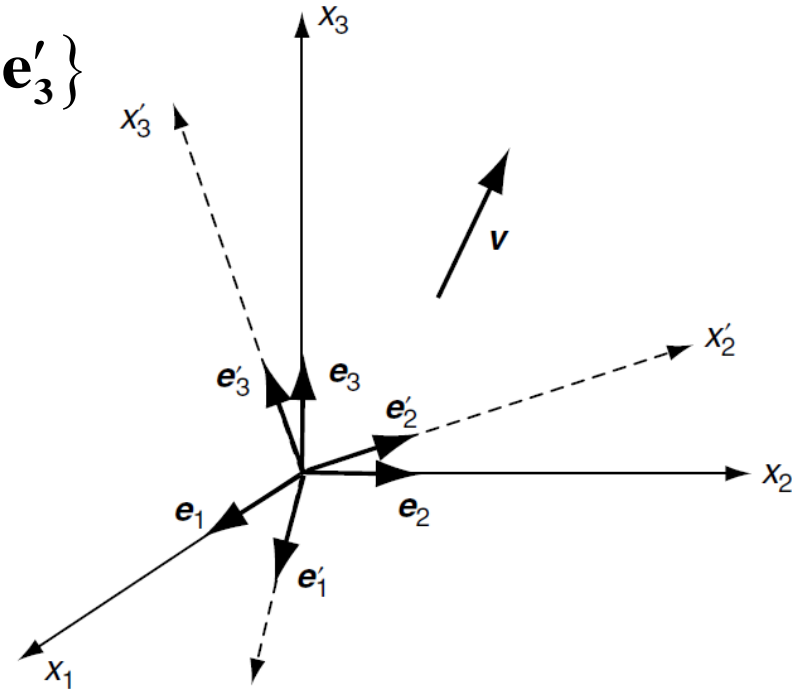
- Transformation matrix

$$\boxed{Q_{ij} = \cos(x_i', x_j)}$$

$$\mathbf{e_1'} = Q_{11}\mathbf{e_1} + Q_{12}\mathbf{e_2} + Q_{13}\mathbf{e_3}$$
$$\mathbf{e_2'} = Q_{21}e_1 + Q_{22}e_2 + Q_{23}\mathbf{e_3}$$
$$\mathbf{e_3'} = Q_{31}\mathbf{e_1} + Q_{32}\mathbf{e_2} + Q_{33}\mathbf{e_3}$$

- Relations between base vectors

$$\mathbf{e_i'} = Q_{ij}\mathbf{e_j}; \qquad \mathbf{e_i} = Q_{ji}\mathbf{e_j'}$$

Coordinate Transformation

- Transformation between vectors

$$\mathbf{v} = v_1 \mathbf{e_1} + v_2 \mathbf{e_2} + v_3 \mathbf{e_3} = v_i \mathbf{e_i} \boxed{= v_1' \mathbf{e_1'} + v_2' \mathbf{e_2'} + v_3' \mathbf{e_3'} = v_i' \mathbf{e_i'}}$$

$$v_i' = Q_{ij} v_j; \qquad v_i = Q_{ji} v_j'$$

- Property of transformation matrix

$$v_i = Q_{ji} v_j' = Q_{ji} Q_{jk} v_k \quad \Rightarrow \boxed{Q_{ji} Q_{jk} = \delta_{ik}} \boxed{\mathbf{Q}^T \mathbf{Q} = \mathbf{I}}$$

$$v_i' = Q_{ij} v_j = Q_{ij} Q_{kj} v_k'; \quad \Rightarrow \boxed{Q_{ij} Q_{kj} = \delta_{ik}} \boxed{\mathbf{Q} \mathbf{Q}^T = \mathbf{I}}$$

$$\boxed{\det\left[Q_{ij} \right] = \pm 1}$$

- For right-handed coordinates

$$\boxed{\det\left[Q_{ij} \right] = 1}$$
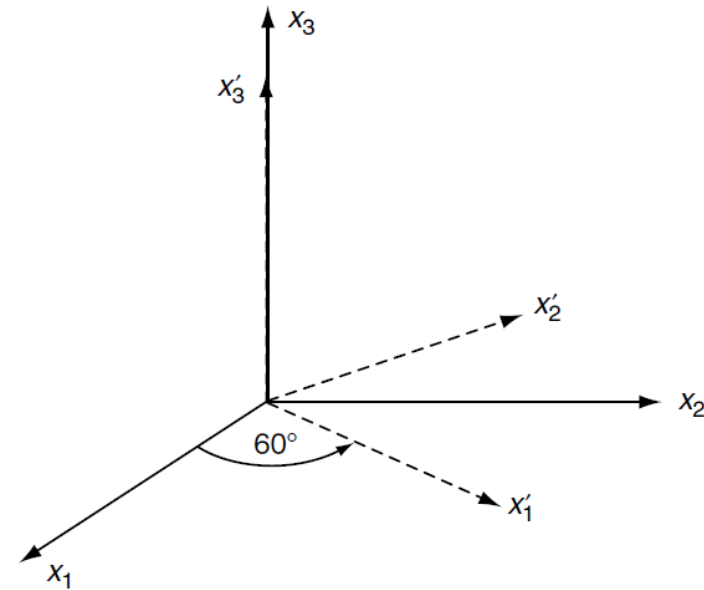
Sample Problem

- The components of a first and second order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, a_{ij} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

- Determine the components of each tensor in a new coordinate system found through a rotation of 60° about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis.

Sample Problem - Solution

$$Q_{ij} = \begin{bmatrix} \cos 60° & \cos 30° & \cos 90° \\ \cos 150° & \cos 60° & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_i' = Q_{ij}a_j = \begin{vmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + 2\sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip}Q_{jq}a_{pq} = \begin{vmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2 + \sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1 - 3\sqrt{3}/2 \\ 3/2 + \sqrt{3} & 1 - 3\sqrt{3}/2 & 4 \end{bmatrix}$$

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Tensor - Definition

- **Those matrices that satisfy the following rule are called second-order tensors.**

$$A = A'_{kl}\mathbf{e}'_{\mathbf{k}}\mathbf{e}'_{\mathbf{l}} = A'_{kl}\left(Q_{ki}\mathbf{e_i}\right)\left(Q_{lj}\mathbf{e_j}\right) = Q_{ki}Q_{lj}A'_{kl}\mathbf{e_i}\mathbf{e_j} \boxed{= A_{ij}\mathbf{e_i}\mathbf{e_j}}$$

$$\Rightarrow \boxed{A_{ij} = Q_{ki}Q_{lj}A'_{kl}} \qquad \Rightarrow \boxed{A'_{ij} = Q_{ik}Q_{jl}A_{kl}}$$

- **Exceptional matrices do exist, i.e. Spinors（旋量）Fortunately, this is not our concern.**

Tensor - Definition

- **A tensor can be any order and must satisfy**

$a' = a$, zero order (scalar)

$a'_i = Q_{ip}a_p$, first order (vector)

$a'_{ij} = Q_{ip}Q_{jq}a_{pq}$, second order (matrix)

$a'_{ijk} = Q_{ip}Q_{jq}Q_{kr}a_{pqr}$, third order

$a'_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}a_{pqrs}$, fourth order

- For general order

$$A'_{ijkl\cdots} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\cdots A_{mnop\cdots}$$

$$A_{ijkl\cdots} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}\cdots A'_{mnop\cdots}$$

Isotropic Tensors

- **Defined as those tensors whose components remain the same under all transformations**

- All scalars (zero order) are isotropic

- No non-trivia isotropic tensor of the first order

- Kronecker Delta is the most general second order isotropic tensor

$$T_{ij} = \alpha \delta_{ij}$$

$$\Rightarrow T'_{ij} = Q_{ik} Q_{jl} T_{kl} = Q_{ik} Q_{jl} \alpha \delta_{kl} = \alpha Q_{ik} Q_{jk} = \alpha \delta_{ij} = T_{ij}$$

Isotropic Tensors

- Levi-Civita Symbol is the most general third order isotropic tensor

$$\det[\mathbf{a}] = a_{1m}a_{2n}a_{3o}\varepsilon_{mno} = a_{m1}a_{n2}a_{o3}\varepsilon_{mno}$$

$$\Rightarrow \varepsilon_{ijk}\det[\mathbf{a}] = a_{im}a_{jn}a_{ko}\varepsilon_{mno}$$

$$T_{ijk} = \alpha\varepsilon_{ijk}$$

$$\Rightarrow T'_{ijk} = Q_{im}Q_{jn}Q_{ko}T_{mno} = \alpha Q_{im}Q_{jn}Q_{ko}\varepsilon_{mno} = \alpha\varepsilon_{ijk}\det[\mathbf{Q}] = T_{ijk}$$

- The most general fourth order isotropic tensor

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$\Rightarrow T'_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}T_{mnop} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\left(\alpha\delta_{mn}\delta_{op} + \beta\delta_{mo}\delta_{np} + \gamma\delta_{mp}\delta_{on}\right)$$

$$= \left(\alpha Q_{im}Q_{jm}Q_{ko}Q_{lo} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}\right) = T_{ijkl}$$

Tensor Algebra

- The distinction between a tensor and its components

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_i e_i$$
$$= v_1' e_1' + v_2' e_2' \vdash v_3' e_3' = v_i' e_i'$$

$$A = A_{11} e_1 e_1 + A_{12} e_1 e_2 + A_{13} e_1 e_3$$
$$+ A_{21} e_2 e_1 + A_{22} e_2 e_2 + A_{23} e_2 e_3$$
$$+ A_{31} e_3 e_1 + A_{32} e_3 e_2 + A_{33} e_3 e_3$$
$$= A_{ij} e_i e_j = A_{ij}' e_i' e_j'$$

- Equality of two tensors: $\mathbf{A} = \mathbf{B} \Rightarrow A_i \mathbf{e_i} = B_i \mathbf{e_i} \Rightarrow A_i = B_i$

- Addition/subtraction

- Scalar multiplication: $\lambda \mathbf{A} = \lambda A_i \mathbf{e_i}$

- Cross product of two vectors

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_j b_k e_i$$

- Triple product of three vectors

$$a \cdot b \times c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k$$

Tensor Algebra

- Tensor product (outer multiplication)

$$\mathbf{A} \otimes \mathbf{B} = A_i \mathbf{e_i} \otimes B_j \mathbf{e_j} \Rightarrow A_i B_j \mathbf{e_i} \otimes \mathbf{e_j} = A_i B_j \mathbf{e_i} \mathbf{e_j}$$

- Contraction on a pair of indices

(Tensor product $\rightarrow$ inner product)

$$\mathbf{A} \cdot \mathbf{B} = A_i \mathbf{e_i} \cdot B_j \mathbf{e_j} \Rightarrow A_i B_j \mathbf{e_i} \cdot \mathbf{e_j} = A_i B_i$$

- Generalized inner product (matrix product)

$$Aa = [A]\{a\} = A_{ij} a_j = a_j A_{ij}$$
$$a^T A = \{a\}^T [A] = a_i A_{ij} = A_{ij} a_i$$
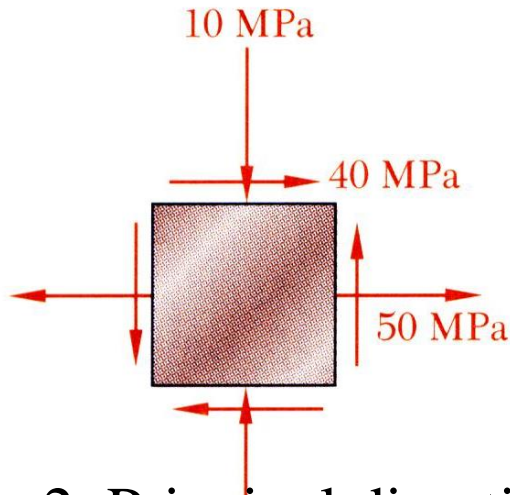
$$AB = [A][B] = A_{ij} B_{jk}$$
$$AB^T = A_{ij} B_{kj}$$
$$A^T B = A_{ji} B_{jk}$$
$$tr(AB) = A_{ij} B_{ji}$$
$$tr(AB^T) = tr(A^T B) = A_{ij} B_{ij}$$

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Principal Values and Directions - 2D

- For the state of plane stress shown, determine the principal stresses and principal directions.

- Solution 1. Principal stresses

$$\begin{bmatrix} 50-\sigma & 40 \\ 40 & -10-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 50-\sigma & 40 \\ 40 & -10-\sigma \end{vmatrix} = 0 \Rightarrow (50-\sigma)(-10-\sigma)-1600=0$$

$$\Rightarrow \sigma^2 - 40\sigma - 2100 = 0 \Rightarrow \sigma_1 = 70, \sigma_2 = -30$$

2. Principal directions

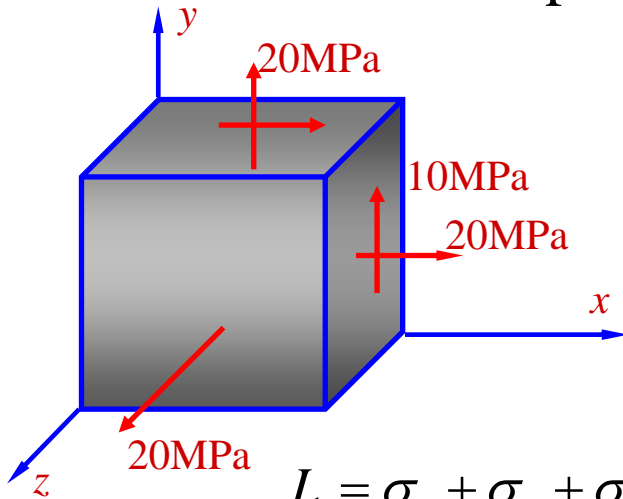$$\begin{bmatrix} 50-70 & 40 \\ 40 & -10-70 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} -20 & 40 \\ 40 & -80 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow n_1 = 2n_2 \Rightarrow \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{Bmatrix}$$

$$\begin{bmatrix} 50+30 & 40 \\ 40 & -10+30 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 80 & 40 \\ 40 & 20 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow n_2 = -2n_1 \Rightarrow \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{Bmatrix}$$

Principal Values and Directions - 3D

- For the state of 3-D stress shown, determine the principal stresses and principal directions.

- Solution   1. Principal stresses:

$$\begin{bmatrix} 20-\sigma & 10 & 0 \\ 10 & 20-\sigma & 0 \\ 0 & 0 & 20-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\det\left[\sigma_{ij} - \sigma\delta_{ij}\right] = 0$$

$$\Rightarrow -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

$$\Rightarrow \sigma_1 = 30, \quad \sigma_1 = 20, \quad \sigma_1 = 10.$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 20 + 20 + 20 = 60$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = 400 + 400 + 400 - 100 = 1100$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = 8000 - 2000 = 6000$$

2. Principal directions:

$$\begin{bmatrix} 20-\sigma & 10 & 0 \\ 10 & 20-\sigma & 0 \\ 0 & 0 & 20-\sigma \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

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Tensor Field and Comma Notation

- The field concept of a tensor component

$$a = a(x_1, x_2, x_3) = a(x_i) = a(\boldsymbol{x})$$

$$a_i = a_i(x_1, x_2, x_3) = a_i(x_i) = a_i(\boldsymbol{x})$$

$$a_{ij} = a_{ij}(x_1, x_2, x_3) = a_{ij}(x_i) = a_{ij}(\boldsymbol{x})$$

- The Comma Notation for partial differentiation

Gradient operator: $\nabla = \dfrac{\partial}{\partial x_1}\mathbf{e}_1 + \dfrac{\partial}{\partial x_2}\mathbf{e}_2 + \dfrac{\partial}{\partial x_3}\mathbf{e}_3 = \dfrac{\partial}{\partial x_i}\mathbf{e}_i$

Laplacian operator: $\nabla^2 = \nabla \cdot \nabla = \dfrac{\partial}{\partial x_i}\mathbf{e}_i \cdot \dfrac{\partial}{\partial x_j}\mathbf{e}_j = \dfrac{\partial^2}{\partial x_i \partial x_i}$

Gradient of a scalar: $\nabla \phi = \dfrac{\partial \phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i$

Laplacian of a scalar: $\nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x_i \partial x_i} = \phi_{,ii}$

Tensor Calculus

Directional derivative: $\dfrac{d\phi}{ds} = \nabla\phi \cdot \mathbf{n} = \left(\dfrac{\partial\phi}{\partial x_1}\mathbf{e_1} + \dfrac{\partial\phi}{\partial x_2}\mathbf{e_2} + \dfrac{\partial\phi}{\partial x_3}\mathbf{e_3}\right) \cdot \left(\dfrac{dx_1}{ds}\mathbf{e_1} + \dfrac{dx_2}{ds}\mathbf{e_2} + \dfrac{dx_3}{ds}\mathbf{e_3}\right)$

Gradient of a vector: $\boxed{\mathbf{u}\overleftarrow{\nabla} = \left(u_i\mathbf{e}_i\right)\left(\dfrac{\overleftarrow{\partial}}{\partial x_j}\mathbf{e}_j\right)} = \dfrac{\partial u_i}{\partial x_j}\mathbf{e}_i\mathbf{e}_j = u_{i,j}\mathbf{e}_i\mathbf{e}_j$

Divergence of a vector: $\nabla \cdot \mathbf{u} = \left(\dfrac{\partial}{\partial x_j}\mathbf{e}_j\right) \cdot \left(u_i\mathbf{e}_i\right) = u_{i,j}\delta_{ij} = u_{i,i}$

Gradient of a tensor: $\boxed{\boldsymbol{\sigma}\overleftarrow{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\dfrac{\overleftarrow{\partial}}{\partial x_k}\mathbf{e}_k\right)} = \dfrac{\partial\sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$

Divergence of a tensor: $\nabla \cdot \boldsymbol{\sigma} = \left(\dfrac{\partial}{\partial x_k}\mathbf{e}_k\right) \cdot \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \dfrac{\partial\sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k \cdot \mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

Curl of a vector: $\nabla \times \mathbf{u} = \left(\dfrac{\partial}{\partial x_j}\mathbf{e}_j\right) \times \left(u_k\mathbf{e}_k\right) = \dfrac{\partial u_k}{\partial x_j}\left(\varepsilon_{ijk}\mathbf{e}_i\right) = \varepsilon_{ijk}u_{k,j}\mathbf{e}_i$

Laplacian of a vector: $\nabla^2\mathbf{u} = \dfrac{\partial^2}{\partial x_i\partial x_i}\left(u_k\mathbf{e}_k\right) = u_{k,ii}\mathbf{e}_k$

Useful Identities

$$\nabla(\phi\psi) = (\nabla\phi)\psi + \phi(\nabla\psi)$$

$$\nabla^2(\phi\psi) = (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi \cdot \nabla\psi$$

$$\nabla \cdot (\phi\boldsymbol{u}) = \nabla\phi \cdot \boldsymbol{u} + \phi(\nabla \cdot \boldsymbol{u})$$

$$\nabla \times (\phi\boldsymbol{u}) = \nabla\phi \times \boldsymbol{u} + \phi(\nabla \times \boldsymbol{u})$$

$$\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$$

$$\nabla \times \nabla\phi = 0$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi$$

$$\nabla \cdot \nabla \times \boldsymbol{u} = 0$$

$$\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2\boldsymbol{u}$$

$$\boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \frac{1}{2}\nabla(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \cdot \nabla\boldsymbol{u}$$

- All can be proved with Indicial and Comma Notations.

Sample Problem

- Scalar and vector field functions are given by
  $\phi = x^2 - y^2$ and $\mathbf{u} = 2x\mathbf{e_1} + 3yz\mathbf{e_2} + xy\mathbf{e_3}$.

  Calculate the following expressions:

  $\nabla\phi, \ \nabla^2\phi, \ \nabla \cdot \mathbf{u}, \ \nabla\mathbf{u}, \ \nabla \times \mathbf{u}.$

- Solution

$$\nabla\phi = 2x\mathbf{e_1} - 2y\mathbf{e_2}$$

$$\nabla^2\phi = 2 - 2 = 0$$

$$\nabla \cdot \mathbf{u} = u_{i,i} = 2 + 3z + 0 = 2 + 3z$$

$$\mathbf{u}\overset{\leftarrow}{\nabla} = u_{i,j}\mathbf{e_i}\mathbf{e_j} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3z & 3y \\ y & x & 0 \end{bmatrix}$$

$$\nabla \times \mathbf{u} = \frac{\partial}{\partial x_i}\mathbf{e_i} \times u_j\mathbf{e_j} = \varepsilon_{kij}u_{j,i}\mathbf{e_k}$$

$$= \mathbf{e_1}\left(u_{3,2} - u_{2,3}\right) + \mathbf{e_2}\left(u_{1,3} - u_{3,1}\right) + \mathbf{e_3}\left(u_{2,1} - u_{1,2}\right)$$

$$= \mathbf{e_1}\left(x - 3y\right) + \mathbf{e_2}\left(-y\right)$$

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Integral Theorems

- Gauss (Divergence) theorem

$$\iint_S \mathbf{u} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{u} dV;$$

$$\iint_S u_k n_k dS = \iiint_V u_{k,k} dV.$$

- Stokes theorem

$$\int_C \mathbf{u} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS;$$

$$\int_C u_k dx_k = \iint_S \varepsilon_{ijk} u_{k,j} n_i dS.$$

- Green's theorem on a plane

$$\int_C (u_1 dx_1 + u_2 dx_2) = \iint_S (u_{2,1} - u_{1,2}) dx_1 dx_2.$$

Cylindrical Coordinates

$$x = r\cos\theta,\, y = r\sin\theta,\, r = \sqrt{x^2 + y^2},\, \theta = \tan^{-1}\frac{y}{x}$$
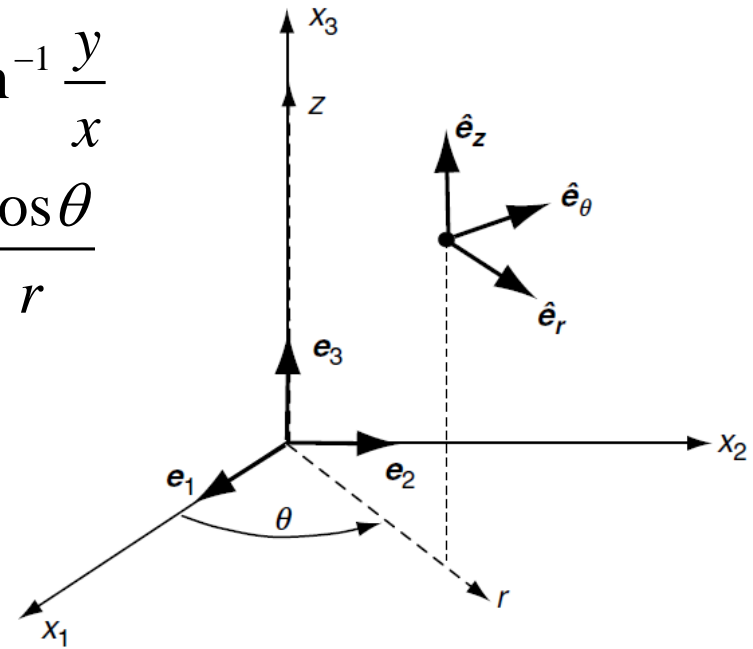
$$\frac{\partial r}{\partial x} = \cos\theta,\, \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r},\, \frac{\partial r}{\partial y} = \sin\theta,\, \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\begin{Bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{Bmatrix} = \begin{Bmatrix} \mathbf{e}_x\cos\theta + \mathbf{e}_y\sin\theta \\ -\mathbf{e}_x\sin\theta + \mathbf{e}_y\cos\theta \\ \mathbf{e}_z \end{Bmatrix}$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta,\, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$$

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Cylindrical Coordinates

$$\nabla_c = \mathbf{Q}\nabla = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} \dfrac{\partial}{\partial x} \\[2mm] \dfrac{\partial}{\partial y} \\[2mm] \dfrac{\partial}{\partial z} \end{array} \right\} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

$$\boxed{\mathbf{u}\bar{\nabla}_c = \left(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z \right)\left(\frac{\bar{\partial}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\bar{\partial}}{\partial\theta}\mathbf{e}_\theta + \frac{\bar{\partial}}{\partial z}\mathbf{e}_z \right)}$$

$$= \begin{bmatrix} \dfrac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial\theta} - u_\theta \right)\mathbf{e}_r\mathbf{e}_\theta + \dfrac{\partial u_r}{\partial z}\mathbf{e}_r\mathbf{e}_z \\[4mm] + \dfrac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\mathbf{e}_r + \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial\theta} \right)\mathbf{e}_\theta\mathbf{e}_\theta + \dfrac{\partial u_\theta}{\partial z}\mathbf{e}_\theta\mathbf{e}_z \\[4mm] + \dfrac{\partial u_z}{\partial r}\mathbf{e}_z\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\mathbf{e}_z\mathbf{e}_\theta + \dfrac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z \end{bmatrix}$$

Cylindrical Coordinates

- Double gradient of a scalar (symmetric as expected)

$$\nabla \phi = \mathbf{e}_r \frac{\partial \phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \mathbf{e}_z \frac{\partial \phi}{\partial z}$$

$$\Rightarrow (\nabla \phi) \overleftarrow{\nabla} = \begin{bmatrix} \dfrac{\partial^2 \phi}{\partial r^2} \mathbf{e}_r \mathbf{e}_r + \dfrac{\partial}{\partial r}\left(\dfrac{1}{r} \dfrac{\partial \phi}{\partial \theta} \right) \mathbf{e}_r \mathbf{e}_\theta + \dfrac{\partial^2 \phi}{\partial r \partial z} \mathbf{e}_r \mathbf{e}_z \\[4mm] + \dfrac{\partial}{\partial r}\left(\dfrac{1}{r} \dfrac{\partial \phi}{\partial \theta} \right) \mathbf{e}_\theta \mathbf{e}_r + \dfrac{1}{r}\left(\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r} \dfrac{\partial^2 \phi}{\partial \theta^2} \right) \mathbf{e}_\theta \mathbf{e}_\theta + \dfrac{1}{r} \dfrac{\partial^2 \phi}{\partial \theta \partial z} \mathbf{e}_\theta \mathbf{e}_z \\[4mm] + \dfrac{\partial^2 \phi}{\partial r \partial z} \mathbf{e}_z \mathbf{e}_r + \dfrac{1}{r} \dfrac{\partial^2 \phi}{\partial \theta \partial z} \mathbf{e}_z \mathbf{e}_\theta + \dfrac{\partial^2 \phi}{\partial z^2} \mathbf{e}_z \mathbf{e}_z \end{bmatrix}$$

$$= \nabla (\nabla \phi)$$

Cylindrical Coordinates

$$\nabla_c \cdot \mathbf{u} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \right)$$

$$= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla_c \times \mathbf{u} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \times \left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \right)$$

$$= \mathbf{e}_r \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) + \mathbf{e}_\theta \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \mathbf{e}_z \left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{1}{r} u_\theta \right)$$

$$\nabla_c^2 = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical Coordinates

$$\boldsymbol{\sigma}\bar{\nabla} = \begin{pmatrix} \sigma_r \mathbf{e}_r \mathbf{e}_r + \tau_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + \tau_{rz}\mathbf{e}_r\mathbf{e}_z + \tau_{\theta r}\mathbf{e}_\theta\mathbf{e}_r + \sigma_\theta\mathbf{e}_\theta\mathbf{e}_\theta \\ +\tau_{\theta z}\mathbf{e}_\theta\mathbf{e}_z + \tau_{zr}\mathbf{e}_z\mathbf{e}_r + \tau_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + \sigma_z\mathbf{e}_z\mathbf{e}_z \end{pmatrix} \left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z \right)$$

$$= \begin{aligned}
&\frac{\partial\sigma_r}{\partial r}\mathbf{e}_r\mathbf{e}_r\mathbf{e}_r + \frac{\partial\tau_{r\theta}}{\partial r}\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_r + \frac{\partial\tau_{rz}}{\partial r}\mathbf{e}_r\mathbf{e}_z\mathbf{e}_r + \frac{\partial\tau_{\theta r}}{\partial r}\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_r + \frac{\partial\sigma_\theta}{\partial r}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_r \\
&+\frac{\partial\tau_{\theta z}}{\partial r}\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_r + \frac{\partial\tau_{zr}}{\partial r}\mathbf{e}_z\mathbf{e}_r\mathbf{e}_r + \frac{\partial\tau_{z\theta}}{\partial r}\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_r + \frac{\partial\sigma_z}{\partial r}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_r \\
&+\left(\frac{1}{r}\frac{\partial\sigma_r}{\partial\theta} - \frac{\tau_{r\theta} + \tau_{\theta r}}{r}\right)\mathbf{e}_r\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r}\right)\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{rz}}{\partial\theta} - \frac{\tau_{\theta z}}{r}\right)\mathbf{e}_r\mathbf{e}_z\mathbf{e}_\theta \\
&+\left(\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_\theta \\
&+\left(\frac{1}{r}\frac{\partial\tau_{zr}}{\partial\theta} - \frac{\tau_{z\theta}}{r}\right)\mathbf{e}_z\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\tau_{zr}}{r}\right)\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial\sigma_z}{\partial\theta}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_\theta \\
&+\frac{\partial\sigma_r}{\partial z}\mathbf{e}_r\mathbf{e}_r\mathbf{e}_z + \frac{\partial\tau_{r\theta}}{\partial z}\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial\tau_{rz}}{\partial z}\mathbf{e}_r\mathbf{e}_z\mathbf{e}_z + \frac{\partial\tau_{\theta r}}{\partial z}\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_z + \frac{\partial\sigma_\theta}{\partial z}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_z \\
&+\frac{\partial\tau_{\theta z}}{\partial z}\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_z + \frac{\partial\tau_{zr}}{\partial z}\mathbf{e}_z\mathbf{e}_r\mathbf{e}_z + \frac{\partial\tau_{z\theta}}{\partial z}\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial\sigma_z}{\partial z}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_z
\end{aligned}$$

Cylindrical Coordinates

Gradient of a tensor: $\boxed{\boldsymbol{\sigma}\overleftarrow{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\frac{\overleftarrow{\partial}}{\partial x_k}\mathbf{e}_k\right) = \frac{\partial\sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k}$

Divergence of a tensor: $\nabla\cdot\boldsymbol{\sigma} = \left(\frac{\partial}{\partial x_k}\mathbf{e}_k\right)\cdot\left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \frac{\partial\sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k\cdot\mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

$\nabla\cdot\boldsymbol{\sigma} =$ contraction on the first and third index of $\boldsymbol{\sigma}\overleftarrow{\nabla}$

$\Rightarrow$
$$
\boxed{
\begin{aligned}
\nabla\cdot\boldsymbol{\sigma} = &\left(\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\partial\tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r}\right)\mathbf{e}_r \\
&+\left(\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial z} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r}\right)\mathbf{e}_\theta \\
&+\left(\frac{\partial\tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\sigma_z}{\partial z} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_z
\end{aligned}
}
$$

Cylindrical Coordinates

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_r}{\partial \theta^2} = \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\frac{\partial \mathbf{e}_r}{\partial \theta} = -\mathbf{e}_\theta$$

$$\frac{\partial}{\partial \theta}\left(u_r \mathbf{e}_r\right) = u_r \mathbf{e}_\theta + \mathbf{e}_r \frac{\partial u_r}{\partial \theta}, \frac{\partial^2}{\partial \theta^2}\left(u_r \mathbf{e}_r\right) = \mathbf{e}_r \frac{\partial^2 u_r}{\partial \theta^2} + 2\mathbf{e}_\theta \frac{\partial u_r}{\partial \theta} - u_r \mathbf{e}_r$$

$$\frac{\partial^2}{\partial \theta^2}\left(u_\theta \mathbf{e}_\theta\right) = \mathbf{e}_\theta \frac{\partial^2 u_\theta}{\partial \theta^2} + 2\frac{\partial u_\theta}{\partial \theta}\frac{\partial \mathbf{e}_\theta}{\partial \theta} + u_\theta \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = \mathbf{e}_\theta \frac{\partial^2 u_\theta}{\partial \theta^2} - 2\mathbf{e}_r \frac{\partial u_\theta}{\partial \theta} - u_\theta \mathbf{e}_\theta$$

$$\nabla_c^2 \mathbf{u} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z\right)$$

$$= \left(\nabla_c^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)\mathbf{e}_r + \left(\nabla_c^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)\mathbf{e}_\theta + \nabla_c^2 u_z \mathbf{e}_z$$

Cylindrical Coordinates

$$\boxed{\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_r}{\partial \theta^2} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\mathbf{e}_\theta}$$

$$\nabla_c^2 \boldsymbol{\sigma} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \left(\begin{array}{l} \sigma_r \mathbf{e}_r \mathbf{e}_r + \tau_{r\theta} \mathbf{e}_r \mathbf{e}_\theta + \tau_{rz} \mathbf{e}_r \mathbf{e}_z + \tau_{\theta r} \mathbf{e}_\theta \mathbf{e}_r + \sigma_\theta \mathbf{e}_\theta \mathbf{e}_\theta \\ +\tau_{\theta z} \mathbf{e}_\theta \mathbf{e}_z + \tau_{zr} \mathbf{e}_z \mathbf{e}_r + \tau_{z\theta} \mathbf{e}_z \mathbf{e}_\theta + \sigma_z \mathbf{e}_z \mathbf{e}_z \end{array} \right)$$

$$= \left[\nabla^2 \sigma_r - \frac{2}{r^2}\left(\sigma_r - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} \right) \right] \mathbf{e}_r \mathbf{e}_r + \left[\nabla^2 \sigma_\theta + \frac{2}{r^2}\left(\sigma_r - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} \right) \right] \mathbf{e}_\theta \mathbf{e}_\theta$$

$$+ \left[\nabla^2 \sigma_z \right] \mathbf{e}_z \mathbf{e}_z + \left[\nabla^2 \tau_{r\theta} + \frac{2}{r^2}\left(\frac{\partial \sigma_r}{\partial \theta} - \frac{\partial \sigma_\theta}{\partial \theta} - \tau_{r\theta} - \tau_{\theta r} \right) \right] \mathbf{e}_r \mathbf{e}_\theta + \left[\nabla^2 \tau_{rz} - \frac{1}{r^2}\left(\tau_{rz} + 2\frac{\partial \tau_{\theta z}}{\partial \theta} \right) \right] \mathbf{e}_r \mathbf{e}_z$$

$$+ \left[\nabla^2 \tau_{\theta r} + \frac{2}{r^2}\left(\frac{\partial \sigma_r}{\partial \theta} - \frac{\partial \sigma_\theta}{\partial \theta} - \tau_{r\theta} - \tau_{\theta r} \right) \right] \mathbf{e}_\theta \mathbf{e}_r + \left[\nabla^2 \tau_{\theta z} + \frac{1}{r^2}\left(2\frac{\partial \tau_{rz}}{\partial \theta} - \tau_{\theta z} \right) \right] \mathbf{e}_\theta \mathbf{e}_z$$

$$+ \left[\nabla^2 \tau_{zr} - \frac{1}{r^2}\left(\tau_{zr} + 2\frac{\partial \tau_{z\theta}}{\partial \theta} \right) \right] \mathbf{e}_z \mathbf{e}_r + \left[\nabla^2 \tau_{z\theta} + \frac{1}{r^2}\left(2\frac{\partial \tau_{zr}}{\partial \theta} - \tau_{z\theta} \right) \right] \mathbf{e}_z \mathbf{e}_\theta$$

- The Laplacian is symmetric if the tensor is.

Spherical Coordinates

$$z = R\cos\varphi, r = R\sin\varphi, R = \sqrt{z^2 + r^2}, \varphi = \tan^{-1}\frac{r}{z}$$

$$\frac{\partial R}{\partial z} = \cos\varphi, \frac{\partial\varphi}{\partial z} = -\frac{\sin\varphi}{R}, \frac{\partial R}{\partial r} = \sin\varphi, \frac{\partial\varphi}{\partial r} = \frac{\cos\varphi}{R}$$
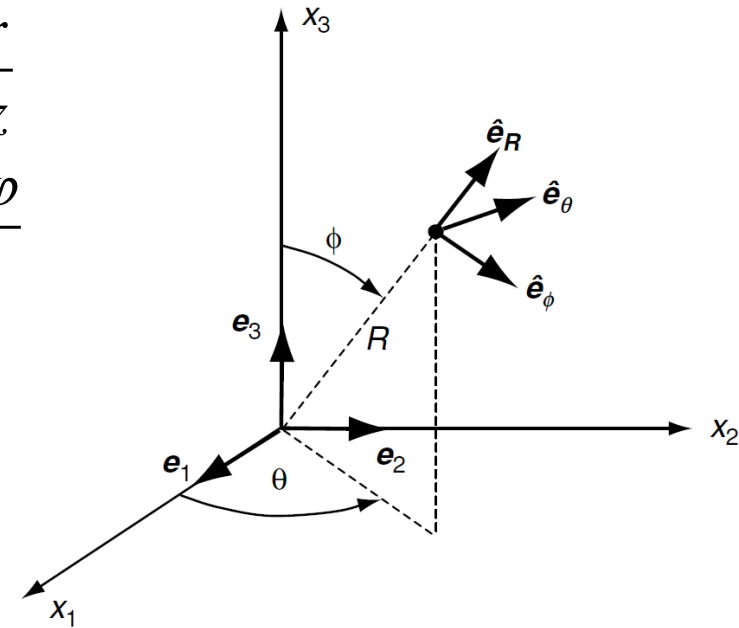
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial R}\frac{\partial R}{\partial z} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial z} = \cos\varphi\frac{\partial}{\partial R} - \frac{\sin\varphi}{R}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial R}\frac{\partial R}{\partial r} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial r} = \sin\varphi\frac{\partial}{\partial R} + \frac{\cos\varphi}{R}\frac{\partial}{\partial\varphi}$$

$$\begin{Bmatrix} \mathbf{e}_R \\ \mathbf{e}_\varphi \\ \mathbf{e}_\theta \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_z \\ \mathbf{e}_r \\ \mathbf{e}_\theta \end{Bmatrix} = \begin{Bmatrix} \sin\varphi\mathbf{e}_r + \cos\varphi\mathbf{e}_z \\ \cos\varphi\mathbf{e}_r - \sin\varphi\mathbf{e}_z \\ \mathbf{e}_\theta \end{Bmatrix}$$

$$\frac{\partial\mathbf{e}_R}{\partial\varphi} = \mathbf{e}_\varphi, \frac{\partial\mathbf{e}_\varphi}{\partial\varphi} = -\mathbf{e}_R, \qquad \frac{\partial\mathbf{e}_R}{\partial\theta} = \sin\varphi\mathbf{e}_\theta, \frac{\partial\mathbf{e}_\varphi}{\partial\theta} = \cos\varphi\mathbf{e}_\theta, \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\sin\varphi\mathbf{e}_R - \cos\varphi\mathbf{e}_\varphi$$

$$\mathbf{e}_r = \sin\varphi\mathbf{e}_R + \cos\varphi\mathbf{e}_\varphi$$

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Spherical Coordinates

$$\nabla_s = \mathbf{Q}\nabla_c = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial R} \\ \dfrac{1}{R}\dfrac{\partial}{\partial\varphi} \\ \dfrac{1}{R\sin\varphi}\dfrac{\partial}{\partial\theta} \end{Bmatrix}$$

$$\boxed{\mathbf{u}\bar{\nabla}_s = \left(u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi + u_\theta\mathbf{e}_\theta\right)\left(\frac{\bar{\partial}}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\bar{\partial}}{\partial\varphi}\mathbf{e}_\varphi + \frac{1}{R\sin\varphi}\frac{\bar{\partial}}{\partial\theta}\mathbf{e}_\theta\right)}$$

$$= \begin{bmatrix} \dfrac{\partial u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \left(\dfrac{1}{R}\dfrac{\partial u_R}{\partial\varphi} - \dfrac{u_\varphi}{R}\right)\mathbf{e}_R\mathbf{e}_\varphi + \left(\dfrac{1}{R\sin\varphi}\dfrac{\partial u_R}{\partial\theta} - \dfrac{u_\theta}{R}\right)\mathbf{e}_R\mathbf{e}_\theta \\[2em] + \dfrac{\partial u_\varphi}{\partial R}\mathbf{e}_\varphi\mathbf{e}_R + \left(\dfrac{u_R}{R} + \dfrac{1}{R}\dfrac{\partial u_\varphi}{\partial\varphi}\right)\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(-\dfrac{\cot\varphi\, u_\theta}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\varphi}{\partial\theta}\right)\mathbf{e}_\varphi\mathbf{e}_\theta \\[2em] + \dfrac{\partial u_\theta}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \dfrac{1}{R}\dfrac{\partial u_\theta}{\partial\varphi}\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\dfrac{u_R}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{\cot\varphi\, u_\varphi}{R}\right)\mathbf{e}_\theta\mathbf{e}_\theta \end{bmatrix}$$

Spherical Coordinates

$$\nabla_s \cdot \mathbf{u} = \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\varphi \frac{1}{R} \frac{\partial}{\partial \varphi} + \mathbf{e}_\theta \frac{1}{R \sin \varphi} \frac{\partial}{\partial \theta} \right) \cdot \left(u_R \mathbf{e}_R + u_\varphi \mathbf{e}_\varphi + u_\theta \mathbf{e}_\theta \right)$$

$$= \frac{\partial u_R}{\partial R} + \frac{2u_R}{R} + \frac{\cot\varphi u_\varphi}{R} + \frac{1}{R} \frac{\partial u_\varphi}{\partial \varphi} + \frac{1}{R \sin \varphi} \frac{\partial u_\theta}{\partial \theta}$$

$$\nabla_s \times \mathbf{u} = \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\varphi \frac{1}{R} \frac{\partial}{\partial \varphi} + \mathbf{e}_\theta \frac{1}{R \sin \varphi} \frac{\partial}{\partial \theta} \right) \times \left(u_R \mathbf{e}_R + u_\varphi \mathbf{e}_\varphi + u_\theta \mathbf{e}_\theta \right)$$

$$= \left(-\frac{1}{R} \frac{\partial u_\theta}{\partial \varphi} - \frac{\cot\varphi u_\theta}{R} + \frac{1}{R \sin \varphi} \frac{\partial u_\varphi}{\partial \theta} \right) \mathbf{e}_R$$

$$+ \left(-\frac{1}{R \sin \varphi} \frac{\partial u_R}{\partial \theta} + \frac{u_\theta}{R} + \frac{\partial u_\theta}{\partial R} \right) \mathbf{e}_\varphi + \left(\frac{1}{R} \frac{\partial u_R}{\partial \varphi} - \frac{u_\varphi}{R} - \frac{\partial u_\varphi}{\partial R} \right) \mathbf{e}_\theta$$

$$\nabla_s^2 = \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\varphi \frac{1}{R} \frac{\partial}{\partial \varphi} + \mathbf{e}_\theta \frac{1}{R \sin \varphi} \frac{\partial}{\partial \theta} \right) \cdot \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\varphi \frac{1}{R} \frac{\partial}{\partial \varphi} + \mathbf{e}_\theta \frac{1}{R \sin \varphi} \frac{\partial}{\partial \theta} \right)$$

$$= \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R} + \frac{\cot\varphi}{R^2} \frac{\partial}{\partial \varphi} + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{R^2 \sin^2 \varphi} \frac{\partial^2}{\partial \theta^2}$$

Spherical Coordinates

$$\boldsymbol{\sigma}\bar{\nabla} = \begin{pmatrix} \sigma_R\mathbf{e}_R\mathbf{e}_R + \tau_{R\varphi}\mathbf{e}_R\mathbf{e}_\varphi + \tau_{R\theta}\mathbf{e}_R\mathbf{e}_\theta + \tau_{\varphi R}\mathbf{e}_\varphi\mathbf{e}_R + \sigma_\varphi\mathbf{e}_\varphi\mathbf{e}_\varphi \\ +\tau_{\varphi\theta}\mathbf{e}_\varphi\mathbf{e}_\theta + \tau_{\theta R}\mathbf{e}_\theta\mathbf{e}_R + \tau_{\theta\varphi}\mathbf{e}_\theta\mathbf{e}_\varphi + \sigma_\theta\mathbf{e}_\theta\mathbf{e}_\theta \end{pmatrix}\left(\frac{\partial}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi + \frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\mathbf{e}_\theta \right)$$

$$
\begin{aligned}
= \ & \frac{\partial\sigma_R}{\partial R}\mathbf{e}_R\mathbf{e}_R\mathbf{e}_R + \frac{\partial\tau_{R\varphi}}{\partial R}\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial\tau_{R\theta}}{\partial R}\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial\tau_{\varphi R}}{\partial R}\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_R + \frac{\partial\sigma_\varphi}{\partial R}\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial\tau_{\varphi\theta}}{\partial R}\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial\tau_{\theta R}}{\partial R}\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_R \\[2mm]
& + \frac{\partial\tau_{\theta\varphi}}{\partial R}\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial\sigma_\theta}{\partial R}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_R + \left(\frac{1}{R}\frac{\partial\sigma_R}{\partial\varphi} - \frac{\tau_{R\varphi}+\tau_{\varphi R}}{R} \right)\mathbf{e}_R\mathbf{e}_R\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial\tau_{R\varphi}}{\partial\varphi} + \frac{\sigma_R-\sigma_\varphi}{R} \right)\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial\tau_{R\theta}}{\partial\varphi} - \frac{\tau_{\varphi\theta}}{R} \right)\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_\varphi \\[2mm]
& + \left(\frac{1}{R}\frac{\partial\tau_{\varphi R}}{\partial\varphi} + \frac{\sigma_R-\sigma_\varphi}{R} \right)\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi} + \frac{\tau_{R\varphi}+\tau_{\varphi R}}{R} \right)\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi} + \frac{\tau_{R\theta}}{R} \right)\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial\tau_{\theta R}}{\partial\varphi} - \frac{\tau_{\theta\varphi}}{R} \right)\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_\varphi \\[2mm]
& + \left(\frac{1}{R}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi} + \frac{\tau_{\theta R}}{R} \right)\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_\varphi + \frac{1}{R}\frac{\partial\sigma_\theta}{\partial\varphi}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\frac{1}{R\sin\varphi}\frac{\partial\sigma_R}{\partial\theta} - \frac{\tau_{R\theta}+\tau_{\theta R}}{R} \right)\mathbf{e}_R\mathbf{e}_R\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{R\varphi}}{\partial\theta} - \frac{\cot\varphi\,\tau_{R\theta}+\tau_{\theta\varphi}}{R} \right)\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_\theta \\[2mm]
& + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{R\theta}}{\partial\theta} + \frac{\sigma_R-\sigma_\theta+\cot\varphi\,\tau_{R\varphi}}{R} \right)\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{\varphi R}}{\partial\theta} - \frac{\cot\varphi\,\tau_{\theta R}+\tau_{\varphi\theta}}{R} \right)\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_\theta \\[2mm]
& + \left(\frac{1}{R\sin\varphi}\frac{\partial\sigma_\varphi}{\partial\theta} - \frac{\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R} \right)\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{\varphi\theta}}{\partial\theta} - \frac{\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)-\tau_{\varphi R}}{R} \right)\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_\theta \\[2mm]
& + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta R}}{\partial\theta} + \frac{\sigma_R-\sigma_\theta}{R} + \frac{\cot\varphi\,\tau_{\varphi R}}{R} \right)\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta} + \frac{\tau_{R\varphi}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R} \right)\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_\theta \\[2mm]
& + \left(\frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\tau_{R\theta}+\tau_{\theta R}+\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R} \right)\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\theta
\end{aligned}
$$

Spherical Coordinates

Gradient of a tensor: $\boldsymbol{\sigma}\overset{\leftarrow}{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\dfrac{\overset{\leftarrow}{\partial}}{\partial x_k}\mathbf{e}_k\right) = \dfrac{\partial\sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$

Divergence of a tensor: $\nabla\cdot\boldsymbol{\sigma} = \left(\dfrac{\partial}{\partial x_k}\mathbf{e}_k\right)\cdot\left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \dfrac{\partial\sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k\cdot\mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

$\nabla\cdot\boldsymbol{\sigma}$ = contraction on the first and third index of $\boldsymbol{\sigma}\overset{\leftarrow}{\nabla}$

$$\Rightarrow \begin{aligned} \nabla\cdot\boldsymbol{\sigma} = &\left(\frac{\partial\sigma_R}{\partial R} + \frac{1}{R}\frac{\partial\tau_{\varphi R}}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta R}}{\partial\theta} + \frac{2\sigma_R - \sigma_\theta - \sigma_\varphi + \cot\varphi\,\tau_{\varphi R}}{R}\right)\mathbf{e}_R \\ + &\left(\frac{\partial\tau_{R\varphi}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta} + \frac{2\tau_{R\varphi} + \tau_{\varphi R} - \cot\varphi\left(\sigma_\theta - \sigma_\varphi\right)}{R}\right)\mathbf{e}_\varphi \\ + &\left(\frac{\partial\tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{R\theta} + \tau_{\theta R} + \cot\varphi\left(\tau_{\theta\varphi} + \tau_{\varphi\theta}\right)}{R}\right)\mathbf{e}_\theta \end{aligned}$$

Spherical Coordinates

$$\frac{\partial \mathbf{e}_R}{\partial \varphi} = \mathbf{e}_\varphi, \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_R, \frac{\partial \mathbf{e}_R}{\partial \theta} = \sin\varphi\mathbf{e}_\theta, \frac{\partial \mathbf{e}_\varphi}{\partial \theta} = \cos\varphi\mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -sin\varphi\mathbf{e}_R - cos\varphi\mathbf{e}_\varphi$$

$$\frac{\partial^2 \mathbf{e}_R}{\partial \varphi^2} = -\mathbf{e}_R, \frac{\partial^2 \mathbf{e}_\varphi}{\partial \varphi^2} = -\mathbf{e}_\varphi$$

$$\frac{\partial^2 \mathbf{e}_R}{\partial \theta^2} = -sin^2\varphi\mathbf{e}_R - \sin\varphi cos\varphi\mathbf{e}_\varphi, \frac{\partial^2 \mathbf{e}_\varphi}{\partial \theta^2} = -sin\varphi\cos\varphi\mathbf{e}_R - cos^2\varphi\mathbf{e}_\varphi, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\mathbf{e}_\theta$$

$$\nabla_s^2 \mathbf{u} = \left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{\cot\varphi}{R^2}\frac{\partial}{\partial \varphi} + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{R^2\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} \right)\left(u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi + u_\theta\mathbf{e}_\theta \right)$$

$$= \left(\nabla_s^2 u_R - \frac{2u_R}{R^2} - \frac{2\cot\varphi u_\varphi}{R^2} - \frac{2}{R^2}\frac{\partial u_\varphi}{\partial \varphi} - \frac{2}{R^2\sin\varphi}\frac{\partial u_\theta}{\partial \theta} \right)\mathbf{e}_R$$

$$+ \left(\nabla_s^2 u_\varphi + \frac{2}{R^2}\frac{\partial u_R}{\partial \varphi} - \frac{u_\varphi}{R^2\sin^2\varphi} - \frac{2\cot\varphi}{R^2\sin\varphi}\frac{\partial u_\theta}{\partial \theta} \right)\mathbf{e}_\varphi$$

$$+ \left(\nabla_s^2 u_\theta - \frac{u_\theta}{R^2\sin^2\varphi} + \frac{2}{R^2\sin\varphi}\frac{\partial u_R}{\partial \theta} + \frac{2\cot\varphi}{R^2\sin\varphi}\frac{\partial u_\varphi}{\partial \theta} \right)\mathbf{e}_\theta$$

Outline

- Scalar, vector and matrix（标量、矢量、矩阵）
- Indicial notation and summation convention（指标记法与求和约定）
- Kronecker delta（克罗内克δ）
- Levi-Civita symbol（轮换记号）
- Coordinate transformation（坐标变换）
- Tensor（张量）
- Principal values and directions（特征值与特征向量）
- Tensor algebra（张量代数）
- Tensor calculus（张量微分）
- Integral theorems（积分定理）
- Cylindrical and Spherical Coordinates（柱与球坐标）