J.R. Barber

Solid Mechanics and its Applications

Elasticity

Springer

Elasticity

SOLID MECHANICS AND ITS APPLICATIONS
Volume 172

Aims and Scope of the Series

The fundamental questions arising in mechanics are: *Why?*, *How?*, and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies: vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

For other titles published in this series, go to
www.springer.com/series/6557

J.R. Barber

Elasticity

3rd Revised Edition

Springer

J.R. Barber
Department of Mechanical Engineering
and Applied Mechanics
University of Michigan
Ann Arbor, MI 48109-2125
USA
jbarber@umich.edu

ISSN 0925-0042
ISBN 978-90-481-3808-1 e-ISBN 978-90-481-3809-8
DOI 10.1007/978-90-481-3809-8
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009940788

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Part I GENERAL CONSIDERATIONS

1 Introduction 3
1.1 Notation for stress and displacement 4
1.1.1 Stress 4
1.1.2 Index and vector notation and the summation convention 6
1.1.3 Vector operators in index notation 7
1.1.4 Vectors, tensors and transformation rules 8
1.1.5 Principal stresses and Von Mises stress 12
1.1.6 Displacement 13
1.2 Strains and their relation to displacements 14
1.2.1 Tensile strain 14
1.2.2 Rotation and shear strain 16
1.2.3 Transformation of coördinates 17
1.2.4 Definition of shear strain 19
1.3 Stress-strain relations 20
1.3.1 Lamé's constants 22
1.3.2 Dilatation and bulk modulus 22
PROBLEMS 23

2 Equilibrium and compatibility 25
2.1 Equilibrium equations 25
2.2 Compatibility equations 26
2.2.1 The significance of the compatibility equations 27
2.3 Equilibrium equations in terms of displacements 31
PROBLEMS 32

Part II TWO-DIMENSIONAL PROBLEMS

3 Plane strain and plane stress 37
3.1 Plane strain 37
3.1.1 The corrective solution 39
3.1.2 Saint-Venant's principle 40
3.2 Plane stress 40
3.2.1 Generalized plane stress 42
3.2.2 Relationship between plane stress and plane strain 42
PROBLEMS 43

4 Stress function formulation 45
4.1 The concept of a scalar stress function 45
4.2 Choice of a suitable form 46
4.3 The Airy stress function 46
4.3.1 Transformation of coördinates 47
4.3.2 Non-zero body forces 48
4.4 The governing equation 48
4.4.1 The compatibility condition 48
4.4.2 Method of solution 49
4.4.3 Reduced dependence on elastic constants 50
PROBLEMS 51

5 Problems in rectangular coördinates 55
5.1 Biharmonic polynomial functions 55
5.1.1 Second and third degree polynomials 57
5.2 Rectangular beam problems 59
5.2.1 Bending of a beam by an end load 59
5.2.2 Higher order polynomials — a general strategy 61
5.2.3 Manual solutions — symmetry considerations 66
5.3 Fourier series and transform solutions 68
5.3.1 Choice of form 69
5.3.2 Fourier transforms 72
PROBLEMS 74

6 End effects 77
6.1 Decaying solutions 77
6.2 The corrective solution 78
6.2.1 Separated-variable solutions 79
6.2.2 The eigenvalue problem 80
6.3 Other Saint-Venant problems 84
6.4 Mathieu's solution 84
PROBLEMS 88

7 Body forces ... 91
7.1 Stress function formulation ... 91
7.1.1 Conservative vector fields ... 92
7.1.2 The compatibility condition ... 93
7.2 Particular cases ... 93
7.2.1 Gravitational loading ... 94
7.2.2 Inertia forces ... 94
7.2.3 Quasi-static problems ... 94
7.2.4 Rigid-body kinematics ... 95
7.3 Solution for the stress function ... 97
7.3.1 The rotating rectangular beam ... 97
7.3.2 Solution of the governing equation ... 99
7.4 Rotational acceleration ... 100
7.4.1 The circular disk ... 100
7.4.2 The rectangular bar ... 102
7.4.3 Weak boundary conditions and the equation of motion ... 105
PROBLEMS ... 105

8 Problems in polar coördinates ... 109
8.1 Expressions for stress components ... 109
8.2 Strain components ... 111
8.3 Fourier series expansion ... 111
8.3.1 Satisfaction of boundary conditions ... 112
8.3.2 Circular hole in a shear field ... 113
8.3.3 Degenerate cases ... 116
8.4 The Michell solution ... 118
8.4.1 Hole in a tensile field ... 119
PROBLEMS ... 121

9 Calculation of displacements ... 123
9.1 The cantilever with an end load ... 123
9.1.1 Rigid-body displacements and end conditions ... 125
9.1.2 Deflection of the free end ... 126
9.2 The circular hole ... 127
9.3 Displacements for the Michell solution ... 129
9.3.1 Equilibrium considerations ... 129
9.3.2 The cylindrical pressure vessel ... 131
PROBLEMS ... 132

10 Curved beam problems ... 135
10.1 Loading at the ends ... 135
10.1.1 Pure bending ... 135
10.1.2 Force transmission ... 138
10.2 Eigenvalues and eigenfunctions ... 139
10.3 The inhomogeneous problem ... 140

10.3.1 Beam with sinusoidal loading . . . 140
10.3.2 The near-singular problem . . . 143
10.4 Some general considerations . . . 146
10.4.1 Conclusions . . . 146
PROBLEMS . . . 147

11 Wedge problems . . . 149
11.1 Power law tractions . . . 149
11.1.1 Uniform tractions . . . 150
11.1.2 The rectangular body revisited . . . 152
11.1.3 More general uniform loading . . . 153
11.1.4 Eigenvalues for the wedge angle . . . 154
11.2 Williams' asymptotic method . . . 154
11.2.1 Acceptable singularities . . . 155
11.2.2 Eigenfunction expansion . . . 157
11.2.3 Nature of the eigenvalues . . . 159
11.2.4 The singular stress fields . . . 162
11.2.5 Other geometries . . . 163
11.3 General loading of the faces . . . 165
PROBLEMS . . . 167

12 Plane contact problems . . . 171
12.1 Self-similarity . . . 171
12.2 The Flamant Solution . . . 172
12.3 The half-plane . . . 173
12.3.1 The normal force F_y . . . 173
12.3.2 The tangential force F_x . . . 174
12.3.3 Summary . . . 175
12.4 Distributed normal tractions . . . 175
12.5 Frictionless contact problems . . . 177
12.5.1 Method of solution . . . 178
12.5.2 The flat punch . . . 179
12.5.3 The cylindrical punch (Hertz problem) . . . 181
12.6 Problems with two deformable bodies . . . 183
12.7 Uncoupled problems . . . 186
12.7.1 Contact of cylinders . . . 186
12.8 Combined normal and tangential loading . . . 187
12.8.1 Cattaneo and Mindlin's problem . . . 188
12.8.2 Steady rolling: Carter's solution . . . 191
PROBLEMS . . . 194

13 Forces dislocations and cracks 199
13.1 The Kelvin solution 199
13.1.1 Body force problems 201
13.2 Dislocations 202
13.2.1 Dislocations in Materials Science 204
13.2.2 Similarities and differences 205
13.2.3 Dislocations as Green's functions 206
13.2.4 Stress concentrations 206
13.3 Crack problems 207
13.3.1 Linear Elastic Fracture Mechanics 207
13.3.2 Plane crack in a tensile field 209
13.3.3 Energy release rate 212
13.4 Method of images 213
PROBLEMS 216

14 Thermoelasticity 219
14.1 The governing equation 219
14.2 Heat conduction 221
14.3 Steady-state problems 222
14.3.1 Dundurs' Theorem 223
PROBLEMS 224

15 Antiplane shear 227
15.1 Transformation of coordinates 228
15.2 Boundary conditions 228
15.3 The rectangular bar 229
15.4 The concentrated line force 230
15.5 The screw dislocation 231
PROBLEMS 232

Part III END LOADING OF THE PRISMATIC BAR

16 Torsion of a prismatic bar 241
16.1 Prandtl's stress function 242
16.1.1 Solution of the governing equation 243
16.2 The membrane analogy 245
16.3 Thin-walled open sections 245
16.4 The rectangular bar 247
16.5 Multiply connected (closed) sections 249
16.5.1 Thin-walled closed sections 250
PROBLEMS 251

17 Shear of a prismatic bar 259
17.1 The semi-inverse method 259
17.2 Stress function formulation 260
17.3 The boundary condition 261
17.3.1 Integrability 261
17.3.2 Relation to the torsion problem 262
17.4 Methods of solution 262
17.4.1 The circular bar 263
17.4.2 The rectangular bar 264
PROBLEMS 266

Part IV COMPLEX VARIABLE FORMULATION

18 Preliminary mathematical results 273
18.1 Holomorphic functions 274
18.2 Harmonic functions 275
18.3 Biharmonic functions 275
18.4 Expressing real harmonic and biharmonic functions in complex form 276
18.4.1 Biharmonic functions 277
18.5 Line integrals 278
18.5.1 The residue theorem 279
18.5.2 The Cauchy integral theorem 281
18.6 Solution of harmonic boundary value problems 283
18.6.1 Direct method for the interior problem for a circle 283
18.6.2 Direct method for the exterior problem for a circle 285
18.6.3 The half plane 286
18.7 Conformal mapping 288
PROBLEMS 290

19 Application to elasticity problems 293
19.1 Representation of vectors 293
19.1.1 Transformation of coördinates 294
19.2 The antiplane problem 295
19.2.1 Solution of antiplane boundary-value problems 296
19.3 In-plane deformations 298
19.3.1 Expressions for stresses 299
19.3.2 Rigid-body displacement 300
19.4 Relation between the Airy stress function and the complex potentials 301
19.5 Boundary tractions 303
19.5.1 Equilibrium considerations 305
19.6 Boundary-value problems 305
19.6.1 Solution of the interior problem for the circle 306

19.6.2 Solution of the exterior problem for the circle 308
19.7 Conformal mapping for in-plane problems 311
19.7.1 The elliptical hole 313
PROBLEMS 316

Part V THREE DIMENSIONAL PROBLEMS

20 Displacement function solutions 321
20.1 The strain potential 321
20.2 The Galerkin vector 322
20.3 The Papkovich-Neuber solution 324
20.3.1 Change of coördinate system 325
20.4 Completeness and uniqueness 325
20.4.1 Methods of partial integration 327
20.5 Body forces 329
20.5.1 Conservative body force fields 330
20.5.2 Non-conservative body force fields 330
PROBLEMS 331

21 The Boussinesq potentials 333
21.1 Solution A : The strain potential 334
21.2 Solution B 334
21.3 Solution E : Rotational deformation 335
21.4 Other coördinate systems 336
21.4.1 Cylindrical polar coördinates 336
21.4.2 Spherical polar coördinates 337
21.5 Solutions obtained by superposition 339
21.5.1 Solution F : Frictionless isothermal contact problems . . . 339
21.5.2 Solution G: The surface free of normal traction 340
21.6 A three-dimensional complex variable solution 342
PROBLEMS 343

22 Thermoelastic displacement potentials 347
22.1 Plane problems 349
22.1.1 Axisymmetric problems for the cylinder 349
22.1.2 Steady-state plane problems 350
22.1.3 Heat flow perturbed by a circular hole 351
22.1.4 Plane stress 354
22.2 The method of strain suppression 354
22.3 Steady-state temperature : Solution T 355
22.3.1 Thermoelastic plane stress 357
PROBLEMS 358

23 Singular solutions . . . 363
23.1 The source solution . . . 363
23.1.1 The centre of dilatation . . . 364
23.1.2 The Kelvin solution . . . 365
23.2 Dimensional considerations . . . 366
23.2.1 The Boussinesq solution . . . 367
23.3 Other singular solutions . . . 370
23.4 Image methods . . . 372
23.4.1 The traction-free half space . . . 373
PROBLEMS . . . 375

24 Spherical harmonics . . . 377
24.1 Fourier series solution . . . 378
24.2 Reduction to Legendre's equation . . . 378
24.3 Axisymmetric potentials and Legendre polynomials . . . 379
24.3.1 Singular spherical harmonics . . . 380
24.3.2 Special cases . . . 380
24.4 Non-axisymmetric harmonics . . . 381
24.5 Cartesian and cylindrical polar coördinates . . . 381
24.6 Harmonic potentials with logarithmic terms . . . 382
24.6.1 Logarithmic functions for cylinder problems . . . 384
24.7 Non-axisymmetric cylindrical potentials . . . 385
24.8 Spherical harmonics in complex notation . . . 387
24.8.1 Bounded cylindrical harmonics . . . 388
24.8.2 Singular cylindrical harmonics . . . 390
PROBLEMS . . . 390

25 Cylinders and circular plates . . . 391
25.1 Axisymmetric problems for cylinders . . . 391
25.1.1 The solid cylinder . . . 392
25.1.2 The hollow cylinder . . . 396
25.2 Axisymmetric circular plates . . . 397
25.2.1 Uniformly loaded plate on a simple support . . . 398
25.3 Non-axisymmetric problems . . . 400
25.3.1 Cylindrical cantilever with an end load . . . 400
PROBLEMS . . . 402

26 Problems in spherical coördinates . . . 405
26.1 Solid and hollow spheres . . . 405
26.1.1 The solid sphere in torsion . . . 406
26.1.2 Spherical hole in a tensile field . . . 406
26.2 Conical bars . . . 408
26.2.1 Conical bar transmitting an axial force . . . 409
26.2.2 Inhomogeneous problems . . . 411
26.2.3 Non-axisymmetric problems . . . 413

PROBLEMS . . . 414

27 Axisymmetric torsion . . . 419
27.1 The transmitted torque . . . 420
27.2 The governing equation . . . 420
27.3 Solution of the governing equation . . . 421
27.4 The displacement field . . . 423
27.5 Cylindrical and conical bars . . . 424
27.5.1 The centre of rotation . . . 425
27.6 The Saint Venant problem . . . 426
PROBLEMS . . . 427

28 The prismatic bar . . . 429
28.1 Power series solutions . . . 429
28.1.1 Superposition by differentiation . . . 430
28.1.2 The problems $\mathcal{P}_0$ and $\mathcal{P}_1$ . . . 431
28.1.3 Properties of the solution to $\mathcal{P}_m$ . . . 432
28.2 Solution of $\mathcal{P}_m$ by integration . . . 432
28.3 The integration process . . . 433
28.4 The two-dimensional problem $\mathcal{P}_0$ . . . 435
28.5 Problem $\mathcal{P}_1$ . . . 436
28.5.1 The corrective antiplane solution . . . 437
28.5.2 The circular bar . . . 438
28.6 The corrective in-plane solution . . . 439
28.7 Corrective solutions using real stress functions . . . 439
28.7.1 Airy function . . . 440
28.7.2 Prandtl function . . . 440
28.8 Solution procedure . . . 441
28.9 Example . . . 442
28.9.1 Problem $\mathcal{P}_1$ . . . 442
28.9.2 Problem $\mathcal{P}_2$ . . . 443
28.9.3 End conditions . . . 445
PROBLEMS . . . 447

29 Frictionless contact . . . 449
29.1 Boundary conditions . . . 449
29.1.1 Mixed boundary-value problems . . . 450
29.2 Determining the contact area . . . 451
29.3 Contact problems involving adhesive forces . . . 454

30 The boundary-value problem . . . 459
30.1 Hankel transform methods . . . 459
30.2 Collins' Method . . . 460
30.2.1 Indentation by a flat punch . . . 460
30.2.2 Integral representation . . . 462

30.2.3 Basic forms and surface values 463
30.2.4 Reduction to an Abel equation 465
30.2.5 Smooth contact problems 468
30.2.6 Choice of form 471
30.3 Non-axisymmetric problems 471
30.3.1 The full stress field 473
PROBLEMS 475

31 The penny-shaped crack 479
31.1 The penny-shaped crack in tension 479
31.2 Thermoelastic problems 482
PROBLEMS 485

32 The interface crack 487
32.1 The uncracked interface 487
32.2 The corrective solution 489
32.2.1 Global conditions 489
32.2.2 Mixed conditions 490
32.3 The penny-shaped crack in tension 492
32.3.1 Reduction to a single equation 494
32.3.2 Oscillatory singularities 495
32.4 The contact solution 496
32.5 Implications for Fracture Mechanics 497

33 Variational methods 499
33.1 Strain energy 500
33.1.1 Strain energy density 500
33.2 Conservation of energy 501
33.3 Potential energy of the external forces 501
33.4 Theorem of minimum total potential energy 502
33.5 Approximate solutions — the Rayleigh-Ritz method 503
33.6 Castigliano's second theorem 506
33.7 Approximations using Castigliano's second theorem 507
33.7.1 The torsion problem 507
33.7.2 The in-plane problem 509
33.8 Uniqueness and existence of solution 511
33.8.1 Singularities 512
PROBLEMS 514

34 The reciprocal theorem 517
34.1 Maxwell's Theorem 517
34.2 Betti's Theorem 518
34.3 Use of the theorem 519
34.3.1 A tilted punch problem 520
34.3.2 Indentation of a half-space 523

34.4 Thermoelastic problems 524
PROBLEMS 526

A Using Maple and Mathematica 529

Preface

The subject of Elasticity can be approached from several points of view, depending on whether the practitioner is principally interested in the mathematical structure of the subject or in its use in engineering applications and, in the latter case, whether essentially numerical or analytical methods are envisaged as the solution method. My first introduction to the subject was in response to a need for information about a specific problem in Tribology. As a practising Engineer with a background only in elementary Mechanics of Materials, I approached that problem initially using the concepts of concentrated forces and superposition. Today, with a rather more extensive knowledge of analytical techniques in Elasticity, I still find it helpful to go back to these roots in the elementary theory and think through a problem physically as well as mathematically, whenever some new and unexpected feature presents difficulties in research. This way of thinking will be found to permeate this book. My engineering background will also reveal itself in a tendency to work examples through to final expressions for stresses and displacements, rather than leave the derivation at a point where the remaining manipulations would be mathematically routine.

The first edition of this book, published in 1992, was based on a one semester graduate course on Linear Elasticity that I have taught at the University of Michigan since 1983. In two subsequent revisions, the amount of material has almost doubled and the character of the book has necessarily changed, but I remain committed to my original objective of writing for those who wish to find the solution of specific practical engineering problems. With this in mind, I have endeavoured to keep to a minimum any dependence on previous knowledge of Solid Mechanics, Continuum Mechanics or Mathematics. Most of the text should be readily intelligible to a reader with an undergraduate background of one or two courses in elementary Mechanics of Materials and a rudimentary knowledge of partial differentiation. Cartesian tensor notation and the index convention are used in a few places to shorten the derivation of some general results, but these sections are presented so as to be as far as possible self-explanatory.

Modern practitioners of Elasticity are necessarily influenced by developments in numerical methods, which promise to solve all problems with no more information about the subject than is needed to formulate the description of a representative element of material in a relatively simple state of stress. As a researcher in Solid Mechanics, with a primary interest in the physical behaviour of the systems I am investigating, rather than in the mathematical structure of the solutions, I have frequently had recourse to numerical methods of all types and have tended to adopt the pragmatic criterion that the best method is that which gives the most convincing and accurate result in the shortest time. In this context, 'convincing' means that the solution should be capable of being checked against reliable closed-form solutions in suitable limiting cases and that it is demonstrably stable and in some sense convergent. Measured against these criteria, the 'best' solution to many practical problems is often not a direct numerical method, such as the finite element method, but rather one involving some significant analytical steps before the final numerical evaluation. This is particularly true in three-dimensional problems, where direct numerical methods are extremely computer-intensive if any reasonably accuracy is required, and in problems involving infinite or semi-infinite domains, discontinuities, bonded or contacting material interfaces or theoretically singular stress fields. By contrast, I would immediately opt for a finite element solution of any two-dimensional problem involving finite bodies with relatively smooth contours, unless it happened to fall into the (surprisingly wide) class of problems to which the solution can be written down in closed form. The reader will therefore find my choice of topics significantly biassed towards those fields identified above where analytical methods are most useful.

As in the second edition, I encourage the reader to become familiar with the use of symbolic mathematical languages such as Maple$^{\mathrm{TM}}$ and Mathematica$^{\mathrm{TM}}$, since these tools open up the possibility of solving considerably more complex and hence interesting and realistic elasticity problems. They also enable the student to focus on the formulation of the problem (e.g. the appropriate governing equations and boundary conditions) rather than on the algebraic manipulations, with a consequent improvement in insight into the subject and in motivation. Finally, they each posess post-processing graphics facilities that enable the user to explore important features of the resulting stress state. The reader can access numerous files for this purpose at the website www.elasticity.org or at my University of Michigan homepage http://www-personal.umich.edu/ jbarber/elasticity/book.html, including the solution of sample problems, electronic versions of the tables in Chapters 21,22, and algorithms for the generation of spherical harmonic potentials. Some hints about the use of this material are contained in Appendix A, and more detailed tips about programming are included at the above websites. Those who have never used Maple or Mathematica will find that it takes only a few hours of trial and error to learn how to write programs to solve boundary-value problems in elasticity.

This new edition contains four additional chapters, including two concerned with the use of complex-variable methods in two-dimensional elasticity. In keeping with the style of the rest of the book, I have endeavoured to present this material in a such a way as to be usable by a reader with minimal previous experience of complex analysis who wishes to solve specific elasticity problems. This necessarily involves glossing over some of the finer points of the underlying mathematics. The reader wishing for a more complete and rigorous treatment will need to refer to more specialized works on this topic. I have emphasised the relation between the complex and real (Airy and Prandtl) stress functions, including algorithms for obtaining the complex function for a stress field for which the real stress function is already known. The complex variable methods and notation developed in Chapter 19 are also used in the development of a hierarchical treatment of three-dimensional problems for prismatic bars of fairly general cross-section in Chapter 28. The other major addition is a new chapter on variational methods, including the use of the Rayleigh-Ritz method and Castigliano's second theorem in developing approximate solutions to elasticity problems.

The new edition contains numerous additional end-of-chapter problems. As with previous editions, a full set of solutions to these problems is available to *bona fide* instructors on request to the author. Some of these problems are quite challenging, indeed several were the subject of substantial technical papers within the not too distant past, but they can all be solved in a few hours using Maple or Mathematica. Many texts on Elasticity contain problems which offer a candidate stress function and invite the student to 'verify' that it defines the solution to a given problem. Students invariably raise the question 'How would we know to choose that form if we were not given it in advance?' I have tried wherever possible to avoid this by expressing the problems in the form they would arise in Engineering — i.e. as a body of a given geometry subjected to prescribed loading. This in turn has required me to write the text in such a way that the student can approach problems deductively. I have also generally opted for explaining difficulties that might arise in an 'obvious' approach to the problem, rather than steering the reader around them in the interests of brevity.

I have taken this opportunity to correct the numerous typographical errors in the second edition, but no doubt despite my best efforts, the new material will contain more. Please communicate any errors to me.

As in previous editions, I would like to thank my graduate students and more generally scientific correspondents worldwide whose questions continue to force me to re-examine my knowledge of the subject. I am also grateful to Professor John Dundurs for permission to use Table 9.1 and to the Royal Society of London for permission to reproduce Figures 13.2, 13.3. Vikram Gavini, David Hills and Alexander Korsunsky were kind enough to read drafts of the complex-variable chapters and made useful suggestions on presentation, but they are in no way responsible for my rather idiosyncratic approach to this subject.

J.R.Barber
Ann Arbor
2009

Part I

GENERAL CONSIDERATIONS

1

INTRODUCTION

The subject of Elasticity is concerned with the determination of the stresses and displacements in a body as a result of applied mechanical or thermal loads, for those cases in which the body reverts to its original state on the removal of the loads. In this book, we shall further restrict attention to the case of linear infinitesimal elasticity, in which the stresses and displacements are linearly proportional to the applied loads and the displacements are small in comparison with the characteristic length dimensions of the body. These restrictions ensure that linear superposition can be used and enable us to employ a wide range of series and transform techniques which are not available for non-linear problems.

Most engineers first encounter problems of this kind in the context of the subject known as Mechanics of Materials, which is an important constituent of most undergraduate engineering curricula. Mechanics of Materials differs from Elasticity in that various plausible but unsubstantiated assumptions are made about the deformation process in the course of the analysis. A typical example is the assumption that plane sections remain plane in the bending of a slender beam. Elasticity makes no such assumptions, but attempts to develop the solution directly and rigorously from its first principles, which are Newton's laws of motion, Euclidian geometry and Hooke's law. Approximations are often introduced towards the end of the solution, but these are mathematical approximations used to obtain solutions of the governing equations rather than physical approximations that impose artificial and strictly unjustifiable constraints on the permissible deformation field.

However, it would be a mistake to draw too firm a distinction between the two approaches, since practitioners of each have much to learn from the other. Mechanics of Materials, with its emphasis on physical reasoning and a full exploration of the practical consequences of the results, is often able to provide insights into the problem that are less easily obtained from a purely mathematical perspective. Indeed, we shall make extensive use of physical parallels in this book and pursue many problems to conclusions relevant to practical applications, with the hope of deepening the reader's understanding

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_1, © Springer Science+Business Media B.V. 2010

of the underlying structure of the subject. Conversely, the mathematical rigour of Elasticity gives us greater confidence in the results, since, even when we have to resort to an approximate solution, we can usually estimate its accuracy with some confidence — something that is very difficult to do with the physical approximations used in Mechanics of Materials[1]. Also, there is little to be said for using an *ad hoc* approach when, as is often the case, a more rigorous treatment presents no serious difficulty.

1.1 Notation for stress and displacement

It is assumed that the reader is more or less familiar with the concept of stress and strain from elementary courses on Mechanics of Materials. This section is intended to introduce the notation used, to refresh the reader's memory about some important ideas, and to record some elementary but useful results.

1.1.1 Stress

Components of stress will all be denoted by the symbol σ with appropriate suffices. The second suffix denotes the direction of the stress component and the first the direction of the *outward* normal to the surface upon which it acts. This notation is illustrated in Figure 1.1 for the Cartesian coördinate system x, y, z.

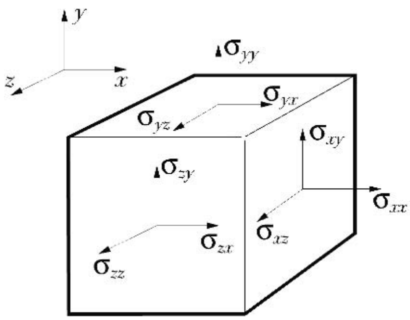

Figure 1.1: Notation for stress components.

Notice that one consequence of this notation is that normal (i.e. tensile and compressive) stresses have both suffices the same (e.g. $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ in Figure 1.1) and are positive when tensile. The remaining six stress components in Figure 1.1 (i.e. $\sigma_{xy}, \sigma_{yx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xz}$) have two *different* suffices and are shear stresses.

[1] In fact, the only practical way to examine the effect of these approximations is to relax them, by considering the same problem, or maybe a simpler problem with similar features, in the context of the theory of Elasticity.

Books on Mechanics of Materials often use the symbol τ for shear stress, whilst retaining σ for normal stress. However, there is no need for a different symbol, since the suffices enable us to distinguish normal from shear stress components. Also, we shall find that the use of a single symbol with appropriate suffices permits matrix methods to be used in many derivations and introduces considerable economies in the notation for general results.

The equilibrium of moments acting on the block in Figure 1.1 requires that

$$\sigma_{xy} = \sigma_{yx} \;\; ; \;\; \sigma_{yz} = \sigma_{zy} \text{ and } \sigma_{zx} = \sigma_{xz} \; . \tag{1.1}$$

This has the incidental advantage of rendering mistakes about the order of suffices harmless! (In fact, a few books use the opposite convention.) Readers who have not encountered three-dimensional problems before should note that there are *two* shear stress components on each surface and one normal stress component. There are some circumstances in which it is convenient to combine the two shear stresses on a given plane into a two-dimensional vector in the plane — i.e. to refer to the *resultant* shear stress on the plane. An elementary situation where this is helpful is in the Mechanics of Materials problem of determining the distribution of shear stress on the cross-section of a beam due to a transverse shear force[2]. For example, we note that in this case, the resultant shear stress on the plane must be tangential to the edge at the boundary of the cross-section, since the shear stress complementary to the component normal to the edge acts on the traction-free surface of the beam and must therefore be zero. This of course is why the shear stress in a thin-walled section tends to follow the direction of the wall.

We shall refer to a plane normal to the x-direction as an 'x-plane' etc. The *only* stress components which act on an x-plane are those which have an x as the first suffix (This is an immediate consequence of the definition).

Notice also that any x-plane can be defined by an equation of the form $x=c$, where c is a constant. More precisely, we can define a 'positive x-plane' as a plane for which the positive x-direction is the outward normal and a 'negative x-plane' as one for which it is the inward normal. This distinction can be expressed mathematically in terms of inequalities. Thus, if part of the boundary of a solid is expressible as $x=c$, the solid must locally occupy one side or the other of this plane. If the domain of the solid is locally described by $x<c$, the bounding surface is a positive x-plane, whereas if it is described by $x>c$, the bounding surface is a negative x-plane.

The discussion in this section suggests a useful formalism for correctly defining the boundary conditions in problems where the boundaries are parallel to the coördinate axes. We first identify the equations which define the boundaries of the solid and then write down the three traction components which act on each boundary. For example, suppose we have a rectangular solid defined by the inequalities $0<x<a,\; 0<y<b,\; 0<z<c$. It is clear that the

[2] For the elasticity solution of this problem, see Chapter 17.

surface $y = b$ is a positive y-plane and we deduce immediately that the corresponding traction boundary conditions will involve the stress components $\sigma_{yx}, \sigma_{yy}, \sigma_{yz}$ — i.e. the three components that have y as the first suffix. This procedure insures against the common student mistake of assuming (for example) that the component σ_{xx} must be zero if the surface $y=b$ is to be traction free. (*Note* : Don't assume that this mistake is too obvious for you to fall into it. When the problem is geometrically or algebraically very complicated, it is only too easy to get distracted.)

Stress components can be defined in the same way for other systems of orthogonal coördinates. For example, components for the system of cylindrical polar coördinates (r, θ, z) are shown in Figure 1.2.

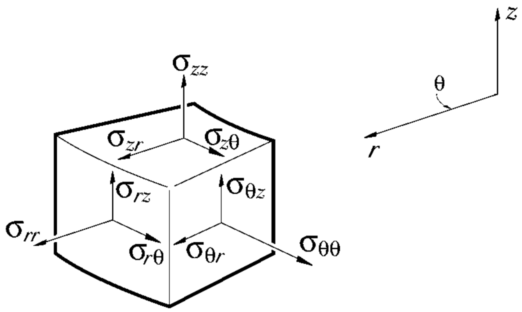

Figure 1.2: Stress components in polar coördinates.

(This is a case where the definition of the 'θ-plane' through an equation, $\theta=\alpha$, is easier to comprehend than 'the plane normal to the θ-direction'. However, note that the θ-direction is the direction in which a particle would move if θ were increased with r, z constant.)

1.1.2 Index and vector notation and the summation convention

In expressing or deriving general results, it is often convenient to make use of vector notation. We shall use bold face symbols to represent vectors and single suffices to define their components in a given coördinate direction. Thus, the vector $\boldsymbol{V}$ can be written

$$\boldsymbol{V} = \boldsymbol{i}V_x + \boldsymbol{j}V_y + \boldsymbol{k}V_z = \{V_x, V_y, V_z\}^T \ , \tag{1.2}$$

where where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors in directions x, y, z respectively.

In the last expression in (1.2), we have used the convenient linear algebra notation for a vector, but to take full advantage of this, we also need to replace V_x, V_y, V_z by V_1, V_2, V_3 or more generally by V_i, where the index i takes the values $1, 2, 3$. Further compression is then achieved by using the *Einstein summation convention* according to which terms containing a repeated latin index are summed over the three values $1, 2, 3$. For example, the expression σ_{ii} is interpreted as

$$\sigma_{ii} \equiv \sum_{i=1}^{3} \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \ . \tag{1.3}$$

Any expression with one free index has a value associated with each coördinate direction and hence represents a vector. A more formal connection between the two notations can be established by writing

$$\boldsymbol{V} = \boldsymbol{e}_1 V_1 + \boldsymbol{e}_2 V_2 + \boldsymbol{e}_3 V_3 = \boldsymbol{e}_i V_i \ , \tag{1.4}$$

where $\boldsymbol{e}_1 = \boldsymbol{i}, \boldsymbol{e}_2 = \boldsymbol{j}, \boldsymbol{e}_3 = \boldsymbol{k}$ and then dropping the implied unit vector $\boldsymbol{e}_i$ in the right-hand side of (1.4).

The position of a point in space is identified by the position vector,

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z \ , \tag{1.5}$$

which in the index notation is simply written as x_i, whilst the Cartesian stress components

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1.6}$$

are written as σ_{ij}.

1.1.3 Vector operators in index notation

All the well-known vector operations can be performed using index notation and the summation convention. For example, if two vectors are represented by P_i, Q_i respectively, their scalar (dot) product can be written concisely as

$$\boldsymbol{P} \cdot \boldsymbol{Q} = P_1 Q_1 + P_2 Q_2 + P_3 Q_3 = P_i Q_i \ , \tag{1.7}$$

because the repeated index i in the last expression implies the summation over all three product terms. The vector (cross) product

$$\boldsymbol{P} \times \boldsymbol{Q} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{vmatrix} \tag{1.8}$$

can be written in index notation as

$$\boldsymbol{P} \times \boldsymbol{Q} = \epsilon_{ijk} P_i Q_j \ , \tag{1.9}$$

where ϵ_{ijk} is the *alternating tensor* which is defined to be 1 if the indices are in cyclic order (e.g. 2,3,1), –1 if they are in *reverse* cyclic order (e.g. 2,1,3) and zero if any two indices are the same. Notice that the only free index in the right-hand side of (1.9) is k, so (for example) the x_1-component of $\boldsymbol{P} \times \boldsymbol{Q}$ is recovered by setting $k = 1$.

The gradient of a scalar function ϕ can be written

$$\boldsymbol{\nabla}\phi = \boldsymbol{i}\frac{\partial\phi}{\partial x} + \boldsymbol{j}\frac{\partial\phi}{\partial y} + \boldsymbol{k}\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial x_i} \ , \tag{1.10}$$

the divergence of a vector $\boldsymbol{V}$ is

$$\text{div } \boldsymbol{V} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{V} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} = \frac{\partial V_i}{\partial x_i} \tag{1.11}$$

using (1.7, 1.10) and the curl of a vector $\boldsymbol{V}$ is

$$\text{curl } \boldsymbol{V} \equiv \boldsymbol{\nabla}\times\boldsymbol{V} = \epsilon_{ijk}\frac{\partial V_j}{\partial x_i} \ , \tag{1.12}$$

using (1.9, 1.10). We also note that the Laplacian of a scalar function can be written in any of the forms

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \text{div } \boldsymbol{\nabla}\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} \ . \tag{1.13}$$

1.1.4 Vectors, tensors and transformation rules

Vectors can be conceived in a mathematical sense as ordered sets of numbers or in a physical sense as mathematical representations of quantities characterized by magnitude and direction. A link between these concepts is provided by the *transformation rules.* Suppose we know the components (V_x, V_y) of the vector $\boldsymbol{V}$ in a given two-dimensional Cartesian coördinate system (x, y) and we wish to determine the components (V_x', V_y') in a new system (x', y') which is inclined to (x, y) at an angle θ in the anticlockwise direction as shown in Figure 1.3. The required components are

$$V_x' = V_x \cos\theta + V_y \sin\theta \ ; \qquad V_y' = V_y \cos\theta - V_x \sin\theta \ . \tag{1.14}$$

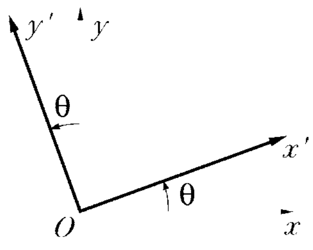

Figure 1.3: The coördinate systems x, y and x', y'.

We could *define* a vector as an entity, described by its components in a specified Cartesian coördinate system, which transforms into other coördinate systems according to rules like equations (1.14) — i.e. as an ordered set of numbers which obey the transformation rules (1.14). The idea of magnitude and direction could then be introduced by noting that we can always choose θ such that (i) $V'_y=0$ and (ii) $V'_x>0$. The corresponding direction x' is then the direction of the resultant vector and the component V'_x is its magnitude.

Now stresses have two suffices and components associated with all possible combinations of two coördinate directions, though we note that equation (1.1) shows that the order of the suffices is immaterial. (Another way of stating this is that the matrix of stress components (1.6) is always symmetric). The stress components satisfy a more complicated set of transformation rules which can be determined by considering the equilibrium of an infinitesimal wedge-shaped piece of material. The resulting equations in the two-dimensional case are those associated with Mohr's circle — i.e.

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \tag{1.15}$$

$$\sigma'_{xy} = \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right) + (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta \tag{1.16}$$

$$\sigma'_{yy} = \sigma_{yy}\cos^2\theta + \sigma_{xx}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta\ , \tag{1.17}$$

where we use the notation σ'_{xx} rather than $\sigma_{x'x'}$ in the interests of clarity.

As in the case of vectors we can define a mathematical entity which has a matrix of components in any given Cartesian coördinate system and which transforms into other such coördinate systems according to rules like (1.15–1.17). Such quantities are called *second order Cartesian tensors*. Notice incidentally that the alternating tensor introduced in equation (1.9) is *not* a second order Cartesian tensor.

We know from Mohr's circle that we can always choose θ such that $\sigma'_{xy}=0$, in which case the directions x', y' are referred to as principal directions and the components $\sigma'_{xx}, \sigma'_{yy}$ as principal stresses. Thus another way to characterize a second order Cartesian tensor is as a quantity defined by a set of orthogonal principal directions and a corresponding set of principal values.

As with vectors, a pragmatic motivation for abstracting the mathematical properties from the physical quantities which exhibit them is that many different physical quantities are naturally represented as second order Cartesian tensors. Apart from stress and strain, some commonly occurring examples are the second moments of area of a beam cross-section (I_x, I_{xy}, I_y), the second partial derivatives of a scalar function $(\partial^2 f/\partial x^2; \partial^2 f/\partial x\partial y; \partial^2 f/\partial y^2)$ and the influence coefficient matrix C_{ij} defining the displacement $\boldsymbol{u}$ due to a force $\boldsymbol{F}$ for a linear elastic system, i.e.

$$u_i = C_{ij}F_j\ , \tag{1.18}$$

where the summation convention is implied.

It is a fairly straightforward matter to prove that each of these quantities obeys transformation rules like (1.15–1.17). It follows immediately (for example) that every beam cross-section has two orthogonal *principal axes* of bending about which the two principal second moments are respectively the maximum and minimum for the cross-section.

A special tensor of some interest is that for which the Mohr's circle degenerates to a point. In the case of stresses, this corresponds to a state of *hydrostatic stress*, so-called because a fluid at rest cannot support shear stress (the constitutive law for a fluid relates *velocity gradient* to shear stress) and hence $\sigma'_{xy}=0$ for all θ. The only Mohr's circle which satisfies this condition is one of zero radius, from which we deduce immediately that all directions are principal directions and that the principal values are all equal. In the case of the fluid, we obtain the well-known result that the pressure in a fluid at rest is equal in all directions.

It is instructive to consider this result in the context of other systems involving tensors. For example, consider the second moments of area for the square cross-section shown in Figure 1.4. By symmetry, we know that Ox, Oy are principal directions and that the two principal second moments are both equal to $a^4/12$. It follows that the Mohr's circle of second moments has zero radius and hence (i) that the second moment about any other axis must also be $a^4/12$ and (ii) that the product inertia I'_{xy} must be zero about all axes, showing that the axis of curvature is always aligned with the applied moment. These results are not at all obvious merely from an examination of the section.

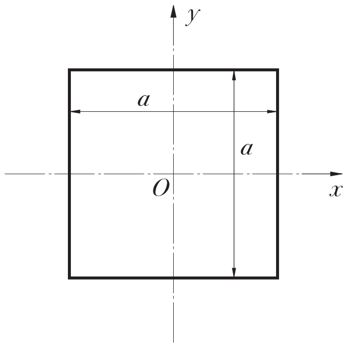

Figure 1.4: A beam of square cross-section.

As a second example, Figure 1.5 shows an elastic system consisting of three identical but arbitrary structures connecting a point, P, to a rigid support, the structures being inclined to each other at angles of 120^o.

The structures each have elastic properties expressible in the form of an influence function matrix as in equation (1.18) and are generally such that the displacement $\boldsymbol{u}$ is not colinear with the force $\boldsymbol{F}$. However, the *overall* influence function matrix for the system has the same properties in three different coördinate systems inclined to each other at 120^o, since a rotation of the Figure through 120^o leaves the system unchanged. The only Mohr's circle which gives equal components after a rotation of 120^o is that of zero radius. We therefore conclude that the support system of Figure 1.5 is such that (i) the displacement of P always has the same direction as the force $\boldsymbol{F}$ and (ii) the stiffness or compliance of the system is the same in all directions[3]. These two examples illustrate that there is sometimes an advantage to be gained from considering a disparate physical problem that shares a common mathematical structure with that under investigation.

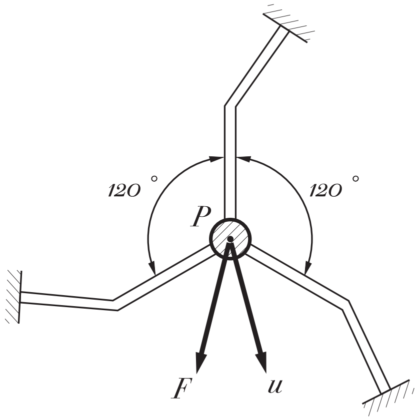

Figure 1.5: Support structure with three similar but unsymmetrical components.

Three-dimensional transformation rules

In three dimensions, the vector transformation rules (1.14) are most conveniently written in terms of *direction cosines* l_{ij}, defined as the cosine of the angle between unit vectors in the directions x_i' and x_j respectively. We then obtain

$$V_i' = l_{ij} V_j \ . \tag{1.19}$$

For the two-dimensional case of Figure 1.3 and equations (1.14), the matrix[4] formed by the components l_{ij} takes the form

$$l_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \ . \tag{1.20}$$

We already remarked in §1.1.1 that only those stress components with an x-suffix act on the surface $x = c$ and hence that if this surface is traction-free (for example if it is an unloaded boundary of a body), then $\sigma_{xx} = \sigma_{xy} =$

[3] A similar argument can be used to show that if a laminated fibre-reinforced composite is laid up with equal numbers of identical, but not necessarily symmetrical, laminæ in each of 3 or more equispaced orientations, it must be elastically isotropic within the plane. This proof depends on the properties of the *fourth order* Cartesian tensor c_{ijkl} describing the stress-strain relation of the laminæ (see equation (1.55) below).

[4] Notice that the matrix of direction cosines is not symmetric.

$\sigma_{xz}=0$, but no restrictions are placed on the components $\sigma_{yy}, \sigma_{yz}, \sigma_{zz}$. More generally, the *traction* $\boldsymbol{t}$ on a specified plane is a vector with three independent components t_i, in contrast to the stress σ_{ij} which is a second order tensor with six independent components. Mathematically, we can define the orientation of a specified inclined plane by a unit vector $\boldsymbol{n}$ in the direction of the outward normal. By considering the equilibrium of an infinitesimal tetrahedron whose surfaces are perpendicular to the directions x_1, x_2, x_3 and $\boldsymbol{n}$ respectively, we then obtain

$$t_i = \sigma_{ij} n_j \ , \tag{1.21}$$

for the traction vector on the inclined surface. The components t_i defined by (1.21) are aligned with the Cartesian axes x_i, but we can use the vector transformation rule (1.19) to resolve $\boldsymbol{t}$ into a new coördinate system in which one axis (x_1' say) is aligned with $\boldsymbol{n}$. We then have

$$n_j = l_{1j} \ ; \quad t_i = \sigma_{ij} l_{1j} \quad \text{and} \quad t_k' = l_{ki} t_i = l_{ki} l_{1j} \sigma_{ij}$$

If we perform this operation for each of the three surfaces orthogonal to x_1', x_2', x_3' respectively, we shall recover the complete stress tensor in the rotated coördinate system, which is therefore given by

$$\sigma_{ij}' = l_{ip} l_{jq} \sigma_{pq} \ . \tag{1.22}$$

Comparison of (1.22) with (1.15–1.17) provides a good illustration of the efficiency of the index notation in condensing general expressions in Cartesian coördinates.

1.1.5 Principal stresses and Von Mises stress

One of the principal reasons for performing elasticity calculations is to determine when and where an engineering component will fail. Theories of material failure are beyond the scope of this book, but the most widely used criteria are the Von Mises distortion energy criterion for ductile materials and the maximum tensile stress criterion for brittle materials[5].

Brittle materials typically fracture when the maximum tensile stress reaches a critical value and hence we need to be able to calculate the maximum principal stress σ_1. For two dimensional problems, the principal stresses can be found by using the condition $\sigma_{xy}'=0$ in equation (1.16) to determine the inclination θ of the principal directions. Substituting into (1.15, 1.17) then yields the well known results

$$\sigma_1, \sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \ . \tag{1.23}$$

[5] J.R.Barber, *Intermediate Mechanics of Materials*, McGraw-Hill, New York, 2000, §2.2.

In most problems, the maximum stresses occur at the boundaries where shear tractions are usually zero. Thus, even in three-dimensional problems, the determination of the maximum tensile stress often involves only a two-dimensional stress transformation. However, the principal stresses are easily obtained in the fully three-dimensional case from the results

$$\sigma_1 = \frac{I_1}{3} + \frac{2}{3}\left(\sqrt{I_1^2 - 3I_2}\right)\cos\phi \tag{1.24}$$

$$\sigma_2 = \frac{I_1}{3} + \frac{2}{3}\left(\sqrt{I_1^2 - 3I_2}\right)\cos\left(\phi + \frac{2\pi}{3}\right) \tag{1.25}$$

$$\sigma_3 = \frac{I_1}{3} + \frac{2}{3}\left(\sqrt{I_1^2 - 3I_2}\right)\cos\left(\phi + \frac{4\pi}{3}\right) , \tag{1.26}$$

where

$$\phi = \frac{1}{3}\arccos\left(\frac{2I_1^3 - 9I_1I_2 + 27I_3}{2(I_1^2 - 3I_2)^{3/2}}\right) \tag{1.27}$$

and

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{1.28}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{xy}^2 - \sigma_{yz}^2 - \sigma_{zx}^2 \tag{1.29}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} . \tag{1.30}$$

The quantities I_1, I_2, I_3 are known as *stress invariants* because for a given stress state they are the same in all coördinate systems. If the principal value of $\theta = \arccos(x)$ is defined such that $0 \le \theta < \pi$, the principal stresses defined by equations (1.24–1.26) will always satisfy the inequality $\sigma_1 \ge \sigma_3 \ge \sigma_2$.

Von Mises theory states that a ductile material will yield when the strain energy of distortion per unit volume reaches a certain critical value. This enables us to define an *equivalent tensile stress* or *Von Mises stress* σ_E as

$$\begin{aligned}\sigma_E &\equiv \sqrt{I_1^2 - 3I_2} \\ &= \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} + 3\sigma_{xy}^2 + 3\sigma_{yz}^2 + 3\sigma_{zx}^2} .\end{aligned} \tag{1.31}$$

The Von Mises yield criterion then states that yield will occur when σ_E reaches the yield stress in uniaxial tension, S_Y. The Maple and Mathematica files 'principalstresses' use equations (1.24–1.31) to calculate the principal stresses and the Von Mises stress from a given set of stress components.

1.1.6 Displacement

The displacement of a particle P is a vector

$$\boldsymbol{u} = \boldsymbol{i}u_x + \boldsymbol{j}u_y + \boldsymbol{k}u_z \tag{1.32}$$

representing the difference between the final and the initial position of P — i.e. it is the distance that P moves during the deformation. In index notation, the displacement is represented as u_i.

The deformation of a body is completely defined if we know the displacement of its every particle. Notice however that there is a class of displacements which do not involve deformation — the so-called 'rigid-body displacements'. A typical case is where all the particles of the body have the same displacement. The name arises, of course, because rigid-body displacement is the *only* class of displacement that can be experienced by a rigid body.

1.2 Strains and their relation to displacements

Components of strain will be denoted by the symbol, e, with appropriate suffices (e.g. e_{xx}, e_{xy}). As in the case of stress, no special symbol is required for shear strain, though we shall see below that the quantity defined in most elementary texts (and usually denoted by γ) differs from that used in the mathematical theory of Elasticity by a factor of 2. A major advantage of this definition is that it makes the strain, e, a second order Cartesian Tensor (see §1.1.4 above). We shall demonstrate this by establishing transformation rules for strain similar to equations (1.15–1.17) in §1.2.4 below.

1.2.1 Tensile strain

Students usually first encounter the concept of strain in elementary Mechanics of Materials as the ratio of extension to original length and are sometimes confused by the apparently totally different definition used in more mathematical treatments of solid mechanics. We shall discuss here the connection between the two definitions — partly for completeness, and partly because the physical insight that can be developed in the simple problems of Mechanics of Materials is very useful if it can be carried over into more difficult problems.

Figure 1.6 shows a bar of original length L and density ρ hanging from the ceiling. Suppose we are asked to find how much it increases in length under the loading of its own weight.

It is easily shown that the tensile stress σ_{xx} at the point P, distance x from the ceiling, is

$$\sigma_{xx} = \rho g(L - x) , \tag{1.33}$$

where g is the acceleration due to gravity, and hence from Hooke's law,

$$e_{xx} = \frac{\rho g(L - x)}{E} . \tag{1.34}$$

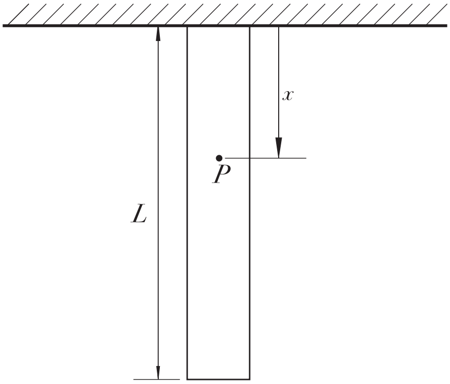

Figure 1.6: The bar suspended from the ceiling.

However, the strain varies continuously over the length of the bar and hence we can only apply the Mechanics of Materials definition if we examine an infinitesimal piece of the bar over which the strain can be regarded as sensibly constant.

We describe the deformation in terms of the downward displacement u_x which depends upon x and consider that part of the bar between x and $x+\delta x$, denoted by PQ in Figure 1.7.

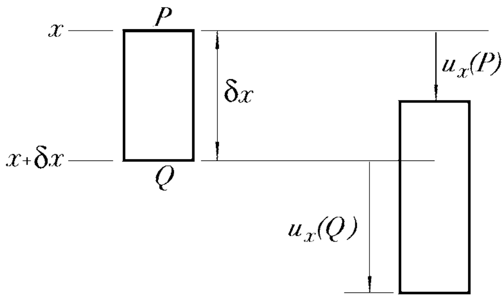

Figure 1.7: Infinitesimal section of the bar.

After the deformation, PQ must have extended by $u_x(Q)-u_x(P)$ and hence the local value of 'Mechanics of Materials' tensile strain is

$$e_{xx} = \frac{u_x(Q) - u_x(P)}{\delta x} = \frac{u_x(x + \delta x) - u_x(x)}{\delta x} \; . \tag{1.35}$$

Taking the limit as $\delta x \rightarrow 0$, we obtain the definition

$$e_{xx} = \frac{\partial u_x}{\partial x} \; . \tag{1.36}$$

Corresponding definitions can be developed in three-dimensional problems for the other normal strain components. i.e.

$$e_{yy} = \frac{\partial u_y}{\partial y} \;\; ; \;\; e_{zz} = \frac{\partial u_z}{\partial z} \; . \tag{1.37}$$

Notice how easy the problem of Figure 1.6 becomes when we use these definitions. We get

$$\frac{\partial u_x}{\partial x} = \frac{\rho g(L-x)}{E} \; , \tag{1.38}$$

from (1.34, 1.36) and hence

$$u_x = \frac{\rho g(2Lx - x^2)}{2E} + A \; , \tag{1.39}$$

where A is an arbitrary constant of integration which expresses the fact that our knowledge of the stresses and hence the strains in the body is not sufficient to determine its position in space. In fact, A represents an arbitrary rigid-body displacement. In this case we need to use the fact that the top of the bar is joined to a supposedly rigid ceiling — i.e. $u_x(0)=0$ and hence $A=0$ from (1.39).

1.2.2 Rotation and shear strain

Noting that the two x's in e_{xx} correspond to those in its definition $\partial u_x/\partial x$, it is natural to seek a connection between the shear strain e_{xy} and one or both of the derivatives $\partial u_x/\partial y, \partial u_y/\partial x$. As a first step, we shall discuss the geometrical interpretation of these derivatives.

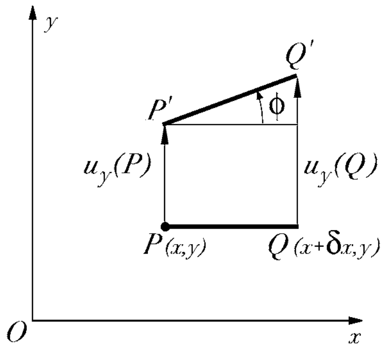

Figure 1.8: Rotation of a line segment.

Figure 1.8 shows a line segment PQ of length δx, aligned with the x-axis, the two ends of which are displaced in the y-direction. Clearly if $u_y(Q) \neq u_y(P)$, the line PQ will be rotated by these displacements and if the angle of rotation is small it can be written

$$\phi = \frac{u_y(x+\delta x) - u_y(x)}{\delta x} \; , \tag{1.40}$$

(anticlockwise positive).

Proceeding to the limit as $\delta x \to 0$, we have

$$\phi = \frac{\partial u_y}{\partial x} \ . \tag{1.41}$$

Thus, $\partial u_y/\partial x$ is the angle through which a line originally in the x-direction rotates towards the y-direction during the deformation[6].

Now, if PQ is a line drawn on the surface of an elastic solid, the occurence of a rotation ϕ does not necessarily indicate that the solid is deformed — we could rotate the line simply by rotating the solid as a rigid body. To investigate this matter further, we imagine drawing a series of lines at different angles through the point P as shown in Figure 1.9(a).

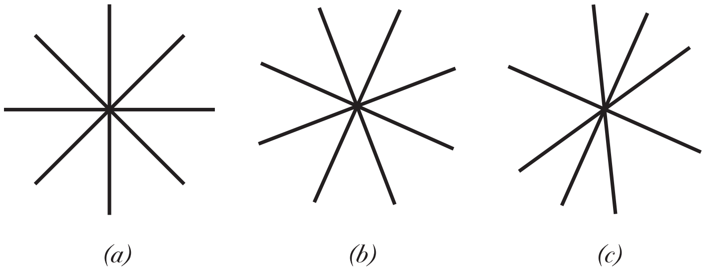

Figure 1.9: Rotation of lines at point P; (a) Original state, (b) Rigid-body rotation; (c) Rotation and deformation.

If the vicinity of the point P suffers merely a local rigid-body rotation, all the lines will rotate through the same angle and retain the same relative inclinations as shown in 1.9(b). However, if different lines rotate through *different* angles, as in 1.9(c), the body must have been deformed. We shall show in the next section that the rotations of the lines in Figure 1.9(c) are not independent and a consideration of their interdependency leads naturally to a definition of shear strain.

1.2.3 Transformation of coördinates

Suppose we knew the displacement components u_x, u_y throughout the body and wished to find the rotation, ϕ, of the line PQ in Figure 1.10, which is inclined at an angle θ to the x-axis.

We construct a new axis system $Ox'y'$ with Ox' parallel to PQ as shown, in which case we can argue as above that PQ rotates anticlockwise through the angle

$$\phi(PQ) = \frac{\partial u'_y}{\partial x'} \ . \tag{1.42}$$

[6] Students of Mechanics of Materials will have already used a similar result when they express the slope of a beam as du/dx, where x is the distance along the axis of the beam and u is the transverse displacement.

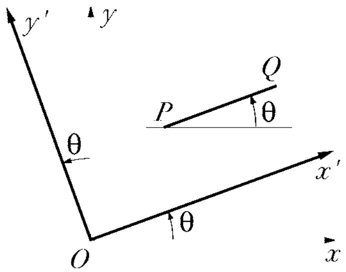

Figure 1.10: Rotation of a line inclined at angle θ.

Furthermore, we have

$$\frac{\partial}{\partial x'} = \boldsymbol{\nabla}\cdot\boldsymbol{i}' = \left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}\right)\cdot\boldsymbol{i}' = \boldsymbol{i}\cdot\boldsymbol{i}'\frac{\partial}{\partial x}+\boldsymbol{j}\cdot\boldsymbol{i}'\frac{\partial}{\partial y}$$
$$= \cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\;, \qquad (1.43)$$

and by similar arguments,

$$u'_x = u_x\cos\theta + u_y\sin\theta\;; \qquad u'_y = u_y\cos\theta - u_x\sin\theta\;. \qquad (1.44)$$

In fact equations (1.43,1.44) are simply restatements of the vector transformation rules (1.14), since both $\boldsymbol{u}$ and the gradient operator are vectors.

Substituting these results into equation (1.42), we find

$$\begin{aligned}\phi(PQ) &= \left(\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right)(u_y\cos\theta - u_x\sin\theta)\\ &= \frac{\partial u_y}{\partial x}\cos^2\theta - \frac{\partial u_x}{\partial y}\sin^2\theta + \left(\frac{\partial u_y}{\partial y}-\frac{\partial u_x}{\partial x}\right)\sin\theta\cos\theta\\ &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\cos(2\theta)\\ &\quad +\frac{1}{2}\left(\frac{\partial u_y}{\partial y}-\frac{\partial u_x}{\partial x}\right)\sin(2\theta)\;. \end{aligned} \qquad (1.45)$$

In the final expression (1.45), the first term is independent of the inclination θ of the line PQ and hence represents a rigid-body rotation as in Figure 1.9(b). We denote this rotation by the symbol ω, which with the convention illustrated in Figure 1.8 is anticlockwise positive.

Notice that as this term is independent of θ in equation (1.45), it is the same for any right-handed set of axes — i.e.

$$\omega = \frac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u'_y}{\partial x'}-\frac{\partial u'_x}{\partial y'}\right)\;, \qquad (1.46)$$

for any x', y'.

In three dimensions, ω represents a small positive rotation about the z-axis and is therefore more properly denoted by ω_z to distinguish it from the corresponding rotations about the x and y axes — i.e.

$$\omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \;\; ; \;\; \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \;\; ; \;\; \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) . \tag{1.47}$$

The rotation $\boldsymbol{\omega}$ is therefore a vector in three-dimensional problems and is more compactly defined in vector or index notation as

$$\boldsymbol{\omega} = \frac{1}{2}\text{curl } \boldsymbol{u} \equiv \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{u} \quad \text{or} \quad \omega_k = \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\right) , \tag{1.48}$$

where ϵ_{ijk} is defined in §1.1.3. In two dimensions $\boldsymbol{\omega}$ behaves as a scalar since two of its components degenerate to zero[7].

1.2.4 Definition of shear strain

We are now in a position to *define* the shear strain e_{xy} as the difference between the rotation of a line drawn in the x-direction and the corresponding rigid-body rotation, ω_z, i.e.

$$e_{xy} = \frac{\partial u_y}{\partial x} - \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{1.49}$$

and similarly

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \;\; ; \;\; e_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) . \tag{1.50}$$

Note that e_{xy} so defined is one half of the quantity γ_{xy} used in Mechanics of Materials and in many older books on Elasticity.

The strain-displacement relations (1.36, 1.37, 1.49, 1.50) can be written in the concise form

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{1.51}$$

With the notation of equations (1.36, 1.37, 1.47, 1.49), we can now write (1.45) in the form

$$\phi(PQ) = \omega_z + e_{xy}(\cos^2\theta - \sin^2\theta) + (e_{yy} - e_{xx})\sin\theta\cos\theta \tag{1.52}$$

and hence

[7] In some books, $\omega_x, \omega_y, \omega_z$ are denoted by $\omega_{yz}, \omega_{zx}, \omega_{xy}$ respectively. This notation is not used here because it gives the erroneous impression that $\boldsymbol{\omega}$ is a second order tensor rather than a vector.

$$\begin{aligned} e'_{xy} &= \phi(PQ) - \omega_z \qquad\qquad \text{(by definition)} \\ &= e_{xy}(\cos^2\theta - \sin^2\theta) + (e_{yy} - e_{xx})\sin\theta\cos\theta \ . \end{aligned} \tag{1.53}$$

This is one of the coördinate transformation relations for strain, the other one

$$e'_{xx} = e_{xx}\cos^2\theta + e_{yy}\sin^2\theta + 2e_{xy}\sin\theta\cos\theta \tag{1.54}$$

being obtainable from equations (1.36, 1.43, 1.44) in the same way. A comparison of equations (1.53, 1.54) and (1.15, 1.16) confirms that, with these definitions, the strain e_{ij} is a second order Cartesian tensor. The corresponding three-dimensional strain transformation equations have the same form as (1.22) and can be obtained using the strain-displacement relation (1.51) and the vector transformation rule (1.19). This derivation is left as an excercise for the reader (Problem 1.9).

1.3 Stress-strain relations

The fundamental assumption of linear elasticity is that the material obeys Hooke's law, implying that there is a linear relation between stress and strain. The most general such relation can be written in index notation as

$$\sigma_{ij} = c_{ijkl}e_{kl} \ ; \quad e_{ij} = s_{ijkl}\sigma_{kl} \ , \tag{1.55}$$

where c_{ijkl}, s_{ijkl} are the *elasticity tensor* and the *compliance tensor* respectively. Both the elasticity tensor and the compliance tensor must satisfy the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij} = c_{ijlk} \ , \tag{1.56}$$

which follow from (i) the symmetry of the stress and strain tensors (e.g. $\sigma_{ij} = \sigma_{ji}$) and (ii) the reciprocal theorem, which we shall discuss in Chapter 34. Substituting the strain-displacement relations (1.51) into the first of (1.55) and using (1.56) to combine appropriate terms, we obtain

$$\sigma_{ij} = c_{ijkl}\frac{\partial u_k}{\partial x_l} \ . \tag{1.57}$$

In this book, we shall restrict attention to the case where the material is *isotropic*, meaning that the tensors c_{ijkl}, s_{ijkl} are invariant under coördinate transformation. In other words, a test specimen of the material would show identical behaviour regardless of its orientation relative to the original piece of material from which it was manufactured.

We shall develop the various forms of Hooke's law for the isotropic medium starting from the results of the uniaxial tensile test, in which the only non-zero stress component is σ_{xx}. Symmetry considerations then show that the shear strains e_{xy}, e_{yz}, e_{zx} must be zero and that the transverse strains e_{yy}, e_{zz} must be equal. Thus, the most general relation can be written in the form

$$e_{xx} = \frac{\sigma_{xx}}{E} \; ; \quad e_{yy} = e_{zz} = -\frac{\nu\sigma_{xx}}{E} \; ,$$

where *Young's modulus* E and *Poisson's ratio* ν are experimentally determined constants. The strain components due to a more general triaxial state of stress can then be written down by linear superposition as

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} \tag{1.58}$$

$$e_{yy} = -\frac{\nu\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} \tag{1.59}$$

$$e_{zz} = -\frac{\nu\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} \; . \tag{1.60}$$

The relation between e_{xy} and σ_{xy} can then be obtained by using the coördinate transformation relations. We know that there are three principal directions such that if we align them with x, y, z, we have

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \tag{1.61}$$

and hence by symmetry

$$e_{xy} = e_{yz} = e_{zx} = 0 \; . \tag{1.62}$$

Using a system aligned with the principal directions, we write

$$e'_{xy} = (e_{yy} - e_{xx}) \sin\theta \cos\theta \tag{1.63}$$

from equations (1.53, 1.62), and hence

$$\begin{aligned} e'_{xy} &= \frac{(\sigma_{yy} - \sigma_{xx})(1+\nu)\sin\theta\cos\theta}{E} \\ &= \frac{(1+\nu)\sigma'_{xy}}{E} \; , \end{aligned} \tag{1.64}$$

from (1.58, 1.59, 1.16, 1.61).

We define

$$\mu = \frac{E}{2(1+\nu)} \; , \tag{1.65}$$

so that equation (1.64) takes the form

$$e'_{xy} = \frac{\sigma'_{xy}}{2\mu} \; . \tag{1.66}$$

1.3.1 Lamé's constants

It is often desirable to solve equations (1.58–1.60) to express σ_{xx} in terms of e_{xx} etc. The solution is routine and leads to the equation

$$\sigma_{xx} = \frac{E\nu(e_{xx} + e_{yy} + e_{zz})}{(1+\nu)(1-2\nu)} + \frac{Ee_{xx}}{(1+\nu)} \tag{1.67}$$

and similar equations, which are more concisely written in the form

$$\sigma_{xx} = \lambda e + 2\mu e_{xx} \tag{1.68}$$

etc., where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)} \tag{1.69}$$

and

$$e \equiv e_{xx} + e_{yy} + e_{zz} \equiv e_{ii} \equiv \text{div } \boldsymbol{u} \tag{1.70}$$

is known as the *dilatation.*

The stress-strain relations (1.66, 1.68) can be written more concisely in the index notation in the form

$$\sigma_{ij} = \lambda\delta_{ij}e_{mm} + 2\mu e_{ij} \ , \tag{1.71}$$

where δ_{ij} is the *Kronecker delta*, defined as 1 if $i=j$ and 0 if $i \neq j$. Equivalently, we can use equation (1.55) with

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) \ . \tag{1.72}$$

The constants λ, μ are known as Lamé's constants. Young's modulus and Poisson's ratio can be written in terms of Lamé's constants through the equations

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)} \ ; \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \ . \tag{1.73}$$

1.3.2 Dilatation and bulk modulus

The dilatation, e, is easily shown to be invariant as to coördinate transformation and is therefore a scalar quantity. In physical terms it is the local volumetric strain, since a unit cube increases under strain to a block of dimensions $(1+e_{xx}), (1+e_{yy}), (1+e_{zz})$ and hence the volume change is

$$\varDelta V = (1 + e_{xx})(1 + e_{yy})(1 + e_{zz}) - 1 = e_{xx} + e_{yy} + e_{zz} + O(e_{xx}e_{yy}) \ . \tag{1.74}$$

It can be shown that the dilatation e and the rotation vector $\boldsymbol{\omega}$ are harmonic — i.e. $\nabla^2 e = \nabla^2 \boldsymbol{\omega} = 0$. For this reason, many early solutions of elasticity problems were formulated in terms of these variables, so as to make use of the wealth of mathematical knowledge about harmonic functions. We now have other more convenient ways of expressing elasticity problems in terms of harmonic functions, which will be discussed in Chapter 20 *et seq.*.

The dilatation is proportional to the mean stress $\overline{\sigma}$ through a constant known as the *bulk modulus* K_b. Thus, from equation (1.68),

$$\overline{\sigma} \equiv \frac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3} \equiv \frac{1}{3}\sigma_{ii} = K_b e \; , \tag{1.75}$$

where

$$K_b = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} \; . \tag{1.76}$$

We note that $K_b \rightarrow \infty$ if $\nu \rightarrow 0.5$, in which case the material is described as *incompressible*.

PROBLEMS

1. Show that

$$\text{(i)} \quad \frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad \text{and} \quad \text{(ii)} \quad R = \sqrt{x_i x_i} \; ,$$

where $R = |\boldsymbol{R}|$ is the distance from the origin. Hence find $\partial R/\partial x_j$ in index notation. Confirm your result by finding $\partial R/\partial x$ in x, y, z notation.

2. Prove that the partial derivatives $\partial^2 f/\partial x^2; \partial^2 f/\partial x \partial y; \partial^2 f/\partial y^2$ of the scalar function $f(x, y)$ transform into the rotated coördinate system x', y' by rules similar to equations (1.15–1.17).

3. Show that the direction cosines defined in (1.19) satisfy the identity

$$l_{ij} l_{ik} = \delta_{jk} \; .$$

Hence or otherwise, show that the product $\sigma_{ij}\sigma_{ij}$ is invariant under coördinate transformation.

4. By restricting the indices i, j etc. to the values 1,2 only, show that the two-dimensional stress transformation relations (1.15–1.17) can be obtained from (1.22) using the two-dimensional direction cosines (1.20).

5. Use the index notation to develop concise expressions for the three stress invariants I_1, I_2, I_3 and the equivalent tensile stress σ_E.

6. Choosing a local coördinate system x_1, x_2, x_3 aligned with the three principal axes, determine the tractions on the *octahedral plane* defined by the unit vector

$$\boldsymbol{n} = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}^T$$

which makes equal angles with all three principal axes, if the principal stresses are $\sigma_1, \sigma_2, \sigma_3$. Hence show that the magnitude of the resultant shear stress on this plane is $\sqrt{2}\sigma_E/3$, where σ_E is given by equation (1.31).

7. A rigid body is subjected to a small rotation $\omega_z = \Omega \ll 1$ about the z-axis. If the displacement of the origin is zero, find expressions for the three displacement components u_x, u_y, u_z as functions of x, y, z.

8. Use the index notation to develop a general expression for the derivative

$$\frac{\partial u_i}{\partial x_j}$$

in terms of strains and rotations.

9. Use the three-dimensional vector transformation rule (1.19) and the index notation to prove that the strain components (1.51) transform according to the equation

$$e'_{ij} = l_{ip} l_{jq} e_{pq} \; .$$

Hence show that the dilatation e_{ii} is invariant under coördinate transformation.

10. Find an index notation expression for the compliance tensor s_{ijkl} of equation (1.55) for the isotropic elastic material in terms of the elastic constants E, ν.

11. Show that equations (1.58–1.60, 1.64) can be written in the concise form

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\delta_{ij}\sigma_{mm}}{E} \; . \tag{1.77}$$

2

EQUILIBRIUM AND COMPATIBILITY

We can think of an elastic solid as a highly redundant framework — each particle is built-in to its neighbours. For such a framework, we expect to get some equations from considerations of *equilibrium*, but not as many as there are unknowns. The deficit is made up by *compatibility* conditions — statements that the deformed components must fit together. These latter conditions will relate the dimensions and hence the strains of the deformed components and in order to express them in terms of the same unknowns as the stresses (forces) we need to make use of the stress-strain relations as applied to each component separately.

If we were to approximate the continuous elastic body by a system of interconnected elastic bars, this would be an exact description of the solution procedure. The only difference in treating the continuous medium is that the system of algebraic equations is replaced by partial differential equations describing the same physical or geometrical principles.

2.1 Equilibrium equations

We consider a small rectangular block of material — side $\delta x, \delta y, \delta z$ — as shown in Figure 2.1. We suppose that there is a body force[1], $\boldsymbol{p}$ per unit volume and that the stresses vary with position so that the stress components on opposite faces of the block differ by the differential quantities $\delta\sigma_{xx}, \delta\sigma_{xy}$ etc. Only those stress components which act in the x-direction are shown in Figure 2.1 for clarity.

[1] A body force is one that acts directly on every particle of the body, rather than being applied by tractions at its boundaries and transmitted to the various particles by means of internal stresses. This is an important distinction which will be discussed further in Chapter 7, below. The commonest example of a body force is that due to gravity.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_2, © Springer Science+Business Media B.V. 2010

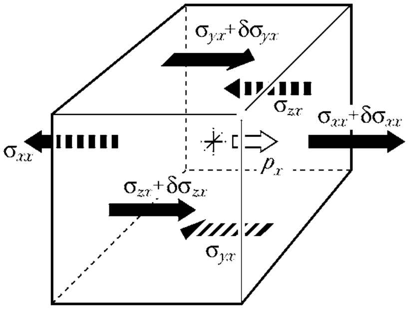

Figure 2.1: Forces in the x-direction on an elemental block.

Resolving forces in the x-direction, we find

$$\begin{aligned}(\sigma_{xx} + \delta\sigma_{xx} - \sigma_{xx})\delta y\delta z + (\sigma_{xy} + \delta\sigma_{xy} - \sigma_{xy})\delta z\delta x \\ +(\sigma_{xz} + \delta\sigma_{xz} - \sigma_{xz})\delta x\delta y + p_x\delta x\delta y\delta z = 0\ .\end{aligned} \tag{2.1}$$

Hence, dividing through by $(\delta x\delta y\delta z)$ and proceeding to the limit as these infinitesimals tend to zero, we obtain

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + p_x = 0\ . \tag{2.2}$$

Similarly, we have

$$\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} + p_y = 0 \tag{2.3}$$

$$\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + p_z = 0\ , \tag{2.4}$$

or in index notation

$$\frac{\partial\sigma_{ij}}{\partial x_j} + p_i = 0\ . \tag{2.5}$$

These are the *differential equations of equilibrium.*

2.2 Compatibility equations

The easiest way to satisfy the equations of compatibility — as in framework problems — is to express all the strains in terms of the displacements. In a framework, this ensures that the components fit together by identifying the displacement of points in two links which are pinned together by the same symbol. If the framework is redundant, the number of pin displacements thereby introduced is less than the number of component lengths (and hence extensions) determined by them — and the number of unknowns is therefore reduced.

For the continuum, the process is essentially similar, but much more straightforward. We define the six components of strain in terms of displacements through equations (1.51). These six equations introduce only three unknowns (u_x, u_y, u_z) and hence the latter can be eliminated to give equations constraining the strain components.

For example, from (1.36, 1.37) we find

$$\frac{\partial^2 e_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2} \; ; \; \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial y \partial x^2} \tag{2.6}$$

and hence

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \,, \tag{2.7}$$

from (1.49) — i.e.

$$\frac{\partial^2 e_{xx}}{\partial y^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} + \frac{\partial^2 e_{yy}}{\partial x^2} = 0 \,. \tag{2.8}$$

Two more equations of the same form may be obtained by permuting suffices. It is tempting to pursue an analogy with algebraic equations and argue that, since the six strain components are defined in terms of three independent displacement components, we must be able to develop three (i.e. $6-3$) independent compatibility equations. However, it is easily verified that, in addition to the three equations similar to (2.8), the strains must satisfy three more equations of the form

$$\frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} - \frac{\partial e_{xy}}{\partial z}\right) \,. \tag{2.9}$$

The resulting six equations are independent in the sense that no one of them can be derived from the other five, which all goes to show that arguments for algebraic equations do not always carry over to partial differential equations.

A concise statement of the six compatibility equations can be written in the index notation in the form

$$\epsilon_{ijk}\epsilon_{pqr}\frac{\partial^2 e_{jq}}{\partial x_k \partial x_r} = 0 \,. \tag{2.10}$$

The full set of six equations makes the problem very complicated. In practice, therefore, most three-dimensional problems are treated in terms of displacements instead of strains, which satisfies the requirement of compatibility identically. However, in two dimensions, all except one of the compatibility equations degenerate to identities, so that a formulation in terms of stresses or strains is more practical.

2.2.1 The significance of the compatibility equations

The physical meaning of equilibrium is fairly straightforward, but people often get mixed up about just what is being guaranteed by the compatibility equations.

Single-valued displacements

Mathematically, we might say that the strains are compatible *when they are definable in terms of a single-valued, continuously differentiable displacement.*

We could imagine reversing this process — i.e. integrating the strains (displacement gradients) to find the relative displacement $(\boldsymbol{u}_B - \boldsymbol{u}_A)$ of two points A, B in the solid (see Figure 2.2).

Figure 2.2: Path of the integral in equation (2.11).

Formally we can write[2]

$$\boldsymbol{u}_B - \boldsymbol{u}_A = \int_A^B \frac{\partial \boldsymbol{u}}{\partial S} ds \ , \tag{2.11}$$

where s is a coordinate representing distance along a line S between A and B. If the displacements are to be single-valued, it mustn't make any difference if we change the line S provided it remains within the solid. In other words, the integral should be *path-independent.*

The compatibility equations are not quite sufficient to guarantee this.

They *do* guarantee

$$\oint \frac{\partial \boldsymbol{u}}{\partial S} ds = 0 \tag{2.12}$$

around an *infinitesimal* closed loop.

Now, we could make infinitesimal changes in our line from A to B by taking in such small loops until the whole line was sensibly changed, thus satisfying the requirement that the integral (2.11) is path-independent, but the fact that the line is changed infinitesimally stops us from taking a qualitatively (topologically) different route through a multiply connected body. For example, in Figure 2.3, it is impossible to move S_1 to S_2 by infinitesimal changes without passing outside the body[3].

[2] Explicit forms of the integral (2.11) in terms of the strain components were developed by E.Cesaro and are known as *Cesaro integrals*. See, for example, A.E.H.Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th edn., Dover, 1944, §156A.

[3] For a more rigorous discussion of this question, see A.E.H.Love, *loc.cit.* or B.A.Boley and J.H.Weiner, *Theory of Thermal Stresses*, John Wiley, New York, 1960, §§3.6–3.8.

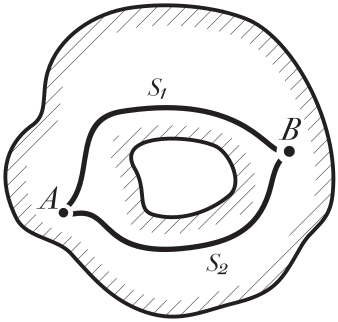

Figure 2.3: Qualitatively different integration paths in a multiply connected body.

In practice, if a solid is multiply connected it is usually easier to work in terms of displacements and by-pass this problem. Otherwise the equivalence of topologically different paths in integrals like (2.11, 2.12) must be explicitly enforced.

Compatibility of deformed shapes

A more 'physical' way of thinking of compatibility is to state that *the separate particles of the body must deform under load in such a way that they fit together after deformation*. This interpretation is conveniently explored by way of a 'jig-saw' analogy.

We consider a two-dimensional body cut up as a jig-saw puzzle (Figure 2.4), of which the pieces are deformable. Figure 2.4(a) shows the original puzzle and 2.4(b) the puzzle after deformation by some external loads $\boldsymbol{F_i}$. (The pieces are shown as initially rectangular to aid visualization.)

The deformation of the puzzle must satisfy the following conditions:-

(i) The forces on any given piece (including external forces if any) must be in equilibrium.
(ii) The deformed pieces must be the right *shape* to fit together to make the deformed puzzle (Figure 2.4(b)).

For the continuous solid, the compatibility conditions (2.10) guarantee that (ii) is satisfied, since shape is defined by displacement derivatives and hence by strains. However, if the puzzle is multiply connected — e.g. if it has a central hole — condition (ii) is *not* sufficient to ensure that the deformed pieces can be assembled into a coherent body. Suppose we imagine assembling the 'deformed' puzzle working from one piece outwards. The partially completed puzzle is simply connected and the shape condition is sufficient to ensure the success of our assembly until we reach a piece which would convert the partial puzzle to a multiply connected body. This piece will be the right *shape* for both sides, but may be the wrong *size* for the separation. If so, it will be possible to leave the puzzle in a state with a discontinuity as shown in Figure 2.4(c) at any arbitrarily chosen position[4], but there will be no way in which Figure 2.4(b) can be constructed.

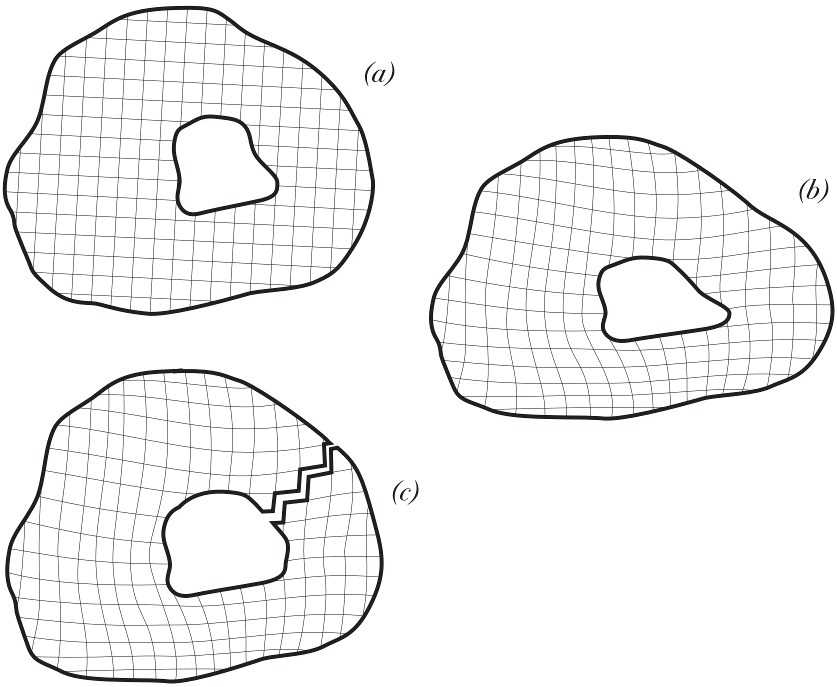

Figure 2.4: A multiply connected, deformable jig-saw puzzle: (a) Before deformation; (b) After deformation; (c) Result of attempting to assemble the deformed puzzle if equation (2.12) is not satisfied for any closed path encircling the hole.

The lines defining the two sides of the discontinuity in Figure 2.4(c) are the same shape and hence, in the most general case, the discontinuity can be defined by six arbitrary constants corresponding to the three rigid-body translations and three rotations needed to move one side to coincide with the other[5]. We therefore get six additional algebraic conditions for each hole in a multiply connected body.

[4] since we could move one or more pieces from one side of the discontinuity to the other.

[5] This can be proved by evaluating the relative displacements of corresponding points on opposite sides of the cut, using an integral of the form (2.11) whose path does not cross the cut.

2.3 Equilibrium equations in terms of displacements

Recalling that it is often easier to work in terms of displacements rather than stresses to avoid complications with the compatibility conditions, it is convenient to express the equilibrium equations in terms of displacements. This is most easily done in index notation. Substituting for the stresses from (1.57) into the equilibrium equation (2.5), we obtain

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + p_i = 0 \; . \tag{2.13}$$

For the special case of isotropy, we can substitute for c_{ijkl} from (1.72), obtaining

$$\Big[\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right)\Big]\frac{\partial^2 u_k}{\partial x_j \partial x_l} + p_i = 0$$

or

$$\lambda\frac{\partial^2 u_l}{\partial x_i \partial x_l} + \mu\frac{\partial^2 u_i}{\partial x_l \partial x_l} + \mu\frac{\partial^2 u_k}{\partial x_k \partial x_i} + p_i = 0 \; .$$

The first and third terms in this equation have the same form, since k, l are dummy indices. We can therefore combine them and write

$$(\lambda + \mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + p_i = 0 \; . \tag{2.14}$$

Also, noting that

$$\lambda + \mu = \mu\left(1 + \frac{2\nu}{1-2\nu}\right) = \frac{\mu}{(1-2\nu)} \tag{2.15}$$

from (1.69), we have

$$\frac{\partial^2 u_k}{\partial x_i \partial x_k} + (1-2\nu)\frac{\partial^2 u_i}{\partial x_k \partial x_k} + \frac{(1-2\nu)p_i}{\mu} = 0 \; , \tag{2.16}$$

or in vector notation

$$\boldsymbol{\nabla}\text{div}\; \boldsymbol{u} + (1-2\nu)\nabla^2\boldsymbol{u} + \frac{(1-2\nu)\boldsymbol{p}}{\mu} = 0 \; . \tag{2.17}$$

PROBLEMS

1. Show that, if there are no body forces, the dilatation e must satisfy the condition

$$\nabla^2 e = 0 \ .$$

2. Show that, if there are no body forces, the rotation $\boldsymbol{\omega}$ must satisfy the condition

$$\nabla^2 \boldsymbol{\omega} = 0 \ .$$

3. One way of satisfying the compatibility equations in the absence of rotation is to define the components of displacement in terms of a potential function ψ through the relations

$$u_x = \frac{\partial \psi}{\partial x} \ \ ; \ \ u_y = \frac{\partial \psi}{\partial y} \ \ ; \ \ u_z = \frac{\partial \psi}{\partial z} \ .$$

Use the stress-strain relations to derive expressions for the stress components in terms of ψ.

Hence show that the stresses will satisfy the equilibrium equations in the absence of body forces if and only if

$$\nabla^2 \psi = \text{constant} \ .$$

4. Plastic deformation during a manufacturing process generates a state of residual stress in the large body $z > 0$. If the residual stresses are functions of z only and the surface $z = 0$ is not loaded, show that the stress components $\sigma_{yz}, \sigma_{zx}, \sigma_{zz}$ must be zero everywhere.

5. By considering the equilibrium of a small element of material similar to that shown in Figure 1.2, derive the three equations of equilibrium in cylindrical polar coördinates r, θ, z.

6. In cylindrical polar coördinates, the strain-displacement relations for the 'in-plane' strains are

$$e_{rr} = \frac{\partial u_r}{\partial r} \ \ ; \ \ e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \ \ ; \ \ e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \ .$$

Use these relations to obtain a compatibility equation that must be satisfied by the three strains.

7. If no stresses occur in a body, an increase in temperature T causes unrestrained thermal expansion defined by the strains

$$e_{xx} = e_{yy} = e_{zz} = \alpha T \ \ ; \ \ e_{xy} = e_{yz} = e_{zx} = 0 \ .$$

Show that this is possible only if T is a linear function of x, y, z and that otherwise stresses must be induced in the body, regardless of the boundary conditions.

8. If there are no body forces, show that the equations of equilibrium and compatibility imply that

$$(1+\nu)\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = 0 \; .$$

9. Using the strain-displacement relation (1.51), show that an alternative statement of the compatibility condition is that the tensor

$$\mathcal{C}_{ijkl} \equiv \frac{\partial^2 e_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 e_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 e_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i \partial x_l} = 0 \; .$$

Part II

TWO-DIMENSIONAL PROBLEMS

3

PLANE STRAIN AND PLANE STRESS

A problem is two-dimensional if the field quantities such as stress and displacement depend on only two coördinates (x, y) and the boundary conditions are imposed on a line $f(x, y) = 0$ in the xy-plane.

In this sense, there are strictly no two-dimensional problems in elasticity. There *are* circumstances in which the stresses are independent of the z-coördinate, but all real bodies must have some bounding surfaces which are not represented by a line in the xy-plane. The two-dimensionality of the resulting fields depends upon the boundary conditions on such surfaces being of an appropriate form.

3.1 Plane strain

It might be argued that a closed line in the xy-plane *does* define a solid body — namely an infinite cylinder of the appropriate cross-section whose axis is parallel to the z-direction. However, making a body infinite does not really dispose of the question of boundary conditions, since there are usually some implied boundary conditions 'at infinity'. For example, the infinite cylinder could be in a state of uniaxial tension, $\sigma_{zz} = C$, where C is an arbitrary constant. However, a unique two-dimensional infinite cylinder problem can be defined by demanding that u_x, u_y be independent of z and that $u_z = 0$ for all x, y, z, in which case it follows that

$$e_{zx} = e_{zy} = e_{zz} = 0 \; . \tag{3.1}$$

This is the two-dimensional state known as *plane strain*.

In view of the stress-strain relations, an equivalent statement to equation (3.1) is

$$\sigma_{zx} = \sigma_{zy} = 0 \;\; ; \;\; u_z = 0 \; , \tag{3.2}$$

and hence a condition of plane strain will exist in a *finite* cylinder provided that (i) any tractions or displacements imposed on the sides of the cylinder

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_3, © Springer Science+Business Media B.V. 2010

are independent of z and have no component in the z-direction and (ii) the cylinder has plane ends (e.g. $z=\pm c$) which are in frictionless contact with two plane rigid walls.

From the condition $e_{zz}=0$ (3.1) and the stress-strain relations, we can deduce

$$0=\frac{\sigma_{zz}}{E}-\frac{\nu(\sigma_{xx}+\sigma_{yy})}{E}$$

— i.e.

$$\sigma_{zz}=\nu(\sigma_{xx}+\sigma_{yy}) \tag{3.3}$$

and hence

$$\begin{aligned} e_{xx} &= \frac{\sigma_{xx}}{E}-\frac{\nu\sigma_{yy}}{E}-\frac{\nu^2(\sigma_{xx}+\sigma_{yy})}{E} \\ &= \frac{(1-\nu^2)\sigma_{xx}}{E}-\frac{\nu(1+\nu)\sigma_{yy}}{E} \; . \end{aligned} \tag{3.4}$$

Of course, there are comparatively few practical applications in which a cylinder with plane ends is constrained between frictionless rigid walls, but fortunately the plane strain solution can be used in an approximate sense for a cylinder with any end conditions, provided that the length of the cylinder is large compared with its cross-sectional dimensions.

We shall illustrate this with reference to the long cylinder of Figure 3.1, for which the ends, $z=\pm c$ are traction-free and the sides are loaded by tractions which are independent of z.

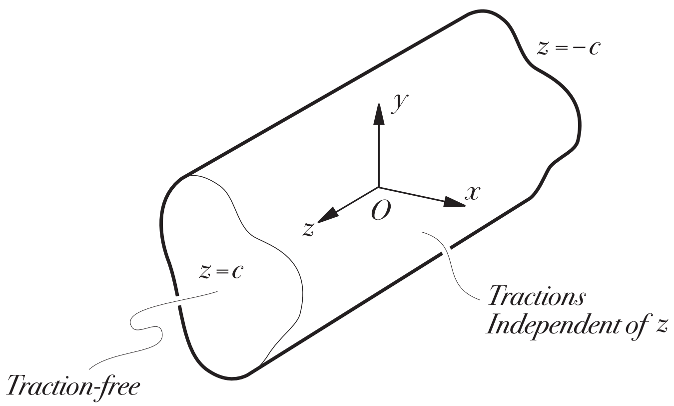

Figure 3.1: The long cylinder with traction-free ends.

We first solve the problem under the plane strain assumption, obtaining an exact solution in which all the stresses are independent of z and in which there exists a normal stress σ_{zz} on all z-planes, which we can calculate from equation (3.3).

The plane strain solution satisfies all the boundary conditions of the problem *except* that σ_{zz} also acts on the end faces, $z=\pm c$, where it appears as an unwanted normal traction. We therefore seek a *corrective* solution which, when superposed on the plane strain solution, removes the unwanted normal tractions on the surfaces $z=\pm c$, without changing the boundary conditions on the sides of the cylinder.

3.1.1 The corrective solution

The corrective solution must have zero tractions on the sides of the cylinder and a prescribed normal traction (equal and opposite to that obtained in the plane strain solution) on the end faces, A.

This is a fully three-dimensional problem which generally has no closed-form solution. However, if the prescribed tractions have the linear form

$$\sigma_{zz} = B + Cx + Dy \; , \tag{3.5}$$

the solution can be obtained in the context of Mechanics of Materials by treating the long cylinder as a beam subjected to an axial force

$$F = \int_A \sigma_{zz} dxdy \tag{3.6}$$

and bending moments

$$M_x = \int_A \sigma_{zz} y dxdy \;\; ; \;\; M_y = -\int_A \sigma_{zz} x dxdy \tag{3.7}$$

about the axes Ox, Oy respectively, where the origin O is chosen to coincide with the centroid of the cross-section[1].

If the original plane strain solution does not give a distribution of σ_{zz} of this convenient linear form, we can still use equations (3.6, 3.7) to define the force and moments for an *approximate* Mechanics of Materials corrective solution. The error involved in using this approximate solution will be that associated with yet another corrective solution corresponding to the problem in which the end faces of the cylinder are loaded by tractions equal to the *difference* between those in the plane strain solution and the Mechanics of Materials linear form (3.5) associated with the force resultants (3.6, 3.7).

In this final corrective solution, the ends of the cylinder are loaded by *self-equilibrated tractions*, since the Mechanics of Materials approximation is carefully chosen to have the same force and moment resultants as the required exact solution and in such cases, we anticipate that significant stresses will only be generated in the immediate vicinity of the ends — or more precisely, in regions whose distance from the ends is comparable with the cross-sectional

[1] The Mechanics of Materials solution for axial force and pure bending is in fact *exact* in the sense of the Theory of Elasticity, since the stresses clearly satisfy the equilibrium equations (2.5) and the resulting strains are linear functions of x, y and hence satisfy the compatibility equations (2.10) identically.

dimensions of the cylinder. If the cylinder is many times longer than its cross-sectional dimensions, there will be a substantial portion near the centre where the final corrective solution gives negligible stresses and hence where the sum of the original plane strain solution and the Mechanics of Materials correction is a good approximation to the actual three-dimensional stress field.

3.1.2 Saint-Venant's principle

The thesis that a self-equilibrating system of loads produces only local effects is known as *Saint-Venant's principle*. It seems intuitively reasonable, but has not been proved rigorously except for certain special cases — some of which we shall encounter later in this book (see for example Chapter 6). It can be seen as a consequence of the rule that alternate load paths through a structure share the load in proportion with their stiffnesses. If a region of the boundary is loaded by a self-equilibrating system of tractions, the stiffest paths are the shortest — i.e. those which do not penetrate far from the loaded region. Hence, the longer paths — which are those which contribute to stresses distant from the loaded region — carry relatively little load.

Note that if the local tractions are *not* self-equilibrating, some of the load paths must go to other distant parts of the boundary and hence there will be significant stresses in intermediate regions. For this reason, it is important to superpose the Mechanics of Materials approximate corrective solution when solving plane strain problems for long cylinders with traction free ends. By contrast, the final stage of solving the 'Saint-Venant problem' to calculate the correct stresses near the ends is seldom of much importance in practical problems, since the ends, being traction-free, are not generally points of such high stress as the interior.

As a point of terminology, we shall refer to problems in which the boundary conditions on the ends have been corrected only in the sense of force and moment resultants as being solved in the *weak* sense with respect to these boundaries. Boundary conditions are satisfied in the *strong* sense when the tractions are specified in a pointwise rather than a force resultant sense.

3.2 Plane stress

Plane stress is an approximate solution, in contrast to plane strain, which is exact. In other words, plane strain is a special solution of the complete three-dimensional equations of elasticity, whereas plane stress is only approached in the limit as the thickness of the loaded body tends to zero.

It is argued that if the two bounding z-planes of a thin plate are sufficiently close in comparison with the other dimensions, and if they are also free of tractions, the stresses on all parallel z-planes will be sufficiently small to be neglected in which case we write

$$\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0 \; , \tag{3.8}$$

for all x, y, z.

It then follows that

$$e_{zx} = e_{zy} = 0 \; , \tag{3.9}$$

but $e_{zz} \neq 0$, being given in fact by

$$e_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \; . \tag{3.10}$$

The two-dimensional stress-strain relations are then

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \tag{3.11}$$

$$e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \; . \tag{3.12}$$

The fact that plane stress is not an exact solution can best be explained by considering the compatibility equation

$$\frac{\partial^2 e_{yy}}{\partial z^2} - 2\frac{\partial^2 e_{yz}}{\partial y \partial z} + \frac{\partial^2 e_{zz}}{\partial y^2} = 0 \; . \tag{3.13}$$

Since *ex-hypothesi* none of the stresses vary with z, the first two terms in this equation are identically zero and hence the equation will be satisfied if and only if

$$\frac{\partial^2 e_{zz}}{\partial y^2} = 0 \; . \tag{3.14}$$

This in turn requires

$$\frac{\partial^2(\sigma_{xx} + \sigma_{yy})}{\partial y^2} = 0 \; , \tag{3.15}$$

from equation (3.10).

Applying similar arguments to the other compatibility equations, we conclude that the plane stress assumption is exact if and only if $(\sigma_{xx}+\sigma_{yy})$ is a linear function of x, y. i.e. if

$$\sigma_{xx} + \sigma_{yy} = B + Cx + Dy \; . \tag{3.16}$$

The attentive reader will notice that this condition is exactly equivalent to (3.3, 3.5). In other words, the plane stress solution is exact if and only if the 'weak form' solution of the corresponding plane strain solution is exact. Of course this is not a coincidence. In this special case, the process of off-loading the ends of the long cylinder in the plane strain solution has the effect of making σ_{zz} zero throughout the cylinder and hence the resulting solution also satisfies the plane stress assumption, whilst remaining exact.

3.2.1 Generalized plane stress

The approximate nature of the plane stress formulation is distasteful to elasticians of a more mathematical temperament, who prefer to preserve the rigour of an exact theory. This can be done by the contrivance of defining the *average* stresses across the thickness of the plate — e.g.

$$\overline{\sigma}_{xx} \equiv \frac{1}{2c}\int_{-c}^{c} \sigma_{xx} dx \ . \tag{3.17}$$

It can then be shown that the average stresses so defined satisfy the plane stress equations *exactly*. This is referred to as the *generalized plane stress* formulation.

In practice, of course, the gain in rigour is illusory unless we can also establish that the stress variation across the section is small, so that the local values are reasonably close to the average. A fully three-dimensional theory of thin plates under in-plane loading shows that the plane stress assumption is a good approximation except in regions whose distance from the boundary is comparable with the plate thickness[2].

3.2.2 Relationship between plane stress and plane strain

The solution of a problem under either the plane strain or plane stress assumptions involves finding a two-dimensional stress field, defined in terms of the components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$, which satisfies the equilibrium equations (2.5), and for which the corresponding strains, e_{xx}, e_{xy}, e_{yy}, satisfy the only non-trivial compatibility equation (2.8). The equilibrium and compatibility equations are the same in both formulations, the only difference being in the relation between the stress and strain components, which for normal stresses are given by (3.4) for plane strain and (3.11, 3.12) for plane stress. The relation between the shear stress σ_{xy} and the shear strain e_{xy} is the same for both formulations and is given by equation (1.66). Thus, from a mathematical perspective, the plane strain solution simply looks like the plane stress solution for a material with different elastic constants and *vice versa*. In fact, it is easily verified that equation (3.4) can be obtained from (3.11) by making the substitutions

$$E = \frac{E'}{(1-\nu'^2)} \ ; \ \ \nu = \frac{\nu'}{(1-\nu')} \ , \tag{3.18}$$

and then dropping the primes. This substitution also leaves the shear stress-shear strain relation unchanged as required.

An alternative approach is to write the two-dimensional stress-strain relations in the form

[2] See for example S.P.Timoshenko and J.N.Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 3rd. edn., 1970, §98.

$$e_{xx} = \left(\frac{\kappa+1}{8\mu}\right)\sigma_{xx} - \left(\frac{3-\kappa}{8\mu}\right)\sigma_{yy} \; ; \quad e_{yy} = \left(\frac{\kappa+1}{8\mu}\right)\sigma_{yy} - \left(\frac{3-\kappa}{8\mu}\right)\sigma_{xx}$$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu} \,, \tag{3.19}$$

where κ is *Kolosov's constant*, defined as

$$\begin{aligned} \kappa = & \; (3-4\nu) && \text{for plane strain} \\ = & \left(\frac{3-\nu}{1+\nu}\right) && \text{for plane stress.} \end{aligned} \tag{3.20}$$

In the following chapters, we shall generally treat two-dimensional problems with the plane stress assumptions, noting that results for plane strain can be recovered by the substitution (3.18) when required.

PROBLEMS

1. The plane strain solution for the stresses in the rectangular block $0<x<a, -b<y<b, -c<z<c$ with a given loading is

$$\sigma_{xx} = \frac{3Fxy}{2b^3} \; ; \; \sigma_{xy} = \frac{3F(b^2-y^2)}{4b^3} \; ; \; \sigma_{yy} = 0 \; ; \; \sigma_{zz} = \frac{3\nu Fxy}{2b^3} \, .$$

Find the tractions on the surfaces of the block and illustrate the results on a sketch of the block.

We wish to use this solution to solve the corresponding problem in which the surfaces $z = \pm c$ are traction-free. Determine an approximate corrective solution for this problem by offloading the unwanted force and moment resultants using the elementary bending theory. Find the maximum error in the stress σ_{zz} in the corrected solution and compare it with the maximum tensile stress in the plane strain solution.

2. For a solid in a state of plane stress, show that if there are body forces p_x, p_y per unit volume in the direction of the axes x, y respectively, the compatibility equation can be expressed in the form

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = -(1+\nu)\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right) \, .$$

Hence deduce that the stress distribution for any particular case is independent of the material constants and the body forces, provided the latter are constant.

3.(i) Show that the compatibility equation (2.8) is satisfied by unrestrained thermal expansion ($e_{xx} = e_{yy} = \alpha T, e_{xy} = 0$), provided that the temperature, T, is a two-dimensional harmonic function — i.e.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \; .$$

(ii) Hence deduce that, subject to certain restrictions which you should explicitly list, no thermal stresses will be induced in a thin body with a steady-state, two-dimensional temperature distribution and no boundary tractions.

(iii) Show that an initially straight line on such a body will be distorted by the heat flow in such a way that its curvature is proportional to the local heat flux across it.

4. Find the inverse relations to equations (3.18) — i.e. the substitutions that should be made for the elastic constants E, ν in a plane strain solution if we want to recover the solution of the corresponding plane stress problem.

5. Show that in a state of plane stress without body forces, the in-plane displacements must satisfy the equations

$$\nabla^2 u_x + \left(\frac{1+\nu}{1-\nu}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \;\; ; \;\; \nabla^2 u_y + \left(\frac{1+\nu}{1-\nu}\right)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \; .$$

6. Show that in a state of plane strain without body forces,

$$\frac{\partial e}{\partial x} = \left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial y} \;\; ; \;\; \frac{\partial e}{\partial y} = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial x} \; .$$

7. If a material is incompressible ($\nu = 0.5$), a state of hydrostatic stress $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ produces no strain. One way to write the corresponding stress-strain relations is

$$\sigma_{ij} = 2\mu e_{ij} - q\delta_{ij} \; ,$$

where q is an unknown hydrostatic pressure which will generally vary with position. Also, the condition of incompressibility requires that the dilatation

$$e \equiv e_{kk} = 0 \; .$$

Show that under plain strain conditions, the stress components and the hydrostatic pressure q must satisfy the equations

$$\nabla^2 q = \operatorname{div} \boldsymbol{p} \;\; \text{and} \;\; \sigma_{xx} + \sigma_{yy} = -2q \; ,$$

where $\boldsymbol{p}$ is the body force.

4

STRESS FUNCTION FORMULATION

4.1 The concept of a scalar stress function

Newton's law of gravitation states that two heavy bodies attract each other with a force proportional to the inverse square of their distance — thus it is essentially a vector theory, being concerned with forces. However, the idea of a scalar gravitational potential can be introduced by defining the work done in moving a unit mass from infinity to a given point in the field. The principle of conservation of energy requires that this be a unique function of position and it is easy to show that the gravitational force at any point is then proportional to the gradient of this scalar potential. Thus, the original vector problem is reduced to a problem about a scalar potential and its derivatives.

In general, scalars are much easier to deal with than vectors. In particular, they lend themselves very easily to coördinate transformations, whereas vectors (and to an even greater extent tensors) require a set of special transformation rules (e.g. Mohr's circle).

In certain field theories, the scalar potential has an obvious physical significance. For example, in the conduction of heat, the temperature is a scalar potential in terms of which the vector heat flux can be defined. However, it is not necessary to the method that such a physical interpretation can be given. The gravitational potential can be given a physical interpretation as discussed above, but this interpretation may never feature in the solution of a particular problem, which is simply an excercise in the solution of a certain partial differential equation with appropriate boundary conditions. In the theory of elasticity, we make use of scalar potentials called *stress functions* or *displacement functions* which have no obvious physical meaning other than their use in defining stress or displacement components in terms of derivatives.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_4, © Springer Science+Business Media B.V. 2010

4.2 Choice of a suitable form

In the choice of a suitable form for a stress or displacement function, there is only one absolute rule — that the operators which define the relationship between the scalar and vector (or tensor) quantities should indeed define a vector (or tensor).

For example, it is appropriate to define the displacement in terms of the first derivatives (the gradient) of a scalar or to define the stress components in terms of the second derivatives of a scalar, since the second derivatives of a scalar form the components of a Cartesian tensor.

In effect, what we are doing in requiring this similarity of form between the definitions and the defined quantity is ensuring that the relationship is preserved in coördinate transformations. It would be quite possible to work out an elasticity problem in terms of the displacement components u_x, u_y, u_z, treating these as essentially scalar quantities which vary with position — indeed this was a technique which was used in early theories. However, we would then get into trouble as soon as we tried to make any statements about quantities in other coördinate directions. By contrast, if we define (for example) $u_x = \partial\psi/\partial x$; $u_y = \partial\psi/\partial y$; $u_z = \partial\psi/\partial z$, (i.e. $\boldsymbol{u} = \boldsymbol{\nabla}\psi$), it immediately follows that $u'_x = \partial\psi/\partial x'$ for any x'.

4.3 The Airy stress function

If there are no body forces and the stress components are independent of z, they must satisfy the two equilibrium equations

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \; ; \quad \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0 \; . \tag{4.1}$$

The two terms in each of these equations are therefore equal and opposite. Suppose we take the first term in the first equation and integrate it once with respect to x and once with respect to y to define a new function $\psi(x, y)$. We can then write

$$\frac{\partial\sigma_{xx}}{\partial x} = -\frac{\partial\sigma_{xy}}{\partial y} \equiv \frac{\partial^2\psi}{\partial x\partial y} \; , \tag{4.2}$$

and a particular solution is

$$\sigma_{xx} = \frac{\partial\psi}{\partial y} \; ; \quad \sigma_{xy} = -\frac{\partial\psi}{\partial x} \; . \tag{4.3}$$

Substituting the second of these results in the second equilibrium equation and using the same technique to define another new function $\phi(x, y)$, we obtain

$$\frac{\partial\sigma_{yy}}{\partial y} = -\frac{\partial\sigma_{yx}}{\partial x} = \frac{\partial^2\psi}{\partial x^2} \equiv \frac{\partial^3\phi}{\partial x^2\partial y} \; , \tag{4.4}$$

so a particular solution is

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \psi = \frac{\partial \phi}{\partial y} \; . \tag{4.5}$$

Using (4.3, 4.5) we can then express all three stress components in terms of ϕ as

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \; \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \; \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; . \tag{4.6}$$

This representation was introduced by G.B.Airy[1] in 1862 and ϕ is therefore generally referred to as the *Airy stress function*. It is clear from the derivation that the stress components (4.6) will satisfy the two-dimensional equilibrium equations (4.1) for all functions ϕ. Also, the mathematical operations involved in obtaining (4.6) can be performed for any two-dimensional stress distribution satisfying (4.1) and hence the representation (4.6) is *complete*, meaning that it is capable of representing the most general such stress distribution. It is worth noting that the derivation makes no reference to Hooke's law and hence the Airy stress function also provides a complete representation for two-dimensional problems involving inelastic constitutive laws — for example plasticity theory — where its satisfaction of the equilibrium equations remains an advantage.

The technique introduced in this section can be used in other situations to define a representation of a field satisfying one or more partial differential equations. The trick is to rewrite the equation so that terms involving different unknowns are on opposite sides of the equals sign and then equate each side of the equation to a derivative that contains the lowest common denominator of the derivatives on the two sides, as in equations (4.2, 4.4). For other applications of this method, see Problems 4.3, 4.4 and 21.2.

4.3.1 Transformation of coördinates

It is easily verified that equation (4.6) transforms as a Cartesian tensor as required. For example, using (1.43) we can write

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x'^2} &= \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right)^2 \phi \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2\theta + \frac{\partial^2 \phi}{\partial y^2} \sin^2\theta + 2\frac{\partial^2 \phi}{\partial x \partial y} \sin\theta\cos\theta \; , \end{aligned} \tag{4.7}$$

from which using (1.17) and (4.6) we deduce that $\sigma'_{yy} = \partial^2\phi/\partial x'^2$ as required.

If we were simply to seek a representation of a two-dimensional Cartesian tensor in terms of the derivatives of a scalar function without regard to the

[1] For a good historical survey of the development of potential function methods in Elasticity, see H.M.Westergaard, *Theory of Elasticity and Plasticity*, Dover, New York, 1964, Chapter 2.

equilibrium equations, the Airy stress function is not the most obvious form. It would seem more natural to write $\sigma_{xx} = \partial^2\psi/\partial x^2$; $\sigma_{yy} = \partial^2\psi/\partial y^2$; $\sigma_{xy} = \partial^2\psi/\partial x\partial y$ and indeed this also leads to a representation which is widely used as part of the general three-dimensional solution (see Chapter 20 below). In some books, it is stated or implied that the Airy function is used because it satisfies the equilibrium equation, but this is rather misleading, since the 'more obvious form' — although it requires some constraints on ψ in order to satisfy equilibrium — can be shown to define displacements which automatically satisfy the compatibility condition which is surely equally useful[2]. The real reason for preferring the Airy function is that it is complete, whilst the alternative is more restrictive, as we shall see in §20.1.

4.3.2 Non-zero body forces

If the body force $\boldsymbol{p}$ is not zero, but is of a restricted form such that it can be written $\boldsymbol{p} = -\boldsymbol{\nabla} V$, where V is a two-dimensional scalar potential, equations (4.6) can be generalized by including an extra term $+V$ in each of the normal stress components whilst leaving the shear stress definition unchanged. It is easily verified that, with this modification, the equilibrium equations are again satisfied.

Problems involving body forces will be discussed in more detail in Chapter 7 below. For the moment, we restrict attention to the the case where $\boldsymbol{p} = 0$ and hence where the representation (4.6) is appropriate.

4.4 The governing equation

As discussed in Chapter 2, the stress or displacement field has to satisfy the equations of equilibrium and compatibility if they are to describe permissible states of an elastic body. In stress function representations, this generally imposes certain constraints on the choice of stress function, which can be expressed by requiring it to be a solution of a certain partial differential equation.

We have already seen that the equilibrium condition is satisfied identically by the use of the Airy stress function ϕ, so it remains to determine the governing equation for ϕ by substituting the representation (4.6) into the compatibility equation in two dimensions.

4.4.1 The compatibility condition

We first express the compatibility condition in terms of stresses using the stress-strain relations, obtaining

[2] See Problem 2.3.

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu \frac{\partial^2 \sigma_{yy}}{\partial y^2} - 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = 0 \ , \tag{4.8}$$

where we have cancelled a common factor of $1/E$.

We then substitute for the stress components from (4.6) obtaining

$$\frac{\partial^4 \phi}{\partial y^4} - \nu \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + 2(1+\nu)\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial x^4} - \nu \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0 \ , \tag{4.9}$$

i.e.

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi = 0 \ . \tag{4.10}$$

This equation is known as the *biharmonic equation* and is usually written in the concise form

$$\nabla^4 \phi = 0 \ . \tag{4.11}$$

The biharmonic equation is the governing equation for the Airy stress function in elasticity problems. Thus, by using the Airy stress function representation, the problem of determining the stresses in an elastic body is reduced to that of finding a solution of equation (4.11) (i.e. a *biharmonic function*) whose derivatives satisfy certain boundary conditions on the surfaces.

4.4.2 Method of solution

Historically, the boundary-value problem for the Airy stress function has been approached in a semi-inverse way — i.e. by using the variation of tractions along the boundaries to give a clue to the kind of function required, but then exploring the stress fields developed from a wide range of such functions and selecting a combination which can be made to satisfy the required conditions. The disadvantage with this method is that it requires a wide experience of particular solutions and even then is not guaranteed to be successful.

A more modern method which has the advantage of always developing an appropriate stress function if the boundary conditions are unmixed (i.e. all specified in terms of stresses or all in terms of displacements) is based on representing the stress function in terms of analytic functions of the complex variable. This method will be discussed in Chapter 19. However, although it is powerful and extremely elegant, it requires a certain familarity with the properties of functions of the complex variable and generally involves the evaluation of a contour integral, which may not be as convenient a numerical technique as a direct series or finite difference attack on the original problem using real stress functions. There are some exceptions — notably for bodies which are susceptible of a simple conformal transformation into the unit circle.

4.4.3 Reduced dependence on elastic constants

It is clear from dimensional considerations that, when the compatibility equation is expressed in terms of ϕ, Young's modulus must appear in every term and can therefore be cancelled as in equation (4.8), but it is an unexpected bonus that the Poisson's ratio terms also cancel in (4.9), leaving an equation that is independent of elastic constants. It follows that the stress field in a simply connected elastic body[3] in a state of plane strain or plane stress is independent of the material properties if the boundary conditions are expressed in terms of tractions and, in particular, that the plane stress and plane strain fields are identical.

Dundurs[4] has shown that a similar reduced dependence on elastic constants occurs in plane problems involving interfaces between two dissimilar elastic materials. In such cases, three independent dimensionless parameters (e.g. $\mu_1/\mu_2, \nu_1, \nu_2$) can be formed from the elastic constants $\mu_1, \nu_1, \mu_2, \nu_2$ of the materials 1, 2 respectively, but Dundurs proved that the stress field can be written in terms of only two parameters which he defined as

$$\alpha = \left(\frac{\kappa_1 + 1}{\mu_1} - \frac{\kappa_2 + 1}{\mu_2}\right) \Big/ \left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}\right) \tag{4.12}$$

$$\beta = \left(\frac{\kappa_1 - 1}{\mu_1} - \frac{\kappa_2 - 1}{\mu_2}\right) \Big/ \left(\frac{\kappa_1 + 1}{\mu_1} + \frac{\kappa_2 + 1}{\mu_2}\right) , \tag{4.13}$$

where κ is Kolosov's constant (see equation (3.20)). It can be shown that Dundurs' parameters must lie in the range $-1 \leq \alpha \leq 1$; $-0.5 \leq \beta \leq 0.5$ if $0 \leq \nu_1, \nu_2 \leq 0.5$.

[3] The restriction to simply connected bodies is necessary, since in a multiply connected body there is an implied displacement condition, as explained in §2.2 above.

[4] J.Dundurs, Discussion on 'Edge bonded dissimilar orthogonal elastic wedges under normal and shear loading', *ASME Journal of Applied Mechanics,* Vol. 36 (1969), pp.650–652.

PROBLEMS

1. Newton's law of gravitation states that two heavy particles of mass m_1, m_2 respectively will experience a mutual attractive force

$$F = \frac{\gamma m_1 m_2}{R^2} ,$$

where R is the distance between the particles and γ is the gravitational constant. Use an energy argument and superposition to show that the force acting on a particle of mass m_0 can be written

$$\boldsymbol{F} = -\gamma m_0 \boldsymbol{\nabla} V ,$$

where

$$V(x, y, z) = -\iiint_{\Omega} \frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} ,$$

Ω represents the volume of the universe and ρ is the density of material in the universe, which will generally be a function of position (ξ, η, ζ).

Could a similar method have been used if Newton's law had been of the more general form

$$F = \frac{\gamma m_1 m_2}{R^\lambda} .$$

If so, what would have been the corresponding expression for V? If not, why not?

2. An ionized liquid in an electric field experiences a body force $\boldsymbol{p}$. Show that the liquid can be in equilibrium only if $\boldsymbol{p}$ is a conservative vector field. **Hint:** Remember that a stationary liquid must be everywhere in a state of hydrostatic stress.

3. An *antiplane* state of stress is one for which the only non-zero stress components are σ_{zx}, σ_{zy} and these are independent of z. Show that two of the three equilibrium equations are then satisfied identically if there is no body force. Use a technique similar to that of §4.3 to develop a representation of the non-zero stress components in terms of a scalar function, such that the remaining equilibrium equation is satisfied identically.

4. If a body of fairly general axisymmetric shape is loaded in torsion, the only non-zero stress components in cylindrical polar coördinates are $\sigma_{\theta r}, \sigma_{\theta z}$ and these are required to satisfy the equilibrium equation

$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{2\sigma_{\theta r}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z} = 0 .$$

Use a technique similar to that of §4.3 to develop a representation of these stress components in terms of a scalar function, such that the equilibrium equation is satisfied identically.

5.(i) Show that the function

$$\phi = y\omega + \psi$$

satisfies the biharmonic equation if ω, ψ are both *harmonic* (i.e. $\nabla^2\omega = 0, \nabla^2\psi = 0$).

(ii) Develop expressions for the stress components in terms of ω, ψ, based on the use of ϕ as an Airy stress function.

(iii) Show that a solution suitable for the half-plane $y > 0$ subject to normal surface tractions only (i.e. $\sigma_{xy} = 0$ on $y = 0$) can be obtained by writing

$$\omega = -\frac{\partial\psi}{\partial y}$$

and hence that under these conditions the normal stress σ_{xx} near the surface $y = 0$ is equal to the applied traction σ_{yy}.

(iv) Do you think this is a rigorous proof? Can you think of any exceptions? If so, at what point in your proof of section (iii) can you find a lack of generality?

6. The constitutive law for an orthotropic elastic material in plane stress can be written

$$e_{xx} = s_{11}\sigma_{xx} + s_{12}\sigma_{yy} \;\; ; \;\; e_{yy} = s_{12}\sigma_{xx} + s_{22}\sigma_{yy} \;\; ; \;\; e_{xy} = s_{44}\sigma_{xy} \; ,$$

where $s_{11}, s_{12}, s_{22}, s_{44}$ are elastic constants.

Using the Airy stress function ϕ to represent the stress components, find the equation that must be satisfied by ϕ.

7. Show that if the two-dimensional function $\omega(x, y)$ is harmonic ($\nabla^2\omega = 0$), the function

$$\phi = (x^2 + y^2)\omega$$

will be biharmonic.

8. The constitutive law for an isotropic incompressible elastic material can be written

$$\sigma_{ij} = \overline{\sigma}\delta_{ij} + 2\mu e_{ij} \; ,$$

where

$$\overline{\sigma} = \frac{\sigma_{kk}}{3}$$

represents an arbitrary hydrostatic stress field. Some soils can be approximated as an incompressible material whose modulus varies linearly with depth, so that

$$\mu = Mz$$

for the half space $z > 0$.

Use the displacement function representation

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi$$

to develop a potential function solution for the stresses in such a body. Show that the functions $\phi, \overline{\sigma}$ must satisfy the relations

$$\nabla^2\phi = 0 \ ; \ \overline{\sigma} = -2M\frac{\partial\phi}{\partial z}$$

and hence obtain expressions for the stress components in terms of the single harmonic function ϕ.

If the half-space is loaded by a normal pressure

$$\sigma_{zz}(x,y,0) = -p(x,y) \ ; \ \sigma_{zx}(x,y,0) = \sigma_{zy}(x,y,0) = 0 \ ,$$

show that the corresponding normal surface displacement $u_z(x,y,0)$ is linearly proportional to the local pressure $p(x,y)$ and find the constant of proportionality[5].

9. Show that Dundurs' constant $\beta \to 0$ for plane strain in the limit where $\nu_1 = 0.5$ and $\mu_1/\mu_2 \to 0$ — i.e. material 1 is incompressible and has a much lower shear modulus[6] than material 2. What is the value of α in this limit?

10. Solve Problem 3.7 for the case where there is no body force, using the Airy stress function ϕ to represent the stress components. Hence show that the governing equation is $\nabla^4\phi=0$, as in the case of compressible materials.

[5] For an alternative proof of this result see C.R.Calladine and J.A.Greenwood, Line and point loads on a non-homogeneous incompressible elastic half-space, *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 31 (1978), pp.507–529.

[6] This is a reasonable approximation for the important case of rubber (material 1) bonded to steel (material 2).

5

PROBLEMS IN RECTANGULAR COÖRDINATES

The Cartesian coördinate system (x, y) is clearly particularly suited to the problem of determining the stresses in a rectangular body whose boundaries are defined by equations of the form $x = a, y = b$. A wide range of such problems can be treated using stress functions which are polynomials in x, y. In particular, polynomial solutions can be obtained for 'Mechanics of Materials' type beam problems in which a rectangular bar is bent by an end load or by a distributed load on one or both faces.

5.1 Biharmonic polynomial functions

In rectangular coördinates, the biharmonic equation takes the form

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{5.1}$$

and it follows that any polynomial in x, y of degree less than four will be biharmonic and is therefore appropriate as a stress function. However, for higher order polynomial terms, equation (5.1) is not identically satisfied. Suppose, for example, that we consider just those terms in a general polynomial whose combined degree (the sum of the powers of x and y) is N. We can write these terms in the form

$$P_N(x, y) = A_0 x^N + A_1 x^{N-1} y + A_2 x^{N-2} y^2 + \ldots + A_N y^N \tag{5.2}$$

$$= \sum_{i=0}^{N} A_i x^{N-i} y^i \ , \tag{5.3}$$

where we note that there are $(N+1)$ independent coefficients, $A_i (i = 0, N)$. If we now substitute $P_N(x, y)$ into equation (5.1), we shall obtain a new polynomial of degree $(N-4)$, since each term is differentiated four times. We can denote this new polynomial by $Q_{N-4}(x, y)$ where

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_5, © Springer Science+Business Media B.V. 2010

$$Q_{N-4}(x,y) = \nabla^4 P_N(x,y) \tag{5.4}$$

$$= \sum_{i=0}^{N-4} B_i x^{(N-4-i)} y^i \; . \tag{5.5}$$

The $(N-3)$ coefficients $B_0, \ldots, B_{N-4}$ are easily obtained by expanding the right-hand side of equation (5.4) and equating coefficients. For example,

$$B_0 = N(N-1)(N-2)(N-3)A_0 + 4(N-2)(N-3)A_2 + 24A_4 \; . \tag{5.6}$$

Now the original function $P_N(x,y)$ will be biharmonic if and only if $Q_{N-4}(x,y)$ is zero for all x, y and this in turn is only possible if every term in the series (5.5) is identically zero, since the polynomial terms are all linearly independent of each other. In other words

$$B_i = 0 \;\; ; \;\; i = 0 \text{ to } (N-4) \; . \tag{5.7}$$

These conditions can be converted into a corresponding set of $(N-3)$ equations for the coefficients A_i. For example, the equation $B_0 = 0$ gives

$$N(N-1)(N-2)(N-3)A_0 + 4(N-2)(N-3)A_2 + 24A_4 = 0 \; , \tag{5.8}$$

from (5.6). We shall refer to the $(N-3)$ equations of this form as *constraints* on the coefficients A_i, since the coefficients are constrained to satisfy them if the original polynomial is to be biharmonic.

One approach would be to use the constraint equations to eliminate $(N–3)$ of the unknown coefficients in the original polynomial — for example, we could treat the first four coefficients, A_0, A_1, A_2, A_3, as unknown constants and use the constraint equations to define all the remaining coefficients in terms of these unknowns. Equation (5.8) would then be treated as an equation for A_4 and the subsequent constraint equations would each define one new constant in the series. It may help to consider a particular example at this stage. Suppose we consider the fifth degree polynomial

$$P_5(x,y) = A_0 x^5 + A_1 x^4 y + A_2 x^3 y^2 + A_3 x^2 y^3 + A_4 x y^4 + A_5 y^5 \; , \tag{5.9}$$

which has six independent coefficients. Substituting into equation (5.4), we obtain the first degree polynomial

$$Q_1(x,y) = (120A_0 + 24A_2 + 24A_4)x + (24A_1 + 24A_3 + 120A_5)y \; . \tag{5.10}$$

The coefficients of x and y in Q_1 must both be zero if P_5 is to be biharmonic and we can write the resulting two constraint equations in the form

$$A_4 = -5A_0 - A_2 \tag{5.11}$$

$$A_5 = -A_1/5 - A_3/5 \; . \tag{5.12}$$

Finally, we use (5.11, 5.12) to eliminate A_4, A_5 in the original definition of P_5, obtaining the definition of the most general biharmonic fifth degree polynomial

$$P_5(x,y) = A_0(x^5 - 5xy^4) + A_1(x^4y - y^5/5) \\ + A_2(x^3y^2 - xy^4) + A_3(x^2y^3 - y^5/5) \; . \tag{5.13}$$

This function will be biharmonic for any values of the four independent constants A_0, A_1, A_2, A_3. We can express this by stating that the biharmonic polynomial P_5 has four degrees of freedom.

In general, the polynomial Q is of degree 4 less than P because the biharmonic equation is of degree 4. It follows that there are always four fewer constraint equations than there are coefficients in the original polynomial P and hence that they can be satisfied leaving a polynomial with 4 degrees of freedom. However, the process degenerates if $N < 3$.

In view of the above discussion, it might seem appropriate to write an expression for the general polynomial of degree N in the form of equation (5.13) as a preliminary to the solution of polynomial problems in rectangular coördinates. However, as can be seen from equation (5.13), the resulting expressions are algebraically messy and this approach becomes unmanageable for problems of any complexity. Instead, it turns out to be more straightforward algebraically to define problems in terms of the simpler unconstrained polynomials like equation (5.2) and to impose the constraint equations at a later stage in the solution.

5.1.1 Second and third degree polynomials

We recall that the stress components are defined in terms of the stress function ϕ through the relations

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \tag{5.14}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \tag{5.15}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; . \tag{5.16}$$

It follows that when the stress function is a polynomial of degree N in x, y, the stress components will be polynomials of degree $(N-2)$. In particular, constant and linear terms in ϕ correspond to null stress fields (zero stress everywhere) and can be disregarded.

The second degree polynomial

$$\phi = A_0x^2 + A_1xy + A_2y^2 \tag{5.17}$$

yields the stress components

$$\sigma_{xx} = 2A_2 \;\; ; \;\; \sigma_{xy} = -A_1 \;\; ; \;\; \sigma_{yy} = 2A_0 \tag{5.18}$$

and hence corresponds to the most general state of biaxial uniform stress.

The third degree polynomial

$$\phi = A_0x^3 + A_1x^2y + A_2xy^2 + A_3y^3 \tag{5.19}$$

yields the stress components

$$\sigma_{xx} = 2A_2x + 6A_3y \;\; ; \;\; \sigma_{xy} = -2A_1x - 2A_2y \;\; ; \;\; \sigma_{yy} = 6A_0x + 2A_1y \; . \tag{5.20}$$

If we arbitrarily set $A_0, A_1, A_2 = 0$, the only remaining non-zero stress will be

$$\sigma_{xx} = 6A_3y \; , \tag{5.21}$$

which corresponds to a state of pure bending, when applied to the rectangular beam $-a<x<a, -b<y<b$, as shown in Figure 5.1.

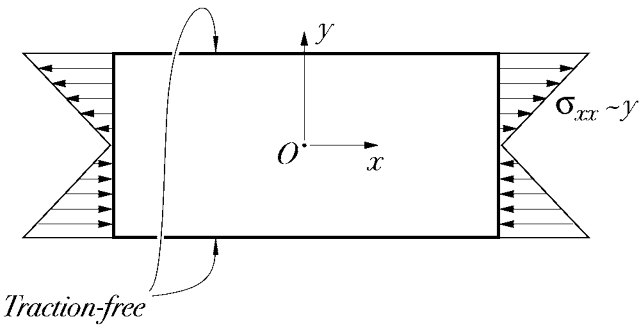

Figure 5.1: The rectangular beam in pure bending.

The other terms in equation (5.19) correspond to a more general state of bending. For example, the constant A_0 describes bending of the beam by tractions σ_{yy} applied to the boundaries $y = \pm b$, whilst the terms involving shear stresses σ_{xy} could be obtained by describing a general state of biaxial bending with reference to a Cartesian coördinate system which is not aligned with the axes of the beam.

The above solutions are of course very elementary, but we should remember that, in contrast to the Mechanics of Materials solutions for simple bending, they are obtained without making any simplifying assumptions about the stress fields. For example, we have not assumed that plane sections remain plane, nor have we demanded that the beam be long in comparison with its depth. Thus, the present section could be taken as verifying the *exactness* of the Mechanics of Materials solutions for uniform stress and simple bending, as applied to a rectangular beam.

5.2 Rectangular beam problems

5.2.1 Bending of a beam by an end load

Figure 5.2 shows a rectangular beam, $0 < x < a, -b < y < b$, subjected to a transverse force, F at the end $x = 0$, and built-in at the end $x = a$, the horizontal boundaries $y = \pm b$ being traction free. The boundary conditions for this problem are most naturally written in the form

$$\sigma_{xy} = 0 \; ; \; y = \pm b \tag{5.22}$$

$$\sigma_{yy} = 0 \; ; \; y = \pm b \tag{5.23}$$

$$\sigma_{xx} = 0 \; ; \; x = 0 \tag{5.24}$$

$$\int_{-b}^{b} \sigma_{xy} dy = F \; ; \; x = 0 \; . \tag{5.25}$$

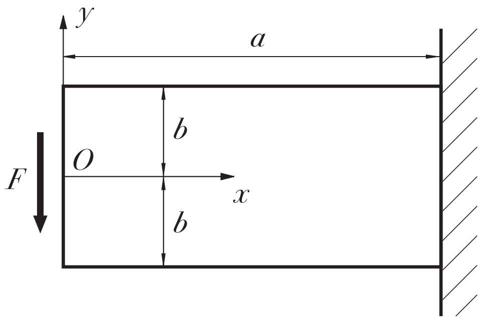

Figure 5.2: Cantilever with an end load.

The boundary condition (5.25) is imposed in the *weak* form, which means that the value of the traction is not specified at each point on the boundary — only the force resultant is specified. In general, we shall find that problems for the rectangular beam have finite polynomial solutions when the boundary conditions on the ends are stated in the weak form, but that the *strong* (i.e. pointwise) boundary condition can only be satisfied on all the boundaries by an infinite series or transform solution. This issue is further discussed in Chapter 6.

Mechanics of Materials considerations suggest that the bending moment in this problem will vary linearly with x and hence that the stress component σ_{xx} will have a leading term proportional to xy. This in turn suggests a fourth degree polynomial term xy^3 in the stress function ϕ. Our procedure is therefore to start with the trial stress function

$$\phi = C_1 xy^3 \; , \tag{5.26}$$

examine the corresponding tractions on the boundaries and then seek a *corrective* solution which, when superposed on equation (5.26), yields the solution to the problem. Substituting (5.26) into (5.14–5.16), we obtain

$$\sigma_{xx} = 6C_1xy \tag{5.27}$$
$$\sigma_{xy} = -3C_1y^2 \tag{5.28}$$
$$\sigma_{yy} = 0 \ , \tag{5.29}$$

from which we note that the boundary conditions (5.23, 5.24) are satisfied identically, but that (5.22) is not satisfied, since (5.28) implies the existence of an unwanted uniform shear traction $-3C_1b^2$ on both of the edges $y=\pm b$. This unwanted traction can be removed by superposing an appropriate uniform shear stress, through the additional stress function term C_2xy. Thus, if we define

$$\phi = C_1xy^3 + C_2xy \ , \tag{5.30}$$

equations (5.27, 5.29) remain unchanged, whilst (5.28) is modified to

$$\sigma_{xy} = -3C_1y^2 - C_2 \ . \tag{5.31}$$

The boundary condition (5.22) can now be satisfied if we choose C_2 to satisfy the equation

$$C_2 = -3C_1b^2 \ , \tag{5.32}$$

so that

$$\sigma_{xy} = 3C_1(b^2 - y^2) \ . \tag{5.33}$$

The constant C_1 can then be determined by substituting (5.33) into the remaining boundary condition (5.25), with the result

$$C_1 = \frac{F}{4b^3} \ . \tag{5.34}$$

The final stress field is therefore defined through the stress function

$$\phi = \frac{F(xy^3 - 3b^2xy)}{4b^3} \ , \tag{5.35}$$

the corresponding stress components being

$$\sigma_{xx} = \frac{3Fxy}{2b^3} \tag{5.36}$$
$$\sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \tag{5.37}$$
$$\sigma_{yy} = 0 \ . \tag{5.38}$$

The solution of this problem is given in the Mathematica and Maple files ‘S521’.

We note that no boundary conditions have been specified on the built-in end, $x=a$. In the weak form, these would be

$$\int_{-b}^{b} \sigma_{xx} dy = 0 \;\; ; \;\; x = a \tag{5.39}$$

$$\int_{-b}^{b} \sigma_{xy} dy = F \;\; ; \;\; x = a \tag{5.40}$$

$$\int_{-b}^{b} \sigma_{xx} y dy = Fa \;\; ; \;\; x = a \; . \tag{5.41}$$

However, if conditions (5.22–5.25) are satisfied, (5.39–5.41) are merely equivalent to the condition that the whole beam be in equilibrium. Now the Airy stress function is so defined that whatever stress function is used, the corresponding stress field will satisfy equilibrium in the local sense of equations (2.5). Furthermore, if every particle of a body is separately in equilibrium, it follows that the whole body will also be in equilibrium. It is therefore not necessary to enforce equations (5.39–5.41), since if we were to check them, we should necessarily find that they are satisfied identically.

5.2.2 Higher order polynomials — a general strategy

In the previous section, we developed the solution by trial and error, starting from the leading term whose form was dictated by equilibrium considerations. A more general technique is to identify the highest order polynomial term from equilibrium considerations and then write down the most general polynomial of that degree and below. The constant multipliers on the various terms are then obtained by imposing boundary conditions and biharmonic constraint equations.

The only objection to this procedure is that it involves a lot of algebra. For example, in the problem of §5.2.1, we would have to write down the most general polynomial of degree 4 and below, which involves 12 separate terms even when we exclude the linear and constant terms as being null. However, this is not a serious difficulty if we are using Maple or Mathematica, so we shall first develop the steps needed for this general strategy. Shortcuts which would reduce the complexity of the algebra in a manual calculation will then be discussed in §5.2.3.

Order of the polynomial

Suppose we have a normal traction on the surface $y=b$ varying with x^n. In Mechanics of Materials terms, this corresponds to a distributed load proportional to x^n and elementary equilibrium considerations show that the bending moment can then be expected to contain a term proportional to x^{n+2}. This in turn implies a bending stress σ_{xx} proportional to $x^{n+2}y$ and a term in the stress function proportional to $x^{n+2}y^3$ — i.e. a term of polynomial order $(n+5)$. A corresponding argument for shear tractions proportional to x^m shows that we require a polynomial order of $(m+4)$.

We shall show in Chapter 28 that these arguments from equilibrium and elementary bending theory define the highest order of polynomial required to satisfy any polynomial boundary conditions on the lateral surfaces of a beam, even in three-dimensional problems. A sufficient polynomial order can therefore be selected by the following procedure:-

(i) Identify the highest order polynomial term n in the normal tractions σ_{yy} on the surfaces $y=\pm b$.
(ii) Identify the highest order polynomial term m in the shear tractions σ_{yx} on the surfaces $y=\pm b$.
(iii) Use a polynomial for ϕ including all polynomial terms of order $\max(m+4, n+5)$ and below, but excluding constant and linear terms.

In the special case where both surfaces $y=\pm b$ are traction-free, it is sufficient to use a polynomial of 4th degree and below (as in §5.2.1).

Solution procedure

Once an appropriate polynomial has been identified for ϕ, we proceed as follows:-

(i) Substitute ϕ into the biharmonic equation (5.1), leading to a set of constraint equations, as in §5.1.
(ii) Substitute ϕ into equations (5.14–5.16), to obtain the stress components as functions of x, y.
(iii) Substitute the equations defining the boundaries (e.g. $x=0, y=b, y=-b$ in the problem of §5.2.1) into appropriate[1] stress components, to obtain the tractions on each boundary.
(iv) For the longer boundaries (where strong boundary conditions will be enforced), sort the resulting expressions into powers of x or y and equate coefficients with the corresponding expression for the prescribed tractions.
(v) For the shorter boundaries, substitute the tractions into the appropriate weak boundary conditions, obtaining three further independent algebraic equations.

The equations so obtained will generally not all be linearly independent, but they will be sufficient to determine all the coefficients uniquely. The solvers in Maple and Mathematica can handle this redundancy.

[1] Recall from §1.1.1 that the only stress components that act on (e.g.) $y=b$ are those which contain y as one of the suffices.

Example

We illustrate this procedure with the example of Figure 5.3, in which a rectangular beam $-a < x < a$, $-b < y < b$ is loaded by a uniform compressive normal traction p on $y=b$ and simply supported at the ends.

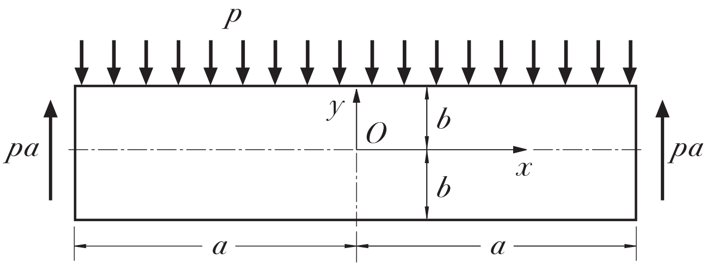

Figure 5.3: Simply supported beam with a uniform load.

The boundary conditions on the surfaces $y=\pm b$ can be written

$$\sigma_{yx} = 0 \;\; ; \;\; y = \pm b \tag{5.42}$$

$$\sigma_{yy} = -p \;\; ; \;\; y = b \tag{5.43}$$

$$\sigma_{yy} = 0 \;\; ; \;\; y = -b\,. \tag{5.44}$$

These boundary conditions are to be satisfied in the strong sense. To complete the problem definition, we shall require three linearly independent weak boundary conditions on one or both of the ends $x=\pm a$. We might use symmetry and equilibrium to argue that the load will be equally shared between the supports, leading to the conditions[2]

$$F_x(a) = \int_{-b}^{b} \sigma_{xx}(a,y)dy = 0 \tag{5.45}$$

$$F_y(a) = \int_{-b}^{b} \sigma_{xy}(a,y)dy = pa \tag{5.46}$$

$$M(a) = \int_{-b}^{b} \sigma_{xx}(a,y)ydy = 0 \tag{5.47}$$

[2] It is not necessary to use symmetry arguments to obtain three linearly independent weak conditions. Since the beam is simply supported, we know that

$$M(a) = \int_{-b}^{b} \sigma_{xx}(a,y)ydy = 0 \;\; ; \;\; M(-a) = \int_{-b}^{b} \sigma_{xx}(-a,y)ydy = 0$$

$$F_x(a) = \int_{-b}^{b} \sigma_{xx}(a,y)dy = 0\,.$$

It is easy to verify that these conditions lead to the same solution as (5.45–5.47).

on the end $x = a$. As explained in §5.2.1, we do not need to enforce the additional three weak conditions on $x = -a$.

The normal traction is uniform — i.e. it varies with x^0 ($n = 0$), so the above criterion demands a polynomial of order $(n+5) = 5$. We therefore write

$$\begin{aligned}\phi = {} & C_1x^2 + C_2xy + C_3y^2 + C_4x^3 + C_5x^2y + C_6xy^2 + C_7y^3 + C_8x^4 \\ & + C_9x^3y + C_{10}x^2y^2 + C_{11}xy^3 + C_{12}y^4 + C_{13}x^5 + C_{14}x^4y \\ & + C_{15}x^3y^2 + C_{16}x^2y^3 + C_{17}xy^4 + C_{18}y^5 \; . \end{aligned} \tag{5.48}$$

This is a long expression, but remember we only have to type it in once to the computer file. We can cut and paste the expression in the solution of subsequent problems (and the reader can indeed cut and paste from the web file 'polynomial'). Substituting (5.48) into the biharmonic equation (5.1), we obtain

$$\begin{aligned}(120C_{13} + 24C_{15} + 24C_{17})x + (24C_{14} + 24C_{16} + 120C_{18})y & \\ + (24C_8 + 8C_{10} + 24C_{12}) = 0 & \end{aligned} \tag{5.49}$$

and this must be zero for all x, y leading to the three constraint equations

$$120C_{13} + 24C_{15} + 24C_{17} = 0 \tag{5.50}$$

$$24C_{14} + 24C_{16} + 120C_{18} = 0 \tag{5.51}$$

$$24C_8 + 8C_{10} + 24C_{12} = 0 \; . \tag{5.52}$$

The stresses are obtained by substituting (5.48) into (5.14–5.16) with the result

$$\begin{aligned}\sigma_{xx} = {} & 2C_3 + 2C_6x + 6C_7y + 2C_{10}x^2 + 6C_{11}xy + 12C_{12}y^2 + 2C_{15}x^3 \\ & + 6C_{16}x^2y + 12C_{17}xy^2 + 20C_{18}y^3 \end{aligned} \tag{5.53}$$

$$\begin{aligned}\sigma_{xy} = {} & -C_2 - 2C_5x - 2C_6y - 3C_9x^2 - 4C_{10}xy - 3C_{11}y^2 - 4C_{14}x^3 \\ & - 6C_{15}x^2y - 6C_{16}xy^2 - 4C_{17}y^3 \end{aligned} \tag{5.54}$$

$$\begin{aligned}\sigma_{yy} = {} & 2C_1 + 6C_4x + 2C_5y + 12C_8x^2 + 6C_9xy + 2C_{10}y^2 + 20C_{13}x^3 \\ & + 12C_{14}x^2y + 6C_{15}xy^2 + 2C_{16}y^3 \; . \end{aligned} \tag{5.55}$$

The tractions on $y = b$ are therefore

$$\begin{aligned}\sigma_{yx} = {} & -4C_{14}x^3 - (3C_9 + 6C_{15}b)x^2 - (2C_5 + 4C_{10}b + 6C_{16}b^2)x \\ & - (C_2 + 2C_6b + 3C_{11}b^2 + 4C_{17}b^3) \end{aligned} \tag{5.56}$$

$$\begin{aligned}\sigma_{yy} = {} & 20C_{13}x^3 + (12C_8 + 12C_{14}b)x^2 + (6C_4 + 6C_9b + 6C_{15}b^2)x \\ & + (2C_1 + 2C_5b + +2C_{10}b^2 + 2C_{16}b^3) \end{aligned} \tag{5.57}$$

and these must satisfy equations (5.42, 5.43) for all x, giving

$$4C_{14} = 0 \tag{5.58}$$
$$3C_9 + 6C_{15}b = 0 \tag{5.59}$$
$$2C_5 + 4C_{10}b + 6C_{16}b^2 = 0 \tag{5.60}$$
$$C_2 + 2C_6b + 3C_{11}b^2 + 4C_{17}b^3 = 0 \tag{5.61}$$
$$20C_{13} = 0 \tag{5.62}$$
$$12C_8 + 12C_{14}b = 0 \tag{5.63}$$
$$6C_4 + 6C_9b + 6C_{15}b^2 = 0 \tag{5.64}$$
$$2C_1 + 2C_5b + 2C_{10}b^2 + 2C_{16}b^3 = -p\ . \tag{5.65}$$

A similar procedure for the edge $y=-b$ yields the additional equations

$$3C_9 - 6C_{15}b = 0 \tag{5.66}$$
$$2C_5 - 4C_{10}b + 6C_{16}b^2 = 0 \tag{5.67}$$
$$C_2 - 2C_6b + 3C_{11}b^2 - 4C_{17}b^3 = 0 \tag{5.68}$$
$$12C_8 - 12C_{14}b = 0 \tag{5.69}$$
$$6C_4 - 6C_9b + 6C_{15}b^2 = 0 \tag{5.70}$$
$$2C_1 - 2C_5b + 2C_{10}b^2 - 2C_{16}b^3 = 0\ . \tag{5.71}$$

On $x=a$, we have

$$\sigma_{xx} = 2C_3 + 2C_6a + 6C_7y + 2C_{10}a^2 + 6C_{11}ay + 12C_{12}y^2 + 2C_{15}a^3 + 6C_{16}a^2y + 12C_{17}ay^2 + 20C_{18}y^3 \tag{5.72}$$
$$\sigma_{xy} = -C_2 - 2C_5a - 2C_6y - 3C_9a^2 - 4C_{10}ay - 3C_{11}y^2 - 4C_{14}a^3 - 6C_{15}a^2y - 6C_{16}ay^2 - 4C_{17}y^3\ . \tag{5.73}$$

Substituting into the weak conditions (5.45–5.47) and evaluating the integrals, we obtain the three additional equations

$$4C_3b + 4C_6ab + 4C_{10}a^2b + 8C_{12}b^3 + 4C_{15}a^3b + 8C_{17}ab^3 = 0 \tag{5.74}$$
$$-2C_2b - 4C_5ab - 6C_9a^2b - 2C_{11}b^3 - 8C_{14}a^3b - 4C_{16}ab^3 = pa \tag{5.75}$$
$$4C_7b^3 + 4C_{11}ab^3 + 4C_{16}a^2b^3 + 8C_{18}b^5 = 0\ . \tag{5.76}$$

Finally, we solve equations (5.50–5.52, 5.58–5.71, 5.74–5.76) for the unknown constants $C_1, ..., C_{18}$ and substitute back into (5.48), obtaining

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3 - 10b^3x^2)\ . \tag{5.77}$$

The corresponding stress field is

$$\sigma_{xx} = \frac{p}{20b^3}(15x^2y - 10y^3 - 15a^2y + 6b^2y) \tag{5.78}$$
$$\sigma_{xy} = \frac{3px}{4b^3}(b^2 - y^2) \tag{5.79}$$
$$\sigma_{yy} = \frac{p}{4b^3}(y^3 - 3b^2y - 2b^3)\ . \tag{5.80}$$

The reader is encouraged to run the Maple or Mathematica files 'S522', which contain the above solution procedure. Notice that most of the algebraic operations are generated by quite simple and repetitive commands. These will be essentially similar for any polynomial problem involving rectangular coördinates, so it is a simple matter to modify the program to cover other cases.

5.2.3 Manual solutions — symmetry considerations

If the solution is to be obtained manually, the complexity of the algebra makes the process time consuming and increases the likelihood of errors. Fortunately, the complexity can be reduced by utilizing the natural symmetry of the rectangular beam. In many problems, the loading has some symmetry which can be exploited in limiting the number of independent polynomial terms and even when this is not the case, some saving of complexity can be achieved by representing the loading as the sum of symmetric and antisymmetric parts. We shall illustrate this procedure by repeating the solution of the problem of Figure 5.3.

The problem is symmetrical about the mid-point of the beam and hence, taking the origin there, we deduce that the resulting stress function will contain only *even* powers of x. This immediately reduces the number of terms in the general stress function to 10.

The beam is also symmetrical about the axis $y=0$, but the loading is not. We therefore decompose the problem into the two sub-problems illustrated in Figure 5.4(a,b).

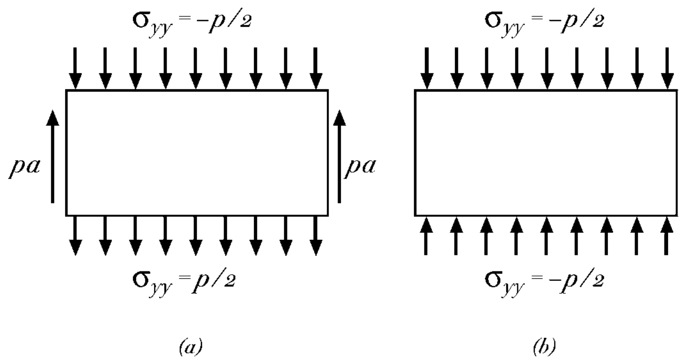

Figure 5.4: Decomposition of the problem into (a) antisymmetric and (b) symmetric parts.

The problem in Figure 5.4(a) is *antisymmetric* in y and hence requires a stress function with only *odd* powers of y, whereas that of Figure 5.4(b) is *symmetric* and requires only *even* powers. In fact, the problem of Figure 5.4(b)

clearly has the trivial solution corresponding to uniform uniaxial compression, $\sigma_{yy}=-p/2$, the appropriate stress function being $\phi=-px^2/4$.

For the problem of Figure 5.4(a), the most general fifth degree polynomial which is even in x and odd in y can be written

$$\phi = C_5x^2y + C_7y^3 + C_{14}x^4y + C_{16}x^2y^3 + C_{18}y^5 \ , \tag{5.81}$$

which has just five degrees of freedom. We have used the same notation for the remaining constants as in (5.48) to aid in comparing the two solutions. The appropriate boundary conditions for this sub-problem are

$$\sigma_{xy} = 0 \ ; \ \ y = \pm b \tag{5.82}$$

$$\sigma_{yy} = \mp\frac{p}{2} \ ; \ \ y = \pm b \tag{5.83}$$

$$\int_{-b}^{b} \sigma_{xx}dy = 0 \ ; \ \ x = \pm a \tag{5.84}$$

$$\int_{-b}^{b} \sigma_{xx}ydy = 0 \ ; \ \ x = \pm a \ . \tag{5.85}$$

Notice that, in view of the symmetry, it is only necessary to satisfy these conditions on one of each pair of edges (e.g. on $y=b, x=a$). For the same reason, we do not have to impose a condition on the vertical force at $x=\pm a$, since the symmetry demands that the forces be equal at the two ends and the *total* force must be $2pa$ to preserve global equilibrium, this being guaranteed by the use of the Airy stress function, as in the problem of §5.2.1.

It is usually better strategy to start a manual solution with the strong boundary conditions (equations (5.82, 5.83)), and in particular with those conditions that are homogeneous (in this case equation (5.82)), since these will often require that one or more of the constants be zero, reducing the complexity of subsequent steps. Substituting (5.81) into (5.15, 5.16), we find

$$\sigma_{xy} = -2C_5x - 4C_{14}x^3 - 6C_{16}xy^2 \tag{5.86}$$

$$\sigma_{yy} = 2C_5y + 12C_{14}x^2y + 2C_{16}y^3 \ . \tag{5.87}$$

Thus, condition (5.82) requires that

$$4C_{14}x^3 + (2C_5 + 6C_{16}b^2)x = 0 \ ; \ \ \text{for all } x \tag{5.88}$$

and this condition is satisfied if and only if

$$C_{14} = 0 \quad \text{and} \quad 2C_5 + 6C_{16}b^2 = 0 \ . \tag{5.89}$$

A similar procedure with equation (5.87) and boundary condition (5.83) gives the additional equation

$$2C_5b + 2C_{16}b^3 = -\frac{p}{2} \ . \tag{5.90}$$

Equations (5.89, 5.90) have the solution

$$C_5 = -\frac{3p}{8b} \ ; \ C_{16} = \frac{p}{8b^3} \ . \tag{5.91}$$

We next determine C_{18} from the condition that the function ϕ is biharmonic, obtaining

$$(24C_{14} + 24C_{16} + 120C_{18})y = 0 \tag{5.92}$$

and hence

$$C_{18} = -\frac{p}{40b^3} \ , \tag{5.93}$$

from (5.89, 5.91, 5.92).

It remains to satisfy the two weak boundary conditions (5.84, 5.85) on the ends $x = \pm a$. The first of these is satisfied identically in view of the antisymmetry of the stress field and the second gives the equation

$$4C_7b^3 + 4C_{16}a^2b^3 + 8C_{18}b^5 = 0 \ , \tag{5.94}$$

which, with equations (5.91, 5.93), serves to determine the remaining constant,

$$C_7 = \frac{p(2b^2 - 5a^2)}{40b^3} \ . \tag{5.95}$$

The final solution of the complete problem (the sum of that for Figures 5.4(a) and (b)) is therefore obtained from the stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3 - 10b^3x^2) \ , \tag{5.96}$$

as in the 'computer solution' (5.77), and the stresses are therefore given by (5.78–5.80) as before.

5.3 Fourier series and transform solutions

Polynomial solutions can, in principle, be extended to more general loading of the beam edges, as long as the tractions are capable of a power series expansion. However, the practical use of this method is limited by the algebraic complexity encountered for higher order polynomials and by the fact that many important traction distributions do not have convergent power series representations.

A more useful method in such cases is to build up a general solution by components of Fourier form. For example, if we write

$$\phi = f(y)\cos(\lambda x) \quad \text{or} \quad \phi = f(y)\sin(\lambda x) \ , \tag{5.97}$$

substitution in the biharmonic equation (5.1) shows that $f(y)$ must have the general form

$$f(y) = (A + By)e^{\lambda y} + (C + Dy)e^{-\lambda y} \ , \tag{5.98}$$

where A, B, C, D are arbitrary constants. Alternatively, by defining new arbitrary constants A', B', C', D' through the relations $A = (A'+C')/2$, $B = (B'+D')/2$, $C=(A'-C')/2$, $D=(B'-D')/2$, we can group the exponentials into hyperbolic functions, obtaining the equivalent form

$$f(y) = (A' + B'y)\cosh(\lambda y) + (C' + D'y)\sinh(\lambda y) \ . \tag{5.99}$$

The hyperbolic form enables us to take advantage of any symmetry about $y=0$, since $\cosh(\lambda y), y\sinh(\lambda y)$ are even functions of y and $\sinh(\lambda y), y\cosh(\lambda y)$ are odd functions.

More general biharmonic stress functions can be constructed by superposition of terms like (5.98, 5.99), leading to Fourier series expansions for the tractions on the surfaces $y=\pm b$. The theory of Fourier series can then be used to determine the coefficients in the series, using strong boundary conditions on $y=\pm b$. Quite general traction distributions can be expanded in this way, so Fourier series solutions provide a methodology applicable to any problem for the rectangular bar.

5.3.1 Choice of form

The stresses due to the stress function $\phi=f(y)\cos(\lambda x)$ are

$$\sigma_{xx} = f''(y)\cos(\lambda x) \ ; \ \sigma_{xy} = \lambda f'(y)\sin(\lambda x) \ ; \ \sigma_{yy} = -\lambda^2 f(y)\cos(\lambda x) \tag{5.100}$$

and the tractions on the edge $x=a$ are

$$\sigma_{xx}(a,y) = f''(y)\cos(\lambda a) \ ; \ \sigma_{xy}(a,y) = \lambda f'(y)\sin(\lambda a) \ . \tag{5.101}$$

It follows that we can satisfy homogeneous boundary conditions on one (but not both) of these tractions in the strong sense, by restricting the Fourier series to specific values of λ. In equation (5.101), the choice $\lambda=n\pi/a$ will give $\sigma_{xy}=0$ on $x=\pm a$, whilst $\lambda=(2n-1)\pi/2a$ will give $\sigma_{xx}=0$ on $x=\pm a$, where n is any integer.

Example

We illustrate this technique by considering the rectangular beam $-a<x<a$, $-b<y<b$, simply supported at $x=\pm a$ and loaded by compressive normal tractions $p_1(x)$ on the upper edge $y=b$ and $p_2(x)$ on $y=-b$ — i.e.

$$\sigma_{xy} = 0 \ \ ; \ \ y = \pm b \tag{5.102}$$

$$\sigma_{yy} = -p_1(x) \ \ ; \ \ y = b \tag{5.103}$$

$$= -p_2(x) \ \ ; \ \ y = -b \tag{5.104}$$

$$\sigma_{xx} = 0 \ \ ; \ \ x = \pm a \ . \tag{5.105}$$

Notice that we have replaced the weak conditions (5.84, 5.85) by the strong condition (5.105). As in §5.2.2, it is not necessary to enforce the remaining weak conditions (those involving the vertical forces on $x=\pm a$), since these will be identically satisfied by virtue of the equilibrium condition.

The algebraic complexity of the problem will be reduced if we use the geometric symmetry of the beam to decompose the problem into four sub-problems. For this purpose, we define

$$f_1(x) = f_1(-x) \equiv \frac{1}{4}\{p_1(x)+p_1(-x)+p_2(x)+p_2(-x)\} \quad (5.106)$$

$$f_2(x) = -f_2(-x) \equiv \frac{1}{4}\{p_1(x)-p_1(-x)+p_2(x)-p_2(-x)\} \quad (5.107)$$

$$f_3(x) = f_3(-x) \equiv \frac{1}{4}\{p_1(x)+p_1(-x)-p_2(x)-p_2(-x)\} \quad (5.108)$$

$$f_4(x) = -f_4(-x) \equiv \frac{1}{4}\{p_1(x)-p_1(-x)-p_2(x)+p_2(-x)\} \quad (5.109)$$

and hence

$$p_1(x) = f_1(x)+f_2(x)+f_3(x)+f_4(x) \;\;;\;\; p_2(x) = f_1(x)+f_2(x)-f_3(x)-f_4(x)\;. \quad (5.110)$$

The boundary conditions now take the form

$$\sigma_{xy} = 0 \;\;;\;\; y=\pm b \quad (5.111)$$

$$\sigma_{yy} = -f_1(x)-f_2(x)-f_3(x)-f_4(x) \;\;;\;\; y=b \quad (5.112)$$

$$= -f_1(x)-f_2(x)+f_3(x)+f_4(x) \;\;;\;\; y=-b \quad (5.113)$$

$$\sigma_{xx} = 0 \;\;;\;\; x=\pm a \quad (5.114)$$

and each of the functions f_1, f_2, f_3, f_4 defines a separate problem with either symmetry or antisymmetry about the x- and y-axes. We shall here restrict attention to the loading defined by the function $f_3(x)$, which is symmetric in x and antisymmetric in y. The boundary conditions of this sub-problem are

$$\sigma_{xy} = 0 \;\;;\;\; y=\pm b \quad (5.115)$$

$$\sigma_{yy} = \mp f_3(x) \;\;;\;\; y=\pm b \quad (5.116)$$

$$\sigma_{xx} = 0 \;\;;\;\; x=\pm a\;. \quad (5.117)$$

The problem of equations (5.115–5.117) is even in x and odd in y, so we use a cosine series in x with only the odd terms from the hyperbolic form (5.99) — i.e.

$$\phi = \sum_{n=1}^{\infty}\{A_n y\cosh(\lambda_n y)+B_n\sinh(\lambda_n y)\}\cos(\lambda_n x)\;, \quad (5.118)$$

where A_n, B_n are arbitrary constants. The strong condition (5.117) on $x=\pm a$ can then be satisfied in every term by choosing

$$\lambda_n = \frac{(2n-1)\pi}{2a} \ . \tag{5.119}$$

The corresponding stresses are

$$\sigma_{xx}=\sum_{n=1}^{\infty} \Big\{2A_n\lambda_n \sinh(\lambda_n y)+A_n\lambda_n^2 y \cosh(\lambda_n y)+B_n\lambda_n^2 \sinh(\lambda_n y)\Big\} \cos(\lambda_n x)$$

$$\sigma_{xy}=\sum_{n=1}^{\infty} \Big\{A_n\lambda_n \cosh(\lambda_n y)+A_n\lambda_n^2 y \sinh(\lambda_n y)+B_n\lambda_n^2 \cosh(\lambda_n y)\Big\} \sin(\lambda_n x)$$

$$\sigma_{yy}=-\sum_{n=1}^{\infty} \Big\{A_n\lambda_n^2 y \cosh(\lambda_n y) + B_n\lambda_n^2 \sinh(\lambda_n y)\Big\} \cos(\lambda_n x) \tag{5.120}$$

and hence the boundary conditions (5.115, 5.116) on $y=\pm b$ require that

$$\sum_{n=1}^{\infty}\{A_n\lambda_n \cosh(\lambda_n b)+A_n\lambda_n^2 b \sinh(\lambda_n b)+B_n\lambda_n^2 \cosh(\lambda_n b)\} \sin(\lambda_n x) = 0 \tag{5.121}$$

$$\sum_{n=1}^{\infty} \Big\{A_n\lambda_n^2 b \cosh(\lambda_n b)+B_n\lambda_n^2 \sinh(\lambda_n b)\Big\} \cos(\lambda_n x) = f_3(x) \ . \tag{5.122}$$

To invert the series, we multiply (5.122) by $\cos(\lambda_m x)$ and integrate from $-a$ to a, obtaining

$$\sum_{n=1}^{\infty} \int_{-a}^{a} \Big\{A_n\lambda_n^2 b \cosh(\lambda_n b) + B_n\lambda_n^2 \sinh(\lambda_n b)\Big\} \cos(\lambda_n x) \cos(\lambda_m x) dx$$

$$= \int_{-a}^{a} f_3(x) \cos(\lambda_m x) dx \ . \tag{5.123}$$

The integrals on the left-hand side are all zero except for the case $m=n$ and hence, evaluating the integrals, we find

$$\Big\{A_m\lambda_m^2 b \cosh(\lambda_m b) + B_m\lambda_m^2 \sinh(\lambda_m b)\Big\} a = \int_{-a}^{a} f_3(x) \cos(\lambda_m x) dx \ . \tag{5.124}$$

The homogeneous equation (5.121) is clearly satisfied if

$$A_m\lambda_m \cosh(\lambda_m b) + A_m\lambda_m^2 b \sinh(\lambda_m b) + B_m\lambda_m^2 \cosh(\lambda_m b) = 0 \ . \tag{5.125}$$

Solving (5.124, 5.125) for A_m, B_m, we have

$$A_m = \frac{\cosh(\lambda_m b)}{\lambda_m a\{\lambda_m b - \sinh(\lambda_m b) \cosh(\lambda_m b)\}} \int_{-a}^{a} f_3(x) \cos(\lambda_m x) dx$$

$$B_m = -\frac{(\cosh(\lambda_m b) + \lambda_m b \sinh(\lambda_m b))}{\lambda_m^2 a\{\lambda_m b - \sinh(\lambda_m b) \cosh(\lambda_m b)\}} \int_{-a}^{a} f_3(x) \cos(\lambda_m x) dx \ , \tag{5.126}$$

where λ_m is given by (5.119). The stresses are then recovered by substitution into equations (5.120).

The corresponding solutions for the functions f_1, f_2, f_4 are obtained in a similar way, but using a sine series for the odd functions f_2, f_4 and the even terms $y\sinh(\lambda y), \cosh(\lambda y)$ in ϕ for f_1, f_2. The complete solution is then obtained by superposing the solutions of the four sub-problems.

The Fourier series method is particularly useful in problems where the traction distribution on the long edges has no power series expansion, typically because of discontinuities in the loading. For example, suppose the beam is loaded only by a concentrated compressive force F on the upper edge at $x=0$, corresponding to the loading $p_1(x)=F\delta(x),\ p_2(x)=0$. For the symmetric/antisymmetric sub-problem considered above, we then have

$$f_3(x) = \frac{F\delta(x)}{4} \tag{5.127}$$

from (5.108) and the integral in equations (5.126) is therefore

$$\int_{-a}^{a} f_3(x)\cos(\lambda_m x)dx = \frac{F}{4}\ , \tag{5.128}$$

for all m.

This solution satisfies the end condition on σ_{xx} in the strong sense, but the condition on σ_{xy} only in the weak sense. In other words, the tractions σ_{xy} on the ends add up to the forces required to maintain equilibrium, but we have no control over the exact distribution of these tractions. This represents an improvement over the polynomial solution of §5.2.3, where weak conditions were used for both end tractions, so we might be tempted to use a Fourier series even for problems with continuous polynomial loading. However, this improvement is made at the cost of an infinite series solution. If the series were truncated at a finite value of n, errors would be obtained particularly near the ends or any discontinuities in the loading.

5.3.2 Fourier transforms

If the beam is infinite or semi-infinite ($a \to \infty$), the series (5.118) must be replaced by the integral representation

$$\phi(x,y) = \int_0^\infty f(\lambda,y)\cos(\lambda x)d\lambda\ , \tag{5.129}$$

where

$$f(\lambda,y) = A(\lambda)y\cosh(\lambda y) + B(\lambda)\sinh(\lambda y)\ . \tag{5.130}$$

Equation (5.129) is introduced here as a generalization of (5.97) by superposition, but $\phi(x,y)$ is in fact the Fourier cosine transform of $f(\lambda,y)$, the corresponding inversion being

$$f(\lambda, y) = \frac{2}{\pi}\int_0^\infty \phi(x,y)\cos(\lambda x)dx \; . \tag{5.131}$$

The boundary conditions on $y=\pm b$ will also lead to Fourier integrals, which can be inverted in the same way to determine the functions $A(\lambda), B(\lambda)$. For a definitive treatment of the Fourier transform method, the reader is referred to the treatise by Sneddon[3]. Extensive tables of Fourier transforms and their inversions are given by Erdelyi[4]. The cosine transform (5.129) will lead to a symmetric solution. For more general loading, the complex exponential transform can be used.

It is worth remarking on the way in which the series and transform solutions are natural generalizations of the elementary solution (5.97). One of the most powerful techniques in Elasticity — and indeed in any physical theory characterized by linear partial differential equations — is to seek a simple form of solution (often in separated-variable form) containing a parameter which can take a range of values. A more general solution can then be developed by superposing arbitrary multiples of the solution with different values of the parameter.

For example, if a particular solution can be written symbolically as $\phi = f(x,y,\lambda)$, where λ is a parameter, we can develop a general series form

$$\phi(x,y) = \sum_{i=0}^{\infty} A_i f(x,y,\lambda_i) \tag{5.132}$$

or an integral form

$$\phi(x,y) = \int_a^b A(\lambda) f(x,y,\lambda) d\lambda \; . \tag{5.133}$$

The series form will naturally arise if there is a discrete set of *eigenvalues*, λ_i for which $f(x,y,\lambda_i)$ satisfies some of the boundary conditions of the problem. Additional examples of this kind will be found in §§6.2, 11.2. In this case, the series (5.132) is most properly seen as an eigenfunction expansion. Integral forms arise most commonly (but not exclusively) in problems involving infinite or semi-infinite domains (see, for example, §§11.3, 30.2.2.).

Any particular solution containing a parameter can be used in this way and, since transforms are commonly named after their originators, the reader desirous of instant immortality might like to explore some of those which have not so far been used. Of course, the usefulness of the resulting solution depends upon its *completeness* — i.e. its capacity to represent all stress fields of a given class — and upon the ease with which the transform can be inverted.

---

[3] I.N.Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[4] A.Erdelyi, **ed.**, *Tables of Integral Transforms*, Bateman Manuscript Project, California Institute of Technology, Vol.1, McGraw-Hill, New York, 1954.

PROBLEMS

1. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=0$ and loaded by a uniform shear traction $\sigma_{xy}=S$ on the upper edge, $y=b$, the remaining edges, $x=L,\ y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=L$.

2. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a shear traction $\sigma_{xy}=Sx/L$ on the lower edge, $y=-b$, the upper edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=\pm L$.

3. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a linearly-varying compressive normal traction $p(x)=Sx/L$ on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=0$.

4. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a compressive normal traction

$$p(x) = S\cos\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the lower edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem.

5. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a compressive normal traction

$$p(x) = S\sin\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Use a combination of the stress function (5.97) and an appropriate polynomial to find the stress components for this problem, using the weak boundary conditions on $x=0$.

6. A large plate defined by $y>0$ is subjected to a sinusoidally varying load

$$\sigma_{yy} = S\sin\lambda x\ \ ;\ \ \sigma_{xy} = 0$$

at its plane edge $y=0$.

Find the complete stress field in the plate and hence estimate the depth y at which the amplitude of the variation in σ_{yy} has fallen to 10% of S.

Hint: You might find it easier initially to consider the case of the layer $0<y<h$, with $y=h$ traction-free, and then let $h \to \infty$.

7. The beam $-a<x<a, -b<y<b$ is loaded by a uniform compressive traction p in the central region $-a/2<x<a/2$ of both of the edges $y=\pm b$, as shown in Figure 5.5. The remaining edges are traction-free. Use a Fourier series with the appropriate symmetries to obtain a solution for the stress field, using the weak condition on σ_{xy} on the edges $x=\pm a$ and the strong form of all the remaining boundary conditions.

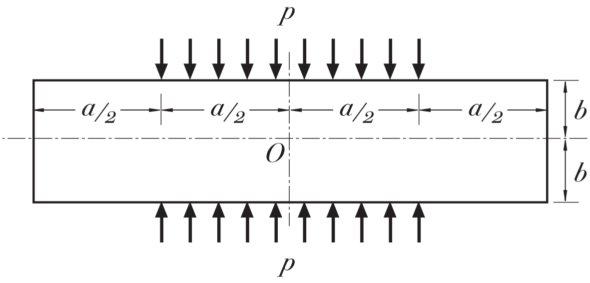

Figure 5.5

8. Use a Fourier series to solve the problem of Figure 5.4(a) in §5.2.3. Choose the terms in the series so as to satisfy the condition $\sigma_{xx}(\pm a, y) = 0$ in the strong sense.

If you are solving this problem in Maple or Mathematica, compare the solution with that of §5.2.3 by making a contour plot of the difference between the truncated Fourier series stress function and the polynomial stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3) \ .$$

Examine the effect of taking different numbers of terms in the series.

9. The large plate $y>0$ is loaded at its remote boundaries so as to produce a state of uniform tensile stress

$$\sigma_{xx} = S \ \ ; \ \ \sigma_{xy} = \sigma_{yy} = 0 \ ,$$

the boundary $y=0$ being traction-free. We now wish to determine the perturbation in this simple state of stress that will be produced if the traction-free boundary had a slight waviness, defined by the line

$$y = \epsilon \cos(\lambda x) \ ,$$

where $\lambda\epsilon \ll 1$. To solve this problem

(i) Start with the stress function

$$\phi = \frac{Sy^2}{2} + f(y)\cos(\lambda x)$$

and determine $f(y)$ if the function is to be biharmonic.

(ii) The perturbation will be localized near $y=0$, so select only those terms in $f(y)$ that decay as $y \to \infty$.

(iii) Find the stress components and use the stress transformation equations to determine the tractions on the wavy boundary. Notice that the inclination of the wavy surface to the plane $y=0$ will be everywhere small if $\lambda\epsilon \ll 1$ and hence the trigonometric functions involving this angle can be approximated using $\sin(x) \approx x,\ \cos(x) \approx 1, x \ll 1$.

(iv) Choose the free constants in $f(y)$ to satisfy the traction-free boundary condition on the wavy surface.

(v) Determine the maximum tensile stress and hence the stress concentration factor as a function of $\lambda\epsilon$.

6

END EFFECTS

The solution of §5.2.2 must be deemed approximate insofar as the boundary conditions on the ends $x = \pm a$ of the rectangular beam are satisfied only in the weak sense of force resultants, through equations (5.45–5.47). In general, if a rectangular beam is loaded by tractions of finite polynomial form, a finite polynomial solution can be obtained which satisfies the boundary conditions in the strong (i.e. pointwise) sense on two edges and in the weak sense on the other two edges.

The error involved in such an approximation corresponds to the solution of a corrective problem in which the beam is loaded by the difference between the actual tractions applied and those implied by the approximation. These tractions will of course be confined to the edges on which the weak boundary conditions were applied and will be self-equilibrated, since the weak conditions imply that the tractions in the approximate solution have the same force resultants as the actual tractions.

For the particular problem of §5.2.2, we note that the stress field of equations (5.78–5.80) satisfies the boundary conditions on the edges $y = \pm b$, but that there is a self-equilibrated normal traction

$$\sigma_{xx} = \frac{p}{10b^3}(3b^2y - 5y^3) \tag{6.1}$$

on the ends $x = \pm a$, which must be removed by superposing a corrective solution if we wish to satisfy the boundary conditions of Figure 5.3 in the strong sense.

6.1 Decaying solutions

In view of Saint-Venant's theorem, we anticipate that the stresses in the corrective solution will decay as we move away from the edges where the self-equilibrated tractions are applied. The decay rate is likely to be related to the width of the loaded region and hence we anticipate that the stresses in

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_6, © Springer Science+Business Media B.V. 2010

the corrective solution will be significant only in two regions near the ends, of linear dimensions comparable to the width of the beam. These regions are, shown shaded in Figure 6.1. It follows that the solution of §5.2.2 will be a good approximation in the *unshaded* region in Figure 6.1.

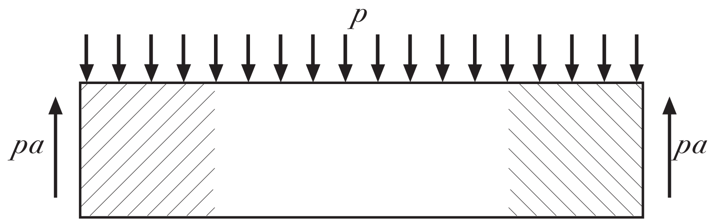

Figure 6.1: Regions of the beam influenced by end effects.

It also follows that the corrective solutions for the two ends are *uncoupled*, since the corrective field for the end $x=-a$ has decayed to negligible proportions before we reach the end $x=+a$ and vice versa. This implies that, so far as the left end is concerned, the corrective solution is essentially identical to that which would be required in the *semi-infinite* beam, $x>-a$. We can therefore simplify the statement of the problem by considering the corrective solution for the left end only and shifting the origin to $-a$, so that the semi-infinite beam under consideration is defined by $x>0,\ -b<y<b$.

It is also now clear why we chose to satisfy the strong boundary conditions on the *long* edges, $y=\pm b$. If instead we had imposed strong conditions on $x=\pm a$ and weak conditions on $y=\pm b$, the shaded regions would have overlapped and there would be no region in which the finite polynomial solution was a good approximation to the stresses in the beam. It also follows that the approximation will only be useful when the beam has an aspect ratio significantly different from unity — i.e. $b/a \gg 1$.

6.2 The corrective solution

We recall that the stress function, ϕ, for the corrective solution must (i) satisfy the biharmonic equation (5.1), (ii) have zero tractions on the boundaries $y=\pm b$ — i.e.

$$\sigma_{yy} = \sigma_{yx} = 0 \ ; \ \ y = \pm b \tag{6.2}$$

and (iii) have prescribed non-zero tractions (such as those defined by equation (6.1)) on the end(s), which we can write in the form

$$\sigma_{xx}(0,y) = f_1(y) \ ; \quad \sigma_{xy}(0,y) = f_2(y) \ , \tag{6.3}$$

where

$$\int_{-b}^{b} f_1(y)dy = \int_{-b}^{b} f_1(y)ydy = \int_{-b}^{b} f_2(y)dy = 0 \ , \tag{6.4}$$

since the corrective tractions are required to be self-equilibrated.

We cannot generally expect to find a solution to satisfy all these conditions in closed form and hence we seek a series or transform (integral) solution as suggested in §5.3. However, the solution will be simpler if we can find a *class* of solutions, all of which satisfy some of the conditions. We can then write down a more general solution as a superposition of such solutions and choose the coefficients so as to satisfy the remaining condition(s).

Since the traction boundary condition on the end will vary from problem to problem, it is convenient to seek solutions which satisfy (i) and (ii) — i.e. biharmonic functions for which

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial x \partial y} = 0 \;\; ; \;\; y = \pm b \; . \tag{6.5}$$

6.2.1 Separated-variable solutions

One way to obtain functions satisfying conditions (6.5) is to write them in the separated-variable form

$$\phi = f(x)g(y) \; , \tag{6.6}$$

in which case, (6.5) will be satisfied for all x, provided that

$$g(y) = g'(y) = 0 \;\; ; \;\; y = \pm b \; . \tag{6.7}$$

Notice that the final corrective solution cannot be expected to be of separated-variable form, but we shall see that it can be represented as the sum of a series such terms.

If the functions (6.6) are to be biharmonic, we must have

$$g\frac{d^4 f}{dx^4} + 2\frac{d^2 f}{dx^2}\frac{d^2 g}{dy^2} + f\frac{d^4 g}{dy^4} = 0 \; , \tag{6.8}$$

and this equation must be satisfied for all values of x, y. Now, if we consider the subset of points (x, c), where c is a constant, it is clear that $f(x)$ must satisfy an equation of the form

$$A\frac{d^4 f}{dx^4} + B\frac{d^2 f}{dx^2} + Cf = 0 \; , \tag{6.9}$$

where A, B, C, are constants, and hence $f(x)$ must consist of exponential terms such as $f(x) = \exp(\lambda x)$. Similar considerations apply to the function $g(y)$. Notice incidentally that λ might be complex or imaginary, giving sinusoidal functions, and there are also degenerate cases where C and/or $B = 0$ in which case $f(x)$ could also be a polynomial of degree 3 or below.

Since we are seeking to represent a field which decays with x, we select terms of the form

$$\phi = g(y)e^{-\lambda x} \; , \tag{6.10}$$

in which case, (6.8) reduces to

$$\frac{d^4g}{dy^4} + 2\lambda^2 \frac{d^2g}{dy^2} + \lambda^4 g = 0 \; , \tag{6.11}$$

which is a fourth order ordinary differential equation for $g(y)$ with general solution

$$g(y) = (A_1 + A_2 y)\cos(\lambda y) + (A_3 + A_4 y)\sin(\lambda y) \; . \tag{6.12}$$

6.2.2 The eigenvalue problem

The arbitrary constants A_1, A_2, A_3, A_4 are determined from the boundary conditions (6.2), which in view of (6.7) lead to the four simultaneous equations

$$(A_1 + A_2 b)\cos(\lambda b) + (A_3 + A_4 b)\sin(\lambda b) = 0 \tag{6.13}$$
$$(A_1 - A_2 b)\cos(\lambda b) - (A_3 - A_4 b)\sin(\lambda b) = 0 \tag{6.14}$$
$$(A_2 + A_3\lambda + A_4\lambda b)\cos(\lambda b) - (A_1\lambda + A_2\lambda b - A_4)\sin(\lambda b) = 0 \tag{6.15}$$
$$(A_2 + A_3\lambda - A_4\lambda b)\cos(\lambda b) + (A_1\lambda - A_2\lambda b - A_4)(\sin \lambda b) = 0 \; . \tag{6.16}$$

This set of equations is homogeneous and will generally have only the trivial solution $A_1 = A_2 = A_3 = A_4 = 0$. However, there are some eigenvalues of the exponential decay rate, λ, for which the determinant of coefficients is singular and the solution is non-trivial.

A more convenient form of the equations can be obtained by taking sums and differences in pairs — i.e. by constructing the equations (6.13 + 6.14), (6.13 − 6.14), (6.15 + 6.16), (6.15 − 6.16), which after rearrangement and cancellation of non-zero factors yields the set

$$A_1\cos(\lambda b) + A_4 b\sin(\lambda b) = 0 \tag{6.17}$$
$$A_1\lambda\sin(\lambda b) - A_4\{\sin(\lambda b) + \lambda b(\cos(\lambda b)\} = 0 \tag{6.18}$$
$$A_2 b\cos(\lambda b) + A_3\sin(\lambda b) = 0 \tag{6.19}$$
$$A_2\{\cos(\lambda b) - \lambda b\sin(\lambda b)\} + A_3\lambda\cos(\lambda b) = 0 \; . \tag{6.20}$$

What we have done here is to use the symmetry of the system to partition the matrix of coefficients. The terms $A_1\cos(\lambda y),\ A_4 y\sin(\lambda y)$ are symmetric, whereas $A_2 y\cos(\lambda y),\ A_3\sin(\lambda y)$ are antisymmetric. The boundary conditions are also symmetric and hence the symmetric and antisymmetric terms must *separately* satisfy them.

We conclude that the set of equations (6.17–6.20) has two sets of eigenvalues, for one of which the resulting eigenfunction is symmetric and the other antisymmetric. The symmetric eigenvalues $\lambda^{(S)}$ are obtained by eliminating A_1, A_4 from (6.17, 6.18) with the result

$$\sin(2\lambda^{(S)}b) + 2\lambda^{(S)}b = 0 \; , \tag{6.21}$$

whilst the antisymmetric eigenvalues $\lambda^{(A)}$ are obtained in the same way from (6.19, 6.20) with the result

$$\sin(2\lambda^{(A)}b) - 2\lambda^{(A)}b = 0 \,. \tag{6.22}$$

Figure 6.2 demonstrates graphically that the only *real* solution of equations (6.21, 6.22) is the trivial case $\lambda = 0$ (which in fact corresponds to the non-decaying solutions in which an axial force or moment resultant is applied at the end and transmitted along the beam).

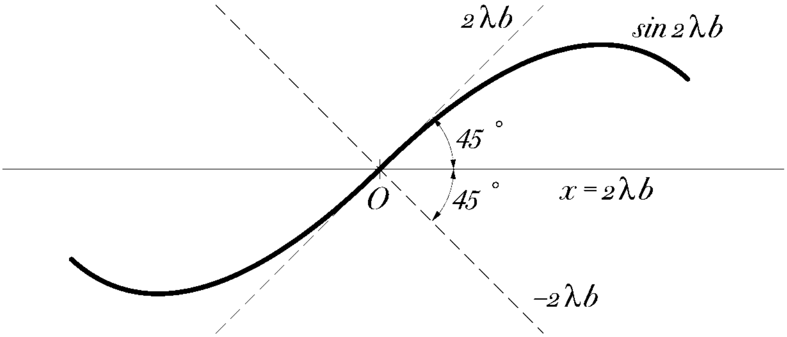

Figure 6.2: Graphical solution of equations (6.21, 6.22).

However, there is a denumerably infinite set of non-trivial *complex* solutions, corresponding to stress fields which oscillate whilst decaying along the beam. These solutions are fairly easy to find by writing $\lambda b = c + \imath d$, separating real and imaginary parts in the complex equation, and solving the resulting two simultaneous equations for the real numbers, c, d, using a suitable numerical algorithm.

Once the eigenvalues have been determined, the corresponding eigenfunctions $g^{(S)}, g^{(A)}$ are readily recovered using (6.12, 6.21, 6.22). We obtain

$$g^{(S)} = C\left[y\sin(\lambda^{(S)}y)\cos(\lambda^{(S)}b) - b\sin(\lambda^{(S)}b)\cos(\lambda^{(S)}y)\right] \tag{6.23}$$

$$g^{(A)} = D\left[y\sin(\lambda^{(S)}b)\cos(\lambda^{(S)}y) - b\sin(\lambda^{(S)}y)\cos(\lambda^{(S)}b)\right] \,, \tag{6.24}$$

where C, D are new arbitrary constants related to A_1, A_4 and A_2, A_3 respectively. We can then establish a more general solution of the form

$$\phi = \sum_{i=1}^{\infty} C_i g_i^{(S)}(y)e^{-\lambda_i^{(S)}x} + \sum_{i=1}^{\infty} D_i g_i^{(A)}(y)e^{-\lambda_i^{(A)}x} \,, \tag{6.25}$$

where $\lambda_i^{(S)}, \lambda_i^{(A)}$ represent the eigenvalues of equations (6.21, 6.22) respectively.

The final step is to choose the constants C_i, D_i so as to satisfy the prescribed boundary conditions (6.3) on the end $x = 0$. An improved but still approximate solution of this problem can be obtained by truncating the infinite series at some finite value $i = N$, defining a scalar measure of the error $\mathcal{E}$ in the boundary conditions and then imposing the $2N$ conditions

$$\frac{\partial \mathcal{E}}{\partial C_i} = 0 \; ; \quad \frac{\partial \mathcal{E}}{\partial D_i} = 0 \quad \text{for} \quad i = 1, N \tag{6.26}$$

to determine the unknown constants. An appropriate non-negative quadratic error measure is

$$\mathcal{E} = \int_{-b}^{b} \left[\left\{ f_1(y) - \sigma_{xx}(0, y) \right\}^2 + \left\{ f_2(y) - \sigma_{xy}(0, y) \right\}^2 \right] dy \;, \tag{6.27}$$

where σ_{xx}, σ_{xy} are the stress components defined by the truncated series. Since biharmonic boundary-value problems arise in many areas of mechanics, techniques of this kind have received a lot of attention. Convergence of the truncated series is greatly improved if the boundary conditions are continuous in the corners[1]. It can be shown that the use of the weak boundary conditions automatically defines continuous values of ϕ and its first derivatives in the corners $(0, -b), (0, b)$ in the corrective solution. However, convergence problems are still likely to occur if the shear tractions on the two orthogonal edges in the corner are different, for example if

$$\lim_{x \to 0} \sigma_{yx}(x, b) \neq \lim_{y \to b} \sigma_{xy}(0, y) \; . \tag{6.28}$$

Since the tractions on the boundaries are independent, this is a perfectly legitimate physical possiblity, but it involves an infinite stress gradient in the corner and leads to a modified Gibbs phenomenon in the series solution, where increase of N leads to greater accuracy over most of the boundary, but to an oscillation of finite amplitude and decreasing wavelength in the immediate vicinity of the corner. The problem can be avoided at the cost of a more complex fundamental problem by extracting a closed-form solution respresenting the discontinuous tractions. We shall discuss this special solution in §11.1.2 below.

Completeness

It is clear that the accuracy of the solution (however defined) can always be improved by taking more terms in the series (6.25) and in particular cases this is easily established numerically. However, it is more challenging to prove that the eigenfunction expansion is complete in the sense that any prescribed self-equilibrated traction on $x = 0$ can be described to within an arbitrarily

[1] M.I.G.Bloor and M.J.Wilson, An approximate solution method for the biharmonic problem, *Proceedings of the Royal Society of London*, Vol. A462, pp.1107–1121.

small accuracy by taking a sufficient number of terms in the series, though experience with other eigenfunction expansions (e.g. with expansion of elastodynamic states of a structure in terms of normal modes) suggests that this will always be true. In fact, although the analysis described in this section has been known since the investigations by Papkovich and Fadle in the early 1940s, the formal proof of completeness was only completed by Gregory[2] in 1980.

It is worth noting that, as in many related problems, the eigenfunctions oscillate in y with increasing frequency as λ_i increases and in fact every time we increase i by 1, an extra zero appears in the function $g_i(y)$ in the range $0<y<b$. Thus, there is a certain similarity to the process of approximating functions by Fourier series and in particular, the residual error in case of truncation will always cross zero once more than the last eigenfunction included.

This is also helpful in that it enables us to estimate the decay rate of the first excluded term. We see from equation (6.12) that the distance between zeros in y in any of the separate terms would be (π/λ_R), where λ_R is the real part of λ. It follows that over a corresponding distance in the x-direction, the field would decay by the factor $\exp(-\pi)=0.0432$. This suggests that we might estimate the decay rate of the end field by noting the distance between zeros in the corresponding tractions[3]. For example, the traction of equation (6.1) has zeros at $y=0, 3b/5$, corresponding to $\lambda_R=(5\pi/3b)=5.23/b$.

An alternative way of estimating the decay rate is to note that the decay rate for the various terms in (6.25) increases with i and hence as x increases, the leading term will tend to predominate. The tractions of (6.1) are antisymmetric and hence the leading term corresponds to the real part of the first eigenvalue of equation (6.22), which is found numerically to be $\lambda_R b=3.7$.

Either way, we can conclude that the error associated with the end tractions in the approximate solution of §5.2.2 has decayed to around e^{-4}, i.e. to about 2% of the values at the end, within a distance b of the end. Thus the region affected by the end condition — the shaded region in Figure 6.1 — is quite small.

For problems which are *symmetric* in y, the leading self-equilibrated term is likely to have the form of Figure 6.3(a), which has a longer wavelength than the corresponding antisymmetric form, 6.3(b). The end effects in symmetric

---

[2] R.D.Gregory, Green's functions, bi-linear forms and completeness of the eigenfunctions for the elastostatic strip and wedge, *J.Elasticity,* Vol. 9 (1979), pp.283–309; R.D.Gregory, The semi-infinite strip $x \geq 0, -1 \leq y \leq 1$; completeness of the Papkovich-Fadle eigenfunctions when $\phi_{xx}(0,y), \phi_{yy}(0,y)$ are prescribed, *J.Elasticity,* Vol. 10 (1980), pp.57–80; R.D.Gregory, The traction boundary value problem for the elastostatic semi-infinite strip; existence of solution and completeness of the Papkovich-Fadle eigenfunctions, *J.Elasticity,* Vol. 10 (1980), pp.295-327. These papers also include extensive references to earlier investigations of the problem.

[3] This assumes that the wavelength of the tractions is the same as that of ϕ, which of course is an approximation, since neither function is purely sinusoidal.

problems therefore decay more slowly and this is confirmed by the fact that the real part of the first eigenvalue of the symmetric equation (6.21) is only $\lambda_R b = 2.1$.

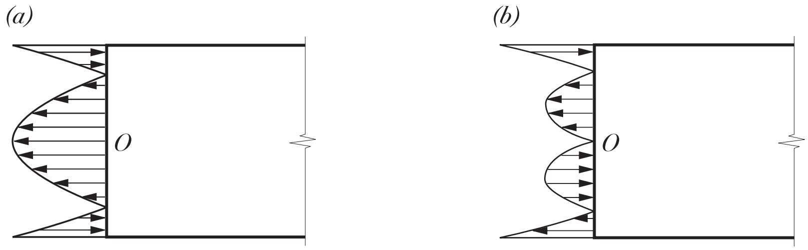

Figure 6.3: Leading term in self-equilibrated tractions for (a) symmetric loading and (b) antisymmetric loading.

6.3 Other Saint-Venant problems

The general strategy used in §6.2 can be applied to other curvilinear coördinate systems to correct the errors incurred by imposing the weak boundary conditions on appropriate edges. The essential steps are:-

(i) Define a coördinate system (ξ, η) such that the boundaries on which the strong conditions are applied are of the form, $\eta =$ constant.
(ii) Find a class of separated-variable biharmonic functions containing a parameter (λ in the above case).
(iii) Set up a system of four homogeneous equations for the coefficients of each function, based on the four traction-free boundary conditions for the corrective solution on the edges $\eta =$ constant.
(iv) Find the eigenvalues of the parameter for which the system has a non-trivial solution and the corresponding eigenfunctions, which are then used as the terms in an eigenfunction expansion to define a general form for the corrective field.
(v) Determine the coefficients in the eigenfunction expansion from the prescribed inhomogeneous boundary conditions on the end $\xi =$ constant.

6.4 Mathieu's solution

The method described in §6.2 is particularly suitable for rectangular bodies of relatively large aspect ratio, since the weak solution is then quite accurate over a substantial part of the domain and the corrections at the two ends are essentially independent of each other. In other cases, and particularly for the square $b = a$, we lose these advantages and the inconvenience of solving the eigenvalue equations (6.21, 6.22) tilts the balance in favour of an alternative method due originally to Mathieu, who represented the solution of the entire problem as the sum of two orthogonal Fourier series of the form (5.118). The general problem can be decomposed into four sub-problems which are respectively either symmetric or antisymmetric with respect to the x- and y-axes. Following Meleshko and Gomilko[4], we restrict attention to the symmetric/symmetric case, defined by the boundary conditions

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = f(y) \ ; \quad \sigma_{xy}(a,y) = -\sigma_{xy}(-a,y) = g(y) \qquad (6.29)$$

$$\sigma_{yx}(x,b) = -\sigma_{yx}(x,-b) = h(x) \ ; \quad \sigma_{yy}(x,b) = \sigma_{yy}(x,-b) = \ell(x) \ , \qquad (6.30)$$

where f, ℓ are even and g, h odd functions of their respective variables. We shall also assume that there is no discontinuity in shear traction of the form (6.28) at the corners and hence that $g(b)=h(a)$.

We define a biharmonic stress function with the appropriate symmetries as

$$\phi = A_0 y^2 + \sum_{m=1}^{\infty} (A_m y \sinh(\alpha_m y) + C_m \cosh(\alpha_m y)) \cos(\alpha_m x)$$
$$+ B_0 x^2 + \sum_{n=1}^{\infty} (B_n x \sinh(\beta_n x) + D_n \cosh(\beta_n x)) \cos(\beta_n y) \ . \qquad (6.31)$$

As in §5.3.1, the α_m, β_n can be chosen so as to ensure that each term gives either zero shear tractions or zero normal tractions on two opposite edges, but not both. For example, with

$$\alpha_m = \frac{m\pi}{a} \ ; \quad \beta_n = \frac{n\pi}{b} \ , \qquad (6.32)$$

the second series makes no contribution to the shear traction on $y=b$, whilst the first series (that involving the coefficients A_m, C_m) makes no contribution to the shear traction on $x=a$. The shear traction on $x=a$ is then given by

$$\sigma_{xy}(a,y) = \sum_{n=1}^{\infty} \beta_n^2 \left(B_n \left[a \coth(\beta_n a) + \frac{1}{\beta_n} \right] + D_n \right) \sinh(\beta_n a) \sin(\beta_n y) \ ,$$

using (4.6), and substituting into $(6.29)_2$ and applying the Fourier inversion theorem[5], we have

$$B_n \left[a \coth(\beta_n a) + \frac{1}{\beta_n} \right] + D_n = \frac{1}{\beta_n^2 b \sinh(\beta_n a)} \int_{-b}^{b} g(y) \sin(\beta_n y) dy \ ,$$

[4] V.V.Meleshko and A.M.Gomilko, Infinite systems for a biharmonic problem in a rectangle, *Proceeedings of the Royal Society of London,* Vol. A453 (1997), pp.2139–2160. The reader should be warned that there are several typographical errors in this paper, some of which are corrected in V.V.Meleshko and A.M.Gomilko, Infinite systems for a biharmonic problem in a rectangle: further discussion, *Proceeedings of the Royal Society of London,* Vol. A460 (2004), pp.807–819.

[5] See for example equation (5.123).

which defines a linear relation between each individual pair of constants B_n, D_n. A similar relation between A_m, C_m can be obtained from $(6.30)_1$ and these conditions can then be used to eliminate C_m, D_n in (6.31), giving

$$\begin{aligned}\phi = A_0 y^2 - \sum_{m=1}^{\infty} A_m B(y, \alpha_m, b) \sinh(\alpha_m b) \cos(\alpha_m x) \\ + B_0 x^2 - \sum_{n=1}^{\infty} B_n B(x, \beta_n, a) \sinh(\beta_n a) \cos(\beta_n y) \\ + \sum_{n=1}^{\infty} \frac{g_n \cosh(\beta_n x) \cos(\beta_n y)}{\beta_n^2 b \sinh(\beta_n a)} + \sum_{m=1}^{\infty} \frac{h_m \cosh(\alpha_m y) \cos(\alpha_m x)}{\alpha_m^2 a \sinh(\alpha_m b)} ,\end{aligned} \tag{6.33}$$

where

$$g_n = \int_{-b}^{b} g(y) \sin(\beta_n y) dy \; ; \quad h_m = \int_{-a}^{a} h(x) \sin(\alpha_m x) dx \; ,$$

and

$$B(z, \lambda, h) = \left[h \coth(\lambda h) + \frac{1}{\lambda} \right] \frac{\cosh(\lambda z)}{\sinh(\lambda h)} - z \frac{\sinh(\lambda z)}{\sinh(\lambda h)} \; .$$

The normal stress components are then obtained as

$$\begin{aligned}\sigma_{xx} = 2A_0 - \sum_{m=1}^{\infty} A_m \alpha_m^2 A(y, \alpha_m, b) \sinh(\alpha_m b) \cos(\alpha_m x) \\ + \sum_{n=1}^{\infty} B_n \beta_n^2 B(x, \beta_n, a) \sinh(\beta_n a) \cos(\beta_n y) \\ - \sum_{n=1}^{\infty} \frac{g_n \cosh(\beta_n x) \cos(\beta_n y)}{b \sinh(\beta_n a)} + \sum_{m=1}^{\infty} \frac{h_m \cosh(\alpha_m y) \cos(\alpha_m x)}{a \sinh(\alpha_m b)} \; .\end{aligned}$$

$$\begin{aligned}\sigma_{yy} = 2B_0 + \sum_{m=1}^{\infty} A_m \alpha_m^2 B(y, \alpha_m, b) \sinh(\alpha_m b) \cos(\alpha_m x) \\ - \sum_{n=1}^{\infty} B_n \beta_n^2 A(x, \beta_n, a) \sinh(\beta_n a) \cos(\beta_n y) \\ + \sum_{n=1}^{\infty} \frac{g_n \cosh(\beta_n x) \cos(\beta_n y)}{b \sinh(\beta_n a)} - \sum_{m=1}^{\infty} \frac{h_m \cosh(\alpha_m y) \cos(\alpha_m x)}{a \sinh(\alpha_m b)} \; ,\end{aligned}$$

where

$$A(z, \lambda, h) = \left[h \coth(\lambda h) - \frac{1}{\lambda} \right] \frac{\cosh(\lambda z)}{\sinh(\lambda h)} - z \frac{\sinh(\lambda z)}{\sinh(\lambda h)} \; .$$

Imposing the remaining two boundary conditions $(6.29)_1$, $(6.30)_2$ and applying the Fourier inversion theorem to the resulting equations, we then obtain the infinite set of simultaneous equations

$$X_m bB(b,\alpha_m,b) = \sum_{n=1}^{\infty} \frac{4Y_n\alpha_m^2}{(\alpha_m^2+\beta_n^2)^2} + H_m \ ,$$

$$Y_n aB(a,\beta_n,a) = \sum_{m=1}^{\infty} \frac{4X_m\beta_n^2}{(\alpha_m^2+\beta_n^2)^2} + K_n \ , \tag{6.34}$$

for $m,n \geq 1$, where

$$X_m = \frac{(-1)^{m+1}A_m\alpha_m^2\sinh(\alpha_m b)}{b} \ ; \quad Y_n = \frac{(-1)^n B_n\beta_n^2\sinh(\beta_n a)}{a}$$

$$H_m = \frac{(-1)^{m+1}\left[\ell_m+\coth(\alpha_m b)h_m\right]}{a} + \sum_{n=1}^{\infty}\frac{2(-1)^n\beta_n g_n}{(\alpha_m^2+\beta_n^2)ab}$$

$$K_n = \frac{(-1)^n\left[f_n+\coth(\beta_n a)g_n\right]}{b} - \sum_{m=1}^{\infty}\frac{2(-1)^m\alpha_m h_m}{(\alpha_m^2+\beta_n^2)ba}$$

and

$$f_n = \int_{-b}^{b} f(y)\cos(\beta_n y)dy \ ; \quad \ell_m = \int_{-a}^{a}\ell(x)\cos(\alpha_m x)dx \ .$$

Also, the zeroth-order Fourier inversion yields the constants A_0, B_0 as

$$A_0 = \frac{f_0}{4b} - \sum_{m=1}^{\infty}\frac{(-1)^m h_m}{2\alpha_m ba} \ ; \quad B_0 = \frac{\ell_0}{4a} - \sum_{n=1}^{\infty}\frac{(-1)^n g_n}{2\beta_n ab} \ .$$

With this solution, the coefficients X_m, Y_n generally tend to a common constant value G at large m,n and the series solution exhibits a non-vanishing error near the corners of the rectangle[6]. A more convergent solution can be obtained by defining new constants through the relations

$$X_m = \tilde{X}_m + G \ ; \quad Y_n = \tilde{Y}_n + G \ .$$

so that $\tilde{X}_m$,$\tilde{Y}_n$ decay with increasing m,n. The terms involving G can then be summed explicitly yielding the additional polynomial term

$$\phi_0 = \frac{G}{24}\left[(y^2-b^2)^2-(x^2-a^2)^2\right] \tag{6.35}$$

in the stress function ϕ. Unfortunately, the required value of G cannot be obtained in closed form except in certain special cases. However, an approximation can be obtained using the Rayleigh-Ritz method[7], leading to the result

$$G = \frac{45}{4(a^4+b^4)}\left[\frac{1}{b}\int_{-b}^{b} f(y)\left(y^2-\frac{b^2}{3}\right)dy - \frac{1}{a}\int_{-a}^{a}\ell(x)\left(x^2-\frac{a^2}{3}\right)dx\right] \ . \tag{6.36}$$

This agrees with the exact value cited by Meleshko and Gomilko for the special case of the square $b=a$ with $\ell(x)$ a quadratic function of x.

[6] Meleshko and Gomilko *loc. cit.*

[7] See §33.5 and Problem 33.5 below.

Alternative series solutions

In the technique described above, the parameters α_m, β_n were chosen so as to simplify the satisfaction of the shear traction boundary conditions, after which the normal traction conditions led to an infinite set of algebraic equations. An alternative approach is to reverse this procedure by choosing

$$\alpha_m = \frac{(2m-1)\pi}{2a} \; ; \quad \beta_n = \frac{(2n-1)\pi}{2b} \tag{6.37}$$

and omitting the terms with A_0, B_0. Equation (6.31) then defines a stress field in which the first series makes no contribution to the *normal* tractions on $x=a$ and the second series makes no contribution to the normal tractions on $y=b$. These boundary conditions imply a one-to-one relation between A_m, C_m, after which the *shear* traction boundary conditions lead to an infinite set of algebraic equations. One advantage of this version is that it leads to a convergent solution even in the case where $g(b) \neq h(a)$ and hence the shear tractions are discontinuous in the corners[8], though convergence is slow in such cases. An alternative method of treating this discontinuity is to extract it explicitly, as discussed in §11.1.2 below, and then use 'even' series defined by equations (6.32) for the corrective problem.

PROBLEMS

1. Show that if $\zeta = x + \imath y$ and $\sin(\zeta) - \zeta = 0$, where x, y are real variables, then

$$f(x) \equiv \cos x \sqrt{x^2 - \sin^2 x} + \sin x \ln(\sin x) - \sin x \ln\left(x + \sqrt{x^2 - \sin^2 x}\right) = 0 \; .$$

Using Maple or Mathematica to plot the function $f(x)$, find the first six roots of this equation and hence determine the first six values of $\lambda_R b$ for the antisymmetric mode.

2. Devise a method similar to that outlined in Problem 6.1 to determine the first six values of $\lambda_R b$ for the symmetric mode.

3. A displacement function representation for plane strain problems can be developed[9] in terms of two harmonic functions ϕ, ω in the form

$$2\mu u_x = \frac{\partial \phi}{\partial x} + y\frac{\partial \omega}{\partial x} \; ; \; 2\mu u_y = \frac{\partial \phi}{\partial y} + y\frac{\partial \omega}{\partial y} - (3 - 4\nu)\omega$$

[8] Meleshko and Gomilko *loc. cit.*

[9] This solution is a two-dimensional version of a three-dimensional solution developed in Chapter 19 and tabulated in Table 19.1 as solutions A and B.

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} + y\frac{\partial^2 \omega}{\partial x^2} - 2\nu\frac{\partial \omega}{\partial y} \;\; ; \;\; \sigma_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} + y\frac{\partial^2 \omega}{\partial x \partial y} - (1-2\nu)\frac{\partial \omega}{\partial x}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} + y\frac{\partial^2 \omega}{\partial y^2} - 2(1-\nu)\frac{\partial \omega}{\partial y} \; .$$

Use this representation to formulate the eigenvalue problem of the long strip $x>0, -b<y<b$ whose edges $y=\pm b$ are both bonded to a rigid body. Find the eigenvalue equation for symmetric and antisymmetric modes and comment on the expected decay rates for loading of the strip on the end $x=0$.

4. Use the displacement function representation of Problem 6.3 to formulate the eigenvalue problem for the long strip $x>0, -b<y<b$ whose edges $y=\pm b$ are in frictionless contact with a rigid body (so that the normal displacement is zero, but the frictional (tangential) traction is zero). Find the eigenvalue equation for symmetric and antisymmetric modes and comment on the expected decay rates for loading of the strip on the end $x=0$.

5. Use the displacement function representation of Problem 6.3 to formulate the eigenvalue problem for the long strip $x>0, -b<y<b$ which is bonded to a rigid surface at $y=-b$, the other long edge $y=b$ being traction-free. Notice that this problem is not symmetrical, so the problem will not partition into symmetric and antisymmetric modes.

6. Use Mathieu's method to approximate the stresses in the square $-a<x<a$, $-a<y<a$ subjected to the tractions

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = \frac{Sy^2}{a^2} \; ,$$

all the remaining tractions being zero. Start by using (6.35, 6.36) to define a polynomial first approximation to the solution. Then use the series (6.31) to define a *corrective solution* — i.e. the stress function which when added to ϕ_0 defines the complete solution. The constants X_m, Y_n in the corrective solution will then decay with increasing m, n and a good approximation can be found by truncating the series at $m=n=2$.

7

BODY FORCES

A *body force* is defined as one which acts directly on the interior particles of the body, rather than on the boundary. Since the interior of the body is not accessible, it follows necessarily that body forces can only be produced by some kind of physical process which acts 'at a distance'. The commonest examples are forces due to gravity and magnetic or electrostatic attraction. In addition, we can formulate quasi-static elasticity problems for accelerating bodies in terms of body forces, using D'Alembert's principle (see §7.2.2 below).

7.1 Stress function formulation

We noted in §4.3.2 that the Airy stress function formulation satisfies the equilibrium equations if and only if the body forces are identically zero, but the method can be extended to the case of non-zero body forces provided the latter can be expressed as the gradient of a scalar potential, V.

We adopt the new definitions

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V \tag{7.1}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V \tag{7.2}$$

$$\sigma_{xy} = = -\frac{\partial^2 \phi}{\partial x \partial y} \ , \tag{7.3}$$

in which case the two-dimensional equilibrium equations will be satisfied if and only if

$$p_x = -\frac{\partial V}{\partial x} \ \ ; \ \ p_y = -\frac{\partial V}{\partial y} \ , \tag{7.4}$$

i.e.

$$\boldsymbol{p} = -\boldsymbol{\nabla} V \ . \tag{7.5}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_7,

Notice that the body force potential V appears in the definitions of the two normal stress components σ_{xx}, σ_{yy} (7.1, 7.2), but not in the shear stress σ_{xy} (7.3). Thus, the modification in these equations is equivalent to the addition of a biaxial hydrostatic tension of magnitude V, which is of course invariant under coördinate transformation (see §1.1.4).

7.1.1 Conservative vector fields

If a force field is capable of being represented as the gradient of a scalar potential, as in equation (7.5), it is referred to as *conservative*. This terminology arises from gravitational theory, since if we move a particle around in a gravitational field and if the force on the particle varies with position, the principle of conservation of energy demands that the work done in moving the particle from point A to point B should be path-independent — or equivalently, the work done in moving it around a closed path should be zero. If this were not the case, we could choose a direction for the particle to move around the path which would release energy and hence have an inexhaustible source of energy.

If all such integrals *are* path-independent, we can use the work done in bringing a particle from infinity to a given point as the definition of a unique local potential. Then, by equating the work done in an infinitesimal motion to the corresponding change in potential energy, we can show that the local force is proportional to the gradient of the potential, thus demonstrating that a conservative force field must be capable of a representation like (7.5). Conversely, if a given force field can be represented in this form, we can show by integration that the work done in moving a particle from A to B is proportional to $V(A)-V(B)$ and is therefore path-independent.

Not all body force fields are conservative and hence the formulation of §7.1 is not sufficiently general for all problems. However, we shall show below that most of the important problems involving body forces can be so treated.

We can develop a condition for a vector field to be conservative in the same way as we developed the compatibility conditions for strains. We argue that the two independent body force components p_x, p_y are defined in terms of a single scalar potential V and hence we can obtain a constraint equation on p_x, p_y by eliminating V between equations (7.4) with the result

$$\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y} = 0 \; . \tag{7.6}$$

In three dimensions there are three equations like (7.6), from which we conclude that a vector field $\boldsymbol{p}$ is conservative if and only if curl $\boldsymbol{p}=0$. Another name for such fields is *irrotational*, since we note that if we replace $\boldsymbol{p}$ by $\boldsymbol{u}$, the conditions like (7.6) are equivalent to the statement that the rotation $\boldsymbol{\omega}$ is identically zero (*cf* equation (1.47)).

If the body force field satisfies equation (7.6), the corresponding potential can be recovered by partial integration. We shall illustrate this procedure in §7.2 below.

7.1.2 The compatibility condition

We demonstrated in §4.4.1 that, in the absence of body forces, the compatibility condition reduces to the requirement that the Airy stress function ϕ be biharmonic. This condition is modified when body forces are present.

We follow the same procedure as in §4.4.1, but use equations (7.1–7.3) in place of (4.6). Substituting into the compatibility equation in terms of stresses (4.8), we obtain

$$\frac{\partial^4\phi}{\partial y^4} + \frac{\partial^2 V}{\partial y^2} - \nu\frac{\partial^4\phi}{\partial x^2\partial y^2} - \nu\frac{\partial^2 V}{\partial y^2} + 2(1+\nu)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^2 V}{\partial x^2} - \nu\frac{\partial^4\phi}{\partial x^2\partial y^2} - \nu\frac{\partial^2 V}{\partial x^2} = 0 \; , \qquad (7.7)$$

i.e.

$$\nabla^4\phi = -(1-\nu)\nabla^2 V \; . \qquad (7.8)$$

Methods of obtaining suitable functions which satisfy this equation will be discussed in §7.3 below.

7.2 Particular cases

It is worth noting that the vast majority of mechanical engineering components are loaded principally by boundary tractions rather than body forces. Of course, most components are subject to gravity loading, but the boundary loads are generally so much larger that gravity can be neglected. This is less true for civil engineering structures such as buildings, where the self-weight of the structure may be much larger than the weight of the contents or wind loads, but even in this case it is important to distinguish between the gravity loading on the individual component and that transmitted to the component by way of boundary tractions.

It might be instructive at this point for the reader to draw free-body diagrams for a few common engineering components and identify the sources and relative magnitudes of the forces acting upon them. There are really comparatively few ways of applying a load to a body. By far the commonest is to push against it with another body — in other words to apply the load by *contact*. This is why contact problems occupy a central place in elasticity theory[1]. Significant loads may also be applied by fluid pressure as in the case of turbine blades or aircraft wings. Notice that it is fairly easy to apply a compressive normal traction to a boundary, but much harder to apply tension or shear.

This preamble might be taken as a justification for not studying the subject of body forces at all, but there are a few applications in which they are

[1] See Chapters 12, 29.

of critical importance, most notably those dynamic problems in which large accelerations occur. We shall develop expressions for the body force potential for some important cases in the following sections.

7.2.1 Gravitational loading

The simplest type of body force loading is that due to a gravitational field. If the problem is to remain two-dimensional, the direction of the gravitational force must lie in the xy-plane and we can choose it to be in the negative y-direction without loss of generality. The magnitude of the force will be ρg per unit volume, where ρ is the density of the material, so in the notation of §2.1 we have

$$p_x = 0 \;\; ; \;\; p_y = -\rho g \; . \tag{7.9}$$

This force field clearly satisfies condition (7.6) and is therefore conservative and by inspection we note that it can be derived from the body force potential

$$V = \rho g y \; . \tag{7.10}$$

It also follows that $\nabla^2 V = 0$ and hence the stress function, ϕ for problems involving gravitational loading is biharmonic, from (7.6).

7.2.2 Inertia forces

It might be argued that D'Alembert achieved immortality simply by moving a term from one side of an equation to the other, since *D'Alembert's principle* consists merely of writing Newton's second law of motion in the form $\boldsymbol{F} - m\boldsymbol{a} = 0$ and treating the term $-m\boldsymbol{a}$ as a fictitious force in order to reduce the dynamic problem to one in statics.

This simple process enables us to formulate elasticity problems for accelerating bodies as elastostatic body force problems, the corresponding body forces being

$$p_x = -\rho a_x \;\; ; \;\; p_y = -\rho a_y \; , \tag{7.11}$$

where a_x, a_y are the local components of acceleration, which may of course vary with position through the body.

7.2.3 Quasi-static problems

If the body were rigid, the accelerations of equation (7.11) would be restricted to those associated with rigid-body translation and rotation, but in a deformable body, the distance between two points can change, giving rise to additional, stress-dependent terms in the accelerations.

These two effects give qualitatively distinct behaviour, both mathematically and physically. In the former case, the accelerations will generally be defined *a priori* from the kinematics of the problem, which therefore reduces to an elasticity problem with body forces. We shall refer to such problems as *quasi-static*.

By contrast, when the accelerations associated with deformations are important, they are not known *a priori*, since the stresses producing the deformations are themselves part of the solution. In this case the kinematic and elastic problems are coupled and must be solved together. The resulting equations are those governing the propagation of elastic waves through a solid body and their study is known as *Elastodynamics*.

In this chapter, we shall restrict attention to quasi-static problems. As a practical point, we note that the characteristic time scale of elastodynamic problems is very short. For example it generally takes only a very short time for an elastic wave to traverse a solid. If the applied loads are applied gradually in comparison with this time scale, the quasi-static assumption generally gives good results. A measure of the success of this approximation is that it works quite well even for the case of elastic impact between bodies, which may have a duration of the order of a few milliseconds[2].

7.2.4 Rigid-body kinematics

The most general acceleration for a rigid body in the plane involves arbitrary translation and rotation. We choose a coördinate system fixed in the body and suppose that, at some instant, the origin has velocity $\boldsymbol{v_0}$ and acceleration $\boldsymbol{a_0}$ and the body is rotating in the clockwise sense with absolute angular velocity Ω and angular acceleration $\dot{\Omega}$. The instantaneous acceleration of the point (r, θ) *relative to the origin* can then be written

$$a_r = -\Omega^2 r \ \ ; \ \ a_\theta = -\dot{\Omega} r \ . \tag{7.12}$$

Transforming these results into the x, y-coördinate system and adding the acceleration of the origin, we obtain the components of acceleration of the point (x, y) as

$$a_x = a_{0x} - \Omega^2 x + \dot{\Omega} y \tag{7.13}$$
$$a_y = a_{0y} - \Omega^2 y - \dot{\Omega} x \tag{7.14}$$

and hence the corresponding body force field is

$$p_x = -\rho(a_{0x} - \Omega^2 x + \dot{\Omega} y) \tag{7.15}$$
$$p_y = -\rho(a_{0y} - \Omega^2 y - \dot{\Omega} x) \ . \tag{7.16}$$

[2] For more information about elastodynamic problems, the reader is referred to the classical texts of J.D.Achenbach, *Wave Propagation in Elastic Solids*, North Holland, Amsterdam, 1973 and A.C.Eringen and E.S.Şuhubi, *Elastodynamics*, Academic Press, New York, 1975. For a more detailed discussion of the impact of elastic bodies, see K.L.Johnson, *Contact Mechanics*, Cambridge University Press, Cambridge, 1985, §11.4, W.Goldsmith, *Impact*, Arnold, London, 1960.

The astute reader will notice that the case of gravitational loading can be recovered as a special case of these results by writing $a_{0y}=g$ and setting all the other terms to zero. In fact, a reasonable interpretation of the gravitational force is as a D'Alembert force consequent on resisting the gravitational acceleration. Notice that if a body is in free fall — i.e. if it is accelerating freely in a gravitational field and not rotating — there is no body force and hence no internal stress unless the boundaries are loaded.

Substitution of (7.15, 7.16) into (7.6) shows that the inertia forces due to rigid-body accelerations are conservative if and only if $\dot{\Omega}=0$ — i.e. if the angular velocity is constant. We shall determine the body force potential for this special case. Methods of treating the problem with non-zero angular acceleration are discussed in §7.4 below.

From equations (7.4, 7.15, 7.16) with $\dot{\Omega}=0$ we have

$$\frac{\partial V}{\partial x} = \rho(a_{0x} - \Omega^2 x) \tag{7.17}$$

$$\frac{\partial V}{\partial y} = \rho(a_{0y} - \Omega^2 y) \; , \tag{7.18}$$

and hence, on partial integration of (7.17)

$$V = \rho\left(a_{0x}x - \frac{1}{2}\Omega^2 x^2\right) + h(y) \; , \tag{7.19}$$

where $h(y)$ is an arbitrary function of y only. Substituting this result into (7.18) we obtain the ordinary differential equation

$$\frac{dh}{dy} = \rho(a_{0y} - \Omega^2 y) \tag{7.20}$$

for $h(y)$, which has the general solution

$$h(y) = \rho\left(a_{0y}y - \frac{\Omega^2 y^2}{2}\right) + C \; , \tag{7.21}$$

where C is an arbitrary constant which can be taken to be zero without loss of generality, since we are only seeking a *particular* potential function V.

The final expression for V is therefore

$$V = \rho\left(a_{0x}x + a_{0y}y - \frac{1}{2}\Omega^2(x^2 + y^2)\right) \; . \tag{7.22}$$

The reader might like to try this procedure on a set of body forces which *do not* satisfy the condition (7.6). It will be found that the right-hand side of the ordinary differential equation like (7.20) then contains terms which depend on x and hence this equation cannot be solved for $h(y)$.

7.3 Solution for the stress function

Once the body force potential V has been determined, the next step is to find a suitable function ϕ, which satisfies the compatibility condition (7.8) and which defines stresses through equations (7.1–7.3) satisfying the boundary conditions of the problem. There are broadly speaking two ways of doing this. One is to choose some suitable form (such as a polynomial) without regard to equation (7.8) and then satisfy the constraint conditions resulting from (7.8) in the same step as those arising from the boundary conditions. The other is to seek a general solution of the inhomogeneous equation (7.8) and then determine the resulting arbitrary constants from the boundary conditions.

7.3.1 The rotating rectangular beam

As an illustration of the first method, we consider the problem of the rectangular beam $-a < x < a$, $-b < y < b$, rotating about the origin at constant angular velocity Ω, all the boundaries being traction-free (see Figure 7.1).

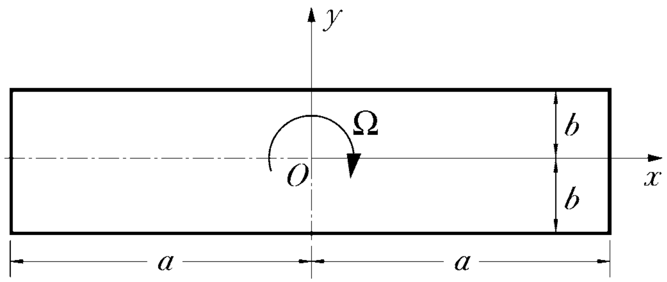

Figure 7.1: The rotating rectangular bar.

The body force potential for this problem is obtained from equation (7.22) as

$$V = -\frac{1}{2}\rho\Omega^2(x^2 + y^2) \;, \tag{7.23}$$

and hence the stress function must satisfy the equation

$$\nabla^4\phi = 2\rho(1-\nu)\Omega^2 \;, \tag{7.24}$$

from (7.8, 7.23).

The geometry suggests a formulation in Cartesian coördinates and equation (7.24) leads us to expect a polynomial of degree 4 in x, y. We also note that V is even in both x and y and that the boundary conditions are homogeneous, so we propose the candidate stress function

$$\phi = A_1x^4 + A_2x^2y^2 + A_3y^4 + A_4x^2 + A_5y^2 \;, \tag{7.25}$$

which contains all the terms of degree 4 and below with the required symmetry.

The constants $A_1, \ldots, A_5$ will be determined from equation (7.24) and from the boundary conditions

$$\sigma_{yx} = 0 \ ; \ \ y = \pm b \tag{7.26}$$

$$\sigma_{yy} = 0 \ ; \ \ y = \pm b \tag{7.27}$$

$$\int_{-b}^{b} \sigma_{xx} dy = 0 \ ; \ \ x = \pm a \ , \tag{7.28}$$

where we have applied the weak boundary conditions only on $x=\pm a$. Note also that the other two weak boundary conditions — that there should be no moment and no shear force on the ends — are satisfied identically in view of the symmetry of the problem about $y=0$.

As in the problem of §5.2.3, it is algebraically simpler to start by satisfying the strong boundary conditions (7.26, 7.27). Substituting (7.25) into (7.1–7.3), we obtain

$$\sigma_{xx} = 2A_2x^2 + 12A_3y^2 + 2A_5 - \frac{1}{2}\rho\Omega^2(x^2+y^2) \tag{7.29}$$

$$\sigma_{yy} = 12A_1x^2 + 2A_2y^2 + 2A_4 - \frac{1}{2}\rho\Omega^2(x^2+y^2) \tag{7.30}$$

$$\sigma_{yx} = -4A_2xy \ . \tag{7.31}$$

It follows that conditions (7.26, 7.27) will be satisfied for all x if and only if

$$A_1 = \rho\Omega^2/24 \ ; \ \ A_2 = 0 \ ; \ \ A_4 = \rho\Omega^2b^2/4 \ . \tag{7.32}$$

The constant A_3 can now be determined by substituting (7.25) into (7.24), with the result

$$24(A_1 + A_3) = 2\rho\Omega^2(1-\nu) \ , \tag{7.33}$$

which is the inhomogeneous equivalent of the constraint equations (see §5.1), and hence

$$A_3 = \rho\Omega^2(1-2\nu)/24 \ , \tag{7.34}$$

using (7.32).

Finally, we determine the remaining constant A_5 by substituting (7.29) into the weak boundary condition (7.28) and evaluating the integral, with the result

$$A_5 = \rho\Omega^2\left(\frac{\nu b^2}{6} + \frac{a^2}{4}\right) \ . \tag{7.35}$$

The final stress field is therefore

$$\sigma_{xx} = \rho\Omega^2\left(\frac{(a^2-x^2)}{2} + \frac{\nu(b^2-3y^2)}{3}\right) \tag{7.36}$$

$$\sigma_{yy} = \frac{\rho\Omega^2}{2}(b^2-y^2) \tag{7.37}$$

$$\sigma_{yx} = 0 \ , \tag{7.38}$$

from equations (7.29–7.31).

Notice that the boundary conditions on the ends $x = \pm a$ agree with those of the physical problem except for the second term in σ_{xx}, which represents a symmetric self-equilibrated traction. From §6.2.2, we anticipate that the error due to this disagreement will be confined to regions near the ends of length comparable with the half-length, b.

7.3.2 Solution of the governing equation

Equation (7.24) is an inhomogeneous partial differential equation — i.e. it has a known function of x, y on the right-hand side — and it can be solved in the same way as an inhomogeneous *ordinary* differential equation, by first finding a particular solution of the equation and then superposing the general solution of the corresponding homogeneous equation.

In this context, a particular solution is *any* function ϕ that satisfies (7.24). It contains no arbitrary constants. The generality in the general solution comes from arbitrary constants in the *homogeneous* solution. Furthermore, the homogeneous solution is the solution of equation (7.24) modified to make the right-hand side zero — i.e.

$$\nabla^4\phi = 0 \tag{7.39}$$

Thus, the homogeneous solution is a general biharmonic function and we can summarize the solution method as containing the three steps:-

(i) finding *any* function ϕ which satisfies (7.24);
(ii) superposing a sufficiently general biharmonic function (which therefore contains several arbitrary constants);
(iii) choosing the arbitrary constants to satisfy the boundary conditions.

In the problem of the preceding section, the particular solution would be any fourth degree polynomial satisfying (7.24) and the homogeneous solution, the most general fourth degree *biharmonic* function with the appropriate symmetry.

Notice that the particular solution (i) is itself a solution of a different physical problem — one in which the body forces are correctly represented, but the correct boundary conditions are not (usually) satisfied. Thus, the function of the homogeneous solution is to introduce additional degrees of freedom to enable us to satisfy the boundary conditions.

Physical superposition

It is often helpful to think of this superposition process in a physical rather than a mathematical sense. In other words, we devise a related problem in which the body forces are the same as in the real problem, but where the boundary conditions are simpler. For example, if a beam is subjected to gravitational loading, a simple physical 'particular' solution would correspond to the problem in which the beam is resting on a rigid foundation and hence the stress field is one-dimensional. To complete the real problem, we would then have to superpose the solution of a corrective problem with tractions equal to the difference between the required tractions and those implied by the particular solution, but with no body force, since this has already been taken into account in the particular solution.

One advantage of this way of thinking is that it is not restricted to problems in which the body force can be represented by a potential. We can therefore use it to solve the problem of the rotationally accelerating beam in the next section.

7.4 Rotational acceleration

We saw in §7.2.4 that the body force potential cannot be used in problems where the angular acceleration is non-zero. In this section, we shall generate a particular solution for this problem and then generalize it, using the results without body force from Chapter 5.

7.4.1 The circular disk

Consider the rotationally symmetric problem of Figure 7.2. A solid circular disk, radius a, is initially at rest ($\Omega = 0$) and at time $t = 0$ it is caused to accelerate in a clockwise direction with angular acceleration $\dot{\Omega}$ by tractions uniformly distributed around the edge $r = a$. Note that we could determine the magnitude of these tractions by writing the equation of motion for the disk, but it will not be necessary to do this — the result will emerge from the analysis of the stress field.

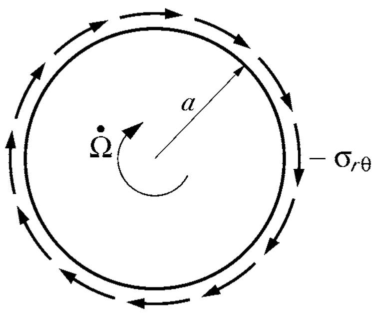

Figure 7.2: The disk with rotational acceleration.

At any given time, the body forces and hence the resulting stress field will be the sum of two parts, one due to the instantaneous angular velocity and the other to the angular acceleration. The former can be obtained using the body force potential and we therefore concentrate here on the contribution due to the angular acceleration. This is the *only* body force at the beginning of the process, so the following solution can also be regarded as the solution for the instant, $t=0$.

The problem is clearly axisymmetric, so the stresses and displacements must depend on the radius r only, but it is also antisymmetric, since if the sign of the tractions were changed, the problem would become a mirror image of that illustrated. Now suppose some point in the disk had a non-zero outward radial displacement, u_r. Changing the sign of the tractions would change the sign of this displacement — i.e. make it be directed inwards — but this is impossible if the new problem is to be a mirror image of the old. We can therefore conclude from symmetry that $u_r = 0$ throughout the disk. In the same way, we can conclude that the stress components $\sigma_{rr}, \sigma_{\theta\theta}$ are zero everywhere. Incidentally, we also note that if the problem is conceived as being one of plane strain, symmetry demands that σ_{zz} be everywhere zero. It follows that this is one of those lucky problems in which plane stress and plane strain are the same and hence the plane stress assumption involves no approximation.

We conclude that there is only one non-zero displacement component, u_θ, and one non-zero strain component, $e_{r\theta}$. Thus, the number of strain and displacement components is equal and no non-trivial compatibility conditions can be obtained by eliminating the displacement components. (An alternative statement is that all the compatibility equations are satisfied identically, as can be verified by substitution — the compatibility equations in cylindrical polar coördinates are given by Saada[3]).

It follows that the only non-zero stress component, $\sigma_{r\theta}$ can be determined from equilibrium considerations alone. Considering the equilibrium of a small element due to forces in the θ-direction and dropping terms which are zero due to the symmetry of the system, we obtain

$$\frac{d\sigma_{r\theta}}{dr} + \frac{2\sigma_{r\theta}}{r} = -\rho r \dot{\Omega} \;, \tag{7.40}$$

which has the general solution

$$\sigma_{r\theta} = -\frac{\rho \dot{\Omega} r^2}{4} + \frac{A}{r^2} \;, \tag{7.41}$$

where A is an arbitrary constant which must be set to zero to retain continuity at the origin. Thus, the stress field in the disk of Figure 7.2 is

$$\sigma_{r\theta} = -\frac{\rho \dot{\Omega} r^2}{4} \;. \tag{7.42}$$

In particular, the traction at the surface $r = a$ is

[3] A.S.Saada, *Elasticity*, Pergamon Press, New York, 1973, §6.9.

$$-\sigma_{r\theta}(a) = \frac{\rho\dot{\Omega}a^2}{4} \tag{7.43}$$

and the applied moment about the axis of rotation is therefore

$$M = -2\pi a^2 \sigma_{r\theta}(a) = \frac{\pi\rho\dot{\Omega}a^4}{2} \, , \tag{7.44}$$

since the traction acts over a length $2\pi a$ and the moment arm is a. We know from elementary dynamics that $M = I\dot{\Omega}$, where I is the moment of inertia and we can therefore deduce from (7.44) that

$$I = \frac{\pi\rho a^4}{2} \, , \tag{7.45}$$

which is of course the correct expression for the moment of inertia of a solid disk of density ρ, radius a and unit thickness. Notice however that we were able to deduce the relation between M and $\dot{\Omega}$ without using the equations of rigid-body dynamics. Equation (7.40) ensures that every particle of the body obeys Newton's law and this is sufficient to ensure that the complete body satisfies the equations of rigid-body dynamics.

7.4.2 The rectangular bar

We can use the stress field (7.42) as a particular solution for determining the stresses in a body of any shape due to angular acceleration. One way to think of this is to imagine cutting out the real body from an imaginary disk of sufficiently large radius. The stresses in the cut-out body will be the same as those in the disk, provided we arrange to apply tractions to the boundaries of the body that are equal to the stress components on those surfaces before the cut was made. These will not generally be the correct boundary tractions for the real problem, but we can correct the boundary tractions by superposing a homogeneous solution — i.e. a corrective solution for the actual body with prescribed boundary tractions (equal to the difference between those applied and those obtained in the disk solution) but no body forces (since these have already been taken care of in the disk solution).

As an illustration, we consider the rectangular bar, $-a<x<a$, $-b<y<b$, accelerated by two equal shear forces, F, applied at the ends $x=\pm a$ as shown in Figure 7.3, the other boundaries, $y=\pm b$ being traction free.

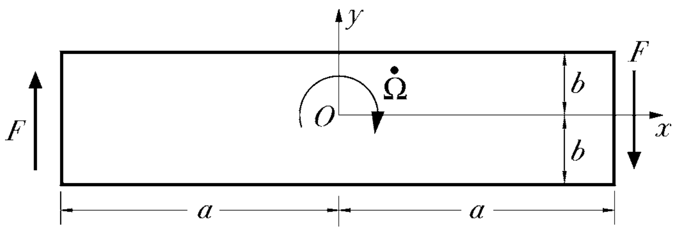

Figure 7.3: The rectangular bar with rotational acceleration.

We first transform the disk solution into rectangular coördinates, using the transformation relations (1.15–1.17), obtaining

$$\sigma_{xx} = -2\sigma_{r\theta}\sin\theta\cos\theta = \frac{\rho\dot{\Omega}r^2\sin\theta\cos\theta}{2} = \frac{\rho\dot{\Omega}xy}{2} \tag{7.46}$$

$$\sigma_{yy} = 2\sigma_{r\theta}\sin\theta\cos\theta = -\frac{\rho\dot{\Omega}r^2\sin\theta\cos\theta}{2} = -\frac{\rho\dot{\Omega}xy}{2} \tag{7.47}$$

$$\sigma_{xy} = \sigma_{r\theta}(\cos^2\theta - \sin^2\theta) = -\frac{\rho\dot{\Omega}r^2(\cos^2\theta - \sin^2\theta)}{2} = -\frac{\rho\dot{\Omega}(x^2-y^2)}{4} \; . \tag{7.48}$$

This stress field is clearly odd in both x and y and involves normal tractions on $y=\pm b$ which vary linearly with x. The bending moment will therefore vary with x^3 suggesting a stress function ϕ with a leading term x^5y — i.e. a sixth degree polynomial.

The most general polynomial with the appropriate symmetry is

$$\phi = A_1x^5y + A_2x^3y^3 + A_3xy^5 + A_4x^3y + A_5xy^3 + A_6xy \; . \tag{7.49}$$

The stress components are the sum of those obtained from the homogeneous stress function (7.49) *using the definitions (4.6)* — remember the homogeneous solution here is one without body force — and those from the disk problem given by equations (7.46–7.48). We find

$$\sigma_{xx} = \frac{1}{2}\rho\dot{\Omega}xy + 6A_2x^3y + 20A_3xy^3 + 6A_5xy \tag{7.50}$$

$$\sigma_{yy} = -\frac{1}{2}\rho\dot{\Omega}xy + 20A_1x^3y + 6A_2xy^3 + 6A_4xy \tag{7.51}$$

$$\sigma_{xy} = -\frac{1}{4}\rho\dot{\Omega}(x^2-y^2) - 5A_1x^4 - 9A_2x^2y^2 - 5A_3y^4 - 3A_4x^2 - 3A_5y^2 - A_6 \; . \tag{7.52}$$

The boundary conditions are

$$\sigma_{yx} = 0 \;\; ; \;\; y = \pm b \tag{7.53}$$

$$\sigma_{yy} = 0 \;\; ; \;\; y = \pm b \tag{7.54}$$

$$\int_{-b}^{b} y\sigma_{xx}dy = 0 \;\; ; \;\; x = \pm a \; , \tag{7.55}$$

where we note that weak boundary conditions are imposed on the ends $x=\pm a$. Since the solution is odd in y, only the moment and shear force conditions are non-trivial and the latter need not be explicitly imposed, since they will be satisfied by global equilibrium (as in the example in §5.2.2).

Conditions (7.53, 7.54) have to be satisfied for all x and hence the corresponding coefficients of all powers of x must be zero. It follows immediately that $A_1=0$ and the remaining conditions can be written

$$-\frac{1}{2}\rho\dot{\Omega} + 6A_2b^2 + 6A_4 = 0 \tag{7.56}$$

$$-\frac{1}{4}\rho\dot{\Omega} - 9A_2b^2 - 3A_4 = 0 \tag{7.57}$$

$$\frac{1}{4}\rho\dot{\Omega}b^2 - 5A_3b^4 - 3A_5b^2 - A_6 = 0\ . \tag{7.58}$$

We get one additional condition from the requirement that ϕ (equation (7.49)) be biharmonic

$$72A_2 + 120A_3 = 0 \tag{7.59}$$

and another from the boundary condition (7.55), which with (7.50) yields

$$\frac{1}{6}\rho\dot{\Omega} + 2A_2a^2 + 4A_3b^2 + 2A_5 = 0\ . \tag{7.60}$$

The solution of these equations is routine, giving the stress function

$$\phi = -\frac{\rho\dot{\Omega}}{60b^2}(5x^3y^3 - 3xy^5 - 10b^2x^3y + 11b^2xy^3 - 5a^2xy^3 + 15a^2b^2xy - 33b^4xy)\ . \tag{7.61}$$

The complete stress field, including the particular solution terms, is

$$\sigma_{xx} = \rho\dot{\Omega}xy\left(\frac{y^2}{b^2} - \frac{3}{5} + \frac{(a^2 - x^2)}{2b^2}\right) \tag{7.62}$$

$$\sigma_{yy} = \frac{\rho\dot{\Omega}xy}{2}\left(1 - \frac{y^2}{b^2}\right) \tag{7.63}$$

$$\sigma_{xy} = \rho\dot{\Omega}(b^2 - y^2)\left(\frac{y^2 + a^2 - 3x^2}{4b^2} - \frac{11}{20}\right)\ , \tag{7.64}$$

from (7.50–7.52).

Finally, we can determine the forces F on the ends by integrating the shear traction, σ_{xy} over either end as

$$F = -\int_{-b}^{b}\sigma_{xy}(a)dy = \frac{2}{3}\rho\dot{\Omega}b(a^2 + b^2)\ . \tag{7.65}$$

Maple and Mathematica solutions of this problem are given in the files 'S742'. As in §7.4.1, the relation between the applied loading and the angular acceleration has been obtained without recourse to the equations of rigid-body dynamics. However, we note that the moment of inertia of the rectangular bar for rotation about the origin is $I = 4\rho ab(a^2 + b^2)/3$ and the applied moment is $M = 2Fa$, so application of the equation $M = I\dot{\Omega}$ leads to the same expression for F as that found in (7.65).

7.4.3 Weak boundary conditions and the equation of motion

We saw in §5.2.1 that weak boundary conditions need only be applied at one end of a stationary rectangular bar, since the stress field defined in terms of the Airy stress function must involve tractions that maintain the body in equilibrium. Similarly, in dynamic problems involving prescribed accelerations, an appropriate set of weak boundary conditions can be omitted. For example, we did not have to specify the value of F in the problem of Figure 7.3. Instead, it's value was calculated after the stress field had been determined and necessarily proved to be consistent with the equations of rigid-body dynamics.

If a body is prevented from moving by a statically determinate support, it is natural to treat the support reactions as the 'neglected' weak boundary conditions. Thus, if the body is attached to a rigid support at one boundary, we apply no weak conditions at that boundary, as in §5.2.1. If the body is simply supported at a boundary, we impose the weak boundary condition that there be zero moment applied at the support, leading for example to the conditions stated in footnote 2 on Page 63.

Similar considerations apply in a dynamic problem if the body is attached to a support which moves in such a way as to prescribe the acceleration of the body. However, we may also wish to solve problems in which specified non-equilibrated loads are applied to an unsupported body. For example, we may be asked to determine the stresses in the body of Figure 7.3 due to prescribed end loads F. One way to do this would be first to solve a rigid-body dynamics problem to determine the accelerations and then proceed as in §7.4.2. However, in view of the present discussion, a more natural approach would be to include the angular acceleration $\dot{\Omega}$ as an unknown and use equation (7.65) to determine it in terms of F at the end of the solution procedure.

More generally, if we have a two-dimensional body subjected to prescribed tractions on all edges, we could assume the most general accelerations (7.13, 7.14) and solve the body force problem, treating $a_{0x}, a_{0y}, \dot{\Omega}$ as if they were known. If strong boundary conditions are imposed on two opposite edges and weak boundary conditions on *both* the remaining edges, it will then be found that there are three extra conditions which serve to determine the unknown accelerations.

PROBLEMS

1. Every particle of an elastic body of density ρ experiences a force

$$F = \frac{C\delta m}{r^2}$$

directed towards the origin, where C is a constant, r is the distance from the origin and δm is the mass of the particle. Find a body force potential V that satisfies these conditions.

2. Verify that the body force distribution

$$p_x = Cy \;\; ; \;\; p_y = -Cx$$

is non-conservative, by substituting into equation (7.6). Use the technique of §7.2.4 to attempt to construct a body force potential V for this case. Identify the step at which the procedure breaks down.

3. To construct a particular solution for the stress components in plane strain due to a non-conservative body force distribution, it is proposed to start by representing the displacement components in the form

$$u_x = \frac{1}{2\mu}\frac{\partial\psi}{\partial y} \;\; ; \;\; u_y = -\frac{1}{2\mu}\frac{\partial\psi}{\partial x} \;\; ; \;\; u_z = 0 \; .$$

Use the strain-displacement equations (1.51) and Hooke's law (1.71) to find expressions for the stress components in terms of ψ. Substitute these results into the equilibrium equations (2.2, 2.3) to find the governing equations for the stress function ψ.

What is the condition that must be satisfied by the body force distribution $\boldsymbol{p}$ if these equations are to have a solution? Show that this condition is satisfied if the body force distribution can be written in terms of a potential function W as

$$p_x = \frac{\partial W}{\partial y} \;\; ; \;\; p_y = -\frac{\partial W}{\partial x} \; .$$

For the special case

$$p_x = Cy \;\; ; \;\; p_y = -Cx \; ,$$

find a particular solution for ψ in the axisymmetric polynomial form

$$\psi = A(x^2 + y^2)^n \; ,$$

where A is a constant and n is an appropriate integer power. Show that this solution can be used to obtain the stress components (7.46–7.48). Suggest ways in which this method might be adapted to give a particular solution for more general non-conservative body force distributions.

4. If the elastic displacement $\boldsymbol{u}$ varies in time, there will generally be accelerations $\boldsymbol{a} = \ddot{\boldsymbol{u}}$ and hence body forces $\boldsymbol{p} = -\rho\ddot{\boldsymbol{u}}$, from equation (7.11). Use this result and equation (2.17) to develop the general equation of linear Elastodynamics.

Show that this equation is satisfied by a displacement field of the form

$$u_x = f(x - c_1 t) \;\; ; \;\; u_y = g(x - c_2 t) \;\; ; \;\; u_z = 0 \; ,$$

where f, g are any functions and c_1, c_2 are two constants that depend on the material properties. Find the values of c_1, c_2 and comment on the physical significance of this solution.

5. The beam $-b<y<b$, $0<x<L$ is built-in at the edge $x=L$ and is loaded merely by its own weight, ρg per unit volume.

Find a solution for the stress field, using weak conditions on the end $x=0$.

6. One wall of a multistory building of height H is approximated as a thin plate $-b < x < b$, $0 < y < H$. During an earthquake, ground motion causes the building to experience a uniform acceleration a in the x-direction. Find the resulting stresses in the wall if the material has density ρ and the edges $x=\pm b$, $y=H$ can be regarded as traction-free.

7. A tall thin rectangular plate $-a<x<a$, $-b<y<b$ $(b\gg a)$ is supported on the vertical edges $x=\pm a$ and loaded only by its own weight (density ρ). Find the stresses in the plate using weak boundary conditions on the horizontal edges $y=\pm b$ and assuming that the support tractions consist only of uniform shear.

8. Figure 7.4(a) shows a triangular cantilever, defined by the boundaries $y=0$, $y = x\tan\alpha$ and built-in at $x = a$. It is loaded by its own weight, ρg per unit volume. Find a solution for the complete stress field and compare the maximum tensile bending stress with that predicted by the Mechanics of Materials theory.

Would the maximum tensile stress be lower if the alternative configuration of Figure 7.4(b) were used?

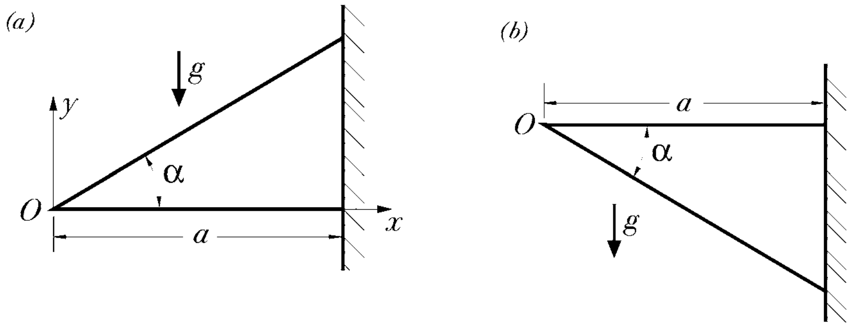

Figure 7.4

9. The thin rectangular plate $-a<x<a, -b<y<b$ with $a\gg b$ rotates about the y-axis at constant angular velocity Ω. All surfaces of the plate are traction-free. Find a solution for the stress field, using strong boundary conditions on the long edges $y=\pm b$ and weak boundary conditions on the ends $x=\pm a$.

10. Solve Problem 7.9 for the case where $a\ll b$. In this case you should use strong boundary conditions on $x = \pm a$ and weak boundary conditions on $y=\pm b$.

11. A thin triangular plate bounded by the lines $y = \pm x \tan\alpha$, $x = a$ rotates about the axis $x = a$ at constant angular velocity Ω. The inclined edges of the plate $y = \pm x \tan\alpha$ are traction-free. Find a solution for the stress field, using strong boundary conditions on the inclined edges (weak boundary conditions will then be implied on $x = a$).

12. A thin triangular plate bounded by the lines $y = \pm x \tan\alpha$, $x = a$ is initially at rest, but is subjected to tractions at the edge $x = a$ causing an angular acceleration $\dot{\Omega}$ about the perpendicular axis through the point $(a, 0)$. The inclined surfaces $y = \pm x \tan\alpha$ are traction-free. Find a solution for the instantaneous stress field, using the technique of §7.4.2.

13. The thin square plate $-a < x < a$, $-a < y < a$ rotates about the z-axis at constant angular velocity Ω. Use Mathieu's method (§6.4 and Problem 6.6) to develop a series solution that satisfies all the boundary conditions in the strong sense. Make a contour plot of the maximum principal stress based on truncating the series at $m = n = 2$, and identify the location and magnitude of the maximum tensile stress. By what percentage does this value differ from the approximation obtained from §7.3.1?

8

PROBLEMS IN POLAR COÖRDINATES

Polar coördinates (r, θ) are particularly suited to problems in which the boundaries can be expressed in terms of equations like $r = a, \theta = \alpha$. This includes the stresses in a circular disk or around a circular hole, the curved beam with circular boundaries and the wedge, all of which will be discussed in this and subsequent chapters.

8.1 Expressions for stress components

We first have to transform the biharmonic equation (4.11) and the expressions for stress components (4.6) into polar coördinates, using the relations

$$x = r\cos\theta \;\;;\;\; y = r\sin\theta \tag{8.1}$$

$$r = \sqrt{x^2+y^2} \;\;;\;\; \theta = \arctan\left(\frac{y}{x}\right) \;. \tag{8.2}$$

The derivation is tedious, but routine. We first note by differentiation that

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{8.3}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \;. \tag{8.4}$$

It follows that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) \\ &= \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \\ &\quad +2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial \theta}\right)\end{aligned} \tag{8.5}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_8, © Springer Science+Business Media B.V. 2010

and by a similar process, we find that

$$\frac{\partial^2}{\partial y^2} = \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$
$$-2\sin\theta\cos\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \quad (8.6)$$
$$\frac{\partial^2}{\partial x\partial y} = \sin\theta\cos\theta\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$
$$-\left(\cos^2\theta - \sin^2\theta\right)\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) . \quad (8.7)$$

Finally, we can determine the expressions for stress components, noting for example that

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \quad (8.8)$$
$$= \cos^2\theta\frac{\partial^2\phi}{\partial y^2} + \sin^2\theta\frac{\partial^2\phi}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2\phi}{\partial x\partial y}$$
$$= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} , \quad (8.9)$$

after substituting for the partial derivatives from (8.5–8.7) and simplifying.

The remaining stress components, $\sigma_{\theta\theta}, \sigma_{r\theta}$ can be obtained by a similar procedure. We find

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \; ; \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \quad (8.10)$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) . \quad (8.11)$$

If there is a conservative body force $\boldsymbol{p}$ described by a potential V, the stress components are modified to

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + V \; ; \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} + V \quad (8.12)$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) , \quad (8.13)$$

where

$$p_r = -\frac{\partial V}{\partial r} \; ; \quad p_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} . \quad (8.14)$$

We also note that the Laplacian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} , \quad (8.15)$$

from equations (8.5, 8.6) and hence

$$\nabla^4\phi \equiv \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) . \tag{8.16}$$

Notice that in applying the second Laplacian operator in (8.16) it is necessary to differentiate by parts. The two differential operators cannot simply be multiplied together. In Mathematica and Maple, the easiest technique is to define a new function

$$f \equiv \nabla^2\phi = \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{8.17}$$

and then obtain $\nabla^4\phi$ from

$$\nabla^4\phi = \nabla^2 f = \frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} . \tag{8.18}$$

8.2 Strain components

A similar technique can be used to obtain the strain-displacement relations in polar coördinates. Writing

$$u_x = u_r\cos\theta - u_\theta\sin\theta \tag{8.19}$$

$$u_y = u_r\sin\theta + u_\theta\cos\theta \tag{8.20}$$

and substituting in (8.3), we find

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_r}{\partial r}\cos^2\theta + \left(\frac{u_\theta}{r}-\frac{\partial u_\theta}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\sin\theta\cos\theta + \left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)\sin^2\theta . \tag{8.21}$$

Using the same method to obtain expressions for e_{xy}, e_{yy} and substituting the results in the strain transformation relations analogous to (8.8) etc., we obtain the polar coördinate strain-displacement relations

$$e_{rr} = \frac{\partial u_r}{\partial r} \;\; ; \;\; e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right) \;\; ; \;\; e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r} . \tag{8.22}$$

8.3 Fourier series expansion

The simplest problems in polar coördinates are those in which there are no θ-boundaries, the most general case being the disk with a central hole illustrated in Figure 8.1 and defined by $a<r<b$.

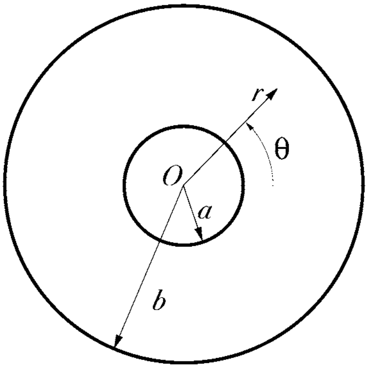

Figure 8.1: The disk with a central hole.

The stresses and displacements must be single-valued and continuous and hence they must be periodic functions of θ, since $(r, \theta+2m\pi)$ defines the same point as (r, θ), when m is any integer. It is therefore natural to seek a general solution of the problem of Figure 8.1 in the form

$$\phi = \sum_{n=0}^{\infty} f_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} g_n(r)\sin(n\theta) \; . \tag{8.23}$$

Substituting this expression into the biharmonic equation, using (8.16), we find that the functions f_n, g_n must satisfy the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} - \frac{n^2 f_n}{r^2}\right) = 0 \; , \tag{8.24}$$

which, for $n \neq 0, 1$ has the general solution

$$f_n(r) = A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n} \; , \tag{8.25}$$

where $A_{n1}, \dots A_{n4}$ are four arbitrary constants.

When $n = 0, 1$, the solution (8.25) develops repeated roots and equation (8.24) has a different form of solution given by

$$f_0(r) = A_{01}r^2 + A_{02}r^2\ln(r) + A_{03}\ln(r) + A_{04} \tag{8.26}$$

$$f_1(r) = A_{11}r^3 + A_{12}r\ln(r) + A_{13}r + A_{14}r^{-1} \; . \tag{8.27}$$

In all the above equations, we note that the stress functions associated with the constants A_{n3}, A_{n4} are *harmonic* and hence biharmonic *a fortiori*, whereas those associated with A_{n1}, A_{n2} are biharmonic but not harmonic.

8.3.1 Satisfaction of boundary conditions

The boundary conditions on the surfaces $r = a, b$ will generally take the form

$$\sigma_{rr} = F_1(\theta) \;\; ; \;\; r = a \tag{8.28}$$
$$= F_2(\theta) \;\; ; \;\; r = b \tag{8.29}$$
$$\sigma_{r\theta} = F_3(\theta) \;\; ; \;\; r = a \tag{8.30}$$
$$= F_4(\theta) \;\; ; \;\; r = b \; , \tag{8.31}$$

each of which has to be satisfied for all values of θ. This can conveniently be done by expanding the functions $F_1, \ldots F_4$ as Fourier series in θ. i.e.

$$F_j(\theta) = \sum_{n=0}^{\infty} C_{nj} \cos(n\theta) + \sum_{n=1}^{\infty} D_{nj} \sin(n\theta) \;\; ; \;\; j = 1, \ldots, 4 \; . \tag{8.32}$$

In combination with the stress function of equation (8.23), (8.28–8.31) will then give four independent equations for each trigonometric term, $\cos(n\theta)$, $\sin(n\theta)$ in the series and hence serve to determine the four constants A_{n1}, A_{n2}, A_{n3}, A_{n4}. The problem of Figure 8.1 is therefore susceptible of a general solution.

8.3.2 Circular hole in a shear field

The general solution has some anomolous features for the special values $n = 0, 1$, but before discussing these we shall illustrate the method by solving a simple example. Figure 8.2 shows a large plate in a state of pure shear $\sigma_{xy} = S$, perturbed by a hole of radius a.

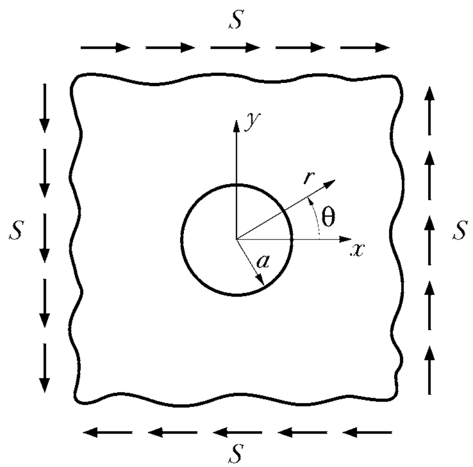

Figure 8.2: Circular hole in a shear field.

A formal statement of the boundary conditions for this problem is

$$\sigma_{rr} = 0 \;\; ; \;\; r = a \tag{8.33}$$

$$\sigma_{r\theta} = 0 \;\; ; \;\; r = a \tag{8.34}$$

$$\sigma_{xx}, \sigma_{yy} \to 0 \;\; ; \;\; r \to \infty \tag{8.35}$$

$$\sigma_{xy} \to S \;\; ; \;\; r \to \infty \; . \tag{8.36}$$

Notice that we describe a body as 'large' when we mean to interpret the boundary condition at the outer boundary as being applied at $r \to \infty$. Notice also that we have stated these 'infinite' boundary conditions in (x, y) coördinates, since this is the most natural way to describe a state of uniform stress. We shall see that the stress function approach makes it possible to use both rectangular and polar coördinates in the same problem without any special difficulty.

This is a typical *perturbation* problem in which a simple state of stress is perturbed by a local geometric feature (in this case a hole). It is reasonable in such cases to anticipate that the stresses distant from the hole will be unperturbed and that the effect of the hole will only be felt at moderate values of r/a. Perturbation problems are most naturally approached by first solving the simpler problem in which the perturbation is absent (in this case the plate *without* a hole) and then seeking a corrective solution which will describe the influence of the hole on the stress field. With such a formulation, we anticipate that the corrective solution will decay with increasing r.

The unperturbed field is clearly a state of uniform shear, $\sigma_{xy} = S$, and this in turn is conveniently described by the stress function

$$\phi = -Sxy = -Sr^2 \sin\theta \cos\theta = -\frac{Sr^2 \sin(2\theta)}{2} \; , \tag{8.37}$$

from equation (4.6). Notice that although the stress function is originally determined in rectangular coördinates, it is easily transformed into polar coördinates using (8.1) and then into the Fourier form of equation (8.23).

The unperturbed solution satisfies the 'infinite' boundary conditions (8.35, 8.36), but will violate the conditions at the hole surface (8.33, 8.34). However, we can correct the stress field by superposing those terms from the series (8.23) which (i) have the same Fourier dependence as (8.37) and (ii) lead to stresses which decay as r increases. There will always be two such terms for any given Fourier component, permitting the two traction boundary conditions to be satisfied. In the present instance, the required terms are those derived from the constants A_{22}, A_{24} in equation (8.25), giving the stress function[1]

$$\phi = -\frac{Sr^2 \sin(2\theta)}{2} + A\sin(2\theta) + \frac{B\sin(2\theta)}{r^2} \; . \tag{8.38}$$

The corresponding stress components are obtained by substituting in (8.10, 8.11) with the result

[1] It might be thought that the term involving A_{22} is inappropriate because it does not decay with r. However, it leads to *stresses* which decay with r, which is of course what we require.

$$\sigma_{rr} = \left(S - \frac{4A}{r^2} - \frac{6B}{r^4} \right) \sin(2\theta) \tag{8.39}$$

$$\sigma_{r\theta} = \left(S + \frac{2A}{r^2} + \frac{6B}{r^4} \right) \cos(2\theta) \tag{8.40}$$

$$\sigma_{\theta\theta} = \left(-S + \frac{6B}{r^4} \right) \sin(2\theta) \; . \tag{8.41}$$

The two boundary conditions at the hole surface (8.33, 8.34) then yield the two equations

$$\frac{4A}{a^2} + \frac{6B}{a^4} = S \tag{8.42}$$

$$\frac{2A}{a^2} + \frac{6B}{a^4} = -S \tag{8.43}$$

for the constants A, B, with solution

$$A = Sa^2 \;\; ; \;\; B = -\frac{Sa^4}{2} \tag{8.44}$$

and the final stress field is

$$\sigma_{rr} = S \left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4} \right) \sin(2\theta) \tag{8.45}$$

$$\sigma_{r\theta} = S \left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4} \right) \cos(2\theta) \tag{8.46}$$

$$\sigma_{\theta\theta} = S \left(-1 - 3\frac{a^4}{r^4} \right) \sin(2\theta) \; . \tag{8.47}$$

Notice incidentally that the maximum stress is the *hoop* stress, $\sigma_{\theta\theta} = 4S$ at the point $(a, 3\pi/4)$. At this point there is a state of uniaxial tension, so the maximum *shear* stress is $2S$. Since the unperturbed shear stress has a magnitude S, we say that the hole produces a *stress concentration* of 2. However, for a brittle material, we might be inclined to define the stress concentration factor as the ratio of the maximum tensile stresses in the perturbed and unperturbed solutions, which in the present problem is 4. In general, a stress concentration factor implies a measure of the severity of the stress field — usually a failure theory — which serves as a standard of comparison and the magnitude of the stress concentration factor will depend upon the measure used.

The determination of the stress concentration due to holes, notches and changes of section under various loading conditions is clearly a question of considerable practical importance. An extensive discussion of problems of this kind is given by Savin[2] and stress concentration factors for a wide range of geometries are tabulated by Peterson[3] in a form suitable for use in engineering design.

[2] G.N.Savin, *Stress Concentration around Holes*, Pergamon Press, Oxford, 1961.

[3] R.E.Peterson, *Stress Concentration Design Factors*, John Wiley, New York, 1974.

8.3.3 Degenerate cases

We have already remarked in §8.3 above that the solution (8.25) degenerates for $n=0,1$ and must be supplemented by some additional terms. In fact, even the modified stress function of (8.26, 8.27) is degenerate because the stress function

$$\phi = A + Bx + Cy = A + Br\cos\theta + Cr\sin\theta \qquad (8.48)$$

defines a trivial null state of stress (see §5.1.1) and hence the constants A_{04}, A_{13} in equations (8.26, 8.27) correspond to null stress fields.

This question of degeneracy arises elsewhere in Elasticity and indeed in mathematics generally, so we shall take this opportunity to develop a general technique for resolving it. As we saw in §8.3, the degeneracy often arises from the occurrence of repeated roots to an equation. Thus, in equation (8.25), the terms $A_{n2}r^{-n+2}, A_{n3}r^n$ degenerate to the same form when $n=1$.

Suppose for the moment that we relax the restriction that n be an integer. Clearly the degeneracy in equation (8.25) only arises exactly at the values $n=0,1$. There is no degeneracy for $n=1+\epsilon$ for any non-zero ϵ, however small. We shall show therefore that we can recover the extra solution at the degenerate point by allowing the solution to tend smoothly to the limit $n\rightarrow 1$, rather than setting $n=1$ *ab initio.*

For $n=1+\epsilon$, the two offending terms in (8.25) can be written

$$f(r) = Ar^{1-\epsilon} + Br^{1+\epsilon} \; , \qquad (8.49)$$

where A, B, are two arbitrary constants.

Clearly the two terms tend to the same form as $\epsilon\rightarrow 0$. However, suppose we construct a new function from the sum and difference of these functions in the form

$$f(r) = C(r^{1+\epsilon} + r^{1-\epsilon}) + D(r^{1+\epsilon} - r^{1-\epsilon}) \; . \qquad (8.50)$$

In this form, the first function tends to $2Cr$ as $\epsilon\rightarrow 0$, whilst the second tends to zero. However, we can prevent the second term degenerating to zero, since the constant D is arbitrary. We can therefore choose $D=E\epsilon^{-1}$ in which case the second term will tend to the limit

$$\lim_{\epsilon\rightarrow 0} E\frac{r^{1+\epsilon} - r^{1-\epsilon}}{\epsilon} = 2Er\ln(r) \; , \qquad (8.51)$$

where we have used L'Hôpital's rule to evaluate the limit. Notice that this result agrees with the special term included in equation (8.27) above to resolve the degeneracy in ϕ for $n=1$.

This procedure can be generalized as follows: Whenever a degeneracy occurs at a denumerable set of values of a parameter (such as n in equation (8.25)), we can always make up the defecit using additional terms obtained by differentiating the original form with respect to the parameter *before* allowing it to take the special value.

The pairs r^{n+2}, r^{-n+2} and r^n, r^{-n} both degenerate to the same form for $n=0$, so the new functions are of the form

$$\lim_{n\to 0} \frac{d(r^{n+2})}{dn} = r^2 \ln(r) \tag{8.52}$$

$$\lim_{n\to 0} \frac{d(r^n)}{dn} = \ln(r) \;, \tag{8.53}$$

again agreeing with the special terms introduced in (8.26).

Sometimes the degeneracy is of a higher order, corresponding to more than two identical roots, in which case L'Hôpital's rule has to be applied more than once — i.e. we have to differentiate more than once with respect to the parameter. It is always a straightforward matter to check the resulting solutions to make sure they are of the required form.

We now turn our attention to the degeneracy implied by the triviality of the stress function of equation (8.48). Here, the stress function *itself* is not degenerate — i.e. it is a legitimate function of the required Fourier form and it is linearly independent of the other functions of the same form. The trouble is that it gives a null stress field. We must therefore look for stress functions that are *not* of the standard Fourier form, but which give Fourier type stress components.

Once again, the method is to approach the solution as a limit using L'Hôpital's rule, but this time we have to operate on the complete stress function — not just on the part that varies with r. Suppose we consider the axisymmetric degenerate term, $\phi=A$, which can be regarded as the limit of a stress function of the form

$$\phi = Ar^\epsilon \cos(\epsilon\theta) + Br^\epsilon \sin(\epsilon\theta) \;, \tag{8.54}$$

as $\epsilon\to 0$.

Differentiating this function with respect to ϵ and then letting ϵ tend to zero,we obtain the new function

$$\phi = A\ln(r) + B\theta \;. \tag{8.55}$$

The first term is the same one that we found by operating on the function $f_n(r)$ and is of the correct Fourier form, but the second term is not appropriate to a Fourier series. However, when we substitute the second term into equations (8.10, 8.11), we obtain the stress components

$$\sigma_{rr} = \sigma_{\theta\theta} = 0 \quad ; \quad \sigma_{r\theta} = \frac{B}{r^2} \;, \tag{8.56}$$

which *are* of the required Fourier form (for $n=0$), since the stress components do not vary with θ.

In the same way, we can develop special terms to make up the deficit due to the two null terms with $n=1$ (equation (8.48), obtaining the new stress functions

$$\phi = Br\theta\sin\theta + Cr\theta\cos\theta \ , \tag{8.57}$$

which generate the Fourier-type stress components

$$\sigma_{r\theta} = \sigma_{\theta\theta} = 0 \quad ; \quad \sigma_{rr} = \frac{2B\cos\theta}{r} - \frac{2C\sin\theta}{r} \ . \tag{8.58}$$

8.4 The Michell solution

The preceding results now permit us to write down a general solution of the elasticity problem in polar coördinates, such that the stress components form a Fourier series in θ. We have

$$\begin{aligned}\phi = {} & A_{01}r^2 + A_{02}r^2\ln(r) + A_{03}\ln(r) + A_{04}\theta \\ & +(A_{11}r^3 + A_{12}r\ln(r) + A_{14}r^{-1})\cos\theta + A_{13}r\theta\sin\theta \\ & +(B_{11}r^3 + B_{12}r\ln(r) + B_{14}r^{-1})\sin\theta + B_{13}r\theta\cos\theta \\ & +\sum_{n=2}^{\infty}(A_{n1}r^{n+2} + A_{n2}r^{-n+2} + A_{n3}r^n + A_{n4}r^{-n})\cos(n\theta) \\ & +\sum_{n=2}^{\infty}(B_{n1}r^{n+2} + B_{n2}r^{-n+2} + B_{n3}r^n + B_{n4}r^{-n})\sin(n\theta)\end{aligned} \tag{8.59}$$

This solution is due to Michell[4]. The corresponding stress components are easily obtained by substituting into equations (8.10, 8.11). For convenience, we give them in tabular form in Table 8.1, since we shall often wish to select a few components from the general solution in the solution of specific problems. The terms in the Michell solution are also given in the Maple and Mathematica files 'Michell', from which appropriate terms can be cut and pasted as required.

Notice that there are four independent stress functions for each term in the Fourier series, as required by the argument of §8.3.1. If the disk is solid, there is no inner boundary and we must exclude those components in Table 8.1 that give stresses which go to infinity as $r\to 0$.

In addition to the special stress functions necessitated by the degeneracy discussed in §8.3.3, the cases $n=0,1$ exhibit other anomolies. In particular, some of the stress functions for $n=0,1$ correspond to multiple-valued displacement fields and cannot be used for the complete annulus of Figure 8.1. Also, equilibrium requirements place restrictions on the permissible boundary conditions for the terms $n=0,1$ in the Fourier series (8.32).

Detailed discussion of these difficulties will be postponed to §9.3.1, after we have introduced methods of determining the displacements associated with a given stress field. In the present chapter, we shall restrict attention to cases where these difficulties do not arise.

[4] J.H.Michell, On the direct determination of stress in an elastic solid, with application to the theory of plates, *Proceedings of the London Mathematical Society,* Vol. 31 (1899), pp.100–124.

Table 8.1. The Michell solution — stress components

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\ln(r)$	$2\ln(r)+1$	0	$2\ln(r)+3$
$\ln(r)$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln(r)\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln(r)\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\sin n\theta$

8.4.1 Hole in a tensile field

To illustrate the use of Table 8.1, we consider the case where the the body of Figure 8.2 is subjected to uniform tension at infinity instead of shear, so that the boundary conditions become

$$\sigma_{rr} = 0 \ ; \ \ r = a \tag{8.60}$$

$$\sigma_{r\theta} = 0 \ ; \ \ r = a \tag{8.61}$$

$$\sigma_{xy}, \sigma_{yy} \to 0 \ ; \ \ r \to \infty \tag{8.62}$$

$$\sigma_{xx} \to S \ ; \ \ r \to \infty \ . \tag{8.63}$$

The unperturbed problem in this case can clearly be described by the stress function

$$\phi = \frac{Sy^2}{2} = \frac{Sr^2\sin^2\theta}{2} = \frac{Sr^2}{4} - \frac{Sr^2\cos(2\theta)}{4} \ . \tag{8.64}$$

This function contains both an axisymmetric term and a $\cos(2\theta)$ term, so to complete the solution, we supplement it with those terms from Table 8.1 which have the same form and for which the stresses decay as $r\to\infty$.

The resulting stress function is

$$\phi = \frac{Sr^2}{4} - \frac{Sr^2\cos(2\theta)}{4} + A\ln(r) + B\theta + C\cos(2\theta) + \frac{D\cos(2\theta)}{r^2} \tag{8.65}$$

and the corresponding stress components are

$$\sigma_{rr} = \frac{S}{2} + \frac{S\cos(2\theta)}{2} + \frac{A}{r^2} - \frac{4C\cos(2\theta)}{r^2} - \frac{6D\cos(2\theta)}{r^4} \tag{8.66}$$

$$\sigma_{r\theta} = -\frac{S\sin(2\theta)}{2} + \frac{B}{r^2} - \frac{2C\sin(2\theta)}{r^2} - \frac{6D\sin(2\theta)}{r^4} \tag{8.67}$$

$$\sigma_{\theta\theta} = \frac{S}{2} - \frac{S\cos(2\theta)}{2} - \frac{A}{r^2} + \frac{6D\cos(2\theta)}{r^4}\ . \tag{8.68}$$

The boundary conditions (8.60, 8.61) will be satisfied if and only if the coefficients of both Fourier terms are zero on $r=a$ and hence we obtain the equations

$$\frac{S}{2} + \frac{A}{a^2} = 0 \tag{8.69}$$

$$\frac{B}{a^2} = 0 \tag{8.70}$$

$$\frac{S}{2} - \frac{4C}{a^2} - \frac{6D}{a^4} = 0 \tag{8.71}$$

$$-\frac{S}{2} - \frac{2C}{a^2} - \frac{6D}{a^4} = 0\ , \tag{8.72}$$

which have the solution

$$A = -\frac{Sa^2}{2}\ \ ;\ \ B = 0\ \ ;\ \ C = \frac{Sa^2}{2}\ \ ;\ \ D = -\frac{Sa^4}{4}\ . \tag{8.73}$$

The final stress field is then obtained as

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S\cos(2\theta)}{2}\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2} + 1\right) \tag{8.74}$$

$$\sigma_{r\theta} = \frac{S\sin(2\theta)}{2}\left(\frac{3a^4}{r^4} - \frac{2a^2}{r^2} - 1\right) \tag{8.75}$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S\cos(2\theta)}{2}\left(\frac{3a^4}{r^4} + 1\right)\ . \tag{8.76}$$

The maximum tensile stress is $\sigma_{\theta\theta}=3S$ at $(a,\pi/2)$ and hence the stress concentration factor for tensile loading is 3. The maximum stress point and the unperturbed region at infinity are both in uniaxial tension and hence the stress concentration factor will be the same whatever criterion is used to measure the severity of the stress state. This contrasts with the problem of §8.3.2, where the maximum stress state is uniaxial tension, but the unperturbed field is pure shear.

PROBLEMS

1. A large plate with a small central hole of radius a is subjected to in-plane hydrostatic compression $\sigma_{xx} = \sigma_{yy} = -S$, $\sigma_{xy} = 0$ at the remote boundaries. Find the stress field in the plate if the surface of the hole is traction-free.

2. A large rectangular plate is loaded in such a way as to generate the unperturbed stress field

$$\sigma_{xx} = Cy^2 \ ; \ \sigma_{yy} = -Cx^2 \ ; \ \sigma_{xy} = 0 \ .$$

The plate contains a small traction-free circular hole of radius a centred on the origin. Find the perturbation in the stress field due to the hole.

3. Figure 8.3 shows a thin uniform circular disk, which rotates at constant speed Ω about the diametral axis $y = 0$, all the surfaces being traction-free. Determine the complete stress field in the disk.

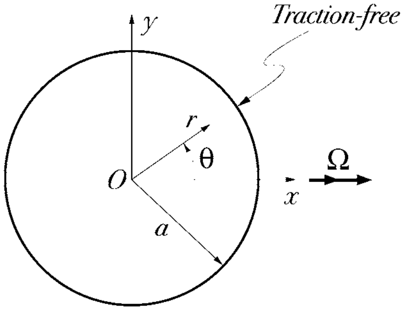

Figure 8.3: Thin disk rotating about a diametral axis.

4. A series of experiments is conducted in which a thin plate is subjected to biaxial tension/compression, σ_1, σ_2, the plane surface of the plate being traction-free (i.e. $\sigma_3 = 0$).

Unbeknown to the experimenter, the material contains microscopic defects which can be idealized as a sparse distribution of small holes through the thickness of the plate. Show graphically the relation which will hold at yield between the tractions σ_1, σ_2 applied to the defective plate, if the Tresca (maximum shear stress) criterion applies for the undamaged material.

5. The circular disk $0 \leq r < a$ is subjected to uniform compressive tractions $\sigma_{rr} = -S$ in the two arcs $-\pi/4 < \theta < \pi/4$ and $3\pi/4 < \theta < 5\pi/4$, the remainder of the surface $r = a$ being traction-free. Expand these tractions as a Fourier series in θ and hence develop a series solution for the stress field. Use Maple or Mathematica to produce a contour plot of the Von Mises stress σ_E, using a series truncated at 10 terms.

6. A hole of radius a in a large elastic plate is loaded only by a self-equilibrated distribution of normal pressure $p(\theta)$ that varies around the circumference of the hole. By expanding $p(\theta)$ as a Fourier series in θ and using Table 8.1, show that the hoop stress $\sigma_{\theta\theta}$ at the edge of the hole is given by

$$\sigma_{\theta\theta}(a, \theta) = 2\overline{p} - p(\theta) \ ,$$

where

$$\overline{p} = \frac{1}{2\pi} \int_0^{2\pi} p(\theta) d\theta$$

is the mean value of $p(\theta)$.

9

CALCULATION OF DISPLACEMENTS

So far, we have restricted attention to the calculation of stresses and to problems in which the boundary conditions are stated in terms of tractions or force resultants, but there are many problems in which displacements are also of interest. For example, we may wish to find the deflection of the rectangular beams considered in Chapter 5, or calculate the stress concentration factor due to a rigid circular inclusion in an elastic matrix, for which a displacement boundary condition is implied at the bonded interface.

If the stress components are known, the strains can be written down from the stress-strain relations (1.77) and these in turn can be expressed in terms of displacement gradients through (1.51). The problem is therefore reduced to the integration of these gradients to recover the displacement components.

The method is most easily demonstrated by examples, of which we shall give two — one in rectangular and one in polar coördinates.

9.1 The cantilever with an end load

We first consider the cantilever beam loaded by a transverse force, F, at the free end (Figure 5.2), for which the stress components were calculated in §5.2.1, being

$$\sigma_{xx} = \frac{3Fxy}{2b^3} \;\; ; \;\; \sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \;\; ; \;\; \sigma_{yy} = 0 \; , \tag{9.1}$$

from (5.36–5.38).

The corresponding strain components are therefore

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} = \frac{3Fxy}{2Eb^3} \tag{9.2}$$

$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E} = \frac{3F(1+\nu)(b^2 - y^2)}{4Eb^3} \tag{9.3}$$

$$e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} = -\frac{3F\nu xy}{2Eb^3} \; , \tag{9.4}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_9, © Springer Science+Business Media B.V. 2010

for plane stress.

We next make use of the strain-displacement relation to write

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{3Fxy}{2Eb^3} \,, \tag{9.5}$$

which can be integrated with respect to x to give

$$u_x = \frac{3Fx^2y}{4Eb^3} + f(y) \,, \tag{9.6}$$

where we have introduced an arbitrary function $f(y)$ of y, since any such function would make no contribution to the partial derivative in (9.5). A similar operation on (9.4) yields the result

$$u_y = -\frac{3F\nu xy^2}{4Eb^3} + g(x) \,, \tag{9.7}$$

where $g(x)$ is an arbitrary function of x.

To determine the two functions $f(y), g(x)$, we use the definition of shear strain (1.49) and (9.3) to write

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \frac{3F(1+\nu)(b^2-y^2)}{4Eb^3} \,. \tag{9.8}$$

Substituting for u_x, u_y, from (9.6, 9.7) and rearranging the terms, we obtain

$$\frac{3Fx^2}{8Eb^3} + \frac{1}{2}\frac{dg}{dx} = \frac{3F\nu y^2}{8Eb^3} - \frac{1}{2}\frac{df}{dy} + \frac{3F(1+\nu)(b^2-y^2)}{4Eb^3} \,. \tag{9.9}$$

Now, the left-hand side of this equation is independent of y and the right-hand side is independent of x. Thus, the equation can only be satisfied for all x, y, if *both* sides are independent of both x and y — i.e. if they are equal to a constant, which we shall denote by $\frac{1}{2}C$.

Equation (9.9) can then be partitioned into the two ordinary differential equations

$$\frac{dg}{dx} = -\frac{3Fx^2}{4Eb^3} + C \tag{9.10}$$

$$\frac{df}{dy} = \frac{3F\nu y^2}{4Eb^3} + \frac{3F(1+\nu)(b^2-y^2)}{2Eb^3} - C \,, \tag{9.11}$$

which have the solution

$$g(x) = -\frac{Fx^3}{4Eb^3} + Cx + B \tag{9.12}$$

$$f(y) = \frac{F\nu y^3}{4Eb^3} + \frac{F(1+\nu)(3b^2y-y^3)}{2Eb^3} - Cy + A \,, \tag{9.13}$$

where A, B are two new arbitrary constants.

The final expressions for the displacements are therefore

$$u_x = \frac{3Fx^2y}{4Eb^3} + \frac{3F(1+\nu)y}{2Eb} - \frac{F(2+\nu)y^3}{4Eb^3} + A - Cy \tag{9.14}$$

$$u_y = -\frac{3F\nu xy^2}{4Eb^3} - \frac{Fx^3}{4Eb^3} + B + Cx \; . \tag{9.15}$$

This partial integration process can be performed for any biharmonic function ϕ in Cartesian coördinates, using the Maple and Mathematica files 'uxy'.

9.1.1 Rigid-body displacements and end conditions

The three constants A, B, C define the three degrees of freedom of the cantilever as a rigid body, A, B corresponding to translations in the x-, y-directions respectively and C to a small anticlockwise rotation about the origin.

These rigid-body terms always arise in the integration of strains to determine displacements and they reflect the fact that a complete knowledge of the stresses and hence the strains throughout the body is sufficient to determine its deformed shape, but not its location in space.

As in Mechanics of Materials, the rigid-body displacements can be determined from appropriate information about the way in which the structure is supported. Since the cantilever is built-in at $x=a$, we would ideally like to specify

$$u_x = u_y = 0 \;\; ; \;\; x = a \; , \; -b < y < b \; , \tag{9.16}$$

but the three constants in (9.14, 9.15) do not give us sufficient freedom to satisfy such a strong boundary condition. We should anticipate this deficiency, since the complete solution procedure only permitted us to satisfy weak boundary conditions on the ends of the beam and (9.16), in addition to locating the beam in space, implies a pointwise traction distribution sufficient to keep the end plane and unstretched.

Many authors adapt the Mechanics of Materials support conditions for a cantilever by demanding that the mid-point $(a, 0)$ of the end have zero displacement and the axis of the beam $(y=0)$ have zero slope at that point. This leads to the three conditions

$$u_x = u_y = 0 \;\; ; \;\; \frac{\partial u_y}{\partial x} = 0 \;\; ; \;\; x = a \; , \; y = 0 \tag{9.17}$$

and yields the values

$$A = 0 \;\; ; \;\; B = -\frac{Fa^3}{2Eb^3} \;\; ; \;\; C = \frac{3Fa^2}{4Eb^3} \; , \tag{9.18}$$

when (9.14, 9.15) are substituted in (9.17).

However, the displacement u_x at $x=a$ corresponds to the deformed shape of Figure 9.1(a). This might be an appropriate end condition if the cantilever is supported in a horizontal groove, but other built-in support conditions might approximate more closely to the configuration of Figure 9.1(b), where the third of conditions (9.17) is replaced by

$$\frac{\partial u_x}{\partial y} = 0 \ ; \ \ x=a \ , \ y=0 \ . \tag{9.19}$$

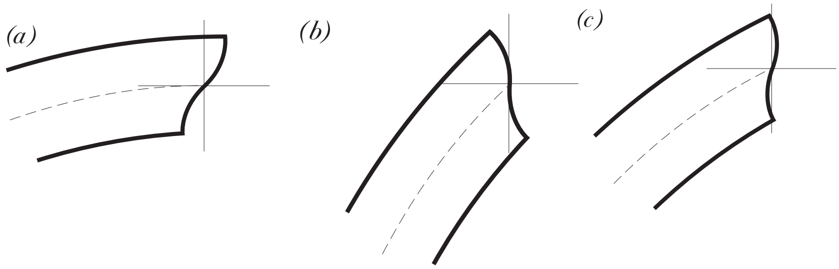

Figure 9.1: End conditions for the cantilever.

This modified boundary condition leads to the values

$$A=0 \ ; \ \ B=-\frac{Fa^3}{2Eb^3}\left(1+3(1+\nu)\frac{b^2}{a^2}\right) \ \ ; \ \ C=\frac{3Fa^2}{4Eb^3}\left(1+2(1+\nu)\frac{b^2}{a^2}\right) \tag{9.20}$$

for the rigid-body displacement coefficients.

The configurations 9.1(a,b) are clearly extreme cases and we might anticipate that the best approximation to (9.16) would be obtained by an intermediate case such as Figure 9.1(c). In fact, the displacement equivalent of the weak boundary conditions of Chapter 5 would be to prescribe

$$\int_{-b}^{b} u_x dy = 0 \ ; \ \ \int_{-b}^{b} u_y dy = 0 \ ; \ \ \int_{-b}^{b} y u_x dy = 0 \ ; \ \ x=a \ . \tag{9.21}$$

The substitution of (9.14, 9.15) into (9.21) gives a new set of rigid-body displacement coefficients, which are

$$A=0 \ ; \ \ B=-\frac{Fa^3}{2Eb^3}\left(1+\frac{(12+11\nu)}{5}\frac{b^2}{a^2}\right) \ \ ; \ \ C=\frac{3Fa^2}{4Eb^3}\left(1+\frac{(8+9\nu)}{5}\frac{b^2}{a^2}\right) \ . \tag{9.22}$$

9.1.2 Deflection of the free end

The different end conditions corresponding to (a,b,c) of Figure 9.1 will clearly lead to different estimates for the deflection of the cantilever. We can investigate this effect by considering the displacement of the mid-point of the free end, $x=0, y=0$, for which

$$u_y(0,0) = B \ , \tag{9.23}$$

from equation (9.15).

Thus, the end deflection predicted with the boundary conditions of (a, b, c) respectively of Figure 9.1 are

$$\begin{aligned} u_y(0,0) &= -\frac{Fa^3}{2Eb^3} && (a) \\ &= -\frac{Fa^3}{2Eb^3}\left(1+3(1+\nu)\frac{b^2}{a^2}\right) && (b) \\ &= -\frac{Fa^3}{2Eb^3}\left(1+\frac{(12+11\nu)}{5}\frac{b^2}{a^2}\right) && (c)\ . \end{aligned} \tag{9.24}$$

The first of these results (a) is also that predicted by the elementary Mechanics of Materials solution and the second (b) is that obtained when a correction is made for the 'shear deflection', based on the shear stress at the beam axis. Case (c), which is intermediate between (a) and (b), but closer to (b), is probably the closest approximation to the true built-in boundary condition (9.16).

All three expressions have the same leading term and the corrective term in (b, c) will be small as long as $b \ll a$ — i.e. as long as the cantilever can reasonably be considered as a slender beam.

9.2 The circular hole

The procedure of §9.1 has some small but critical differences in polar coördinates, which we shall illustrate using the problem of §8.3.2, in which a traction-free circular hole perturbs a uniform shear field. The stresses are given by equations (8.45–8.47) and, in plane stress, the strains are therefore

$$e_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu\sigma_{\theta\theta}}{E} = \frac{S}{E}\left(1+\nu-\frac{4a^2}{r^2}+\frac{3(1+\nu)a^4}{r^4}\right)\sin(2\theta) \tag{9.25}$$

$$e_{r\theta} = \frac{(1+\nu)\sigma_{r\theta}}{E} = \frac{S}{E}\left(1+\nu+\frac{2(1+\nu)a^2}{r^2}-\frac{3(1+\nu)a^4}{r^4}\right)\cos(2\theta) \tag{9.26}$$

$$e_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu\sigma_{rr}}{E} = \frac{S}{E}\left(-1-\nu+\frac{4\nu a^2}{r^2}-\frac{3(1+\nu)a^4}{r^4}\right)\sin(2\theta)\ . \tag{9.27}$$

Two of the three strain-displacement relations (8.22) contain both displacement components, so we start with the simpler relation

$$e_{rr} = \frac{\partial u_r}{\partial r}\ , \tag{9.28}$$

substitute for e_{rr} from (9.25) and integrate, obtaining

$$u_r = \frac{S}{E}\left((1+\nu)r+\frac{4a^2}{r}-\frac{(1+\nu)a^4}{r^3}\right)\sin(2\theta)+f(\theta)\ , \tag{9.29}$$

where $f(\theta)$ is an arbitrary function of θ.

Writing the expression (8.22) for $e_{\theta\theta}$ in the form

$$\frac{\partial u_\theta}{\partial \theta} = re_{\theta\theta} - u_r \ , \tag{9.30}$$

we can substitute for $e_{\theta\theta}, u_r$ from (9.27, 9.29) respectively and integrate with respect to θ, obtaining

$$u_\theta = \frac{S}{E}\left((1+\nu)r + \frac{2(1-\nu)a^2}{r} + \frac{(1+\nu)a^4}{r^3}\right)\cos(2\theta) - F(\theta) + g(r) \ , \tag{9.31}$$

where $g(r)$ is an arbitrary function of r and we have written $F(\theta)$ for $\int f(\theta)d\theta$.

Finally, we substitute for $u_r, u_\theta, e_{r\theta}$ from (9.29, 9.31, 9.26) into the second of equations (8.22) to obtain an equation for the arbitrary functions $F(\theta), g(r)$, which reduces to

$$F(\theta) + F''(\theta) = g(r) - rg'(r) \ . \tag{9.32}$$

As in §9.1, this equation can only be satisfied for all r, θ if both sides are equal to a constant[1]. Solving the two resulting ordinary differential equations for $F(\theta), g(r)$ and substituting into equations (9.29, 9.31), remembering that $f(\theta) = F'(\theta)$ we obtain

$$u_r = \frac{S}{E}\left((1+\nu)r + \frac{4a^2}{r} - \frac{(1+\nu)a^4}{r^3}\right)\sin(2\theta) + A\cos\theta + B\sin\theta \tag{9.33}$$

$$u_\theta = \frac{S}{E}\left((1+\nu)r + \frac{2(1-\nu)a^2}{r} + \frac{(1+\nu)a^4}{r^3}\right)\cos(2\theta) - A\sin\theta + B\cos\theta + Cr \ , \tag{9.34}$$

for the displacement components.

This partial integration process can be performed for any biharmonic function ϕ in polar coördinates, using the Maple and Mathematica files 'urt'. As before, the three arbitrary constants A, B, C correspond to rigid-body displacements, A, B being translations in the x-, y-directions respectively, whilst C describes a small anticlockwise rotation about the origin.

In the present problem, we might reasonably set A, B, C to zero to preserve symmetry. At the hole surface, the radial displacement is then

$$u_r = \frac{4Sa\sin(2\theta)}{E} \ , \tag{9.35}$$

which implies that the hole distorts into the ellipse shown in Figure 9.2.

[1] It turns out in polar coördinate problems that the value of this constant does not affect the final expressions for the displacement components.

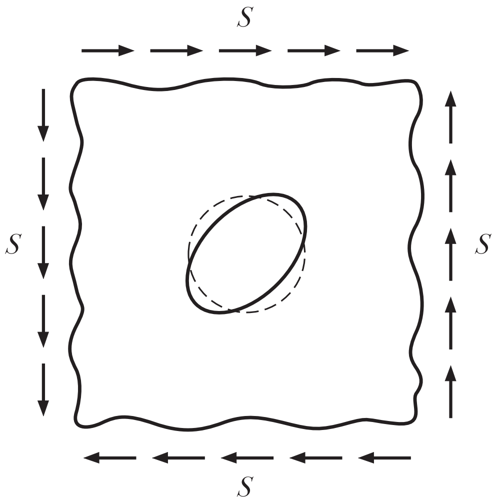

Figure 9.2: Distortion of a circular hole in a shear field.

The reader should note that, in both of the preceding examples, the third strain-displacement relation gives an equation for two arbitrary functions of one variable which can be partitioned into two ordinary differential equations. The success of the calculation depends upon this partition, which in turn is possible if and only if the strains satisfy the compatibility conditions. In the present examples, this is of course ensured through the derivation of the stress field from a biharmonic stress function.

9.3 Displacements for the Michell solution

Displacements for all the stress functions of Table 8.1 can be obtained[2] by the procedure of §9.2 and the results[3] are tabulated in Table 9.1. In this Table, results for plane strain can be recovered by setting $\kappa = (3-4\nu)$, whereas for plane stress, $\kappa = (3-\nu)/(1+\nu)$ (see equation (3.20)). Note that it is always possible to superpose a rigid-body displacement defined by $u_r = A\cos\theta + B\sin\theta,\ u_\theta = -A\sin\theta + B\cos\theta + Cr$, where A, B, C are arbitrary constants.

9.3.1 Equilibrium considerations

In the geometry of Figure 8.1, the boundary conditions (8.28–8.31) on the surfaces $r = a, b$ are not completely independent, since they must satisfy the

[2] The reader might like to confirm this in a few cases by using the Maple and Mathematica files 'urt'. Notice however that the results for the '$n = 0, 1$' functions may differ by a rigid-body displacement from those given in the table.

[3] These results were first compiled by Professor J.Dundurs of Northwestern University and his research students, and are here reprinted with his permission.

Table 9.1. The Michell solution — displacement components

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln(r)$	$(\kappa-1)r\ln(r)-r$	$(\kappa+1)r\theta$
$\ln(r)$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta$ $+(\kappa+1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$
$r\ln(r)\cos\theta$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta$ $-(\kappa-1)\ln(r)\sin\theta\}$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta$ $-(\kappa+1)\ln(r)\cos\theta\}$
$r\ln(r)\sin\theta$	$\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta$ $+(\kappa-1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

condition that the body be in equilibrium. This requires that

$$\int_0^{2\pi}(F_1(\theta)\cos\theta-F_3(\theta)\sin\theta)\,ad\theta - \int_0^{2\pi}(F_2(\theta)\cos\theta-F_4(\theta)\sin\theta)bd\theta=0 \tag{9.36}$$

$$\int_0^{2\pi}(F_1(\theta)\sin\theta+F_3(\theta)\cos\theta)ad\theta - \int_0^{2\pi}(F_2(\theta)\sin\theta+F_4(\theta)\cos\theta)bd\theta=0 \tag{9.37}$$

$$\int_0^{2\pi} F_3(\theta)a^2 d\theta - \int_0^{2\pi} F_4(\theta)b^2 d\theta = 0 \ . \tag{9.38}$$

The orthogonality of the terms of the Fourier series ensures that these relations only concern the Fourier terms for $n=0,1$, but for these cases, there are then only three independent algebraic equations to determine each of the corresponding sets of four constants, $A_{01},\ldots A_{04}, A_{11},\ldots A_{14}, B_{11},\ldots B_{14}$.

The key to this paradox is to be seen in the displacements of Table 9.1. In the terms for $n=0,1$, some of the displacements include a θ-multiplier. For example, we find $2\mu u_\theta = (\kappa+1)r\theta$ for the stress function $\phi = r^2\ln(r)$. Now the function θ is multi-valued. We can make it single-valued by defining a *principal value* — e.g., by restricting θ to the range $0<\theta<2\pi$, but we would then have a discontinuity at the line $\theta=0, 2\pi$ which is unacceptable for the continuous body of Figure 8.1. We must therefore restrict our choice of stress functions to a set which defines a single-valued continuous displacement and it turns out that this imposes precisely one additional condition on each of the sets of four constants for $n=0,1$.

Notice incidentally that if the annulus were incomplete, or if it were cut along the line $\theta=0$, the principal value of θ would be continuous and single-valued throughout the body and the above restrictions would be removed. However, we would then also lose the equilibrium restrictions (9.36–9.38), since the body would have two new edges (e.g. $\theta=0, 2\pi$, for the annulus with a cut on $\theta=0$) on which there may be non-zero tractions.

We see then that the complete annulus has some complications which the incomplete annulus lacks. This arises of course because the complete annulus is *multiply connected*. The results of this section are a direct consequence of the discussion of compatibility in multiply connected bodies in §2.2.1. These questions and problems in which they are important will be discussed further in Chapter 13.

9.3.2 The cylindrical pressure vessel

Consider the plane strain problem of a long cylindrical pressure vessel of inner radius a and outer radius b, subjected to an internal pressure p_0. The boundary conditions for this problem are

$$\sigma_{rr} = -p_0 \ \ ; \ \ \sigma_{r\theta} = 0 \ \ ; \ \ r = a \tag{9.39}$$

$$\sigma_{rr} = \sigma_{r\theta} = 0 \ \ ; \ \ r = b \ . \tag{9.40}$$

The loading is clearly independent of θ, so we seek a suitable stress function in the $n=0$ row of Tables 8.1, 9.1. However, we must exclude the term $r^2\ln(r)$, since it gives a multivalued expression for u_θ. We therefore start with

$$\phi = Ar^2 + B\ln(r) + C\theta \ , \tag{9.41}$$

for which the stress components are

$$\sigma_{rr} = 2A + \frac{B}{r^2} \;\; ; \;\; \sigma_{r\theta} = \frac{C}{r^2} \;\; ; \;\; \sigma_{\theta\theta} = 2A - \frac{B}{r^2} \; . \tag{9.42}$$

The boundary conditions (9.39, 9.40) then give the equations

$$2A + \frac{B}{a^2} = -p_0 \tag{9.43}$$

$$\frac{C}{a^2} = 0 \tag{9.44}$$

$$2A + \frac{B}{b^2} = 0 \tag{9.45}$$

$$\frac{C}{b^2} = 0 \; , \tag{9.46}$$

with solution

$$A = \frac{p_0 a^2}{2(b^2 - a^2)} \;\; ; \;\; B = -\frac{p_0 a^2 b^2}{2(b^2 - a^2)} \;\; ; \;\; C = 0 \; . \tag{9.47}$$

The final stress field is therefore

$$\sigma_{rr} = \frac{p_0 a^2}{(b^2 - a^2)} \left(1 - \frac{b^2}{r^2} \right) \tag{9.48}$$

$$\sigma_{r\theta} = 0 \tag{9.49}$$

$$\sigma_{\theta\theta} = \frac{p_0 a^2}{(b^2 - a^2)} \left(1 + \frac{b^2}{r^2} \right) \; , \tag{9.50}$$

from (9.42, 9.47). Notice that the equations (9.43–9.46) are not linearly independent. This occurs because the loading must satisfy the global equilibrium condition (9.38), which here reduces to

$$a^2 \sigma_{r\theta}(a) = b^2 \sigma_{r\theta}(b) \; . \tag{9.51}$$

If this condition had not been satisfied — for example if there had been a uniform shear traction on only one of the two surfaces $r = a, b$, the above boundary-value problem would have no solution. Of course, with this loading, the cylinder would experience rotational acceleration about the axis and this would need to be taken into account in formulating the problem, as in the problem of §7.4.1.

PROBLEMS

1. Find the displacement field corresponding to the stress field of equations (5.78–5.80). The beam is simply-supported at $x = \pm a$. You will need to impose this in the form of appropriate weak conditions on the displacements.

Find the vertical displacement at the points $(0, 0)$, $(0, -b)$, $(0, b)$ and compare them with the predictions of the elementary bending theory.

2. The rectangular plate $-a<x<a$, $-b<y<b$ is bonded to rigid supports at $x=\pm a$, the edges $y=\pm b$ being traction-free. The plate is loaded only by its own weight in the negative y-direction.

(i) Find a solution for the stress field assuming uniform shear tractions on $x=\pm a$. Use strong boundary conditions on $x=\pm a$ and weak conditions on $y=\pm b$.
(ii) Find the displacements corresponding to this stress field. Do the resulting expressions permit the built-in boundary condition at $x=\pm a$ to be satisfied in the strong sense?
(iii) What are the restrictions on the ratio a/b for this solution to be a reasonable approximation to the physical problem?

3. A state of pure shear, $\sigma_{xy}=S$ in a large plate is perturbed by the presence of a rigid circular inclusion in the region $r<a$. The inclusion is perfectly bonded to the plate and is prevented from moving, so that $u_r=u_\theta=0$ at $r=a$.

Find the complete stress field in the plate and hence determine the stress concentration factor due to the inclusion based on (i) the maximum shear stress criterion or (ii) the maximum tensile stress criterion.

Is it necessary to apply a force or a moment to the inclusion to prevent it from moving?

4. A thin annular disk, inner radius b and outer radius a rotates at constant speed Ω about the axis of symmetry, all the surfaces being traction-free. Determine the stress field in the disk.

5. A large rectangular plate is subjected to simple bending, such that the stress field is given by

$$\sigma_{xx}=Cy \;\;;\;\; \sigma_{xy}=\sigma_{yy}=0\;.$$

The plate contains a small traction-free circular hole of radius a centred on the origin. Find the perturbation in the stress field due to the hole.

6. A heavy disk of density ρ and radius a is bonded to a rigid support at $r=a$. The gravitational force acts in the direction $\theta=-\pi/2$. Find the stresses and displacements in the disk.

Hint: The easiest method is to find the stresses and displacements for a simple particular solution for the body force and then superpose 'homogeneous' terms using Tables 8.1 and 9.1.

7. A heavy disk of density ρ, elastic properties μ,ν and radius $a+\epsilon$ is pressed into a frictionless hole of radius a in a rigid body.

What is the minimum value of ϵ if the disk is to remain in contact with the hole at all points when the gravitational force acts in the direction $\theta=-\pi/2$.

8. A rigid circular inclusion of radius a in a large elastic plate is subjected to a force F in the x-direction.

Find the stress field in the plate if the inclusion is perfectly bonded to the plate at $r=a$ and the stresses tend to zero as $r\to\infty$.

9. A rubber bushing comprises a hollow cylinder of rubber $a<r<b$, bonded to concentric thin-walled steel cylinders at $r=a$ and $r=b$, as shown in Figure 9.3. The steel cylinders may be considered as rigid compared with the rubber.

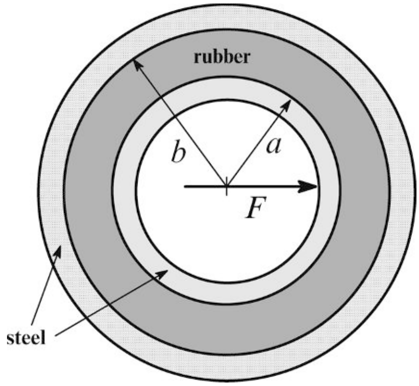

Figure 9.3

The outer cylinder is held fixed, whilst a force F per unit axial length is applied to the inner cylinder in the x-direction. Find the stiffness of the bushing under this loading — i.e. the relationship between F and the resulting displacement δ of the inner cylinder in the x-direction.

In particular, find the stiffness for a long cylinder (plane strain conditions) and a short cylinder (plane stress), assuming that rubber has a shear modulus μ and is incompressible ($\nu=0.5$).

Plot a graph of the dimensionless stiffness $F/\mu\delta$ as a function of the ratio a/b in the range $F/\mu\delta<100$. Can you provide a simple physical explanation of the difference between the plane stress and plane strain results?

Hint: The hollow cylinder is a multiply connected body.

10

CURVED BEAM PROBLEMS

If we cut the circular annulus of Figure 8.1 along two radial lines, $\theta=\alpha,\beta$, we generate a curved beam. The analysis of such beams follows that of Chapter 8, except for a few important differences — notably that (i) the ends of the beam constitute two new boundaries on which boundary conditions (usually weak boundary conditions) are to be applied and (ii) it is no longer necessary to enforce continuity of displacements (see §9.3.1), since a suitable principal value of θ can be defined which is both continuous and single-valued.

10.1 Loading at the ends

We first consider the case in which the curved surfaces of the beam are traction-free and only the ends are loaded. As in §5.2.1, we only need to impose boundary conditions on one end — the Airy stress function formulation will ensure that the tractions on the other end have the correct force resultants to guarantee global equilibrium.

10.1.1 Pure bending

The simplest case is that illustrated in Figure 10.1, in which the beam $a<r<b,\ 0<\theta<\alpha$ is loaded by a bending moment M_0, but no forces. Equilibrium considerations demand that the bending moment M_0 and zero axial force and shear force will be transmitted across all for θ-surfaces and hence that the stress field will be independent of θ. We therefore seek the solution in the axisymmetric terms in Table 8.1, using the stress function

$$\phi = Ar^2 + Br^2\ln(r) + C\ln(r) + D\theta \ , \qquad (10.1)$$

the corresponding stress components being

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_10, © Springer Science+Business Media B.V. 2010

$$\sigma_{rr} = 2A + B(2\ln(r) + 1) + \frac{C}{r^2} \tag{10.2}$$

$$\sigma_{r\theta} = \frac{D}{r^2} \tag{10.3}$$

$$\sigma_{\theta\theta} = 2A + B(2\ln(r) + 3) - \frac{C}{r^2} \ . \tag{10.4}$$

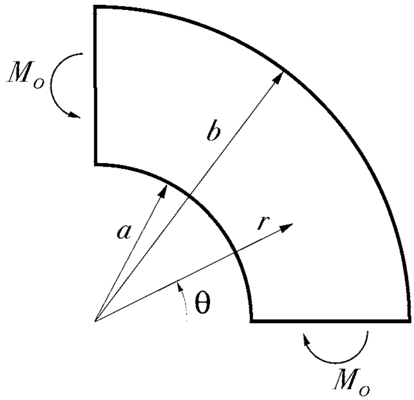

Figure 10.1: Curved beam in pure bending.

The boundary conditions for the problem of Figure 10.1 are

$$\sigma_{rr} = 0 \ \ ; \ \ r = a, b \tag{10.5}$$

$$\sigma_{r\theta} = 0 \ \ ; \ \ r = a, b \tag{10.6}$$

$$\int_a^b \sigma_{\theta\theta} dr = 0 \ \ ; \ \ \theta = 0 \tag{10.7}$$

$$\int_a^b \sigma_{\theta r} dr = 0 \ \ ; \ \ \theta = 0 \tag{10.8}$$

$$\int_a^b \sigma_{\theta\theta} r dr = M_0 \ \ ; \ \ \theta = 0 \ . \tag{10.9}$$

Notice that in practice it is not necessary to impose the zero force conditions (10.7, 10.8), since if there were a non-zero force on the end $\theta = 0$, it would have to be transmitted around the beam and this would result in a non-axisymmetric stress field. Thus, our assumption of axisymmetry automatically rules out there being any force resultant.

We first impose the strong boundary conditions (10.5, 10.6), which with (10.2, 10.3) yield the four algebraic equations

$$2A + B(2\ln a + 1) + \frac{C}{a^2} = 0 \tag{10.10}$$

$$2A + B(2\ln b + 1) + \frac{C}{b^2} = 0 \tag{10.11}$$

$$\frac{D}{a^2} = 0 \quad (10.12)$$

$$\frac{D}{b^2} = 0 \; , \quad (10.13)$$

and an additional equation is obtained by substituting (10.4) into (10.9), giving

$$(A+B)(b^2 - a^2) + B(b^2 \ln b - a^2 \ln a) - C \ln(b/a) = M_0 \; . \quad (10.14)$$

These equations have the solution

$$A = -\frac{M_0}{N_0}(b^2 - a^2 + 2b^2 \ln b - 2a^2 \ln a)$$

$$B = \frac{2M_0}{N_0}(b^2 - a^2) \;\; ; \;\; C = \frac{4M_0}{N_0} a^2 b^2 \ln \frac{b}{a} \;\; ; \;\; D = 0 \; , \quad (10.15)$$

where

$$N_0 \equiv (b^2 - a^2)^2 - 4a^2 b^2 \ln^2(b/a) \quad (10.16)$$

The corresponding stress field is readily obtained by substituting these values back into equations (10.2–10.4).

The principal practical interest in this solution lies in the extent to which the classical Mechanics of Materials bending theory underestimates the bending stress $\sigma_{\theta\theta}$ at the inner edge, $r = a$ when a/b is small. Timoshenko and Goodier give some numerical values and show that the elementary theory starts to deviate significantly from the correct result for $a/b < 0.5$. The theory of curved beams — based on the application of the principle that plane sections remain plane, but allowing for the fact that the beam elements increase in length as r increases due to the curvature — gives a much better agreement. However, the exact result is quite easy to compute.

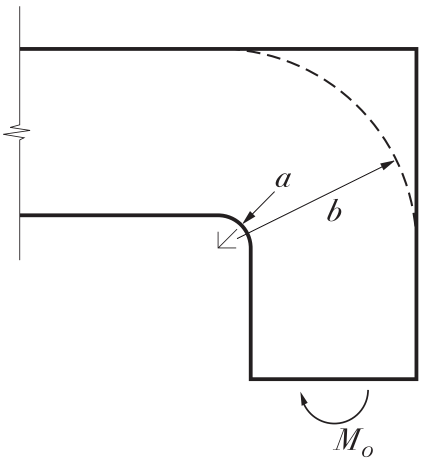

Figure 10.2: Bend in a rectangular beam with a small fillet radius.

An important special case is that of a 'rectangular' bend in a beam, with a small inner fillet radius (see Figure 10.2). This is reasonably modelled as the curved beam shown dotted, since there will be very little stress in the outer corner of the bend.

10.1.2 Force transmission

A similar type of solution can be generated for the problem in which a force is applied at the end of the beam. We consider the case illustrated in Figure 10.3, where a shear force is applied at $\theta=0$, resulting in the boundary conditions

$$\int_a^b \sigma_{\theta\theta} dr = 0 \;\; ; \;\; \theta = 0 \tag{10.17}$$

$$\int_a^b \sigma_{\theta r} dr = F \;\; ; \;\; \theta = 0 \tag{10.18}$$

$$\int_a^b \sigma_{\theta\theta} r dr = 0 \;\; ; \;\; \theta = 0 \; , \tag{10.19}$$

which replace (10.7–10.9) in the problem of §10.1.1.

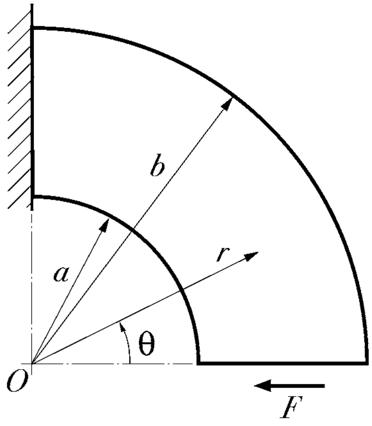

Figure 10.3: Curved beam with an end load.

Equilibrium considerations show that the shear force on any section $\theta=\alpha$ will vary as $F\cos\alpha$ and hence we seek the solution in those terms in Table 8.1 in which the shear stress $\sigma_{r\theta}$ depends on $\cos\theta$ — i.e.

$$\phi = (Ar^3 + Br^{-1} + Cr\ln(r))\sin\theta + Dr\theta\cos\theta \; , \tag{10.20}$$

for which the stress components are

$$\sigma_{rr} = (2Ar - 2Br^{-3} + Cr^{-1} - 2Dr^{-1})\sin\theta \tag{10.21}$$

$$\sigma_{r\theta} = (-2Ar + 2Br^{-3} - Cr^{-1})\cos\theta \tag{10.22}$$

$$\sigma_{\theta\theta} = (6Ar + 2Br^{-3} + Cr^{-1})\sin\theta \; . \tag{10.23}$$

The hoop stress $\sigma_{\theta\theta}$ is identically zero on $\theta=0$ and hence the end conditions (10.17, 10.19) are satisfied identically in the strong sense, whilst the strong boundary conditions (10.5, 10.6) lead to the set of equations

$$2Aa - \frac{2B}{a^3} + \frac{C}{a} - \frac{2D}{a} = 0 \tag{10.24}$$

$$2Ab - \frac{2B}{b^3} + \frac{C}{b} - \frac{2D}{b} = 0 \tag{10.25}$$

$$2Aa - \frac{2B}{a^3} + \frac{C}{a} = 0 \tag{10.26}$$

$$2Ab - \frac{2B}{b^3} + \frac{C}{b} = 0 \; . \tag{10.27}$$

The final solution that also satisfies the inhomogeneous condition (10.18) is obtained as

$$A = \frac{F}{2N_1} \;\; ; \;\; B = -\frac{Fa^2b^2}{2N_1} \;\; ; \;\; C = -\frac{F(a^2+b^2)}{N_1} \;\; ; \;\; D = 0 \; , \tag{10.28}$$

where

$$N_1 = (a^2 - b^2) + (a^2 + b^2)\ln(b/a) \; . \tag{10.29}$$

As before, the final stress field is easily obtained by back substitution into equations (10.21–10.23).

The corresponding solution for an axial force at the end $\theta = 0$ is obtained in the same way except that we interchange sine and cosine in the stress function (10.20). Alternatively, we can use the present solution and measure the angle from the end $\theta=\pi/2$ instead of from $\theta=0$. Notice however that the axial force at $\theta=\pi/2$ must have the same line of action as the shear force at $\theta=0$ in Figure 10.3. In other words, it must act through the origin of coördinates. If we wish to solve a problem in which an axial force is applied with some other, parallel, line of action — e.g. if the force acts through the mid-point of the beam, $r=(a+b)/2$ — we can do so by superposing an appropriate multiple of the bending solution of §10.1.1.

10.2 Eigenvalues and eigenfunctions

A remarkable feature of the preceding solutions is that in each case we had a set of four simultaneous homogeneous algebraic equations for four unknown constants — equations (10.10–10.13) in §10.1.1 and (10.24–10.27) in §10.1.2 — but in each we were able to obtain a non-trivial solution, because the equations were not linearly independent[1].

Suppose we were to define an *inhomogeneous* problem for the curved beam in which the curved edges $r=a, b$ were loaded by arbitrary tractions $\sigma_{rr}, \sigma_{r\theta}$.

[1] In particular, (10.12, 10.13) can both be satisfied by setting $D=0$ and (10.24–10.27) reduce to only two independent equations if $D=0$.

We could then decompose these tractions into appropriate Fourier series — as indicated in §8.3.1 — and use the general solution of equation (8.59) and Table 8.1. There would then be four linearly independent stress function terms for each Fourier component and hence the four boundary conditions would lead to a well-conditioned set of four algebraic equations for four arbitrary constants for each separate Fourier component.

In the special case where there are no tractions of the appropriate Fourier form on the curved surfaces, we should get four homogeneous algebraic equations and we anticipate only the trivial solution in which the corresponding four constants — and hence the stress components — are zero.

This is exactly what happens for all values of n *other than* $0, 1$, but, as we have seen above, for these two special cases, there is a non-trivial solution to the homogeneous problem. In a sense, $0, 1$ are *eigenvalues* of the general Fourier problem and the corresponding stress fields are *eigenfunctions.*

10.3 The inhomogeneous problem

Of course, eigenvalues can be viewed from two different perspectives. They are the values of a parameter at which the homogeneous problem has a non-trivial solution — e.g. the frequency at which a dynamic system will vibrate without excitation — but they are also the values at which the *inhomogeneous* problem has an unbounded solution — excitation at the natural frequency predicts an unbounded amplitude.

In the present instance, we must therefore anticipate difficulties in the inhomogeneous problem if the boundary tractions contain Fourier terms with $n=0$ or 1.

10.3.1 Beam with sinusoidal loading

As an example, we consider the problem of Figure 10.4, in which the curved beam is subjected to a radial normal traction

$$\sigma_{rr} = S\sin\theta \;\; ; \;\; r = b \; , \tag{10.30}$$

the other boundary tractions being zero.

A 'naïve' approach to this problem would be to use a stress function composed of those terms in Table 8.1 for which the stress component σ_{rr} varies with $\sin\theta$. This of course would lead to the formulation of equations (10.20–10.23) and the strong boundary conditions on the curved edges would give the four equations

$$2Aa - \frac{2B}{a^3} + \frac{C}{a} - \frac{2D}{a} = 0 \tag{10.31}$$

$$2Ab - \frac{2B}{b^3} + \frac{C}{b} - \frac{2D}{b} = S \tag{10.32}$$

$$2Aa - \frac{2B}{a^3} + \frac{C}{a} = 0 \tag{10.33}$$

$$2Ab - \frac{2B}{b^3} + \frac{C}{b} = 0\ . \tag{10.34}$$

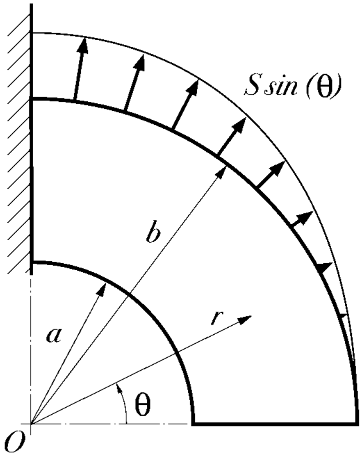

Figure 10.4: Curved beam with sinusoidal loading.

This looks like a well-behaved set of four equations for four unknown constants, but we know that the coefficient matrix has a zero determinant, since we successfully obtained a non-trivial solution to the corresponding homogeneous problem in §10.1.2 above.

In such cases, we must seek an additional special solution in which the dependence of the stress field on θ differs from that in the boundary conditions. Such solutions are known for a variety of geometries, but are usually presented as a *fait accompli* by the author, so it is difficult to see why they work, or how we might have been expected to determine them without guidance. In this section, we shall present the appropriate function and explain why it works, but we shall then develop a more rational approach to determining the class of solution required.

The special solution can be generated by expressing the original function (10.20) in complex variable form and then multiplying it by $\ln\zeta$, where $\zeta = x + \imath y$ is the complex variable and the conjugate $x - \imath y$ is denoted by $\bar{\zeta}$. Thus, (10.20) consists of a linear combination of the terms $\Im(\bar{\zeta}\zeta^2; \zeta^{-1}; \zeta\ln\zeta; \bar{\zeta}\ln\zeta)$, so we can generate a new biharmonic function from the terms $\Im(\bar{\zeta}\zeta^2\ln\zeta; \zeta^{-1}\ln\zeta; \zeta\ln^2\zeta; \bar{\zeta}\ln^2\zeta)$, which can be written in the form

$$\begin{aligned}\phi = A'r^3(\ln(r)\sin\theta + \theta\cos\theta) + B'r^{-1}(\theta\cos\theta - \ln r\sin\theta)\\ +C'r\ln(r)\theta\cos\theta + D'r(\ln^2 r\sin\theta - \theta^2\sin\theta)\ .\end{aligned} \tag{10.35}$$

The corresponding stress components are

$$\begin{aligned}\sigma_{rr} &= (2A'r - 2B'r^{-3} + C'r^{-1} - 4D'r^{-1})\theta\cos\theta \\ &\quad +(2A'r\ln(r) - A'r + 2B'r^{-3}\ln(r) - 3B'r^{-3} \\ &\quad -2C'r^{-1}\ln(r) + 2D'r^{-1}\ln(r) - 2D'r^{-1})\sin\theta \qquad (10.36)\\ \sigma_{r\theta} &= (2A'r - 2B'r^{-3} + C'r^{-1})\theta\sin\theta \\ &\quad +(-2A'r\ln(r) - 3A'r - 2B'r^{-3}\ln(r) + 3B'r^{-3} \\ &\quad -C'r^{-1} - 2D'r^{-1}\ln(r))\cos\theta \qquad (10.37)\\ \sigma_{\theta\theta} &= (6A'r + 2B'r^{-3} + C'r^{-1})\theta\cos\theta \\ &\quad +(6A'r\ln(r) + 5A'r - 2B'r^{-3}\ln(r) + 3B'r^{-3} \\ &\quad +2D'r^{-1}\ln(r) + 2D'r^{-1})\sin\theta\ . \qquad (10.38)\end{aligned}$$

Notice in particular that these expressions contain some terms of the required form for the problem of Figure 10.4, but they also contain terms with multipliers of the form $\theta\sin\theta, \theta\cos\theta$, which are inappropriate. We therefore get four homogeneous algebraic equations for the coefficients A', B', C', D' from the requirement that these inappropriate terms should vanish in the components $\sigma_{rr}, \sigma_{r\theta}$ on the boundaries $r=a, b$, i.e.

$$2A'a - \frac{2B'}{a^3} + \frac{C'}{a} - \frac{4D'}{a} = 0 \qquad (10.39)$$

$$2A'b - \frac{2B'}{b^3} + \frac{C'}{b} - \frac{4D'}{b} = 0 \qquad (10.40)$$

$$2A'a - \frac{2B'}{a^3} + \frac{C'}{a} = 0 \qquad (10.41)$$

$$2A'b - \frac{2B'}{b^3} + \frac{C'}{b} = 0\ . \qquad (10.42)$$

Clearly these equations are identical[2] with (10.24–10.27) and are not linearly independent. There is therefore a non-trivial solution to (10.39–10.42), which leaves us with a stress function that can supplement that of equation (10.20) to make the problem well-posed.

From here on, the solution is algebraically tedious, but routine. Adding the two stress functions (10.20, 10.35), we obtain a function with 8 unknown constants which is required to satisfy 9 boundary conditions comprising (i) equations (10.39–10.42), (ii) equations (10.31–10.34) modified to include the $\sin\theta, \cos\theta$ terms from equations (10.36, 10.37) respectively and (iii) the weak traction-free condition

$$\int_a^b \sigma_{\theta r} = 0 \ \ ; \ \ \theta = 0 \qquad (10.43)$$

on the end of the beam. Since two of equations (10.39–10.42) are not independent, the system reduces to a set of 8 equations for 8 constants whose solution is

[2] except that $4D'$ replaces $2D$.

$$
\begin{aligned}
A' &= \frac{Sb}{4N_1} \ ; \ \ B' = -\frac{Sa^2b^3}{4N_1} \ ; \ \ C' = -\frac{Sb(a^2+b^2)}{2N_1} \ ; \ \ D' = 0 \\
A &= \frac{Sb}{8N_1^2}\{2(b^2-a^2) - (3a^2+b^2)\ln b + (3b^2+a^2)\ln a \\
&\quad -2(a^2\ln a + b^2 \ln b)\ln(b/a)\} \ ; \\
B &= \frac{Sa^2b^3}{8N_1^2}\{-2(b^2-a^2) + (3b^2+a^2)\ln b - (3a^2+b^2)\ln a \\
&\quad -2(a^2\ln b + b^2\ln a)\ln(b/a)\} \\
C &= \frac{Sb}{4N_1^2}\{2(b^4+a^4)\ln(b/a) - (b^4-a^4)\} \\
D &= \frac{Sb}{4N_1}\{(b^2-a^2) + 2(b^2+a^2)\ln a\} \ ,
\end{aligned}
\tag{10.44}
$$

where N_1 is given by equation (10.29).

Of course, it is not fortuitous that the stress function (10.35) leads to a set of equations (10.39–10.42) identical with (10.24–10.27). When we differentiate the function $f(\zeta,\bar{\zeta})\ln(\zeta)$ by parts to determine the stresses, the extra multiplier $\ln(\zeta)$ is only preserved in those terms where it is not differentiated and hence in which all the differential operations are performed on $f(\zeta,\bar{\zeta})$. But these operations on $f(\zeta,\bar{\zeta})$ are precisely those leading to the stresses in the original solution and hence to equations (10.24–10.27). The reader will notice a parallel here with the procedure for determining the general solution of a differential equation with repeated differential multipliers and with that for dealing with degeneracy of solutions discussed in §8.3.3.

10.3.2 The near-singular problem

Suppose we next consider a more general version of the problem of Figure 10.4 in which the inhomogeneous boundary condition (10.30) is replaced by

$$\sigma_{rr} = S\sin(\lambda\theta) \ ; \quad r = b \ , \tag{10.45}$$

where λ is a constant. In the special case where $\lambda = 1$, this problem reduces to that of §10.3.1. For all other values (excluding $\lambda = 0$), we can use the stress function

$$\phi = (Ar^{\lambda+2} + Br^{\lambda} + Cr^{-\lambda} + Dr^{-\lambda+2})\sin(\lambda\theta) \ , \tag{10.46}$$

with stress components

$$
\begin{aligned}
\sigma_{rr} &= -\{A(\lambda-2)(\lambda+1)r^{\lambda} + B\lambda(\lambda-1)r^{\lambda-2} \\
&\quad + C\lambda(\lambda+1)r^{-\lambda-2} + D(\lambda+2)(\lambda-1)r^{-\lambda}\}\sin(\lambda\theta) \\
\sigma_{r\theta} &= \{-A\lambda(\lambda+1)r^{\lambda} - B\lambda(\lambda-1)r^{\lambda-2} \\
&\quad + C\lambda(\lambda+1)r^{-\lambda-2} + D\lambda(\lambda-1)r^{-\lambda}\}\cos(\lambda\theta) \\
\sigma_{\theta\theta} &= \{A(\lambda+1)(\lambda+2)r^{\lambda} + B\lambda(\lambda-1)r^{\lambda-2} \\
&\quad + C\lambda(\lambda+1)r^{-\lambda-2} + D(\lambda-1)(\lambda-2)r^{-\lambda}\}\sin(\lambda\theta) \ .
\end{aligned}
\tag{10.47}
$$

The boundary conditions on the curved edges lead to the four equations

$$\begin{aligned}
A(\lambda-2)(\lambda+1)a^{\lambda}+B\lambda(\lambda-1)a^{\lambda-2}+C\lambda(\lambda+1)a^{-\lambda-2}+D(\lambda+2)(\lambda-1)a^{-\lambda} &= 0\\
A(\lambda-2)(\lambda+1)b^{\lambda}+B\lambda(\lambda-1)b^{\lambda-2}+C\lambda(\lambda+1)b^{-\lambda-2}+D(\lambda+2)(\lambda-1)b^{-\lambda} &= -S\\
-A\lambda(\lambda+1)a^{\lambda}-B\lambda(\lambda-1)a^{\lambda-2}+C\lambda(\lambda+1)a^{-\lambda-2}+D\lambda(\lambda-1)a^{-\lambda} &= 0\\
-A\lambda(\lambda+1)b^{\lambda}-B\lambda(\lambda-1)b^{\lambda-2}+C\lambda(\lambda+1)b^{-\lambda-2}+D\lambda(\lambda-1)b^{-\lambda} &= 0\ ,
\end{aligned} \tag{10.48}$$

which have the solution

$$\begin{aligned}
A &= \frac{Sb^{\lambda}}{2(\lambda+1)N(\lambda)}[(\lambda+1)f(\lambda)-\lambda b^2 f(\lambda-1)\}\\
B &= \frac{-Sb^{\lambda}}{2(\lambda-1)N(\lambda)}\{\lambda f(\lambda+1)-(\lambda-1)b^2 f(\lambda)\}\\
C &= \frac{-Sa^{2\lambda}b^{\lambda+2}}{2(\lambda+1)N(\lambda)}\{\lambda b^{2\lambda-2}f(1)+f(\lambda)\}\\
D &= \frac{Sa^{2\lambda-2}b^{\lambda}}{2(\lambda-1)N(\lambda)}\{\lambda b^{2\lambda}f(1)+a^2 f(\lambda)\}\ ,
\end{aligned} \tag{10.49}$$

where

$$N(\lambda) = \lambda^2 f(\lambda-1)f(\lambda+1)-(\lambda^2-1)f(\lambda)^2 \tag{10.50}$$

$$f(p) = b^{2p}-a^{2p}\ . \tag{10.51}$$

On casual inspection, it seems that this problem behaves rather remarkably as λ passes through unity. The stress field is sinusoidal for all $\lambda \neq 1$ and increases without limit as λ approaches unity from either side, since $N(\lambda)=0$ when $\lambda=1$. (Notice that an additional singularity is introduced through the factor $(\lambda-1)$ in the denominator of the coefficients B, D.) However, when λ is exactly equal to unity, the problem has the bounded solution derived in §10.3.1, in which the stress field is not sinusoidal.

However, we shall demonstrate that the solution is not as discontinuous as it looks. The stress field obtained by substituting (10.49) into (10.46) is not a complete solution to the problem since the end condition (10.43) is not satisfied. The solution from (10.46) will generally involve a non-zero force on the end, given by

$$\begin{aligned}
F(\lambda) &= \int_a^b \sigma_{r\theta}dr\\
&= -\lambda\left\{Af\left(\frac{\lambda+1}{2}\right)+Bf\left(\frac{\lambda-1}{2}\right)+Cf\left(\frac{-\lambda-1}{2}\right)+Df\left(\frac{-\lambda+1}{2}\right)\right\}\ .
\end{aligned} \tag{10.52}$$

To restore the traction-free condition on the end (in the weak sense of zero force resultant) we must subtract the solution of the homogeneous problem of §10.1.2, with F given by (10.52). The final solution therefore involves the superposition of the stress functions (10.20) and (10.46). As λ approaches unity, both components of the solution increase without limit, since (10.52) contains the singular terms (10.49), but they also tend to assume the same form since, for example, $r^{2+\lambda}\sin(\lambda\theta)$ approaches $r^3\sin\theta$. In the limit, the corresponding term takes the form

$$\lim_{\lambda\to 1}\frac{A_1(\lambda)r^{2+\lambda}\sin(\lambda\theta)-A_2(\lambda)r^3\sin\theta}{N(\lambda)}\ , \tag{10.53}$$

where

$$A_1(\lambda) = \frac{Sb^\lambda}{2(\lambda+1)}\{(\lambda+1)f(\lambda)-\lambda b^2 f(\lambda-1)\} \tag{10.54}$$

$$A_2(\lambda) = \frac{F(\lambda)N(\lambda)}{2N_1}\ . \tag{10.55}$$

Setting $\lambda=1$ in equations (10.54, 10.55), we find

$$A_1(1) = A_2(1) = \frac{Sb(b^2-a^2)}{2}\ . \tag{10.56}$$

The zero in $N(\lambda)$ is therefore cancelled and (10.53) has a bounded limit which can be recovered by using L'Hôpital's rule. We obtain

$$\frac{\{A_1'(1)-A_2'(1)\}r^3\sin\theta}{N'(1)}+\frac{A_1(1)g(r,\theta,1)}{N'(1)}\ , \tag{10.57}$$

where

$$g(r,\theta,\lambda) = \frac{\partial}{\partial\lambda}r^{2+\lambda}\sin(\lambda\theta) = r^{2+\lambda}\{\ln(r)\sin(\lambda\theta)+\theta\cos(\lambda\theta)\}\ , \tag{10.58}$$

which in the limit $\lambda=1$ reduces to $r^3\{\ln(r)\sin\theta+\theta\cos\theta\}$. The coefficient of this term will be

$$A' = \frac{A_1(1)}{N'(1)} = \frac{Sb}{4N_1}\ , \tag{10.59}$$

agreeing with the corresponding coefficient, A', in the special solution of §10.3.1 (see equations (10.44)).

A similar limiting process yields the form and coefficients of the remaining 7 stress functions in the solution of §10.3.1. In particular, we note that the coefficients B, D in equations (10.49) have a second singular term in the denominator and hence L'Hôpital's rule has to be applied twice, leading to stress functions of the form of the last two terms in equation (10.35).

Thus, we find that the solution to the more general problem of equation (10.45) includes that of §10.3.1 as a limiting case and there are no values of λ for which the stress field is singular. The limiting process also shows us a more general way of obtaining the special stress functions required for the problem of §10.3.1. We simply take the stress function (10.46), which degenerates when $\lambda=1$, and differentiate it with respect to the parameter, λ, before setting $\lambda=1$. Since the last two terms of (10.20) are already of this differentiated form, a second differentiation is required to generate the last two terms of (10.35).

10.4 Some general considerations

Similar examples could be found for other geometries in the two-dimensional theory of elasticity. To fix ideas, suppose we consider the body $a < \xi < b$, $c<\eta<d$ in the general system of curvilinear coördinates, ξ, η. In general we might anticipate a class of separated-variable solutions of the form

$$\phi = f(\lambda, \xi) g(\lambda, \eta) \ , \tag{10.60}$$

where λ is a parameter. An important physical problem is that in which the boundaries $\eta = a, b$ are traction-free and the tractions on the remaining boundaries have a non-zero force or moment resultant. In such cases, we can generally use dimensional and/or equilibrium arguments to determine the form of the stress variation in the ξ-direction and hence the appropriate function $f(\lambda_0, \xi)$. The biharmonic equation will then reduce to a fourth order ordinary differential equation for $g(\lambda_0, \eta)$, with four linearly independent solutions. Enforcement of the traction-free boundary conditions will give a set of four homogeneous equations for the unknown multipliers of these four solutions.

If the solution is to be possible, the equations must have a non-trivial solution, implying that the matrix of coefficients is singular. It therefore follows that the corresponding *inhomogeneous* problem cannot be solved by the stress function of equation (10.60) if the tractions on $\eta=a, b$ vary with $f(\lambda_0, \xi)$ — i.e. in the same way as the stresses in the homogeneous (end loaded) problem. However, special stress functions appropriate to this limiting case can be obtained by differentiating (10.60) with respect to the parameter λ and then setting $\lambda=\lambda_0$.

The special solution for $\lambda = \lambda_0$ is not qualitatively different from that at more general values of λ, but appears as a regular limit *once appropriate boundary conditions are imposed on the edges* $\xi=c, d$.

Other simple examples in the two-dimensional theory of elasticity include the curved beam in bending (§10.1.1) and the wedge with traction-free faces (§11.2). Similar arguments can also be applied to three-dimensional axisymmetric problems — e.g. to problems of the cone or cylinder with traction-free curved surfaces.

10.4.1 Conclusions

Whenever we find a classical solution in which there are two traction-free boundaries, we can formulate a related inhomogeneous problem for which the equation system resulting from these boundary conditions will be singular. Special solutions for these cases can be obtained by parametric differentiation of more general solutions for the same geometry. Alternatively, the special case can be recovered as a limit as the parameter tends to its eigenvalue, provided that the remaining boundary conditions are satisfied in an appropriate sense (e.g. in the weak sense of force resultants) before the limit is taken.

PROBLEMS

1. A curved beam of inner radius a and outer radius $5a$ is subjected to a force F at its end as shown in Figure 10.5. The line of action of the force passes through the mid-point of the beam.

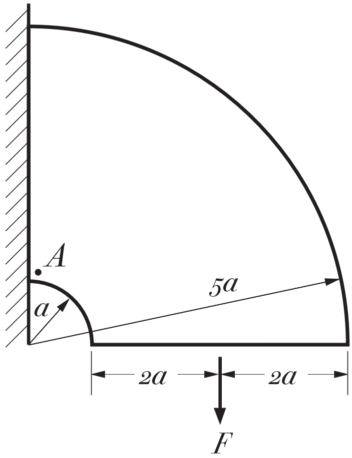

Figure 10.5: Curved beam loaded by a central force.

By superposing the solutions for the curved beam subjected to an end force and to pure bending respectively, find the hoop stress $\sigma_{\theta\theta}$ at the point $A(a, \pi/2)$ and compare it with the value predicted by the elementary Mechanics of Materials bending theory.

2. The curved beam $a<r<b$, $0<\theta<\pi/2$ is built in at $\theta=\pi/2$ and loaded only by its own weight, which acts in the direction $\theta=-\pi/2$. Find the stress field in the beam.

3. The curved beam $a<r<b$, $0<\theta<\pi/2$ is built in at $\theta=\pi/2$ and loaded by a uniform normal pressure $\sigma_{rr}=-S$ at $r=b$, the other edges being traction-free. Find the stress field in the beam.

4. The curved beam $a<r<b$, $0<\theta<\pi/2$ is built in at $\theta=\pi/2$ and loaded by a uniform shear traction $\sigma_{r\theta}=S$ at $r=b$, the other edges being traction-free. Find the stress field in the beam.

5. The curved beam $a<r<b$, $0<\theta<\pi/2$ rotates about the axis $\theta=\pi/2$ at speed Ω, the edges $r=a,b$ and $\theta=0$ being traction-free. Find the stresses in the beam. **Hint:** The easiest method is probably to solve first the problem of the complete annular ring $a<r<b$ rotating about $\theta=\pi/2$ and then correct the boundary condition at $\theta=0$ (in the weak sense) by superposing a suitable homogeneous solution.

6. Figure 10.6 shows a crane hook of thickness t that is loaded by a force F acting through the centre of the curved section. Find the stress field in the curved portion of the hook and compare the maximum tensile stress with that predicted by the elementary bending theory. Neglect the self-weight of the hook.

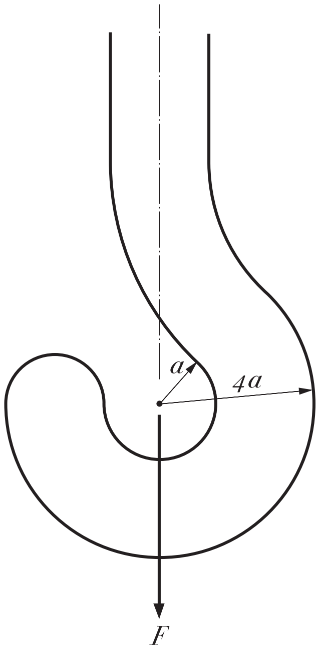

Figure 10.6: The crane hook.

11

WEDGE PROBLEMS

In this chapter, we shall consider a class of problems for the semi-infinite wedge defined by the lines $\alpha<\theta<\beta$, illustrated in Figure 11.1.

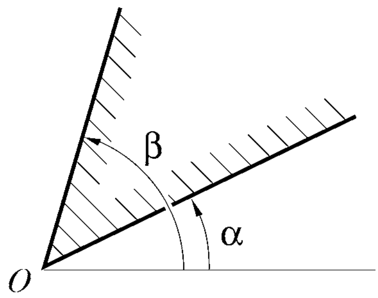

Figure 11.1: The semi-infinite wedge.

11.1 Power law tractions

We first consider the case in which the tractions on the boundaries vary with r^n, in which case equations (8.10, 8.11) suggest that the required stress function will be of the form

$$\phi = r^{n+2} f(\theta) \; . \tag{11.1}$$

The function $f(\theta)$ can be found by substituting (11.1) into the biharmonic equation (8.16), giving the ordinary differential equation

$$\left(\frac{d^2}{d\theta^2} + (n+2)^2\right)\left(\frac{d^2}{d\theta^2} + n^2\right) f = 0 \; . \tag{11.2}$$

For $n \neq 0, -2$, the four solutions of this equation define the stress function

$$\phi = r^{n+2}\{A_1 \cos(n+2)\theta + A_2 \cos(n\theta) + A_3 \sin(n+2)\theta + A_4 \sin(n\theta)\} \tag{11.3}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_11, © Springer Science+Business Media B.V. 2010

and the corresponding stress and displacement components can be taken from Tables 8.1, 9.1 respectively, with appropriate values of n.

11.1.1 Uniform tractions

Timoshenko and Goodier, in a footnote in §45, state that the uniform terms in the four boundary tractions are not independent and that only three of them may be prescribed. This is based on the argument that in a more general non-singular series solution, the uniform terms represent the stress in the corner, $r=0$ and the stress at a point only involves three independent components in two-dimensions.

However, the differential equation (11.2) has four independent non-trivial solutions for $n=0$, corresponding to the stress function[1]

$$\phi = r^2\{A_1\cos(2\theta) + A_2 + A_3\sin(2\theta) + A_4\theta\} \; . \qquad (11.4)$$

The corresponding stress components are

$$\sigma_{rr} = -2A_1\cos(2\theta) + 2A_2 - 2A_3\sin(2\theta) + 2A_4\theta \qquad (11.5)$$

$$\sigma_{r\theta} = 2A_1\sin(2\theta) - 2A_3\cos(2\theta) - A_4 \qquad (11.6)$$

$$\sigma_{\theta\theta} = 2A_1\cos(2\theta) + 2A_2 + 2A_3\sin(2\theta) + 2A_4\theta \; , \qquad (11.7)$$

and, in general, these permit us to solve the problem of the wedge with any combination of four independent traction components on the faces[2]. Timoshenko's assertion is therefore incorrect.

Uniform shear on a right-angle wedge

To explore this paradox further, it is convenient to consider the problem of Figure 11.2, in which the right-angle corner $x>0,\; y>0$ is subjected to uniform shear on one face defined by

$$\sigma_{xy} = S \;\; ; \;\; \sigma_{xx} = 0 \;\; ; \;\; x = 0 \qquad (11.8)$$

$$\sigma_{yy} = 0 \;\; ; \;\; \sigma_{yx} = 0 \;\; ; \;\; y = 0 \; . \qquad (11.9)$$

[1] Another way of generating this special solution is to note that the term in (11.2) which degenerates when $n=0$ is $A_4\sin(n\theta)$. Following §8.3.3, we can find the special solution by differentiating with respect to n, before proceeding to the limit $n\rightarrow 0$, giving the result $A_4\theta$.

[2] This problem was investigated by E.Reissner, Note on the theorem of the symmetry of the stress tensor, *Journal of Mathematics and Physics* Vol. 23 (1944), pp.192–194. See also D.B.Bogy and E.Sternberg, The effect of couple-stresses on the corner singularity due to an asymmetric shear loading, *International Journal of Solids and Structures*, Vol. 4 (1968), pp.159–174.

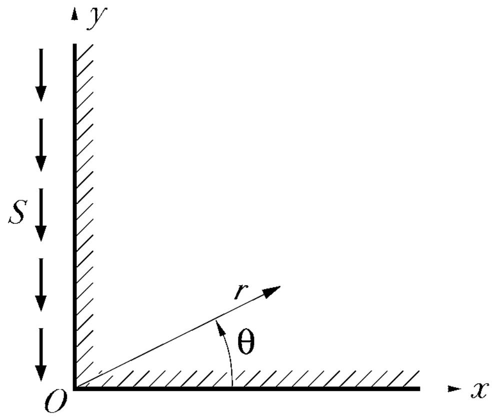

Figure 11.2: Uniform shear on a right-angle wedge.

Timoshenko's footnote would seem to argue that this is not a well-posed problem, since it implies that $\sigma_{xy} \neq \sigma_{yx}$ at $x = y = 0$.

Casting the problem in polar coördinates, (r, θ), the boundary conditions become

$$\sigma_{\theta r} = -S \;\; ; \;\; \sigma_{\theta\theta} = 0 \;\; ; \;\; \theta = \frac{\pi}{2} \tag{11.10}$$

$$\sigma_{\theta r} = 0 \;\; ; \;\; \sigma_{\theta\theta} = 0 \;\; ; \;\; \theta = 0 \; . \tag{11.11}$$

Using the stress field of equations (11.5–11.7), this reduces to the system of four algebraic equations

$$2A_3 - A_4 = -S \tag{11.12}$$

$$-2A_1 + 2A_2 + A_4\pi = 0 \tag{11.13}$$

$$-2A_3 - A_4 = 0 \tag{11.14}$$

$$2A_1 + 2A_2 = 0 \; , \tag{11.15}$$

with solution

$$A_1 = \frac{\pi S}{8} \;\; ; \;\; A_2 = -\frac{\pi S}{8} \;\; ; \;\; A_3 = -\frac{S}{4} \;\; ; \;\; A_4 = \frac{S}{2} \; , \tag{11.16}$$

giving the stress function

$$\phi = S\left(\frac{\pi r^2 \cos(2\theta)}{8} - \frac{\pi r^2}{8} - \frac{r^2 \sin(2\theta)}{4} + \frac{r^2\theta}{2}\right) \; . \tag{11.17}$$

Timoshenko's 'paradox' is associated with the apparent inconsistency in the stress components σ_{xy}, σ_{yx} at $x = y = 0$, so it is convenient to recast the stress function in Cartesian coördinates as

$$\phi = S\left(\frac{\pi}{8}(x^2 - y^2) - \frac{\pi(x^2 + y^2)}{8} - \frac{xy}{2} + \frac{(x^2 + y^2)}{2} \arctan\frac{y}{x}\right) \; . \tag{11.18}$$

We then determine the shear stress σ_{xy} as

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{Sy^2}{x^2 + y^2} \; . \tag{11.19}$$

It is easily verified that this expression tends to zero for $y=0$ and to S for $x=0$, except of course that it is indeterminate at the point $x=y=0$. The first three terms in the stress function (11.18) are second degree polynomials in x, y and therefore define a uniform state of stress throughout the wedge, but the fourth term, resulting from $A_4 r^2 \theta$ in (11.4), defines stresses which are uniform along any line θ=constant, but which vary with θ. Thus, any stress component σ is a bounded function of θ only. It follows that the corner of the wedge is not a singular point, but the stress *gradients* in the θ-direction $\left(\frac{1}{r}\frac{\partial \sigma}{\partial \theta}\right)$ increase with r^{-1} as $r \rightarrow 0$. This arises because lines of constant θ meet at $r=0$.

11.1.2 The rectangular body revisited

We noted in Chapter 6 that convergence problems are encountered in series solutions for the rectangular body when the shear tractions are discontinuous in the corners. One way to circumvent this difficulty is to extract the discontinuity explicitly, using appropriate multiples of the stress function (11.4) centred on each corner. In fact, it is clear from the Cartesian coördinate expression (11.18) that only the term

$$\frac{(x^2 + y^2)}{2} \arctan \frac{y}{x}$$

contributes to the discontinuity. For example, we can construct a particular solution for the symmetric/symmetric problem of equations (6.29, 6.30) in the form

$$\phi_P = C\left[f(a,b) + f(-a,b) + f(a,-b) + f(-a,-b)\right] \tag{11.20}$$

where

$$f(c,d) = \frac{((x-c)^2 + (y-d)^2)}{2} \arctan\left(\frac{y-d}{x-c}\right) \tag{11.21}$$

and C is a constant. The corresponding shear tractions in the corner (a, b) then have a discontinuity defined by

$$\lim_{y \rightarrow b} \sigma_{xy}(a,y) - \lim_{x \rightarrow a} \sigma_{yx}(x,b) = C \; . \tag{11.22}$$

Thus, if we choose $C=g(b)-h(a)$, the corrective boundary value problem defined by the stress function $\phi_C=\phi-\phi_P$ will involve continuous shear tractions at all four corners (because of symmetry) and the series solution of §6.4 will converge.

It is interesting to note that the form of the corner stress field defined by equation (11.20) was obtained by Meleshko and Gomilko[3] using the series (6.31) with parameters (6.38), by examining the limiting form of the coefficients at large m, n.

11.1.3 More general uniform loading

More generally, there is no inconsistency in specifying four independent uniform traction components on the faces of a wedge, though values which are inconsistent with the conditions for stress at a point will give infinite stress gradients in the corner.

For the general case, it is convenient to choose a coördinate system symmetric with respect to the two faces of the wedge, which is then bounded by the lines $\theta = \pm\alpha$.

Writing the boundary conditions

$$\sigma_{\theta r} = T_1 \;\; ; \;\; \theta = \alpha \tag{11.23}$$

$$\sigma_{\theta r} = T_2 \;\; ; \;\; \theta = -\alpha \tag{11.24}$$

$$\sigma_{\theta\theta} = N_1 \;\; ; \;\; \theta = \alpha \tag{11.25}$$

$$\sigma_{\theta\theta} = N_2 \;\; ; \;\; \theta = -\alpha \; , \tag{11.26}$$

and using the solution (11.5–11.7), we obtain the four algebraic equations

$$2A_1 \sin(2\alpha) - 2A_3 \cos(2\alpha) - A_4 = T_1 \tag{11.27}$$

$$-2A_1 \sin(2\alpha) - 2A_3 \cos(2\alpha) - A_4 = T_2 \tag{11.28}$$

$$2A_1 \cos(2\alpha) + 2A_2 + 2A_3 \sin(2\alpha) + 2A_4\alpha = N_1 \tag{11.29}$$

$$2A_1 \cos(2\alpha) + 2A_2 - 2A_3 \sin(2\alpha) - 2A_4\alpha = N_2 \; . \tag{11.30}$$

As in §6.2.2, we can exploit the symmetry of the problem to partition the coefficient matrix by taking sums and differences of these equations in pairs, obtaining the simpler set

$$-4A_3 \cos(2\alpha) - 2A_4 = T_1 + T_2 \tag{11.31}$$

$$4A_1 \sin(2\alpha) = T_1 - T_2 \tag{11.32}$$

$$4A_1 \cos(2\alpha) + 4A_2 = N_1 + N_2 \tag{11.33}$$

$$4A_3 \sin(2\alpha) + 4A_4\alpha = N_1 - N_2 \; , \tag{11.34}$$

where we note that the terms involving A_1, A_2 correspond to a symmetric stress field and those involving A_3, A_4 to an antisymmetric field.

[3] V.V.Meleshko and A.M.Gomilko, Infinite systems for a biharmonic problem in a rectangle, *Proceeedings of the Royal Society of London,* Vol. A453 (1997), pp.2139–2160.

11.1.4 Eigenvalues for the wedge angle

The solution of these equations is routine, but we note that there are two eigenvalues for the wedge angle 2α at which the matrix of coefficients is singular. In particular, the solution for the symmetric terms (equations (11.32, 11.33) is singular if

$$\sin(2\alpha) = 0 \;, \tag{11.35}$$

i.e. at $2\alpha = 180^o$ or 360^o, whilst the antisymmetric terms are singular when

$$\tan(2\alpha) = 2\alpha \;, \tag{11.36}$$

which occurs at $2\alpha = 257.4^o$. The 180^o wedge is the half plane $x > 0$, whilst the 257.4^o wedge is a reëntrant corner.

As in the problem of §10.3, special solutions are needed for the inhomogeneous problem if the wedge has one of these two special angles. As before, they can be obtained by differentiating the general solution (11.3) with respect to n before setting $n=0$, leading to the new terms, $r^2\{\ln(r)\cos(2\theta) - \theta\sin(2\theta)\}$ and $r^2\ln(r)$. A particular problem for which these solutions are required is that of the half plane subjected to a uniform shear traction over half of its boundary (see Problem 11.1, below).

The homogeneous solution

For wedges of 180^o and 257.4^o, the *homogeneous* problem, where $N_1 = N_2 = T_1 = T_2 = 0$ has a non-trivial solution. For the 180^o wedge (i.e. the half plane $x > 0$), this solution can be seen by inspection to be one of uniaxial tension, $\sigma_{yy} = S$, which is non-trivial, but involves no tractions on the free surface $x = 0$. The same state of stress is also a non-trivial solution for the 360^o wedge, which corresponds to a semi-infinite crack (see Figure 11.6 and §11.2.3 below).

11.2 Williams' asymptotic method

Figure 11.3 shows a body with a notch, loaded by tractions on the remote boundaries. Intuitively we anticipate a stress concentration at the notch and we shall show in this section that the stress field there is generally singular — i.e. that the stress components tend to infinity as we approach the sharp corner of the notch.

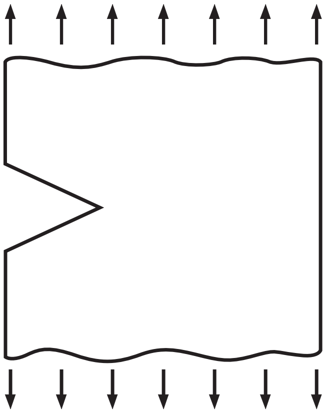

Figure 11.3: The notched bar in tension.

Williams[4] developed a method of exploring the nature of the stress field near this singularity by defining a set of polar coördinates centred on the corner and expanding the stress field as an asymptotic series in powers of r.

11.2.1 Acceptable singularities

Before developing the asymptotic solution in detail, we must first address the question as to whether singular stress fields are ever acceptable in elasticity problems.

Engineering criteria

This question can be approached from various points of view. The engineer is usually inclined to argue that no real materials are capable of sustaining an infinite stress and hence any situation in which such a stress is predicted by an elastic analysis will in practice lead to yielding or some other kind of failure.

We also note that stress singularities are always associated with discontinuities in the geometry or the boundary conditions — for example, a sharp corner, as in the present instance, or a concentrated (delta function) load. In practice, there are no sharp corners and loads can never be perfectly concentrated. If we are to give any meaning to these solutions, they must be considered as limits of more practical situations, such as a corner with a very small fillet radius or a load applied over a very small region of the boundary.

Yet another practical limitation is that of the continuum theory. Real materials are not continua — they have atomic structure and often a larger

[4] M.L.Williams, Stress singularities resulting from various boundary conditions in angular corners of plates in extension, *ASME Journal of Applied Mechanics,* Vol. 19 (1952), pp.526–528.

scale granular structure as well. Thus, it doesn't make much practical sense to talk about the value of quantities nearer to a corner than (say) one atomic distance, since the theory is going to break down there anyway.

Of course, we find this kind of argument whenever we try to idealize a physical system, which means essentially whenever we try to describe its behaviour in mathematical terms. Because there are so many idealizations or approximations involved in the modeling process, it is never certain which one is responsible when mathematical difficulties are encountered in a problem and often there are several ways of reformulating the problem to introduce more physical reality and avoid the difficulty.

Mathematical criteria

A totally different approach to the question — more favoured by mathematicians — is to make choices based on questions of uniqueness, convergence and existence of solutions, rather than on the physical grounds discussed above. In terms of the functional analysis, we choose to locate the theory of elasticity in a function space which guarantees that problems are mathematically well-posed. A function space here denotes a set of functions in which the solution is to be sought, and generally this involves some statement about the strength of acceptable singularities in the solution.

In this context, the engineer's objections outlined above can be addressed by limiting arguments. It is not logically impossible that a material should have a yield strength of any arbitrarily large (but finite) magnitude, that the radius of a corner should be arbitrarily small, but finite, etc. Solutions of problems involving singularities are then to be conceived as the result of allowing these quantities to tend to their limits. Generally the difference between the real problem and the limiting case will be only localized.

In regions of a body or its boundary where no concentrated traction etc. is applied, we adopt the criterion that the only acceptable singularities are those for which the total strain energy in a small region surrounding the singular point vanishes as the size of that region tends to zero.

If the stresses and hence the strains vary with r^a as we approach the point $r=0$, the strain energy in a two-dimensional problem will be an integral of the form

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^r \sigma_{ij}e_{ij} r dr d\theta = C\int_0^r r^{2a+1} dr \ , \tag{11.37}$$

where C is a constant which depends on the elastic constants and the nature of the stress variation with θ. This integral is bounded if $a>-1$ and otherwise unbounded. Thus, singular stress fields are acceptable if and only if the exponent on the stress components exceeds -1.

Concentrated forces and dislocations[5] involve stress singularities with exponent $a=-1$ and are therefore excluded by this criterion, since the integral

[5] See Chapter 13 below.

in (11.37) would lead to a logarithm, which is unbounded at $r=0$. However, these solutions can be permitted on the understanding that they are really idealizations of explicitly specified distributions. In this case, it is important to recognize that the resulting solutions are not meaningful at or near the singular point. We also use these and other singular solutions as Green's functions — i.e. as the kernels of convolution integrals to define the solution of a non-singular problem. This technique will be illustrated in Chapters 12 and 13, below.

11.2.2 Eigenfunction expansion

We are concerned only with the stress components in the notch at very small values of r and hence we imagine looking at the corner through a strong microscope, so that we see the wedge of Figure 11.4. The magnification is so large that the other surfaces of the body, including the loaded boundaries, appear far enough away for us to treat the wedge as infinite, with 'loading at infinity'.

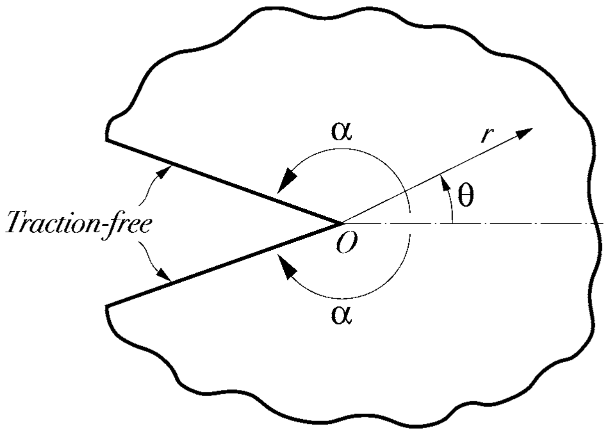

Figure 11.4: The semi-infinite notch.

The stress field at the notch is of course a complicated function of r, θ, but as in Chapter 6, we seek to expand it as a series of separated-variable terms, each of which satisfies the traction-free boundary conditions on the wedge faces. The appropriate separated-variable form is of course that given by equation (11.3), except that we increase the generality of the solution by relaxing the requirement that the exponent n be an integer — indeed, we shall find that most of the required solutions have *complex* exponents.

Following Williams' notation, we replace n by $(\lambda-1)$ in equation (11.3), obtaining the stress function

$$\phi = r^{\lambda+1}\{A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\}\ , \tag{11.38}$$

with stress components

$$\sigma_{rr} = r^{\lambda-1}\{-A_1\lambda(\lambda+1)\cos(\lambda+1)\theta - A_2\lambda(\lambda-3)\cos(\lambda-1)\theta$$
$$-A_3\lambda(\lambda+1)\sin(\lambda+1)\theta - A_4\lambda(\lambda-3)\sin(\lambda-1)\theta\} \tag{11.39}$$
$$\sigma_{r\theta} = r^{\lambda-1}\{A_1\lambda(\lambda+1)\sin(\lambda+1)\theta + A_2\lambda(\lambda-1)\sin(\lambda-1)\theta$$
$$-A_3\lambda(\lambda+1)\cos(\lambda+1)\theta - A_4\lambda(\lambda-1)\cos(\lambda-1)\theta\} \tag{11.40}$$
$$\sigma_{\theta\theta} = r^{\lambda-1}\{A_1\lambda(\lambda+1)\cos(\lambda+1)\theta + A_2\lambda(\lambda+1)\cos(\lambda-1)\theta$$
$$+A_3\lambda(\lambda+1)\sin(\lambda+1)\theta + A_4\lambda(\lambda+1)\sin(\lambda-1)\theta\}\ . \tag{11.41}$$

For this solution to satisfy the traction-free boundary conditions

$$\sigma_{\theta r} = \sigma_{\theta\theta} = 0;\quad \theta = \pm\alpha \tag{11.42}$$

we require

$$\lambda\{A_1(\lambda+1)\sin(\lambda+1)\alpha + A_2(\lambda-1)\sin(\lambda-1)\alpha$$
$$-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\lambda-1)\cos(\lambda-1)\alpha\} = 0 \tag{11.43}$$
$$\lambda\{-A_1(\lambda+1)\sin(\lambda+1)\alpha - A_2(\lambda-1)\sin(\lambda-1)\alpha$$
$$-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\lambda-1)\cos(\lambda-1)\alpha\} = 0 \tag{11.44}$$
$$\lambda\{A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\lambda+1)\cos(\lambda-1)\alpha$$
$$+A_3(\lambda+1)\sin(\lambda+1)\alpha + A_4(\lambda+1)\sin(\lambda-1)\alpha\} = 0 \tag{11.45}$$
$$\lambda\{A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\lambda+1)\cos(\lambda-1)\alpha$$
$$-A_3(\lambda+1)\sin(\lambda+1)\alpha - A_4(\lambda+1)\sin(\lambda-1)\alpha\} = 0\ . \tag{11.46}$$

This is a set of four homogeneous equations for the four constants A_1, A_2, A_3, A_4 and will have a non-trivial solution only for certain eigenvalues of the exponent λ. Since all the equations have a λ multiplier, $\lambda = 0$ must be an eigenvalue for all wedge angles. We can simplify the equations by canceling this factor and taking sums and differences in pairs to expose the symmetry of the system, with the result

$$A_1(\lambda+1)\sin(\lambda+1)\alpha + A_2(\lambda-1)\sin(\lambda-1)\alpha = 0 \tag{11.47}$$
$$A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\lambda+1)\cos(\lambda-1)\alpha = 0 \tag{11.48}$$
$$A_3(\lambda+1)\cos(\lambda+1)\alpha + A_4(\lambda-1)\cos(\lambda-1)\alpha = 0 \tag{11.49}$$
$$A_3(\lambda+1)\sin(\lambda+1)\alpha + A_4(\lambda+1)\sin(\lambda-1)\alpha = 0\ . \tag{11.50}$$

This procedure partitions the coefficient matrix, to yield the two independent matrix equations

$$\begin{bmatrix} (\lambda+1)\sin(\lambda+1)\alpha & (\lambda-1)\sin(\lambda-1)\alpha \\ (\lambda+1)\cos(\lambda+1)\alpha & (\lambda+1)\cos(\lambda-1)\alpha \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \tag{11.51}$$

and

$$\begin{bmatrix} (\lambda+1)\cos(\lambda+1)\alpha & (\lambda-1)\cos(\lambda-1)\alpha \\ (\lambda+1)\sin(\lambda+1)\alpha & (\lambda+1)\sin(\lambda-1)\alpha \end{bmatrix} \begin{Bmatrix} A_3 \\ A_4 \end{Bmatrix} = 0 \; . \tag{11.52}$$

The symmetric terms A_1, A_2 have a non-trivial solution if and only if the determinant of the coefficient matrix in (11.51) is zero, leading to the characteristic equation

$$\lambda \sin 2\alpha + \sin(2\lambda\alpha) = 0 \; , \tag{11.53}$$

whilst the antisymmetric terms A_3, A_4 have a non-trivial solution if and only if

$$\lambda \sin 2\alpha - \sin(2\lambda\alpha) = 0 \; , \tag{11.54}$$

from (11.52).

11.2.3 Nature of the eigenvalues

We first note from equations (11.39–11.41) that the stress components are proportional to $r^{\lambda-1}$ and hence λ is restricted to positive values by the energy criterion of §11.2.1. Furthermore, we shall generally be most interested in the lowest possible eigenvalue[6], since this will define the strongest singularity that can occur at the notch and will therefore dominate the stress field at sufficiently small r.

Both equations (11.53, 11.54) are satisfied for all α if $\lambda=0$ and (11.54) is also satisfied for all α if $\lambda=1$.

The case $\lambda=0$ is strictly excluded by the energy criterion, but we shall find in the next chapter that it corresponds to the important case in which a concentrated force is applied at the origin. It can legitimately be regarded as describing the stress field sufficiently distant from a load distributed on the wedge faces near the origin, but as such it is clearly not appropriate to the unloaded problem of Figure 11.3.

The solution $\lambda=1$ of equation (11.54) is a spurious eigenvalue, since if we compute the corresponding eigenfunction, it turns out to have the null form $\phi=A_4\sin(0)$. The correct limiting form for $\lambda=1$ requires the modified stress function (11.4) and leads to the condition (11.36) for antisymmetric fields, which is satisfied only for $2\alpha=257.4^o$.

Some insight into the nature of the eigenvalues for more general wedge angles can be gained from the graphical representation of Figure 11.5, where we plot the two terms in equations (11.53, 11.54) against $x=2\lambda\alpha$. The sine wave represents the term $\sin(2\lambda\alpha)$ $(=\sin x)$ and the straight lines represent various possible positions for the terms $\pm\lambda\sin(2\alpha)=\pm x(\sin(2\alpha)/2\alpha)$.

---

[6] More rigorously, a general stress field near the corner can be expressed as an eigenfunction expansion, but the term with the smallest eigenvalue will be arbitrarily larger than the next term in the expansion at sufficiently small values of r.

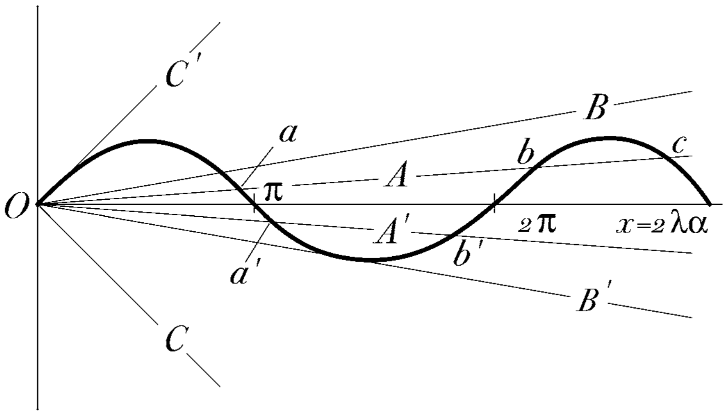

Figure 11.5: Graphical solution of equations (11.53, 11.54).

The simplest case is that in which $2\alpha = 2\pi$ and the wedge becomes the crack illustrated in Figure 11.6. We then have $\sin(2\alpha) = 0$ and the solutions of both equations (11.53, 11.54) correspond to the points where the sine wave crosses the horizontal axis — i.e.

$$\lambda = \frac{1}{2}, 1, \frac{3}{2}, \ldots \tag{11.55}$$

This is an important special case because of its application to fracture mechanics. In particular, we note that the lowest positive eigenvalue of λ is $\frac{1}{2}$ and hence the crack-tip field is square-root singular for both symmetric and antisymmetric loading.

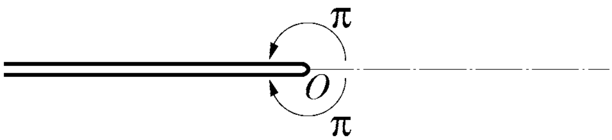

Figure 11.6: Tip of a crack considered as a 360^o wedge.

Suppose we now consider a wedge of somewhat less than 2π, in which case equations (11.53, 11.54) are represented by the intersection between the sine wave and the lines A, A' respectively in Figure 11.5.

As the wedge angle is reduced, the slope $(\sin(2\alpha)/2\alpha)$ of the lines first becomes increasingly negative until a maximum is reached at $2\alpha = 257.4^o$, corresponding to the tangential line B' and its mirror image B. Further reduction causes the slope to increase monotonically, passing through zero again at $2\alpha = \pi$ (corresponding to the case of the half plane) and reaching the limit C, C' at $2\alpha = 0$.

Clearly, if the lines A, A' are even slightly inclined to the horizontal, they will only intersect a finite number of waves, corresponding to a finite set of real roots, the number of real roots increasing as the slope approaches zero. Increasing the slope causes initially equal-spaced real roots to become closer in pairs such as b, c in Figure 11.5. For any given pair, there will be a critical slope at which the roots coalesce and for further increase in slope, a pair of complex conjugate roots will be developed.

In the antisymmetric case for $360^o > 2\alpha > 257.4^o$, the intersection b' in Figure 11.5 corresponds to the spurious eigenvalue of (11.54), but a' is a meaningful eigenvalue corresponding to a stress singularity that weakens from square-root at $2\alpha = 360^o$ to zero at $2\alpha = 257.4^o$. For smaller wedge angles, a' is the spurious eigenvlaue and b' corresponds to a real but non-singular root.

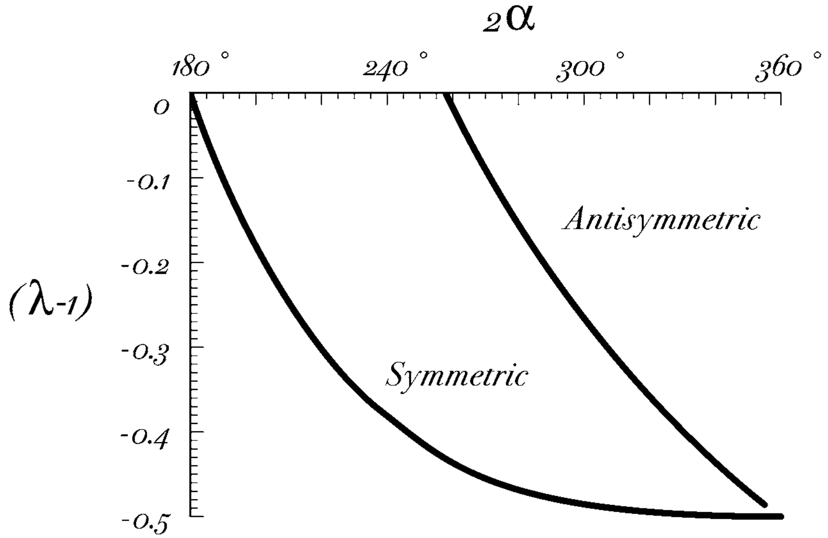

Figure 11.7: Strength of the singularity in a reëntrant corner.

In the symmetric case, the root a gives a real eigenvalue corresponding to a singular stress field whose strength falls monotonically from square-root to zero as the wedge angle is reduced from 2π to π. The strength of the singularity $(\lambda - 1)$ for both symmetric and antisymmetric terms is plotted against total wedge angle[7] in Figure 11.7. The singularity associated with the symmetric field is always stronger than that for the antisymmetric field except in the limit $2\alpha = 2\pi$, where they are equal, and there is a range of angles $(257.4^o > 2\alpha > 180^o)$ where the symmetric field is singular, but the antisymmetric field is not.

For non-reëntrant wedges $(2\alpha < \pi)$, both fields are bounded, the dominant eigenvalue for the symmetric field becoming complex for $2\alpha < 146^o$. Since bounded solutions correspond to eigenfunctions with r raised to a power with

[7] Figure 11.7 is reproduced by courtesy of Professor D.A.Hills of Oxford University.

positive real part, the stress field always tends to zero in a non-reëntrant corner.

11.2.4 The singular stress fields

If $\lambda=\lambda_S$ is an eigenvalue of (11.53), the two equations (11.47, 11.48) will not be linearly independent and we can satisfy both of them by choosing A_1, A_2 such that

$$A_1 = A(\lambda_S - 1)\sin(\lambda_S - 1)\alpha \tag{11.56}$$

$$A_2 = -A(\lambda_S + 1)\sin(\lambda_S + 1)\alpha\ , \tag{11.57}$$

where A is a new arbitrary constant. The corresponding (symmetric) singular stress field is then defined through the stress function

$$\phi_S = Ar^{\lambda_S+1}\left\{(\lambda_S - 1)\sin(\lambda_S - 1)\alpha\cos(\lambda_S + 1)\theta - (\lambda_S + 1)\sin(\lambda_S + 1)\alpha\cos(\lambda_S - 1)\theta\right\}\ . \tag{11.58}$$

A similar procedure with equations (11.49, 11.50) defines the antisymmetric singular field through the stress function

$$\phi_A = Br^{\lambda_A+1}\left\{(\lambda_A + 1)\sin(\lambda_A - 1)\alpha\sin(\lambda_A + 1)\theta - (\lambda_A + 1)\sin(\lambda_A + 1)\alpha\sin(\lambda_A - 1)\theta\right\}\ , \tag{11.59}$$

where λ_A is an eigenvalue of (11.54) and B is an arbitrary constant.

For the crack tip of Figure 11.6, the eigenvalues of both equations are given by equation (11.55) and the dominant singular fields correspond to the values $\lambda_S=\lambda_A=\frac{1}{2}$. Substituting this value into (11.58) and the resulting expression into (8.10, 8.11), we obtain the symmetric singular field as

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \tag{11.60}$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right\} \tag{11.61}$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left\{\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right\}\ , \tag{11.62}$$

where we have introduced the new constant

$$K_I = 3A\sqrt{\frac{\pi}{2}}\ , \tag{11.63}$$

known as the *mode I stress intensity factor*[8].

[8] Brittle materials typically fail when the stress intensity factor at a crack tip reaches a critical value. This will be discussed in more detail in §13.3.

The corresponding antisymmetric singular crack-tip field is obtained from (11.59) as

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad (11.64)$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right\} \quad (11.65)$$

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left\{\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right\}, \quad (11.66)$$

where K_{II} is the mode II stress intensity factor.

11.2.5 Other geometries

Williams' method can also be applied to other discontinuities in elasticity and indeed in other fields in mechanics. For example, we can extend it to determine the strength of the singularity in a composite wedge of two different materials[9]. A special case is illustrated in Figure 11.8, where two dissimilar materials are bonded[10], leaving a composite wedge of angle $3\pi/2$ at the points A, B. Another composite wedge is the tip of a crack (or debonded zone) at the interface between two dissimilar materials (see Chapter 32).

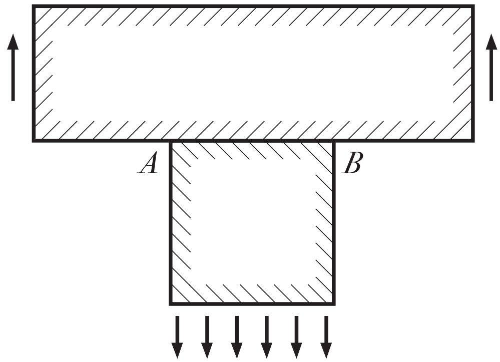

Figure 11.8: Composite T-bar, with composite notches at the reëntrant corners A, B.

[9] D.B.Bogy, Two edge-bonded elastic wedges of different materials and wedge angles under surface tractions, *ASME Journal of Applied Mechanics,* Vol. 38 (1971), pp.377–386.

[10] A problem of this kind was solved by G.G.Adams, A semi-infinite elastic strip bonded to an infinite strip, *ASME Journal of Applied Mechanics,* Vol. 47 (1980), pp.789–794.

The method has also been applied to determine the asymptotic fields in contact problems, with and without friction[11] (see Chapter 12 below).

These problems generally involve both traction and displacement boundary conditions. For example, if the dissimilar wedges $\alpha_1 < \theta < \alpha_0$ and $\alpha_0 < \theta < \alpha_2$ are bonded together at the common plane $\theta = \alpha_0$ and the other faces $\theta = \alpha_1, \alpha_2$ are traction-free, the boundary conditions are

$$\sigma^1_{\theta\theta}(r, \alpha_1) = 0 \tag{11.67}$$
$$\sigma^1_{\theta r}(r, \alpha_1) = 0 \tag{11.68}$$
$$\sigma^1_{\theta\theta}(r, \alpha_0) - \sigma^2_{\theta\theta}(r, \alpha_0) = 0 \tag{11.69}$$
$$\sigma^1_{\theta r}(r, \alpha_0) - \sigma^2_{\theta r}(r, \alpha_0) = 0 \tag{11.70}$$
$$u^1_r(r, \alpha_0) - u^2_r(r, \alpha_0) = 0 \tag{11.71}$$
$$u^1_\theta(r, \alpha_0) - u^2_\theta(r, \alpha_0) = 0 \tag{11.72}$$
$$\sigma^2_{\theta\theta}(r, \alpha_2) = 0 \tag{11.73}$$
$$\sigma^2_{\theta r}(r, \alpha_2) = 0 \tag{11.74}$$

where superscripts $1, 2$ refer to the two wedges, respectively. Equations (11.69, 11.70) can be seen as statements of Newton's third law for the tractions transmitted between the two wedges, whilst (11.71, 11.72) express the fact that if two bodies are bonded together, adjacent points on opposite sides of the bond must have the same displacement.

To formulate asymptotic problems of this type, the stress equations (11.39–11.41) must be supplemented by the corresponding expressions for the displacements u_r, u_θ. These can be written down from Table 9.1, since in that table the parameter n does not necessarily have to take integer values. Alternatively, they can be obtained directly by substituting the expression (11.38) into the Maple or Mathematica file 'urt'. Omitting the rigid-body displacements, we obtain

$$\begin{aligned} 2\mu u_r = r^\lambda\{&-A_1(\lambda+1)\cos(\lambda+1)\theta + A_2(\kappa-\lambda)\cos(\lambda-1)\theta \\ &-A_3(\lambda+1)\sin(\lambda+1)\theta + A_4(\kappa-\lambda)\sin(\lambda-1)\theta\} \end{aligned} \tag{11.75}$$
$$\begin{aligned} 2\mu u_\theta = r^\lambda\{&A_1(\lambda+1)\sin(\lambda+1)\theta + A_2(\kappa+\lambda)\sin(\lambda-1)\theta \\ &-A_3(\lambda+1)\cos(\lambda+1)\theta - A_4(\kappa+\lambda)\cos(\lambda-1)\theta\} \end{aligned} \tag{11.76}$$

The results of asymptotic analysis are very useful in numerical methods, since they enable us to predict the nature of the local stress field and hence devise appropriate special elements or meshes. Failure to do this in problems with singular fields will always lead to numerical inefficiency and sometimes to lack of convergence, mesh sensitivity or instability.

[11] J.Dundurs and M.S.Lee, Stress concentration at a sharp edge in contact problems, *Journal of Elasticity*, Vol. 2 (1972), pp.109–112, M.Comninou, Stress singularities at a sharp edge in contact problems with friction, *Zeitschrift für angewandte Mathematik und Physik,* Vol. 27 (1976), pp.493–499.

To look ahead briefly to three-dimensional problems, we note that when we concentrate our attention on a very small region at a corner, the resulting magnification makes all other dimensions (including radii of curvature etc) look very large. Thus, a notch in a three-dimensional body will generally have a two-dimensional asymptotic field at small r. The above results and the underlying method are therefore of very general application. Another consequence is that the out-of-plane dimension becomes magnified indefinitely, so that plane strain (rather than plane stress) conditions are appropriate in all three-dimensional asymptotic problems.

11.3 General loading of the faces

If the faces of the wedge $\theta=\pm\alpha$ are subjected to tractions that can be expanded as power series in r, a solution can be obtained using a series of terms like (11.1). However, as with the rectangular beam, power series are of limited use in representing a general traction distribution, particularly when it is relatively localized. Instead, we can adapt the Fourier transform representation (5.129) to the wedge, by noting that the stress function (11.38) will oscillate along lines of constant θ if we choose complex values for λ. This leads to a representation as a *Mellin transform* defined as

$$\phi(r,\theta) = \frac{1}{2\pi i}\int_{c-\imath\infty}^{c+\imath\infty} f(s,\theta) r^{-s} ds \tag{11.77}$$

for which the inversion is[12]

$$f(s,\theta) = \int_0^\infty \phi(r,\theta) r^{s-1} dr \; . \tag{11.78}$$

The integrand of (11.77) is the function (11.38) with $\lambda = -s-1$. The integral is path-independent in any strip of the complex plane in which the integrand is a holomorphic[13] function of s. If we evaluate it along the straight line $s=c+\imath\omega$, it can be written in the form

$$\begin{aligned} r^c\phi(r,\theta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(c+\imath\omega,\theta) r^{-\imath\omega} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(c+\imath\omega) e^{\imath\omega\ln(r)} d\omega \; , \end{aligned} \tag{11.79}$$

which is readily converted to a Fourier integral[14] by the change of variable $t=\ln(r)$. The constant c has to be chosen to ensure regularity of the integrand.

[12] I.N.Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[13] See Chapter 18 and in particular, §18.5.

[14] The basic theory of the Mellin transform and its relation to the Fourier transform is explained by I.N.Sneddon, *loc. cit.*. For applications to elasticity problems for

If there are no unacceptable singularities in the tractions and the latter are bounded at infinity, it can be shown that a suitable choice is $c=-1$.

The Mellin transform and power series methods can be applied to a wedge of any angle, but it should be remarked that, because of the singular asymptotic fields obtained in §11.2, meaningful results for the reëntrant wedge ($2\alpha > \pi$) can only be obtained if the conditions at infinity are also prescribed and satisfied.

More precisely, we could formulate the problem for the large but finite sector $-\alpha < \theta < \alpha$, $0 < r < b$, with precribed tractions on all edges[15]. The procedure would be first to develop a particular solution for the loading on the faces $\theta = \pm\alpha$ as if the wedge were infinite and then to correct the boundary conditions at $r=b$ by superposing an infinite sequence of eigenfunctions from §11.2 with appropriate multipliers[16]. For sufficiently large b, the stress field near the apex of the wedge would be adequately described by the particular solution and the singular terms from the eigenfunction expansion.

---

the wedge, see E.Sternberg and W.T.Koiter, The wedge under a concentrated couple: A paradox in the two-dimensional theory of elasticity, *ASME Journal of Applied Mechanics,* Vol. 25 (1958), pp.575–581, W.J.Harrington and T.W.Ting, Stress boundary-value problems for infinite wedges, *Journal of Elasticity,* Vol. 1 (1971), pp.65–81.

[15] A problem of this kind was considered by G.Tsamasphyros and P.S.Theocaris, On the solution of the sector problem, *Journal of Elasticity,* Vol. 9 (1979), pp.271–281.

[16] This requires that the eigenfunction series is complete for this problem. The proof is given by R.D.Gregory, Green's functions, bi-linear forms and completeness of the eigenfunctions for the elastostatic strip and wedge, *Journal of Elasticity,* Vol. 9 (1979), pp.283–309.

PROBLEMS

1. Figure 11.9 shows a half plane, $y<0$, subjected to a uniform shear traction, $\sigma_{xy}=S$ on the half-line, $x>0,\ y=0$, the remaining tractions on $y=0$ being zero.

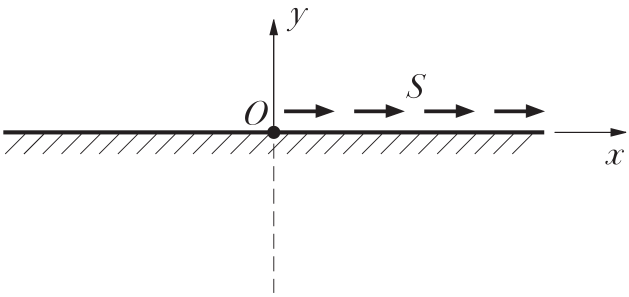

Figure 11.9: The half plane with shear tractions.

Find the complete stress field in the half plane. **Note**: This is a problem requiring a special stress function with a logarithmic multiplier (see §11.1.4 above).

2. The half plane $y<0$ is subjected to a uniform normal pressure $\sigma_{yy}=-S$ on the half-line, $x>0,\ y=0$, the remaining tractions on $y=0$ being zero. Find the complete stress field in the half plane.

3. The wedge $-\alpha<\theta<\alpha$ is loaded by a concentrated moment M_0 at the apex, the plane edges being traction free. Use dimensional arguments to show that the stress components must all have the separated-variable form

$$\sigma = \frac{f(\theta)}{r^2}\ .$$

Use this result to choose a suitable stress function and hence find the complete stress field in the wedge.

4. Show that $\phi = Ar^2\theta$ can be used as a stress function and determine the tractions which it implies on the boundaries of the region $-\pi/2 < \theta < \pi/2$. Hence show that the stress function appropriate to the loading of Figure 11.10 is

$$\phi = -\frac{F}{4\pi a}(r_1^2\theta_1 + r_2^2\theta_2)$$

where $r_1, r_2, \theta_1, \theta_2$ are defined in the Figure.

Determine the principal stresses at the point B.

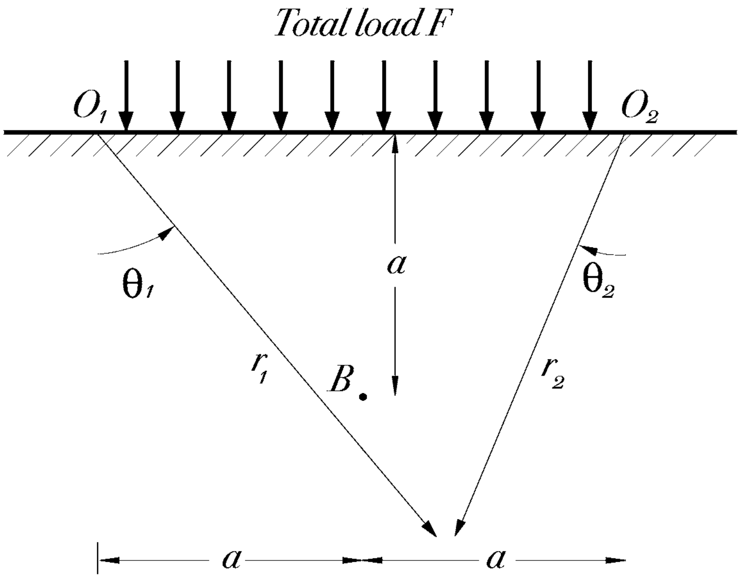

Figure 11.10: Uniform loading over a discrete region.

5. A wedge-shaped concrete dam is subjected to a hydrostatic pressure ρgh varying with depth h on the vertical face $\theta=0$ as shown in Figure 11.11, the other face $\theta=\alpha$ being traction-free. The dam is also loaded by self-weight, the density of concrete being $\rho_c=2.3\rho$.

Find the minimum wedge angle α if there is to be no tensile stress in the dam.

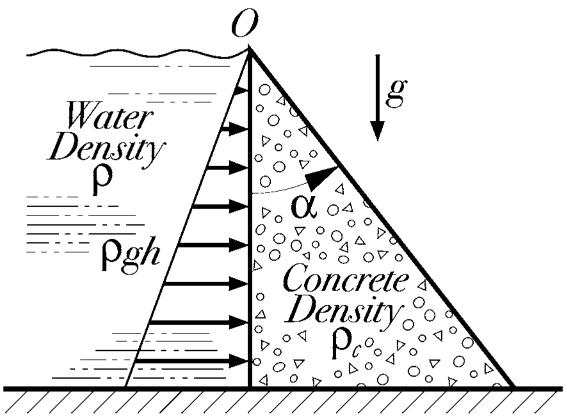

Figure 11.11: The wedge-shaped dam.

6. Figure 11.12 shows a 45^o triangular plate ABC built in at BC and loaded by a uniform pressure p_0 on the upper edge AB. The inclined edge AC is traction-free.

Find the stresses in the plate, using weak boundary conditions on the edge BC (which do not need to be explicitly enforced). Hence compare the maximum tensile stress with the prediction of the elementary bending theory.

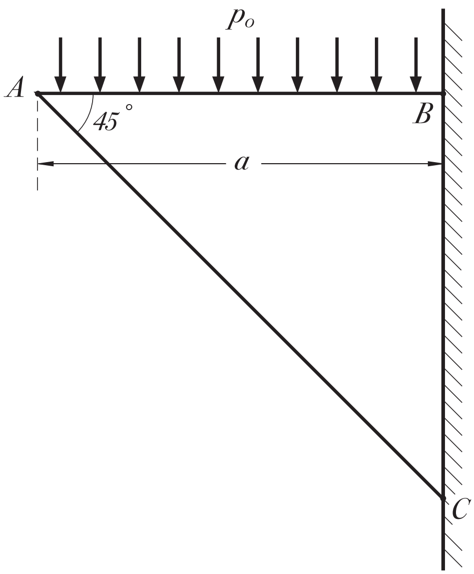

Figure 11.12

7. The wedge $-\alpha < \theta < \alpha$ is bonded to a rigid body on both edges $\theta = \pm\alpha$. Use the eigenfunction expansion of §11.2.2 to determine the characteristic equations that must be satisfied by the exponent λ in the stress function (11.38) for symmetric and antisymmetric stress fields. Show that these equations reduce to (11.53, 11.54) if $\nu = 0.5$ and plane strain conditions are assumed.

8. Find the equation that must be satisfied by λ if the stress function (11.38) is to define a non-trivial solution of the problem of the half plane $0 < \theta < \pi$, traction-free on $\theta = 0$ and in frictionless contact with a rigid plane surface at $\theta = \pi$. Find the lowest value of λ that satisfies this equation and obtain explicit expressions for the form of the corresponding singular stress field near the corner. This solution is of importance in connection with the frictionless indentation of a smooth elastic body by a rigid body with a sharp corner.

9. Two large bodies of similar materials with smooth, continuously differentiable curved surfaces make frictionless contact. Use the asymptotic method of §11.2 to examine the asymptotic stress fields near the edge of the resulting

contact area. The solution of this problem must also satisfy two *inequalities*: that the contact tractions in the contact region must be compressive and that the normal displacements in the non-contact region must cause a non-negative gap. Show that one or other of these conditions is violated, whatever sign is taken for the multiplier on the first eigenfunction. What conclusion do you draw from this result?

10. Find the equation that must be satisfied by λ if the stress function (11.38) is to define a non-trivial solution of the problem of the wedge $0 < \theta < \pi/2$, traction-free on $\theta = 0$ and bonded to a rigid plane surface at $\theta = \pi/2$. Do not attempt to solve the equation.

12

PLANE CONTACT PROBLEMS[1]

In the previous chapter, we considered problems in which the infinite wedge was loaded on its faces or solely by tractions on the infinite boundary. A related problem of considerable practical importance concerns the wedge with traction-free faces, loaded by a concentrated force $\boldsymbol{F}$ at the vertex, as shown in Figure 12.1.

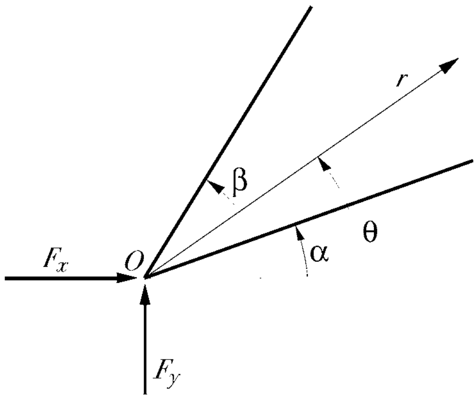

Figure 12.1: The wedge loaded by a force at the vertex.

12.1 Self-similarity

An important characteristic of this problem is that there is no inherent length scale. An enlarged photograph of the problem would look the same as the

[1] For a more detailed discussion of elastic contact problems, see K.L.Johnson, *Contact Mechanics*, Cambridge University Press, 1985 and G.M.L.Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_12,

original. The solution must therefore share this characteristic and hence, for example, contours of the stress function ϕ must have the same geometric shape at all distances from the vertex. Problems of this type — in which the solution can be mapped into itself after a change of length scale — are described as *self-similar*.

An immediate consequence of the self-similarity is that all the stress components must be capable of expression in the separated-variable form

$$\sigma = f(r)g(\theta) \ . \tag{12.1}$$

Furthermore, since the tractions on the line $r=a$ must balance a constant force $\boldsymbol{F}$ for all a, we can deduce that $f(r)=r^{-1}$, because the area available for transmitting the force increases linearly with radius[2].

12.2 The Flamant Solution

Choosing those terms in Table 8.1 that give stresses proportional to r^{-1}, we obtain

$$\phi = C_1 r\theta \sin\theta + C_2 r\theta \cos\theta + C_3 r \ln(r) \cos\theta + C_4 r \ln(r) \sin\theta \ , \tag{12.2}$$

for which the stress components are

$$\begin{aligned} \sigma_{rr} &= r^{-1}(2C_1 \cos\theta - 2C_2 \sin\theta + C_3 \cos\theta + C_4 \sin\theta) \\ \sigma_{r\theta} &= r^{-1}(C_3 \sin\theta - C_4 \cos\theta) \\ \sigma_{\theta\theta} &= r^{-1}(C_3 \cos\theta + C_4 \sin\theta) \ . \end{aligned} \tag{12.3}$$

For the wedge faces to be traction-free, we require

$$\sigma_{\theta r} = \sigma_{\theta\theta} = 0 \ \ ; \ \ \theta = \alpha, \beta \ , \tag{12.4}$$

which can be satisfied by taking $C_3, C_4=0$.

To determine the remaining constants C_1, C_2, we consider the equilibrium of the region $0<r<a$, obtaining

$$F_x + 2\int_\alpha^\beta \left(\frac{C_1 \cos\theta - C_2 \sin\theta}{a}\right) a \cos\theta d\theta = 0 \tag{12.5}$$

$$F_y + 2\int_\alpha^\beta \left(\frac{C_1 \cos\theta - C_2 \sin\theta}{a}\right) a \sin\theta d\theta = 0 \ . \tag{12.6}$$

These two equations permit us to choose the constants C_1, C_2 to give the applied force $\boldsymbol{F}$ any desired magnitude and direction. Notice that the radius a

[2] In the same way, we can deduce that the stresses in a cone subjected to a force at the vertex must vary with r^{-2}.

cancels in equations (12.5, 12.6) and hence, if they are satisfied for any radius, they are satisfied for all.

This is known as the *Flamant solution*, or sometimes as the *simple radial distribution*, since the act of setting $C_3 = C_4 = 0$ clears all θ-surfaces of tractions, leaving

$$\sigma_{rr} = \frac{2C_1 \cos\theta}{r} - \frac{2C_2 \sin\theta}{r} \tag{12.7}$$

as the only non-zero stress component.

12.3 The half-plane

If we take $\alpha = -\pi$ and $\beta = 0$ in the above solution, we obtain the special case of a force acting at a point on the surface of the half-plane $y < 0$ as shown in Figure 12.2.

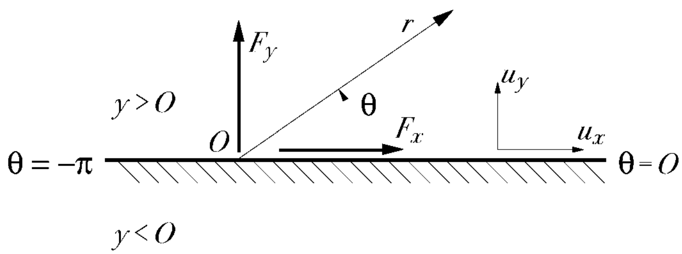

Figure 12.2: Force acting on the surface of a half-plane.

For this case, performing the integrals in equations (12.5, 12.6), we find

$$F_x + 2\int_{-\pi}^{0} (C_1 \cos^2\theta - C_2 \sin\theta\cos\theta)d\theta = F_x + \pi C_1 = 0 \tag{12.8}$$

$$F_y + 2\int_{-\pi}^{0} (C_1 \cos\theta\sin\theta - C_2 \sin^2\theta)d\theta = F_y - \pi C_2 = 0 \tag{12.9}$$

and hence

$$C_1 = -\frac{F_x}{\pi} \; ; \; C_2 = \frac{F_y}{\pi} \; . \tag{12.10}$$

It is convenient to consider these terms separately.

12.3.1 The normal force F_y

The normal (tensile) force F_y produces the stress field

$$\sigma_{rr} = -\frac{2F_y \sin\theta}{\pi r} \; , \tag{12.11}$$

which reaches a maximum tensile value on the negative y-axis ($\theta=-\pi/2$) — i.e. directly beneath the load — and falls to zero as we approach the surface along any line other than one which passes through the point of application of the force.

The displacement field corresponding to this stress can conveniently be taken from Table 9.1 and is

$$2\mu u_r = \frac{F_y}{2\pi}\{(\kappa-1)\theta\cos\theta - (\kappa+1)\ln r\sin\theta + \sin\theta\} \tag{12.12}$$

$$2\mu u_\theta = \frac{F_y}{2\pi}\{-(\kappa-1)\theta\sin\theta - (\kappa+1)\ln(r)\cos\theta - \cos\theta\}\ . \tag{12.13}$$

These solutions will be used as Green's functions for problems in which a half-plane is subjected to various surface loads and hence the surface displacements are of particular interest.

On $\theta=0$, ($y=0$, $x>0$), we have

$$2\mu u_r = 2\mu u_x = 0 \tag{12.14}$$

$$2\mu u_\theta = 2\mu u_y = \frac{F_y}{2\pi}\{-(\kappa+1)\ln(r) - 1\}\ , \tag{12.15}$$

whilst on $\theta=-\pi$, ($y=0$, $x<0$), we have

$$2\mu u_r = -2\mu u_x = \frac{F_y}{2}(\kappa-1) \tag{12.16}$$

$$2\mu u_\theta = -2\mu u_y = \frac{F_y}{2\pi}\{(\kappa+1)\ln(r) + 1\}\ . \tag{12.17}$$

It is convenient to impose symmetry on the solution by superposing a rigid-body displacement

$$2\mu u_x = \frac{F_y(\kappa-1)}{4}\ \ ;\ \ 2\mu u_y = \frac{F_y}{2\pi}\ , \tag{12.18}$$

after which equations (12.14–12.17) can be summarised in the form

$$u_x = \frac{F_y(\kappa-1)\mathrm{sgn}(x)}{8\mu} \tag{12.19}$$

$$u_y = -\frac{F_y(\kappa+1)\ln|x|}{4\pi\mu}\ , \tag{12.20}$$

where the function sgn(x) is defined to be +1 for $x>0$ and −1 for $x<0$.

Note incidentally that $r=|x|$ on $y=0$.

12.3.2 The tangential force F_x

The tangential force F_x produces the stress field

$$\sigma_{rr} = -\frac{2F_x \cos\theta}{\pi r} \, , \tag{12.21}$$

which is compressive ahead of the force ($\theta = 0$) and tensile behind it ($\theta = -\pi$) as we might expect.

The corresponding displacement field is found from Table 9.1 as before and after superposing an appropriate rigid-body displacement as in §12.3.1, the surface displacements can be written

$$u_x = -\frac{F_x(\kappa+1)\ln|x|}{4\pi\mu} \tag{12.22}$$

$$u_y = -\frac{F_x(\kappa-1)\mathrm{sgn}(x)}{8\mu} \, . \tag{12.23}$$

12.3.3 Summary

We summarize the results of §§12.3.1, 12.3.2 in the equations

$$u_x = -\frac{F_x(\kappa+1)\ln|x|}{4\pi\mu} + \frac{F_y(\kappa-1)\mathrm{sgn}(x)}{8\mu} \tag{12.24}$$

$$u_y = -\frac{F_x(\kappa-1)\mathrm{sgn}(x)}{8\mu} - \frac{F_y(\kappa+1)\ln|x|}{4\pi\mu} \, , \tag{12.25}$$

see Figure 12.3.

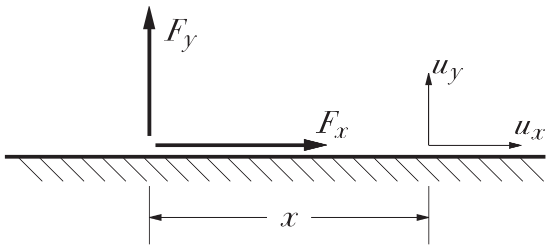

Figure 12.3: Surface displacements due to a surface force.

12.4 Distributed normal tractions

Now suppose that the surface of the half-plane is subjected to a distributed normal load $p(\xi)$ per unit length as shown in Figure 12.4. The stress and displacement field can be found by superposition, using the Flamant solution as a Green's function — i.e. treating the distributed load as the limit of a set of point loads of magnitude $p(\xi)\delta\xi$.

Of particular interest is the distortion of the surface, defined by the normal displacement u_y. At the point $P(x, 0)$, this is given by

$$u_y = -\frac{(\kappa+1)}{4\pi\mu} \int_A p(\xi) \ln|x-\xi| d\xi \; , \tag{12.26}$$

from equation (12.20), where A is the region over which the load acts.

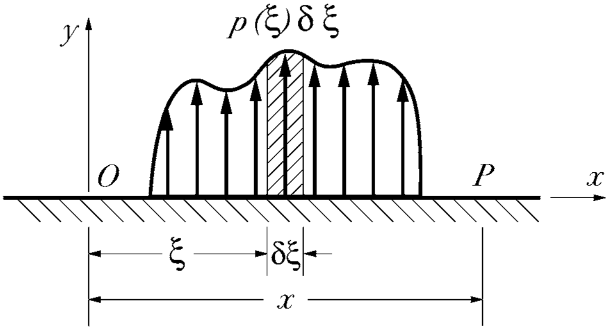

Figure 12.4: Half-plane subjected to a distributed traction.

The displacement defined by equation (12.26) is logarithmically unbounded as x tends to infinity. In general, if a finite region of the surface of the half-plane is subjected to a non-self-equilibrated system of loads, the stress and displacement fields at a distance $r \gg A$ will approximate those due to the force resultants applied as concentrated forces — i.e. to the expressions of equations (12.11–12.13) for normal loading. In other words, the stresses will decay with $1/r$ and the displacements (being integrals of the strains) will vary logarithmically.

This means that the half-plane solution can be used for the stresses in a finite body loaded on a region of the surface which is small compared with the linear dimensions of the body, since $1/r$ is arbitrarily small for sufficiently large r. However, the logarithm is not bounded at infinity and hence the rigid-body displacement of the loaded region with respect to distant parts of the body cannot be found without a more exact treatment taking into account the finite dimensions of the body. For this reason, contact problems for the half-plane are often formulated in terms of the displacement *gradient*

$$\frac{du_y}{dx} = -\frac{(\kappa+1)}{4\pi\mu} \int_A \frac{p(\xi)d\xi}{(x-\xi)} \; , \tag{12.27}$$

thereby avoiding questions of rigid-body translation.

When the point $x=\xi$ lies within A, the integral in (12.27) is interpreted as a Cauchy principal value — for example

$$\int_a^b \frac{p(\xi)d\xi}{(x-\xi)} = \lim_{\epsilon\to 0} \left(\int_a^{x-\epsilon} + \int_{x+\epsilon}^b \right) \frac{p(\xi)d\xi}{(x-\xi)} \quad ; \quad a < x < b \; . \tag{12.28}$$

12.5 Frictionless contact problems

The results of the last section permit us to develop a solution to the problem of a half-plane indented by a frictionless rigid punch of known profile (see Figure 12.5).

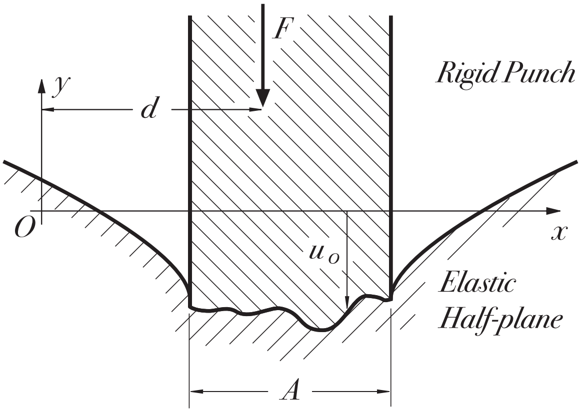

Figure 12.5: The elastic half-plane indented by a frictionless rigid punch.

We suppose that the load F is sufficient to establish contact over the whole of the contact area A, in which case, the displacement of the half-plane must satisfy the equation

$$u_y = -u_0(x) + C_1 x + C_0 \tag{12.29}$$

in A, where u_0 is a known function of x which describes the profile of the punch.

The constants C_0, C_1 define an unknown rigid-body translation and rotation of the punch respectively, which can generally be assigned in such a way as to give a pressure distribution $p(\xi)$ which is statically equivalent to the force F — i.e.

$$\int_A p(\xi) d\xi = -F \tag{12.30}$$

$$\int_A p(\xi)\xi d\xi = -Fd \, , \tag{12.31}$$

where d defines the line of action of F as shown in Figure 12.5 and the negative signs are a consequence of the tensile positive convention for $p(\xi)$. These two equations are sufficient to determine the constants C_0, C_1.

If the contact region, A, is connected, we can define a coördinate system with origin at the mid-point and denote the half-length of the contact by a. Equations (12.27, 12.29) then give

$$-\frac{du_0}{dx} + C_1 = -\frac{(\kappa+1)}{4\pi\mu} \int_{-a}^{a} \frac{p(\xi) d\xi}{(x-\xi)} \;\; ; \;\; -a < x < a \, , \tag{12.32}$$

which is a Cauchy singular integral equation for the unknown pressure $p(\xi)$.

12.5.1 Method of solution

Integral equations of this kind arise naturally in the application of the complex variable method to two-dimensional potential problems and they are extensively discussed in a classical text by Muskhelishvili[3]. However, for our purposes, a simpler solution will suffice, based on the change of variable

$$x = a\cos\phi \;\; ; \;\; \xi = a\cos\theta \tag{12.33}$$

and expansion of the two sides of the equation as Fourier series.

Substituting (12.33) into (12.32), we find

$$\frac{1}{a\sin\phi}\frac{du_0}{d\phi} + C_1 = -\frac{(\kappa+1)}{4\pi\mu}\int_0^\pi \frac{p(\theta)\sin\theta d\theta}{(\cos\phi - \cos\theta)} \;\; ; \;\; 0 < \phi < \pi \, , \tag{12.34}$$

which can be simplified using the result[4]

$$\int_0^\pi \frac{\cos(n\theta)d\theta}{(\cos\phi - \cos\theta)} = -\frac{\pi\sin(n\phi)}{\sin\phi} \, . \tag{12.35}$$

Writing

$$p(\theta) = \sum_{n=0}^\infty \frac{p_n\cos(n\theta)}{\sin\theta} \tag{12.36}$$

$$\frac{du_0}{d\phi} = \sum_{n=1}^\infty u_n\sin(n\phi) \, , \tag{12.37}$$

we find

$$\sum_{n=1}^\infty u_n\sin(n\phi) + C_1 a\sin\phi = \frac{(\kappa+1)a\sin\phi}{4\pi\mu}\sum_{n=1}^\infty \frac{\pi p_n\sin(n\phi)}{\sin\phi}$$
$$= \frac{(\kappa+1)a}{4\mu}\sum_{n=1}^\infty p_n\sin(n\phi) \tag{12.38}$$

and hence, equating Fourier coefficients,

$$p_n = \frac{4\mu u_n}{(\kappa+1)a} \;\; ; \;\; n > 1 \tag{12.39}$$

$$p_1 = \frac{4\mu(C_1 a + u_1)}{(\kappa+1)a} \, . \tag{12.40}$$

[3] N.I.Muskhelishvili, *Singular Integral Equations*, (English translation by J.R.M.Radok, Noordhoff, Groningen, 1953). For applications to elasticity problems, including the above contact problem, see N.I.Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, (English translation by J.R.M.Radok, Noordhoff, Groningen, 1953). For a simpler discussion of the solution of contact problems involving singular integral equations, see K.L.Johnson, *loc. cit*, §2.7.

[4] For a method of proving this result, see Problem 18.7.

Thus, if the shape of the punch u_0 is expanded as in equation (12.37), the corresponding coefficients in the pressure series (12.36) can be written down using (12.39), except for $n=0,1$.

The two coefficients p_0, p_1 are found by substituting (12.36) into the equilibrium conditions (12.30, 12.31). From (12.30), we have

$$-F = \int_{-a}^{a} p(x')dx' = \int_0^{\pi} p(\theta)a\sin\theta d\theta$$
$$= a\sum_{n=0}^{\infty} p_n \int_0^{\pi} \cos(n\theta)d\theta = \pi a p_0 \tag{12.41}$$

and from (12.31),

$$-Fd = \int_{-a}^{a} p(x')x'dx' = \int_0^{\pi} p(\theta)a^2\sin\theta\cos\theta d\theta$$
$$= a^2\sum_{n=0}^{\infty} p_n \int_0^{\pi} \cos\theta\cos(n\theta)d\theta = \frac{\pi a^2 p_1}{2} \ . \tag{12.42}$$

Hence,

$$p_0 = -\frac{F}{\pi a} \ ; \ p_1 = -\frac{2Fd}{\pi a^2} \ . \tag{12.43}$$

The rigid-body rotation of the punch C_1 can then be found by eliminating p_1 between equations (12.40, 12.43), with the result

$$C_1 = -\frac{Fd(\kappa+1)}{2\pi\mu a^2} - \frac{u_1}{a} \ . \tag{12.44}$$

Alternatively, if the punch is constrained to move vertically without rotation, C_1 will be zero and we can solve for the applied moment Fd as

$$Fd = -\frac{2\pi\mu u_1 a}{(\kappa+1)} \ , \tag{12.45}$$

from (12.44).

12.5.2 The flat punch

We first consider the case of a flat punch with a symmetric load as shown in Figure 12.6. The profile of the punch is described by the equation

$$u_0 = C \ ; \ \frac{du_0}{dx} = 0 \tag{12.46}$$

and hence $p_n=0$ for $n>1$ from equation (12.39). Furthermore, since the load is symmetric $(d=0)$, p_1 is also zero (equation (12.43)) and p_0 is given by (12.43).

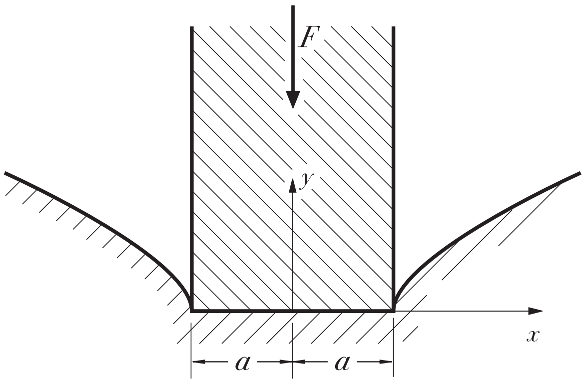

Figure 12.6: Half-plane indented by a flat rigid punch.

The traction distribution under the punch is therefore

$$p(x) = \frac{p_0}{\sin\phi} = -\frac{F}{\pi a\sqrt{1-x^2/a^2}}$$
$$= -\frac{F}{\pi\sqrt{a^2-x^2}} \; . \tag{12.47}$$

We note that the sharp corners of the punch cause a square root singularity in traction at the edges, $x = \pm a$. This is typical of indentation problems where the punch has a sharp corner and could have been predicted by performing an asymptotic analysis of the local stress field, using the methods developed in §11.2 (see Problem 11.8). The traction distribution is illustrated in Figure 12.7.

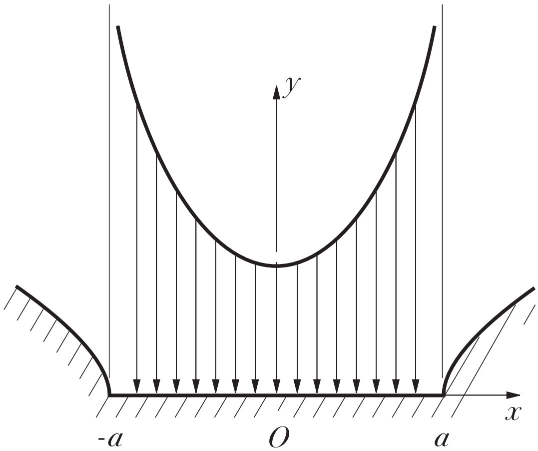

Figure 12.7: Contact pressure distribution under the flat rigid punch.

12.5.3 The cylindrical punch (Hertz problem)

We next consider the indentation of a half-plane by a frictionless rigid cylinder (Figure 12.8), which is a special case of the contact problem associated with the name of Hertz.

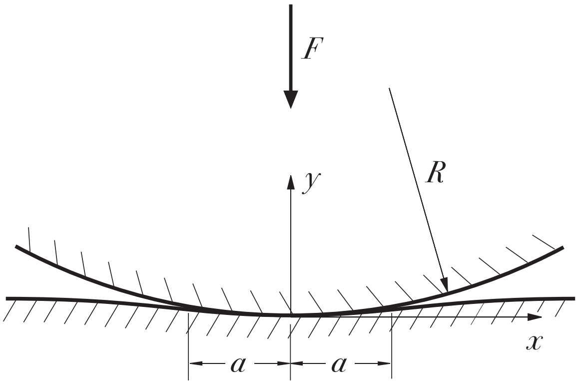

Figure 12.8: The Hertzian contact problem.

This problem differs from the previous example in that the contact area semi-width a depends on the load F and cannot be determined *a priori*. Strictly, a must be determined from the pair of inequalities

$$p(x) \leq 0 \ \ ; \ \ -a < x < a \tag{12.48}$$

$$-u_y(x) \geq u_0(x) \ \ ; \ \ |x| > a \ , \tag{12.49}$$

which express the physical requirements that the contact traction should not be tensile (12.48) and that there should be no interference between the bodies outside the contact area (12.49). However, it can be shown that these conditions are in most cases formally equivalent to the requirement that the contact traction should not be singular at $x = \pm a$. This question will be discussed more rigorously in connection with three-dimensional contact problems in Chapter 29. Here it is sufficient to remark that a traction singularity such as that obtained at the sharp corner of the flat punch (equation (12.47)) causes a discontinuity in the displacement gradient du_y/dx, which will cause a violation of (12.49) if the punch is smooth.

If the radius of the cylinder is R, we have

$$\frac{d^2 u_0}{dx^2} = -\frac{1}{R} \tag{12.50}$$

and hence

$$u_0 = C_0 - \frac{x^2}{2R} = C_0 - \frac{a^2 \cos(2\phi)}{4R} - \frac{a^2}{4R} \ . \tag{12.51}$$

Thus

$$\frac{du_0}{d\phi} = \frac{a^2 \sin 2\phi}{2R} \tag{12.52}$$

and the only non-zero coefficient in the series (12.37) is

$$u_2 = \frac{a^2}{2R} \; . \tag{12.53}$$

It follows that

$$p_2 = \frac{2\mu a}{R(\kappa + 1)} \; , \tag{12.54}$$

from (12.39).

For the cylinder, the load must be symmetrical to retain equilibrium and hence $p_1 = 0$. As before, p_0 is given by (12.43) and hence

$$p(\theta) = \left(-\frac{F}{\pi a} + \frac{2\mu a}{R(\kappa + 1)} \cos(2\theta) \right) \Big/ \sin\theta \; . \tag{12.55}$$

Now this expression will be singular at $\theta = 0, \pi$ $(x = \pm a)$ unless we choose a such that

$$\frac{F}{\pi a} = \frac{2\mu a}{R(\kappa + 1)} \; ,$$

i.e.

$$a = \sqrt{\frac{F(\kappa + 1)R}{2\pi\mu}} \; . \tag{12.56}$$

Note that for plane strain, $\kappa = (3 - 4\nu)$ and

$$\frac{2\mu}{(\kappa + 1)} = \frac{E}{4(1 - \nu^2)} \; , \tag{12.57}$$

whilst for plane stress, $\kappa = (3 - \nu)/(1 + \nu)$ and

$$\frac{2\mu}{(\kappa + 1)} = \frac{E}{4} \; . \tag{12.58}$$

With the value of a from (12.56), we find

$$p(\theta) = -\frac{F\{1 - \cos(2\theta)\}}{\pi a \sin\theta} = -\frac{2F \sin\theta}{\pi a} \tag{12.59}$$

i.e.

$$p(x) = -\frac{2F\sqrt{a^2 - x^2}}{\pi a^2} \; . \tag{12.60}$$

This pressure distribution is illustrated in Figure 12.9.

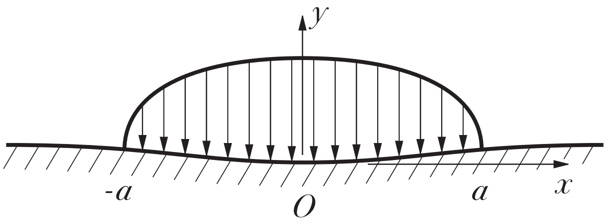

Figure 12.9: The Hertzian pressure distribution.

12.6 Problems with two deformable bodies

The same method can be used to treat problems involving the contact of two deformable bodies, provided that they have sufficiently large radii in the vicinity of the contact area to be approximated by half-planes — i.e. $R_1, R_2 \gg a$, where R_1, R_2 are the local radii[5] (see Figure 12.10). We shall also take the opportunity to generalize the formulation to the case where, in addition to the normal tractions p_y, there are tangential tractions p_x at the interface due to friction.

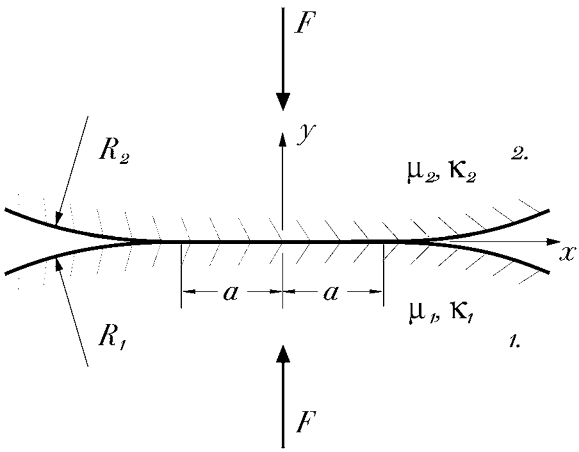

Figure 12.10: Contact of two curved deformable bodies.

We first note that, by Newton's third law, the tractions must be equal and opposite on the two surfaces, so the appropriate Green's function corresponds to the force pairs F_x, F_y of Figure 12.11.

[5] In fact the same condition must be satisfied even for the case where the curved body is rigid, since otherwise the small strain assumption of linear elasticity ($e_{ij} \ll 1$) will be violated near the contact area.

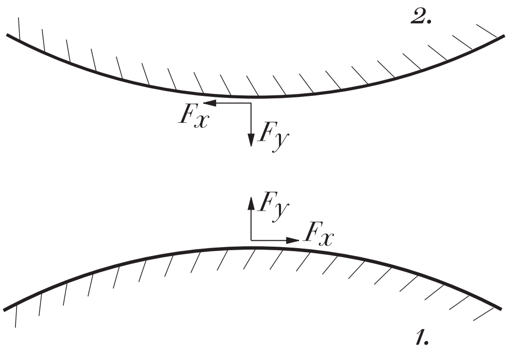

Figure 12.11: Green's function for contact problems.

Suppose that the bodies are placed lightly in contact, as shown in Figure 12.12, and that the initial gap between their surfaces is a known function $g_0(x)$.

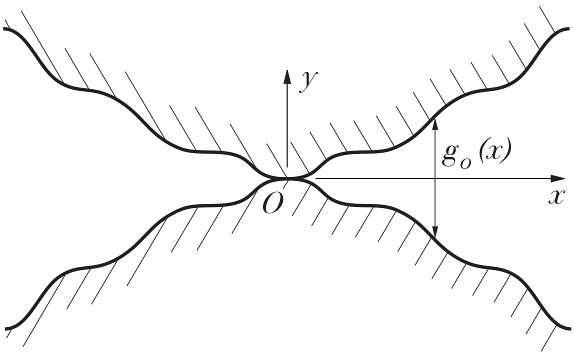

Figure 12.12: Initial gap between two unloaded contacting bodies.

We now give the upper body a vertical rigid-body translation C_0 and a small clockwise rotation C_1 such that, in the absence of deformation, the gap would become

$$g(x) = g_0(x) - C_0 - C_1 x \ . \tag{12.61}$$

In addition, we assume that the contact tractions will cause some elastic deformation, represented by the displacements $\boldsymbol{u_1}, \boldsymbol{u_2}$ in bodies 1,2 respectively.

As a result of these operations, the gap will be modified to

$$g(x) = g_0(x) - u_{y1}(x,0) + u_{y2}(x,0) - C_0 - C_1 x \tag{12.62}$$

and it follows that the contact condition analogous to (12.29) can be written

$$u_{y1}(x,0) - u_{y2}(x,0) = g_0(x) - C_0 - C_1 x \ \ ; \ \ \text{in } A \ , \tag{12.63}$$

since, in a contact region, the gap is by definition zero.

The surface displacement $u_{y1}(x, 0)$ of body 1 is simply (12.26) generalized to include the effect of the tangential traction p_x (see equation (12.25)) — i.e.

$$u_{y1}(x,0) = -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a} p_y(\xi)\ln|x-\xi|d\xi \\ -\frac{(\kappa-1)}{8\mu}\int_{-a}^{a} p_x(\xi)\text{sgn}(x-\xi)d\xi \tag{12.64}$$

and hence[6]

$$\frac{du_{y1}}{dx} = -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a} \frac{p_y(\xi)d\xi}{(x-\xi)} - \frac{(\kappa-1)}{4\mu}p_x(x)\ . \tag{12.65}$$

We also record the corresponding expression for the tangential displacement u_{x1}, which is

$$\frac{du_{x1}}{dx} = -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a} \frac{p_x(\xi)d\xi}{(x-\xi)} + \frac{(\kappa-1)}{4\mu}p_y(x)\ , \tag{12.66}$$

from (12.24).

Comparing Figure 12.11 with Figure 12.3, we see that equations (12.65, 12.66) can be used for the displacements u_x, u_y of body 1, but for the corresponding displacements of body 2 we have to take account of the fact that the tractions p_x, p_y are reversed and that the y-axis is now directed *into* the body. It is easily verified that this can be achieved by changing the signs in the expressions involving u_y, p_x, whilst leaving those expressions with u_x, p_y unchanged. It then follows that

$$\frac{d}{dx}(u_{y1} - u_{y2}) = -\frac{A}{4\pi}\int_{-a}^{a} \frac{p_y(\xi)d\xi}{(x-\xi)} - \frac{B}{4}p_x(x) \tag{12.67}$$

$$\frac{d}{dx}(u_{x1} - u_{x2}) = -\frac{A}{4\pi}\int_{-a}^{a} \frac{p_x(\xi)d\xi}{(x-\xi)} + \frac{B}{4}p_y(x)\ , \tag{12.68}$$

where

$$A = \frac{(\kappa_1+1)}{\mu_1} + \frac{(\kappa_2+1)}{\mu_2} \tag{12.69}$$

$$B = \frac{(\kappa_1-1)}{\mu_1} - \frac{(\kappa_2-1)}{\mu_2}\ . \tag{12.70}$$

[6] Notice that an alternative representation of $\text{sgn}(x)$ is $2H(x)-1$, where $H(x)$ is the Heaviside step function. It follows that the derivative of $\text{sgn}(x)$ is $2\delta(x)$ and that the derivative of the integral in the second term of (12.64) is simply twice the value of $p_x(\xi)$ at the point $\xi=x$.

12.7 Uncoupled problems

We can now substitute (12.67) into the derivative of the contact condition (12.63), obtaining

$$g_0'(x) - C_1 = -\frac{A}{4\pi}\int_{-a}^{a}\frac{p_y(\xi)d\xi}{(x-\xi)} - \frac{B}{4}p_x(x) \;\; ; \;\; -a < x < a \; . \qquad (12.71)$$

This equation is similar in form to (12.32), except for the presence of the term involving p_x. It can therefore be solved in the same way if for any reason this term is identically zero. Four important cases where this condition is satisfied are:-

(i) The contact is frictionless, so $p_x = 0$.
(ii) The materials are similar ($\kappa_1 = \kappa_2,\ \mu_1 = \mu_2$) and hence $B = 0$.
(iii) Both materials are incompressible ($\nu_1 = \nu_2 = 0.5,\ \kappa_1 = \kappa_2 = 1,\ \mu_1 \neq \mu_2$).
(iv) One body is rigid ($\mu_1 = \infty$) and the other incompressible ($\kappa_2 = 1$).

Of course, no real materials are even approximately rigid, but the coupling terms in equations (12.67, 12.68) — i.e. those connecting u_y, p_x and u_x, p_y — can reasonably be neglected provided that the ratio

$$\beta = \frac{B}{A} = \left(\frac{(\kappa_1-1)}{\mu_1} - \frac{(\kappa_2-1)}{\mu_2}\right) \Big/ \left(\frac{(\kappa_1+1)}{\mu_1} + \frac{(\kappa_2+1)}{\mu_2}\right) \ll 1 \; . \qquad (12.72)$$

This will be true if $\mu_1 \gg \mu_2$ and $(\kappa_2 - 1) \ll 1$ and it is a reasonable approximation to the practically important case of rubber in contact with steel. The dimensionless parameter β is one of Dundurs' bimaterial parameters (see §4.4.3).

In the rest of this chapter, we shall restrict attention to problems in which the coupling terms can be neglected for one of the reasons given above.

12.7.1 Contact of cylinders

If the two bodies are cylinders with radii R_1, R_2 (see Figure 12.10), we have

$$g_0''(x) = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1+R_2}{R_1R_2} \equiv \frac{1}{R} \; . \qquad (12.73)$$

Hence, comparing equations (12.73, 12.50) and (12.71, 12.32), we find that the solution can be written down from equations (12.56, 12.60) by replacing $(\kappa{+}1)/\mu$ by A of equation (12.69) and R by $R_1R_2/(R_1{+}R_2)$ — i.e.

$$a = \sqrt{\frac{FR_1R_2}{2\pi(R_1+R_2)}\left(\frac{(\kappa_1+1)}{\mu_1} + \frac{(\kappa_2+1)}{\mu_2}\right)} \qquad (12.74)$$

$$p_y(x) = -\frac{2F\sqrt{a^2-x^2}}{\pi a^2} \; . \qquad (12.75)$$

12.8 Combined normal and tangential loading

Tangential tractions can be transmitted between contacting bodies only by means of friction and the complete specification of the problem then requires an assumption about the friction law relating the tangential traction to the relative tangential motion at the interface.

As in the normal contact analysis, tangential relative displacement or *shift*, $h(x)$, can result from rigid-body motion, C, and/or elastic deformation, being given by

$$h(x) = u_{x2}(x,0) - u_{x1}(x,0) + C \; , \tag{12.76}$$

where a positive shift corresponds to displacement of the upper body to the right relative to the lower body.

We shall define a state of *stick* as one in which the time derivative of the shift, $\dot{h}(x)=0$. A state with $\dot{h}(x)\neq 0$ will be referred to as *positive* or *negative slip*, depending on the sign of $\dot{h}(x)$.

The simplest frictional assumption is that usually referred to as Coulomb's law, which, in terms of the above notation, can be defined as

$$\dot{h}(x) = \dot{u}_{x2} - \dot{u}_{x1} + \dot{C} = 0 \;\; ; \;\; fp_y(x) < p_x(x) < -fp_y(x) \tag{12.77}$$

in stick regions and

$$p_x(x) = -fp_y(x)\text{sgn}(\dot{h}(x)) \tag{12.78}$$

in slip regions, where f is a constant known as the *coefficient of friction*[7]. We make no distinction between dynamic and static friction coefficients. In interpreting these equations, the reader should recall that the normal traction $p_y(x)$ is always compressive and hence negative in contact problems.

The function $\text{sgn}(\dot{h}(x))$ in (12.78) ensures that the frictional traction opposes the relative motion and hence dissipates energy. This condition and the inequality in (12.77) serve to determine the division of the contact region into positive slip, negative slip and stick zones in much the same way as the inequalities (12.48, 12.49) determine the contact area in the normal contact problem.

Notice that both of equations (12.77, 12.78) involve time derivatives. Thus, frictional contact problems are *incremental* in nature. It is not generally sufficient to know the final loading condition — we also need to know how that condition was reached. In other words, frictional contact problems are *history-dependent*. They share this and other properties with problems involving another well-known dissipative mechanism — plastic deformation.

[7] The more usual symbol μ for the coefficient of friction would lead to confusion with Lamé's constant.

12.8.1 Cattaneo and Mindlin's problem

Consider the problem of two elastic cylinders which are first pressed together by a normal compressive force F and then subjected to a monotonically increasing tangential force T as shown in Figure 12.13. We restrict attention to the uncoupled case, $\beta=0$.

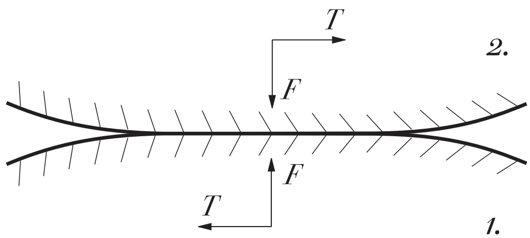

Figure 12.13: Loading for Cattaneo and Mindlin's problem.

The first phase of the loading is described by the analysis of §12.7.1, the contact semi-width a and the normal tractions $p_y(x)$ being defined by equations (12.74, 12.75). The condition $\beta=0$ ensures that these normal tractions produce no tendency for slip (see equations (12.68, 12.72)) and it follows that no tangential tractions are induced and that the whole contact area remains in a state of stick as the normal force F is applied.

The absence of coupling also ensures that the contact area and the normal tractions remain constant during the tangential loading phase. Suppose we first assume that stick prevails everywhere during this phase as well.

Differentiating (12.77) with respect to x and substituting for the displacement derivatives from (12.68) with $B=0$, we obtain

$$-\frac{A}{4\pi}\int_{-a}^{a}\frac{\dot{p}_x(\xi)d\xi}{(x-\xi)} = 0 \;\; ; \;\; -a < x < a \; . \tag{12.79}$$

This equation is identical in form to (12.32) and is solved in the same way. The solution is easily shown to be

$$\dot{p}_x(x) = \frac{\dot{T}}{\pi\sqrt{a^2-x^2}} \; , \tag{12.80}$$

by analogy with (12.47) and hence

$$p_x(x) = \frac{T}{\pi\sqrt{a^2-x^2}} \; , . \tag{12.81}$$

since a is independent of time during the tangential loading phase.

This result shows that the assumption of stick throughout the contact area $-a<x<a$ leads to a singularity in p_x at the edges $x=\pm a$ and hence the frictional inequality (12.77) must be violated there for any f, since p_y is

bounded. We deduce that some slip will occur near the edges of the contact region for any non-zero T, however small.

The problem with slip zones was first solved apparently independently by Cattaneo[8] and Mindlin[9]. In the slip zones, the tractions satisfy the condition $p_x = -fp_y$. We therefore consider the solution for the tangential tractions as the sum of two parts:-

(i) a shear traction

$$p_x(x) = -fp_y(x) = \frac{2fF\sqrt{a^2 - x^2}}{\pi a^2}\ , \tag{12.82}$$

from equation (12.75) *throughout* the contact area $-a < x < a$.

(ii) a *corrective* shear traction p_x^*, which must be zero in the slip zones and which is sufficient to restore the condition (12.77) in the stick zone.

As a first step towards finding the corrective traction p_x^*, we find the shift due to the traction distribution (12.82), which is defined by

$$h'(x) = \frac{d}{dx}(u_{x1} - u_{x2}) = \frac{A}{4\pi}\int_{-a}^{a} \frac{2fF\sqrt{a^2 - \xi^2}d\xi}{\pi a^2(x - \xi)} = -\frac{fFA}{2\pi a^2}\int_0^{\pi} \frac{\sin^2\theta d\theta}{(\cos\phi - \cos\theta)}$$
$$= -\frac{fFA}{2\pi a}\cos\phi = -\frac{fFAx}{2\pi a^2}\ ; \quad -a < x < a\ . \tag{12.83}$$

Now, in the *stick* zone, we require the shift to be independent of x and hence we seek corrective shear tractions in the stick zone that will cancel the right-hand side of equation (12.83). By analogy with equations (12.82, 12.83), it is clear that this cancellation can be achieved by the distribution[10]

$$p_x^*(x) = -\frac{2fF\sqrt{c^2 - x^2}}{\pi a^2}\ \ ; \ \ -c < x < c\ , \tag{12.84}$$

— i.e. a traction similar in form to equation (12.82), but distributed over a smaller centrally located slip zone of width $2c$.

---

[8] C.Cattaneo, Sul contatto di corpi elastici, *Accademia dei Lincei, Rendiconti,* Ser 6, Vol. 27 (1938), pp.342–348, 433–436, 474–478.

[9] R.D.Mindlin, Compliance of elastic bodies in contact, *ASME Journal of Applied Mechanics,* Vol. 17 (1949), pp.259–268.

[10] Notice that the assumption is that all points in $-c < x < c$ are in a state of stick *throughout the loading process.* It is therefore possible to integrate (12.77) and write the boundary condition in terms of $h(x)$ instead of $\dot{h}(x)$. In general, this is possible as long as no point passes from a state of slip to one of stick during the loading. In particular, the stick zone must not *advance* into the slip zone during loading. For an exhaustive study of the effect of loading history in frictional contact problems, see J.Dundurs and M.Comninou, An educational elasticity problem with friction: Part 1, Loading and unloading paths for weak friction, *ASME Journal of Applied Mechanics,* Vol. 48 (1981), pp.841–845; Part 2: Unloading for strong friction and reloading, *ibid.*, Vol. 49 (1982), pp.47–51; Part 3: General load paths, *ibid*, Vol. 50 (1983), pp.77–84.

Thus, the complete shear traction distribution is

$$p_x(x) = \frac{2fF}{\pi a^2}\left\{\sqrt{a^2-x^2} - H(c^2-x^2)\sqrt{c^2-x^2}\right\} \quad ; \quad -a<x<a\ , \qquad (12.85)$$

which is illustrated in Figure 12.14.

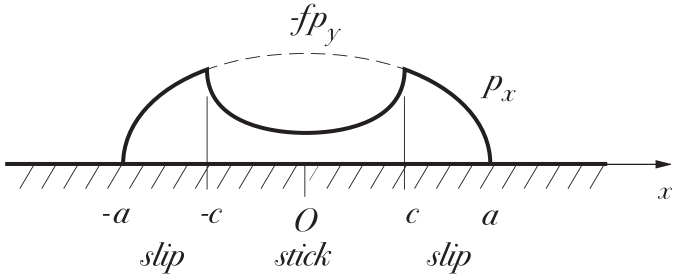

Figure 12.14: Shear traction distribution for Cattaneo and Mindlin's problem.

The stick zone semi-width c can be determined by requiring

$$T = \int_{-a}^{a} p_x(x)dx = fF - fF\left(\frac{c}{a}\right)^2 , \qquad (12.86)$$

from (12.85) and hence

$$c = a\sqrt{1-\frac{T}{fF}} . \qquad (12.87)$$

As we would expect, the stick zone shrinks to zero as the applied tangential force T approaches fF, after which gross slip (i.e. large scale rigid-body motion) occurs.

Finally, we note that it can be shown that the signs of the displacement derivatives satisfy the condition (12.78), provided that T increases monotonically in time (i.e. $\dot{T}>0$ for all t)[11].

[11] The effect of non-monotonic loading in the related problem of two contacting spheres was considered by R.D.Mindlin and H.Deresiewicz, Elastic spheres in contact under varying oblique forces, *ASME Journal of Applied Mechanics,* Vol. 21 (1953), pp.327-344. The history-dependence of the friction law leads to quite complex arrangements of slip and stick zones and consequent variation in the load-compliance relation. These results also find application in the analysis of oblique impact, where neither normal nor tangential loading is monotonic (see N.Maw, J.R.Barber and J.N.Fawcett, The oblique impact of elastic spheres, *Wear,* Vol. 38 (1976), pp.101-114).

The related solution for two spheres in contact (also due to Mindlin) has been verified experimentally by observing the damaged regions produced by cyclic microslip between tangentially loaded spheres[12].

A significant generalization of these results has recently been discovered independently by Jäger[13] and Ciavarella[14], who showed that the frictional traction distribution satisfying both equality and inequality conditions for *any* plane frictional contact problem (not necessarily Hertzian) will consist of a superposition of the limiting friction distribution and an opposing distribution equal to the coefficient of friction multiplied by the normal contact pressure distribution at some smaller value of the normal load. Thus, as the tangential force is increased at constant normal force, the stick zone shrinks, passing monotonically through the same sequence of areas as the normal contact area passed through during the normal loading process. These results can be used to predict the size of the slip zone in conditions of fretting fatigue[15]. One consequence of this result is that wear in the sliding regions due to an oscillating tangential load will not change the extent of the adhesive region, so that in the limit the contact is pure adhesive and a singularity develops in the normal traction at the edge of this region[16].

12.8.2 Steady rolling: Carter's solution

A final example of considerable practical importance is that in which two cylinders roll over each other whilst transmitting a constant tangential force, T. If we assume that the rolling velocity is V, the solution will tend to a steady-state which is invariant with respect to the moving coördinate system

$$\xi = x - Vt \; . \tag{12.88}$$

In this system, we can write the 'stick' condition (12.77) as

$$\dot{h}(x,t) = \frac{d}{dt}\left\{u_{x2}(x-Vt) - u_{x1}(x-Vt) + C\right\} = V\frac{d}{d\xi}(u_{x1} - u_{x2}) + \dot{C} = 0 \; , \tag{12.89}$$

[12] K.L.Johnson, Energy dissipation at spherical surfaces in contact transmitting oscillating forces, *Journal of Mechanical Engineering Science,* Vol. 3 (1961), pp.362–368.

[13] J.Jäger, Half-planes without coupling under contact loading. *Archive of Applied Mechanics* Vol. 67 (1997), pp.247–259.

[14] M.Ciavarella, The generalized Cattaneo partial slip plane contact problem. I-Theory, II-Examples. *International Journal of Solids and Structures.* Vol. 35 (1998), pp.2349–2378.

[15] D.A.Hills and D.Nowell, D. *Mechanics of Fretting Fatigue.* Kluwer, Dordrecht, 1994, M.P.Szolwinski and T.N.Farris, Mechanics of fretting fatigue crack formation. *Wear* Vol. 198 (1996), pp.93–107.

[16] M.Ciavarella and D.A.Hills, Some observations on the oscillating tangential forces and wear in general plane contacts, *European Journal of Mechanics A–Solids.* Vol. 18 (1999), pp.491–497.

where $\dot{C}$ is an arbitrary but constant rigid-body slip (or creep) velocity. Similarly, the slip condition (12.78) becomes

$$p_x(\xi) = -fp_y(\xi)\text{sgn}\left(V\frac{d}{d\xi}(u_{x1}-u_{x2})\right)+\dot{C}\;. \tag{12.90}$$

At first sight, we might think that the Mindlin traction distribution (12.85) satisfies this condition, since it gives

$$\frac{d}{d\xi}(u_{x1}-u_{x2}) = 0 \tag{12.91}$$

in the central stick zone and $\dot{C}$ can be chosen arbitrarily. However, if we substitute the resulting displacements into the slip condition (12.90), we find a sign error in the leading slip zone. This can be explained as follows: In the Mindlin problem, as T is increased, positive slip (i.e. $\dot{h}(x) = \dot{u}_{x2} - \dot{u}_{x1} + \dot{C} > 0$) occurs in both slip zones and the magnitude of $h(x)$ increases from zero at the stick-slip boundary to a maximum at $x = \pm a$. It follows that $\frac{d}{d\xi}(u_{x1}-u_{x2})$ is negative in the right slip zone and positive in the left slip zone. Thus, if this solution is used for the steady rolling problem, a violation of (12.90) will occur in the right zone if V is positive and in the left zone if V is negative. In each case there is a violation in the *leading* slip zone — i.e. in that zone next to the edge where contact is being established.

Now in frictional problems, when we make an assumption that a given region slips and then find that it leads to a sign violation, it is usually an indication that we made the wrong assumption and that the region in question should be in a state of stick. Thus, in the rolling problem, there is no leading slip zone[17]. Carter[18] has shown that the same kind of superposition can be used for the rolling problem as for Mindlin's problem, except that the corrective traction is displaced to a zone adjoining the leading edge. A corrective traction

$$p_x^*(\xi) = \frac{2fF}{\pi a^2}\sqrt{\left(\frac{a-c}{2}\right)^2 - \left(\xi - \frac{a+c}{2}\right)^2} = \frac{2fF}{\pi a^2}\sqrt{(a-\xi)(\xi-c)} \tag{12.92}$$

produces a displacement distribution

$$\frac{d}{d\xi}(u_{x1}-u_{x2}) = \frac{fFA}{2\pi a^2}\left(\xi - \frac{a+c}{2}\right) \quad ; \quad c<\xi<a\;, \tag{12.93}$$

[17] Note that this applies to the uncoupled problem ($\beta = 0$) only. With dissimilar materials, there is generally a leading slip zone and there can also be an additional slip zone contained within the stick zone. This problem is treated by R.H.Bentall and K.L.Johnson, Slip in the rolling contact of dissimilar rollers, *International Journal of Mechanical Sciences,* Vol. 9 (1967), pp.389–404.

[18] F.W.Carter, On the action of a locomotive driving wheel, *Proceedings of the Royal Society of London,* Vol. A112 (1926), pp.151–157.

which cancels the x-varying term in (12.83), leaving only an admissable constant $\dot{C}$ (see equation (12.89)). Hence, the traction distribution for positive V is

$$p_x(\xi) = \frac{2fF}{\pi a^2}\left\{\sqrt{a^2-\xi^2} - H(\xi-c)\sqrt{(a-\xi)(\xi-c)}\right\} \quad ; \quad -a < \xi < a \; , \tag{12.94}$$

which is illustrated in Figure 12.15.

Stick occurs in the leading zone $c<\xi<a$ and positive slip in the trailing zone $-a<\xi<c$.

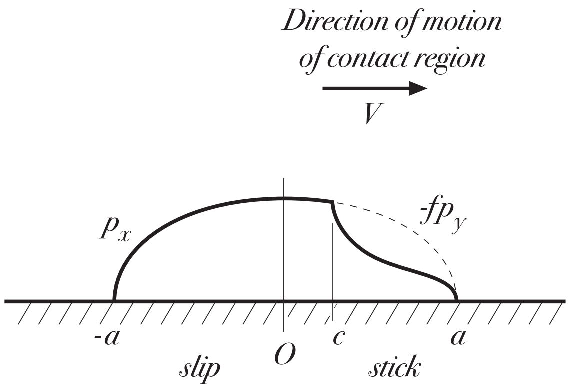

Figure 12.15: Shear traction distribution for Carter's problem.

The corresponding total tangential load is

$$T = fF\left\{1 - \left(\frac{a-c}{2a}\right)^2\right\} \tag{12.95}$$

and hence the stick-slip boundary $\xi=c$ is given by

$$c = a\left(1 - 2\sqrt{1-\frac{T}{fF}}\right) \; . \tag{12.96}$$

An interesting feature of this solution is that the creep velocity $\dot{C}$ is not zero — i.e. there is a small steady-state relative tangential velocity between the two bodies. This has the effect of making the driven roller rotate slightly more slowly than a rigid-body kinematic analysis would lead us to expect. This in turn means that more energy is provided to the driving roller than is recovered from the driven roller, the balance of course being dissipated in the microslip regions in the form of heat. The creep velocity can be calculated by substituting the superposition of (12.83) and (12.93) into (12.89), with the result

$$\dot{C} = -\frac{fVa}{R}\left(1 - \sqrt{1 - \frac{T}{fF}}\right) , \tag{12.97}$$

where R is given by (12.73).

It is interesting to note that Carter's solution was published 20 years before Mindlin's, but there is no evidence that either Mindlin or Cattaneo was aware of it, despite the similarity of the techniques used[19].

PROBLEMS

1. Figure 12.16 shows a disk of radius a subjected to two equal and opposite forces F at the points A, B, the rest of the boundary $r=a$ being traction-free.

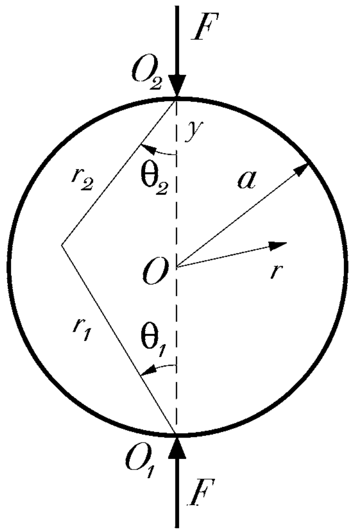

Figure 12.16: Disk loaded by concentrated forces.

The stress function

$$\phi = -\frac{F}{\pi}(r_1\theta_1 \sin\theta_1 + r_2\theta_2 \sin\theta_2)$$

is proposed to account for the localized effect of the forces. Find the stress field due to this function and, in particular, find the tractions implied upon the boundary $r=a$. Then complete the solution by superposing appropriate stress functions from Table 8.1, so as to satisfy the traction-free boundary condition.

[19] For a more extensive discussion of frictional problems of this type, see K.L.Johnson, *loc. cit.*, Chapters 5,7,8.

2. The disk $0 \le r < a$ is accelerated from rest by two concentrated forces F acting in the positive θ-direction at the points $(a, \pm\pi/2)$, as shown in Figure 12.17. Use the Flamant solution in appropriate local coördinates to describe the concentrated forces and the solution of §7.4.1 to describe the inertia forces. Complete the solution by superposing appropriate stress functions from Table 8.1, so as to satisfy the traction-free boundary condition.

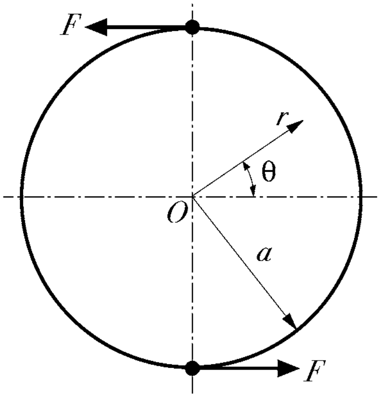

Figure 12.17: Disk accelerated by concentrated tangential forces.

3. Figure 12.18 shows a heavy disk of radius a and density ρ supported by a concentrated force $\pi a^2 \rho g$. Find a solution for the stress field in the disk by combining the Flamant solution with appropriate terms from Table 8.1 in a coordinate system centred on the disk.

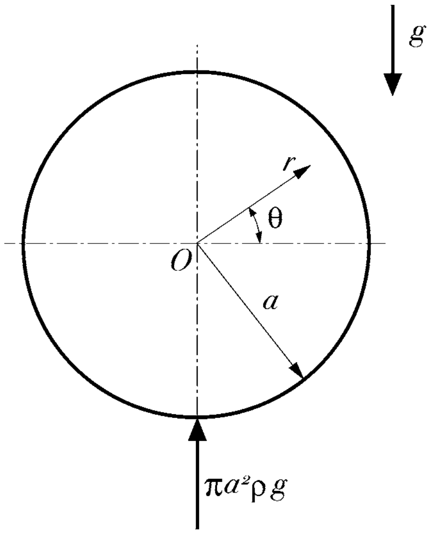

Figure 12.18: Heavy disk supported by a concentrated force.

4. A rigid flat punch of width $2a$ is pressed into an elastic half-plane by a force F whose line of action is displaced a distance b from the centreline, as shown in Figure 12.19.

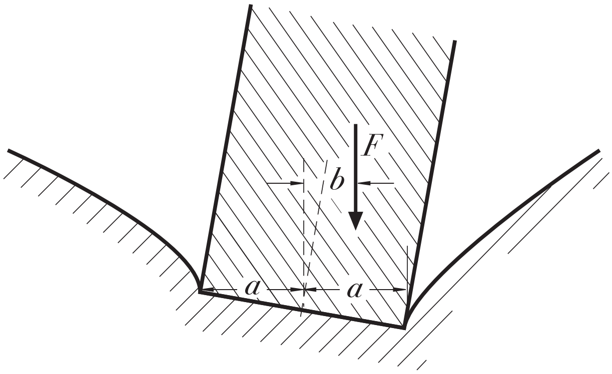

Figure 12.19: Punch with an eccentric load.

(i) Assuming that the punch makes contact over the entire face $-a<x<a$, find the pressure distribution $p(x)$ and the angle of tilt of the punch. Assume the half-plane is prevented from rotating at infinity.
(ii) Hence find the maximum value of b for which there is contact throughout $-a<x<a$.
(iii) Re-solve the problem, assuming that b is larger than the critical value found in (ii). Contact will now occur in the range $c < x < a$ and the unknown left hand end of the contact region ($x=c$) must be found from a smoothness condition on $p(x)$. Express c and $p(x)$ as functions of F, x and b.

5. Two half-planes $y>0$ and $y<0$ of the same material are welded together along the section $-a<x<a$ of their common interface $y=0$. Equal and opposite forces F are now applied at infinity tending to load the weld in tension. Use a symmetry argument to deduce conditions that must be satisfied on the symmetry line $y=0$ and hence determine the tensile stresses transmitted by the weld as a function of x. Find also the *stress intensity factor* K_I, defined as

$$K_I \equiv \lim_{x \to a^-} \sigma_{yy}(x,0)\sqrt{2\pi(a-x)} \;.$$

6. A flat rigid punch is pressed into the surface $y=0$ of the elastic half-plane $y>0$ by a force F. A tangential force T is then applied to the punch. If Coulomb friction conditions apply at the interface with coefficient f and Dundurs constant $\beta=0$, show that no microslip will occur until T reaches the value fF at which point gross slip starts.

7. Express the stress function

$$\phi = \frac{F_y r\theta \cos\theta}{\pi}$$

in Cartesian coördinates and hence find the stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ due to the point force F_y in Figure 12.2.

Use this result and the integration procedure of §12.4 (Figure 12.4) to determine the stress field in the half-plane $y<0$ due to the Hertzian traction distribution of equation (12.60). Make a contour plot of the Von Mises stress σ_E of equation (1.31) and identify the maximum value and its location.

8. From §12.5.2 and equation (12.47), it follows that the contact traction distribution

$$\begin{aligned} p(x,a) &= \frac{1}{\sqrt{a^2-x^2}} \; ; \quad |x|<a \\ &= 0 \qquad\qquad ; \quad |x|>a \end{aligned}$$

produces zero surface slope du_y/dx in $|x|<a$. Use equation (12.27) to determine the corresponding value of slope *outside* the contact area ($|x|>a$) and hence construct the discontinuous function

$$u(x,a) \equiv \frac{du_y}{dx} \; ,$$

such that $p(x,a)$ produces $u(x,a)$.

Linear superposition then shows that the more general traction distribution

$$p(x) = \int_0^b g(a)p(x,a)da$$

produces the surface slope

$$\frac{du_y}{dx} = \int_0^b g(a)u(x,a)da \; ,$$

where $g(a)$ is any function of a. In effect this is a superposition of a range of 'flat punch' traction distributions over different width strips up to a maximum semi-width of b, so the traction will still be zero for $|x|>b$.

Use this representation to solve the problem of the indentation by a wedge of semi-angle $\pi/2-\alpha$ ($\alpha \ll 1$), for which

$$\frac{du_0}{dx} = -|\alpha| \; ; \quad |x|<b \; ,$$

where b is the semi-width of the contact area. In particular, find the contact traction distribution and the relation between b and the applied force F.

Hint: You will find that the boundary condition leads to an Abel integral equation, whose solution is given in Table 30.2 in Chapter 30.

9. Modify the method outlined in Problem 8 to solve the problem of the indentation of a half plane by an unsymmetrical wedge, for which

$$\begin{aligned}\frac{du_0}{dx} &= -\alpha \; ; \quad 0 < x < c \\ &= \beta \; ; \quad -d < x < 0 \; ,\end{aligned}$$

where the unsymmetrical contact area extends from $-d$ to c. In particular, find the contact traction distribution and both c and d as functions of the applied force F.

The modification involves moving the origin to destroy the symmetry once the function $u(x, a)$ has been determined. In other words, instead of superposing a set of symmetrically disposed 'flat punch' distributions, superpose a similar set arranged so that the common point is displaced from the origin.

13

FORCES, DISLOCATIONS AND CRACKS

In this chapter, we shall discuss the applications of two solutions which are singular at an *interior* point of a body, and which can be combined to give the stress field due to a concentrated force (the *Kelvin solution*) and a dislocation. Both solutions involve a singularity in stress with exponent -1 and are therefore inadmissable according to the criterion of §11.2.1. However, like the Flamant solution considered in Chapter 12, they can be used as Green's functions to describe distributions, resulting in convolution integrals in which the singularity is integrated out. The Kelvin solution is also useful for describing the far field (i.e. the field a long way away from the loaded region) due to a force distributed over a small region.

13.1 The Kelvin solution

We consider the problem in which a concentrated force, F acts in the x-direction at the origin in an infinite body. This is not a perturbation problem like the stress field due to a hole in an otherwise uniform stress field, since the force has a non-zero resultant. Thus, no matter how far distant we make the boundary of the body, there will have to be some traction to oppose the force. In fact, self-similarity arguments like those used in §§12.1, 12.2 show that the stress field must decay with r^{-1}.

The Flamant solution has this behaviour and it corresponds to a concentrated force (see §12.2), but it cannot be used at an interior point in the body, since the corresponding displacements (equations (12.12, 12.13)) are multivalued[1]. However, we can construct a solution with the same character and with single-valued displacements from the more general stress function (12.2) by choosing the coefficients in such a way that the multivalued terms cancel.

[1] This was not a problem for the surface loading problem, since the wedge of Figure 12.1 only occupies a part of the θ-domain and hence a suitable principal value of θ can be chosen to be both single-valued and continuous.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_13,

In view of the symmetry of the problem about $\theta=0$, we restrict attention to the symmetric terms

$$\phi = C_1 r\theta \sin\theta + C_3 r \ln(r) \cos\theta \tag{13.1}$$

of (12.2), for which the stress components are

$$\sigma_{rr} = \frac{2C_1 \cos\theta}{r} + \frac{C_3 \cos\theta}{r} \tag{13.2}$$

$$\sigma_{r\theta} = \frac{C_3 \sin\theta}{r} \tag{13.3}$$

$$\sigma_{\theta\theta} = \frac{C_3 \cos\theta}{r} \;, \tag{13.4}$$

from Table 8.1 and the displacement components are

$$2\mu u_r = \frac{C_1}{2}\{(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\ln r\cos\theta\} + \frac{C_3}{2}\{(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\ln(r)\cos\theta\} \tag{13.5}$$

$$2\mu u_\theta = \frac{C_1}{2}\{(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\ln r\sin\theta\} + \frac{C_3}{2}\{(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\ln(r)\sin\theta\} \;, \tag{13.6}$$

from Table 9.1.

Suppose we make an imaginary cut in the plane at $\theta = 0, 2\pi$ and define a principal value of θ such that $0 \le \theta < 2\pi$. This makes θ discontinuous at $\theta=2\pi$, but the trigonometric functions of course remain continuous. The only potential difficulty is associated with the expressions $\theta\sin\theta, \theta\cos\theta$.

Now $\theta\sin\theta=0$ at $\theta=2\pi$ and hence this expression has the same value at the two sides of the cut and is continuous. We can therefore make the whole displacement field continuous by choosing C_1, C_3 such that the terms $\theta\cos\theta$ in u_θ cancel — i.e. by setting[2]

$$C_1(\kappa-1) + C_3(\kappa+1) = 0 \;, \tag{13.7}$$

which for plane stress is equivalent to

$$(1-\nu)C_1 + 2C_3 = 0 \;. \tag{13.8}$$

This leaves us with one degree of freedom (one free constant) to satisfy the condition that the force at the origin is equal to F. Considering the equilibrium of a small circle of radius r surrounding the origin, we have

[2] Notice that this choice also has the effect of cancelling the $\theta\sin\theta$ terms in (13.5), so that the complete displacement field, like the stress field, depends on θ only through sine and cosine terms.

$$F + \int_0^{2\pi} (\sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta) r d\theta = 0 \; . \tag{13.9}$$

We now substitute for the stress components from equations (13.2, 13.3), obtaining

$$C_1 = -\frac{F}{2\pi} \; , \tag{13.10}$$

after which we recover the constant C_3 from equation (13.8) as

$$C_3 = \frac{(1-\nu)F}{4\pi} \; . \tag{13.11}$$

Finally, we substitute these constants back into (13.2–13.4) to obtain

$$\sigma_{rr} = -\frac{(3+\nu)F\cos\theta}{4\pi r} \tag{13.12}$$

$$\sigma_{r\theta} = \frac{(1-\nu)F\sin\theta}{4\pi r} \tag{13.13}$$

$$\sigma_{\theta\theta} = \frac{(1-\nu)F\cos\theta}{4\pi r} \; . \tag{13.14}$$

These are the stress components for Kelvin's problem, where the force acts in the x-direction. The corresponding results for a force in the y-direction can be obtained in the same way, using the antisymmetric terms in (12.2). Alternatively, we can simply rotate the axis system in the above solution by redefining $\theta \to (\theta - \pi/2)$.

13.1.1 Body force problems

Kelvin's problem is a special case of a body force problem — that in which the body force is a delta function at the origin. The solution can also be used to solve more general body force problems by convolution. We consider the body force $p_x \delta x \delta y$ acting on the element $\delta x \delta y$ as a concentrated point force and use the above solution to determine its effect on the stress components at an arbitrary point. Treating the component p_y in the same way and summing over all the elements of the body then gives a double integral representation of the stress field.

This method is not restricted to the infinite body, since we only seek a particular solution of the body force problem. We can therefore use the convolution method to develop a solution for the stresses in an infinite body with the appropriate body force distribution, after which we 'cut out' the shape of the real body and correct the boundary conditions as required, using an appropriate homogeneous solution (i.e. a solution without body forces).

Any body force distribution can be treated this way — the method is not restricted to conservative vector fields. It is generally more algebraically tedious than the methods developed in Chapter 7, but it lends itself naturally to numerical implementation. For example, it can be used to extend the boundary integral method to body force problems.

13.2 Dislocations

The term *dislocation* has related, but slightly different meanings in Elasticity and in Materials Science. In Elasticity, the material is assumed to be a continuum — i.e. to be infinitely divisible. Suppose we take an infinite continuous body and make a cut along the half-plane $x>0,\ y=0$. We next apply equal and opposite tractions to the two surfaces of the cut such as to open up a gap of constant thickness, as illustrated in Figure 13.1. We then slip a thin slice of the same material into the space to keep the surfaces apart and weld the system up, leaving a new continuous body which will now be in a state of residual stress.

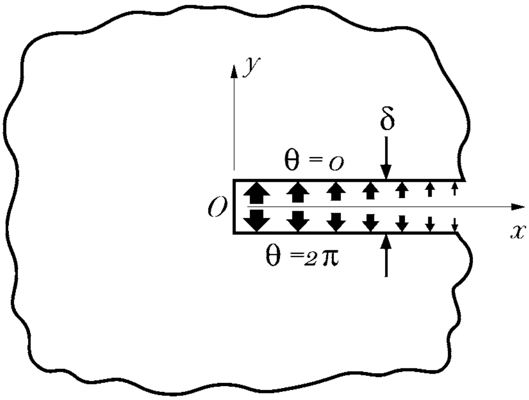

Figure 13.1: The climb dislocation solution.

The resulting stress field is referred to as the *climb dislocation* solution. It can be obtained from the stress function of equation (13.1) by requiring that there be no net force at the origin, with the result $C_1=0$ (see equation 13.10). We therefore have

$$\phi = C_3 r \ln(r) \cos\theta \ . \tag{13.15}$$

The strength of the dislocation can be defined in terms of the discontinuity in the displacement u_θ on $\theta=0, 2\pi$, which is also the thickness of the slice of extra material which must be inserted to restore continuity of material. This thickness is

$$\delta = u_\theta(0) - u_\theta(2\pi) = -\frac{\pi(\kappa+1)C_3}{2\mu} \ . \tag{13.16}$$

Thus, we can define a climb dislocation of strength B_y as one which opens a gap $\delta=B_y$, corresponding to

$$C_3 = -\frac{2\mu B_y}{\pi(\kappa+1)} \ , \tag{13.17}$$

where

$$\frac{2\mu}{(\kappa+1)} = \frac{\mu(1+\nu)}{2} = \frac{E}{4} \qquad \text{(plane stress)}$$
$$= \frac{\mu}{2(1-\nu)} = \frac{E}{4(1-\nu^2)} \qquad \text{(plane strain)}, \tag{13.18}$$

using equation (3.20).

The stress field due to the climb dislocation is

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{2\mu B_y \cos\theta}{\pi(\kappa+1)r} \tag{13.19}$$
$$\sigma_{r\theta} = -\frac{2\mu B_y \sin\theta}{\pi(\kappa+1)r} \; . \tag{13.20}$$

We also record the stress components at $y=0$, — i.e. $\theta = 0, \pi$ — in rectangular coördinates, which are

$$\sigma_{xx} = \sigma_{yy} = -\frac{2\mu B_y}{\pi(\kappa+1)x} \tag{13.21}$$
$$\sigma_{yx} = 0 \; . \tag{13.22}$$

This solution is called a climb dislocation, because it opens a gap on the cut at $\theta = 0, 2\pi$. A corresponding solution can be obtained from the stress function

$$\phi = \frac{2\mu B_x r \ln(r) \sin\theta}{\pi(\kappa+1)} \tag{13.23}$$

which is discontinuous in the displacement component u_r — i.e. for which the two surfaces of the cut experience a relative tangential displacement but do not separate. This is called a *glide dislocation*. The stress field due to a glide dislocation is given by

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{2\mu B_x \sin\theta}{\pi(\kappa+1)r} \tag{13.24}$$
$$\sigma_{r\theta} = -\frac{2\mu B_x \cos\theta}{\pi(\kappa+1)r} \tag{13.25}$$

and on the surface $y=0$, these reduce to

$$\sigma_{xx} = \sigma_{yy} = 0 \tag{13.26}$$
$$\sigma_{yx} = -\frac{2\mu B_x}{\pi(\kappa+1)x} \; , \tag{13.27}$$

where $B_x = u_r(0) - u_r(2\pi)$ is the strength of the dislocation.

The solutions (13.19, 13.20) and (13.24, 13.25) actually differ only in orientation. The climb dislocation solution defines a glide dislocation if we choose to make the cut along the line $\theta = -3\pi/2, \pi/2$ (i.e. along the y-axis instead of the x-axis[3]). The dislocation strengths B_x, B_y can be regarded as the components of a vector, known as the *Burgers vector*.

[3] In fact, the cut can be made along any (not necessarily straight) line from the origin to infinity. The solution of equations (13.19, 13.20) will then exhibit a

13.2.1 Dislocations in Materials Science

Real materials have a discrete atomic or molecular structure. However, we can follow a similar procedure by imagining cleaving the solid between two sheets of molecules up to the line $x = y = 0$ and inserting *one extra layer* of molecules. When the system is released, there will be some motion of the molecules, mostly concentrated at the end of the added layer, resulting in an imperfection in the regular molecular array. This is what is meant by a dislocation in Materials Science.

Considerable insight into the role of dislocations in material behaviour was gained by the work of Bragg[4] with bubble models. Bragg devised a method of generating a two-dimensional collection of identical size bubbles. Attractive forces between the bubbles ensured that they adopted a regular array wherever possible, but dislocations are identifiable as can be seen in Figure 13.2.

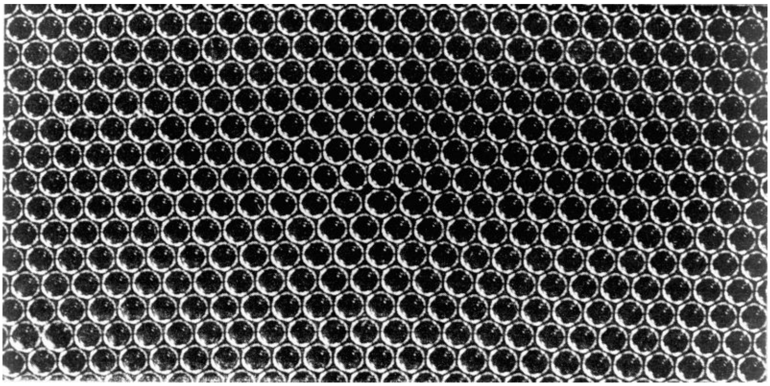

Figure 13.2: The bubble model — a dislocation[5].

When forces are applied to the edges of the bubble assembly, the dislocations are found to move in such a way as to permit the boundaries to move. This dislocation motion is believed to be the principal mechanism of plastic deformation in ductile materials. Larger bubble assemblies exhibit discrete regions in which the arrays are differently aligned as in Figure 13.3. These are analogous with grains in multigranular materials. When the structure is deformed, the dislocations typically move until they reach a grain boundary, but the misalignment prohibits further motion and the stiffness of the assembly

discontinuity in u_y of magnitude B_y at all points along the cut, as long as the principal value of θ is appropriately defined.

[4] L.Bragg and J.F.Nye, A dynamical model of a crystal structure, *Proceedings of the Royal Society of London,* Vol. A190 (1947), pp.474–481.

[5] Figures 13.2 and 13.3 are reproduced by kind permission of the Royal Society of London.

increases. This pile-up of dislocations at grain boundaries is responsible for work-hardening in ductile materials. Also, the accumulated dislocations coalesce into larger disturbances in the crystal structure such as voids or cracks, which function as initiation points for failure by fatigue or fracture.

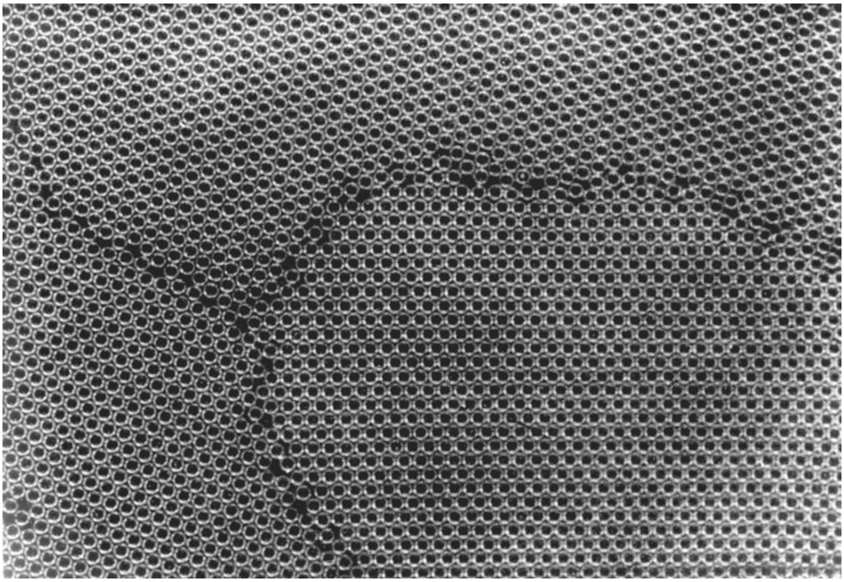

Figure 13.3: The bubble model — grain boundaries.

13.2.2 Similarities and differences

It is tempting to deduce that the elastic solution of §13.2 describes the stresses due to the molecular structure dislocation, if we multiply by a constant defining the thickness of a single layer of molecules. However, the concept of stress is rather vague over dimensions comparable with interatomic distances. In fact, this is preëminently a case where the apparent singularity of the mathematical solution is moderated in reality by the discrete structure of the material.

Notice also that the continuum dislocation of §13.2 can have any strength, corresponding to the fact that B_x, B_y are arbitrary real constants. By contrast, the thickness of the inserted layer is restricted to one layer of molecules in the discrete theory. This thickness is sufficiently small to ensure that the 'stresses' due to a material dislocation are very small in comparison with typical engineering magnitudes, except in the immediate vicinity of the defect. Furthermore, although a stress-relieved metallic component will generally contain numerous dislocations, they will be randomly oriented, leaving the components essentially stress-free on the macroscopic scale.

However, if the component is plastically deformed, the dislocation motion will lead to a more systematic structure, which can result in residual stress. Indeed, one way to represent the stress field due to plastic deformation is as a distribution of mathematical dislocations. For example, the motion of a single dislocation can be represented by introducing a dislocation pair comprising a negative dislocation at the original location and an equal positive dislocation at the final location. This implies a *closure condition* as in §§13.2.3, 13.2.4 below.

13.2.3 Dislocations as Green's functions

In the Theory of Elasticity, the principal use of the dislocation solution is as a Green's function to represent localized processes. Suppose we place a distribution of dislocations in some interior domain Ω of the body, which is completely surrounded by elastic material. The resulting stress field obtained by integration will satisfy the conditions of equilibrium everywhere (since the dislocation solution involves no force) and will satisfy the compatibility condition everywhere *except* in Ω. There is of course the possibility that the displacement may be multiple-valued outside Ω, but we can prevent this by enforcing the two *closure conditions*

$$\iint_\Omega B_x(x,y)dxdy = 0 \;\; ; \;\; \iint_\Omega B_y(x,y)dxdy = 0 \; , \tag{13.28}$$

which state that the total strength of the dislocations in Ω is zero[6].

13.2.4 Stress concentrations

A special case of some importance is that in which the enclosed domain Ω represents a hole which perturbs the stress field in an elastic body.

It may seem strange to place dislocations in a region which is strictly not a part of the body. However, we might start with an infinite body with no hole, place dislocations in Ω generating a stress field and then make a cut along the boundary of Ω producing the body with a hole. The stress field will be unchanged by the cut provided we place tractions on its boundary equal to those which were transmitted across the same surface in the original continuous body. In particular, if we choose the dislocation distribution so as to make the boundaries of Ω traction-free, we can cut out the hole without changing the stress distribution, which is therefore the solution of the original problem for the body with a hole.

The general idea of developing perturbation solutions by placing singularities in a region where the governing equations (here the compatibility condition) are not required to be enforced is well-known in many branches of

[6] This does *not* mean that the stress field is null, since the various self-cancelling dislocations have different locations.

Applied Mechanics. For example, the solution for the flow of a fluid around a rigid body can be developed in many cases by placing an appropriate distribution of sources and sinks in the region occupied by the body, the distribution being chosen so as to make the velocity component normal to the body surface be everywhere zero.

The closure conditions (13.28) ensure that acceptable dislocation distributions can be represented in terms of dislocation pairs — i.e. as matched pairs of dislocations of equal magnitude and opposite direction. It follows that acceptable distributions can be represented as distributions of *dislocation derivatives*, since, for example, a dislocation at P and an equal negative dislocation at Q is equivalent to a uniform distribution of derivatives on the straight line joining P and Q[7].

Now the stress components in the dislocation solution (equations (13.19, 13.20)) decay with r^{-1} and hence those in the *dislocation derivative* solution will decay with r^{-2}. It therefore follows that the dominant term in the perturbation (or corrective) solution due to a hole will decay at large r with r^{-2}. We see this in the particular case of the circular hole in a uniform stress field (§§8.3.2, 8.4.1). The same conclusion follows for the perturbation in the stress field due to an inclusion — i.e. a localized region whose properties differ from those of the bulk material.

13.3 Crack problems

Crack problems are particularly important in Elasticity because of their relevance to the subject of Fracture Mechanics, which broadly speaking is the study of the stress conditions under which cracks grow. For our purposes, a crack will be defined as the limiting case of a hole whose volume (at least in the unloaded case) has shrunk to zero, so that opposite faces touch. It might also be thought of as an interior surface in the body which is incapable of transmitting tension.

In practice, cracks will generally have some small thickness, but if this is small, the crack will behave unilaterally with respect to tension and compression — i.e. it will open if we try to transmit tension, but close in compression, transmitting the tractions by means of contact. For this reason, a cracked body will appear stiffer in compression than it does in tension. Also, the crack acts as a stress concentration in tension, but not in compression, so cracks do not generally propagate in compressive stress fields.

13.3.1 Linear Elastic Fracture Mechanics

We saw in §11.2.3 that the asymptotic stress field at the tip of a crack has a square-root singularity and we shall find this exemplified in the particular

[7] To see this, think of the derivative as the limit of a pair of equal and opposite dislocations separated by a distance δS whose magnitude is proportional to $1/\delta S$.

solutions that follow. The simplest and most prevalent theory of brittle fracture — that due to Griffith — states in essence that crack propagation will occur when the scalar multiplier on this singular stress field exceeds a certain critical value. More precisely, Griffith proposed the thesis that a crack would propagate when propagation caused a reduction in the total energy of the system. Crack propagation causes a reduction in strain energy in the body, but also generates new surfaces which have *surface energy*. Surface energy is related to the force known as surface tension in fluids and follows from the fact that to cleave a solid body along a plane involves doing work against the interatomic forces across the plane. When this criterion is applied to the stress field in particular cases, it turns out that for a small change in crack length, propagation is predicted when the multiplier on the singular term, known as the *stress intensity factor*, exceeds a certain critical value, which is a constant for the material known as the *fracture toughness*.

It may seem paradoxical to found a theory of real material behaviour on properties of a singular elastic field, which clearly cannot accurately represent conditions in the precise region where the failure is actually to occur. However, if the material is brittle, non-linear effects will be concentrated in a relatively small *process zone* surrounding the crack tip. Furthermore, the certainly very complicated conditions in this process zone can only be influenced by the surrounding elastic material and hence the conditions for failure must be expressible in terms of the characteristics of the much simpler surrounding elastic field. As long as the process zone is small compared with the other linear dimensions of the body[8] (notably the crack length), it will have only a very localized effect on the surrounding elastic field, which will therefore be adequately characterized by the dominant singular term in the linear elastic solution, whose multiplier (the stress intensity factor) then determines the conditions for crack propagation.

It is notable that this argument requires no assumption about or knowledge of the actual mechanism of failure in the process zone and, by the same token, the success of Linear Elastic Fracture Mechanics (LEFM) as a predictor of the strength of brittle components provides no evidence for or against any particular failure theory[9].

[8] This is often referred to as the *small scale yielding condition*.

[9] For more details of the extensive development of the field of Fracture Mechanics, the reader is referred to the many excellent texts on the subject, such as M.F.Kanninen and C.H.Popelar, *Advanced Fracture Mechanics*, Clarendon Press, Oxford, 1985, H.Leibowitz, **ed.**, *Fracture, An Advanced Treatise*, 7 Vols., Academic Press, New York, 1971. Stress intensity factors for a wide range of geometries are tabulated by G.C.Sih, *Handbook of Stress Intensity Factors*, Institute of Fracture and Solid Mechanics, Lehigh University, Bethlehem, PA, 1973.

13.3.2 Plane crack in a tensile field

Two-dimensional crack problems are very conveniently formulated using the methods outlined in §§13.2.3, 13.2.4. Thus, we seek a distribution of dislocations on the plane of the crack (which appears as a line in two-dimensions), which, when superposed on the unperturbed stress field, will make the surfaces of the crack traction-free. We shall illustrate the method for the simple case of a plane crack in a tensile stress field.

Figure 13.4 shows a plane crack of width $2a$ occupying the region $-a < x < a,\ y=0$ in a two-dimensional body subjected to uniform tension $\sigma_{yy}=S$ at its remote boundaries.

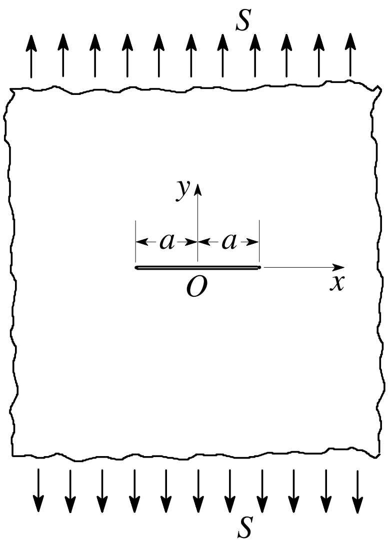

Figure 13.4: Plane crack in a tensile field.

Assuming that the crack opens, the boundary conditions for this problem can be stated in the form

$$\sigma_{yx} = \sigma_{yy} = 0 \ ; \ \ -a < x < a,\ y = 0 \ ; \tag{13.29}$$

$$\sigma_{yy} \rightarrow S \ ; \ \ \sigma_{xy}, \sigma_{xx} \rightarrow 0 \ ; \ \ r \rightarrow \infty \ . \tag{13.30}$$

Following the procedure of §8.3.2, we represent the solution as the sum of the stress field in the corresponding body without a crack — here a uniform uniaxial tension $\sigma_{yy}=S$ — and a corrective solution, for which the boundary conditions are therefore

$$\sigma_{yx} = 0 \ ; \ \ \sigma_{yy} = -S \ ; \ \ -a < x < a,\ y = 0 \ ; \tag{13.31}$$

$$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \rightarrow 0 \ ; \ \ r \rightarrow \infty \ . \tag{13.32}$$

Notice that the corrective solution corresponds to the problem of a crack in an otherwise stress-free body opened by compressive normal tractions of magnitude S.

The most general stress field due to the crack would involve both climb and glide dislocations, but in view of the symmetry of the problem about the plane $y=0$, we conclude that there is no relative tangential motion between the crack faces and hence that the solution can be constructed with a distribution of climb dislocations alone. More precisely, we respresent the solution in terms of a distribution $B_y(x)$ of dislocations per unit length in the range $-a<x<a,\ y=0$.

We consider first the traction σ_{yy} at the point $(x,0)$ due to those dislocations between ξ and $\xi+\delta\xi$ on the line $y=0$. If $\delta\xi$ is small, they can be considered as a concentrated dislocation of strength $B_y(\xi)\delta\xi$ and hence they produce a traction

$$\sigma_{yy} = -\frac{2\mu B_y(\xi)\delta\xi}{\pi(\kappa+1)(x-\xi)} \ , \tag{13.33}$$

from equation (13.21), since the distance from $(\xi,0)$ to $(x,0)$ is $(x-\xi)$.

The traction due to the whole distribution of dislocations can therefore be written as the integral

$$\sigma_{yy} = -\frac{2\mu}{\pi(\kappa+1)}\int_{-a}^{a}\frac{B_y(\xi)d\xi}{(x-\xi)} \tag{13.34}$$

and the boundary condition (13.31) leads to the following Cauchy singular integral equation for $B_y(\xi)$

$$\int_{-a}^{a}\frac{B_y(\xi)d\xi}{(x-\xi)} = \frac{\pi(\kappa+1)S}{2\mu} \quad ; \quad -a<x<a \ . \tag{13.35}$$

This is of exactly the same form as equation (12.32) and can be solved in the same way. Writing

$$x = a\cos\phi \quad ; \quad \xi = a\cos\theta \ , \tag{13.36}$$

we have

$$\int_0^{\pi}\frac{B_y(\theta)\sin\theta d\theta}{(\cos\phi-\cos\theta)} = \frac{\pi(\kappa+1)S}{2\mu} \quad ; \quad 0<\phi<\pi \ . \tag{13.37}$$

Now (12.35) with $n=1$ gives

$$\int_0^{\pi}\frac{\cos\theta d\theta}{(\cos\phi-\cos\theta)} = -\pi \quad ; \quad 0<\phi<\pi \tag{13.38}$$

and hence

$$B_y(\theta) = -\frac{(\kappa+1)S\cos\theta}{2\mu\sin\theta} + \frac{A}{\sin\theta} \ , \tag{13.39}$$

— i.e.

$$B_y(\xi) = -\frac{(\kappa+1)S\xi}{2\mu\sqrt{a^2-\xi^2}} + \frac{Aa}{\sqrt{a^2-\xi^2}} \ . \tag{13.40}$$

The arbitrary constant A is determined from the closure condition (13.28) which here takes the form

$$\int_{-a}^{a} B_y(\xi)d\xi = 0 \tag{13.41}$$

and leads to the result $A=0$.

From a Fracture Mechanics perspective, we are particularly interested in the stress field surrounding the crack tip. For example, the stress component σ_{yy} in $|x|>a,\ y=0$ is given by

$$\sigma_{yy} = -\frac{2\mu}{\pi(\kappa+1)}\int_{-a}^{a}\frac{B_y(\xi)d\xi}{(x-\xi)} \tag{13.42}$$

$$= \frac{S}{\pi}\int_{-a}^{a}\frac{\xi d\xi}{(x-\xi)\sqrt{a^2-\xi^2}}$$

$$= S\left(-1+\frac{|x|}{\sqrt{x^2-a^2}}\right) \quad ; \quad |x|>a,\ y=0 \tag{13.43}$$

using (13.40) and 3.228.2 of Gradshteyn and Ryzhik[10].

Remembering that this is the *corrective* solution, we add the uniform stress field $\sigma_{yy}=S$ to obtain the complete stress field, which on the line $y=0$ gives

$$\sigma_{yy} = \frac{S|x|}{\sqrt{x^2-a^2}} \quad ; \quad |x|>a \ . \tag{13.44}$$

This tends to the uniform field as it should as $x\rightarrow\infty$ and is singular as $x\rightarrow a^+$. We define[11] the *mode I stress intensity factor*, K_I as

$$K_I \equiv \lim_{x\rightarrow a^+}\sigma_{yy}(x)\sqrt{2\pi(x-a)} \tag{13.45}$$

$$= \lim_{x\rightarrow a^+}\frac{Sx\sqrt{2\pi(x-a)}}{\sqrt{x^2-a^2}}$$

$$= S\sqrt{\pi a} \ . \tag{13.46}$$

We can also calculate the crack opening displacement

$$\delta(x) \equiv u_y(x,0^+) - u_y(x,0^-) = \int_{-a}^{x} B_y(\xi)d\xi = \frac{(\kappa+1)S}{2\mu}\sqrt{a^2-x^2} \ . \tag{13.47}$$

Thus, the crack is opened to the shape of a long narrow ellipse as a result of the tensile field.

[10] I.S.Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980.

[11] Notice that the factor 2π in this definition is conventional, but essentially arbitrary. In applying fracture mechanics arguments, it is important to make sure that the results used for the stress intensity factor (a theoretical or numerical calculation) and the fracture toughness (from experimental data) are based on the same definition of K.

13.3.3 Energy release rate

If a crack extends as a result of the local stress field, there will generally be a reduction in the total strain energy U of the body. We shall show that the *strain energy release rate*, defined as

$$G = -\frac{\partial U}{\partial S} \tag{13.48}$$

where S is the extent of the crack, is a unique function of the stress intensity factor. We first note that in the immediate vicinity of the crack tip $x = a$, the stress component σ_{yy} can be approximated by the dominant singular term as

$$\sigma_{yy}(s) = \frac{K_I}{\sqrt{2\pi s}} \; ; \quad s > 0 \; , \tag{13.49}$$

from equation (13.45) with $s = (x - a)$. A similar approximation for the crack opening displacement $\delta(s)$ is

$$\delta(s) = \frac{S(\kappa+1)}{2\mu}\sqrt{2a(-s)} = \frac{K_I(\kappa+1)}{2\mu}\sqrt{\frac{2(-s)}{\pi}} \; , \tag{13.50}$$

using (13.46, 13.47).

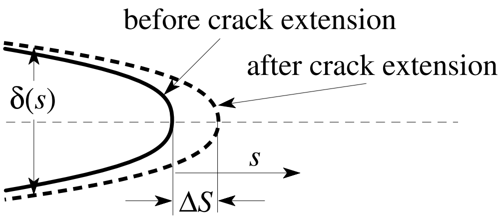

Figure 13.5: Geometry of crack extension.

Figure 13.5 shows the configuration of the open crack before and after extension of the crack tip by an infinitesimal distance ΔS. The tractions on the crack plane before crack extension are given by (13.49), where s is measured from the initial position of the crack tip. The crack opening displacement in ΔS *after* extension is given by

$$\delta(s) = \frac{K_I(\kappa+1)}{2\mu}\sqrt{\frac{2(\Delta S - s)}{\pi}} \; , \tag{13.51}$$

since the point $s < \Delta S$ is now a distance ΔS–s to the left of the new crack tip. The reduction in strain energy during crack extension can be found by following a scenario in which the tractions (13.49) are gradually released, allowing work $W = -\Delta U$ to be done on them in moving through the displacements (13.51). We obtain

$$W = \frac{1}{2}\int_0^{\Delta S} \sigma_{yy}(s)\delta(s)ds = \frac{K_I^2(\kappa+1)}{4\pi\mu}\int_0^{\Delta S}\sqrt{\frac{(\Delta S - s)}{s}}ds = \frac{K_I^2 \Delta S(\kappa+1)}{8\mu} \tag{13.52}$$

and hence

$$G = -\frac{\partial U}{\partial S} = \frac{\partial W}{\partial S} = \frac{K_I^2(\kappa+1)}{8\mu} \; . \tag{13.53}$$

A similar calculation can be performed for a crack loaded in shear, causing a mode II stress intensity factor K_{II}. The two deformation modes are orthogonal to each other, so that the energy release rate for a crack loaded in both modes I and II is given by

$$G = \frac{(K_I^2 + K_{II}^2)(\kappa+1)}{8\mu} \; . \tag{13.54}$$

This expression can be written in terms of E, ν, using (13.18). We obtain

$$G = \frac{(K_I^2 + K_{II}^2)}{E} \qquad \text{(plane stress)} \tag{13.55}$$

$$= \frac{(K_I^2 + K_{II}^2)(1-\nu^2)}{E} \qquad \text{(plane strain).} \tag{13.56}$$

As long as the process zone is small in the sense defined in §13.3.1, a component containing a crack loaded in tension (mode I) will fracture when

$$K_I = K_{Ic} \quad \text{or} \quad G = G_c \; ,$$

where K_{Ic}, G_c are interrelated material properties, the former being known as the fracture toughness. The critical energy release rate G_c seems to imply a single failure criterion under combined mode I and mode II loading, but this is illusory, since experiments show that the value of G_c varies with the mode mixity ratio K_{II}/K_I.

13.4 Method of images

The method described in §13.3.2 applies strictly to the case of a crack in an infinite body, but it will provide a reasonable approximation if the length of the crack is small in comparison with the shortest distance to the boundary of a finite body or to some other geometric feature such as another crack or an interface to a different material. However, the same methodology could be applied to other problems if we could obtain the solution for a dislocation located at an arbitrary point in the uncracked body.

Closed-form solutions exist for bodies of a variety of shapes, including the traction-free half plane, the infinite body with a traction-free circular hole and the infinite body containing a circular inclusion of a different elastic material[12]. These solutions all depend on the location of image singularities at appropriate points outside the body. The nature and strength of these singularities can be chosen so as to satisfy the required boundary conditions on the boundary of the original body. Earlier treatments of the subject proceeded in an essentially *ad hoc* way until the required boundary conditions were satisfied, but a unified treatment for the case of a plane interface was developed by Aderogba[13].

We consider the case in which the half plane $y>0$ with elastic constants μ_1, κ_1 is bonded to the half plane $y<0$ with constants μ_2, κ_2. Suppose that a singularity (typically a dislocation or a concentrated force) exists somewhere in $y>0$ and that the same singularity in an infinite plane would correspond to the Airy stress function $\phi_0(x,y)$ defined throughout the infinite plane. Aderogba showed that the resulting stress field in the bonded bi-material plane is defined by the stress functions

$$\begin{aligned} \phi_1(x,y) &= \phi_0(x,y) + \mathcal{L}\{\phi_0(x,-y)\} && y>0 \\ \phi_2(x,y) &= \mathcal{J}\{\phi_0(x,y)\} && y<0\ , \end{aligned} \tag{13.57}$$

where the linear operators $\mathcal{L}, \mathcal{J}$ are defined by

$$\mathcal{L}\{\cdot\} = A\left[1 - 2y\frac{\partial}{\partial y} + y^2\nabla^2\right] + \frac{(A-B)}{4}\iint \nabla^2\{\cdot\}dydy$$

$$\mathcal{J}\{\cdot\} = 1 + A + \frac{(A-B)}{4}\left[\iint \nabla^2\{\cdot\}dydy - 2y\int \nabla^2\{\cdot\}dy\right]\ ,$$

with

$$A = \frac{(\Gamma-1)}{(\Gamma\kappa_1+1)}\ ; \quad B = \frac{(\Gamma\kappa_1-\kappa_2)}{(\Gamma+\kappa_2)}\ ; \quad \Gamma = \frac{\mu_2}{\mu_1}\ . \tag{13.58}$$

These bimaterial constants are related to Dundurs' constants (4.12, 4.13) through the equations

$$A = \frac{\alpha-\beta}{1+\beta}\ ; \quad B = \frac{\alpha+\beta}{1-\beta}\ .$$

Notice that the arbitrary functions of x implied in the indefinite integrals with respect to y in (13.57) must be chosen so as to ensure that the resulting stress functions ϕ_1, ϕ_2 are biharmonic.

The special case of the traction-free half plane can be recovered in the limit $\Gamma \to 0$ and hence $A, B \to -1$, giving

$$\mathcal{L}\{\cdot\} = -1 + 2y\frac{\partial}{\partial y} - y^2\nabla^2\ . \tag{13.59}$$

[12] See for example J.Dundurs, Concentrated force in an elastically embedded disk, *ASME Journal of Applied Mechanics,* Vol.30 (1963), pp.568–570, J.Dundurs and G.P.Sendeckyj, Edge dislocation inside a circular inclusion, *Journal of the Mechanics and Physics of Solids*, Vol.13 (1965), pp.141–147.

[13] K.Aderogba, An image treatment pf elastostatic transmission from an interface layer, *Journal of the Mechanics and Physics of Solids*, Vol. 51 (2003), pp.267–279.

Example

As an example, we consider the case where a glide dislocation is situated at the point $(0,a)$ in the traction-free half plane $y>0$. The corresponding infinite plane solution is given by equation (13.23) as

$$\phi_0 = \frac{2\mu B_x r \ln(r)\sin\theta}{\pi(\kappa+1)} \ ,$$

where r,θ are measured from the dislocation. Converting to Cartesian coördinates and moving the origin to the surface of the half plane, we have

$$\phi_0(x,y) = \frac{\mu B_x(y-a)}{\pi(\kappa+1)} \ln\left(x^2+(y-a)^2\right) \ .$$

Substituting into equation (13.59), we have

$$\mathcal{L}\left\{\phi_0(x,-y)\right\} = -\frac{\mu B_x}{\pi(\kappa+1)}\left[(y-a)\ln\left(x^2+(y+a)^2\right) + \frac{4ay(y+a)}{x^2+(y+a)^2}\right]$$

and hence the complete stress function is

$$\phi = \frac{\mu B_x}{\pi(\kappa+1)}\left[(y-a)\ln\left(\frac{x^2+(y-a)^2}{x^2+(y+a)^2}\right) - \frac{4ay(y+a)}{x^2+(y+a)^2}\right] \ .$$

The circular hole

A similar procedure can be used to find the perturbation in a stress field due to the presence of the traction-free circular hole $r=a$. If the unperturbed field is defined by the function $\phi_0(r,\theta)$, the perturbed field can be obtained as

$$\phi = \phi_0(r,\theta) - \mathcal{L}\left[\frac{r^2}{a^2}\phi_0\left(\frac{a^2}{r},\theta\right)\right] + \frac{C(r^2-2a^2\ln(r))}{4} \ , \qquad (13.60)$$

where

$$\mathcal{L}\{\cdot\} = \frac{r^2}{a^2} + \left(1-\frac{r^2}{a^2}\right) r\frac{\partial}{\partial r} + \frac{a^2}{4}\left(1-\frac{r^2}{a^2}\right)^2 \nabla^2$$

and the constant C is defined as

$$C = \lim_{r\longrightarrow 0} \nabla^2\phi_0(r,\theta) \ .$$

These mathematical operations are performed by the Maple and Mathematica files 'holeperturbation' and the reader can verify (for example) that they generate the solutions of §8.3.2, §8.4.1, starting from the stress functions defining the corresponding unperturbed uniform stress fields.

PROBLEMS

1. Figure 13.6 shows a large body with a circular hole of radius a, subjected to a concentrated force F, tangential to the hole. The remainder of the hole surface is traction-free and the stress field is assumed to tend to zero as $r \to \infty$.

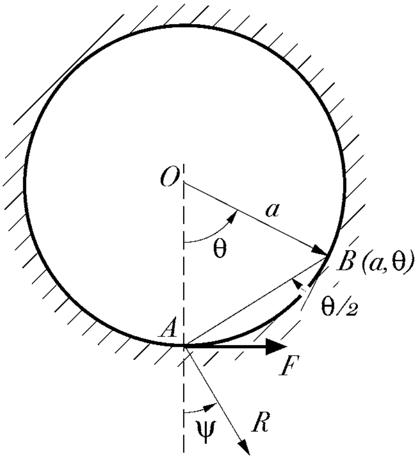

Figure 13.6

(i) Find the stress components at the point $B(a, \theta)$ on the surface of the hole, due to the candidate stress function

$$\phi = \frac{F}{\pi} R\psi \cos\psi$$

in polar coördinates R, ψ centred on the point A as shown.

(ii) Transform these stress components into the polar coördinate system r, θ with origin at the centre of the hole, O.

Note: *For points on the surface of the hole* $(r = a)$, we have $R = 2a\sin(\theta/2)$, $\sin\psi = \cos(\theta/2)$, $\cos\psi = -\sin(\theta/2)$.

(iii) Complete the solution by superposing stress functions (in r, θ) from Tables 8.1, 9.1 with appropriate Fourier components in the tractions and determining the multiplying constants from the conditions that (a) the surface of the hole be traction-free (except at A) and (b) the displacements be everywhere single-valued.

2. Use a method similar to that outlined in Problem 13.1 to find the stress field if the force F acts in the direction *normal* to the surface of the otherwise traction-free hole $r = a$.

3. A large plate is in a state of pure bending such that $\sigma_{xx} = \sigma_{xy} = 0$, $\sigma_{yy} = Cx$, where C is a constant.

We now introduce a crack in the range $-a<x<a,\ y=0$.

(i) Find a suitable corrective solution which, when superposed on the simple bending field, will make the surfaces of the crack free of tractions. (**Hint:** represent the corrective solution by a distribution of climb dislocations in $-a<x<a$. Don't forget the closure condition (13.41)).
(ii) Find the corresponding crack opening displacement as a function of x and show that it has an unacceptable negative value in $-a<x<0$.
(iii) Re-solve the problem assuming that there is frictionless contact in some range $-a<x<b$, where b is a constant to be determined. The dislocations will now have to be distributed only in $b<x<a$ and b is found from a continuity condition at $x=b$. (Move the origin to the mid-point of the range $b<x<a$.)
(iv) For the case with contact, find expressions for (a) the crack opening displacement, (b) the stress intensity factor at $x=a$, (c) the dimension b and (d) the contact traction in $-a<x<b$ and hence verify that the contact inequalities are satisfied.

4. A state of uniform shear $\sigma_{xy}=S, \sigma_{xx}=\sigma_{yy}=0$ in a large block of material is perturbed by the presence of the plane crack $-a<x<a, y=0$. Representing the perturbation due to the crack by a distribution of glide dislocations along the crack line, find the shear stress distribution on the line $x>a, y=0$, the relative motion between the crack faces in $-a<x<a, y=0$ and the mode II stress intensity factor K_{II} defined as

$$K_{II} \equiv \lim_{x\to a^+} \sigma_{yx}(x,0)\sqrt{2\pi(x-a)}\ .$$

5. A state of uniform general bi-axial stress $\sigma_{xx}=S_{xx}, \sigma_{yy}=S_{yy}, \sigma_{xy}=S_{xy}$ in a large block of material is perturbed by the presence of the plane crack $-a<x<a, y=0$. Note that the crack will close completely if $S_{yy}<0$ and will be fully open if $S_{yy}>0$. Use the method proposed in Problem 13.4 to find the mode II stress intensity factor K_{II} for both cases, assuming that Coulomb friction conditions hold in the closed crack with coefficient f.

If the block contains a large number of widely separated similar cracks of all possible orientations and if the block fails when at any one crack

$$\sqrt{K_I^2+K_{II}^2} = K_{Ic}\ ,$$

where the fracture toughness K_{Ic} is a material constant, sketch the biaxial failure surface for the material — i.e. the locus of all failure points in principal biaxial stress space (σ_1, σ_2).

6. Find the stress field due to a climb dislocation of unit strength located at the point $(0,a)$ in the traction-free half-plane $y>0$, using the following method:

(i) Write the stress function (13.15) in Cartesian coördinates.
(ii) Find the solution for a dislocation at $(0, a)$ in the full plane $-\infty < y < \infty$ by making a change of origin.
(iii) Superpose additional singularities centred on the image point $(0, -a)$ to make the surface $y = 0$ traction-free. Notice that these singularities are outside the actual body and are therefore admissible. Appropriate functions (in polar coördinates centred on $(0, -a)$) are

$$C_1 r \ln(r) \cos\theta \ ; \ \frac{C_2 \cos\theta}{r} \ ; \ C_3 \sin(2\theta) \ .$$

7. Solve Problem 6 using Aderogba's result (13.59, 13.57) for step (iii).

8. Find the stress field due to a concentrated force F applied in the x-direction at the point $(0, a)$ in the half-plane $y > 0$, if the surface of the half-plane is traction-free. Verify that the results reduce to those of §12.3 in the limit as $a \to 0$.

9. Two dissimilar elastic half planes are bonded on the interface $y = 0$ and have material properties as defined in §13.4. Use Aderogba's formula (13.57) to determine the stress field due to a climb dislocation at the point $(0, a)$ in the half-plane $y > 0$. In particular, find the location and magnitude of the maximum shear traction at the interface.

10. Use equation (13.60) to solve Problem 8.2.

14

THERMOELASTICITY

Most materials tend to expand if their temperature rises and, to a first approximation, the expansion is proportional to the temperature change. If the expansion is unrestrained, all dimensions will expand equally — i.e. there will be a uniform dilatation described by

$$e_{xx} = e_{yy} = e_{zz} = \alpha T \tag{14.1}$$

$$e_{xy} = e_{yz} = e_{zx} = 0 \;, \tag{14.2}$$

where α is the *coefficient of linear thermal expansion*. Notice that no shear strains are induced in unrestrained thermal expansion, so that a body which is heated to a uniformly higher temperature will get larger, but will retain the same shape.

Thermal strains are additive to the elastic strains due to local stresses, so that Hooke's law is modified to the form

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} + \alpha T \tag{14.3}$$

$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E} \;. \tag{14.4}$$

14.1 The governing equation

The Airy stress function can be used for two-dimensional thermoelasticity, but the governing equation will generally include additional terms associated with the temperature field. Repeating the derivation of §4.4.1, but using (14.3) in place of (1.58), we find that the compatibility condition demands that

$$\frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} + E\alpha\frac{\partial^2 T}{\partial y^2} - 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2} + E\alpha\frac{\partial^2 T}{\partial x^2} = 0 \tag{14.5}$$

and after substituting for the stress components from (4.6) and rearranging, we obtain

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_14, © Springer Science+Business Media B.V. 2010

$$\nabla^4\phi = -E\alpha\nabla^2 T \;, \tag{14.6}$$

for plane stress.

The corresponding plane strain equations can be obtained by a similar procedure, noting that the restraint of the transverse strain e_{zz} (14.1) will induce a stress $\sigma_{zz} = -E\alpha T$ and hence additional in-plane strains $\nu\alpha T$. Equation (14.6) is therefore modified to

$$\nabla^4\phi = -\frac{E\alpha}{(1-\nu)}\nabla^2 T \;, \tag{14.7}$$

for plane strain and we can supplement the plane stress to plane strain conversions (3.18) with the relation

$$\alpha = \alpha'(1+\nu') \;. \tag{14.8}$$

Equations (14.6, 14.7) are similar in form to that obtained in the presence of body forces (7.8) and can be treated in the same way. Thus, we can seek any particular solution of (14.6) and then satisfy the boundary conditions of the problem by superposing a more general biharmonic function, since the biharmonic equation is the complementary or homogeneous equation corresponding to (14.6, 14.7).

Example

As an example, we consider the case of the thin circular disk, $r < a$, with traction-free edges, raised to the temperature

$$T = T_0 y^2 = T_0 r^2 \sin^2\theta \;, \tag{14.9}$$

where T_0 is a constant.

Substituting this temperature distribution into equation (14.6), we obtain

$$\nabla^4\phi = -2E\alpha T_0 \tag{14.10}$$

and a simple particular solution is

$$\phi_0 = -\frac{E\alpha T_0 r^4}{32} \;. \tag{14.11}$$

The stresses corresponding to ϕ_0 are

$$\sigma_{rr} = -\frac{E\alpha T_0 r^2}{8} \;\;;\;\; \sigma_{\theta\theta} = -\frac{3E\alpha T_0 r^2}{8} \;\;;\;\; \sigma_{r\theta} = 0 \tag{14.12}$$

and the boundary $r=a$ can be made traction-free by superposing a uniform hydrostatic tension $E\alpha T_0 a^2/8$, resulting in the final stress field[1]

$$\sigma_{rr} = \frac{E\alpha T_0(a^2-r^2)}{8} \;\;;\;\; \sigma_{\theta\theta} = \frac{E\alpha T_0(a^2-3r^2)}{8} \;\;;\;\; \sigma_{r\theta} = 0 \;. \tag{14.13}$$

[1] It is interesting to note that the stress field in this case is axisymmetric, even though the temperature field (14.9) is not.

14.2 Heat conduction

The temperature field might be a given quantity — for example, it might be measured using thermocouples or radiation methods — but more often it has to be calculated from thermal boundary conditions as a separate boundary-value problem. Most materials approximately satisfy the Fourier heat conduction law, according to which the heat flux per unit area $\boldsymbol{q}$ is linearly proportional to the local temperature gradient. i.e.

$$\boldsymbol{q} = -K\boldsymbol{\nabla}T \; , \tag{14.14}$$

where K is the thermal conductivity of the material. The conductivity is usually assumed to be constant, though for real materials it depends upon temperature. However, the resulting non-linearity is only important when the range of temperatures under consideration is large.

We next apply the principle of conservation of energy to a small cube of material. Equation (14.14) governs the flow of heat across each face of the cube and there may also be heat generated, Q per unit volume, within the cube due to some mechanism such as electrical resistive heating or nuclear reaction etc. If the sum of the heat flowing into the cube and that generated within it is positive, the temperature will rise at a rate which depends upon the *thermal capacity* of the material. Combining these arguments we find that the temperature T must satisfy the equation[2]

$$\rho c \frac{\partial T}{\partial t} = K\nabla^2 T + Q \; , \tag{14.15}$$

where ρ, c are respectively the density and specific heat of the material, so that the product ρc is the amount of heat needed to increase the temperature of a unit volume of material by one degree.

In equation (14.15), the first term on the right-hand side is the net heat flow into the element per unit volume and the second term, Q is the rate of heat generated per unit volume. The algebraic sum of these terms gives the heat available for raising the temperature of the cube.

It is convenient to divide both sides of the equation by K, giving the more usual form of the heat conduction equation

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t} - \frac{Q}{K} \; , \tag{14.16}$$

where

$$\kappa = \frac{K}{\rho c} \tag{14.17}$$

[2] More detail about the derivation of this equation and other information about the linear theory of heat conduction can be found in the classical text H.S.Carslaw and J.C.Jaeger, *Conduction of Heat in Solids*, 2nd.ed., Clarendon Press, Oxford, 1959.

is the *thermal diffusivity* of the material. Thermal diffusivity has the dimensions area/time and its magnitude gives some indication of the rate at which a thermal disturbance will propagate through the body.

We can substitute (14.16) into (14.6) obtaining

$$\nabla^4\phi = -E\alpha\left(\frac{1}{\kappa}\frac{\partial T}{\partial t} - \frac{Q}{K}\right) \tag{14.18}$$

for plane stress.

14.3 Steady-state problems

Equation (14.18) shows that if the temperature is independent of time and there is no internal source of heat in the body ($Q=0$), ϕ will be biharmonic. It therefore follows that in the steady-state problem without heat generation, the stress field is unaffected by the temperature distribution. In particular, if the boundaries of the body are traction-free, a steady-state (and hence harmonic) temperature field will not induce any thermal stresses.

Furthermore, if there are internal heat sources — i.e. if $Q\neq 0$ — the stress field can be determined directly from equation (14.18), using Q, without the necessity of first solving a boundary value problem for the temperature T. This also implies that the thermal boundary conditions in two-dimensional steady-state problems have no effect on the thermoelastic stress field.

All of these deductions rest on the assumption that the elastic problem is defined in terms of *tractions*. If some of the boundary conditions are stated in terms of displacements, the resulting thermal distortion will induce boundary tractions and the stress field will be affected, though it will still be the same as that which would have been produced by the same tractions if they had been applied under isothermal conditions.

Recalling the arguments of §2.2.1, we conclude that similar considerations apply to the multiply connected body, for which there exists an implied displacement boundary condition. In other words, multiply connected bodies will generally develop non-zero thermal stresses even under steady-state conditions with no boundary tractions. However, the resulting stress field is essentially that associated with the presence of a dislocation in the hole and hence can be characterized by relatively few parameters[3].

Appropriate conditions for multiply connected bodies can be explicitly imposed, but in general it is simpler to obviate the need for such a condition by reverting to a displacement function representation. We shall therefore postpone discussion of thermoelastic problems for multiply connected bodies until Chapter 22 where such a formulation is introduced.

[3] See for example J.Dundurs, Distortion of a body caused by free thermal expansion, *Mechanics Research Communications,* Vol. 1 (1974), pp.121-124.

14.3.1 Dundurs' Theorem

If the conditions discussed in the last section are satisfied and the temperature field therefore induces no thermal stress, the strains will be given by equations (14.1, 14.2). It then follows that

$$\frac{\partial^2 u_y}{\partial x^2} = -\frac{\partial^2 u_x}{\partial x \partial y} \, , \tag{14.19}$$

because of (14.2)

$$= -\frac{\partial e_{xx}}{\partial y} = -\alpha \frac{\partial T}{\partial y} = \frac{\alpha q_y}{K} \, , \tag{14.20}$$

from (14.1, 14.14). In view of (14.8), the corresponding result for plane strain can be written

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{\alpha(1+\nu)q_y}{K} \, . \tag{14.21}$$

In this equation, the constant of proportionality $\alpha(1+\nu)/K$ is known as the *thermal distortivity* of the material and is denoted by the symbol δ.

Equations (14.20, 14.21) state that the curvature of an initially straight line segment in the x-direction ($\partial^2 u_y/\partial x^2$) is proportional to the local heat flux across that line segment. This result was first proved by Dundurs[4] and is referred to as *Dundurs' Theorem*. It is very useful as a guide to determining the effect of thermal distortion on a structure. Figure 14.1 shows some simple bodies with various thermal boundary conditions and the resulting steady-state thermal distortion.

Notice that straight boundaries that are unheated remain straight, those that are heated become convex outwards, whilst those that are cooled become concave. The angles between the edges are unaffected by the distortion, because there is no shear strain. Since the thermal field is in the steady-state and there are no heat sources, the algebraic sum of the heat input around the boundary must be zero. Thus, although the boundary is locally rotated by the cumulative heat input from an appropriate starting point, this does not lead to incompatibility at the end of the circuit.

Many three-dimensional structures such as box-sections, tanks, rectangular hoppers etc., are fabricated from plate elements. If the unrestrained thermal distortions of these elements are considered separately, the incompatibilities of displacement developed at the junctions between elements permits the thermal stress problem to be described in terms of dislocations, the physical effects of which are more readily visualized.

[4] J.Dundurs *loc. cit.*

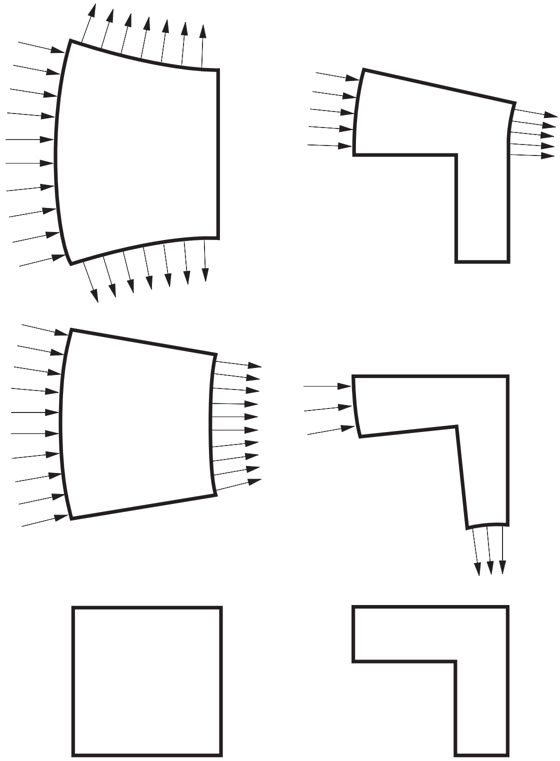

Figure 14.1: Distortion due to thermal expansion.

Dundurs' Theorem can also be used to obtain some useful simplifications in two-dimensional contact and crack problems involving thermal distortion[5].

PROBLEMS

1. A direct electric current I flows along a conductor of rectangular cross-section $-4a < x < 4a$, $-a < y < a$, all the surfaces of which are traction free. The conductor is made of copper of electrical resistivity ρ, thermal conductivity K, Young's modulus E, Poisson's ratio ν and coefficient of thermal expansion α. Assuming the current density to be uniform and neglecting electromagnetic effects, estimate the thermal stresses in the conductor when the temperature has reached a steady state.

2. A fuel element in a nuclear reactor can be regarded as a solid cylinder of radius a. During operation, heat is generated at a rate $Q_0(1+Ar^2/a^2)$ per

[5] For more details, see J.R.Barber, Some implications of Dundurs' Theorem for thermoelastic contact and crack problems, *Journal of Strain Analysis,* Vol. 22 (1980), pp.229-232.

unit volume, where r is the distance from the axis of the cylinder and A is a constant.

Assuming that the element is immersed in a fluid at pressure p and that *axial* expansion is prevented, find the radial and circumferential thermal stresses produced in the steady state.

3. The instantaneous temperature distribution in the thin plate $-a < x < a, -b<y<b$ is defined by

$$T(x,y) = T_0 \left(\frac{x^2}{a^2} - 1 \right) ,$$

where T_0 is a positive constant. Find the magnitude and location of (i) the maximum tensile stress and (ii) the maximum shear stress in the plate if the edges $x=\pm a, y=\pm b$ are traction-free and $a \gg b$.

4. The half plane $y>0$ is subject to periodic heating at the surface $y=0$, such that the surface temperature is

$$T(0,t) = T_0 \cos(\omega t) .$$

Show that the temperature field

$$T(y,t) = T_0 e^{-\lambda y} \cos(\omega t - \lambda y)$$

satisfies the heat conduction equation (14.16) with no internal heat generation, provided that

$$\lambda = \sqrt{\frac{\omega}{2k}} .$$

Find the corresponding thermal stress field as a function of y, t if the surface of the half plane is traction-free.

Using appropriate material properties, estimate the maximum tensile stress generated in a large rock due to diurnal temperature variation, with a maximum daytime temperature of 30^{o}C and minimum nighttime temperature of 10^{o}C.

5. The layer $0<y<h$ rests on a frictionless rigid foundation at $y=0$ and the surface $y=h$ is traction-free. The foundation is a thermal insulator and the free surface is subjected to the steady state heat input

$$q_y = q_0 \cos(mx) .$$

Use Dundurs' theorem to show that the layer will not separate from the foundation and find the amplitude of the sinusoidal perturbation in the free surface due to thermal distortion.

15

ANTIPLANE SHEAR

In Chapters 3–14, we have considered two-dimensional states of stress involving the in-plane displacements u_x, u_y and stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$. Another class of two-dimensional stress states that satisfy the elasticity equations exactly is that in which the in-plane displacements u_x, u_y are everywhere zero, whilst the out-of-plane displacement u_z is independent of z — i.e.

$$u_x = u_y = 0 \quad ; \quad u_z = f(x, y) \; . \tag{15.1}$$

Substituting these results into the strain-displacement relations (1.51) yields

$$e_{xx} = e_{yy} = e_{zz} = 0 \tag{15.2}$$

and

$$e_{xy} = 0 \quad ; \quad e_{yz} = \frac{1}{2}\frac{\partial f}{\partial y} \quad ; \quad e_{zx} = \frac{1}{2}\frac{\partial f}{\partial x} \; . \tag{15.3}$$

It then follows from Hooke's law (1.71) that

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \tag{15.4}$$

and

$$\sigma_{xy} = 0 \quad ; \quad \sigma_{yz} = \mu\frac{\partial f}{\partial y} \quad ; \quad \sigma_{zx} = \mu\frac{\partial f}{\partial x} \; . \tag{15.5}$$

In other words, the only non-zero stress components are the two shear stresses σ_{zx}, σ_{zy} and these are functions of x, y only. Such a stress state is known as *antiplane shear* or *antiplane strain*.

The in-plane equilibrium equations (2.2, 2.3) are identically satisfied by the stress components (15.4, 15.5) if and only if the in-plane body forces p_x, p_y are zero. However, substituting (15.5) into the out-of-plane equilibrium equation (2.4) yields

$$\mu\nabla^2 f + p_z = 0 \; . \tag{15.6}$$

Thus, antiplane problems can involve body forces in the axial direction, provided these are independent of z.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_15, © Springer Science+Business Media B.V. 2010

In the absence of body forces, equation (15.6) reduces to the Laplace equation

$$\nabla^2 f = 0 \; . \tag{15.7}$$

15.1 Transformation of coordinates

It is often convenient to regard the two stress components on the z-plane as components of a shear stress vector

$$\boldsymbol{\tau} = \boldsymbol{i}\sigma_{zx} + \boldsymbol{j}\sigma_{zy} = \mu\boldsymbol{\nabla} f \tag{15.8}$$

from (15.5). The vector transformation equations (1.14) then show that the stress components in the rotated coordinate system x', y' of Figure 1.3 are

$$\sigma_{zx'} = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta \tag{15.9}$$

$$\sigma_{zy'} = \sigma_{zy}\cos\theta - \sigma_{zx}\sin\theta \; . \tag{15.10}$$

Equation (15.8) can also be used to define the expressions for the stress components in polar coordinates as

$$\sigma_{zr} = \mu\frac{\partial f}{\partial r} \quad ; \quad \sigma_{z\theta} = \frac{\mu}{r}\frac{\partial f}{\partial \theta} \; . \tag{15.11}$$

The maximum shear stress at any given point is the magnitude of the vector $\boldsymbol{\tau}$ and is given by

$$|\tau| = \mu\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \; . \tag{15.12}$$

15.2 Boundary conditions

Only one boundary condition can be imposed at each point on the boundary. The axial displacement $u_z = f$ may be prescribed, or the traction

$$\sigma_{nz} = \mu\frac{\partial f}{\partial n} \tag{15.13}$$

may be prescribed, where n is the local outward normal to the cross-section. Thus, antiplane problems reduce to the solution of the Poisson equation (15.6) (or the Laplace equation (15.7) when there is no body force) with prescribed values of f or $\partial f/\partial n$ on the boundary. This is a classical boundary-value problem that is somewhat simpler than that involved in the solution for the Airy stress function, but essentially similar methods can be used for both. Simple examples can be defined for all of the geometries considered in Chapters 5–13. In the present chapter, we shall consider a few special cases, but a wider range of examples can be found in the problems at the end of this chapter.

15.3 The rectangular bar

Figure 15.1 shows the cross-section of a long bar of rectangular cross-section $2a \times b$, where $a \gg b$. The short edges $x = \pm a$ are built in to rigid supports, the edge $y = b$ is loaded by a uniform shear traction $\sigma_{yz} = S$ and the remaining edge $y = 0$ is traction-free.

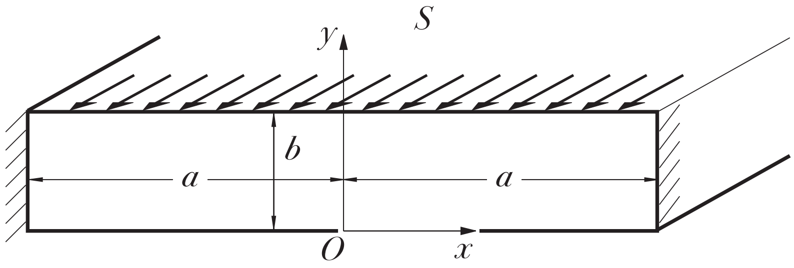

Figure 15.1: Rectangular bar with shear loading on one edge.

Using equations (15.4, 15.5), the boundary conditions can be written in the mathematical form

$$f = 0 \quad ; \quad x = \pm a \tag{15.14}$$

$$\frac{\partial f}{\partial y} = \frac{S}{\mu} \quad ; \quad y = b \tag{15.15}$$

$$\frac{\partial f}{\partial y} = 0 \quad ; \quad y = 0 \; . \tag{15.16}$$

However, as in Chapter 5, a finite polynomial solution can be found only if the boundary conditions (15.14) on the shorter edges are replaced by the weak form, which in this case take the form

$$\int_0^b f(x, y) dy = 0 \quad ; \quad x = \pm a \; . \tag{15.17}$$

This problem is even in x and a solution can be obtained using the trial function

$$f = C_1 x^2 + C_2 y^2 + C_3 y + C_4 \; . \tag{15.18}$$

Substitution into equations (15.7, 15.15–15.17) yields the conditions

$$2C_1 + 2C_2 = 0 \tag{15.19}$$

$$2C_2 b + C_3 = \frac{S}{\mu} \tag{15.20}$$

$$C_3 = 0 \tag{15.21}$$

$$C_1 a^2 b + \frac{C_2 b^3}{3} + \frac{C_3 b^2}{2} + C_4 = 0 \tag{15.22}$$

with solution

$$C_1 = -\frac{S}{2\mu b} \quad ; \quad C_2 = \frac{S}{2\mu b} \quad ; \quad C_3 = 0 \quad ; \quad C_4 = \frac{S(3a^2 - b^2)}{6\mu b} \; . \tag{15.23}$$

The final solution for the stresses and displacements is therefore

$$u_z = \frac{S}{2\mu b}\left(y^2 - x^2 + a^2 - \frac{b^2}{3}\right) \tag{15.24}$$

$$\sigma_{zx} = -\frac{Sx}{b} \tag{15.25}$$

$$\sigma_{zy} = \frac{Sy}{b} \; , \tag{15.26}$$

from equations (15.23, 15.1, 15.5).

15.4 The concentrated line force

Consider an infinite block of material loaded by a force F per unit length acting along the z-axis. This problem is axisymmetric, in contrast to that solved in §13.1, since the force this time is directed along the axis rather than perpendicular to it. It follows that the resulting displacement function f must be axisymmetric and equation (15.7) reduces to the ordinary differential equation

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 0 \; , \tag{15.27}$$

since there is no body force except at the origin. Equation (15.27) has the general solution

$$f = C_1 \ln(r) + C_2 \; , \tag{15.28}$$

leading to the stress field

$$\sigma_{rz} = \frac{\mu C_1}{r} \quad ; \quad \sigma_{\theta z} = 0 \; , \tag{15.29}$$

from (15.11). The constant C_1 can be determined by considering the equilibrium of a cylinder of material of radius a and unit length. We obtain

$$F + \int_0^{2\pi} \sigma_{rz}(a,\theta) a d\theta = 0 \tag{15.30}$$

and hence

$$C_1 = -\frac{F}{2\pi\mu} \; . \tag{15.31}$$

The corresponding stresses are then obtained from (15.29) as

$$\sigma_{rz} = -\frac{F}{2\pi r} \quad ; \quad \sigma_{\theta z} = 0 \; . \tag{15.32}$$

Of course, the same result could have been obtained without recourse to elasticity arguments, appealing simply to axisymmetry and equilibrium and hence the same stresses would be obtained for a fairly general non-linear or inelastic material, as long as it is isotropic.

The elastic displacement is

$$u_z = f = -\frac{F\ln(r)}{2\pi\mu} + C_2 \ , \tag{15.33}$$

where the constant C_2 represents an arbitrary rigid-body displacement.

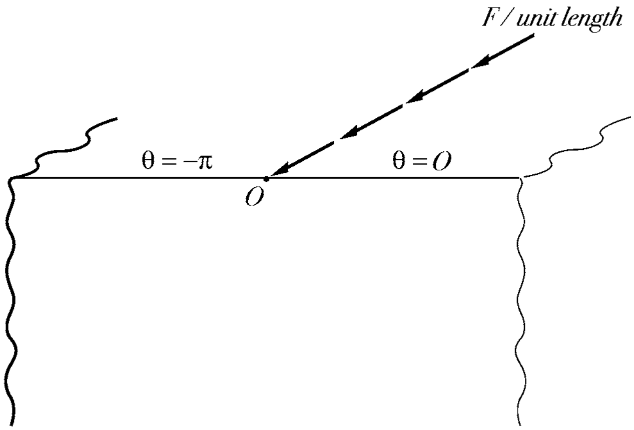

Figure 15.2: Half-space loaded by an out-of-plane line force.

All θ-surfaces are traction-free and hence the same solution can be used for a wedge of any angle loaded by a uniform force per unit length acting at and parallel to the apex, though the factor of 2π in C_1 will then be replaced by the subtended wedge angle. In particular, Figure 15.2 shows the half-space $-\pi<\theta<0$ loaded by a uniformly distributed tangential force along the z-axis, for which it is easily shown that

$$\sigma_{rz} = -\frac{F}{\pi r} \quad ; \quad u_z = -\frac{F\ln(r)}{\pi\mu} + C_2 \ . \tag{15.34}$$

This result can be used to solve frictional contact problems analogous to those considered in Chapter 12 (see Problems 15.11, 15.12).

15.5 The screw dislocation

The dislocation solution in antiplane shear corresponds to the situation in which the infinite body is cut on the half-plane $x > 0, y = 0$ and the two faces of the cut experience a relative displacement $\delta = B_z$ in the z-direction.

Describing the cut space as the wedge $0<\theta<2\pi$, the boundary conditions for this problem can be written

$$u_z(r,0) - u_z(r,2\pi) = B_z \qquad (15.35)$$

and a suitable solution satisfying equation (15.7) is

$$u_z = -\frac{B_z\theta}{2\pi} \;\; ; \;\; \sigma_{zr} = 0 \;\; ; \;\; \sigma_{z\theta} = \frac{\mu B_z}{2\pi r} \; . \qquad (15.36)$$

The corresponding material defect is known as a *screw dislocation*. Notice that the stress components due to a screw dislocation decay with r^{-1} with distance from the origin, as do those for the in-plane climb and glide dislocations discussed in §13.2.

A distribution of screw dislocations can be used to solve problems involving cracks in an antiplane shear field, using the technique introduced in §13.3.

PROBLEMS

1. The long rectangular bar $0 < x < a, 0 < y < b$, $a \gg b$ is built in to a rigid support at $x = a$ and loaded by a uniform shear traction $\sigma_{yz} = S$ at $y = b$. The remaining surfaces are free of traction. Find a solution for the displacement and stress fields, using strong boundary conditions on the edges $y = 0, b$.

2. Figure 15.3 shows the cross-section of a long bar of equilateral triangular cross-section of side a. The three faces of the bar are built in to rigid supports and the axis of the bar is vertical, resulting in gravitational loading $p_z = -\rho g$.

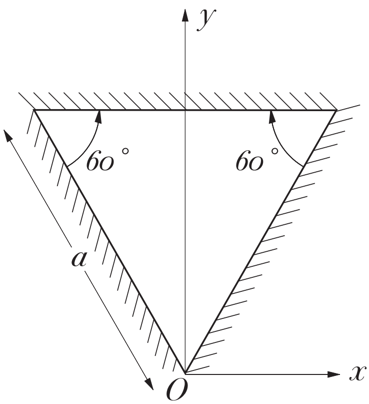

Figure 15.3: Cross-section of the triangular bar.

Show that an exact solution for the stress and displacement fields can be obtained using a stress function of the form

$$f = C(y - \sqrt{3}x)(y + \sqrt{3}x)\left(y - \frac{\sqrt{3}a}{2}\right) .$$

Find the value of the constant C and the location and magnitude of the maximum shear stress.

3. A state of uniform shear $\sigma_{xz} = S, \sigma_{yz} = 0$ in a large block of material is perturbed by the presence of a small traction-free hole whose boundary is defined by the equation $r = a$ in cylindrical polar coordinates r, θ, z. Find the complete stress field in the block and hence determine the appropriate stress concentration factor.

4. A state of uniform shear $\sigma_{xz} = S, \sigma_{yz} = 0$ in a large block of material is perturbed by the presence of a rigid circular inclusion whose boundary is defined by the equation $r = a$ in cylindrical polar coordinates r, θ, z. The inclusion is perfectly bonded to the elastic material and is prevented from moving, so that $u_z = 0$ at $r = a$. Find the complete stress field in the block and hence determine the appropriate stress concentration factor.

5. The body defined by $a < r < b, 0 < \theta < \pi/2, -\infty < z < \infty$ is built in at $\theta = \pi/2$ and loaded by a shear force F per unit length in the z-direction on the surface $\theta = 0$. The curved surfaces $r = a, b$ are traction-free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere.

6. The body defined by $a < r < b, 0 < \theta < \pi/2, -\infty < z < \infty$ is built in at $\theta = \pi/2$, such that the z-axis is vertical. It is loaded only by its own weight with density ρ, the curved surfaces $r = a, b$ being traction-free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere.

7. The body defined by $a < r < b, 0 < \theta < \pi/2, -\infty < z < \infty$ is built in at $\theta = \pi/2$ and loaded by the uniform shear traction $\sigma_{rz} = S$ on $r = a$, the other surfaces being traction-free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere. **Hint:** This is a degenerate problem in the sense of §10.3.1 and requires a special stress function. Equilibrium arguments suggest that the stress component $\sigma_{\theta z}$ must increase with θ, implying a function f varying with θ^2. You can construct a suitable function by starting with the harmonic function $r^n \cos(n\theta)$, differentiating twice with respect to n and then setting $n = 0$.

8. The body $-\alpha < \theta < \alpha, 0 \leq r < a$ is supported at $r = a$ and loaded only by a uniform antiplane shear traction $\sigma_{\theta z} = S$ on the surface $\theta = \alpha$, the other

surface being traction-free. Find the complete stress field in the body, using strong boundary conditions on $\theta = \pm\alpha$ and weak conditions on $r = a$.

9. The half-space $-\pi < \theta < 0$ is bonded to a rigid body in the region $\theta = 0$, the remaining surface $\theta = -\pi$ being traction-free. If the rigid body is loaded in the z-direction (out-of-plane), use an asymptotic argument analogous to that in §11.2 to determine the exponent of the most singular term in the stress field near the origin.

10. The wedge $-\alpha < \theta < \alpha$ has traction-free surfaces $\theta = \pm\alpha$ and is loaded at large r in antiplane shear. Develop an asymptotic (eigenfunction) expansion for the most general stress field near the apex $r = 0$ and plot the exponent of the dominant term for (i) symmetric and (ii) antisymmetric stress fields as functions of the complete wedge angle 2α.

11. Two large parallel cylinders each of radius R are pressed together by a normal force P per unit length. An axial tangential force Q per unit length is then applied, tending to make the cylinders slide relative to each other along the axis as shown in Figure 15.4. If the coefficient of friction is f and $Q < fP$, find the extent of the contact region, the extent of the slip and stick regions and the distribution of normal and tangential traction in the contact area[1]

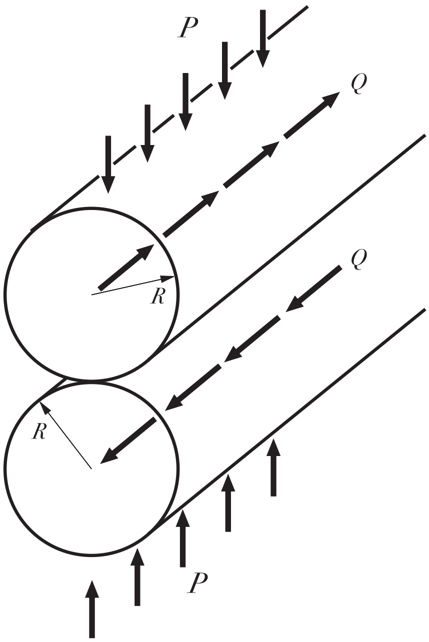

Figure 15.4: Contacting cylinders loaded out of plane.

[1] This problem is analogous to Mindlin's problem discussed in §12.8.1.

12. Solve Problem 15.11 for the case where the cylinders are rolling against each other, as in Carter's problem (§12.8.2). If the rolling speed is Ω, find the relative (axial) creep velocity.

Now suppose that the axes of the cylinders are very slightly misaligned through some angle ϵ which is insufficient to make the contact conditions vary significantly along the axis. If the cylinders run in bearings which prevent relative axial motion, find the axial bearing force per unit length and the maximum misalignment which can occur without there being gross slip.

13. A state of uniform shear $\sigma_{xz} = 0, \sigma_{yz} = S$ in a large block of material is perturbed by the presence of the plane crack $-a < x < a, y = 0$. Representing the perturbation due to the crack by a distribution of screw dislocations along the crack line, find the shear stress distribution on the line $x > a, y = 0$, the relative motion between the crack faces in $-a < x < a, y = 0$ and the mode III stress intensity factor K_{III} defined as

$$K_{III} \equiv \lim_{x \to a^+} \sigma_{yz}(x, 0)\sqrt{2\pi(x - a)} \; .$$

Part III

END LOADING OF THE PRISMATIC BAR

One of the most important problems in the technical application of Elasticity concerns a long[2] bar of uniform cross-section loaded only at the ends. Recalling Saint Venant's principle[3], we anticipate a region near the ends where the stress field is influenced by the exact local traction distribution, but these end effects should decay with distance from the ends, leaving the stress field in a central region depending only upon the force resultants transmitted along the bar. In this section we shall consider the problem of determining this 'preferred' form of stress distribution associated with various force resultant end loadings of the bar.

The most general end loading of the bar will comprise an axial force F, shear forces V_x, V_y in the x- and y-directions respectively, bending moments M_x, M_y about the x- and y-axes and a torque T. The axial force and bending moments cause normal stresses σ_{zz} on the cross-section, whilst the torque and shear forces cause shear stresses σ_{zx}, σ_{zy}. Notice however that the existence of a shear force implies a bending moment varying linearly with z. These problems are strictly three-dimensional in that the displacements vary with z, but this variation is at most linear or quadratic and the mathematical formulation of the problem therefore reduces to the solution of a boundary value problem on the two-dimensional cross-sectional domain.

Elementary Mechanics of Materials solutions of these problems are based on *ad hoc* assumptions, most notably the assumption that plane sections remain plane during bending. In Elasticity it is legitimate to use the same approach provided that the final stress field is checked to ensure that it satisfies the equilibrium equations (2.5) and that the corresponding strains satisfy the six compatibility equations (2.10). However, a more satisfactory starting point is to note that in the central region the stress state is independent[4] of z and hence each 'slice' of bar defined by initially parallel cross-sectional planes must deform in exactly the same way. Thus, if initially plane sections deform out of plane, this deformation is independent of z and the distance between corresponding points on two such sections after deformation will be defined by a general rigid-body translation and rotation. This argument is sufficient to establish that the strain e_{zz} is a linear function of x, y, as assumed in the classical theory of pure bending. Notice incidentally that the constitutive law is not required for this argument and hence the conclusion holds for a general non-linear or inelastic material[5].

Using these arguments, it is easily shown that the elementary Mechanics of Materials results are exact for the case where the bar is loaded by an axial force and/or a bending moment. It is convenient to locate the origin at the centroid of the cross-section, in which case a tensile force F acting along the

---

[2] Here 'long' implies that the length of the bar is at least several times larger than the largest dimension in the cross-section.

[3] See §3.1.2 and Chapter 6.

[4] Except for the linear increase of bending stresses due to a shear force.

[5] See J.R.Barber, *Intermediate Mechanics of Materials*, McGraw-Hill, Burr Ridge, 2000, Chapter 5.

z-axis will generate the uniform uniaxial stress field

$$\sigma_{zz} = \frac{F}{A} \;\; ; \;\; \sigma_{yy} = \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; ,$$

where A is the area of the cross-section. For the bending problem, some simplification results if we choose the coördinate axes to coincide with the principal axes of bending of the bar[6]. The bending stress field is then given by

$$\sigma_{zz} = \frac{M_1 y}{I_1} - \frac{M_2 x}{I_2} \;\; ; \;\; \sigma_{yy} = \sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; ,$$

where I_1, I_2 are the principal second moments of area of the cross-section and M_1, M_2 are the bending moments about the corresponding axes, chosen to coincide with the x, y-directions respectively.

The problem of the bar loaded in torsion or shear cannot be treated exactly using elementary Mechanics of Materials arguments[7] and will be discussed in the next two chapters.

[6] See for example J.R.Barber, *Intermediate Mechanics of Materials*, McGraw-Hill, Burr Ridge, 2000, Chapter 4.

[7] Except for the special case of a circular bar loaded only in torsion.

16

TORSION OF A PRISMATIC BAR

If a bar is loaded by equal and opposite torques T on its ends, we anticipate that the relative rigid-body displacement of initially plane sections will consist of rotation, leading to a twist per unit length β. These sections may also deform out of plane, but this deformation must be the same for all values of z. These kinematic considerations lead to the candidate displacement field

$$u_x = -\beta zy \; ; \quad u_y = \beta zx \; ; \quad u_z = \beta f(x,y) \; , \tag{16.1}$$

where f is an unknown function of x, y describing the out-of-plane deformation. Notice that it is convenient to extract the factor β explicitly in u_z, since whatever the exact nature of f, it is clear that the complete deformation field must be linearly proportional to the applied torque and hence to the twist per unit length.

Substituting these results into the strain-displacement relations (1.51) yields

$$e_{xx} = e_{yy} = e_{zz} = e_{xy} = 0 \; ; \quad e_{zx} = \frac{\beta}{2}\left(\frac{\partial f}{\partial x} - y\right) \; ; \quad e_{zy} = \frac{\beta}{2}\left(\frac{\partial f}{\partial y} + x\right) \tag{16.2}$$

and it follows from Hooke's law (1.71) that

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \tag{16.3}$$

and

$$\sigma_{xy} = 0 \; ; \quad \sigma_{zx} = \mu\beta\left(\frac{\partial f}{\partial x} - y\right) \; ; \quad \sigma_{zy} = \mu\beta\left(\frac{\partial f}{\partial y} + x\right) \; . \tag{16.4}$$

There are no body forces, so substitution into the equilibrium equations (2.5) yields

$$\nabla^2 f = 0 \; . \tag{16.5}$$

The torsion problem is therefore reduced to the determination of a harmonic function f such that the stresses (16.4) satisfy the traction-free condition

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_16, © Springer Science+Business Media B.V. 2010

condition on the curved surfaces of the bar. The twist per unit length β can then be determined by evaluating the torque on the cross-section Ω

$$T = \iint_{\Omega} (x\sigma_{zy} - y\sigma_{zx}) dxdy \; . \tag{16.6}$$

16.1 Prandtl's stress function

The imposition of the boundary condition is considerably simplified by introducing Prandtl's stress function φ defined such that

$$\boldsymbol{\tau} \equiv \boldsymbol{i}\sigma_{zx} + \boldsymbol{j}\sigma_{zy} = \text{curl}\; \boldsymbol{k}\varphi \tag{16.7}$$

or

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} \; ; \quad \sigma_{zy} = -\frac{\partial \varphi}{\partial x} \; . \tag{16.8}$$

With this representation, the traction-free boundary condition can be written

$$\boldsymbol{\tau} \cdot \boldsymbol{n} = \sigma_{zn} = \frac{\partial \varphi}{\partial t} = 0 \; , \tag{16.9}$$

where $\boldsymbol{n}$ is the local normal to the boundary of Ω and n, t are a corresponding set of local orthogonal coördinates respectively normal and tangential to the boundary. Thus φ must be constant around the boundary and for simply connected bodies this constant can be taken as zero without loss of generality giving the simple condition

$$\varphi = 0 \; . \tag{16.10}$$

on the boundary.

The expressions (16.8) satisfy the equilibrium equations (2.5) identically. To obtain the governing equation for φ, we eliminate σ_{zx}, σ_{zy} between equations (16.4, 16.8), obtaining

$$\frac{\partial \varphi}{\partial y} = \mu\beta \left(\frac{\partial f}{\partial x} - y \right) \; ; \quad \frac{\partial \varphi}{\partial x} = -\mu\beta \left(\frac{\partial f}{\partial y} + x \right) \tag{16.11}$$

and then eliminate f between these equations, obtaining

$$\nabla^2 \varphi = -2\mu\beta \; . \tag{16.12}$$

Thus, in Prandtl's formulation, the torsion problem reduces to the determination of a function φ satisfying the Poisson equation (16.12), such that $\varphi = 0$ on the boundary of the cross-section. The final stage is to determine the constant β from (16.6), which here takes the form

$$T = -\iint_{\Omega} \left(x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y} \right) dxdy \; . \tag{16.13}$$

Integrating by parts and using the fact that $\varphi = 0$ on the boundary of Ω, we obtain the simple expression

$$T = 2\iint_{\Omega} \varphi dxdy \; . \tag{16.14}$$

16.1.1 Solution of the governing equation

The Poisson equation (16.12) is an inhomogeneous partial differential equation and its general solution can be written

$$\varphi = \varphi_P + \varphi_H \; , \tag{16.15}$$

where φ_P is any particular solution of (16.12) and φ_H is the general solution of the corresponding homogeneous equation

$$\nabla^2 \varphi_H = 0 \tag{16.16}$$

— i.e. Laplace's equation. Since (16.12) is a second order equation, it will yield a constant when substituted with any quadratic polynomial function and hence suitable particular solutions in Cartesian or polar coördinates are

$$\varphi_P = -\mu\beta x^2 \; ; \quad \varphi_P = -\mu\beta y^2 \; ; \quad \varphi_P = -\frac{\mu\beta r^2}{2} \; . \tag{16.17}$$

The homogeneous equation (16.16) is satisfied by both the real and imaginary parts of any analytic function of the complex variable

$$\zeta = x + \imath y = re^{\imath\theta}$$

— i.e.

$$\varphi_H = \Re(f(\zeta)) \quad \text{or} \quad \varphi_H = \Im(f(\zeta)) \; , \tag{16.18}$$

where f is any function. Simple functions obtained by taking $f = \zeta^n$ are

$$C_1 r^n \cos(n\theta) \; ; \quad C_2 r^n \sin(n\theta) \; ; \quad C_3 r^{-n} \cos(n\theta) \; ; \quad C_4 r^{-n} \sin(n\theta) \; , \tag{16.19}$$

where $C_1, ...C_4$ are arbitrary constants. The first two of these correspond to polynomials in Cartesian coördinates, such as

$$x \; ; \quad y \; ; \quad x^2 - y^2 \; ; \quad xy \; ; \quad x^3 - 3y^2 x \; ; \quad y^3 - 3x^2 y \tag{16.20}$$

etc.

Approximate solutions of the torsion problem for particular cross-sections can be obtained by combining functions such as (16.17, 16.18) and adjusting the multiplying constants so as to achieve a contour approximating the required shape. A relatively small number of simple shapes permit exact closed form solutions, including the circle (Problem 16.1), the equilateral triangle (Problem 16.4), a circle with a semicircular groove (Problem 16.5) and the ellipse, which we solve here as an example.

Example — the elliptical cross-section

We consider the bar of elliptical cross-section defined by the boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \, , \tag{16.21}$$

loaded by a torque T. The quadratic function

$$\varphi = C \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \tag{16.22}$$

clearly satisfies the boundary condition (16.10). Substituting into (16.12), we obtain

$$\nabla^2 \varphi = C \left(\frac{2}{a^2} + \frac{2}{b^2} \right) = -2\mu\beta \, , \tag{16.23}$$

which will be satisfied for all x, y if

$$C = -\frac{\mu\beta a^2 b^2}{a^2 + b^2} \, . \tag{16.24}$$

The torque T is obtained from (16.14) as

$$T = 2C \int_{-b}^{b} \int_{-a(1-y^2/b^2)}^{a(1-y^2/b^2)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) dxdy = -\pi abC \tag{16.25}$$

and hence

$$C = -\frac{T}{\pi ab} \; ; \quad \beta = \frac{T(a^2 + b^2)}{\pi \mu a^3 b^3} \, . \tag{16.26}$$

The stresses are then obtained from (16.8) as

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} = \frac{2Cy}{b^2} = -\frac{2Ty}{\pi ab^3} \tag{16.27}$$

$$\sigma_{zy} = -\frac{\partial \varphi}{\partial x} = -\frac{2Cx}{a^2} = \frac{2Tx}{\pi a^3 b} \, . \tag{16.28}$$

The torsional rigidity of the section K, defined such that

$$T = \mu K \beta \, , \tag{16.29}$$

is

$$K = \frac{\pi a^3 b^3}{(a^2 + b^2)} \, . \tag{16.30}$$

16.2 The membrane analogy

Equation (16.12) with the boundary condition (16.10) is similar in form to that governing the displacement of an elastic membrane stretched between the boundaries of the cross-sectional curve and loaded by a uniform normal pressure. This analogy is useful as a means of estimating the effect of the cross-section on the maximum shear stress and the torsional rigidity of the bar. You can perform a simple experiment by making a plane wire frame in the shape of the cross-section, dipping it into a soap solution to develop a soap film and then blowing gently against one side to see the shape of the deformed film. The stress function is then proportional to the displacement of the film from the plane of the frame. The transmitted torque is proportional to the integral of the stress function over the cross-section (see equation (16.14)) and hence the stiffest cross-sections are those which permit the maximum volume to be developed between the deformed film and the plane of the frame for a given pressure. The shear stress is proportional to the slope of the film. For sections such as squares, rectangles, circles, ellipses etc, the maximum displacement will occur near the centre of the section and the maximum slope (and hence shear stress) will be at the point on the boundary nearest the centre. For thin-walled sections, the maximum displacement will occur where the section is thickest and the maximum stress will be at the corresponding point on the boundary. However, if there are reëntrant corners in the section, these will cause locally increased stresses which are unbounded if the corner is sharp.

Notice that it is not really necessary to perform the experiment to come up with these conclusions. A 'thought experiment' is generally sufficient.

16.3 Thin-walled open sections

Many structural components consist of thin-walled sections, for example I-beams, channel sections and turbine blades. Figure 16.1 shows a generic thin-walled section of length b, whose thickness t varies with distance ξ along the length. As long as the thickness t of the section does not vary too rapidly with ξ, it is clear from the membrance analogy that we can neglect the curvature of the membrane (and hence of the stress function) in the ξ-direction, reducing the governing equation to

$$\frac{d^2\varphi}{d\eta^2} = -2\mu\beta \; , \tag{16.31}$$

where η is a coördinate orthogonal to ξ measured from the local mid-point of the cross-section. The solution of this equation satisfying the condition

$$\varphi = 0 \quad ; \quad \eta = \pm\frac{t}{2} \tag{16.32}$$

is

$$\varphi = \mu\beta\left(\frac{t^2}{4} - \eta^2\right) \tag{16.33}$$

and the stress field is

$$\sigma_{z\xi} = \frac{\partial\varphi}{\partial\eta} = -2\mu\beta\eta\ ; \quad \sigma_{z\eta} = 0\ . \tag{16.34}$$

Thus, the shear stress is everywhere parallel to the centreline of the section and varies linearly from that line reaching a maximum

$$\tau_{\text{max}}(\xi) = \mu\beta t(\xi) \tag{16.35}$$

at the edges of the section $\xi = \pm t(\xi)/2$. It follows that the overall maximum shear stress must occur at the point where the thickness is a maximum and is

$$\tau_{\text{max}} = \mu\beta t_{\text{max}}\ . \tag{16.36}$$

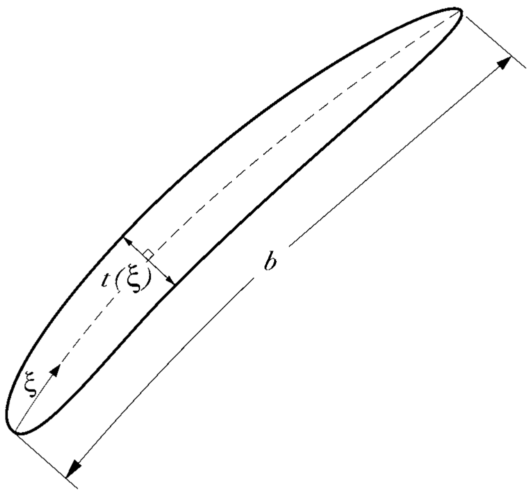

Figure 16.1: Thin-walled open section with variable thickness.

The torque transmitted by the section is obtained from equations (16.14, 16.33) as

$$T = 2\int_0^b \int_{-t(\xi)/2}^{t(\xi)/2} \varphi d\eta d\xi = \frac{\mu\beta}{3}\int_0^b t(\xi)^3 d\xi \tag{16.37}$$

and hence the torsional rigidity of the section is

$$K = \frac{1}{3}\int_0^b t(\xi)^3 d\xi\ . \tag{16.38}$$

In the special case where the thickness is constant, we have

$$K = \frac{bt^3}{3}\ ; \quad \tau_{\text{max}} = \mu\beta t = \frac{3T}{bt^2}\ . \tag{16.39}$$

16.4 The rectangular bar

The results of the previous section can be used to obtain an approximate solution for the torsion of the rectangular bar $-a < x < a, -b < y < b$, when $b \gg a$. Taking the origin at the centre of the cross-section, we obtain

$$\varphi = \mu\beta(a^2 - x^2) \tag{16.40}$$

from (16.33). The corresponding torsional rigidity and maximum shear stress are

$$K = \frac{16a^3b}{3} \; ; \quad \tau_{\max} = 2\mu\beta a = \frac{3T}{8a^2b} \; . \tag{16.41}$$

The stress function (16.40) satisfies the governing equation (16.12) and the boundary condition on the long edges

$$\varphi = 0 \; ; \quad x = \pm a \; , \tag{16.42}$$

but it does not satisfy the corresponding boundary condition on the short edges

$$\varphi = 0 \; ; \quad y = \pm b \; . \tag{16.43}$$

To satisfy this condition and hence to obtain an exact solution for all values of the ratio a/b, we need to supplement the stress function (16.40) with appropriate solutions of the homogeneous equation (16.16). These additional functions must preserve the condition (16.42) and we therefore seek separated-variable solutions, as in §6.2.1. Indeed the whole procedure of 'correcting' the approximate solution (16.40) is exactly analogous to that discussed in §6.2. For harmonic functions, separation of variables demands that the functions be trigonometric in one coordinate and exponential in the other and the symmetry of the boundary-value problem about both axes suggests harmonic functions of the form $\cos(\lambda x)\cosh(\lambda y)$. Selecting those values of λ that satisfy (16.42), we obtain the stress function

$$\varphi = \mu\beta(a^2 - x^2) + \sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi x}{2a}\right) \cosh\left(\frac{(2n-1)\pi y}{2a}\right) \; , \tag{16.44}$$

where C_n is a set of arbitrary constants. The function (16.44) will satisfy (16.43) if

$$\sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi x}{2a}\right) \cosh\left(\frac{(2n-1)\pi b}{2a}\right) = -\mu\beta(a^2 - x^2) \; . \tag{16.45}$$

To evaluate the constants C_n, we multiply both sides of (16.45) by $\cos((2m-1)\pi x/2a)$ and integrate in $-a < x < a$, obtaining

$$\sum_{n=1}^{\infty} C_n \int_{-a}^{a} \cos\left(\frac{(2n-1)\pi x}{2a}\right) \cosh\left(\frac{(2n-1)\pi b}{2a}\right) \cos\left(\frac{(2m-1)\pi x}{2a}\right) dx$$

$$= -\mu\beta \int_{-a}^{a} (a^2 - x^2) \cos\left(\frac{(2m-1)\pi x}{2a}\right) dx \qquad (16.46)$$

and hence

$$C_m a \cosh\left(\frac{(2m-1)\pi b}{2a}\right) = \frac{32\mu\beta(-1)^m a^3}{\pi^3(2m-1)^3} \qquad (16.47)$$

or

$$C_m = \frac{32\mu\beta(-1)^m a^2}{\pi^3(2m-1)^3 \cosh((2m-1)\pi b/2a)} , \qquad (16.48)$$

where we have used the orthogonality condition

$$\int_{-a}^{a} \cos\left(\frac{(2n-1)\pi x}{2a}\right) \cos\left(\frac{(2m-1)\pi x}{2a}\right) dx = a\delta_{mn} . \qquad (16.49)$$

The transmitted torque is obtained from equations (16.14, 16.44) as

$$T = \frac{16\mu\beta a^3 b}{3} - \sum_{n=1}^{\infty} \frac{32(-1)^n C_n a^2 \sinh((2n-1)\pi b/2a)}{\pi^2(2n-1)^2}$$

$$= \frac{16\mu\beta a^3 b}{3} \left(1 - \sum_{n=1}^{\infty} \frac{192a}{\pi^5(2n-1)^5 b} \tanh\left(\frac{(2n-1)\pi b}{2a}\right)\right) , \qquad (16.50)$$

using (16.48) to substitute for the constants C_n. The maximum shear stress occurs at the points $(\pm a, 0)$ and is

$$\tau_{\max} = \left|\frac{\partial\varphi}{\partial x}(\pm a, 0)\right| = 2\mu\beta a - \sum_{n=1}^{\infty} \frac{16\mu\beta a}{\pi^2(2n-1)^2 \cosh((2n-1)\pi b/2a)} . \qquad (16.51)$$

We can define the coördinate system such that $0 < a/b < 1$ without loss of generality and in this range the series (16.50, 16.51) converge extremely rapidly. In fact, taking just one term of each series gives better than 0.5% accuracy. Figure 16.2 shows the effect of the aspect ratio on the torsional rigidity and the maximum shear stress, normalized with respect to the thin-walled expressions (16.41).

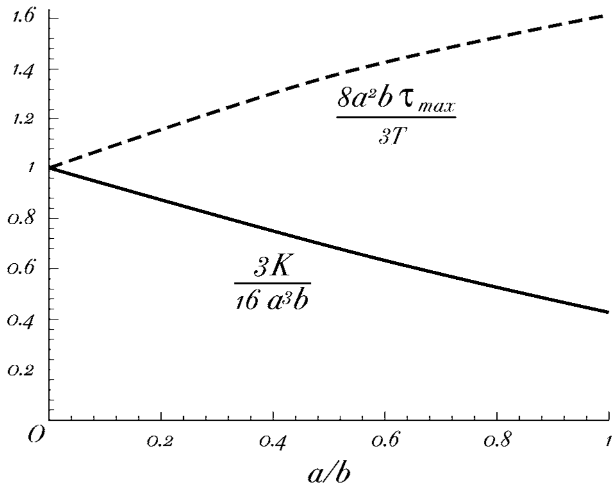

Figure 16.2: Effect of aspect ratio a/b on the torsional rigidity and maximum shear stress in a rectangular bar.

16.5 Multiply connected (closed) sections

The traction-free boundary condition (16.9) implies that the stress function remains constant as we move around the boundary Γ and we chose to take this constant as zero in equation (16.10). However, if the bar contains one or more enclosed holes, the boundary will be defined by more than one closed curve $\Gamma_0, \Gamma_1, \Gamma_2$ etc. The condition (16.9) implies that the value of φ on each such curve will be constant — e.g.

$$\varphi = \varphi_0 \text{ on } \Gamma_0 \; ; \quad \varphi = \varphi_1 \text{ on } \Gamma_1 \; ,$$

but we cannot conclude that $\varphi_1 = \varphi_0$, since the curves Γ_1, Γ_0 do not connect.

Since the expressions for the stresses involve derivatives of φ, one of the values $\varphi_0, \varphi_1, ...$ can be selected arbitrarily and it is convenient to define $\varphi_0 = 0$, where the corresponding curve Γ_0 defines the external boundary of the bar. If there is a single hole with boundary Γ_1, the constant φ_1 is an additional unknown which must be determined by imposing a Cesaro integral condition, as discussed in §2.2.1. In the present case, this condition is easy to define, since we started from a displacement formulation in equations (16.1).

We require that the displacements be single-valued functions of x, y and hence that the integral

$$\oint_S \frac{\partial f}{\partial s} ds = \oint_S \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) = 0 \tag{16.52}$$

for all closed curves S. The stress function representation guarantees this for infinitesimal curves, but as in §2.2.1, we must explicitly enforce the condition around a representative curve S that encircles the hole.

Eliminating f from (16.52) using (16.4), we obtain

$$\frac{1}{\mu}\oint_S (\sigma_{zx}dx + \sigma_{zy}dy) - \beta\oint_S (xdy - ydx) = 0 \ . \tag{16.53}$$

The second integral in (16.53) is equal to twice the area A enclosed by S, so we obtain the condition

$$\oint_S \sigma_{zs}ds = 2\mu\beta A \ , \tag{16.54}$$

which serves to determine the free constant φ_1.

If the section contains several holes, there will be a corresponding number of additional unknowns, but for each hole we can write an integral of the form (16.54) where the corresponding path S is taken to encircle that particular hole.

The torque transmitted by a multiply connected section can still be calculated by (16.13), provided that we recognize that there will be no contribution from those parts of the total domain Ω within the holes. This can be achieved by adopting the convention that $\varphi = \varphi_1$ throughout the area enclosed by Γ_1 as well as along the boundary. The derivatives in the integrand in (16.13) in this area will then be zero as required. With this convention we can still use the simple expression (16.14) for the torque T, with the result

$$T = 2\iint_{\bar{\Omega}} \varphi dxdy + 2\sum A_i\varphi_i \ , \tag{16.55}$$

where A_i is the area of the hole enclosed by the boundary Γ_i on which $\varphi = \varphi_i$ and $\bar{\Omega} = \Omega - \sum A_i$ — i.e. the 'solid' area of the bar excluding the holes.

16.5.1 Thin-walled closed sections

Most practical applications of multiply connected sections have thin walls, relative to the other dimensions of the section, as shown for example in Figure 16.3. In this case, the stress function φ can reasonably be approximated as a linear function between φ_1 and zero on the two adjacent boundaries, implying a local shear stress

$$\sigma_{zs} = \frac{\varphi_1}{t} \ , \tag{16.56}$$

where t is the local wall thickness. The value of φ_1 can then be determined from (16.54) as

$$\varphi_1 = 2\mu\beta A \Bigg/ \oint_S \frac{ds}{t} \tag{16.57}$$

and the torque is approximately

$$T = 2A\varphi_1 \tag{16.58}$$

from (16.55), where A is the area enclosed by the mean line equidistant between the inner and outer boundary.

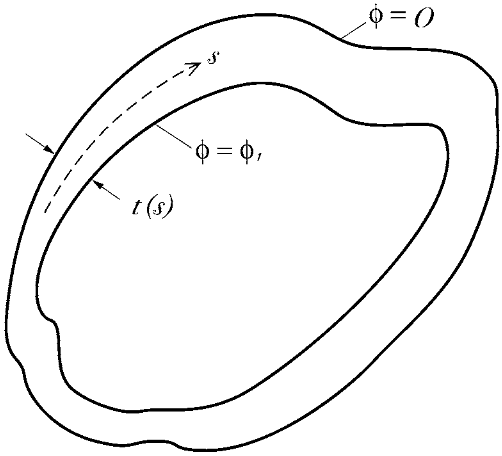

Figure 16.3: A thin walled closed section.

It follows that the torsional rigidity of a thin-walled closed section is approximately

$$K = 4A^2 \Bigg/ \oint_S \frac{ds}{t} \quad . \tag{16.59}$$

These results can also be derived without reference to a stress function, using Mechanics of Materials arguments[1]. For more details, including practical examples, the reader is referred to the many texts on Advanced Mechanics of Materials[2].

PROBLEMS

1. Find the solution for torsion of a solid circular shaft of radius a by setting $b=a$ in the solution for the elliptical bar. Express the stress function and hence the stress components in polar coördinates and verify that these results and the relation between T and β agree with those given by the elementary

[1] J.R.Barber, *Intermediate Mechanics of Materials*, McGraw-Hill, Burr Ridge, 2000, Chapter 6

[2] See for example A.P.Boresi, R.J.Schmidt and O.M.Sidebottom, *Advanced Mechanics of Materials,* John Wiley, New York, 5th edn., 1993, Chapter 6, W.B.Bickford, *Advanced Mechanics of Materials,*, Addison Wesley, Menlo Park, 1998, Chapter 3.

Mechanics of Materials theory of torsion. Also, verify that there is no warping of the cross-section $f=0$ in this case.

2. A bar transmitting a torque T has the cross-section of a thin sector $0 < r < a, -\alpha < \theta < \alpha$, where $\alpha \ll 1$. Develop an approximate solution for the shear stress distribution and the twist per unit length, using strong boundary conditions on the edges $\theta=\pm\alpha$ only.

3. A bar transmitting a torque T has the cross-section of an equilateral triangle of side a. Find the equations of the three sides of the triangle in Cartesian coördinates, taking the origin at one corner and the x-axis to bisect the opposite side. Express these equations in the form $f_i(x,y)=0$, $i=1,2,3$.

Show that the function

$$\varphi = Cf_1(x,y)f_2(x,y)f_3(x,y)$$

satisfies the boundary condition (16.10) and can be made to satisfy (16.12) with a suitable choice of the constant C. Hence find the stress field in the bar and make a contour plot of the maximum shear stress. Why can this method not be used for a more general triangular cross-section?

4. A bar transmitting a torque T has the cross-section of a sector $0 < r < a, -\alpha < \theta < \alpha$, where α is not small compared with unity.

(i) Find a combination of the functions r^2 and $r^2 \cos(2\theta)$ that satisfies the governing equation (16.12) and the boundary condition (16.10) *on the straight edges* $\theta=\pm\alpha$ *only.*
(ii) Superpose a series of harmonic functions of the form $r^\gamma \cos(\gamma\theta)$ where γ is chosen to satisfy (16.10) on $\theta=\pm\alpha$.
(iii) Choose the coefficients on these terms so as to satisfy the boundary condition on the curved edge $r=a$ and hence obtain expressions for the torsional rigidity K and the maximum shear stress.
(iv) Evaluate your results for the special case $\alpha=\pi/6$ and compare them with those obtained using the thin-walled approximation of §16.3.

5. Figure 16.4 shows the cross-section of a circular shaft of radius a with a circular groove of radius b whose centre lies on the surface of the ungrooved shaft. Construct a suitable stress function for this problem using the particular solution

$$\varphi_P = C(x^2 + y^2 - b^2)$$

and a homogeneous solution φ_H comprising appropriate multipliers of the harmonic functions $r\cos\theta$ and $r^{-1}\cos\theta$ centred on the point B, which in Cartesian coördinates can be written

$$x \text{ and } \frac{x}{x^2+y^2} .$$

Remember that the resulting function must be zero at all points on both the circles $x^2+y^2=b^2$ and $(x-a)^2+y^2=a^2$. Hence find the stress field in the bar and make a contour plot of the maximum shear stress for the case where $b=0.1a$.

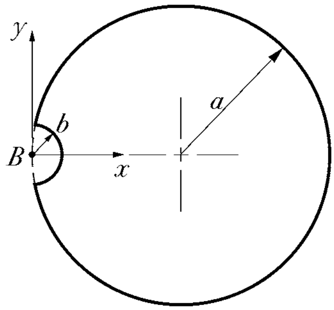

Figure 16.4: Circular shaft with a semi-circular groove.

6. Figure 16.5 shows part of a thin-walled open section that is loaded in torsion. Find the stress function φ in the curved section, assuming that it depends upon r only and hence estimate[3] the maximum shear stress in the corner if the twist per unit length is β and the shear modulus is μ.

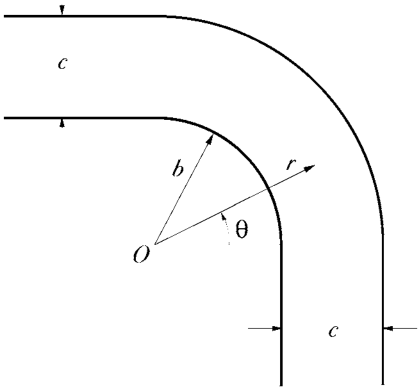

Figure 16.5: Stress concentration due to a bend.

Find the corresponding maximum shear stress in the straight segment, distant from the corner and hence find the stress concentration factor due to the corner. Plot the stress concentration factor as a function of the ratio of the inner radius to the section thickness (b/c).

[3] This solution is approximate because the stress function is not independent of θ in the curved region. Otherwise there would be a discontinuity at the curved/straight transition.

7. Approximations to the stresses in a square bar in torsion can be obtained by adding harmonic functions of the form $r^n\cos(n\theta)$ to the axisymmetric particular solution (16.17).

Express the function

$$\varphi = r^2 + r^4\cos(4\theta)$$

in Cartesian coördinates and make a contour plot. Select the contour which is the best approximation to a square with rounded corners and scale the resulting function to obtain an approximate solution for the maximum shear stress in a square bar of side a when the twist per unit length is β. Compare the result with that obtained from Figure 16.2 when $b/a=1$.

8. We know from Mechanics of Materials that a bar cross-section posesses a *shear centre* which is the point through which a shear force must act if there is to be no twist of the section. Maxwell's reciprocal theorem[4] implies that the shear centre must also be the only point in the section that has no in-plane displacement when the bar is loaded in torsion and hence that equations (16.1) are correct if and only if the shear centre is taken as origin.

Rederive the torsion theory using a more general origin and hence show that the formulation in terms of Prandtl's stress function remains true for all choices of origin.

9. A titanium alloy turbine blade is idealized by the section of Figure 16.6. The leading edge is a semicircle of radius 2 mm and the inner and outer edges of the trailing section are cylindrical surfaces of radius 43 mm and 35 mm respectively, with centres of curvature on the line Oy. Find the twist per unit length β and the maximum shear stress when the section is loaded by a torque of 5 Nm. The shear modulus for the alloy is 43 GPa.

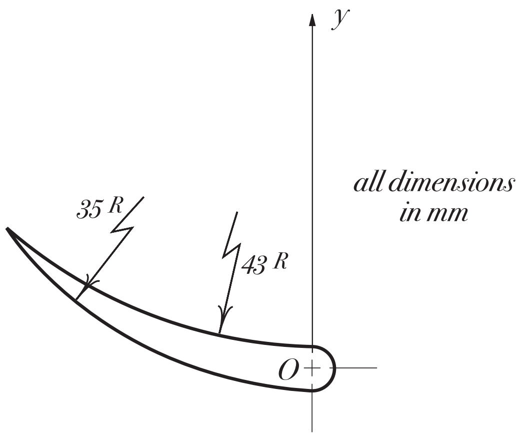

Figure 16.6: Turbine blade cross-section.

[4] See Chapter 34.

10. The W200×22 I-beam of Figure 16.7 is made of steel ($G = 80$ GPa) and is 15 m long. One end is built in and the other is loaded by two forces F constituting a torque, as shown. Find the torsional rigidity K for the beam section and hence determine the value of F needed to cause the end to twist through an angle of 20^o. Does the result surprise you? How large an angle of twist[5] could a person of average strength produce in such a beam using his or her hands only?

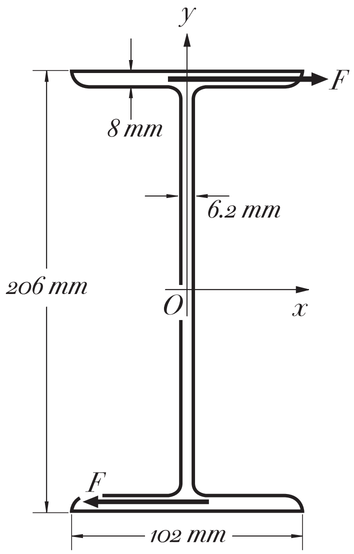

Figure 16.7: I-beam twisted by two forces.

11. Show that the torsional rigidity of a closed thin-walled steel tube of uniform thickness is reduced by the factor

$$\frac{12A^2}{A_s^2}$$

when a longitudinal slit is made down the whole length of the tube to form an open section. In this expression, A is the area enclosed by the mean line of the closed section and A_s is the cross-sectional area of the steel.

Find also by what ratio the maximum shear stress will increase for a given applied torque.

[5] If you want to impress your unsuspecting colleagues with your engineering expertise (or your immense strength), ask them to guess the answer to this question and then perform the experiment.

12. The outer and inner circular walls of a closed thin-walled circular tube have slightly different centres, as shown in Figure 16.9, so that the thickness of the section varies according to

$$t = t_0(1 - \epsilon\cos\theta) ,$$

where $\epsilon < 1$. The mean radius of the tube is a. Find the torsional rigidity K and the maximum shear stress when the bar is loaded by a torque T. Plot your results as functions of ϵ in the range $0 \leq \epsilon < 1$.

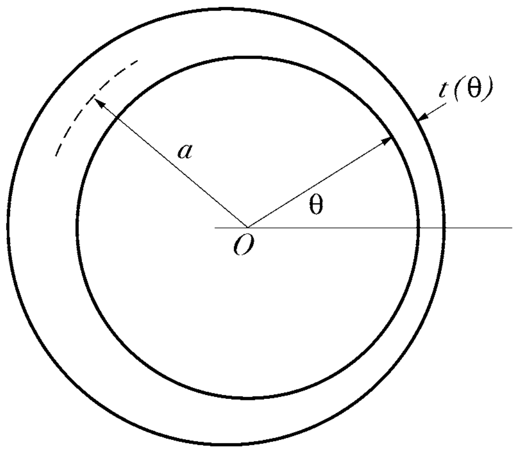

Figure 16.8: Cylindrical tube with variable wall thickness.

13. Show that the stress function

$$\varphi = -\frac{\mu\beta}{2}\left(x^2 + y^2 - 2a^2 - 2ax + \frac{4a^3x}{(x^2+y^2)}\right)$$

defines the exact solution for the cross-section defined by the intersection of the circles

$$x^2 + y^2 = 2a^2 \quad \text{and} \quad (x-a)^2 + y^2 = a^2 .$$

Find the corresponding torque T and the maximum shear stress τ_{max} and express the latter as a function of T, a only.

Now find an approximate solution to this problem by assuming that it is a 'thin-walled section' — i.e. using equations (16.36, 16.38). What is the percentage error in the approximate solution (i) in the stiffness K and (ii) in τ_{max}? Are these quantities overestimated or underestimated by the approximate solution?

14. Figure 16.9 shows the cross-section of a solid circular bar of radius a which is slit to the centre along the line $\theta=\pm\pi$. The bar is subjected to a prescribed twist β per unit length.

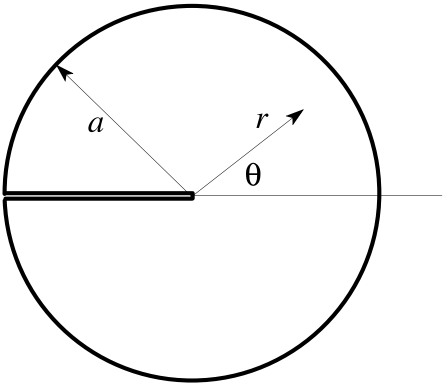

Figure 16.9

(i) Show that the particular solution

$$\varphi_P = -\mu\beta y^2 = -\mu\beta r^2 \sin^2(\theta)$$

satisfies the governing equation and also satisfies the condition that $\varphi=0$ on the slit $\theta=\pm\pi$.

(ii) To complete the solution, we need to superpose a series of stress functions each of which satisfies the homogeneous equation

$$\nabla^2 \varphi_H = 0$$

and $\varphi = 0$ on $\theta = \pm\pi$. Show that the functions $r^\gamma \cos(\gamma\theta)$ meet these conditions with a suitable choice of γ.

(iii) Use these solutions to construct a series solution of the problem and use the boundary condition on the curved edge $r=a$ to define equations that the coefficients of the series must satisfy.

(iv) Solve the equations and hence determine the mode III stress-intensity factor K_{III} at $r=0$, defined as

$$K_{III} = \lim_{r\to 0} \sigma_{z\theta}(r,0)\sqrt{2\pi r}\ .$$

17

SHEAR OF A PRISMATIC BAR[1]

In this chapter, we shall consider the problem in which a prismatic bar occupying the region $z > 0$ is loaded by transverse forces F_x, F_y in the negative x- and y-directions respectively on the end $z = 0$, the sides of the bar being unloaded. Equilibrium considerations then show that there will be shear forces

$$V_x \equiv \iint_\Omega \sigma_{zx} dxdy = F_x \ \ ; \ \ V_y \equiv \iint_\Omega \sigma_{zy} dxdy = F_y \tag{17.1}$$

and bending moments

$$M_x \equiv \iint_\Omega \sigma_{zz} y dxdy = zF_y \ \ ; \ \ M_y \equiv -\iint_\Omega \sigma_{zz} x dxdy = -zF_x \ , \tag{17.2}$$

at any given cross-section Ω of the bar. In other words, the bar transmits constant shear forces, but the bending moments increase linearly with distance from the loaded end.

17.1 The semi-inverse method

To solve this problem, we shall use a semi-inverse approach — i.e. we shall make certain assumptions about the stress distribution and later verify that these assumptions are correct by showing that they are sufficiently general to define a solution that satisfies all the equilibrium and compatibility equations. The assumptions we shall make are

(i) The stress component σ_{zz} varies linearly with x, y, as in the elementary bending theory.

[1] In many texts, this topic is referred to as 'bending' or 'flexure' of prismatic bars. However, it should not be confused with 'pure bending' due to the application of equal and opposite bending moments at the two ends of the bar. The present topic is the Elasticity counterpart of that referred to in Mechanics of Materials as 'shear stress distribution in beams'.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_17, © Springer Science+Business Media B.V. 2010

(ii) The shear stresses σ_{zx}, σ_{zy} due to the shear forces are indpendent of z.
(iii) The in-plane stresses $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0$.

A more direct justification for these assumptions can be established using results from Chapter 28, where we discuss the more general problem of the prismatic bar loaded on its lateral surfaces (see §28.5).

The bending moments vary linearly with z, so assumption (i) implies that

$$\sigma_{zz} = (Ax + By)z \; , \tag{17.3}$$

where A, B are two unknown constants and we have chosen to locate the origin at the centroid of Ω. To determine A, B, we substitute (17.3) into (17.2) obtaining

$$zF_y = (AI_{xy} + BI_x)z \;\; ; \;\; -zF_x = -(AI_y + BI_{xy})z \; , \tag{17.4}$$

where

$$I_x = \iint_\Omega y^2 dxdy \;\; ; \;\; I_y = \iint_\Omega x^2 dxdy \;\; ; \;\; I_{xy} = \iint_\Omega xydxdy \tag{17.5}$$

are the second moments of area of the section. Solving (17.4), we have

$$A = \frac{F_x I_x - F_y I_{xy}}{I_x I_y - I_{xy}^2} \;\; ; \;\; B = \frac{F_y I_y - F_x I_{xy}}{I_x I_y - I_{xy}^2} \; . \tag{17.6}$$

17.2 Stress function formulation

Assumptions (ii,iii) imply that the first two equilibrium equations (2.2, 2.3) are satisfied identically, whilst (2.4) requires that

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + Ax + By = 0 \; . \tag{17.7}$$

This equation will be satisfied identically if the shear stresses are defined in terms of a stress function φ through the relations

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} - \frac{Ax^2}{2} \;\; ; \;\; \sigma_{zy} = -\frac{\partial \varphi}{\partial x} - \frac{By^2}{2} \; . \tag{17.8}$$

This is essentially a generalization of Prandtl's stress function of equations (16.8, 16.9). The governing equation for φ is then obtained by using Hooke's law to determine the strains and substituting the resulting expressions into the six compatibility equations (2.10). The procedure is algebraically lengthy but routine and will be omitted here. It is found that the three equations of the form (2.8) and one of the three like (2.9) are satisfied identically by the assumed stress field. The two remaining equations yield the conditions

$$\frac{\partial}{\partial x}\nabla^2\varphi = -\frac{B\nu}{(1+\nu)} \quad ; \quad \frac{\partial}{\partial y}\nabla^2\varphi = \frac{A\nu}{(1+\nu)} \tag{17.9}$$

which imply that

$$\nabla^2\varphi = \frac{\nu}{(1+\nu)}(Ay - Bx) + C \ , \tag{17.10}$$

where C is an arbitrary constant of integration.

17.3 The boundary condition

We require the boundaries of Ω to be traction-free and hence

$$\sigma_{zn} = \sigma_{zx}\frac{dy}{dt} - \sigma_{zy}\frac{dx}{dt} = 0 \ , \tag{17.11}$$

where n, t is a Cartesian coördinate system locally normal and tangential to the boundary. Substituting for the stresses from equation (17.8), we obtain

$$\frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial x}\frac{dx}{dt} = \frac{Ax^2}{2}\frac{dy}{dt} - \frac{By^2}{2}\frac{dx}{dt} \ , \tag{17.12}$$

or

$$\frac{\partial\varphi}{\partial t} = \frac{Ax^2}{2}\frac{dy}{dt} - \frac{By^2}{2}\frac{dx}{dt} \ . \tag{17.13}$$

17.3.1 Integrability

In order to define a well-posed boundary-value problem for φ, we need to integrate (17.13) with respect to t to determine the value of φ at all points on the boundary. This process will yield a single-valued expression except for an arbitrary constant *if and only if* the integral

$$\oint_\Gamma \left(\frac{Ax^2}{2}\frac{dy}{dt} - \frac{By^2}{2}\frac{dx}{dt}\right) dt = 0 \ , \tag{17.14}$$

where Γ is the boundary of the cross-sectional domain Ω, which we here assume to be simply connected. Using Green's theorem in the form

$$\iint_\Omega \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dxdy = \int_\Gamma \left(f\frac{dy}{dt} - g\frac{dx}{dt}\right) dt \ , \tag{17.15}$$

we obtain

$$\oint_\Gamma \left(\frac{Ax^2}{2}\frac{dy}{dt} - \frac{By^2}{2}\frac{dx}{dt}\right) dt = \iint_\Omega (Ax + By)dxdy \ . \tag{17.16}$$

The integral on the right-hand side of (17.16) is a linear combination of the two first moments of area for Ω and it is zero because we chose the origin to be at the centroid.

We conclude that (17.13) can always be integrated and the problem is therefore reduced to the solution of the Poisson equation (17.10) with prescribed boundary values of φ. This problem has a unique solution for all simply connected[2] domains, thus justifying the assumptions made in §17.1.

17.3.2 Relation to the torsion problem

In the special case where $F_x = F_y = 0$, we have $A = B = 0$ from (17.6) and the problem is defined by the equations

$$\nabla^2 \varphi = C \quad \text{in } \Omega \tag{17.17}$$

$$\frac{\partial \varphi}{\partial t} = 0 \quad \text{on } \Gamma \tag{17.18}$$

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} \;\; ; \;\; \sigma_{zy} = -\frac{\partial \varphi}{\partial x} \, , \tag{17.19}$$

which are identical with (16.12, 16.9, 16.8) of the previous chapter if we write $C = -2\mu\beta$. Thus, the present formulation includes the solution of the torsion problem as a special case and the constant C is proportional to the twist of the section and hence to the applied torque.

When the resultant shear force $\boldsymbol{F} \neq 0$, superposing a torque on the loading is equivalent to changing the line of action of $\boldsymbol{F}$. The constant C in equation (17.10) provides the degree of freedom needed for this generalization. It is generally easier to decompose the problem into two parts — a shear loading problem in which C is set arbitrarily to zero and a torsion problem (in which A and B are zero). These can be combined at the end to ensure that the resultant force has the required line of action. In the special case where the section is symmetric about the axes, setting $C = 0$ will generate a stress field with the same symmetry, corresponding to loading by a shear force acting through the origin.

17.4 Methods of solution

The problem defined by equations (17.10, 17.13) can be solved by methods similar to those used in the torsion problem and discussed in §16.1.1. A particular solution to (17.10) is readily found as a third degree polynomial and more general solutions can then be obtained by superposing appropriate harmonic functions. Closed form solutions can be obtained only for a limited number of cross-sections, including the circle, the ellipse and the equilateral triangle. Other cases can be treated by series methods.

[2] If the cross-section of the bar has one or more enclosed holes, additional unknowns will be introduced corresponding to the constants of integration of (17.13) on the additional boundaries. A corresponding set of additional algebraic conditions is obtained from the Cesaro integrals around each hole, as in §16.5.

17.4.1 The circular bar

We consider the case of the circular cylindrical bar bounded by the surface

$$x^2 + y^2 = a^2 \tag{17.20}$$

and loaded by a shear force $F_y = F$ acting through the origin[3]. For the circular section, we have

$$I_x = I_y = \frac{\pi a^4}{4} \;\; ; \;\; I_{xy} = 0 \tag{17.21}$$

and hence

$$A = 0 \;\; ; \;\; B = \frac{4F}{\pi a^4} \; , \tag{17.22}$$

from (17.6). Substituting into (17.13) and integrating, we obtain

$$\varphi = -\int \frac{2Fy^2 dx}{\pi a^4} \; , \tag{17.23}$$

where the integration must be performed on the line (17.20). Substituting for y from (17.20) and integrating, we obtain

$$\varphi = -\int \frac{2F(a^2 - x^2)dx}{\pi a^4} = -\frac{2F(3a^2 x - x^3)}{3\pi a^4} \quad \text{on } x^2 + y^2 = a^2 \; , \tag{17.24}$$

where we have set the constant of integration to zero to preserve antisymmetry about $x=0$.

Equations (17.10, 17.24) both suggest a third order polynomial for φ with odd powers of x and even powers of y, so we use the trial function

$$\varphi = C_1 x^3 + C_2 x y^2 + C_3 x \; . \tag{17.25}$$

Substitution into (17.10, 17.24) yields the conditions

$$\begin{aligned} 6C_1 x + 2C_2 x &= -\frac{4F\nu x}{\pi a^4 (1+\nu)} \\ C_1 x^3 + C_2 x(a^2 - x^2) + C_3 x &= -\frac{2F(3a^2 x - x^3)}{3\pi a^4} \; , \end{aligned} \tag{17.26}$$

which will be satisfied for all x if

$$\begin{aligned} 6C_1 + 2C_2 &= -\frac{4F\nu}{\pi a^4 (1+\nu)} \\ C_1 - C_2 &= \frac{2F}{3\pi a^4} \\ C_2 a^2 + C_3 &= -\frac{2F}{\pi a^2} \end{aligned} \tag{17.27}$$

[3] This problem is solved by a different method in §25.3.1.

with solution

$$C_1 = \frac{F(1-2\nu)}{6\pi a^4(1+\nu)} \;\; ; \;\; C_2 = -\frac{F(1+2\nu)}{2\pi a^4(1+\nu)} \;\; ; \;\; C_3 = -\frac{F(3+2\nu)}{2\pi a^2(1+\nu)} \; . \quad (17.28)$$

The stress components are then recovered from (17.8) as

$$\sigma_{zx} = -\frac{F(1+2\nu)xy}{\pi a^4(1+\nu)} \;\; ; \;\; \sigma_{zy} = \frac{F\{(3+2\nu)(a^2-y^2)-(1-2\nu)x^2\}}{2\pi a^4(1+\nu)} \; . \quad (17.29)$$

The maximum shear stress occurs at the centre and is

$$\tau_{\max} = \frac{F(3+2\nu)}{2\pi a^2(1+\nu)} \; . \quad (17.30)$$

The elementary Mechanics of Materials solution gives

$$\tau^*_{\max} = \frac{4F}{3\pi a^2} \; , \quad (17.31)$$

which is exact for $\nu = 0.5$ and 12% lower than the exact value for $\nu = 0$.

17.4.2 The rectangular bar

As a second example, we consider the rectangular bar whose cross-section is bounded by the lines $x = \pm a, y = \pm b$ loaded by a force $F_y = F$ in the negative y-direction acting through the origin. As in the corresponding torsion problem of §16.4, we cannot obtain a closed form solution to this problem. The strategy is to seek a simple solution that satisfies the boundary conditions on the two longer edges in the strong sense and then superpose a series of harmonic functions that leaves this condition unchanged, whilst providing extra degrees of freedom to satisfy the condition on the remaining edges.

For the rectangular section, we have

$$I_x = \frac{4ab^3}{3} \;\; ; \;\; I_y = \frac{4a^3b}{3} \;\; ; \;\; I_{xy} = 0 \quad (17.32)$$

and hence

$$A = 0 \;\; ; \;\; B = \frac{3F}{4ab^3} \; , \quad (17.33)$$

from (17.6). The boundary condition (17.12) then requires that

$$\frac{\partial \varphi}{\partial x}(x, \pm b) = -\frac{Bb^2}{2} = -\frac{3F}{8ab} \;\; ; \;\; \frac{\partial \varphi}{\partial y}(\pm a, y) = 0 \; . \quad (17.34)$$

The governing equation (17.10) is satisfied by a third degree polynomial and the most general such function with the required symmetry is

$$\varphi = C_1 x^3 + C_2 xy^2 + C_3 x \; . \quad (17.35)$$

Substitution in (17.10) shows that

$$6C_1 + 2C_2 = -\frac{3F\nu}{4ab^3(1+\nu)} \tag{17.36}$$

and on the boundaries of the rectangle we have

$$\frac{\partial\varphi}{\partial x}(x,\pm b) = 3C_1x^2 + C_2b^2 + C_3 \;\; ; \;\; \frac{\partial\varphi}{\partial y}(\pm a, y) = \pm C_2 ay \; . \tag{17.37}$$

This solution permits us to satisfy the conditions (17.34) in the strong sense on any two opposite edges, but not on all four edges. In the interests of convergence of the final series solution, it is preferable to use the strong boundary conditions on the long edges at this stage. For example, if $b>a$, we satisfy the condition on $x=\pm a$ by requiring

$$C_2 = 0 \tag{17.38}$$

and hence

$$C_1 = -\frac{F\nu}{8ab^3(1+\nu)} \; , \tag{17.39}$$

from (17.36). The remaining boundary conditions are satisfied in the weak sense

$$\int_{-a}^{a}\left(3C_1x^2 + C_2b^2 + C_3 + \frac{3F}{8ab}\right)dx = 0 \tag{17.40}$$

from which

$$C_3 = \frac{F\nu a}{8b^3(1+\nu)} - \frac{3F}{8ab} \; . \tag{17.41}$$

To complete the solution, we superpose a series of harmonic functions φ_n chosen to satisfy the condition $\varphi_n = 0$ on $x = \pm a$. In this way, the governing equation and the boundary conditions on $x = \pm a$ are unaffected and the remaining constants can be chosen to satisfy the boundary conditions on $y = \pm b$ in the strong sense. This is exactly the same procedure as was used for the torsion problem in §16.4. Here we need odd functions of x satisfying $\varphi_n(\pm a, y) = 0$, leading to the solution

$$\varphi = -\frac{F\nu x^3}{8ab^3(1+\nu)} + \frac{F\nu ax}{8b^3(1+\nu)} - \frac{3Fx}{8ab} + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) \; , \tag{17.42}$$

where D_n is a set of arbitrary constants. This function satisfies the first of (17.34) if

$$\sum_{n=1}^{\infty}\frac{n\pi D_n}{a}\cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi b}{a}\right) = \frac{F\nu(3x^2-a^2)}{8ab^3(1+\nu)} \;\; ; \;\; -a < x < a \; . \tag{17.43}$$

Multiplying both sides of this equation by $\cos(m\pi x/a)$ and integrating over the range $-a<x<a$, we then obtain

$$m\pi D_m \cosh\left(\frac{m\pi b}{a}\right) = \frac{F\nu}{8ab^3(1+\nu)}\int_{-a}^{a}(3x^2-a^2)\cos\left(\frac{m\pi x}{a}\right)dx = \frac{3(-1)^m F\nu a^2}{2\pi^2 m^2 b^3(1+\nu)} \tag{17.44}$$

and hence

$$D_m = \frac{3(-1)^m F\nu a^2}{2\pi^3 m^3 b^3(1+\nu)} \Bigg/ \cosh\left(\frac{m\pi b}{a}\right) . \tag{17.45}$$

The complete stress field is therefore

$$\sigma_{zx} = \frac{3F\nu a}{2\pi^2 b^3(1+\nu)}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\Bigg/\cosh\left(\frac{n\pi b}{a}\right) \tag{17.46}$$

$$\sigma_{zy} = \frac{3F(b^2-y^2)}{8ab^3} + \frac{F\nu(3x^2-a^2)}{8ab^3(1+\nu)} - \frac{3F\nu a}{2\pi^2 b^3(1+\nu)}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\Bigg/\cosh\left(\frac{n\pi b}{a}\right) , \tag{17.47}$$

from (17.8, 17.33, 17.42, 17.45)

The maximum shear stress occurs at the points $(\pm a, 0)$ and is

$$\tau_{\max} = \frac{3F}{8ab} + \frac{F\nu a}{4b^3(1+\nu)}\left\{1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty} 1 \Bigg/ n^2\cosh\left(\frac{n\pi b}{a}\right)\right\} . \tag{17.48}$$

The first term in this expression is that given by the elementary Mechanics of Materials theory. The error of the elementary theory (in the range $0<a\le b$) is greatest when $\nu=0.5$ and $a=b$, for which the approximation underestimates the exact value by 18%. However, the greater part of this correction is associated with the second (closed form) term in (17.47). If this term is included, but the series is omitted, the error in $\tau_{\max}$ is only about 1%.

This solution can be used for all values of the ratio a/b, but the series terms would make a more significant contribution in the range $b>a$. For this case, it is better to use the constants in (17.35) to satisfy strong conditions on $y=\pm b$ and weak conditions on $x=\pm a$ (see Problem 17.5).

PROBLEMS

1. Find an expression for the *local* twist per unit length of the bar, defined as

$$\beta(x,y) = \frac{\partial \omega_z}{\partial z} ,$$

where ω_z is the rotation defined in equation (1.47). Hence show that the *average* twist per unit length for a general unsymmetrical section will be zero if $C=0$ in equation (17.10).

2. A bar of elliptical cross-section defined by the boundary

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

is loaded by a shear force $F_x = F$ acting along the negative x-axis. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section.

3. A bar of equilateral triangular cross-section defined by the boundaries

$$y=\sqrt{3}x \;\; ; \;\; y=-\sqrt{3}x \;\; ; \;\; y=\frac{\sqrt{3}a}{2}$$

is loaded by a shear force $F_y = F$ acting along the negative y-axis. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section for the special case where $\nu = 0.5$. **Note:** You will need to move the origin to the centroid of the section $(0, a/\sqrt{3})$, since equation (17.3) is based on this choice of origin.

4. A bar of equilateral triangular cross-section defined by the boundaries

$$y=\sqrt{3}x \;\; ; \;\; y=-\sqrt{3}x \;\; ; \;\; y=\frac{\sqrt{3}a}{2}$$

is loaded by a shear force $F_x = F$. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section for the special case where $\nu = 0.5$, assuming the constant $C=0$ in equation (17.10). You will need to move the origin to the centroid of the section $(0, a/\sqrt{3})$, since equation (17.3) is based on this choice of origin. Show that the resultant force $\boldsymbol{F}$ acts through the centroid.

5. Solve the problem of the rectangular bar of §17.4.2 for the case where $a>b$. You will need to satisfy the boundary conditions in the strong sense on $y=\pm b$ and in the weak sense on $x=\pm a$. Compare the solution you get at this stage with the predictions of the elementary Mechanics of Materials theory. Then superpose an appropriate series of harmonic functions (sinusoidal in y and hyperbolic in x) to obtain an exact solution. In what range of the ratio a/b is it necessary to include the series to obtain 1% accuracy in the maximum shear stress?

6. Find the distribution of shear stress in a *hollow* cylindrical bar of inner radius b and outer radius a loaded by a shear force $F_y=F$ acting through the origin. You will need to supplement the polynomial solution by the singular harmonic functions $x/(x^2+y^2)$ and $(x^3-3xy^2)/(x^2+y^2)^3$ with appropriate multipliers[4].

[4] These functions are Cartesian versions of the functions $r^{-1}\cos\theta$ and $r^{-3}\cos 3\theta$ respectively.

7. A bar has the cross-section of a sector $0<r<a, -\alpha<\theta<\alpha$, where α is not small compared with unity. The bar is loaded by a shear force F in the direction $\theta=0$.

(i) Find a closed form solution for the shear stresses in the bar, using the strong boundary conditions on the straight edges $\theta = \pm\alpha$ and a weak boundary condition on the curved edge $r=a$.
(ii) Superpose a series of harmonic functions φ_n of the form $r^\gamma \sin(\gamma\theta)$ where γ is chosen to satisfy $\varphi_n=0$ on $\theta=\pm\alpha$.
(iii) Choose the coefficients on these terms so as to satisfy the boundary condition on the curved edge $r=a$ and hence obtain an expression for the maximum shear stress.

8. Solve Problem 17.7 for the case where the force F acts in the direction $\theta=\pi/2$. Assume that the line of action of F is such as to make the constant C in (17.10) be zero.

Part IV

COMPLEX VARIABLE FORMULATION

In Parts II and III we have shown how to reduce the in-plane and antiplane problems to boundary value problems for an appropriate function of the coördinates (x, y). An alternative and arguably more elegant formulation can be obtained by combining x, y in the form of the complex variable $\zeta = x + \imath y$ and expressing the stresses and displacements as functions of ζ and its complex conjugate $\bar{\zeta}$. A significant advantage of this method is that it enables a connection to be made between harmonic boundary-value problems and the theory of Cauchy contour integrals. In certain cases (notably domains bounded by a circle or a straight line), this permits the stresses in the interior of the body to be written down in terms of integrals of the boundary tractions or displacements, thus removing the 'inspired guesswork' that is sometimes needed in the real stress function approach. Furthermore, the scope of this powerful method can be extended to other geometries using the technique of conformal mapping.

In view of these obvious advantages, the reader might reasonably ask why one should not dispense with the whole real stress function approach of Chapters 4 to 15 and instead rely solely on the complex variable formulation for two-dimensional problems in elasticity. An obvious practical reason is that the latter requires some familiarity with the niceties of complex algebra and the notation tends to reduce the 'transparency' of the physical meaning of the resulting expressions. Also, although we might be able to write down the solution as a definite integral, which is certainly a very satisfactory mathematical outcome, evaluation of the resulting integral can be challenging. Indeed, the cases where the integral is easy to evaluate are usually precisely those where the real stress function solution also permits a very straightforward solution. One final point to bear in mind in this connection is that although Mathematica and Maple have the capacity to handle complex algebra, they do not generally perform well, at least at the present time of writing.

On the positive side, the complex variable approach extends the scope of exact solutions to a broader class of geometries and boundary conditions and it offers significant advantages of elegance and compactness, which as we shall see later can also be used to simplify some three-dimensional problems. It can be said with some confidence that the development of this theory was the single most significant contribution to the theory of linear elasticity during the twentieth century and no-one with serious pretensions as an elastician can afford to be ignorant of it.

In the next two chapters we shall give merely a brief introduction to the complex variable formulation. In Chapter 18, we introduce the notation and establish some essential mathematical results, notably concerning the integrals of functions of the complex variable around closed contours and the technique known as conformal mapping. We then apply these results to both in-plane and antiplane elasticity problems in Chapter 19. We shall endeavour to demonstrate connections to the corresponding real stress function results wherever possible, since the two methods can often be used in combination. Readers who wish to obtain a deeper understanding of the subject are referred to the several classical works on the subject[5]. More approachable treatments for the engineering reader are given by A.H.England, *Complex Variable Methods in Elasticity*, John Wiley, London, 1971, S.P.Timoshenko and J.N.Goodier, *loc. cit.* Chapter 6 and D.S.Dugdale and C.Ruiz, *Elasticity for Engineers*, McGraw-Hill, London, 1971, Chapter 2.

[5] N.I.Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity,* P.Noordhoff, Groningen, 1963, A.C.Stevenson, Some boundary problems of two-dimensional elasticity, *Philosophical Magazine*, Vol. 34 (1943), pp.766–793, A.C.Stevenson, Complex potentials in two-dimensional elasticity, *Proceedings of the Royal Society of London*, Vol. A184 (1945), pp.129–179, A.E.Green and W.Zerna, *Theoretical Elasticity,* Clarendon Press, Oxford, 1954, I.S.Sokolnikoff, *Mathematical Theory of Elasticity,* 2nd.ed., McGraw-Hill, New York, 1956. L.M.Milne-Thomson, *Antiplane Elastic Systems*, Springer, Berlin, 1962, L.M.Milne-Thomson, *Plane Elastic Systems*, 2nd edn, Springer, Berlin, 1968.

18

PRELIMINARY MATHEMATICAL RESULTS

The position of a point in the plane is defined by the two independent coördinates (x, y) which we here combine to form the complex variable ζ and its conjugate $\bar{\zeta}$, defined as

$$\zeta = x + \imath y \; ; \quad \bar{\zeta} = x - \imath y \; . \tag{18.1}$$

We can recover the Cartesian coördinates by the relations $x = \Re(\zeta)$, $y = \Im(\zeta)$, but a more convenient algebraic relationship between the real and complex formulations is obtained by solving equations (18.1) to give

$$x = \frac{1}{2}(\zeta + \bar{\zeta}) \;\; ; \;\; y = -\frac{\imath}{2}(\zeta - \bar{\zeta}) \; . \tag{18.2}$$

At first sight, this seems a little paradoxical, since if we know ζ, we already know its real and imaginary parts x and y and hence $\bar{\zeta}$. However, for the purpose of the complex analysis, we regard ζ as the indissoluble combination of $x + \imath y$, and hence ζ and $\bar{\zeta}$ act as two independent variables defining position. In this chapter, we shall always make this explicit by writing $f(\zeta, \bar{\zeta})$ for a function that has fairly general dependence on position in the plane.

In polar coördinates (r, θ), we have the alternative expressions

$$\zeta = r \exp(\imath\theta) \; ; \quad \bar{\zeta} = r \exp(-\imath\theta) \tag{18.3}$$

for the complex variable, with solution

$$r = \left(\zeta\bar{\zeta}\right)^{1/2} \;\; ; \quad \theta = \frac{\imath}{2} \ln\left(\frac{\bar{\zeta}}{\zeta}\right) \; . \tag{18.4}$$

Any real or complex function of x, y can be expressed as a function of $\zeta, \bar{\zeta}$ using (18.2). For example

$$x^2 + y^2 = \frac{1}{4}(\zeta + \bar{\zeta})^2 - \frac{1}{4}(\zeta - \bar{\zeta})^2 = \zeta\bar{\zeta} \; ; \quad xy = -\frac{\imath}{4}(\zeta + \bar{\zeta})(\zeta - \bar{\zeta}) = \frac{\imath}{4}(\bar{\zeta}^2 - \zeta^2) \; .$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_18,

The derivative of a function $f(\zeta,\bar{\zeta})$ with respect to ζ can be related to its derivatives with respect to x and y by

$$\frac{\partial f}{\partial \zeta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath\frac{\partial f}{\partial y}\right) \tag{18.5}$$

and by a similar argument

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \imath\frac{\partial f}{\partial y}\right) . \tag{18.6}$$

Substituting (18.1) into (18.5, 18.6), we find that

$$\frac{\partial \bar{\zeta}}{\partial \zeta} = \frac{\partial \zeta}{\partial \bar{\zeta}} = 0 , \tag{18.7}$$

showing that $(\zeta,\bar{\zeta})$ represent an orthogonal coördinate set.

18.1 Holomorphic functions

If a function f depends *only* on ζ, so that $\partial f/\partial\bar{\zeta}=0$, and if it is also infinitely differentiable at all points in some domain Ω, it is referred to as a *holomorphic function* of ζ in Ω. The complex conjugate $\overline{f}$ of a holomorphic function f can be obtained by replacing ζ by $\bar{\zeta}$ and replacing any complex constants by their conjugates. Thus, $\overline{f}$ is a differentiable function of $\bar{\zeta}$ only and will also be described as holomorphic. We shall see in the next section that holomorphic functions always satisfy the Laplace equation and hence are harmonic.

If $f(\zeta)=f_x+\imath f_y$ is a holomorphic function of ζ whose real and imaginary parts are f_x, f_y respectively, we can write

$$\frac{\partial f}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \imath\frac{\partial f_x}{\partial y} + \imath\frac{\partial f_y}{\partial x} - \frac{\partial f_y}{\partial y}\right) = 0 ,$$

using (18.6). For this to be satisfied, both real and imaginary parts must be separately zero, giving the *Cauchy-Riemann relations*

$$\frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} ; \quad \frac{\partial f_x}{\partial y} = -\frac{\partial f_y}{\partial x} . \tag{18.8}$$

Since a holomorphic function $f(\zeta)$ is infinitely differentiable in Ω, it can be replaced by its Taylor series

$$f(\zeta) = \sum_{n=0}^{\infty} C_n(\zeta-\zeta_0)^n \tag{18.9}$$

in the neighbourhood of any point ζ_0 in Ω, where C_n is a set of complex constants.

18.2 Harmonic functions

It follows from (18.5, 18.6) that

$$4\frac{\partial^2 f}{\partial\zeta\partial\bar{\zeta}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f \tag{18.10}$$

and hence the Laplace equation ($\nabla^2\phi=0$) in two dimensions can be written

$$\frac{\partial^2 \phi}{\partial\zeta\partial\bar{\zeta}} = 0 \ . \tag{18.11}$$

The most general solution of (18.11) can be written

$$\phi = f_1 + \overline{f}_2 \ , \tag{18.12}$$

where f_1, f_2 are arbitrary holomorphic functions of ζ and $\bar{\zeta}$, respectively. Notice that instead of making the argument explicit as in $\overline{f}_2(\bar{\zeta})$, we can identify a holomorphic function of $\bar{\zeta}$ by the condensed notation $\overline{f}_2$.

The function defined by equation (18.12) will generally be complex, but its real and imaginary parts must separately satisfy (18.10) and hence a general *real* harmonic function can be written

$$\phi = \Re\left(f_1 + \overline{f}_2\right) \ . \tag{18.13}$$

As in equation (18.2), it is convenient to avoid the necessity of extracting real and imaginary parts by recognizing that the sum of a function and its complex conjugate will always be real. Thus the function

$$\phi = g + \overline{g} \tag{18.14}$$

represents the most general real solution of the Laplace equation, where g is an arbitrary holomorphic function of ζ only and $\overline{g}$ is its complex conjugate.

18.3 Biharmonic functions

It follows from (18.10) that the biharmonic equation $\nabla^4\phi=0$ reduces to

$$\frac{\partial^4 \phi}{\partial\zeta^2\partial\bar{\zeta}^2} = 0 \tag{18.15}$$

and has the general complex solution

$$\phi = f_1 + \overline{f}_2 + \bar{\zeta} f_3 + \zeta \overline{f}_4 \ , \tag{18.16}$$

where f_1, f_3 are arbitrary holomorphic functions of ζ and $\overline{f}_2, \overline{f}_4$ are arbitrary holomorphic functions of $\bar{\zeta}$.

The most general *real* solution of the biharmonic equation can be expressed in the form

$$\phi = g_1 + \overline{g}_1 + \bar{\zeta} g_2 + \zeta \overline{g}_2 \ , \tag{18.17}$$

where g_1, g_2 are arbitrary holomorphic functions of ζ and $\overline{g}_1, \overline{g}_2$ are their complex conjugates.

Example

As an example, we can generate a real biharmonic polynomial function of degree n by taking $g_1(\zeta)=(A+\imath B)\zeta^n$, $g_2(\zeta)=(C+\imath D)\zeta^{n-1}$, where A, B, C, D are real arbitrary constants. Equation (18.17) then gives

$$\phi = (A+\imath B)\zeta^n + (A-\imath B)\bar{\zeta}^n + (C+\imath D)\bar{\zeta}\zeta^{n-1} + (C-\imath D)\zeta\bar{\zeta}^{n-1} \; . \quad (18.18)$$

This expression can be used in place of the procedure of §5.1 to develop the most general biharmonic polynomial in x, y of degree n. For example, for the special case $n=5$ we have

$$\begin{aligned} \phi = {} & (A+\imath B)(x+\imath y)^5 + (A-\imath B)(x-\imath y)^5 + (C+\imath D)(x-\imath y)(x+\imath y)^4 \\ & +(C-\imath D)(x+\imath y)(x-\imath y)^4 \; . \end{aligned} \quad (18.19)$$

Expanding and simplifying the resulting expressions, we obtain

$$\begin{aligned} \phi = {} & 2A\left(x^5 - 10x^3y^2 + 5xy^4\right) - 2B\left(5x^4y - 10x^2y^3 + y^5\right) \\ & +2C\left(x^5 - 2x^3y^2 - 3xy^4\right) - 2D\left(3x^4y + 2x^2y^3 - y^5\right) \; , \end{aligned} \quad (18.20)$$

which can be converted to the form (5.13) by choosing

$$A = -\frac{(2A_0 + A_2)}{16} \; ; \quad B = \frac{(3A_3 - 2A_1)}{80} \; ; \quad C = \frac{(10A_0 + A_2)}{16}$$

$$D = -\frac{(2A_1 + A_3)}{16} \; . \quad (18.21)$$

Alternatively, using the polar coördinate expressions (18.3) in (18.18) we have

$$\begin{aligned} \phi = {} & (A+\imath B)r^n \exp(\imath n\theta) + (A-\imath B)r^n \exp(-\imath n\theta) + (C+\imath D)r^n \exp(\imath(n-2)\theta) \\ & +(C-\imath D)r^n \exp(-\imath(n-2)\theta) \; , \end{aligned} \quad (18.22)$$

which expands to

$$\phi = 2Ar^n \cos(n\theta) - 2Br^n \sin(n\theta) + 2Cr^n \cos((n-2)\theta) - 2Dr^n \sin((n-2)\theta) \; . \quad (18.23)$$

This is of course identical to the general form of the Michell solution for a given leading power r^n, as shown for example by comparison with equation (11.3) with n replaced by $(n-2)$.

18.4 Expressing real harmonic and biharmonic functions in complex form

Suppose we are given a real harmonic function $f(x, y)$ and wish to find the holomorphic function g such that

$$g + \overline{g} = f(x, y) \; . \tag{18.24}$$

All we need to do is to use (18.2) to eliminate x, y in $f(x, y)$. If f is harmonic and real, the resulting function of $\zeta, \bar{\zeta}$ must reduce to the sum of a holomorphic function of ζ and its conjugate and these functions can then often be identified by inspection. Alternatively, starting from the form

$$g + \overline{g} = f(\zeta, \bar{\zeta}) \; , \tag{18.25}$$

we differentiate with respect to ζ, obtaining

$$g' = \frac{\partial f}{\partial \zeta} \; , \tag{18.26}$$

since $\partial \overline{g}/\partial \zeta = 0$. The right-hand side of (18.26) must be a function of ζ only and hence g can be recovered by integration. This operation is performed by the Mathematica and Maple files 'rtoch'.

Example

Consider the function

$$f(x, y) = \frac{x \ln\left(x^2 + y^2\right)}{2} - y \arctan\left(\frac{y}{x}\right) \; ,$$

which is easily shown to be harmonic by substitution into the Laplace equation. Using (18.2, 18.4) we have

$$x^2 + y^2 = r^2 = \zeta\bar{\zeta} \; ; \quad \arctan\left(\frac{y}{x}\right) = \theta = \frac{\imath}{2} \ln\left(\frac{\bar{\zeta}}{\zeta}\right)$$

and hence

$$g + \overline{g} = \frac{(\bar{\zeta} + \zeta)\ln(\zeta\bar{\zeta})}{4} + \frac{(\bar{\zeta} - \zeta)}{4} \ln\left(\frac{\bar{\zeta}}{\zeta}\right) = \frac{1}{2}\left(\zeta \ln(\zeta) + \bar{\zeta}\ln(\bar{\zeta})\right)$$

We conclude by inspection that

$$g = \frac{\zeta \ln(\zeta)}{2} \; .$$

18.4.1 Biharmonic functions

The same procedure can also be used to express real biharmonic functions in the form (18.17). We first use (18.2) to express the given real function as a function of $\zeta, \bar{\zeta}$. Equation (18.17) can then be written

$$g_1 + \overline{g_1} + \bar{\zeta} g_2 + \zeta \overline{g_2} = f(\zeta, \bar{\zeta}) \; , \tag{18.27}$$

where $f(\zeta,\bar{\zeta})$ is a known function. Differentiating with respect to ζ and $\bar{\zeta}$, we obtain

$$g_2^{'} + \overline{g_2^{'}} = \frac{\partial^2 f}{\partial\zeta\partial\bar{\zeta}} \qquad (18.28)$$

and the function $g_2^{'}$ and hence g_2 can then be obtained using the above procedure for harmonic functions. Once g_2 is known, it can be substituted into (18.27) and the resulting equation solved for the harmonic function g_1 using the same procedure. The Maple and Mathematica files 'rtocb' perform this operation for any biharmonic function specified in Cartesian coördinates.

18.5 Line integrals

Suppose that S is a closed contour and that $f(\zeta)$ is holomorphic on S and at all points in the region enclosed by S. It can then be shown that the contour integral

$$\oint_S f(\zeta)d\zeta = 0\ . \qquad (18.29)$$

To prove this result[1] we write $f(\zeta) = f_x + \imath f_y$, $d\zeta = dx + \imath dy$, separate real and imaginary parts and use Green's theorem to convert the resulting real contour integrals into integrals over the area enclosed by S. The integrand of each of these area integrals will then be found to be identically zero because of the Cauchy-Riemann relations (18.8).

Figure 18.1: Path of the integral in equation (18.30).

As in §2.2.1, we can use (18.29) to show that the line integral

$$\int_A^B f(\zeta)d\zeta = \int_{\zeta_A}^{\zeta_B} f(\zeta)d\zeta \qquad (18.30)$$

along the line S in Figure 18.1 is not changed by making any infinitesimal change in the path S as long as this does not pass outside the region within which $f(\zeta)$ is holomorphic. It follows that (18.30) is path-independent if S lies wholly within a simply connected region Ω in which $f(\zeta)$ is holomorphic.

[1] See for example E.T.Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Clarendon Press, Oxford, 1935, §4.21, G.F.Carrier, M.Krook and C.E.Pearson, *Functions of a Complex Variable*, McGraw-Hill, New York, 1966, §2.2.

18.5.1 The residue theorem

Next suppose that the function $f(\zeta)$ is holomorphic everywhere in a simply connected domain Ω *except* at a point P, where it is singular. As before, we can use (18.29) to show that the value of the contour integral is not changed by making any infinitesimal deviation in the path S. It follows that all contour integrals whose paths pass around P once and only once must have the same value.

Suppose that P is defined by $\zeta=\zeta_0$ and that the singularity has the form

$$f(\zeta)=\frac{f_1(\zeta)}{(\zeta-\zeta_0)^n}+f_2(\zeta)\;, \tag{18.31}$$

where $f_1(\zeta), f_2(\zeta)$ are holomorphic throughout the domain enclosed by S (including at P) and n is an integer. The function $f(\zeta)$ is then said to have a *pole of order* n at $\zeta=\zeta_0$. Since the contour integral is the same for all contours enclosing P, we can evaluate it around an infinitesimal circle of radius ϵ centred on $\zeta=\zeta_0$. For the special case $n=1$, $f(\zeta)$ has a pole of order unity (also known as a *simple pole*) at $\zeta=\zeta_0$ and if ϵ is sufficiently small, we can replace the continuous function $f_1(\zeta)$ by $f_1(\zeta_0)$ giving

$$\begin{aligned}\oint_S f(\zeta)d\zeta &= f_1(\zeta_0)\oint_S\frac{d\zeta}{(\zeta-\zeta_0)} = f_1(\zeta_0)\ln(\zeta-\zeta_0)\Big|_{\zeta_0+\epsilon\exp(\imath\theta_1)}^{\zeta_0+\epsilon\exp(\imath(\theta_1+2\pi))}\\ &= 2\pi\imath f_1(\zeta_0)\;,\end{aligned} \tag{18.32}$$

where θ_1 is defined such that $\zeta=\zeta_0+\epsilon\exp(\imath\theta_1)$ is any point on the path S and the integral is evaluated in the anticlockwise direction (increasing θ). Notice that the second term in (18.31) makes no contribution to the integral, because of (18.29). The quantity

$$f_1(\zeta_0)=\lim_{\zeta\to\zeta_0} f(\zeta)(\zeta-\zeta_0)\equiv \mathrm{Res}(\zeta_0) \tag{18.33}$$

is known as the *residue* at the simple pole P. It then follows from (18.32) that

$$\oint_S f(\zeta)d\zeta = 2\pi\imath\,\mathrm{Res}(\zeta_0)\;. \tag{18.34}$$

For higher order poles $n>1$, the limit (18.33) does not exist, but we can extract the residue by successive partial integrations. We obtain

$$\oint\frac{f_1(\zeta)d\zeta}{(\zeta-\zeta_0)^n} = -\frac{f_1(\zeta)}{(n-1)(\zeta-\zeta_0)^{(n-1)}}\Bigg|_{\epsilon\exp(\imath\theta_1)}^{\epsilon\exp(\imath(\theta_1+2\pi))} + \oint\frac{f_1'(\zeta)d\zeta}{(n-1)(\zeta-\zeta_0)^{(n-1)}}\;, \tag{18.35}$$

where we have used the same circular path of radius ϵ. The first term is zero for $n\neq 1$, so

$$\oint \frac{f_1(\zeta)d\zeta}{(\zeta-\zeta_0)^n} = \oint \frac{f_1'(\zeta)d\zeta}{(n-1)(\zeta-\zeta_0)^{(n-1)}} \; . \tag{18.36}$$

Repeating this procedure until the final integral has only a simple pole, we obtain

$$\oint \frac{f_1(\zeta)d\zeta}{(\zeta-\zeta_0)^n} = \oint \frac{f_1^{[n-1]}(\zeta)d\zeta}{(n-1)!(\zeta-\zeta_0)} \; , \tag{18.37}$$

where

$$f_1^{[n-1]}(\zeta) \equiv \frac{d^{(n-1)}f_1(\zeta)}{d\zeta^{(n-1)}} \; . \tag{18.38}$$

The residue at a pole of order n is therefore

$$\text{Res}(\zeta_0) = \frac{f_1^{[n-1]}(\zeta_0)}{(n-1)!} \; , \tag{18.39}$$

where $f_1^{[n-1]}$ is defined through equations (18.38, 18.31).

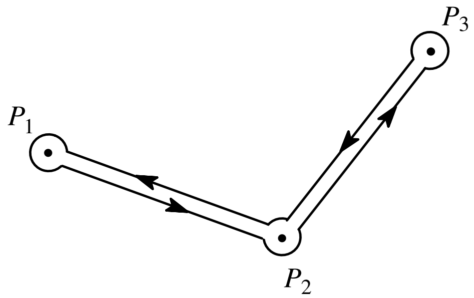

Figure 18.2: Integral path around three poles at P_1, P_2, P_3..

Suppose now that the function $f(\zeta)$ posesses several poles and we wish to determine the value of the contour integral along a path enclosing those at the points $\zeta_1, \zeta_2, ..., \zeta_k$. One such path enclosing three poles at P_1, P_2, P_3 is shown in Figure 18.2. It is clear that the contributions from the straight line segments will be self-cancelling, so the integral is simply the sum of those for the circular paths around each separate pole. Thus, the resulting integral will have the value

$$\oint f(\zeta)d\zeta = 2\pi\imath \sum_{j=1}^{k} \text{Res}(\zeta_j) \; . \tag{18.40}$$

This result is extremely useful as a technique for determining the values of complex line integrals. Similar considerations apply to the evaluation of contour integrals around one or more holes in a multiply connected body.

Example

As a special case, we shall determine the value of the contour integral

$$\oint_S \frac{d\zeta}{\zeta^n(\zeta - a)} ,$$

where the integration path S encloses the points $\zeta = 0, a$. The integrand has a simple pole at $\zeta = a$ and a pole of order n at $\zeta = 0$. For the simple pole, we have

$$\text{Res}(a) = \frac{1}{a^n} ,$$

from (18.33). For the pole at $\zeta = 0$, we have

$$f_1(\zeta) = \frac{1}{(\zeta - a)} \qquad \text{and} \qquad f_1^{[n-1]}(\zeta) = \frac{(-1)^{(n-1)}(n-1)!}{(\zeta - a)^n} ,$$

so

$$f_1^{[n-1]}(0) = \frac{(-1)^{(n-1)}(n-1)!}{(-a)^n} \qquad \text{and} \qquad \text{Res}(0) = \frac{(-1)^{(n-1)}}{(-a)^n} = -\frac{1}{a^n} , \tag{18.41}$$

from (18.39). Using (18.40), we then have

$$\oint_S \frac{d\zeta}{\zeta^n(\zeta - a)} = 2\pi\imath \left(\frac{1}{a^n} - \frac{1}{a^n} \right) = 0 ; \qquad n \geq 1 \tag{18.42}$$

$$= 2\pi\imath ; \qquad n = 0 , \tag{18.43}$$

since if $n = 0$ we have only the simple pole at $\zeta = a$.

18.5.2 The Cauchy integral theorem

Changing the symbols in equation (18.32), we have

$$f(\zeta_0) = \frac{1}{2\pi\imath} \oint_S \frac{f(s)ds}{(s - \zeta_0)} , \tag{18.44}$$

where the path of the integral, S, encloses the point $s = \zeta_0$. If S is chosen to coincide with the boundary of a simply connected region Ω within which f is holomorphic, this provides an expression for f at a general point in Ω in terms of its boundary values. We shall refer to this as the *interior problem*, since the region Ω is interior to the contour. Notice that equation (18.32) and hence (18.44) is based on the assumption that the contour S is traversed in the anticlockwise direction. Changing the direction would lead to a sign change in the result. An equivalent statement of this requirement is that the contour is traversed in a direction such that the enclosed simply connected region Ω always lies on the left of the path S.

A related result can be obtained for the *exterior problem* in which f is holomorphic in the multiply connected region $\overline{\Omega}$ exterior to the closed contour S and extending to and including the point at infinity. In this case, a general holomorphic function f can be expressed as a *Laurent series*

$$f = C_0 + \sum_{k=1}^{\infty} \frac{C_k}{\zeta^k} , \tag{18.45}$$

where the origin lies within the hole excluded by S. For the exterior problem to be well posed, we must specify the value of f at all points on S and also at infinity $\zeta \to \infty$. We shall restrict attention to the case where $f(\infty)=0$ and hence $C_0=0$, since the more general case is most easily handled by first solving a problem for the entire plane (with no hole) and then defining a corrective problem for the conditions at the inner boundary S.

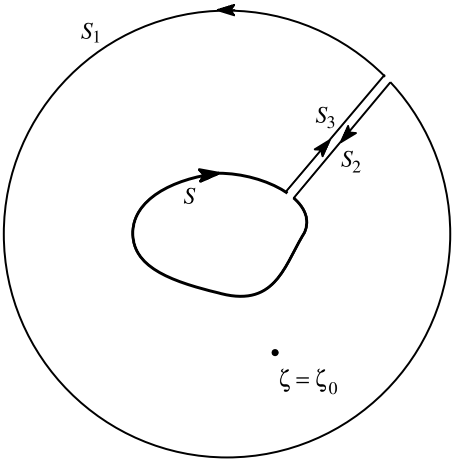

Figure 18.3: Contour for the region exterior to a hole S.

To invoke the Cauchy integral theorem, we need to construct a contour that encloses a simply connected region including the general point $\zeta=\zeta_0$. A suitable contour is illustrated in Figure 18.3, where the path S corresponds to the boundary of the hole and S_1 is a circle of sufficiently large radius to include the point $\zeta=\zeta_0$. Notice that the theorem requires that the contour be traversed in an anticlockwise direction as shown.

Equation (18.44) now gives

$$f(\zeta_0) = \frac{1}{2\pi\imath} \oint_{S+S_1+S_2+S_3} \frac{f(s)ds}{(s-\zeta_0)} . \tag{18.46}$$

In this integral, the contributions from the line segments S_2, S_3 clearly cancel, leaving

$$f(\zeta_0) = -\frac{1}{2\pi\imath} \oint_S \frac{f(s)ds}{(s-\zeta_0)} + \frac{1}{2\pi\imath} \oint_{S_1} \frac{f(s)ds}{(s-\zeta_0)} . \tag{18.47}$$

The remaining terms are two separate contour integrals around S and S_1 respectively in Figure 18.3. Notice that we have changed the sign in the integral around S, since S is now a closed contour and the convention demands that it be evaluated in the anticlockwise direction, which is opposite to the direction implied in (18.46) and Figure 18.3.

Using (18.45) and recalling that $C_0 = 0$, we have

$$\oint_{S_1} \frac{f(s)ds}{(s-\zeta_0)} = \sum_{k=1}^{\infty} \oint_{S_1} \frac{C_k ds}{s^k(s-\zeta_0)} = 0 \; ,$$

from (18.42). Thus, the second term in (18.47) is zero and we have

$$f(\zeta_0) = -\frac{1}{2\pi\imath} \oint_S \frac{f(s)ds}{(s-\zeta_0)} \; , \tag{18.48}$$

which determines the value of the function f at a point in the region exterior to the contour S in terms of the values on this contour.

18.6 Solution of harmonic boundary value problems

At first sight, equations (18.44, 18.48) appear to provide a direct method for finding the value of a harmonic function from given boundary data. However, if the required harmonic function is real, we meet a difficulty, since we know only the real part of the boundary data.

We could write a general real harmonic function as

$$\phi = f + \overline{f} \; , \tag{18.49}$$

in which case we know the boundary values of the function ϕ, but not of the separate functions $f, \overline{f}$. However, we shall show in the next section that a direct solution *can* be developed for both the interior and exterior problems in the special case where the boundary S is circular. Furthermore, this solution can be extended to more general geometries using the technique of conformal mapping.

18.6.1 Direct method for the interior problem for a circle

We consider the simply connected region interior to the circle $r = a$. The boundary of this circle, S, is defined by

$$s = a\exp(\imath\theta) \; ; \quad \overline{s} = a\exp(-\imath\theta) \tag{18.50}$$

and hence

$$s\overline{s} = a^2 \; ; \quad \overline{s} = \frac{a^2}{s} \; ; \quad s \in S \; . \tag{18.51}$$

Consider the problem of determining a real harmonic function $\phi(x,y)$ from real boundary data on $r=a$. We choose to write ϕ as the sum of a holomorphic function f and its conjugate as in (18.49). We then expand f inside the circle as a Taylor series

$$f(\zeta) = C_0 + \sum_{k=1}^{\infty} C_k \zeta^k \tag{18.52}$$

in which case

$$\overline{f(s)} = \overline{C}_0 + \sum_{k=1}^{\infty} \frac{\overline{C}_k a^{2k}}{s^k} \; ; \quad s \in S \; , \tag{18.53}$$

using (18.51).

We now apply the Cauchy operator to the boundary data, obtaining

$$\begin{aligned}
\frac{1}{2\pi\imath} \oint \frac{\phi(s)ds}{(s-\zeta)} &= \frac{1}{2\pi\imath} \oint \frac{f(s)ds}{(s-\zeta)} + \frac{1}{2\pi\imath} \oint \frac{\overline{f(s)}ds}{(s-\zeta)} \\
&= \frac{1}{2\pi\imath} \oint \frac{f(s)ds}{(s-\zeta)} + \frac{\overline{C}_0}{2\pi\imath} \oint \frac{ds}{(s-\zeta)} + \frac{1}{2\pi\imath} \sum_{k=1}^{\infty} \overline{C}_k \oint \frac{a^{2k}ds}{s^k(s-\zeta)} \\
&= f(\zeta) + \overline{C}_0 \; ,
\end{aligned} \tag{18.54}$$

using (18.44) for the first two integrals and noting that every integral in the third term summation is zero in view of (18.42). Thus, we have

$$f(\zeta) = -\overline{C}_0 + \frac{1}{2\pi\imath} \oint \frac{\phi(s)ds}{(s-\zeta)} \; , \tag{18.55}$$

after which the real harmonic function ϕ can be recovered from (18.49) except for the unknown complex constant C_0. To determine this constant, we equate (18.55) to the Taylor series (18.52) and evaluate it at $\zeta=0$, obtaining

$$C_0 = -\overline{C}_0 + \frac{1}{2\pi\imath} \oint \frac{\phi(s)ds}{s} \; ,$$

or

$$C_0 + \overline{C}_0 = \frac{1}{2\pi\imath} \oint \frac{\phi(s)ds}{s} \; . \tag{18.56}$$

Alternatively, writing

$$s = ae^{\imath\theta} \; ; \quad ds = a\imath e^{\imath\theta} d\theta$$

from (18.50), we have

$$C_0 + \overline{C}_0 = \frac{1}{2\pi} \int_0^{2\pi} \phi(a,\theta) d\theta \; . \tag{18.57}$$

Thus, the real part of the constant C_0 can be determined, but not its imaginary part and this is reasonable since if we add an arbitrary imaginary constant into f it will make no contribution to ϕ in (18.49).

Example

Suppose that $\phi(a,\theta) = \phi_2 \cos(2\theta)$, where ϕ_2 is a real constant and we wish to find the corresponding harmonic function $\phi(r,\theta)$ inside the circle of radius a. Using (18.50), we can write

$$\phi_2 \cos(2\theta) = \frac{\phi_2(e^{2\imath\theta} + e^{-2\imath\theta})}{2} = \frac{\phi_2}{2}\left(\frac{s^2}{a^2} + \frac{a^2}{s^2}\right)$$

and it follows that

$$\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = \frac{\phi_2}{4\pi a^2 \imath}\oint \frac{s^2 ds}{(s-\zeta)} + \frac{\phi_2 a^2}{4\pi\imath}\oint \frac{ds}{s^2(s-\zeta)} = \frac{\phi_2 \zeta^2}{2a^2} ,$$

where we have used (18.44) for the first integral and the second integral is zero in view of (18.42).

Using (18.55), we then have

$$f = \frac{\phi_2 \zeta^2}{2a^2} - \overline{C}_0 \; ; \quad \overline{f} = \frac{\phi_2 \bar{\zeta}^2}{2a^2} - C_0$$

and

$$\phi = f + \overline{f} = \frac{\phi_2(\zeta^2 + \bar{\zeta}^2)}{2a^2} - C_0 - \overline{C}_0 \, .$$

Evaluating the integral in (18.57), we obtain

$$C_0 + \overline{C}_0 = \frac{\phi_2}{2\pi}\int_0^{2\pi} \cos(2\theta) d\theta = 0 \, ,$$

and hence the required harmonic function is

$$\phi = \frac{\phi_2(\zeta^2 + \bar{\zeta}^2)}{2a^2} = \frac{\phi_2 r^2 \cos(2\theta)}{a^2} \, ,$$

using (18.3). Of course, this result could also have been obtained by writing a general harmonic function as the Fourier series

$$\phi = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

and equating coefficients on the boundary $r = a$, but the present method, though longer, is considerably more direct.

18.6.2 Direct method for the exterior problem for a circle

For the exterior problem, we again define ϕ as in (18.49) and apply the Cauchy integral operator to the boundary data, obtaining

$$\frac{1}{2\pi\imath}\oint\frac{\phi(s)ds}{(s-\zeta)}=\frac{1}{2\pi\imath}\oint\frac{f(s)ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\overline{f(s)}ds}{(s-\zeta)}\ , \tag{18.58}$$

where points on the contour are defined by (18.50). We assume that the function ϕ is required to satisfy the condition $\phi\to 0, r\to\infty$, in which case the function f, which is holomorphic *exterior* to the circle, can be written as the Laurent expansion

$$f(\zeta)=\sum_{k=1}^{\infty}\frac{C_k}{\zeta^k}\qquad\text{and hence}\qquad \overline{f(s)}=\sum_{k=1}^{\infty}\frac{\overline{C}_k s^k}{a^{2k}}\ ,$$

using (18.51). It follows that the second integral on the right-hand side of (18.58) is

$$\frac{1}{2\pi\imath}\oint\frac{\overline{f(s)}ds}{(s-\zeta)}=\frac{1}{2\pi\imath}\sum_{k=1}^{\infty}\frac{\overline{C}_k}{a^{2k}}\oint\frac{s^k ds}{(s-\zeta)}$$

and this is zero since ζ lies *outside* the circle and hence the integrand defines a holomorphic function in the region *interior* to the circle. Using this result and (18.48) in (18.58), we obtain

$$f(\zeta)=-\frac{1}{2\pi\imath}\oint\frac{\phi(s)ds}{(s-\zeta)}\ , \tag{18.59}$$

after which ϕ is recovered from (18.49).

The reader might reasonably ask why the method of §§18.6.1, 18.6.2 works for circular contours, but not for more general problems. The answer lies in the relation (18.51) which enables us to define the conjugate $\overline{s}$ in terms of s and the constant radius a. For a more general contour, this relation would also involve the coordinate r in $s=r\exp(\imath\theta)$ and this varies around the contour if the latter is non-circular.

18.6.3 The half plane

If the radius of the circle is allowed to grow without limit and we move the origin to a point on the boundary, we recover the problem of the half plane, which can conveniently be defined as the region $y>0$ whose boundary is the infinite line $-\infty<x<\infty, y=0$. The contour for the integral theorem (18.55) then comprises a circle of infinite radius which lies entirely at infinity except for the line segment $-\infty<x<\infty$, $y=0$ and we obtain

$$f(\zeta)=-\overline{C}_0+\frac{1}{2\pi\imath}\int_{-\infty}^{\infty}\frac{\phi(x)dx}{(x-\zeta)}\ . \tag{18.60}$$

In some cases, an even more direct method can be used to solve the boundary-value problem. We first note that the holomorphic (and hence harmonic) function $f(\zeta)$ reduces to $f(x)$ on $y=0$. Thus, if we simply replace x by

ζ in the boundary function $\phi(x)$, we shall obtain a function of ζ that reduces to $\phi(x)$ on the boundary $y=0$.

Unfortunately, the resulting function $\phi(\zeta)$ will often not be holomorphic, either because the boundary function $\phi(x)$ is not continuously differentiable, or because $\phi(\zeta)$ contains one or more poles in the half plane $y>0$. In these cases, the integral theorem (18.60) will always yield the correct solution. To illustrate this procedure, we shall give two examples.

Example 1

Consider the boundary-value problem

$$\begin{aligned}\phi(x,0) &= 1\ ; \quad -a<x<a \\ &= 0\ ; \quad |x|>a\ .\end{aligned} \tag{18.61}$$

Since the function is discontinuous, we must use the integral theorem (18.60), which here yields

$$f(\zeta) = -\overline{C}_0 + \frac{1}{2\pi\imath}\int_{-a}^{a}\frac{dx}{(x-\zeta)} = -\overline{C}_0 + \frac{1}{2\pi\imath}\ln\left(\frac{\zeta-a}{\zeta+a}\right)\ .$$

The corresponding real function ϕ is then obtained from (18.49) and after using (18.1) and simplifying, we obtain

$$\phi(x,y) = \frac{1}{\pi}\left[\arctan\left(\frac{y}{x-a}\right) - \arctan\left(\frac{y}{x+a}\right)\right]\ ,$$

where the real constant $C_0+\overline{C}_0$ can be determined to be zero by equating $\phi(x,y)$ to its value at any convenient point on the boundary.

Example 2

Consider now the case where

$$\phi(x,0) = \frac{1}{(1+x^2)}\ .$$

This function is continuous and differentiable, so a first guess would be to propose the function

$$\phi = \Re\left(f_1(\zeta)\right) \qquad \text{where} \qquad f_1(\zeta) = \frac{1}{(1+\zeta^2)}\ ,$$

which certainly tends to the correct value on $y=0$. However, f_1 has poles at $\zeta=\pm\imath$, i.e. at the points $(0,1),(0,-1)$, the former of which lies in the half plane $y>0$. Thus $f_1(\zeta)$ is not holomorphic throughout the half plane and its real part is therefore not harmonic at $(0,1)$.

As in the previous example, we can resolve the difficulty by using the integral theorem. However, the fact that the boundary function is continuous permits us to evaluate the integral using the residue theorem. We therefore write

$$f(\zeta) = -\overline{C}_0 + \frac{1}{2\pi\imath}\oint_S \frac{ds}{(1+s^2)(s-\zeta)} = -\overline{C}_0 + \frac{1}{2\pi\imath}\oint_S \frac{ds}{(s+\imath)(s-\imath)(s-\zeta)} ,$$

where the contour S is the circle of infinite radius which completely encloses the half plane $y>0$. The integrand has simple poles at $s=\imath, -\imath, \zeta$, but only the first and third of these lie within the contour. We therefore obtain

$$\frac{1}{2\pi\imath}\oint_S \frac{ds}{(s+\imath)(s-\imath)(s-\zeta)} = \frac{1}{(\zeta+\imath)(\zeta-\imath)} + \frac{1}{2\imath(\imath-\zeta)} = \frac{\imath}{2(\zeta+\imath)}$$

and the real harmonic function ϕ is then recovered as

$$\phi(\zeta,\bar{\zeta}) = f + \overline{f} - C_0 - \overline{C}_0 = \frac{\imath}{2(\zeta+\imath)} - \frac{\imath}{2(\bar{\zeta}-\imath)} - C_0 - \overline{C}_0 .$$

The constants can be determined by evaluating this expression at the origin $\zeta = 0$ and equating it to $\phi(0,0)$, giving

$$\frac{1}{2} + \frac{1}{2} - C_0 - \overline{C}_0 = 1 \quad \text{or} \quad C_0 + \overline{C}_0 = 0 .$$

We then use (18.1) to express $\phi(\zeta,\bar{\zeta})$ in terms of (x,y), obtaining

$$\phi(x,y) = \frac{(1+y)}{x^2+(1+y)^2} .$$

This function can be verified to be harmonic and it clearly tends to the required function $\phi(x,0)$ on $y=0$.

18.7 Conformal mapping

In §18.6, we showed that the value of a real harmonic function in the region interior or exterior to a circle or in the half plane can be written down as an explicit integral of the boundary data, using the Cauchy integral theorem. The usefulness of this result is greatly enhanced by the technique of conformal mapping, which permits us to 'map' a holomorphic function in the circular domain or the half plane to one in a more general domain, usually but not necessarily with the same connectivity.

Suppose that $\omega(\zeta)$ is a holomorphic function of ζ in a domain Ω, whose boundary S is the unit circle $|\zeta|=1$. Each point (x,y) in Ω corresponds to a unique value

$$\omega = \xi + \imath\eta ,$$

whose real and imaginary parts (ξ, η) can be used as Cartesian coördinates to define a point in a new domain Ω^* with boundary S^*. Thus the function $\omega(\zeta)$ can be used to map the points in Ω into the more general domain Ω^* and points on the boundary S into S^*. Since $\zeta \in \Omega$ and $\omega \in \Omega^*$, we shall refer to Ω, Ω^* as the ζ-plane and the ω-plane respectively.

Next suppose that $f(\omega)$ is a holomorphic function of ω in Ω^*, implying that both the real and imaginary parts of f are harmonic functions of (ξ, η), from (18.11). Since ω is a holomorphic function of ζ, it follows that

$$f_\zeta \equiv f(\omega(\zeta))$$

is a holomorphic function of ζ in Ω and hence that its real and imaginary parts are also harmonic functons of (x, y). Thus, the mapping function $\omega(\zeta)$ maps holomorphic functions in Ω^* into holomorphic functions in Ω. It follows that a harmonic boundary-value problem for the domain S^* can be mapped into an equivalent boundary-value problem for the unit circle, which in turn can be solved using the direct method of §18.6.1 or §18.6.2.

Example: The elliptical hole

A simple illustration is provided by the mapping function

$$\omega \equiv \xi + \imath\eta = c\left(\zeta + \frac{m}{\zeta}\right) , \qquad (18.62)$$

where c, m are real constants and $m<1$. Substituting $\zeta = r\exp(\imath\theta)$ into (18.62) and separating real and imaginary parts, we find

$$\xi = c\left(r + \frac{m}{r}\right)\cos\theta\ ; \quad \eta = c\left(r - \frac{m}{r}\right)\sin\theta\ . \qquad (18.63)$$

Eliminating θ between these equations, we then obtain

$$\frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} = 1\ , \qquad (18.64)$$

where

$$\alpha = c\left(r + \frac{m}{r}\right)\ ; \quad \beta = c\left(r - \frac{m}{r}\right)\ . \qquad (18.65)$$

Equation (18.64) with (18.65) shows that the set of concentric circles (lines of constant r) in the ζ-plane maps into a set of confocal ellipses in the ω-plane. In particular, the unit circle $r=1$ maps into the ellipse with semi-axes

$$a = c(1 + m)\ ; \quad b = c(1 - m)\ . \qquad (18.66)$$

If $f(\omega)$ is a holomorphic function in the region *interior* to the ellipse, the mapped function f_ζ will generally be unbounded both at $\zeta=0$ and at $\zeta\to\infty$. This mapping cannot therefore be used for the interior problem for the ellipse.

It can however be used for the exterior problem, since in this case, $f(\omega)$ must posess a Laurent series, every term of which maps into a bounded term in f_ζ.

Equation (18.62) can be solved to give ζ as a function of ω. We obtain the two solutions

$$\zeta = \frac{\omega \pm \sqrt{\omega^2 - 4mc^2}}{2c} \ . \tag{18.67}$$

The positive square root maps points outside the unit circle into points outside the ellipse, whereas the negative root maps points *inside* the unit circle into points outside the ellipse. Either mapping can be used when combined with the appropriate solution of the boundary-value problem. Thus we can use §18.6.1 and the negative root in (18.67), or §18.6.2 and the positive root.

The present author favours the exterior-to-exterior mapping implied by the positive root, since this preserves the topology of the original problem and hence tends to a more physically intuitive procedure. We then have

$$\zeta = \frac{\omega + \sqrt{\omega^2 - 4mc^2}}{2c} \ . \tag{18.68}$$

and it is clear that circles of radius $r>1$ in the ζ-plane map to increasingly circular contours in the ω-plane as $r \to \infty$. To solve a given problem, we map the boundary conditions to the unit circle in the ζ-plane using (18.62), solve for the required holomorphic function of ζ using §18.6.2 and then map this function back into the ω-plane using (18.68).

PROBLEMS

1. By writing

$$\zeta^n = e^{n\ln(\zeta)} \ ,$$

show that

$$\frac{\partial^m}{\partial n^m}(\zeta^n) = \zeta^n \ln^m(\zeta) \ .$$

Using equations (18.3), find the real and imaginary parts of the function $\zeta^n \ln^2(\zeta)$ in polar coordinates r, θ and verify that they are both harmonic.

2. Verify that the real stress function $\phi = r^2\theta$ satisfies the biharmonic equation (8.16). Then express it in the complex form (18.27).

3. Express the real biharmonic function $\phi = r\theta\sin\theta$ (from equation (8.59) or Table 8.1) in the complex form (18.27).

4. Find the imaginary part of the holomorphic function whose real part is the harmonic function $\ln(r)$.

5. Use the residue theorem to evaluate the contour integral

$$\oint_S \frac{\zeta^n d\zeta}{(\zeta^2 - a^2)} ,$$

where n is a positive integer and the contour S is the circle $|\zeta| = 2a$.

6. Find the real and imaginary parts of the function $\sin(\zeta)$ and hence find the points in the x, y-plane at which $\sin(\zeta) = 0$.

Use the residue theorem to evaluate the integral

$$\oint_S \frac{d\zeta}{\sin(\zeta)}$$

for a contour S that encloses only the pole at $\zeta = 0$. By choosing S as a rectangle two of whose sides are the lines $x = \pm\pi/2$, $-\infty < y < \infty$, use your result to evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{dy}{\cosh y} .$$

7. Show that on the unit circle,

$$\zeta + \frac{1}{\zeta} = \cos\theta \; ; \quad \zeta^n + \frac{1}{\zeta^n} = \cos(n\theta) .$$

Use this result and the notation $\omega = e^{\imath\phi}$ to express the real integral (12.35) in terms of an integral around the contour shown in Figure 18.4. Use the residue theorem to evaluate this integral and hence prove equation (12.35). Explain in particular how you determine the contribution from the small semi-circular paths around the poles $\zeta = \omega$, $\zeta = 1/\omega$.

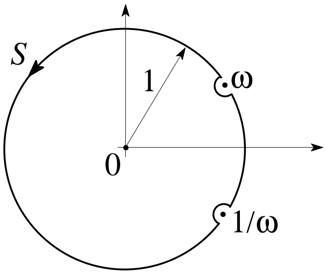

Figure 18.4.

8. Find the real harmonic function $\phi(r, \theta)$ defined inside the circle $0 \le r < a$ whose boundary values are

$$\begin{aligned} \phi(a, \theta) &= 1 \; ; \quad -\alpha < \theta < \alpha \\ &= 0 \; ; \quad \alpha < |\theta| < \pi . \end{aligned}$$

9. Find the real harmonic function $\phi(r,\theta)$ defined in the *external* region $r>a$ satisfying the boundary conditions

$$\phi(a,\theta) = \phi_3 \sin(3\theta) \; ; \quad \phi(r,\theta) \to 0 \; ; \quad r \to \infty \; ,$$

where ϕ_3 is a real constant.

10. Find the real harmonic function $\phi(x,y)$ in the half plane $y \geq 0$ satisfying the boundary conditions

$$\phi(x,0) = \frac{1}{(1+x^2)^2} \; ; \quad \phi(x,y) \to 0 \; ; \quad \sqrt{x^2+y^2} \to \infty \; .$$

11. Use the mapping function (18.62) to find the holomorphic function $f(\omega)$ such that the real harmonic function

$$\phi = f + \overline{f}$$

satisfies the boundary condition $\phi = A\xi$ on the ellipse

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

and $f \to 0$ as $|\omega| \to \infty$, where A is a real constant. Do not attempt to express ϕ as an explicit real function of ξ, η.

12. Show that the function

$$\omega(\zeta) = \zeta^{1/2}$$

maps the quarter plane $\xi \geq 0, \eta \geq 0$ into the half plane $y > 0$. Use this result and appropriate results from the example in §18.6.3 to find the real harmonic function $\phi(\xi,\eta)$ in the quarter plane with boundary values

$$\phi(\xi,0) = \frac{1}{1+\xi^4} \; ; \quad \xi > 0 \; , \qquad \phi(0,\eta) = \frac{1}{1+\eta^4} \; ; \quad \eta > 0 \; .$$

19

APPLICATION TO ELASTICITY PROBLEMS

19.1 Representation of vectors

From a mathematical perspective, both two-dimensional vectors and complex numbers can be characterized as ordered pairs of real numbers. It is therefore a natural step to represent the two components of a vector function $\boldsymbol{V}$ by the real and imaginary parts of a complex function. In other words,

$$\boldsymbol{V} \equiv \boldsymbol{i}V_x + \boldsymbol{j}V_y \tag{19.1}$$

is represented by the complex function

$$V = V_x + \imath V_y \ . \tag{19.2}$$

In the same way, the vector operator

$$\boldsymbol{\nabla} \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{\zeta}} \ , \tag{19.3}$$

from equation (18.6).

If two vectors $\boldsymbol{f}$, $\boldsymbol{g}$ are represented in the form

$$f = f_x + \imath f_y \ ; \quad g = g_x + \imath g_y \tag{19.4}$$

the product

$$\overline{f}g + f\overline{g} = (f_x - \imath f_y)(g_x + \imath g_y) + (f_x + \imath f_y)(g_x - \imath g_y) = 2\,(f_x g_x + f_y g_y) \ . \tag{19.5}$$

It follows that the scalar (dot) product

$$\boldsymbol{f} \cdot \boldsymbol{g} = \frac{1}{2}\left(\overline{f}g + f\overline{g}\right) \ . \tag{19.6}$$

This result can be used to evaluate the divergence of a vector field $\boldsymbol{V}$ as

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_19, © Springer Science+Business Media B.V. 2010

$$\text{div } \boldsymbol{V} \equiv \boldsymbol{\nabla \cdot V} = \frac{\partial V}{\partial \zeta} + \frac{\partial \overline{V}}{\partial \bar{\zeta}} , \tag{19.7}$$

using (19.3, 19.6).

We also record the complex variable expression for the vector (cross) product of two in-plane vectors $\boldsymbol{f}, \boldsymbol{g}$, which is

$$\boldsymbol{f} \times \boldsymbol{g} = (f_x g_y - f_y g_x)\boldsymbol{k} = \frac{\imath}{2} \left(f\overline{g} - \overline{f} g \right) \boldsymbol{k} , \tag{19.8}$$

where $\boldsymbol{k}$ is the unit vector in the out-of-plane direction z. It then follows that

$$\text{curl } \boldsymbol{V} \equiv \boldsymbol{\nabla \times V} = \imath \left(\frac{\partial \overline{V}}{\partial \bar{\zeta}} - \frac{\partial V}{\partial \zeta} \right) \boldsymbol{k} , \tag{19.9}$$

from (19.3, 19.8). Also, the moment of a force $F = F_x + \imath F_y$ about the origin is

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = \frac{\imath}{2} \left(\zeta \overline{F} - \bar{\zeta} F \right) \boldsymbol{k} , \tag{19.10}$$

where $\zeta = x + \imath y$ represents a position vector defining any point on the line of action of F.

19.1.1 Transformation of coördinates

If a vector $V = V_x + \imath V_y$ is transformed into a new coördinate system (x', y') rotated anticlockwise through an angle α with respect to x, y, the transformed components can be combined into the complex quantity

$$V_\alpha = V'_x + \imath V'_y , \tag{19.11}$$

where

$$V'_x = V_x \cos\alpha + V_y \sin\alpha \; ; \quad V'_y = V_y \cos\alpha - V_x \sin\alpha , \tag{19.12}$$

from (1.14). Substituting these expressions into (19.11), we obtain

$$V_\alpha = (\cos\alpha - \imath \sin\alpha) V_x + (\sin\alpha + \imath \cos\alpha) V_y = e^{-\imath\alpha} V . \tag{19.13}$$

In describing the tractions on a surface defined by a contour S, it is sometimes helpful to define the unit vector

$$\hat{t} = \frac{\varDelta s}{|\varDelta s|} \tag{19.14}$$

in the direction of the local tangent. If the contour is traversed in the conventional anticlockwise direction, the outward normal from the enclosed area $\hat{n}$ is then represented by a unit vector rotated through $\pi/2$ clockwise from $\hat{t}$ and hence

$$\hat{n} = \hat{t} e^{-\imath\pi/2} = -\imath \hat{t} . \tag{19.15}$$

For the special case where s represents the circle $s = a\exp(\imath\theta)$, we have

$$\hat{t} = \frac{a\imath \exp(\imath\theta)\varDelta\theta}{a\varDelta\theta} = \imath \exp(\imath\theta) \quad \text{and} \quad \hat{n} = \exp(\imath\theta) .$$

19.2 The antiplane problem

To introduce the application of complex variable methods to elasticity, we first consider the simpler antiplane problem of Chapter 15 and restrict attention to the case where there is no body force ($p_z = 0$). The only non-zero displacement component is u_z, which is constant in direction and hence a scalar function in the xy-plane. From equations (15.1, 15.6), we also deduce that u_z is a real harmonic function, and we can satisfy this condition by writing

$$2\mu u_z = h + \overline{h} , \tag{19.16}$$

where h is a holomorphic function of ζ and $\overline{h}$ is its conjugate. The non-zero stress components σ_{zx}, σ_{zy} can be combined into a vector which we denote as

$$\varPsi \equiv \sigma_{zx} + \imath\sigma_{zy} = \mu\left(\frac{\partial u_z}{\partial x} + \imath\frac{\partial u_z}{\partial y}\right) = 2\mu\frac{\partial u_z}{\partial\bar{\zeta}} , \tag{19.17}$$

from (15.5, 18.6). Substituting (19.16) into (19.17) and noting that h is independent of $\bar{\zeta}$, we then obtain

$$\varPsi = \overline{h'} . \tag{19.18}$$

We can write the complex function

$$h = h_x + \imath h_y , \tag{19.19}$$

where h_x, h_y are real harmonic functions representing the real and imaginary parts of h. Equation (19.18) can then be expanded as

$$\varPsi = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right)(h_x - \imath h_y) = \frac{1}{2}\left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \imath\frac{\partial h_x}{\partial y} - \imath\frac{\partial h_y}{\partial x}\right) \tag{19.20}$$

and hence

$$\sigma_{zx} = \frac{1}{2}\left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y}\right) \; ; \quad \sigma_{zy} = \frac{1}{2}\left(\frac{\partial h_x}{\partial y} - \frac{\partial h_y}{\partial x}\right) . \tag{19.21}$$

However, the Cauchy-Riemann relations (18.8) show that

$$\frac{\partial h_x}{\partial x} = \frac{\partial h_y}{\partial y} \; ; \quad \frac{\partial h_x}{\partial y} = -\frac{\partial h_y}{\partial x} \tag{19.22}$$

so these expressions can be simplified as

$$\sigma_{zx} = \frac{\partial h_x}{\partial x} = \frac{\partial h_y}{\partial y} \; ; \quad \sigma_{zy} = \frac{\partial h_x}{\partial y} = -\frac{\partial h_y}{\partial x} . \tag{19.23}$$

Comparison with equations (16.8) then shows that h_y is identical with the real Prandtl stress function φ — i.e.

$$h_y \equiv \Im(h) = \frac{h - \overline{h}}{2\imath} = \varphi \; . \tag{19.24}$$

It follows as in equation (16.9) that the boundary traction

$$\sigma_{zn} = \frac{\partial h_y}{\partial t} \; , \tag{19.25}$$

where t is a real coordinate measuring distance around the boundary in the anticlockwise direction. Also, from (19.16), we have

$$h_x = \frac{h + \overline{h}}{2} = \mu u_z = \mu f(x, y) \; , \tag{19.26}$$

where $f(x, y)$ is the function introduced in equation (15.1). Thus,

$$h = h_x + \imath h_y = \mu f + \imath \varphi \; , \tag{19.27}$$

showing that the real functions μf and φ from Chapters 15,16 can be combined as the real and imaginary parts of the same holomorphic function, for antiplane problems without body force.

19.2.1 Solution of antiplane boundary-value problems

If the displacement u_z is prescribed throughout the boundary, equation (19.16) reduces the problem to the search for a holomorphic function h such that $h+\overline{h}$ takes specified values on the boundary. If instead the traction σ_{zn} is specified everywhere on the boundary, we must first integrate equation (19.25) to obtain the boundary values of h_y and the problem then reduces to the search for a holomorphic function h such that $h-\overline{h}$ takes specified values on the boundary. This is essentially the same boundary-value problem, since we can define a new holomorphic function g through the equations

$$h_y = g + \overline{g} \quad \text{with} \quad g = -\frac{\imath h}{2} \; . \tag{19.28}$$

Example

As an example, we consider Problem 15.3 in which a uniform antiplane stress field $\sigma_{zx} = S$ is perturbed by the presence of a traction-free hole of radius a. As in §8.3.2, §8.4.1, we start with the unperturbed solution in which $\sigma_{zx} = S$ everywhere. From (19.17, 19.18),

$$\varPsi = \overline{h'} = S$$

and hence a particular solution (omitting a constant of integration which corresponds merely to a rigid-body displacement) is

$$\overline{h} = S\bar{\zeta} \; ; \quad h = S\zeta \; ,$$

where we note that S is a real constant, so $\overline{S} = S$. In particular,

$$h_y = \Im(h) = Sy = Sr\sin\theta \; .$$

To make the boundary of the hole traction-free, we require h_y to be constant around the hole and this constant can be taken to be zero without loss of generality. Thus, the corrective solution must satisfy

$$h_y(a, \theta) = -Sa\sin\theta \; .$$

It is possible to 'guess' the form of the solution, following arguments analogous to those used for the in-plane problem in §8.3.2, but here we shall illustrate the direct method by using equation (18.59). On the boundary $r = a$, we have

$$s = a\exp(\imath\theta) \; ; \quad \sin\theta = \frac{(e^{\imath\theta} - e^{-\imath\theta})}{2\imath} = \frac{1}{2\imath}\left(\frac{s}{a} - \frac{a}{s}\right) \; . \tag{19.29}$$

Since this is an exterior problem, we use (18.59) with (19.28, 19.29) to obtain

$$-\frac{\imath h}{2} = \frac{1}{2\pi\imath}\frac{Sa}{2\imath}\oint\left(\frac{s}{a} - \frac{a}{s}\right)\frac{ds}{(s-\zeta)} \; , \tag{19.30}$$

or

$$h = \frac{S}{2\pi\imath}\oint\left(s - \frac{a^2}{s}\right)\frac{ds}{(s-\zeta)} \; . \tag{19.31}$$

We can use the residue theorem (18.40) to compute the contour integral. Since ζ is outside the contour, only the pole a^2/s at $s = 0$ makes a contribution and we obtain

$$\text{Res}(0) = \frac{Sa^2}{2\pi\imath\zeta} \quad \text{giving} \quad h = \frac{Sa^2}{\zeta} \; . \tag{19.32}$$

The complete stress function (unperturbed + corrective solution) is then

$$h = S\zeta + \frac{Sa^2}{\zeta}$$

and the stress field is obtained from (19.18) as

$$\varPsi = \overline{h'} = S - \frac{Sa^2}{\bar{\zeta}^2} \; .$$

As a check on this result, we note that on $\zeta = s$,

$$h = Ss + \frac{Sa^2}{s} = S(s + \bar{s}) \; ,$$

using $a^2 = s\bar{s}$ from (18.51). Thus, on the boundary, h is the sum of a complex quantity and its conjugate which is real, confirming that the imaginary part $h_y = 0$ as required.

19.3 In-plane deformations

We next turn our attention to the problem of in-plane deformation, which was treated using the Airy stress function in Chapters 4–13. Our strategy will be to express the in-plane displacement

$$u = u_x + \imath u_y \tag{19.33}$$

in terms of complex *displacement functions*, much as the stresses in Chapter 4 were expressed in terms of the real Airy function ϕ. By choosing to represent the displacements rather than the stresses, we automatically satisfy the compatibility conditions, as explained in §2.2, but the stresses must then satisfy the equilibrium equations and this will place restrictions on our choice of displacement functions.

In vector notation, the equilibrium equations in terms of displacements require that

$$\boldsymbol{\nabla}\mathrm{div}\;\boldsymbol{u} + (1-2\nu)\nabla^2\boldsymbol{u} + \frac{(1-2\nu)\boldsymbol{p}}{\mu} = 0\;, \tag{19.34}$$

from (2.17). Using (19.33, 19.3, 19.7, 18.10), we can express the two in-plane components of this condition in the complex-variable form

$$2\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\partial u}{\partial\zeta} + \frac{\partial\overline{u}}{\partial\bar{\zeta}}\right) + 4(1-2\nu)\frac{\partial^2 u}{\partial\zeta\partial\bar{\zeta}} + \frac{(1-2\nu)p}{\mu} = 0\;, \tag{19.35}$$

where the in-plane body force

$$p = p_x + \imath p_y\;. \tag{19.36}$$

Integrating (19.35) with respect to $\bar{\zeta}$, we obtain

$$(3-4\nu)\frac{\partial u}{\partial\zeta} + \frac{\partial\overline{u}}{\partial\bar{\zeta}} = f - \frac{(1-2\nu)}{2\mu}\int p d\bar{\zeta}\;, \tag{19.37}$$

where f is an arbitrary holomorphic function of ζ. This complex equation really comprises two separate equations, one for the real part and one for the imaginary part, both of which must be satisfied by u. It follows that the conjugate equation

$$(3-4\nu)\frac{\partial\overline{u}}{\partial\bar{\zeta}} + \frac{\partial u}{\partial\zeta} = \overline{f} - \frac{(1-2\nu)}{2\mu}\int \overline{p} d\zeta \tag{19.38}$$

must also be satisfied. We can then eliminate $\partial\overline{u}/\partial\bar{\zeta}$ between (19.37, 19.38) to obtain

$$8(1-2\nu)(1-\nu)\frac{\partial u}{\partial\zeta} = (3-4\nu)f - \overline{f} + \frac{(1-2\nu)}{2\mu}\left[\int \overline{p} d\zeta - (3-4\nu)\int p d\bar{\zeta}\right]\;, \tag{19.39}$$

with solution

$$8(1-2\nu)(1-\nu)u = (3-4\nu)\int f d\zeta - \zeta\,\overline{f} + \overline{g}$$
$$+\frac{(1-2\nu)}{2\mu}\int\left[\int \overline{p}d\zeta - (3-4\nu)\int p d\bar{\zeta}\right]d\zeta\ , \quad (19.40)$$

where $\overline{g}$ is an arbitrary function of $\bar{\zeta}$.

The functions f, g are arbitrary and, in particular, $\int f d\zeta$ is simply a new arbitrary function of ζ. We can therefore write a more compact form of (19.40) by defining different arbitrary functions through the relations

$$\int f d\zeta = \frac{4(1-\nu)(1-2\nu)\chi}{\mu}\ ;\quad \overline{g} = -\frac{4(1-\nu)(1-2\nu)\overline{\theta}}{\mu}\ , \quad (19.41)$$

where $\chi, \overline{\theta}$ are holomorphic functions of ζ and $\bar{\zeta}$ only respectively. We then have

$$\overline{f} = \frac{4(1-\nu)(1-2\nu)\overline{\chi'}}{\mu}\ , \quad (19.42)$$

where $\overline{\chi'}$ represents the derivative of $\overline{\chi}$ (which is a function of $\bar{\zeta}$ only) with respect to $\bar{\zeta}$.

With this notation, equation (19.40) takes the form

$$2\mu u = (3-4\nu)\chi - \zeta\,\overline{\chi'} - \overline{\theta} + \frac{1}{8(1-\nu)}\int\left[\int \overline{p}d\zeta - (3-4\nu)\int p d\bar{\zeta}\right]d\zeta\ . \quad (19.43)$$

From this point on, we shall restrict attention to the simpler case where the body forces are zero ($p=\overline{p}=0$), in which case

$$2\mu u = (3-4\nu)\chi - \zeta\,\overline{\chi'} - \overline{\theta}\ . \quad (19.44)$$

It is a simple matter to reintroduce body force terms in the subsequent derivations if required.

19.3.1 Expressions for stresses

The tractions on the x and y-planes are

$$\tau_x = \sigma_{xx} + \imath\sigma_{xy}\ ;\quad \tau_y = \sigma_{yx} + \imath\sigma_{yy} \quad (19.45)$$

and these can be combined to form the functions

$$\Phi \equiv \tau_x + \imath\tau_y = \sigma_{xx} + 2\imath\sigma_{xy} - \sigma_{yy} \quad (19.46)$$
$$\Theta \equiv \tau_x - \imath\tau_y = \sigma_{xx} + \sigma_{yy}\ . \quad (19.47)$$

We note that Θ is a real function and is invariant with respect to coördinate transformation, whilst Φ transforms according to the rule

$$\varPhi_\alpha \equiv \sigma'_{xx} + 2\imath\sigma'_{xy} - \sigma'_{yy} = e^{-2\imath\alpha}\varPhi \;, \tag{19.48}$$

where $\varPhi_\alpha$ and the stress components σ' are defined in a coördinate system x', y' rotated anticlockwise through an angle α with respect to x, y. This result is easily established using the stress transformation equations (1.15–1.17).

Substituting (19.44) into (19.7) and recalling that the state is one of plane strain so $u_z = 0$, we find

$$2\mu \text{div}\, \boldsymbol{u} = 2(1-2\nu)\left(\chi' + \overline{\chi'}\right) \tag{19.49}$$

and hence, using the stress-strain relations (1.71)

$$\begin{aligned} \varTheta &= 2(\lambda+\mu)\text{div}\, \boldsymbol{u} \\ &= 2(\chi' + \overline{\chi'}) \end{aligned} \tag{19.50}$$

$$\begin{aligned} \varPhi &= 2\mu\left\{\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} + \imath\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right\} \\ &= 4\mu\frac{\partial u}{\partial \bar{\zeta}} = -2(\zeta\,\overline{\chi''} + \overline{\theta'}) \;. \end{aligned} \tag{19.51}$$

These expressions can also be written in terms of the single complex function

$$\psi(\zeta, \bar{\zeta}) = \chi + \zeta\overline{\chi'} + \overline{\theta} \;. \tag{19.52}$$

We then have

$$\frac{\partial \psi}{\partial \zeta} = \chi' + \overline{\chi'} \;; \quad \frac{\partial \psi}{\partial \bar{\zeta}} = \zeta\,\overline{\chi''} + \overline{\theta'} \tag{19.53}$$

and hence

$$\varTheta = 2\frac{\partial \psi}{\partial \zeta} \;; \quad \varPhi = -2\frac{\partial \psi}{\partial \bar{\zeta}} \;. \tag{19.54}$$

Notice however that ψ depends on both ζ and $\bar{\zeta}$ and is therefore not holomorphic.

19.3.2 Rigid-body displacement

Equations (19.54) show that all the stress components will be zero everywhere if and only if $\psi(\zeta, \bar{\zeta})$ is a constant. Expanding the holomorphic functions χ, θ as Taylor series

$$\chi = A_0 + A_1\zeta + A_2\zeta^2 + \ldots \;; \quad \theta = B_0 + B_1\zeta + B_2\zeta^2 + \ldots \tag{19.55}$$

and substituting into (19.52), we obtain

$$\psi = A_0 + \bar{B}_0 + (A_1 + \bar{A}_1)\zeta + B_1\bar{\zeta} + A_2\zeta^2 + 2\bar{A}_2\bar{\zeta}\zeta + \bar{B}_2\bar{\zeta}^2 + \ldots \;. \tag{19.56}$$

This will be constant if and only if $(A_1 + \bar{A}_1) = 0$ and all the other coefficients are zero except $A_0, \bar{B}_0$. The condition $(A_1 + \bar{A}_1) = 0$ implies that A_1 is pure imaginary, which can be enforced by writing $A_1 = \imath C_1$, where C_1 is a real constant. Substituting these values into (19.44), we then obtain

$$2\mu u = (3 - 4\nu)A_0 - \bar{B}_0 + 4(1 - \nu)\imath C_1 \zeta \; . \tag{19.57}$$

The first two terms correspond to an arbitrary rigid-body translation and the third to a rigid-body rotation. In fact, either of the complex constants $A_0, \bar{B}_0$ has sufficient degrees of freedom to define an arbitrary rigid-body translation, so one of them can be set to zero without loss of generality. In the following derivations, we shall therefore generally set $B_0 = \bar{B}_0 = 0$, implying that $\theta(0) = 0$.

19.4 Relation between the Airy stress function and the complex potentials

Before considering methods for solving boundary-value problems in complex-variable notation, it is of interest to establish some relationships with the Airy stress function formulation of Chapters 4–13. In particular, we shall show that it is always possible to find complex potentials χ, θ corresponding to a given Airy function ϕ. This has the incidental advantage that the resulting complex potentials can then be used to determine the displacement components, so this procedure provides a method for generating the displacements due to a given Airy stress function, which, as we saw in Chapter 9, is not always straightforward in the real stress function formulation.

We start by using the procedure of §18.4.1 to express the real biharmonic function ϕ in terms of two holomorphic functions g_1, g_2, as in equation (18.27).

The stress components are given in terms of ϕ as

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; , \tag{19.58}$$

from equations (4.6). It follows that

$$\varTheta = \nabla^2 \phi = 4 \frac{\partial^2 \phi}{\partial \zeta \partial \bar{\zeta}} \tag{19.59}$$

and

$$\varPhi = \frac{\partial^2 \phi}{\partial y^2} - 2\imath \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x^2} = -\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y} \right)^2 = -4 \frac{\partial^2 \phi}{\partial \bar{\zeta}^2} \; , \tag{19.60}$$

using (18.10, 18.6).

Substituting for ϕ from (18.27) into (19.59), we then have

$$\varTheta = \nabla^2 \phi = 4 \left(g_2' + \overline{g_2'} \right) \tag{19.61}$$

and comparison with (19.50) shows that

$$\chi = 2g_2 \; . \tag{19.62}$$

Similarly, from (18.27, 19.60) we have

$$\Phi = -4\left(g_1'' + \zeta\,\overline{g_2''}\right) = -2\left(\zeta\,\overline{\chi''} + 2\overline{g_1''}\right) \;, \tag{19.63}$$

using (19.62). Comparison with (19.51) then shows that

$$\theta = 2g_1' \; . \tag{19.64}$$

Example

As an example, we consider the problem of §5.2.1 in which the rectangular beam $x>0, -b<y<b$ is loaded by a shear force F at the end $x=0$. The real stress function for this case is given by (5.35) as

$$\phi = \frac{F(xy^3 - 3b^2xy)}{4b^3} \; .$$

Using (18.2), we have

$$xy = -\frac{\imath}{4}\left(\zeta^2 - \bar{\zeta}^2\right) \qquad \text{and} \qquad xy^3 = \frac{\imath}{16}\left(\zeta^2 - \bar{\zeta}^2\right)\left(\zeta - \bar{\zeta}\right)^2 \;,$$

from which

$$\phi = \frac{\imath F}{64b^3}\left(\zeta^4 - 2\zeta^3\bar{\zeta} + 2\zeta\bar{\zeta}^3 - \bar{\zeta}^4 + 12b^2\zeta^2 - 12b^2\bar{\zeta}^2\right)$$

Comparison with (18.27) shows that

$$g_1 = \frac{\imath F}{64b^3}\left(\zeta^4 + 12b^2\zeta^2\right) \; ; \quad g_2 = -\frac{\imath F\zeta^3}{32b^3} \; .$$

We then have

$$\chi = -\frac{\imath F\zeta^3}{16b^3} \; ; \quad \theta = \frac{\imath F}{8b^3}\left(\zeta^3 + 6b^2\zeta\right) \;, \tag{19.65}$$

from (19.62, 19.64) and hence

$$\begin{aligned} 2\mu u &= 2\mu(u_x + \imath u_y) = (3-4\nu)\chi - \zeta\,\overline{\chi'} - \overline{\theta} \\ &= \frac{\imath F}{16b^3}\left(-(3-4\nu)\zeta^3 - 3\zeta\bar{\zeta}^2 + 2\bar{\zeta}^3 + 12b^2\bar{\zeta}\right) \\ &= \frac{F}{4b^3}\left(\left[3(1-\nu)x^2y - (2-\nu)y^3 + 3b^2y\right] + \imath\left[3b^2x - 3\nu xy^2 - (1-\nu)x^3\right]\right) \;, \end{aligned}$$

agreeing with the plane-strain equivalents of (9.14, 9.15) apart from an arbitrary rigid-body displacement.

19.5 Boundary tractions

We denote the boundary traction (i.e. the force per unit length along the boundary S) as $\boldsymbol{T}(s)$, where s is a real coördinate defining position around S. The traction is a vector which can be expressed in the usual complex form as

$$\boldsymbol{T}(s) \equiv \boldsymbol{i}T_x + \boldsymbol{j}T_y = T_x + \imath T_y \equiv T \ . \tag{19.66}$$

Following the conventions of Chapter 18, the coördinate s increases as we traverse the boundary in the anticlockwise direction as shown in Figure 19.1(a), which also shows the traction components T_x, T_y.

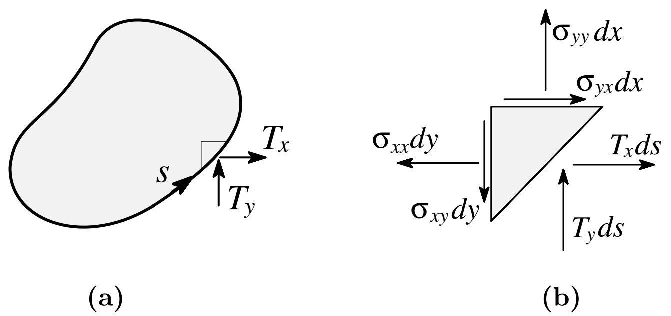

Figure 19.1: (a) The boundary tractions T_x, T_y and (b) equilibrium of a small element at the boundary.

Figure 19.1(b) shows the forces acting on a small triangular element chosen such that x, y, s all increase as we move up the inclined edge[1]. We recall the convention that the material lies always on the left of the boundary as s increases.

Equilibrium of this element then requires that

$$T_x ds = \sigma_{xx} dy - \sigma_{yx} dx \ ; \quad T_y ds = \sigma_{xy} dy - \sigma_{yy} dx \tag{19.67}$$

and hence

$$T ds = (\sigma_{xx} + \imath\sigma_{xy})dy - (\sigma_{yx} + \imath\sigma_{yy})dx = \tau_x dy - \tau_y dx \ , \tag{19.68}$$

using the notation of equation (19.45).

We can solve (19.46, 19.47) for τ_x, τ_y, obtaining

$$\tau_x = \frac{1}{2}(\Theta + \Phi) \ ; \quad \tau_y = \frac{\imath}{2}(\Theta - \Phi) \tag{19.69}$$

and hence

[1] If any other orientation were chosen, one or more of the increments dx, dy, ds would be negative resulting in a change in direction of the corresponding forces, but the same final result would be obtained.

$$Tds = \frac{1}{2}\left(\Theta(dy - \imath dx) + \Phi(dy + \imath dx)\right) = \frac{\imath}{2}\left(\Phi d\bar{\zeta} - \Theta d\zeta\right) . \tag{19.70}$$

Using (19.54), we then have

$$Tds = \imath\left(-\frac{\partial\psi}{\partial\bar{\zeta}}d\bar{\zeta} - \frac{\partial\psi}{\partial\zeta}d\zeta\right) = -\imath d\psi . \tag{19.71}$$

In other words,

$$T = -\imath\frac{d\psi}{ds} = -\imath\frac{d}{ds}\left(\chi + \zeta\,\overline{\chi'} + \overline{\theta}\right) \quad \text{or} \quad \frac{d\psi}{ds} = \imath T . \tag{19.72}$$

Thus, if the tractions are known functions of s, equation (19.72) can be integrated to yield the values of ψ at all points on the boundary. We also note that ψ must be constant in any part of the boundary that is traction-free.

Example

To illustrate these results, we revisit the example in §19.4 in which the rectangular bar $-b<y<b$, $x>0$ is loaded by a transverse force F on the end $x=0$, the edges $y=\pm b$ being traction-free. The complex potentials are given in (19.65). Substituting them into (19.52), we find

$$\psi = -\frac{\imath F}{16b^3}\left\{(\zeta-\bar{\zeta})^3 + 3\bar{\zeta}\left[(\zeta-\bar{\zeta})^2 + 4b^2\right]\right\} . \tag{19.73}$$

In complex coördinates, the boundaries $y=\pm b$ correspond to the lines

$$\zeta - \bar{\zeta} = \pm 2\imath b ,$$

from (18.2) and it follows that the second term in the braces in (19.73) is zero on each boundary, whilst the first term takes the value

$$\mp 8\imath b^3 , \qquad \text{corresponding to} \qquad \psi = \mp\frac{F}{2} .$$

This confirms that ψ is constant along each of the two traction-free boundaries. We also note that as we go down the left end of the bar (thereby preserving the anticlockwise direction for s), ψ increases by F, so

$$F = \int_{\zeta=\imath b}^{\zeta=-\imath b}\frac{d\psi}{ds}ds = \int_{\zeta=\imath b}^{\zeta=-\imath b}\imath T ds ,$$

using (19.72). It follows that the resultant force on the end is

$$\int_{\zeta=\imath b}^{\zeta=-\imath b} T ds = -\imath F ,$$

which corresponds to a force F in the negative y-direction, which of course agrees with Figure 5.2 and equation (5.25).

19.5.1 Equilibrium considerations

The in-plane problem is well posed if and only if the applied loads are self-equilibrated and since we are assuming that there are no body forces, this implies that the resultant force and moment of the boundary tractions must be zero.

The force resultant can be found by integrating the complex traction T around the boundary S, using (19.72). We obtain

$$F_x + \imath F_y = \oint_S T ds = -\imath \oint_S \frac{d\psi}{ds} ds = 0 \;, \tag{19.74}$$

which implies that ψ must be single-valued. This condition applies regardless of the shape of the boundary, assuming the body is simply connected.

The contribution to the resultant moment from the traction Tds on an element of boundary ds can be written

$$dM = \frac{\imath}{2}\left(s\overline{Tds} - \overline{s}Tds\right) = -\frac{1}{2}\left(sd\overline{\psi} + \overline{s}d\psi\right) = -\Re\left(\overline{s}d\psi\right) \;, \tag{19.75}$$

using (19.10) and (19.71). The resultant moment can then be written

$$M = \oint_S dM = -\Re \oint_S \overline{s}\frac{\partial \psi}{\partial \overline{s}} d\overline{s} \tag{19.76}$$

and integrating by parts, we have

$$\oint_S \overline{s}\frac{\partial \psi}{\partial \overline{s}} d\overline{s} = \overline{s}\psi|_S - \oint_S \psi d\overline{s} = -\oint_S \psi d\overline{s} \;,$$

since $\overline{s}\psi$ is single-valued in view of (19.74). Thus, the condition that the resultant moment be zero reduces to

$$M = \Re\left(\oint_S \psi d\overline{s}\right) = 0 \;. \tag{19.77}$$

19.6 Boundary-value problems

Two quantities must be specified at every point on the boundary S. In the *displacement boundary-value problem*, both components of displacement are prescribed and hence we know the value of the complex function

$$2\mu u(s) = \kappa\chi - \zeta\,\overline{\chi'} - \overline{\theta} \;; \quad s \in S \;, \tag{19.78}$$

where we recall that $\kappa = (3-4\nu)$. In the *traction boundary-value problem*, both components of traction $T(s)$ are prescribed and we can integrate equation (19.72) to obtain

$$\psi(s) = \imath \int T(s)ds + C \;, \tag{19.79}$$

where C is an arbitrary constant of integration. The complex potentials must then be chosen to satisfy the boundary condition

$$\chi(s) + s\,\overline{\chi'(s)} + \overline{\theta(s)} = \psi(s) \;; \quad s \in S \;. \tag{19.80}$$

Notice that the variable of integration s in (19.79) is a real curvilinear cöordinate measuring the distance traversed around the boundary S, but since we shall later want to apply the Cauchy integral theorem, it is necessary to express the resulting function ψ as a function of the complex cöordinate s defining a general point on S.

The displacement and traction problems both have the same form and can be combined as

$$\gamma\chi(s) + s\,\overline{\chi'(s)} + \overline{\theta(s)} = f(s) \;; \quad s \in S \;, \tag{19.81}$$

where $\gamma = -\kappa, f(s) = -2\mu u(s)$ for the displacement problem and $\gamma = 1, f(s) = \psi(s)$ for the traction problem.

One significant difference between the displacement and traction problems is that the former is completely defined for all continuous values of the boundary data, whereas the solution of traction problem is indeterminate to within an arbitrary rigid-body displacement as defined in §19.3.2. Also, the permissible traction distributions are restricted by the equilibrium conditions (19.74, 19.76). Notice incidentally that the constant C in (19.79) can be wrapped into the arbitrary rigid-body translation A_0 in (19.57), since a particular solution of the problem $\psi(\zeta, \bar{\zeta}) = C$ on S is $\psi = C$ (constant) throughout Ω.

19.6.1 Solution of the interior problem for the circle

As in §18.6.1, we can use the Cauchy integral theorem to obtain the solution of the in-plane problem for the region Ω bounded by the circle $r = a$ with prescribed boundary tractions or displacements.

Applying the Cauchy integral operator to both sides of equation (19.81), we obtain

$$\frac{\gamma}{2\pi\imath} \oint \frac{\chi(s)ds}{(s-\zeta)} + \frac{1}{2\pi\imath} \oint \frac{s\,\overline{\chi'(s)}ds}{(s-\zeta)} + \frac{1}{2\pi\imath} \oint \frac{\overline{\theta(s)}ds}{(s-\zeta)} = \frac{1}{2\pi\imath} \oint \frac{f(s)ds}{(s-\zeta)} \;. \tag{19.82}$$

Since χ is a holomorphic function of ζ in Ω, equation (18.44) gives

$$\frac{1}{2\pi\imath} \oint \frac{\chi(s)ds}{(s-\zeta)} = \chi(\zeta) \;.$$

Also, writing

$$\chi = \sum_{n=0}^{\infty} A_n \zeta^n \quad ; \quad \overline{\chi'} = \sum_{n=1}^{\infty} n\bar{A}_n \bar{\zeta}^{n-1} \;, \tag{19.83}$$

and noting that $\overline{s}=a^2/s$ on the boundary S, we have

$$s\,\overline{\chi'(s)} = \sum_{n=1}^{\infty} \frac{n\bar{A}_n a^{2n-2}}{s^{n-2}}\ .$$

Using the residue theorem, the second term in (19.82) can then be evaluated as

$$\frac{1}{2\pi\imath}\oint \frac{s\,\overline{\chi'(s)}ds}{(s-\zeta)} = \sum_{n=1}^{\infty} \frac{n\bar{A}_n a^{2n-2}}{2\pi\imath}\oint \frac{ds}{s^{n-2}(s-\zeta)} = \bar{A}_1\zeta + 2\bar{A}_2 a^2\ ,$$

since all the remaining terms in the series integrate to zero as in (18.42).

A similar argument can be applied to the third integral in (19.82). Expanding

$$\theta(\zeta) = \sum_{k=0}^{\infty} B_k\zeta^k\ ,$$

we obtain

$$\frac{1}{2\pi\imath}\oint \frac{\overline{\theta(s)}ds}{(s-\zeta)} = \bar{B}_0\ ,$$

and we note from §19.3.2 that this can be set to zero without loss of generality. Assembling these results, we conclude that

$$\gamma\chi(\zeta) = \frac{1}{2\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)} - \bar{A}_1\zeta - 2\bar{A}_2 a^2\ . \tag{19.84}$$

To determine the constants $\bar{A}_1, \bar{A}_2$, we first replace χ in (19.84) by its Taylor series (19.83) giving

$$\frac{1}{2\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)} = \bar{A}_1\zeta + 2\bar{A}_2 a^2 + \gamma A_0 + \gamma A_1\zeta + \gamma A_2\zeta^2 + \dots\ . \tag{19.85}$$

Differentiating twice with respect to ζ, we obtain

$$-\frac{1}{2\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)^2} = \bar{A}_1 + \gamma A_1 + 2\gamma A_2\zeta + \dots\ .$$
$$\frac{1}{\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)^3} = 2\gamma A_2 + 6\gamma A_3\zeta + \dots \tag{19.86}$$

and evaluating these last two expressions at $\zeta=0$,

$$\gamma A_2 = \frac{1}{2\pi\imath}\oint \frac{f(s)ds}{s^3}\ ;\quad \bar{A}_1 + \gamma A_1 = -\frac{1}{2\pi\imath}\oint \frac{f(s)ds}{s^2}\ . \tag{19.87}$$

Equations (19.87) are sufficient to determine the required constants for the displacement problem ($\gamma=\kappa$), but a degeneracy occurs for the traction problem where $\gamma=1$ and the second equation reduces to

$$\bar{A}_1 + A_1 = -\frac{1}{2\pi\imath}\oint \frac{\psi(s)ds}{s^2} \; . \qquad (19.88)$$

This implies that the right-hand-side must evaluate to a real constant, which (i) places a restriction on the permissible values of $\psi(s)$ and (ii) leaves the imaginary part of A_1 indeterminate. We saw already in §19.3.2 that the imaginary part of A_1 corresponds to an arbitrary rigid-body rotation. The condition that the right-hand side of (19.88) be real is exactly equivalent to the condition (19.77) enforcing moment equilibrium. To establish the connection, we note that for the circular boundary

$$\overline{s} = \frac{a^2}{s} \quad \text{so that} \quad d\overline{s} = -\frac{a^2 ds}{s^2} \; .$$

Using this result in (19.77), we obtain

$$\Re\left(\oint_S \psi d\overline{s}\right) = -\Re\left(a^2 \oint_S \frac{\psi(s)ds}{s^2}\right) = 0 \; ,$$

showing that the integral is pure imaginary and hence that the right-hand side of (19.88) is real.

Once χ is determined, we can apply a similar method to determine θ. We first write the conjugate of equation (19.81) as

$$\theta(s) = \overline{f(s)} - \gamma\overline{\chi(s)} - \overline{s}\chi'(s) \; ; \quad s \in S \; , \qquad (19.89)$$

and apply the Cauchy integral theorem, obtaining

$$\theta(\zeta) = \frac{1}{2\pi\imath}\oint \frac{\theta(s)ds}{(s-\zeta)} = \frac{1}{2\pi\imath}\oint \frac{\overline{f(s)}ds}{(s-\zeta)} - \frac{\gamma}{2\pi\imath}\oint \frac{\overline{\chi(s)}ds}{(s-\zeta)} - \frac{1}{2\pi\imath}\oint \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} \; . \qquad (19.90)$$

The last two integrals on the right-hand side can be evaluated using (19.83) and the residue theorem leading to

$$\theta(\zeta) = \frac{1}{2\pi\imath}\oint \frac{\overline{f(s)}ds}{(s-\zeta)} - \gamma\bar{A}_0 - \frac{a^2\left(\chi'(\zeta) - \chi'(0)\right)}{\zeta} \; . \qquad (19.91)$$

Once this expression has been evaluated, the remaining constant $\bar{A}_0$ can be chosen to satisfy the condition $\theta(0)=0$ as discussed in §19.57. However, there is no essential need to do this, since this constant corresponds merely to a rigid-body displacement and makes no contribution to the stress components.

19.6.2 Solution of the exterior problem for the circle

For the exterior problem, we again apply the Cauchy integral operator, leading to (19.82), which we repeat here for clarity

$$\frac{\gamma}{2\pi\imath}\oint\frac{\chi(s)ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{s\,\overline{\chi'(s)}ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\overline{\theta(s)}ds}{(s-\zeta)}=\frac{1}{2\pi\imath}\oint\frac{f(s)ds}{(s-\zeta)}\;. \tag{19.92}$$

In this equation, the first term

$$\frac{\gamma}{2\pi\imath}\oint\frac{\chi(s)ds}{(s-\zeta)}=-\gamma\chi(\zeta)\;,$$

from (18.48). For the third term, we note that $\theta(\zeta)$ is holomorphic in the exterior region and hence permits a Laurent expansion

$$\theta(\zeta)=\frac{B_1}{\zeta}+\frac{B_2}{\zeta^2}+...$$

so

$$\overline{\theta(s)}=\frac{\bar{B}_1}{\bar{s}}+\frac{\bar{B}_2}{\bar{s}^2}+...=\frac{\bar{B}_1 s}{a^2}+\frac{\bar{B}_2 s^2}{a^4}+...\;.$$

This can be regarded as the surface value of a function that is holomorphic in the interior region and hence, since ζ is outside the contour S,

$$\frac{1}{2\pi\imath}\oint\frac{\overline{\theta(s)}ds}{(s-\zeta)}=0\;. \tag{19.93}$$

For the second term in (19.92), we write

$$\chi(\zeta)=\frac{A_1}{\zeta}+\frac{A_2}{\zeta^2}+\frac{A_3}{\zeta^3}+... \tag{19.94}$$

so

$$\overline{\chi'}=-\frac{\bar{A}_1}{\bar{\zeta}^2}-\frac{2\bar{A}_2}{\bar{\zeta}^3}-\frac{3\bar{A}_3}{\bar{\zeta}^4}+...\;;\quad s\,\overline{\chi'(s)}=-\frac{\bar{A}_1 s^3}{a^4}-\frac{2\bar{A}_2 s^4}{a^6}-\frac{3\bar{A}_3 s^5}{a^8}+...$$

and as before, we conclude that

$$\frac{1}{2\pi\imath}\oint\frac{s\overline{\chi'(s)}ds}{(s-\zeta)}=0\;,$$

so

$$\gamma\chi(\zeta)=-\frac{1}{2\pi\imath}\oint\frac{f(s)ds}{(s-\zeta)}\;. \tag{19.95}$$

To determine θ, the conjugate equation (19.89) and (18.48) lead to

$$\theta(\zeta)=-\frac{1}{2\pi\imath}\oint\frac{\theta(s)ds}{(s-\zeta)}=-\frac{1}{2\pi\imath}\oint\frac{\overline{f(s)}ds}{(s-\zeta)}+\frac{\gamma}{2\pi\imath}\oint\frac{\overline{\chi(s)}ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\bar{s}\chi'(s)ds}{(s-\zeta)}\;,$$

and the second integral on the right-hand side is clearly of the same form as (19.93) and is therefore zero. For the third integral, we have

$$\frac{1}{2\pi\imath}\oint\frac{\bar{s}\chi'(s)ds}{(s-\zeta)}=\frac{1}{2\pi\imath}\oint\frac{a^2\chi'(s)ds}{s(s-\zeta)}=-\frac{a^2\chi'(\zeta)}{\zeta}\;,$$

using (18.48). We conclude that

$$\theta(\zeta)=-\frac{1}{2\pi\imath}\oint\frac{\overline{f(s)}ds}{(s-\zeta)}-\frac{a^2\chi'(\zeta)}{\zeta}\;. \tag{19.96}$$

Example

To illustrate the procedure, we consider the problem shown in Figure 19.2 in which a circular hole of radius a is loaded by equal and opposite compressive forces F on the diameter.

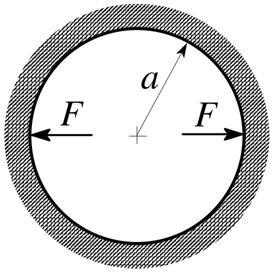

Figure 19.2: Hole in a large body loaded by equal and opposite forces F.

In this case, we have

$$T = \frac{F\delta(\theta)}{a} - \frac{F\delta(\theta-\pi)}{a}$$

and a suitable integral of (19.72) yields

$$\psi(s) = -\frac{\imath F}{2}\text{sgn}(\theta) \quad \text{so} \quad \overline{\psi(s)} = \frac{\imath F}{2}\text{sgn}(\theta)\ , \tag{19.97}$$

where $\text{sgn}(\theta)=+1$ in $0<\theta<\pi$ and -1 in $-\pi<\theta<0$. Notice that in performing this integral, it is necessary to traverse the contour in the direction that keeps the material of the body on the left, as explained in §19.5. Substituting in equation (19.95) with $\gamma=1, f(s)=\psi(s)$, we obtain

$$\begin{aligned}\chi(\zeta) &= -\frac{1}{2\pi\imath}\oint\frac{\psi(s)ds}{(s-\zeta)} = -\frac{F}{4\pi}\left(\ln(s-\zeta)\Big|_{s=-a}^{s=a} - \ln(s-\zeta)\Big|_{s=a}^{s=-a}\right) \\ &= \frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right)\ ,\end{aligned}$$

from which

$$\chi'(\zeta) = -\frac{Fa}{\pi(\zeta^2-a^2)}\ .$$

The integral term in (19.96) is evaluated in the same way as

$$\frac{1}{2\pi\imath}\oint\frac{\overline{\psi(s)}ds}{(s-\zeta)} = \frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right)$$

and it follows that

$$\theta(\zeta) = -\frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) + \frac{Fa^3}{\pi\zeta(\zeta^2-a^2)}\ .$$

Finally, the complex stresses can be recovered by substitution into (19.61, 19.63), giving

$$\Theta = 2\left(\chi' + \overline{\chi'}\right) = -\frac{Fa}{\pi}\left(\frac{1}{(\zeta^2 - a^2)} + \frac{1}{(\bar{\zeta}^2 - a^2)}\right)$$

$$\Phi = -2\left(\zeta\,\overline{\chi''} + \overline{\theta}'\right) = \frac{2Fa}{\pi}\left(\frac{2(a^2 - \zeta\bar{\zeta})}{(\bar{\zeta}^2 - a^2)^2} - \frac{1}{\bar{\zeta}^2}\right) .$$

This problem could have been solved using the real stress function approach of Chapter 13 (for example by superposing the solution of Problem 13.2 for two forces at $\theta = 0, \pi$), but the present method is considerably more direct.

19.7 Conformal mapping for in-plane problems

The technique of conformal mapping introduced in §18.7 can be applied to in-plane problems, but requires some modification because of the presence of the derivative term χ' in the boundary condition (19.81). Suppose that the problem is defined in the domain Ω^* and that we can identify a mapping function $\omega(\zeta)$ that maps each point $\omega \in \Omega^*$ into a point $\zeta \in \Omega$, where Ω represents the domain either interior or exterior to the unit circle $|\zeta| = 1$.

The boundary condition (19.81) requires that

$$\gamma\chi(\omega) + \omega\,\overline{\chi'(\omega)} + \overline{\theta(\omega)} = f(\omega)\ ;\quad \omega \in S^*\ , \tag{19.98}$$

where S^* is the boundary of Ω^*. If we substitute $\omega = \omega(\zeta)$, the functions χ, θ will become functions of ζ, but to avoid confusion, we shall adopt the notation

$$\chi_\zeta(\zeta) = \chi(\omega(\zeta))\ ;\quad \theta_\zeta(\zeta) = \theta(\omega(\zeta))\ ;\quad f_\zeta(\zeta) = f(\omega(\zeta))$$

for the corresponding functions in the ζ-plane. We then have

$$\chi'(\omega) \equiv \frac{d\chi}{d\omega} = \frac{d\chi_\zeta}{d\zeta}\frac{d\zeta}{d\omega} = \frac{\chi'_\zeta}{\omega'}\ .$$

Using the conjugate of this result in (19.98), we obtain

$$\gamma\chi_\zeta(s) + \frac{\omega(s)}{\overline{\omega'(s)}}\overline{\chi'_\zeta(s)} + \overline{\theta_\zeta(s)} = f_\zeta(s)\ ;\quad s \in S\ , \tag{19.99}$$

where S is the unit circle. This defines the boundary-value problem in the ζ-plane.

The mathematical arguments used in §§19.6.1, 19.6.2 can be applied to this equation and will yield identical results except in regard to the second term. To fix ideas, suppose Ω^* is the region exterior to a non-circular hole and $\omega(\zeta)$ maps into the region exterior to the unit circle S. Then, the second term in equation (19.92) must be replaced by

$$\frac{1}{2\pi\imath}\oint_S \frac{\omega(s)\overline{\chi'_\zeta(s)}ds}{\overline{\omega'(s)}(s-\zeta)} \tag{19.100}$$

and after evaluating the remaining terms as in §19.6.2, we obtain

$$\gamma\chi_\zeta(\zeta) = -\frac{1}{2\pi\imath}\oint_S \frac{f_\zeta(s)ds}{(s-\zeta)} - \frac{1}{2\pi\imath}\oint_S \frac{\omega(s)\overline{\chi'_\zeta(s)}ds}{\overline{\omega'(s)}(s-\zeta)} \; . \tag{19.101}$$

The integral (19.100) will be zero if the integrand has no poles inside the contour and can be evaluated using the residue theorem if it has a finite number of poles. As in §19.6.2, we can represent χ_ζ by its Laurent series (19.94) and hence write

$$\overline{\chi'_\zeta(s)} = -\bar{A}_1 s^2 - 2\bar{A}_2 s^3 - 3\bar{A}_3 s^4 + ..., \tag{19.102}$$

since on the unit circle $s\bar{s} = 1$. Equation (19.102) can be regarded as the surface values of a holomorphic function that has no poles within the circle, but the mapping function ω must be holomorphic in the region Ω *exterior* to the circle S, and this generally implies that it will have poles in the interior region.

Since both Ω, Ω^* extend to infinity, it is reasonable to choose a mapping function $\omega(\zeta)$ such that concentric circles in the ζ-plane map into increasingly circular contours as we go further away from the hole. This is achieved by the function

$$\omega(\zeta) = c\zeta + \frac{B_1}{\zeta} + \frac{B_2}{\zeta^2} + \frac{B_3}{\zeta^3} + ... \; , \tag{19.103}$$

since at large ζ, only the first term remains, which represents simply a linear scaling. The series in (19.103) may be infinite or finite, but in either case $\omega(\zeta)$ clearly posesses poles at the origin. If the series is finite, the integrand in (19.100) will have at most a finite number of poles and the integral can be evaluated using the residue theorem. It follows from (18.39) that the resulting expression will contain unknowns representing the derivatives of $\overline{\chi_\zeta}$ evaluated at the origin, which in view of (19.102) comprises a finite number of the unknown coefficients $\bar{A}_1, \bar{A}_2, ...$ However, a sufficient number of linear equations for these coefficients can be obtained by constructing the conjugate $\overline{\chi_\zeta}$ from (19.101) and equating it or its derivatives to the corresponding values at the origin. This is of course precisely the same procedure that was used to determine the constants A_1, A_2 in §19.6.1.

This procedure can be generalized to allow ω to be a rational function

$$\omega(\zeta) = \frac{C_1\zeta + C_2\zeta^2 + ... + C_n\zeta^n}{D_1\zeta + D_2\zeta^2 + ... + D_m\zeta^m} \; ,$$

which will posess simple poles at each of the m zeros of the finite polynomial representing the denominator. In this case, the unknowns appearing in the integral (19.100) will be the values of $\overline{\chi_\zeta}$ at each of these poles and a set of equations for determining them is obtained by evaluating (19.101) at each such pole[2].

Once χ_ζ has been determined, θ_ζ can be obtained exactly as in §19.6.2. We take the conjugate of (19.99) and apply the Cauchy integral operator obtaining

$$\frac{\gamma}{2\pi\imath}\oint\frac{\overline{\chi_\zeta(s)}ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\overline{\omega(s)}\chi'_\zeta(s)ds}{\omega'(s)(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\theta_\zeta(s)ds}{(s-\zeta)}=\frac{1}{2\pi\imath}\oint\frac{\overline{f_\zeta(s)}ds}{(s-\zeta)}$$

The first integral is zero, as in §19.6.2 and applying (18.48) to the third term, we obtain

$$\theta_\zeta(\zeta)=-\frac{1}{2\pi\imath}\oint\frac{\overline{f_\zeta(s)}ds}{(s-\zeta)}+\frac{1}{2\pi\imath}\oint\frac{\overline{\omega(s)}\chi'_\zeta(s)ds}{\omega'(s)(s-\zeta)}\,. \tag{19.104}$$

At this stage, all the quantities on the right-hand side of (19.104) are known and the integrals can generally be evaluated using the residue theorem.

19.7.1 The elliptical hole

A particularly simple case concerns the elliptical hole for which

$$\omega=c\left(\zeta+\frac{m}{\zeta}\right)\,, \tag{19.105}$$

from (18.62), where c, m are real constants. We then have

$$\overline{\omega'(s)}=c\left(1-\frac{m}{\overline{s}^2}\right)=c\left(1-ms^2\right)$$

and

$$\frac{\omega(s)}{\overline{\omega'(s)}}=\frac{1}{s}\left(\frac{s^2+m}{1-ms^2}\right)\,.$$

This expression has a simple pole at the origin, but when introduced into (19.100) this is cancelled by the factor s^2 in (19.102). The simple poles at $s=\pm\sqrt{1/m}$ lie outside S, since $m<1$. It follows that the integral (19.100) is zero and

$$\gamma\chi_\zeta(\zeta)=-\frac{1}{2\pi\imath}\oint_S\frac{f_\zeta(s)ds}{(s-\zeta)}\,, \tag{19.106}$$

from (19.101). Using (19.105), in (19.104), we obtain

[2] For more details of this procedure, the reader is referred to A.H.England, *loc.cit*, §5.3, I.S.Sokolnikoff, *loc.cit*, §84.

$$\theta_\zeta(\zeta) = -\frac{1}{2\pi\imath}\oint \frac{\overline{f_\zeta(s)}ds}{(s-\zeta)} + \frac{1}{2\pi\imath}\oint \frac{s(1+ms^2)\chi'_\zeta(s)ds}{(s^2-m)(s-\zeta)}$$

and since the integrand in the second term has no poles in the region exterior to the unit circle (except for the Cauchy term $(s-\zeta)$), we can evaluate it using the Cauchy integral theorem (18.48), obtaining

$$\theta_\zeta(\zeta) = -\frac{1}{2\pi\imath}\oint \frac{\overline{f_\zeta(s)}ds}{(s-\zeta)} - \frac{\zeta(1+m\zeta^2)\chi'_\zeta(\zeta)}{(\zeta^2-m)} \ . \tag{19.107}$$

Example

The ellipse corresponding to the mapping function (19.105) has semi-axes

$$a = c(1+m) \ ; \quad b = c(1-m) \ ,$$

from equation (18.65), so with $m=1$, we have $a=2c, b=0$. This defines an ellipse with zero minor axis, which therefore degenerates to a plane crack occupying the line $-a<\xi<a, \eta=0$. Equation (19.105) then takes the special form

$$\omega = \frac{a}{2}\left(\zeta+\frac{1}{\zeta}\right) \ ; \quad \overline{\omega} = \frac{a}{2}\left(\bar{\zeta}+\frac{1}{\bar{\zeta}}\right) \ . \tag{19.108}$$

We shall use this and the results of §19.7.1 to determine the complete stress field in a cracked body opened by a far-field uniaxial tensile stress $\sigma_{\eta\eta}=S$. This problem was solved using a distribution of dislocations in §13.3.2.

As usual, we start by considering the unperturbed solution, which is a state of uniform uniaxial tension $\sigma_{\xi\xi}=\sigma_{\xi\eta}=0, \sigma_{\eta\eta}=S$ and hence

$$\Theta = S \ ; \quad \Phi = -S \ ,$$

from (19.47, 19.46) respectively. It is clear from equations (19.50, 19.51) that a particular solution can be obtained in which the unperturbed potentials χ_0, θ_0 are linear functions of ω. Elementary operations show that the required potentials are

$$\chi_0 = \frac{S\omega}{4} \ ; \quad \theta_0 = \frac{S\omega}{2} \qquad \text{and hence} \qquad \psi_0 = \frac{S\omega}{2} + \frac{S\overline{\omega}}{2} \ ,$$

from (19.52).

For the complete (perturbed) solution, we want the crack S^* to be traction-free and hence $\psi(\omega)\equiv\psi_0(\omega)+\psi_1(\omega)=0$ for $\omega\in S^*$, where ψ_1 is the perturbation due to the crack. Using (19.108) to map the perturbation into the ζ-plane and then setting $\zeta=s, \ \bar{\zeta}=1/s$, corresponding to the unit circle $|\zeta|=1$, we have

$$\psi_\zeta(s) = -\frac{Sa}{2}\left(s+\frac{1}{s}\right) \ .$$

Substituting this result into (19.106) with $\gamma=1, f_\zeta=\psi_\zeta$, we obtain

$$\chi_\zeta(\zeta) = \frac{1}{2\pi\imath}\oint \frac{Sa}{2}\left(s+\frac{1}{s}\right)\frac{ds}{(s-\zeta)} \ . \tag{19.109}$$

Since this is an exterior problem, ζ is outside the unit circle and the only pole in the integrand is that at $s=0$. The residue theorem therefore gives

$$\chi_\zeta(\zeta) = \frac{Sa}{2}\frac{1}{(-\zeta)} = -\frac{Sa}{2\zeta} \ . \tag{19.110}$$

Substituting into (19.107) and noting that

$$\overline{\psi_\zeta(s)} = -\frac{Sa}{2}\left(\frac{1}{s}+s\right) \ ; \quad \chi'_\zeta = \frac{Sa}{2\zeta^2} \ ,$$

we then obtain

$$\theta_\zeta(\zeta) = \frac{1}{2\pi\imath}\oint \frac{Sa}{2}\left(\frac{1}{s}+s\right)\frac{ds}{(s-\zeta)} - \frac{Sa(\zeta^2+1)}{2\zeta(\zeta^2-1)} \ .$$

The contour integral is identical with that in (19.109) and after routine calculations we find

$$\theta_\zeta(\zeta) = -\frac{Sa\zeta}{(\zeta^2-1)} \ . \tag{19.111}$$

To move back to the ω-plane, we use (18.68), which with $m=1, c=a/2$ takes the form

$$\zeta = \frac{\omega+\sqrt{\omega^2-a^2}}{a} \ .$$

Substituting this result into (19.110, 19.111), adding in the unperturbed solution χ_0, θ_0 and using the identity

$$\frac{1}{\omega+\sqrt{\omega^2-a^2}} = \frac{\omega-\sqrt{\omega^2-a^2}}{\omega^2-(\omega^2-a^2)} = \frac{\omega-\sqrt{\omega^2-a^2}}{a^2} \ ,$$

we obtain the complete solution to the crack problem as

$$\chi(\omega) = -\frac{S\omega}{4} + \frac{S\sqrt{\omega^2-a^2}}{2} \ ; \quad \theta(\omega) = \frac{S\omega}{2} - \frac{Sa^2}{2\sqrt{\omega^2-a^2}} \ .$$

Finally, the complex stresses are recovered from (19.47, 19.46) as

$$\Theta = 2\left(\chi' + \overline{\chi'}\right) = -S + \frac{S\omega}{\sqrt{\omega^2-a^2}} + \frac{S\overline{\omega}}{\sqrt{\overline{\omega}^2-a^2}}$$

$$\Phi = -2\left(\omega\overline{\chi''} + \overline{\theta'}\right) = -S + \frac{Sa^2(\omega-\overline{\omega})}{2(\overline{\omega}^2-a^2)^{3/2}} \ .$$

Notice that these expressions define the stress field throughout the cracked body; not just the tractions on the plane $y=0$ as in the solution in §13.3.2.

PROBLEMS

1. A state of uniform antiplane shear $\sigma_{xz}=S, \sigma_{yz}=0$ in a large block of material is perturbed by the presence of a rigid circular inclusion whose boundary is defined by the equation $r=a$ in cylindrical polar coordinates r, θ, z. The inclusion is perfectly bonded to the elastic material and is prevented from moving, so that $u_z=0$ at $r=a$. Use the complex-variable formulation to find the complete stress field in the block and hence determine the appropriate stress concentration factor.

2. Show that the function (18.62) maps the surfaces of the crack $-a<\xi<a, \eta=0$ onto the unit circle if $m=1, c=a/2$. Use this result and the complex-variable formulation to solve the antiplane problem of a uniform stress field $\sigma_{\eta z}=S$ perturbed by a traction-free crack. In particular, find the mode III stress intensity factor K_{III}.

3. Show that the function $\omega(\zeta)=\zeta^\lambda$ maps the wedge-shaped region $0<\vartheta<\alpha$ in polar coördinates (ρ, ϑ) into the half-plane $y>0$ if $\lambda=\alpha/\pi$.

Use this mapping to solve the antiplane problem of the wedge of subtended angle α loaded only by a concentrated out-of-plane force F at the point $\rho=a, \vartheta=0$.

4. Use the stress transformation equations (1.15–1.17) to prove the relation (19.48).

5. Express the plane strain equilibrium equations (2.2, 2.3 with $\sigma_{xz}=\sigma_{yz}=0$) in terms of Θ, Φ, p and $\zeta, \bar{\zeta}$.

6. By analogy with equations (19.46, 19.47), we can define the complex strains e, ε as

$$e=e_{xx}+e_{yy}\ ; \quad \varepsilon=e_{xx}+2\imath e_{xy}-e_{yy}\ ,$$

where we note that $e=\operatorname{div}\boldsymbol{u}$ is the dilatation in plane strain. Express the elastic constitutive law (1.71) as a relation between e, ε and Θ, Φ.

7. Express the compatibility equation (2.8) in terms of the complex strains e, ε defined in Problem 6 and $\zeta, \bar{\zeta}$.

8. Use (19.9) and (1.46) to obtain a complex-variable expression for the in-plane rotation ω_z. Then apply your result to equation (19.57) to verify that the term involving C_1 is consistent with a rigid-body rotation.

9. Use the method of §19.4 to determine the displacements due to the Airy stress function $\phi=r^2\theta$. Express the results in terms of the components u_r, u_θ, using the transformation equation (19.13).

10. Substitute the Laurent expansion (19.94) into the contour integral

$$\oint_S \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} \ ,$$

where ζ is a point outside the circular contour S of radius a. Use the residue theorem to evaluate each term in the resulting series and hence verify that

$$\frac{1}{2\pi\imath}\oint \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} = -\frac{a^2\chi'(\zeta)}{\zeta} \ .$$

11. Use the method of §19.6.1 to find the complex stresses $\varTheta, \varPhi$ for Problem 12.1, where a disk of radius a is compressed by equal and opposite forces F acting at the points $\zeta = \pm a$.

12. A state of uniform uniaxial stress, $\sigma_{xx} = S$ is perturbed by the presence of a traction-free circular hole of radius a. Use the method of §19.6.2 to find the complex stresses $\varTheta, \varPhi$. [Note: the Airy stress function solution of this problem is given in §8.4.1.]

13. A state of uniform shear stress, $\sigma_{xy} = S$ is perturbed by the presence of a bonded rigid circular inclusion of radius a. Use the method of §19.6.2 to find the complex stresses $\varTheta, \varPhi$.

14. A state of uniform uniaxial stress, $\sigma_{yy} = S$ is perturbed by the presence of a traction-free elliptical hole of semi-axes a, b. Find the complex potentials χ, θ as functions of ω and hence determine the stress concentration factor as a function of the ratio b/a. Verify that it tends to the value 3 when $b = a$.

15. Use the method of §19.7.1 to find the complete stress field for Problem 13.4 in which a state of uniform shear stress $\sigma_{xy} = S$ is perturbed by the presence of a crack occupying the region $-a < x < a, y = 0$.

Part V

THREE DIMENSIONAL PROBLEMS

20

DISPLACEMENT FUNCTION SOLUTIONS

In Part II, we chose a representation for stress which satisfied the equilibrium equations identically, in which case the compatibility condition leads to a governing equation for the potential function. In three-dimensional elasticity, it is more usual to use the opposite approach — i.e. to define a potential function representation for displacement (which therefore identically satisfies the compatibility condition) and allow the equilibrium condition to define the governing equation.

A major reason for this change of method is simplicity. Stress function formulations of three-dimensional problems are generally more cumbersome than their displacement function counterparts, because of the greater complexity of the three-dimensional compatibility conditions. It is also worth noting that displacement formulations have a natural advantage in both two and three-dimensional problems when displacement boundary conditions are involved — particularly those associated with multiply connected bodies (see §2.2.1).

20.1 The strain potential

The simplest displacement function representation is that in which the displacement is set equal to the gradient of a scalar function — i.e.

$$2\mu\mathbf{u} = \boldsymbol{\nabla}\phi \ . \tag{20.1}$$

The displacement $\boldsymbol{u}$ must satisfy the equilibrium equation (2.17), which in the absence of body force reduces to

$$\boldsymbol{\nabla}\text{div } \boldsymbol{u} + (1-2\nu)\nabla^2\boldsymbol{u} = 0 \ . \tag{20.2}$$

We therefore obtain

$$\boldsymbol{\nabla}\nabla^2\phi = 0 \ , \tag{20.3}$$

and hence

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_20, © Springer Science+Business Media B.V. 2010

$$\nabla^2\phi = C \;, \tag{20.4}$$

where C is an arbitrary constant. The displacement function ϕ is generally referred to as the strain or displacement potential.

The general solution of equation (20.4) is conveniently considered as the sum of a (harmonic) complementary function and a particular integral, and for the latter, we can take an arbitrary second-order polynomial corresponding to a state of uniform stress. Thus, the representation (20.1) can be decomposed into a uniform state of stress superposed on a solution for which ϕ is harmonic.

It is easy to see that this representation is not sufficiently general to represent all possible displacement fields in an elastic body, since, for example, the rotation

$$\omega_z = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} \equiv 0 \;. \tag{20.5}$$

In other words, the strain potential can only be used to describe irrotational deformation fields[1]. It is associated with the name of Lamé (whose name also attaches to the Lamé constants λ, μ), who used it to solve the problems of an elastic cylinder and sphere under axisymmetric loading. In the following work it will chiefly be useful as one term in a more general representation.

20.2 The Galerkin vector

The representation of the previous section is insufficiently general to describe all possible states of deformation of an elastic body and in seeking a more general form it is natural to examine representations in which the displacement is built up from second rather than first derivatives of a potential function. Using the index notation of §1.1.2, the second derivative term

$$\frac{\partial^2 F}{\partial x_i \partial x_j} \;,$$

where F is a scalar, will represent a Cartesian tensor if the indices i, j are distinct. Alternatively if we make i and j the same, we obtain

$$\frac{\partial^2 F}{\partial x_i \partial x_i} \equiv \frac{\partial^2 F}{\partial x_1 \partial x_1} + \frac{\partial^2 F}{\partial x_2 \partial x_2} + \frac{\partial^2 F}{\partial x_3 \partial x_3} = \nabla^2 F \;, \tag{20.6}$$

which defines the Laplace operator and returns a scalar. What we want to obtain is a vector, which has one free index, and this can only be done if F also has an index (and hence represents a vector). Then we have a choice of pairing the index on F with one of the derivatives or leaving it free and pairing the two derivatives. The most general form is a linear combination of the two — i.e.

[1] We encountered the same lack of generality with the body force potential of Chapter 7 (see §7.1.1).

$$2\mu u_i = C\frac{\partial^2 F_i}{\partial x_j \partial x_j} - \frac{\partial^2 F_j}{\partial x_i \partial x_j} \ , \tag{20.7}$$

where C is an arbitrary constant which we shall later assign so as to simplify the representation. The function F_i must be chosen to satisfy the equilibrium equation (20.2), which in index notation takes the form

$$\frac{\partial^2 u_k}{\partial x_i \partial x_k} + (1-2\nu)\frac{\partial^2 u_i}{\partial x_k \partial x_k} = 0 \ , \tag{20.8}$$

from (2.14).

Substituting (20.7) into (20.8), and cancelling non-zero factors, we obtain

$$C\frac{\partial^4 F_k}{\partial x_j \partial x_j \partial x_i \partial x_k} - \frac{\partial^4 F_j}{\partial x_k \partial x_j \partial x_i \partial x_k} + C(1-2\nu)\frac{\partial^4 F_i}{\partial x_j \partial x_j \partial x_k \partial x_k} - (1-2\nu)\frac{\partial^4 F_j}{\partial x_i \partial x_j \partial x_k \partial x_k} = 0 \ . \tag{20.9}$$

Each term in this equation contains one free index i, all the other indices being paired off in an implied sum. In the first, second and fourth term, the index on F is paired with one of the derivatives and the free index appears in one of the derivatives. After expanding the implied summations, these terms are therefore all identical and the representation can be simplified by making them sum to zero. This is achieved by choosing

$$C - 1 - (1-2\nu) = 0 \quad \text{or} \quad C = 2(1-\nu) \ ,$$

after which the representation (20.7) takes the form

$$2\mu u_i = 2(1-\nu)\frac{\partial^2 F_i}{\partial x_j \partial x_j} - \frac{\partial^2 F_j}{\partial x_i \partial x_j} \ , \tag{20.10}$$

or in vector notation

$$2\mu \boldsymbol{u} = 2(1-\nu)\nabla^2 \boldsymbol{F} - \boldsymbol{\nabla} \text{div } \boldsymbol{F} \ . \tag{20.11}$$

With this choice, the equilibrium equation reduces to

$$\frac{\partial^4 F_i}{\partial x_j \partial x_j \partial x_k \partial x_k} = 0 \quad \text{or} \quad \nabla^4 \boldsymbol{F} = 0 \tag{20.12}$$

and the vector function $\boldsymbol{F}$ is biharmonic.

This solution is due to Galerkin and the function $\boldsymbol{F}$ is known as the Galerkin vector. A more detailed derivation is given by Westergaard[2], who gives expressions for the stress components in terms of $\boldsymbol{F}$ and who uses this

[2] H.M.Westergaard, *Theory of Elasticity and Plasticity*, Dover, New York, 1964, §66.

representation to solve a number of classical three-dimensional problems, notably those involving concentrated forces in the infinite or semi-infinite body. We also note that the Galerkin solution was to some extent foreshadowed by Love[3], who introduced a displacement function appropriate for an axisymmetric state of stress in a body of revolution. Love's displacement function is in fact one component of the Galerkin vector $\boldsymbol{F}$.

20.3 The Papkovich-Neuber solution

A closely related solution is that developed independently by Papkovich and Neuber in terms of harmonic functions. In general, solutions in terms of harmonic functions are easier to use than those involving biharmonic functions, because the properties of harmonic functions have been extensively studied in the context of other physical theories such as electrostatics, gravitation, heat conduction and fluid mechanics.

We define the vector function

$$\boldsymbol{\psi} = -\frac{1}{2}\nabla^2\boldsymbol{F} \tag{20.13}$$

and since the function $\boldsymbol{F}$ is biharmonic, $\boldsymbol{\psi}$ must be harmonic.

We also note that

$$\begin{aligned}\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\psi}) &= \nabla^2(x\psi_x + y\psi_y + z\psi_z)\\ &= 2\frac{\partial\psi_x}{\partial x} + 2\frac{\partial\psi_y}{\partial y} + 2\frac{\partial\psi_z}{\partial z}\\ &= 2\ \text{div}\ \boldsymbol{\psi}\ .\end{aligned} \tag{20.14}$$

Substituting for $\boldsymbol{\psi}$ from equation (20.13) into the right-hand side of (20.14), we obtain

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\psi}) = -\text{div}\ \nabla^2\boldsymbol{F} = -\nabla^2\text{div}\ \boldsymbol{F} \tag{20.15}$$

and hence

$$-\text{div}\ \boldsymbol{F} = \boldsymbol{r}\cdot\boldsymbol{\psi} + \phi\ , \tag{20.16}$$

where ϕ is an arbitrary harmonic function.

Finally, we substitute (20.13, 20.16) into the Galerkin vector representation (20.11) obtaining

$$2\mu\boldsymbol{u} = -4(1-\nu)\boldsymbol{\psi} + \boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{\psi} + \phi)\ , \tag{20.17}$$

where $\boldsymbol{\psi}$ and ϕ are both harmonic functions — i.e.

[3] A.E.H.Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th.edn., Dover, New York, 1944, §188.

$$\nabla^2 \boldsymbol{\psi} = 0; \quad \nabla^2 \phi = 0 \; , \tag{20.18}$$

but notice that $\boldsymbol{\psi}$ is a vector function, whilst ϕ is a scalar. This is known as the *Papkovich-Neuber solution* and it is widely used in modern treatments of three-dimensional elasticity problems. We note that the scalar function ϕ is the harmonic strain potential introduced in §20.1 above.

20.3.1 Change of coördinate system

One disadvantage of the Papkovich-Neuber solution is that the term $\boldsymbol{r} \cdot \boldsymbol{\psi}$ introduces a dependence on the origin of coördinates, so that the potentials corresponding to a given displacement field will generally change if the origin is changed. Suppose that a given displacement field is characterized by the potentials $\boldsymbol{\psi}_1, \phi_1$, so that

$$2\mu \boldsymbol{u} = -4(1-\nu)\boldsymbol{\psi}_1 + \boldsymbol{\nabla}(\boldsymbol{r}_1 \cdot \boldsymbol{\psi}_1 + \phi_1) \; , \tag{20.19}$$

where $\boldsymbol{r}_1$ is a position vector defining the distance from an origin O_1. We wish to determine the appropriate potentials in a new coördinate system with origin O_2. If the coordinates of O_1 relative to O_2 are defined by a vector $\boldsymbol{s}$, the position vector $\boldsymbol{r}_2$ in the new coördinate system is

$$\boldsymbol{r}_2 = \boldsymbol{s} + \boldsymbol{r}_1 \; . \tag{20.20}$$

Solving (20.20) for $\boldsymbol{r}_1$ and substituting in (20.19), we have

$$2\mu \boldsymbol{u} = -4(1-\nu)\boldsymbol{\psi}_1 + \boldsymbol{\nabla}[(\boldsymbol{r}_2 - \boldsymbol{s}) \cdot \boldsymbol{\psi}_1 + \phi_1] \; ,$$

or

$$2\mu \boldsymbol{u} = -4(1-\nu)\boldsymbol{\psi}_2 + \boldsymbol{\nabla}(\boldsymbol{r}_2 \cdot \boldsymbol{\psi}_2 + \phi_2) \; ,$$

where

$$\boldsymbol{\psi}_2 = \boldsymbol{\psi}_1 \; ; \quad \phi_2 = \phi_1 - \boldsymbol{s} \cdot \boldsymbol{\psi}_1 \; . \tag{20.21}$$

Thus, the vector potential ψ is unaffected by the change of origin, but there is a change in the scalar potential ϕ. By contrast, the Galerkin vector $\boldsymbol{F}$ is independent of the choice of origin.

20.4 Completeness and uniqueness

It is a fairly easy matter to prove that the representations (20.10, 20.17) are complete — i.e. that they are capable of describing all possible elastic displacement fields in a three dimensional body[4]. The related problem of uniqueness

[4] See for example, H.M.Westergaard, *loc. cit.* §69 and R.D.Mindlin, Note on the Galerkin and Papcovich stress functions, *Bulletin of the American Mathematical Society,* Vol. 42 (1936), pp.373-376.

presents greater difficulties. Both representations are redundant, in the sense that there are infinitely many combinations of functions which correspond to a given displacement field. In a non-rigorous sense, this is clear from the fact that a typical elasticity problem will be reduced to a boundary-value problem with three conditions at each point of the boundary (three tractions or three displacements or some combination of the two). We should normally expect such conditions to be sufficient to determine three harmonic potential functions in the interior, but (20.17) essentially contains four independent functions (three components of $\boldsymbol{\psi}$ and ϕ).

Neuber gave a ‘proof’ that one of these four functions can always be set equal to zero, but it has since been found that his argument only applies under certain restrictions on the shape of the body. A related ‘proof’ for the Galerkin representation is given by Westergaard in his §70. We shall give a summary of one such argument and show why it breaks down if the shape of the body does not meet certain conditions.

We suppose that a solution to a certain problem is known and that it involves all the three components ψ_x, ψ_y, ψ_z of $\boldsymbol{\psi}$ and ϕ. We wish to find another representation *of the same field* in which one of the components is everywhere zero. We could achieve this if we could construct a ‘null’ field — i.e. one which corresponds to zero displacement and stress at all points — for which the appropriate component is the same as in the original solution. The difference between the two solutions would then give the same displacements as the original solution (since we have subtracted a null field only) and the appropriate component will have been reduced to zero.

Equation (20.17) shows that $\boldsymbol{\psi}$, ϕ will define a null field if

$$\boldsymbol{\nabla}(\boldsymbol{r} \cdot \boldsymbol{\psi} + \phi) - 4(1-\nu)\boldsymbol{\psi} = 0 \; . \tag{20.22}$$

Taking the curl of this equation, we obtain

$$\text{curl } \boldsymbol{\nabla}(\boldsymbol{r} \cdot \boldsymbol{\psi} + \phi) - 4(1-\nu)\text{curl } \boldsymbol{\psi} = 0 \tag{20.23}$$

and hence

$$\text{curl } \boldsymbol{\psi} = 0 \; , \tag{20.24}$$

since curl $\boldsymbol{\nabla}$ of any function is identically zero.

We also note that by applying the operator div to equation (20.22), we obtain

$$\nabla^2(\boldsymbol{r} \cdot \boldsymbol{\psi} + \phi) - 4(1-\nu)\text{div } \boldsymbol{\psi} = 0 \tag{20.25}$$

and hence, substituting from equation (20.14) for the first term,

$$2\text{div } \boldsymbol{\psi} - 4(1-\nu)\text{div } \boldsymbol{\psi} = -2(1-2\nu)\text{div } \boldsymbol{\psi} = 0 \; . \tag{20.26}$$

Now equation (20.24) is precisely the condition which must be satisfied for the function $\boldsymbol{\psi}$ to be expressible as the gradient of a scalar function[5]. Hence, we conclude that there exists a function H such that

[5] See §7.1.1.

$$\boldsymbol{\psi} = \boldsymbol{\nabla} H \; . \tag{20.27}$$

Furthermore, substituting (20.27) into (20.26), we see that H must be a harmonic function.

In order to dispense with one of the components (say ψ_z) of $\boldsymbol{\psi}$ in the given solution, we need to be able to construct a harmonic function H to satisfy the equation

$$\frac{\partial H}{\partial z} = \psi_z \; . \tag{20.28}$$

If we can do this, it is easily verified that the appropriate null field can be found, since on substituting into equation (20.22) we get

$$\boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{\nabla} H + \phi - 4(1-\nu)H) = 0 \; , \tag{20.29}$$

which can be satisfied by choosing ϕ so that

$$\phi = 4(1-\nu)H - \boldsymbol{r}\cdot\boldsymbol{\nabla} H \; . \tag{20.30}$$

Noting that

$$\begin{aligned} \nabla^2(\boldsymbol{r}\cdot\boldsymbol{\nabla} H) &= \nabla^2\left(x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z}\right) \\ &= 2\nabla^2 H \; , \end{aligned} \tag{20.31}$$

we see that ϕ will then be harmonic, as required, since H is harmonic.

20.4.1 Methods of partial integration

We therefore consider the problem of integrating equation (20.28) subject to the constraint that the integral be a harmonic function. We shall examine two methods of doing this which give some insight into the problem.

The given solution and hence the function ψ_z is only defined in the finite region occupied by the elastic body which we denote by Ω. We also denote the boundary of Ω (not necessarily connected) by Γ.

One candidate for the required function H is obtained from the integral

$$H_1 = \int_{z_\Gamma}^{z} \psi_z dz \; , \tag{20.32}$$

where the integral is performed from a point on Γ to a point in the body along a line in the z-direction. This function is readily constructed and it clearly satisfies equation (20.28), but it will not in general be harmonic. However,

$$\frac{\partial}{\partial z}\nabla^2 H_1 = \nabla^2\frac{\partial H_1}{\partial z} = 0 \tag{20.33}$$

and hence

$$\nabla^2 H_1 = f(x, y) \ . \tag{20.34}$$

Now a harmonic function H satisfying equation (20.28) can be constructed as

$$H = H_1 + H_2 \ , \tag{20.35}$$

where H_2 is any function of x, y only that satisfies the equation

$$\nabla^2 H_2 = -f(x, y) \ . \tag{20.36}$$

A suitable choice for H_2 is the logarithmic potential due to the source distribution $f(x, y)$, which can be explicitly constructed as a convolution integral on the logarithmic (two-dimensional) point source solution. Notice that H_2 is not uniquely determined by (20.36), but we are looking for a particular integral only, not a general solution. Indeed, this freedom of choice of solution could prove useful in extending the range of conditions under which the integration of (20.28) can be performed.

At first sight, the procedure described above seems to give a general method of constructing the required function H, but a difficulty is encountered with the definition of the lower limit z_Γ in the integral (20.32). The path of integration is along a straight line in the z-direction from Γ to the general point (x, y, z). There may exist points for which such a line cuts the surface Γ at more than two points — i.e. for which z_Γ is multivalued. Such a case is illustrated as the point $P(x, y, z)$ in Figure 20.1. The integrand is undefined outside Ω and hence we can only give a meaning to the definition (20.32) if we choose the point A as the lower limit. However, by the same token, we must choose the point C as the lower limit in the integral for the point $Q(x, y, z')$.

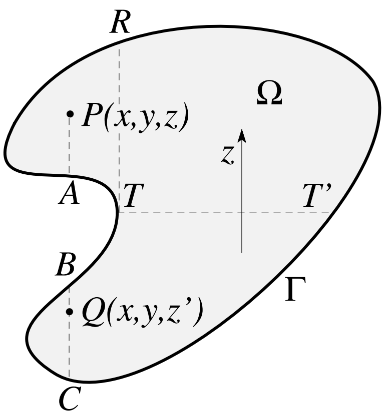

Figure 20.1: Path of integration for equation (20.32).

It follows that the function $f(x, y)$ is not truly a function of x, y only, since it has different values at the points P, Q, which have the same coördinates (x, y). The $f(x, y)$ appropriate to the region above the line TT' will generally have a discontinuity across the line TR, where T is the point of tangency of Γ to lines in the z-direction (see Figure 20.1). Notice, however that the freedom of choice of the integral of equation (20.36) may in certain circumstances permit us to eliminate this discontinuity.

Another way of developing the function H_1 is as follows:- We first find that distribution of sources which when placed on the surface Γ in an infinite space would give the required potential ψ_z in Ω. This distribution is unique and gives a form of continuation of ψ_z into the rest of the infinite space which is everywhere harmonic *except* on the surface Γ. We then write $-\infty$ for the lower limit in equation (20.32).

The function defined by equation (20.32) will now be harmonic except when the path of integration passes through Γ, in which case $\nabla^2 H_1$ will contain a term proportional to the strength of the singularity at the appropriate point(s) of intersection. As these points will be the same for any given value of (x, y), the resulting function will satisfy equation (20.33), but we note that again there will be different values for the function $f(x, y)$ if the appropriate path of integration cuts Γ in more than two points. Furthermore, this method of construction shows that the difference between the two values is proportional to the strength of the singularities on the extra two intersections with Γ (i.e. at points B, C in Figure 20.1). One of the consequences is that $f(x, y)$ in the upper region will then show a square root singular discontinuity to the left of the line TR, assuming that Γ has no sharp corners. This question is still a matter of active interest in elasticity[6].

20.5 Body forces

So far we have concentrated on the development of solutions to the equation of equilibrium (20.2) without body forces. As in Chapter 7, problems involving body forces are conveniently treated by first seeking a particular solution satisfying the equilibrium equation (2.17)

$$\boldsymbol{\nabla}\text{div } \boldsymbol{u} + (1 - 2\nu)\nabla^2 \boldsymbol{u} + \frac{(1 - 2\nu)\boldsymbol{p}}{\mu} = 0 \tag{20.37}$$

and then superposing appropriate homogeneous solutions — i.e. solutions without body force — to satisfy the boundary conditions.

[6] For further discussion of the question, see R.A.Eubanks and E.Sternberg, On the completeness of the Boussinesq-Papcovich stress functions, *Journal of Rational Mechanics and Analysis,* Vol. 5 (1956), pp.735–746., P.M.Naghdi and C.S.Hsu, On a representation of displacements in linear elasticity in terms of three stress functions, *Journal of Mathematics and Mechanics,* Vol. 10 (1961), pp.233–246, T.Tran Cong and G.P.Steven, On the respresentation of elastic displacement fields in terms of three harmonic functions, *Journal of Elasticity,* Vol. 9 (1979), pp.325–333.

20.5.1 Conservative body force fields

If the body force $\boldsymbol{p}$ is conservative, we can write

$$\boldsymbol{p} = -\boldsymbol{\nabla} V \tag{20.38}$$

and a particular solution can always be found using the strain potential (20.1). Substituting (20.1, 20.38) into (20.37) we obtain

$$(1-\nu)\boldsymbol{\nabla}\nabla^2\phi - (1-2\nu)\boldsymbol{\nabla} V = 0 \ , \tag{20.39}$$

which is satisfied if

$$\nabla^2\phi = \frac{(1-2\nu)V}{(1-\nu)} \ . \tag{20.40}$$

Solutions of this equation can be found for all functions V. The corresponding stress components can then be obtained from the stress-strain and strain-displacement relations. For example, we have

$$\sigma_{xx} = \lambda \text{div}\ \boldsymbol{u} + 2\mu\frac{\partial u_x}{\partial x} = \frac{\lambda}{2\mu}\nabla^2\phi + \frac{\partial^2\phi}{\partial x^2} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial x^2} \ , \tag{20.41}$$

using (1.68–1.70, 20.40). For the shear stresses, we have for example

$$\sigma_{xy} = 2\mu e_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \frac{\partial^2\phi}{\partial x\partial y} \ . \tag{20.42}$$

20.5.2 Non-conservative body force fields

If the body force $\boldsymbol{p}$ is non-conservative, a particular solution can always be obtained using the Galerkin vector representation of equation (20.11). Substituting into (2.16), we obtain

$$\frac{\partial^4 F_i}{\partial x_j \partial x_j \partial x_k \partial x_k} = -\frac{p_i}{(1-\nu)} \ , \tag{20.43}$$

or

$$\nabla^4 \boldsymbol{F} = -\frac{\boldsymbol{p}}{(1-\nu)} \ . \tag{20.44}$$

This constitutes three uncoupled equations relating each component of the Galerkin vector to the corresponding component of the body force. Particular solutions can be found for all functions $\boldsymbol{p}$.

PROBLEMS

1. Show that the function

$$\psi = y\frac{\partial \phi}{\partial x} - x\frac{\partial \phi}{\partial y}$$

will be harmonic ($\nabla^2\psi=0$) if ϕ is harmonic. Show also that ψ will be biharmonic if ϕ is biharmonic.

2. Find the most general solution of equation (20.4) that is spherically symmetric — i.e. depends only on the distance $R=\sqrt{x^2+y^2+z^2}$ from the origin. Use your solution to find the stress and displacement field in a hollow spherical container of inner radius b and outer radius a, loaded only by internal pressure p_0 at $R=b$. **Note:** Strain-displacement relations and expressions for the gradient and Laplacian operator in spherical polar coördinates are given in §21.4.2 below.

3. By expressing the vector operators in index notation, show that

$$\boldsymbol{\nabla}\mathrm{div}\ \boldsymbol{\nabla}\mathrm{div}\ \boldsymbol{V} \equiv \nabla^2\boldsymbol{\nabla}\mathrm{div}\ \boldsymbol{V} \equiv \boldsymbol{\nabla}\mathrm{div}\ \nabla^2\boldsymbol{V}\ ,$$

where $\boldsymbol{V}$ is any vector field.

4. Starting from equation (20.10), use (1.51, 1.71, 1.69) to develop general expressions for the stress components σ_{ij} in the Galerkin formulation, in terms of F_i and the elastic constants μ, ν.

5. Verify that the function $\psi=1/\sqrt{x^2+y^2+z^2}$ is harmonic everywhere except at the origin, where it is singular. This function can therefore be used as a Papkovich-Neuber displacement function for a body that does not contain the origin, such as the infinite body with a spherical hole centred on the origin. Show by partial integration that it is impossible to find a function H such that

$$\frac{\partial H}{\partial z} = \psi$$

and $\nabla^2 H=0$, except possibly at the origin.

6. Repeat the derivation of §20.3 using equation (20.44) in place of (20.12) in order to generalize the Papkovich-Neuber solution to include a body force field $\boldsymbol{p}$. In particular, find the governing equations for the functions $\boldsymbol{\psi}, \phi$.

7. Find a particular solution for the Galerkin vector $\boldsymbol{F}$ for the non-conservative body force field

$$p_x=-\rho\dot{\Omega}y\ \ ;\ \ p_y=\rho\dot{\Omega}x\ \ ;\ \ p_z=0\ .$$

8. Prove the result (used in deriving (20.24)) that

$$\text{curl}\ \boldsymbol{\nabla}\phi = 0\ ,$$

for any scalar function ϕ.

9. (i) Verify that the Galerkin vector

$$\boldsymbol{F} = Cx^3 y\boldsymbol{k}$$

satisfies the governing equation (20.12) and determine the corresponding displacement vector $\boldsymbol{u}$.

(ii) Show that

$$\boldsymbol{F}_0 = \Big((1-2\nu)\nabla^2 + \boldsymbol{\nabla}\text{div}\ \Big)\,\boldsymbol{f}$$

defines a null Galerkin vector (i.e. the corresponding displacements are everywhere zero) if $\nabla^4\boldsymbol{f}=0$.

(iii) Using this result or otherwise, find an alternative Galerkin vector for the displacement field of part (i) that is confined to the xy-plane — i.e. such that $F_z=0$.

21

THE BOUSSINESQ POTENTIALS

The Galerkin and Papkovich-Neuber solutions have the advantage of presenting a general solution to the problem of elasticity in a suitably compact notation, but they are not always the most convenient starting point for the solution of particular three-dimensional problems. If the problem has a plane of symmetry or particularly simple boundary conditions, it is often possible to develop a special solution of sufficient generality in one or two harmonic functions, which may or may not be components or linear combinations of components of the Papkovich-Neuber solution. For this reason, it is convenient to record detailed expressions for the displacement and stress components arising from the several terms separately and from certain related displacement potentials.

The first catalogue of solutions of this kind was compiled by Boussinesq[1] and is reproduced by Green and Zerna[2], whose terminology we use here. Boussinesq identified three categories of harmonic potential, one being the strain potential of §20.1, already introduced by Lamé and another comprising a set of three scalar functions equivalent to the three components of the Papkovich-Neuber vector, $\boldsymbol{\psi}$. The third category comprises three solutions particularly suited to torsional deformations about the three axes respectively. In view of the completeness of the Papkovich-Neuber solution, it is clear that these latter functions must be derivable from equation (20.17) and we shall show how this can be done in §21.3 below[3].

[1] J.Boussinesq, *Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*, Gauthier-Villars, Paris, 1885.

[2] A.E.Green and W.Zerna, *Theoretical Elasticity*, 2nd.edn., Clarendon Press, Oxford, 1968, pp.165-176.

[3] Other relations between the Boussinesq potentials are demonstrated by J.P.Bentham, Note on the Boussinesq-Papkovich stress functions, *J.Elasticity,* Vol. 9 (1979), pp.201–206.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_21,

21.1 Solution A : The strain potential

Green and Zerna's Solution A is identical with the strain potential defined in §20.1 above (with the restriction that $\nabla^2\phi=0$) and with the scalar potential ϕ of the Papkovich-Neuber solution (equation (20.17)).

We define

$$2\mu u_x = \frac{\partial \phi}{\partial x} \;\; ; \;\; 2\mu u_y = \frac{\partial \phi}{\partial y} \;\; ; \;\; 2\mu u_z = \frac{\partial \phi}{\partial z} \; , \tag{21.1}$$

and the dilatation

$$2\mu e = 2\mu \text{div}\, \boldsymbol{u} = \nabla^2 \phi = 0 \; . \tag{21.2}$$

Substituting in the stress-strain relations, we have

$$\sigma_{xx} = \lambda e + 2\mu e_{xx} = \frac{\partial^2 \phi}{\partial x^2} \tag{21.3}$$

and similarly

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} \;\; ; \;\; \sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} \; . \tag{21.4}$$

We also have

$$\sigma_{xy} = 2\mu e_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} \tag{21.5}$$

$$\sigma_{yz} = \frac{\partial^2 \phi}{\partial y \partial z} \;\; ; \;\; \sigma_{zx} = \frac{\partial^2 \phi}{\partial z \partial x} \; . \tag{21.6}$$

21.2 Solution B

Green and Zerna's Solution B is obtained by setting to zero all the functions in the Papkovich-Neuber solution except the z-component of $\boldsymbol{\psi}$, which we shall denote by ω. In other words, in equation (20.17), we take

$$\boldsymbol{\psi} = \boldsymbol{k}\omega \;\; ; \quad \phi = 0 \; . \tag{21.7}$$

It follows that

$$2\mu u_x = z\frac{\partial \omega}{\partial x} \;\; ; \;\; 2\mu u_y = z\frac{\partial \omega}{\partial y} \tag{21.8}$$

$$2\mu u_z = z\frac{\partial \omega}{\partial z} - (3-4\nu)\omega \tag{21.9}$$

$$\begin{aligned} 2\mu e &= z\nabla^2\omega + \frac{\partial \omega}{\partial z} - (3-4\nu)\frac{\partial \omega}{\partial z} \\ &= -2(1-2\nu)\frac{\partial \omega}{\partial z} \; , \end{aligned} \tag{21.10}$$

since $\nabla^2\omega=0$.

Substituting these expressions into the stress-strain relations and noting that

$$\lambda = \frac{2\mu\nu}{(1-2\nu)} \ , \tag{21.11}$$

we find

$$\sigma_{xx} = \lambda e + 2\mu e_{xx} = z\frac{\partial^2\omega}{\partial x^2} - 2\nu\frac{\partial\omega}{\partial z} \tag{21.12}$$

$$\sigma_{yy} = z\frac{\partial^2\omega}{\partial y^2} - 2\nu\frac{\partial\omega}{\partial z} \tag{21.13}$$

$$\sigma_{zz} = z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} \tag{21.14}$$

$$\sigma_{xy} = z\frac{\partial^2\omega}{\partial x\partial y} \tag{21.15}$$

$$\begin{aligned}\sigma_{yz} &= \frac{1}{2}\left(\frac{\partial\omega}{\partial y} + z\frac{\partial^2\omega}{\partial y\partial z} + z\frac{\partial^2\omega}{\partial y\partial z} - (3-4\nu)\frac{\partial\omega}{\partial y}\right)\\ &= z\frac{\partial^2\omega}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y}\end{aligned} \tag{21.16}$$

$$\sigma_{zx} = z\frac{\partial^2\omega}{\partial z\partial x} - (1-2\nu)\frac{\partial\omega}{\partial x} \ . \tag{21.17}$$

This solution — if combined with solution A — gives a general solution for the torsionless axisymmetric deformation of a body of revolution. The earliest solution of this problem is due to Love[4] and leads to a single biharmonic function which is actually the component F_z of the Galerkin vector (§20.2). Solution B is also particularly suitable for problems in which the plane surface $z=0$ is a boundary, since the expressions for stresses and displacements take simple forms on this surface.

It is clearly possible to write down similar solutions corresponding to Papkovich-Neuber vectors in the x- and y-directions by permuting suffices. Green and Zerna refer to these as Solutions C and D.

21.3 Solution E : Rotational deformation

Green and Zerna's Solution E can be derived (albeit somewhat indirectly) from the Papkovich-Neuber solution by taking the vector function $\boldsymbol{\psi}$ to be the curl of a vector field oriented in the z-direction — i.e.

$$\boldsymbol{\psi} = \text{curl}\ \boldsymbol{k}\chi \ . \tag{21.18}$$

It follows that

[4] A.E.H.Love, *loc. cit.*

$$\boldsymbol{r}\cdot\boldsymbol{\psi} = x\frac{\partial\chi}{\partial y} - y\frac{\partial\chi}{\partial x} \tag{21.19}$$

and hence

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{\psi}) = 2\frac{\partial^2\chi}{\partial x\partial y} - 2\frac{\partial^2\chi}{\partial y\partial x} = 0\ , \tag{21.20}$$

— i.e. $\boldsymbol{r}\cdot\boldsymbol{\psi}$ is harmonic. We can therefore choose ϕ such that

$$\boldsymbol{r}\cdot\boldsymbol{\psi} + \phi \equiv 0 \tag{21.21}$$

and hence

$$2\mu\boldsymbol{u} = -4(1-\nu)\text{curl}\ \boldsymbol{k}\chi\ . \tag{21.22}$$

To avoid unnecessary multiplying constants, it is convenient to write

$$\Psi = -2(1-\nu)\chi\ , \tag{21.23}$$

in which case

$$2\mu u_x = 2\frac{\partial\Psi}{\partial y}\ \ ;\ \ 2\mu u_y = -2\frac{\partial\Psi}{\partial x}\ \ ;\ \ 2\mu u_z = 0\ , \tag{21.24}$$

giving

$$2\mu e = 0\ . \tag{21.25}$$

Substituting in the stress-strain relations, we find

$$\sigma_{xx} = 2\frac{\partial^2\Psi}{\partial x\partial y}\ \ ;\ \ \sigma_{yy} = -2\frac{\partial^2\Psi}{\partial x\partial y}\ \ ;\ \ \sigma_{zz} = 0 \tag{21.26}$$

$$\sigma_{xy} = \left(\frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}\right)\ \ ;\ \ \sigma_{yz} = -\frac{\partial^2\Psi}{\partial x\partial z}\ \ ;\ \ \sigma_{zx} = \frac{\partial^2\Psi}{\partial y\partial z}\ . \tag{21.27}$$

As with Solution B, two additional solutions of this type can be constructed by permuting suffices.

21.4 Other coördinate systems

The above results have been developed in the Cartesian coördinates x, y, z, but similar expressions are easily obtained in other coördinate systems.

21.4.1 Cylindrical polar coördinates

The cylindrical polar coördinate system r, θ, z is related to the Cartesian system through the equations

$$x = r\cos\theta\ \ \ ;\ \ \ y = r\sin\theta \tag{21.28}$$

as in the two-dimensional transformation (8.1). The gradient operator takes the form

$$\boldsymbol{\nabla} \equiv \boldsymbol{e}_r \frac{\partial}{\partial r} + \frac{\boldsymbol{e}_\theta}{r} \frac{\partial}{\partial \theta} + \boldsymbol{k} \frac{\partial}{\partial z} \ , \tag{21.29}$$

where $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ are unit vectors in the r and θ directions respectively, and the two-dimensional Laplacian operator (8.15) aquires a z-derivative term, becoming

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \ . \tag{21.30}$$

The in-plane strain-displacement relations (8.22) need to be supplemented by

$$e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \quad ; \quad e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \quad ; \quad e_{zz} = \frac{\partial u_z}{\partial z} \ . \tag{21.31}$$

Calculation of the stress and displacement components then proceeds as in §§21.1–21.3. The corresponding expressions for both Cartesian and cylindrical polar coördinates are tabulated in Table 21.1. Also, these expressions are listed in the Maple and Mathematica files 'ABExyz' and 'ABErtz'.

21.4.2 Spherical polar coördinates

The spherical polar coördinate system R, θ, β is related to the cylindrical system by the further change of variables

$$r = R \sin \beta \quad ; \quad z = R \cos \beta \ , \tag{21.32}$$

where

$$R = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2} \tag{21.33}$$

represents the distance from the origin and the two spherical angles θ, β are equivalent to measures of longitude and lattitude respectively, defining position on a sphere of radius R. Notice however that it is conventional to measure the angle of lattitude from the pole (the positive z-axis) rather than from the equator. Thus, the two poles are defined by $\beta = 0, \pi$ and the equator by $\beta = \pi/2$.

The gradient operator takes the form

$$\boldsymbol{\nabla} \equiv \boldsymbol{e}_R \frac{\partial}{\partial R} + \frac{\boldsymbol{e}_\theta}{R \sin \beta} \frac{\partial}{\partial \theta} + \frac{\boldsymbol{e}_\beta}{R} \frac{\partial}{\partial \beta} \ , \tag{21.34}$$

where $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\beta$ are unit vectors in the R, θ and β directions respectively. Transformation of the Laplacian operator, the strain-displacement relations and the expressions for displacement and stress components in Solutions A,B and E is algebraically tedious but mathematically routine and only the principal results are summarized here.

Table 21.1. Green and Zerna's Solutions A, B and E

	Solution A	**Solution B**	**Solution E**
$2\mu u_x$	$\frac{\partial\phi}{\partial x}$	$z\frac{\partial\omega}{\partial x}$	$2\frac{\partial\Psi}{\partial y}$
$2\mu u_y$	$\frac{\partial\phi}{\partial y}$	$z\frac{\partial\omega}{\partial y}$	$-2\frac{\partial\Psi}{\partial x}$
$2\mu u_z$	$\frac{\partial\phi}{\partial z}$	$z\frac{\partial\omega}{\partial z}-(3-4\nu)\omega$	0
σ_{xx}	$\frac{\partial^2\phi}{\partial x^2}$	$z\frac{\partial^2\omega}{\partial x^2}-2\nu\frac{\partial\omega}{\partial z}$	$2\frac{\partial^2\Psi}{\partial x\partial y}$
σ_{xy}	$\frac{\partial^2\phi}{\partial x\partial y}$	$z\frac{\partial^2\omega}{\partial x\partial y}$	$\frac{\partial^2\Psi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x^2}$
σ_{yy}	$\frac{\partial^2\phi}{\partial y^2}$	$z\frac{\partial^2\omega}{\partial y^2}-2\nu\frac{\partial\omega}{\partial z}$	$-2\frac{\partial^2\Psi}{\partial x\partial y}$
σ_{xz}	$\frac{\partial^2\phi}{\partial x\partial z}$	$z\frac{\partial^2\omega}{\partial x\partial z}-(1-2\nu)\frac{\partial\omega}{\partial x}$	$\frac{\partial^2\Psi}{\partial y\partial z}$
σ_{yz}	$\frac{\partial^2\phi}{\partial y\partial z}$	$z\frac{\partial^2\omega}{\partial y\partial z}-(1-2\nu)\frac{\partial\omega}{\partial y}$	$-\frac{\partial^2\Psi}{\partial x\partial z}$
σ_{zz}	$\frac{\partial^2\phi}{\partial z^2}$	$z\frac{\partial^2\omega}{\partial z^2}-2(1-\nu)\frac{\partial\omega}{\partial z}$	0
$2\mu u_r$	$\frac{\partial\phi}{\partial r}$	$z\frac{\partial\omega}{\partial r}$	$\frac{2}{r}\frac{\partial\Psi}{\partial\theta}$
$2\mu u_\theta$	$\frac{1}{r}\frac{\partial\phi}{\partial\theta}$	$\frac{z}{r}\frac{\partial\omega}{\partial\theta}$	$-2\frac{\partial\Psi}{\partial r}$
σ_{rr}	$\frac{\partial^2\phi}{\partial r^2}$	$z\frac{\partial^2\omega}{\partial r^2}-2\nu\frac{\partial\omega}{\partial z}$	$\frac{2}{r}\frac{\partial^2\Psi}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\Psi}{\partial\theta}$
$\sigma_{r\theta}$	$\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}$	$\frac{z}{r}\frac{\partial^2\omega}{\partial r\partial\theta}-\frac{z}{r^2}\frac{\partial\omega}{\partial\theta}$	$\frac{1}{r}\frac{\partial\Psi}{\partial r}-\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}$
$\sigma_{\theta\theta}$	$\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$	$\frac{z}{r}\frac{\partial\omega}{\partial r}+\frac{z}{r^2}\frac{\partial^2\omega}{\partial\theta^2}-2\nu\frac{\partial\omega}{\partial z}$	$-\frac{2}{r}\frac{\partial^2\Psi}{\partial r\partial\theta}+\frac{2}{r^2}\frac{\partial\Psi}{\partial\theta}$
σ_{rz}	$\frac{\partial^2\phi}{\partial r\partial z}$	$z\frac{\partial^2\omega}{\partial r\partial z}-(1-2\nu)\frac{\partial\omega}{\partial r}$	$\frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\partial z}$
$\sigma_{\theta z}$	$\frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial z}$	$\frac{z}{r}\frac{\partial^2\omega}{\partial\theta\partial z}-\frac{1-2\nu}{r}\frac{\partial\omega}{\partial\theta}$	$-\frac{\partial^2\Psi}{\partial r\partial z}$

The Laplacian operator is

$$\nabla^2 \equiv \frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2\sin^2\beta}\frac{\partial^2}{\partial\theta^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\beta^2}+\frac{\cot\beta}{R^2}\frac{\partial}{\partial\beta} \qquad (21.35)$$

and the strain-displacement relations[5] are

$$e_{RR}=\frac{\partial u_R}{\partial R}\ ;\ e_{\theta\theta}=\frac{u_R}{R}+\frac{u_\beta\cot\beta}{R}+\frac{1}{R\sin\beta}\frac{\partial u_\theta}{\partial\theta}$$

[5] See for example A.S.Saada, *Elasticity*, Pergamon Press, New York, 1973, §6.7.

$$e_{\beta\beta} = \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\beta}{\partial \beta} \;\; ; \;\; e_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\beta}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right)$$

$$e_{\theta\beta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial \beta} - \frac{u_\theta \cot\beta}{R} + \frac{1}{R\sin\beta}\frac{\partial u_\beta}{\partial \theta}\right)$$

$$e_{\beta R} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \beta} + \frac{\partial u_\beta}{\partial R} - \frac{u_\beta}{R}\right) . \tag{21.36}$$

The corresponding displacement and stress components in Solutions A,B and E are given in Table 21.2 and are also listed in the Maple and Mathematica files 'ABErtb'.

21.5 Solutions obtained by superposition

The solutions obtained in the above sections can be superposed as required to provide a solution appropriate to the particular problem at hand. A case of particular interest is that in which the plane $z=0$ is one of the boundaries of the body — for example the semi-infinite solid or *half-space* $z>0$. We consider here two special cases.

21.5.1 Solution F : Frictionless isothermal contact problems

If the half-space is indented by a frictionless punch, so that the surface $z=0$ is subjected to normal tractions only, a simple formulation can be obtained by combining solutions A and B and defining a relationship between ϕ and ω in order to satisfy identically the condition $\sigma_{zx}=\sigma_{zy}=0$ on $z=0$.

We write

$$\phi = (1-2\nu)\varphi \;\; ; \;\; \omega = \frac{\partial \varphi}{\partial z} \tag{21.37}$$

in solutions A, B respectively, obtaining

$$\sigma_{zx} = z\frac{\partial^3 \varphi}{\partial x \partial z^2} \;\; ; \;\; \sigma_{zy} = z\frac{\partial^3 \varphi}{\partial y \partial z^2} , \tag{21.38}$$

which vanish as required on the plane $z=0$. The remaining stress and displacement components are listed in Table 21.3 and at this stage we merely note that the normal traction and normal displacement at the surface reduce to the simple expressions

$$\sigma_{zz} = -\frac{\partial^2 \varphi}{\partial z^2} \;\; ; \;\; 2\mu u_z = -2(1-\nu)\frac{\partial \varphi}{\partial z} . \tag{21.39}$$

It follows that indentation problems for the half-space can be reduced to classical boundary-value problems in the harmonic potential function φ. Various examples are considered by Green and Zerna[6], who also show that symmetric problems of the plane crack in an isotropic body can be treated in the same way. These problems are considered in Chapters 29–32 below.

[6] A.E.Green and W.Zerna, *loc. cit.* §§5.8-5.10.

Table 21.2. Solutions A, B and E in spherical polar coördinates

	Solution A	**Solution B**	**Solution E**
$2\mu u_R$	$\dfrac{\partial\phi}{\partial R}$	$R\cos\beta\dfrac{\partial\omega}{\partial R}-(3-4\nu)\omega\cos\beta$	$\dfrac{2}{R}\dfrac{\partial\Psi}{\partial\theta}$
$2\mu u_\theta$	$\dfrac{1}{R\sin\beta}\dfrac{\partial\phi}{\partial\theta}$	$\cot\beta\dfrac{\partial\omega}{\partial\theta}$	$-2\dfrac{\partial\Psi}{\partial R}\sin\beta-\dfrac{2\cos\beta}{R}\dfrac{\partial\Psi}{\partial\beta}$
$2\mu u_\beta$	$\dfrac{1}{R}\dfrac{\partial\phi}{\partial\beta}$	$\cos\beta\dfrac{\partial\omega}{\partial\beta}+(3-4\nu)\omega\sin\beta$	$\dfrac{2\cot\beta}{R}\dfrac{\partial\Psi}{\partial\theta}$
σ_{RR}	$\dfrac{\partial^2\phi}{\partial R^2}$	$R\cos\beta\dfrac{\partial^2\omega}{\partial R^2}-2(1-\nu)\dfrac{\partial\omega}{\partial R}\cos\beta+\dfrac{2\nu}{R}\dfrac{\partial\omega}{\partial\beta}\sin\beta$	$\dfrac{2}{R}\dfrac{\partial^2\Psi}{\partial R\partial\theta}-\dfrac{2}{R^2}\dfrac{\partial\Psi}{\partial\theta}$
$\sigma_{\theta\theta}$	$\dfrac{1}{R}\dfrac{\partial\phi}{\partial R}+\dfrac{\cot\beta}{R^2}\dfrac{\partial\phi}{\partial\beta}+\dfrac{1}{R^2\sin^2\beta}\dfrac{\partial^2\phi}{\partial\theta^2}$	$(1-2\nu)\dfrac{\partial\omega}{\partial R}\cos\beta+\dfrac{2\nu}{R}\dfrac{\partial\omega}{\partial\beta}\sin\beta+\dfrac{\cos^2\beta}{R\sin\beta}\dfrac{\partial\omega}{\partial\beta}+\dfrac{\cot\beta}{R\sin\beta}\dfrac{\partial^2\omega}{\partial\theta^2}$	$\dfrac{2}{R^2\sin^2\beta}\dfrac{\partial\Psi}{\partial\theta}-\dfrac{2}{R}\dfrac{\partial^2\Psi}{\partial R\partial\theta}-\dfrac{2\cot\beta}{R^2}\dfrac{\partial^2\Psi}{\partial\beta\partial\theta}$
$\sigma_{\beta\beta}$	$\dfrac{1}{R}\dfrac{\partial\phi}{\partial R}+\dfrac{1}{R^2}\dfrac{\partial^2\phi}{\partial\beta^2}$	$\dfrac{\cos\beta}{R}\dfrac{\partial^2\omega}{\partial\beta^2}+(1-2\nu)\dfrac{\partial\omega}{\partial R}\cos\beta+\dfrac{2(1-\nu)\sin\beta}{R}\dfrac{\partial\omega}{\partial\beta}$	$\dfrac{2\cot\beta}{R^2}\dfrac{\partial^2\Psi}{\partial\theta\partial\beta}-\dfrac{2\cot^2\beta}{R^2}\dfrac{\partial\Psi}{\partial\theta}$
$\sigma_{\theta\beta}$	$\dfrac{1}{R^2\sin\beta}\dfrac{\partial^2\phi}{\partial\theta\partial\beta}-\dfrac{\cot\beta}{R^2\sin\beta}\dfrac{\partial\phi}{\partial\theta}$	$\dfrac{\cot\beta}{R}\dfrac{\partial^2\omega}{\partial\theta\partial\beta}+\dfrac{2(1-\nu)}{R}\dfrac{\partial\omega}{\partial\theta}-\dfrac{1}{R\sin^2\beta}\dfrac{\partial\omega}{\partial\theta}$	$\dfrac{\cot\beta}{R^2\sin\beta}\dfrac{\partial^2\Psi}{\partial\theta^2}-\dfrac{\sin\beta}{R}\dfrac{\partial^2\Psi}{\partial R\partial\beta}-\dfrac{\cos\beta}{R^2}\dfrac{\partial^2\Psi}{\partial\beta^2}+\dfrac{1}{R^2\sin\beta}\dfrac{\partial\Psi}{\partial\beta}$
$\sigma_{\beta R}$	$\dfrac{1}{R}\dfrac{\partial^2\phi}{\partial\beta\partial R}-\dfrac{1}{R^2}\dfrac{\partial\phi}{\partial\beta}$	$(1-2\nu)\dfrac{\partial\omega}{\partial R}\sin\beta+\cos\beta\dfrac{\partial^2\omega}{\partial\beta\partial R}-\dfrac{2(1-\nu)\cos\beta}{R}\dfrac{\partial\omega}{\partial\beta}$	$\dfrac{1}{R^2}\dfrac{\partial^2\Psi}{\partial\theta\partial\beta}+\dfrac{\cot\beta}{R}\dfrac{\partial^2\Psi}{\partial\theta\partial R}-\dfrac{2\cot\beta}{R^2}\dfrac{\partial\Psi}{\partial\theta}$
$\sigma_{R\theta}$	$\dfrac{1}{R\sin\beta}\dfrac{\partial^2\phi}{\partial R\partial\theta}-\dfrac{1}{R^2\sin\beta}\dfrac{\partial\phi}{\partial\theta}$	$\cot\beta\dfrac{\partial^2\omega}{\partial R\partial\theta}-\dfrac{2(1-\nu)\cot\beta}{R}\dfrac{\partial\omega}{\partial\theta}$	$\dfrac{1}{R^2\sin\beta}\dfrac{\partial^2\Psi}{\partial\theta^2}-\sin\beta\dfrac{\partial^2\Psi}{\partial R^2}-\dfrac{\cos\beta}{R}\dfrac{\partial^2\Psi}{\partial R\partial\beta}+\dfrac{2\cos\beta}{R^2}\dfrac{\partial\Psi}{\partial\beta}+\dfrac{\sin\beta}{R}\dfrac{\partial\Psi}{\partial R}$

21.5.2 Solution G: The surface free of normal traction

The counterpart of the preceding solution is that obtained by combining Solutions A,B and requiring that the *normal* tractions vanish on the surface $z=0$. We shall find this solution useful in problems involving a plane interface between two dissimilar materials (Chapter 32) and also in certain axisymmetric crack and contact problems.

The normal traction on the plane $z=0$ is

$$\sigma_{zz}=\frac{\partial^2\phi}{\partial z^2}-2(1-\nu)\frac{\partial\omega}{\partial z}\ , \tag{21.40}$$

Table 21.3. Solutions F and G

	Solution F Frictionless isothermal	**Solution G**
$2\mu u_x$	$z\dfrac{\partial^2\varphi}{\partial x\partial z} + (1-2\nu)\dfrac{\partial\varphi}{\partial x}$	$2(1-\nu)\dfrac{\partial\chi}{\partial x} + z\dfrac{\partial^2\chi}{\partial x\partial z}$
$2\mu u_y$	$z\dfrac{\partial^2\varphi}{\partial y\partial z} + (1-2\nu)\dfrac{\partial\varphi}{\partial y}$	$2(1-\nu)\dfrac{\partial\chi}{\partial y} + z\dfrac{\partial^2\chi}{\partial y\partial z}$
$2\mu u_z$	$z\dfrac{\partial^2\varphi}{\partial z^2} - 2(1-\nu)\dfrac{\partial\varphi}{\partial z}$	$-(1-2\nu)\dfrac{\partial\chi}{\partial z} + z\dfrac{\partial^2\chi}{\partial z^2}$
σ_{xx}	$z\dfrac{\partial^3\varphi}{\partial x^2\partial z} + \dfrac{\partial^2\varphi}{\partial x^2} + 2\nu\dfrac{\partial^2\varphi}{\partial y^2}$	$2(1-\nu)\dfrac{\partial^2\chi}{\partial x^2} + z\dfrac{\partial^3\chi}{\partial x^2\partial z} - 2\nu\dfrac{\partial^2\chi}{\partial z^2}$
σ_{xy}	$z\dfrac{\partial^3\varphi}{\partial x\partial y\partial z} + (1-2\nu)\dfrac{\partial^2\varphi}{\partial x\partial y}$	$2(1-\nu)\dfrac{\partial^2\chi}{\partial x\partial y} + z\dfrac{\partial^3\chi}{\partial x\partial y\partial z}$
σ_{yy}	$z\dfrac{\partial^3\varphi}{\partial y^2\partial z} + \dfrac{\partial^2\varphi}{\partial y^2} + 2\nu\dfrac{\partial^2\varphi}{\partial x^2}$	$2(1-\nu)\dfrac{\partial^2\chi}{\partial y^2} + z\dfrac{\partial^3\chi}{\partial y^2\partial z} - 2\nu\dfrac{\partial^2\chi}{\partial z^2}$
σ_{xz}	$z\dfrac{\partial^3\varphi}{\partial x\partial z^2}$	$\dfrac{\partial^2\chi}{\partial x\partial z} + z\dfrac{\partial^3\chi}{\partial x\partial z^2}$
σ_{yz}	$z\dfrac{\partial^3\varphi}{\partial y\partial z^2}$	$\dfrac{\partial^2\chi}{\partial y\partial z} + z\dfrac{\partial^3\chi}{\partial y\partial z^2}$
σ_{zz}	$z\dfrac{\partial^3\varphi}{\partial z^3} - \dfrac{\partial^2\varphi}{\partial z^2}$	$z\dfrac{\partial^3\chi}{\partial z^3}$
$2\mu u_r$	$z\dfrac{\partial^2\varphi}{\partial r\partial z} + (1-2\nu)\dfrac{\partial\varphi}{\partial r}$	$2(1-\nu)\dfrac{\partial\chi}{\partial r} + z\dfrac{\partial^2\chi}{\partial r\partial z}$
$2\mu u_\theta$	$\dfrac{z}{r}\dfrac{\partial^2\varphi}{\partial\theta\partial z} + \dfrac{(1-2\nu)}{r}\dfrac{\partial\varphi}{\partial\theta}$	$\dfrac{2(1-\nu)}{r}\dfrac{\partial\chi}{\partial\theta} + \dfrac{z}{r}\dfrac{\partial^2\chi}{\partial\theta\partial z}$
σ_{rr}	$z\dfrac{\partial^3\varphi}{\partial r^2\partial z} + \dfrac{\partial^2\varphi}{\partial r^2} - 2\nu\left(\dfrac{\partial^2\varphi}{\partial r^2} + \dfrac{\partial^2\varphi}{\partial z^2}\right)$	$2(1-\nu)\dfrac{\partial^2\chi}{\partial r^2} + z\dfrac{\partial^3\chi}{\partial r^2\partial z} - 2\nu\dfrac{\partial^2\chi}{\partial z^2}$
$\sigma_{r\theta}$	$\dfrac{z}{r}\dfrac{\partial^3\varphi}{\partial r\partial\theta\partial z} - \dfrac{z}{r^2}\dfrac{\partial^2\varphi}{\partial\theta\partial z}$ $+\dfrac{(1-2\nu)}{r}\left(\dfrac{\partial^2\varphi}{\partial r\partial\theta} - \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta}\right)$	$\dfrac{2(1-\nu)}{r}\left(\dfrac{\partial^2\chi}{\partial r\partial\theta} - \dfrac{1}{r}\dfrac{\partial\chi}{\partial\theta}\right)$ $+\dfrac{z}{r}\dfrac{\partial^3\chi}{\partial z\partial r\partial\theta} - \dfrac{z}{r^2}\dfrac{\partial^2\chi}{\partial z\partial\theta}$
$\sigma_{\theta\theta}$	$-(1-2\nu)\dfrac{\partial^2\varphi}{\partial r^2} - \dfrac{\partial^2\varphi}{\partial z^2}$ $-z\dfrac{\partial^3\varphi}{\partial r^2\partial z} - z\dfrac{\partial^3\varphi}{\partial z^3}$	$-2(1-\nu)\dfrac{\partial^2\chi}{\partial r^2} - 2\dfrac{\partial^2\chi}{\partial z^2}$ $+\dfrac{z}{r}\dfrac{\partial^2\chi}{\partial z\partial r} + \dfrac{z}{r^2}\dfrac{\partial^3\chi}{\partial z\partial\theta^2}$
σ_{rz}	$z\dfrac{\partial^3\varphi}{\partial r\partial z^2}$	$\dfrac{\partial^2\chi}{\partial r\partial z} + z\dfrac{\partial^3\chi}{\partial r\partial z^2}$
$\sigma_{\theta z}$	$\dfrac{z}{r}\dfrac{\partial^3\varphi}{\partial\theta\partial z^2}$	$\dfrac{1}{r}\dfrac{\partial^2\chi}{\partial\theta\partial z} + \dfrac{z}{r}\dfrac{\partial^3\chi}{\partial\theta\partial z^2}$

from Table 21.1 and hence it can be made to vanish by writing

$$\phi = 2(1-\nu)\chi \; ; \; \omega = \frac{\partial \chi}{\partial z} \; , \tag{21.41}$$

where χ is a new harmonic potential function.

The resulting stress and displacement components are given as Solution G in Table 21.3.

21.6 A three-dimensional complex variable solution

We saw in Part IV that the complex variable notation provides an elegant and powerful way to represent two-dimensional stress fields and it is natural to ask whether any similar techniques could be extended to three-dimensional problems. Green[7] used a solution of this kind in the development of higher order solutions for elastic plates and we shall see in Chapter 28 that similar techniques can be used for the prismatic bar with general loading on the lateral surfaces.

We shall combine the displacement and stress components in the same way as in Chapter 19 by defining

$$u = u_x + \imath u_y \; ; \quad \Theta = \sigma_{xx} + \sigma_{yy} \; ; \quad \Phi = \sigma_{xx} + 2\imath\sigma_{xy} - \sigma_{yy} \; ; \quad \Psi = \sigma_{zx} + \imath\sigma_{zy} \; . \tag{21.42}$$

Following the convention of §19.1 and §21.2, we write the components of the Papkovich-Neuber vector potential as

$$\psi = \psi_x + \imath\psi_y \; ; \quad \omega = \psi_z \tag{21.43}$$

and hence

$$\boldsymbol{r} \cdot \boldsymbol{\psi} = \zeta\overline{\psi} + \bar{\zeta}\psi + z\omega \; , \tag{21.44}$$

from (19.6). Using these results and (19.3) in (20.17), we obtain the in-plane displacements as

$$2\mu u = 2\frac{\partial \phi}{\partial \bar{\zeta}} - (3-4\nu)\psi + \zeta\frac{\partial \overline{\psi}}{\partial \bar{\zeta}} + \bar{\zeta}\frac{\partial \psi}{\partial \bar{\zeta}} + 2z\frac{\partial \omega}{\partial \bar{\zeta}} \; . \tag{21.45}$$

and the antiplane displacement as

$$2\mu u_z = \frac{\partial \phi}{\partial z} + \frac{1}{2}\left(\bar{\zeta}\frac{\partial \psi}{\partial z} + \zeta\frac{\partial \overline{\psi}}{\partial z}\right) + z\frac{\partial \omega}{\partial z} - (3-4\nu)\omega \; . \tag{21.46}$$

Once the displacements are defined, the stress components can be found from the constitutive law as in §§21.1,21.2. Combining these as in (21.42), we obtain

[7] A.E.Green, The elastic equilibrium of isotropic plates and cylinders, *Proceedings of the Royal Society of London*, Vol. A195 (1949), pp.533–552.

$$\Theta = -\frac{\partial^2 \phi}{\partial z^2} - 2\left(\frac{\partial \psi}{\partial \zeta} + \frac{\partial \overline{\psi}}{\partial \bar{\zeta}}\right) - \frac{1}{2}\left(\zeta \frac{\partial^2 \overline{\psi}}{\partial z^2} + \bar{\zeta}\frac{\partial^2 \psi}{\partial z^2}\right)$$

$$-z\frac{\partial^2 \omega}{\partial z^2} - 4\nu\frac{\partial \omega}{\partial z} \qquad (21.47)$$

$$\Phi = 4\frac{\partial^2 \phi}{\partial \bar{\zeta}^2} - 4(1-2\nu)\frac{\partial \psi}{\partial \bar{\zeta}} + 2\zeta\frac{\partial^2 \overline{\psi}}{\partial \bar{\zeta}^2} + 2\bar{\zeta}\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} + 4z\frac{\partial^2 \omega}{\partial \bar{\zeta}^2} \qquad (21.48)$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} - 2\nu\left(\frac{\partial \psi}{\partial \zeta} + \frac{\partial \overline{\psi}}{\partial \bar{\zeta}}\right) + \frac{1}{2}\left(\bar{\zeta}\frac{\partial^2 \psi}{\partial z^2} + \zeta\frac{\partial^2 \overline{\psi}}{\partial z^2}\right)$$

$$+z\frac{\partial^2 \omega}{\partial z^2} - 2(1-\nu)\frac{\partial \omega}{\partial z} \qquad (21.49)$$

$$\Psi = 2\frac{\partial^2 \phi}{\partial \bar{\zeta}\partial z} - (1-2\nu)\frac{\partial \psi}{\partial z} + \zeta\frac{\partial^2 \overline{\psi}}{\partial \bar{\zeta}\partial z} + \bar{\zeta}\frac{\partial^2 \psi}{\partial \bar{\zeta}\partial z}$$

$$+2z\frac{\partial^2 \omega}{\partial z \partial \bar{\zeta}} - 2(1-2\nu)\frac{\partial \omega}{\partial \bar{\zeta}}\ , \qquad (21.50)$$

where $\phi(\zeta, \bar{\zeta}, z), \omega(\zeta, \bar{\zeta}, z)$ are real three-dimensional harmonic functions and $\psi(\zeta, \bar{\zeta}, z)$ is a complex three-dimensional harmonic function. Notice that we can write the three-dimensional Laplace equation in the form

$$\frac{\partial^2 \phi}{\partial z^2} + 4\frac{\partial^2 \phi}{\partial \zeta \partial \bar{\zeta}} = 0\ ; \quad \frac{\partial^2 \omega}{\partial z^2} + 4\frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} = 0\ ; \quad \frac{\partial^2 \psi}{\partial z^2} + 4\frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} = 0\ , \qquad (21.51)$$

using (18.9). Thus, if the functions ϕ, ψ depend on z, they will not generally be holomorphic with regard to ζ or $\bar{\zeta}$. We shall explore categories of complex-variable solution of equation (21.51) in §24.8. below.

PROBLEMS

1. Use solution E to solve the elementary torsion problem for a cylindrical bar of radius b transmitting a torque T, the surface $r = b$ being traction-free. You will need to find a suitable harmonic potential function — it will be an axisymmetric polynomial.

2. Find a way of taking a combination of solutions A,B,E so as to satisfy identically the global conditions

$$\sigma_{zy} = \sigma_{zz} = 0 \ \ ; \ \ \text{all } x, y, \ \ z = 0$$

Thus, σ_{zx} should be the only non-zero traction component on the surface.

3. An important class of frictional contact problems involves the steady sliding of an indenter in the x-direction across the surface of the half-space $z > 0$. Assuming Coulomb's friction law to apply with coefficient of friction f, the appropriate boundary conditions are

$$\sigma_{xz} = -f\sigma_{zz} \;\; ; \;\; \sigma_{zy} = 0 \; , \tag{21.52}$$

in the contact area, the rest of the surface $z=0$ being traction-free.

Show that (21.52) are actually *global* conditions for this problem and find a way of combining Solutions A,B,E so as to satisfy them identically.

4. A large number of thin fibres are bonded to the surface $z=0$ of the elastic half space $z>0$, with orientation in the x-direction. The fibres are sufficiently rigid to prevent axial displacement u_x, but sufficiently thin to offer no restraint to lateral motion u_y, u_z.

This composite block is now indented by a frictionless rigid punch. Find a linear combination of Solutions A, B and E that satisfies the global conditions of the problem and in particular find expressions for the surface values of the normal displacement u_z and the normal traction σ_{zz} in terms of the one remaining harmonic potential function.

5. Show that if ϕ, ω are taken to be independent of x, a linear combination of Solutions A and B defines a state of plane strain in the yz-plane (i.e. $u_x=0, \sigma_{xy}=\sigma_{xz}=0$ and all the remaining stress and displacement components are independent of x).

An infinite layer $0<z<h$ rests on a frictionless rigid foundation at $z=0$, whilst the surface $z=h$ is loaded by the normal traction

$$\sigma_{zz} = -p_0 + p_1 \cos(my) \; ,$$

where $p_1<p_0$. Find the distribution of contact pressure at $z=0$.

6. A massive asteroid passes close to the surface of the Earth causing gravitational forces that can be described by a known harmonic body force potential $V(x, y, z)$. Assuming that the affected region of the Earth can be approximated as the traction-free half space $z>0$, find a representation of the resulting stress and displacement fields in terms of a single harmonic potential function. An appropriate strategy is:-

(i) Using the particular solution of §20.5.1, show that the potential ϕ must be biharmonic.
(ii) Show that this condition can be satisfied by writing $\phi=z\chi$, where χ is a *harmonic* function, and find the relation between χ and V.
(iii) Find the tractions on the surface of the half space in terms of derivatives of χ.
(iv) Superpose appropriate combinations of Solutions A and/or B so as to render this surface traction free for all functions χ.

Find the resulting stress field for the special case of a spherical asteroid of radius a, for which

$$V = -\frac{4\pi\gamma\rho\rho_0 a^3}{3\sqrt{x^2+y^2+(z+h)^2}} \; ,$$

where γ is the universal gravitational constant, ρ, ρ_0 are the densities of the asteroid and the Earth respectively, and the centre of the sphere is at the point $(0, 0, -h)$, a height $h > a$ above the Earth's surface.

If the Earth and the asteroid have the same density ($\rho_0 = \rho$) and the self-weight of the Earth is assumed to cause hydrostatic compression of magnitude $\rho g z$, where g is the acceleration due to gravity, is it possible for the passage of the asteroid to cause tensile stresses anywhere in the Earth's crust. If so, where?

22

THERMOELASTIC DISPLACEMENT POTENTIALS

As in the two-dimensional case (Chapter 14), three-dimensional problems of thermoelasticity are conveniently treated by finding a *particular solution* — i.e. a solution which satisfies the field equations without regard to boundary conditions — and completing the general solution by superposition of an appropriate representation for the general isothermal problem, such as the Papkovich-Neuber solution.

In this section, we shall show that a particular solution can always be obtained in the form of a strain potential — i.e. by writing

$$2\mu\boldsymbol{u} = \boldsymbol{\nabla}\phi \; . \tag{22.1}$$

The thermoelastic stress-strain relations (14.3, 14.4) can be solved to give

$$\sigma_{xx} = \lambda e + 2\mu e_{xx} - (3\lambda + 2\mu)\alpha T \tag{22.2}$$

$$\sigma_{xy} = 2\mu e_{xy} \; , \tag{22.3}$$

etc.

Using the strain-displacement relations and substituting for $\boldsymbol{u}$ from (22.1), we then find

$$2\mu e = \nabla^2\phi \tag{22.4}$$

$$\sigma_{xx} = \frac{\lambda}{2\mu}\nabla^2\phi + \frac{\partial^2\phi}{\partial x^2} - (3\lambda + 2\mu)\alpha T \tag{22.5}$$

$$\sigma_{xy} = \frac{\partial^2\phi}{\partial x\partial y} \; , \tag{22.6}$$

etc. and hence the equilibrium equation (2.2) requires that

$$\frac{\partial}{\partial x}\left(\frac{\lambda}{2\mu}\nabla^2\phi + \nabla^2\phi - (3\lambda + 2\mu)\alpha T\right) = 0 \; . \tag{22.7}$$

Two similar equations are obtained from (2.3, 2.4) and, since we are only looking for a particular solution, we can satisfy them by choosing

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_22, © Springer Science+Business Media B.V. 2010

$$\nabla^2\phi = \frac{2\mu(3\lambda+2\mu)\alpha T}{(\lambda+2\mu)} = \frac{2\mu(1+\nu)\alpha T}{(1-\nu)} \; . \tag{22.8}$$

Equation (22.8) defines the potential ϕ due to a source distribution proportional to the given temperature, T. Such a function can be found for all T in any geometric domain[1] and hence a particular solution of the thermoelastic problem in the form (22.1) can always be found.

Substituting for T from equation (22.8) back into the stress equation (22.5), we obtain

$$\begin{aligned}\sigma_{xx} &= \left(\frac{\lambda}{2\mu} - \frac{(\lambda+2\mu)}{2\mu}\right)\nabla^2\phi + \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial x^2} - \nabla^2\phi \\ &= -\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} \; .\end{aligned} \tag{22.9}$$

The particular solution can therefore be summarized in the equations

$$\nabla^2\phi = \frac{2\mu(1+\nu)\alpha T}{(1-\nu)} \tag{22.10}$$

$$2\mu\boldsymbol{u} = \boldsymbol{\nabla}\phi \tag{22.11}$$

$$\begin{gathered}\sigma_{xx} = -\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} \;\; ; \;\; \sigma_{yy} = -\frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial x^2} \;\; ; \;\; \sigma_{zz} = -\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} \\ \sigma_{xy} = \frac{\partial^2\phi}{\partial x\partial y} \;\; ; \;\; \sigma_{yz} = \frac{\partial^2\phi}{\partial y\partial z} \;\; ; \;\; \sigma_{zx} = \frac{\partial^2\phi}{\partial z\partial x} \; .\end{gathered} \tag{22.12}$$

The equivalent expressions in cylindrical polar coördinates are

$$\begin{gathered}2\mu u_r = \frac{\partial\phi}{\partial r} \;\; ; \;\; 2\mu u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \;\; ; \;\; 2\mu u_z = \frac{\partial\phi}{\partial z} \\ \sigma_{rr} = \frac{\partial^2\phi}{\partial r^2} - \nabla^2\phi \;\; ; \;\; \sigma_{\theta\theta} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \nabla^2\phi \;\; ; \;\; \sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} - \nabla^2\phi \\ \sigma_{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \;\; ; \;\; \sigma_{\theta z} = \frac{1}{r}\frac{\partial^2\phi}{\partial\theta\partial z} \;\; ; \;\; \sigma_{zr} = \frac{\partial^2\phi}{\partial z\partial r} \; ,\end{gathered} \tag{22.13}$$

where $\nabla^2\phi$ is given by equation (21.30).

In spherical polar coördinates, we have

$$\begin{gathered}2\mu u_R = \frac{\partial\phi}{\partial R} \;\; ; \;\; 2\mu u_\theta = \frac{1}{R\sin\beta}\frac{\partial\phi}{\partial\theta} \;\; ; \;\; 2\mu u_\beta = \frac{1}{R}\frac{\partial\phi}{\partial\beta} \\ \sigma_{RR} = \frac{\partial^2\phi}{\partial R^2} - \nabla^2\phi \;\; ; \;\; \sigma_{\theta\theta} = \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial\phi}{\partial\beta} - \nabla^2\phi \\ \sigma_{\beta\beta} = \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial\beta^2} - \nabla^2\phi \; ; \;\; \sigma_{R\theta} = \frac{1}{R\sin\beta}\frac{\partial^2\phi}{\partial R\partial\theta} \\ \sigma_{\theta\beta} = \frac{1}{R^2\sin\beta}\frac{\partial^2\phi}{\partial\theta\partial\beta} \;\; ; \;\; \sigma_{\beta R} = \frac{1}{R}\frac{\partial^2\phi}{\partial\beta\partial R} - \frac{1}{R^2}\frac{\partial\phi}{\partial\beta} \; ,\end{gathered} \tag{22.14}$$

where $\nabla^2\phi$ is given by equation (21.35).

[1] For example, as a convolution integral on the point source solution $1/R$.

22.1 Plane problems

If ϕ and T are independent of z, the non-zero stress components reduce to

$$\sigma_{xx} = -\frac{\partial^2 \phi}{\partial y^2} \;\; ; \;\; \sigma_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} \;\; ; \;\; \sigma_{yy} = -\frac{\partial^2 \phi}{\partial x^2} \tag{22.15}$$

$$\sigma_{zz} = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} \tag{22.16}$$

and the solution corresponds to a state of plane strain. In fact, the relations (22.15) are identical to the Airy function definitions, apart from a difference of sign, and similarly, (22.10) reduces to a particular integral of (14.7). However, an important difference is that the present solution also gives explicit relations for the displacements and can therefore be used for problems with displacement boundary conditions, including those arising for multiply connected bodies.

22.1.1 Axisymmetric problems for the cylinder

We consider the cylinder $b<r<a$ in a state of plane strain with a prescribed temperature distribution. The example in §14.1 and the treatment of isothermal problems in Chapters 8, 9 show that this problem could be solved for a fairly general temperature distribution, but a case of particular importance is that in which the temperature is an axisymmetric function $T(r)$.

If ϕ is also taken to depend upon r only, we can write (22.10) in the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\phi}{dr} = \frac{2\mu(1+\nu)\alpha T(r)}{(1-\nu)} \; , \tag{22.17}$$

which has the particular integral

$$\frac{d\phi}{dr} = \frac{2\mu(1+\nu)\alpha}{(1-\nu)r} \int_b^r rT(r)dr \; , \tag{22.18}$$

where we have selected the inner radius b as the lower limit of integration, since the required generality will be introduced later through the superposed isothermal solution.

The corresponding displacement and stress components are then obtained as

$$u_r = \frac{1}{2\mu}\frac{d\phi}{dr} = \frac{(1+\nu)\alpha}{(1-\nu)r}\int_b^r rT(r)dr \tag{22.19}$$

$$\sigma_{rr} = -\frac{1}{r}\frac{d\phi}{dr} = -\frac{2\mu(1+\nu)\alpha}{(1-\nu)r^2}\int_b^r rT(r)dr \tag{22.20}$$

$$\sigma_{\theta\theta} = -\frac{d^2\phi}{dr^2} = \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\left(\frac{1}{r^2}\int_b^r rT(r)dr - T(r)\right) \tag{22.21}$$

$$\sigma_{r\theta} = 0 \;\; ; \;\; u_\theta = 0 \; . \tag{22.22}$$

The general solution is now obtained by superposing those axisymmetric terms[2] from Tables 8.1, 9.1 that give single-valued displacements, with the result

$$u_r = \frac{(1+\nu)\alpha}{(1-\nu)r}\int_b^r rT(r)dr + A(1-2\nu)r - \frac{B}{r} \tag{22.23}$$

$$\sigma_{rr} = -\frac{2\mu(1+\nu)\alpha}{(1-\nu)r^2}\int_b^r rT(r)dr + 2\mu A + \frac{2\mu B}{r^2} \tag{22.24}$$

$$\sigma_{\theta\theta} = \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\left(\frac{1}{r^2}\int_b^r rT(r)dr - T(r)\right) + 2\mu A - \frac{2\mu B}{r^2} \tag{22.25}$$

$$\sigma_{r\theta} = 0 \;\; ; \;\; u_\theta = 0 \; . \tag{22.26}$$

The constants A, B permit arbitrary axisymmetric boundary conditions to be satisfied at $r=a, b$. The same equations can also be used for the solid cylinder or disk ($b=0$), provided the constant B associated with the singular term is set to zero.

A similar method can be used for the hollow or solid sphere with spherically symmetric boundary conditions and whose temperature is a function of radius only. The analysis is given by Timoshenko and Goodier, *loc. cit.*, §152.

22.1.2 Steady-state plane problems

The definitions of the in-plane stress components in equation (22.15) are identical to those of the Airy stress function except for a difference in sign. Furthermore, if the temperature T is in a steady state, it must be harmonic and hence ϕ must be biharmonic, from (22.10). It follows that if we superpose a homogeneous stress field derived from an equal Airy stress function $\varphi=\phi$, the resulting in-plane stress components will be everywhere zero. This provides an alternative proof of the result in §14.3 that for steady-state plane problems in simply connected bodies, there will be no in-plane thermal stress if the boundary tractions are zero.

The displacements of course will not be zero, as we know from §14.3.1. Suppose we are given a temperature distribution T satisfying $\nabla^2 T = 0$ and are asked to determine the corresponding displacement field. If the body is simply connected there are no in-plane stresses, so

$$e_{xx} = \frac{\partial u_x}{\partial x} = \alpha(1+\nu)T \; ; \quad e_{yy} = \frac{\partial u_x}{\partial x} = \alpha(1+\nu)T$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = 0 \; , \tag{22.27}$$

where we have used (14.1, 14.2) modified for plane strain by (14.8). It is convenient to formulate this problem in the complex variable formalism of Chapter 19. We then have

[2] Notice that since this is the plane strain solution, we have used $\kappa = (3-4\nu)$ in Table 9.1.

$$\frac{\partial u}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right)(u_x + \imath u_y) = \frac{1}{2}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) + \frac{\imath}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) ,$$

from (18.6, 19.33). Comparing with (22.27), we see that both real and imaginary parts of the right-hand side are identically zero, so

$$\frac{\partial u}{\partial \bar{\zeta}} = 0 , \tag{22.28}$$

implying that u is a holomorphic function of ζ only. We also have

$$\frac{\partial u}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}\right)(u_x + \imath u_y) = \frac{1}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \frac{\imath}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$
$$= \frac{e}{2} + \imath\omega_z \tag{22.29}$$

and we know the real part of this holomorphic function since

$$\frac{e}{2} = \alpha(1+\nu)T = \alpha(1+\nu)\left(g(\zeta) + \bar{g}(\bar{\zeta})\right) , \tag{22.30}$$

where the (harmonic) temperature is defined as

$$T = g(\zeta) + \bar{g}(\bar{\zeta}) , \tag{22.31}$$

as in equation (18.13). The most general holomorphic function of ζ satisfying (22.29, 22.30) is

$$\frac{\partial u}{\partial \zeta} = 2\alpha(1+\nu)g(\zeta) + \imath\omega_0 , \tag{22.32}$$

where ω_0 is a real constant representing an arbitrary rigid body rotation $\omega_z = \omega_0$. Thus, if we know the temperature and hence the function $g(\zeta)$ in equation (22.31), we can determine the displacement field simply by integrating equation (22.32) with respect to ζ. The translational rigid body terms will arise as an arbitrary complex constant during this integration.

22.1.3 Heat flow perturbed by a circular hole

If the body is multiply connected, the function $\varphi = \phi$ may correspond to multiple-valued displacements and hence not represent a valid homogeneous solution. In this case, additional biharmonic terms terms must be added as in §13.1 and the resulting stress field will generally be non-zero.

Consider the case in which a uniform flow of heat $q_x = q_0$ in a large body is perturbed by the presence of an insulated circular hole of radius a. We can write the temperature field in the form

$$T = T_0 + T_1 ,$$

where T_0 is the temperature that would be obtained if there were no hole and T_1 is the perturbation in temperature due to the hole. Following the strategy of §8.3.2, we first consider the unperturbed problem of uniform heat flow, in order to determine the appropriate Fourier dependence of the terms describing the perturbation.

The unperturbed problem

In the absence of the hole, the body would be simply connected and we know from the previous section that the in-plane stresses would be everywhere zero. However, as in §8.3.2, an examination of the unperturbed problem enables us to determine the appropriate Fourier dependence of the terms describing the perturbation.

The heat flux will then be the same everywhere and we have

$$q_x = -K\frac{\partial T_0}{\partial x} = q_0 \; ; \quad T_0 = -\frac{q_0 x}{K} = -\frac{q_0 r\cos\theta}{K} \; , \tag{22.33}$$

where we have chosen the zero of temperature to be that at the origin. This expression varies with $\cos\theta$ and we conclude that the perturbation in both T and ϕ will exhibit similar dependence on θ.

The temperature perturbation

The temperature perturbation T_1 must be harmonic and decay as $r\rightarrow\infty$ and the only function with the required Fourier form satisfying these conditions is $\cos\theta/r$. We therefore write

$$T = T_0 + T_1 = -\frac{q_0 r\cos\theta}{K} + \frac{C_1\cos\theta}{r} \; , \tag{22.34}$$

where the constant C_1 must be determined from the condition that the radial heat flux $q_r=0$ at $r=a$. In other words,

$$-K\frac{\partial T}{\partial r}(a,\theta) = 0 \; . \tag{22.35}$$

Substituting for T from (22.34) and solving for C_1, we obtain

$$C_1 = -\frac{q_0 a^2}{K} \quad \text{and hence} \quad T_1 = -\frac{q_0 a^2\cos\theta}{Kr} \; . \tag{22.36}$$

The thermoelastic particular solution

We can now obtain a particular thermoelastic solution ϕ_1 for the perturbation problem by substituting T_1 into (22.10), writing $\phi_1 = f(r)\cos\theta$ and solving for $f(r)$. We obtain

$$\phi_1 = -\frac{\mu(1+\nu)\alpha q_0 a^2 r\ln(r)\cos\theta}{K(1-\nu)} \tag{22.37}$$

and the corresponding in-plane stress components are

$$\sigma_{rr} = \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{K(1-\nu)r} \; ; \quad \sigma_{r\theta} = \frac{\mu(1+\nu)\alpha q_0 a^2\sin\theta}{K(1-\nu)r}$$

$$\sigma_{\theta\theta} = \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{K(1-\nu)r} \; , \tag{22.38}$$

from (22.13).

The homogeneous solution

The surface of the hole must be traction-free — i.e.

$$\sigma_{rr} = \sigma_{r\theta} = 0 \; ; \quad r = a \tag{22.39}$$

and these conditions are clearly not satisfied by equations (22.38). We must therefore superpose a homogeneous solution which we construct from the Airy stress function terms

$$\varphi = C_2 r\theta \sin\theta + C_3 r \ln(r)\cos\theta + \frac{C_4 \cos\theta}{r} \; , \tag{22.40}$$

from Table 8.1. The choice of these terms is dictated by the Fourier dependence of the required stress components and the condition that the perturbation must decay as $r \to \infty$. The corresponding in-plane stresses (including the particular solution) are

$$\begin{aligned}
\sigma_{rr} &= \frac{\mu(1+\nu)\alpha q_0 a^2 \cos\theta}{K(1-\nu)r} + \frac{2C_2\cos\theta}{r} + \frac{C_3\cos\theta}{r} - \frac{2C_4\cos\theta}{r^3} \\
\sigma_{r\theta} &= \frac{\mu(1+\nu)\alpha q_0 a^2 \sin\theta}{K(1-\nu)r} + \frac{C_3\sin\theta}{r} - \frac{2C_4\sin\theta}{r^3} \\
\sigma_{\theta\theta} &= \frac{\mu(1+\nu)\alpha q_0 a^2 \cos\theta}{K(1-\nu)r} + \frac{C_3\cos\theta}{r} + \frac{2C_4\cos\theta}{r^3}
\end{aligned} \tag{22.41}$$

and the boundary conditions (22.39) then require that

$$\frac{2C_2}{a} + \frac{C_3}{a} - \frac{2C_4}{a^3} = -\frac{\mu(1+\nu)\alpha q_0 a}{K(1-\nu)} \tag{22.42}$$

$$\frac{C_3}{a} - \frac{2C_4}{a^3} = -\frac{\mu(1+\nu)\alpha q_0 a}{K(1-\nu)} \; . \tag{22.43}$$

A third equation for the three constants C_2, C_3, C_4 is obtained from the requirement that the displacements be single-valued, as in §13.1. This condition is automatically satisfied by the particular solution, since it is derived from a single-valued displacement function ϕ_1, so we simply record the potentially multiple-valued terms arising from φ. The first two terms in (22.40) are identical to (13.1) except that C_1 is replaced by C_2 and hence the required condition is

$$C_2(\kappa-1) + C_3(\kappa+1) = 0 \; , \tag{22.44}$$

by comparison with (13.7). The use of the thermoelastic potential independent of z implies plane strain conditions, so we set $\kappa = 3-4\nu$ obtaining

$$2(1-2\nu)C_2 + 4(1-\nu)C_3 = 0 \; . \tag{22.45}$$

We can then solve (22.42, 22.43, 22.45) for C_2, C_3, C_4, obtaining

$$C_2 = C_3 = 0 \; ; \quad C_4 = \frac{\mu\alpha(1+\nu)q_0 a^4}{2K(1-\nu)} \; . \tag{22.46}$$

The final stress field is then obtained as

$$\begin{aligned} \sigma_{rr} &= \frac{\mu\alpha(1+\nu)q_0 a^2(r^2-a^2)\cos\theta}{K(1-\nu)r^3} \\ \sigma_{r\theta} &= \frac{\mu\alpha(1+\nu)q_0 a^2(r^2-a^2)\sin\theta}{K(1-\nu)r^3} \\ \sigma_{\theta\theta} &= \frac{\mu\alpha(1+\nu)q_0 a^2(r^2+a^2)\cos\theta}{K(1-\nu)r^3} \; . \end{aligned} \tag{22.47}$$

In particular, the maximum tensile stress is

$$\sigma_{\max} = \sigma_{\theta\theta}(a, 0) = \frac{2\mu\alpha(1+\nu)q_0 a}{K(1-\nu)} \; . \tag{22.48}$$

22.1.4 Plane stress

All the results in §22.1 have been obtained from the three dimensional representation (22.1) under the restriction that ϕ be independent of z and hence $u_z = 0$. They therefore define the plane strain solution. Corresponding plane stress results can be obtained by making the substitutions

$$\mu = \mu' \;\; ; \;\; \nu = \frac{\nu'}{(1+\nu')} \;\; ; \;\; \alpha = \frac{\alpha'(1+\nu')}{(1+2\nu')} \tag{22.49}$$

and then removing the primes.

22.2 The method of strain suppression

An alternative approach for obtaining a particular solution for the thermoelastic stress field is to reduce it to a body force problem using the *method of strain suppression*. Suppose we were to constrain every particle of the body so as to prevent it from moving — in other words, we constrain the displacement to be everywhere zero. The strains will then be identically zero everywhere and the constitutive equations (22.2, 22.3) reduce to

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -(3\lambda+2\mu)\alpha T = -\frac{2\mu(1+\nu)\alpha T}{(1-2\nu)} \;\; ; \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; . \tag{22.50}$$

Of course, these stresses will not generally satisfy the equilibrium equations, but we can always find a body force distribution that will restore equilibrium — all we need to do is to substitute the non-equilibrated stresses (22.50) into the equilibrium equation with body forces (2.5) and use the latter as a definition of $\boldsymbol{p}$, obtaining

$$\boldsymbol{p} = \frac{2\mu(1+\nu)}{(1-2\nu)}\alpha\boldsymbol{\nabla}T\ . \tag{22.51}$$

It follows that (22.50) will define the stresses in the body for the specified temperature distribution if (i) body forces (22.51) are applied and (ii) purely normal tractions are applied at the boundaries given by

$$\boldsymbol{t} = -\frac{2\mu(1+\nu)}{(1-2\nu)}\alpha T\boldsymbol{n}\ , \tag{22.52}$$

where T is the local temperature at the boundary and $\boldsymbol{n}$ is the local outward normal unit vector.

To complete the solution, we must now remove the unwanted body forces by superposing the solution of a problem with no thermal strains, but with equal and opposite body forces and with surface tractions that combined with (22.52) satisfy the traction boundary conditions of the original problem.

The resulting body force problem might be solved using the body force potential of §20.5.1, since (22.51) defines a conservative vector field. However, the boundary value problem would then be found to be identical with that obtained by substituting the original temperature distribution into (22.8), so this does not really constitute a new method. An alternative strategy might be to represent the body force as a distribution of point forces as suggested in §13.1.1, but using the three-dimensional Kelvin solution which we shall develop in the next chapter.

22.3 Steady-state temperature : Solution T

If the temperature distribution is in a steady-state and there are no internal heat sources, the temperature T is harmonic, from equation (14.16). It follows from equation (22.10) that the potential function ϕ is biharmonic and it can conveniently be expressed in terms of a harmonic function χ through the relation

$$\phi = z\chi\ . \tag{22.53}$$

We then have

$$\nabla^2\phi = 2\frac{\partial\chi}{\partial z} \tag{22.54}$$

and hence

$$T = \frac{(1-\nu)}{\mu\alpha(1+\nu)}\frac{\partial\chi}{\partial z}\ , \tag{22.55}$$

from equation (22.10). Expressions for the displacements and stresses are easily obtained by substituting (22.53) into equations (22.11–22.12) and they are tabulated as Solution T in Table 22.1 (for Cartesian and cylindrical polar coördinates) and Table 22.2 (for spherical polar coördinates).

We also note that the heat flux $\boldsymbol{q} = -K\boldsymbol{\nabla}T$ and hence, for example,

Table 22.1. Solutions T and P

	Solution T Williams' solution	**Solution P** Thermoelastic Plane Stress
$2\mu u_x$	$z\dfrac{\partial\chi}{\partial x}$	$2(1-\nu)\dfrac{\partial\psi}{\partial x}$
$2\mu u_y$	$z\dfrac{\partial\chi}{\partial y}$	$2(1-\nu)\dfrac{\partial\psi}{\partial y}$
$2\mu u_z$	$\chi + z\dfrac{\partial\chi}{\partial z}$	$-2(1-\nu)\dfrac{\partial\psi}{\partial z}$
σ_{xx}	$z\dfrac{\partial^2\chi}{\partial x^2} - 2\dfrac{\partial\chi}{\partial z}$	$-2(1-\nu)\dfrac{\partial^2\psi}{\partial y^2}$
σ_{xy}	$z\dfrac{\partial^2\chi}{\partial x\partial y}$	$2(1-\nu)\dfrac{\partial^2\psi}{\partial x\partial y}$
σ_{yy}	$z\dfrac{\partial^2\chi}{\partial y^2} - 2\dfrac{\partial\chi}{\partial z}$	$-2(1-\nu)\dfrac{\partial^2\psi}{\partial x^2}$
σ_{xz}	$\dfrac{\partial\chi}{\partial x} + z\dfrac{\partial^2\chi}{\partial x\partial z}$	0
σ_{yz}	$\dfrac{\partial\chi}{\partial y} + z\dfrac{\partial^2\chi}{\partial y\partial z}$	0
σ_{zz}	$z\dfrac{\partial^2\chi}{\partial z^2}$	0
$2\mu u_r$	$z\dfrac{\partial\chi}{\partial r}$	$2(1-\nu)\dfrac{\partial\psi}{\partial r}$
$2\mu u_\theta$	$\dfrac{z}{r}\dfrac{\partial\chi}{\partial\theta}$	$\dfrac{2(1-\nu)}{r}\dfrac{\partial\psi}{\partial\theta}$
σ_{rr}	$z\dfrac{\partial^2\chi}{\partial r^2} - 2\dfrac{\partial\chi}{\partial z}$	$2(1-\nu)\left(\dfrac{1}{r}\dfrac{\partial\psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\psi}{\partial\theta^2}\right)$
$\sigma_{r\theta}$	$\dfrac{z}{r}\dfrac{\partial^2\chi}{\partial r\partial\theta} - \dfrac{z}{r^2}\dfrac{\partial\chi}{\partial\theta}$	$2(1-\nu)\left(\dfrac{1}{r}\dfrac{\partial^2\psi}{\partial r\partial\theta} - \dfrac{1}{r^2}\dfrac{\partial\psi}{\partial\theta}\right)$
$\sigma_{\theta\theta}$	$\dfrac{z}{r}\dfrac{\partial\chi}{\partial r} - \dfrac{z}{r^2}\dfrac{\partial^2\chi}{\partial\theta^2} - 2\dfrac{\partial\chi}{\partial z}$	$-2(1-\nu)\dfrac{\partial^2\psi}{\partial r^2}$
σ_{rz}	$\dfrac{\partial\chi}{\partial r} + z\dfrac{\partial^2\chi}{\partial r\partial z}$	0
$\sigma_{\theta z}$	$\dfrac{z}{r}\dfrac{\partial^2\chi}{\partial\theta\partial z} + \dfrac{1}{r}\dfrac{\partial\chi}{\partial\theta}$	0
T	$\dfrac{1-\nu}{\mu\alpha(1+\nu)}\dfrac{\partial\chi}{\partial z}$	$-\dfrac{(1-\nu)}{\mu\alpha(1+\nu)}\dfrac{\partial^2\psi}{\partial z^2}$
q_z	$-\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\dfrac{\partial^2\chi}{\partial z^2}$	$\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\dfrac{\partial^3\psi}{\partial z^3}$
q_r	$-\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\dfrac{\partial^2\chi}{\partial z\partial r}$	$\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\dfrac{\partial^3\psi}{\partial z^2\partial r}$

Table 22.2. Solution T in spherical polar coördinates

	Solution T
$2\mu u_R$	$R\cos\beta\dfrac{\partial\chi}{\partial R}+\chi\cos\beta$
$2\mu u_\theta$	$\cot\beta\dfrac{\partial\chi}{\partial\theta}$
$2\mu u_\beta$	$\cos\beta\dfrac{\partial\chi}{\partial\beta}-\chi\sin\beta$
σ_{RR}	$R\cos\beta\dfrac{\partial^2\chi}{\partial R^2}+\dfrac{2\sin\beta}{R}\dfrac{\partial\chi}{\partial\beta}$
$\sigma_{\theta\theta}$	$\dfrac{\cot\beta}{R\sin\beta}\dfrac{\partial^2\chi}{\partial\theta^2}-\cos\beta\dfrac{\partial\chi}{\partial R}+\dfrac{(1+\sin^2\beta)}{R\sin\beta}\dfrac{\partial\chi}{\partial\beta}$
$\sigma_{\beta\beta}$	$\dfrac{\cos\beta}{R}\dfrac{\partial^2\chi}{\partial\beta^2}-\dfrac{\partial\chi}{\partial R}\cos\beta$
$\sigma_{\theta\beta}$	$\dfrac{\cot\beta}{R}\dfrac{\partial^2\chi}{\partial\theta\partial\beta}-\dfrac{1}{R\sin^2\beta}\dfrac{\partial\chi}{\partial\theta}$
$\sigma_{\beta R}$	$\cos\beta\dfrac{\partial^2\chi}{\partial R\partial\beta}-\sin\beta\dfrac{\partial\chi}{\partial R}$
$\sigma_{R\theta}$	$\cot\beta\dfrac{\partial^2\chi}{\partial R\partial\theta}$
T	$\dfrac{1-\nu}{\mu\alpha(1+\nu)}\left(\dfrac{\partial\chi}{\partial R}\cos\beta-\dfrac{\sin\beta}{R}\dfrac{\partial\chi}{\partial\beta}\right)$
q_R	$\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\left(\dfrac{\sin\beta}{R}\dfrac{\partial^2\chi}{\partial R\partial\beta}-\dfrac{\partial^2\chi}{\partial R^2}\cos\beta-\dfrac{\sin\beta}{R^2}\dfrac{\partial\chi}{\partial\beta}\right)$
q_θ	$\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\left(\dfrac{1}{R^2}\dfrac{\partial^2\chi}{\partial\beta\partial\theta}-\dfrac{\cot\beta}{R}\dfrac{\partial^2\chi}{\partial R\partial\theta}\right)$
q_β	$\dfrac{(1-\nu)K}{\mu\alpha(1+\nu)}\left(\dfrac{\sin\beta}{R}\dfrac{\partial\chi}{\partial R}-\dfrac{\cos\beta}{R}\dfrac{\partial^2\chi}{\partial R\partial\beta}+\dfrac{\sin\beta}{R^2}\dfrac{\partial^2\chi}{\partial\beta^2}+\dfrac{\cos\beta}{R^2}\dfrac{\partial\chi}{\partial\beta}\right)$

$$q_z=-K\frac{\partial T}{\partial z}=-\frac{(1-\nu)K}{\mu\alpha(1+\nu)}\frac{\partial^2\chi}{\partial z^2}\ , \tag{22.56}$$

where K is the thermal conductivity of the material. These expressions are listed in the Maple and Mathematica files 'Txyz', 'Trtz' 'Trtb' for Cartesian, cylindrical polar and spherical polar coördinates respectively.

The solution here described was first obtained by Williams[3] and was used by him for the solution of some thermoelastic crack problems.

22.3.1 Thermoelastic plane stress

An important thermoelastic solution is that appropriate to the steady-state thermoelastic deformation of a traction-free half-space, which can be obtained

[3] W. E. Williams, A solution of the steady-state thermoelastic equations, *Zeitschrift für angewandte Mathematik und Physik,* Vol. 12 (1961), pp.452–455.

by superposing solutions A,B,T and taking

$$\phi = 2(1-\nu)\psi; \quad \omega = -\chi = \frac{\partial \psi}{\partial z} \; . \tag{22.57}$$

The resulting expressions are given in Table 22.1 as Solution P. A striking result is that the three stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ are zero, not merely on the plane $z=0$, where we have forced them to zero by the relations (22.57), but throughout the half-space. For this reason, the solution is also appropriate to the problem of a thick plate $a<z<b$ with traction free faces. Because of this feature, the solution is henceforth referred to as *thermoelastic plane stress.*

Notice also the similarity of the expressions for the non-zero stresses $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ to the corresponding expressions obtained from the Airy stress function. However, the present solution is not two-dimensional — i.e. the temperature and hence all the non-zero stress and displacement components can be functions of all three coördinates — and it is exact, whereas the two-dimensional plane stress solution is generally approximate.

Another result of interest is

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = -\frac{(1-\nu)}{\mu}\left(\frac{\partial^3 \psi}{\partial x^2 \partial z} + \frac{\partial^3 \psi}{\partial y^2 \partial z}\right) = \frac{(1-\nu)}{\mu}\frac{\partial^3 \psi}{\partial z^3}$$

(because ψ and hence $\partial\psi/\partial z$ is harmonic)

$$= \frac{\alpha(1+\nu)q_z}{K} = \delta q_z \; , \tag{22.58}$$

where δ is the *thermal distortivity* (see §14.3.1).

In other words, the sum of the principal curvatures of any distorted z-plane is proportional to the local heat flux across that plane[4]. The corresponding two-dimensional result was proved in §14.3.1.

PROBLEMS

1. The temperature $T(R)$ in a hollow sphere $b < R < a$ depends only on the radius R and the surfaces of the sphere are traction-free. Use a method similar to that in §22.1.1 to obtain expressions for the stress and displacement components for any function $T(R)$.

Note: The most general spherically symmetrical isothermal state of stress (i.e. the general homogeneous solution) can be written by superposing (i) the function A/R in solution A of Table 21.2 and (ii) a uniform state of hydrostatic stress. The identity

[4] Further consequences of this result are discussed in J.R.Barber, Some implications of Dundurs' theorem for thermoelastic contact and crack problems, *Journal of Strain Analysis,* Vol. 22 (1980), pp.229–232.

$$\frac{d^2}{dR^2} + \frac{2}{R}\frac{d}{dR} \equiv \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\right)$$

may also prove useful.

2. The temperature field in the elastic half-space $z>0$ is defined by

$$T(x,y,z) = f(z)\cos(mx) \ ,$$

where $f(z)$ is a known function of z only. The surface $z=0$ of the half-space is traction-free. Show that the normal surface displacement $u_z(x,y,0)$ can be written

$$u_z(x,y,0) = u_0 \cos(mx) \ ,$$

where

$$u_0 = 2\alpha(1+\nu)\int_0^\infty e^{-ms} f(s)ds \ .$$

Important Note: This problem *cannot* be solved using Solutions T or P, since the temperature field will only be harmonic for certain special functions $f(z)$. To solve the problem, assume a strain potential ϕ of the form

$$\phi = g(z)\cos(mx) \ ,$$

substitute into (22.10) and solve the resulting ordinary differential equation for $g(z)$. You can then superpose appropriate potentials from solutions A and B of Table 21.1 to satisfy the traction-free boundary condition.

3. During a test on an automotive disk brake, it was found that the temperature T of the disk could be approximated by the expression

$$T = \frac{5}{4}(1-e^{-\tau}) - \frac{r^2}{2a^2}(1-e^{-\tau}) + \frac{r^4}{8a^4}\tau e^{-\tau} \ ,$$

where $\tau = t/t_0$ is a dimensionless time, a is the radius of the disk, r is the distance from the axis, t is time and t_0 is a constant. The disk is continuous at $r=0$ and all its surfaces can be assumed to be traction-free.

Derive expressions for the radial and circumferential thermal stresses as functions of radius and time, and hence show that the maximum value of the stress difference $(\sigma_{\theta\theta}-\sigma_{rr})$ at any given time always occurs at the outer edge of the disk.

4. A heat exchanger tube consists of a long hollow cylinder of inner radius a and outer radius b, maintained at temperatures T_a, T_b respectively. Find the steady-state stress distribution in the tube if the curved surfaces are traction-free and plane strain conditions can be assumed.

5. The steady-state temperature $T(\xi,\eta)$ in the quarter plane $\xi \geq 0, \eta \geq 0$ is determined by the boundary conditions

$$\begin{aligned} T(\xi,0) &= 0 \;\;; \quad \xi > 0 \\ T(0,\eta) &= T_0 \,; \quad 0 < \eta < a \\ &= 0 \;\;; \quad \eta > a \,. \end{aligned}$$

Use the function

$$\omega(\zeta) = \zeta^{1/2}$$

to map the problem into the half plane $y > 0$ and solve for T using the method of §18.6.3.

Then determine the displacement throughout the quarter plane, using equation (22.32).

6. Use dimensional arguments and continuity of heat flux to show that the two-dimensional steady-state temperature field in the half plane $y > 0$ due to a line heat source Q per unit length at the origin is

$$T(x,y) = \frac{Q}{K}\ln(r) + C \;,$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the origin, K is the thermal conductivity and C is a constant.

Use this result and equation (22.32) to determine the displacement throughout the half plane. Assume that $C = 0$ and neglect rigid-body displacements. In particular, determine the displacements at the surface $y = 0$ and verify that they satisfy Dundurs' theorem (14.21).

7. A uniform heat flux $q_x = q_0$ in a large body in a state of plane strain is perturbed by the presence of a rigid non-conducting circular inclusion of radius a. The inclusion is bonded to the matrix at $r = a$. Find the complete stress field in the matrix and identify the location and magnitude of the maximum tensile stress. Assume plane strain conditions.

8. A uniform heat flux $q_x = q_0$ in a large body in a state of plane strain with elastic and thermal properties $\mu_1, \nu_1, \alpha_1, K_1$ is perturbed by the presence of a circular inclusion of radius a with properties $\mu_2, \nu_2, \alpha_2, K_2$. The inclusion is bonded to the matrix at $r = a$. Find the complete stress field in the matrix and the inclusion and identify the location and magnitude of the maximum tensile stress in each. Assume plane strain conditions. **Note:** You will find the algebra for this problem formidable if you do not use Maple or Mathematica.

9. A rigid non-conducting body slides at velocity V in the positive x-direction over the surface of the thermoelastic half-space $z > 0$, causing frictional tractions σ_{zx} and frictional heating q_z given by

$$\sigma_{zx} = -f\sigma_{zz} \;\;; \;\; \sigma_{zy} = 0 \;\;; \;\; q_z = V\sigma_{zx} \;,$$

where f is the coefficient of friction. Find a solution in terms of one potential function that satisfies these global boundary conditions in the steady state.

If the half-space remains in contact with the sliding body throughout the plane $z = 0$, we have the additional global boundary condition $u_z = 0$. Show that a non-trivial steady-state solution to this problem can be obtained using the harmonic function

$$Ce^{-mz} \cos(my) ,$$

where C is an arbitrary constant and m is a parameter which must be chosen to satisfy the displacement boundary condition. Comment on the physical significance of this solution.

10. The half-space $z > 0$ is bonded to a rigid body at the surface $z = 0$. Show that in the steady state, the surface temperature $T(x, y, 0)$ and the normal stress $\sigma_{zz}(x, y, 0)$ at the bond are related by the equation

$$\sigma_{zz}(x, y, 0) = -\frac{2\mu\alpha(1+\nu)T(x, y, 0)}{(3 - 4\nu)} .$$

11. The traction-free surface of the half space $z > 0$ is uniformly heated in the annular region $a < r < b$ and unheated elsewhere, so that

$$\begin{aligned} q_z(r, \theta, 0) &= \frac{Q}{\pi(b^2 - a^2)} \quad ; \quad a < r < b \\ &= 0 \quad ; \quad 0 \leq r < a \quad \text{and} \quad r > b . \end{aligned}$$

Use equation (22.58) to determine the slope of the surface

$$\frac{\partial u_z}{\partial r}(r, \theta, 0)$$

in the steady state. By taking the limit of this solution as $a \to b$ or otherwise, find the surface displacement due to a heat input Q uniformly distributed along a circle of radius b. Comment on the implications for steady-state thermoelastic contact problems.

23

SINGULAR SOLUTIONS

Singular solutions have a special place in the historical development of potential theory in general and of Elasticity in particular. Many of the important early solutions were obtained by appropriate superposition of singular potentials, noting that any form of singularity is permitted provided that the singular point is not a point of the body[1].

The generalization of this technique to allow continuous distributions of singularities in space (either at the boundary of the body or in a region of space not occupied by it) is still one of the most widely used methods of treating three-dimensional problems.

In this chapter, we shall consider some elementary forms of singular solution and examine the problems that they solve when used in the various displacement function representations of Chapter 21. The solutions will be developed in a rather *ad hoc* way, starting with those that are mathematically most straightforward and progressing to more complex forms. However, once appropriate forms have been introduced, a systematic way of developing them from first principles will be discussed in the next chapter.

23.1 The source solution

A convenient starting point is the most elementary singular harmonic function

$$\phi = \frac{1}{R} = \frac{1}{\sqrt{x^2+y^2+z^2}} , \tag{23.1}$$

which is easily demonstrated to be harmonic (except at the origin) by substitution into Laplace's equation. We shall refer to this solution as the 'source' solution, since it describes the steady-state temperature in an infinite body

[1] If it *is* a point of the body, certain restrictions must be imposed on the strength of the singularity, as discussed in §11.2.1 and §33.8.1.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_23, © Springer Science+Business Media B.V. 2010

with a point heat source at the origin or alternatively the potential in an infinite space with a point electric charge at the origin. It is clearly spherically symmetric — i.e. the potential depends only on the distance R from the origin — and it can be developed from first principles, starting with Laplace's equation in spherical polar coördinates (21.35) and imposing the condition that ϕ be a function of R only. The governing equation then reduces to

$$\frac{d^2\phi}{dR^2} + \frac{2}{R}\frac{d\phi}{dR} = 0 \tag{23.2}$$

of which (23.1) is clearly the only singular solution.

We shall examine the stress fields corresponding to the use of the source solution in Solutions A and B of Tables 21.1, 21.2.

23.1.1 The centre of dilatation

The strain potential solution (Solution A) is *isotropic* — i.e., the stress and displacement definitions preserve the same form in any Cartesian coördinate transformation — and hence the spherically symmetric function $1/R$ must correspond to a spherically symmetric state of stress and displacement.

Substituting $\phi=1/R$ into Table 21.2, we obtain

$$2\mu u_R = -\frac{1}{R^2} \;\; ; \;\; 2\mu u_\theta = 2\mu u_\beta = 0 \tag{23.3}$$

$$\sigma_{RR} = \frac{2}{R^3} \;\; ; \;\; \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\frac{1}{R^3} \;\; ; \;\; \sigma_{\theta\beta} = \sigma_{\beta R} = \sigma_{R\theta} = 0 \; . \tag{23.4}$$

Consider points on the spherical surface $R=a$, whose area is $4\pi a^2$. These points move radially outwards through a distance $u_R(a)$ and hence the material inside the imaginary sphere increases in volume by

$$\Delta V = 4\pi a^2 u_R(a) = -\frac{2\pi}{\mu} \; ,$$

using (23.3). Notice that ΔV is independent of a, implying that the volume of the imaginary *hollow* sphere $b<R<a$ remains constant. In other words, the volume change is all concentrated at the origin, and we can construct a solution for a prescribed volume change ΔV through the potential

$$\phi = -\frac{\mu \Delta V}{2\pi R} \; . \tag{23.5}$$

This defines the singularity known as the *centre of dilatation* or centre of compression[2]. Combined with a state of uniform hydrostatic stress, it can be used to describe the stresses and displacements in a thick-walled spherical

[2] S.P.Timoshenko and J.N.Goodier, *loc. cit.*, §136.

pressure vessel loaded by arbitrary uniform pressure at the inner and outer surfaces. This solution was first obtained by Lamé.

The centre of dilatation is not admissable at an interior point of a continuous body — it would correspond to an infinitesimal hole containing a fluid at infinite pressure. However, a *distribution* of centres of dilatation can be used to describe dilatational effects such as those arising from thermal expansion. In fact, the thermoelastic potential of equations (22.10–22.12) could be regarded as being derived from Solution A by distributing sources of strength proportional to αT throughout the space occupied by the body, to take account of the thermal dilatation. A similar technique can be used for dilatation arising from other sources, such as phase transformation.

23.1.2 The Kelvin solution

In Solution B, the stress components contain terms that are first derivatives of the potential ω or second derivatives multiplied by z. Hence, if we substitute $\omega=1/R$, we anticipate that the stresses will vary inversely with the square of the distance from the origin and hence that the force resultant over a spherical surface of radius a, with centre at the origin will be independent of a. In fact, this potential corresponds to the three-dimensional Kelvin problem, in which a concentrated force in the z-direction is applied at the origin in an infinite elastic body.

We shall demonstrate this by evaluating the stress components σ_{zz}, σ_{rz} and considering the equilibrium of the region $-h<z<h$, which includes the origin.

Noting that

$$\frac{\partial}{\partial z}\left(\frac{1}{R}\right) = -\frac{z}{R^3} \;\; ; \;\; \frac{\partial}{\partial r}\left(\frac{1}{R}\right) = -\frac{r}{R^3} \; , \tag{23.6}$$

we obtain

$$\begin{aligned} \sigma_{zz} &= z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} = z\left(\frac{3z^2}{R^5} - \frac{1}{R^3}\right) + 2(1-\nu)\frac{z}{R^3} \\ &= (1-2\nu)\frac{z}{R^3} + \frac{3z^3}{R^5} \end{aligned} \tag{23.7}$$

$$\begin{aligned} \sigma_{rz} &= z\frac{\partial^2\omega}{\partial r\partial z} - (1-2\nu)\frac{\partial\omega}{\partial r} \\ &= (1-2\nu)\frac{r}{R^3} + \frac{3rz^2}{R^5} \; . \end{aligned} \tag{23.8}$$

We now consider the equilibrium of the cylinder $r<a, \; -h<z<h$, which is shown in Figure 23.1.

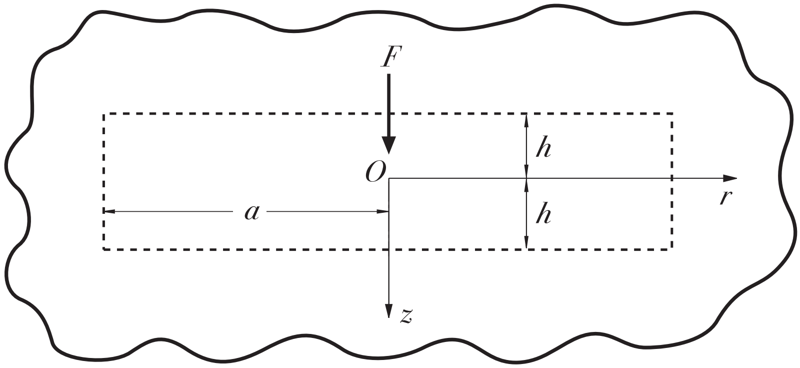

Figure 23.1: The Kelvin problem

The curved surfaces $r = a$ of this cylinder experience a force in the z-direction because of the stress σ_{rz}, but this force decays to zero as $a \to \infty$, since, although the area increases with a (being equal to $2\pi ah$), the stress component decreases with a^2. We can therefore restrict attention to the surfaces $z = \pm h$ if a is sufficiently large.

The stress component σ_{zz} is odd in z and hence the forces transmitted across the two surfaces are equal and have the total value

$$\begin{aligned} F_z &= 2\int_0^\infty 2\pi r \left(\frac{(1-2\nu)h}{R^3} + \frac{3h^3}{R^5} \right) dr \\ &= 4\pi \left[-\frac{(1-2\nu)h}{R} - \frac{h^3}{R^3} \right]_{r=0}^{r=\infty} = 8\pi(1-\nu) \;, \end{aligned} \tag{23.9}$$

where we note that $R = |z|$ when $r = 0$.

It follows that there must be an equal and opposite force at the origin. Hence, the function

$$\omega = -\frac{F}{8\pi(1-\nu)R} \tag{23.10}$$

in Solution B corresponds to the problem of a force F acting in the z-direction at the origin in the infinite elastic body. The complete stress field is easily obtained by substituting (23.10) into the appropriate expressions from Table 21.1 or Table 21.2.

23.2 Dimensional considerations

In the last section, we developed the Kelvin solution ‘accidentally’ by examining the stress field due to the source potential in Solution B. However, a more deductive development of the solution can be made from equilibrium and dimensional considerations.

We first note that the Kelvin problem is self-similar (see §12.1), since there is no inherent length scale. It follows that the form of the variation of the solution with θ, β must be independent of the distance R from the origin and hence that the solution can be expressed in the separated-variable form

$$f(R, \theta, \beta) = g(R)h(\beta, \theta) \; , \tag{23.11}$$

where f is any field quantity such as a stress or displacement component.

Given this result, we can deduce from equilibrium considerations that the function $g(R)$ appropriate to the stress components must be R^{-2}, since the total resultant force across any one of a class of self-similar surfaces (i.e. surfaces of the same shape but different size) must be the same, and the surface area of such surfaces will be proportional to the square of their linear dimensions. Thus, any stress component must decay with distance from the origin according to R^{-2}.

We now examine solutions A and B to see what type of singular function would satisfy this condition. On dimensional grounds, we find that the potential would have to be of order R^0 in Solution A and of order R^{-1} in Solution B. We shall see in the next section that there *are* singular potentials of order R^0, but that they involve singularities not merely at the origin, but along at least half of the z-axis. Thus, they cannot be used for a problem involving the whole infinite body. We are therefore left with the function $1/R$ — which we know to be harmonic — in Solution B, which can then be developed as in §23.1.2.

23.2.1 The Boussinesq solution

We shall now apply similar arguments to solve the Boussinesq problem, in which a point force F in the z-direction is applied at the origin $R=0$ on the surface $z=0$ of the half-space $z>0$, as shown in Figure 23.2. In particular, it is clear that this problem is also self-similar and equilibrium arguments again demand that the stress components decay with distance from the origin with R^{-2}.

This problem is treated by Timoshenko and Goodier[3] by superposing further singular solutions on the Kelvin solution so as to make the surface $z=0$ free of tractions except at the origin. A more direct (and modern) approach is to seek a special potential function solution which identically satisfies two of the three boundary conditions at the surface.

[3] *loc. cit.*, §138.

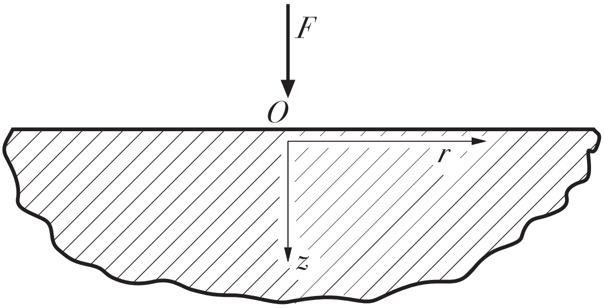

Figure 23.2: The Boussinesq problem

We note that the force is normal to the surface, so that there is no *tangential* traction at any point on the surface[4] — i.e.

$$\sigma_{zx} = \sigma_{zy} = 0 \;\; ; \;\; \text{all } x, y,\; z = 0 \; . \tag{23.12}$$

Since this condition applies at all points of the surface $z = 0$, we shall refer to it as a *global boundary condition* and it is appropriate to satisfy it by constructing a potential function solution which satisfies it identically, this being Solution F of Table 21.3. It then remains to find a suitable potential function which when used in Solution F will yield stress components which decay with R^{-2}.

Examination of Solution F shows that $1/R$ will not be suitable in the present instance, since the stresses are obtained by two differentiations of the stress function and hence we require φ to vary with R^0. We therefore seek a suitable partial integral of $1/R$ — to be dimensionless in R and singular at the origin, but otherwise to be continuous and harmonic in $z > 0$.

It is easily verified that the function

$$\varphi = \int_{-\infty}^{0} \frac{d\zeta}{\sqrt{x^2 + y^2 + (z - \zeta)^2}} = \ln(R + z) \tag{23.13}$$

satisfies these requirements. Notice that this function is singular on the negative z-axis, where $R = -z$, but not on the positive z-axis, where $R = z$. In fact, it can be regarded as the potential due to a uniform distribution of sources on the negative z-axis $(r = 0,\; z < 0)$.

Substituting into Solution F, we find

$$\sigma_{zz} = z \frac{\partial^3 \varphi}{\partial z^3} - \frac{\partial^2 \varphi}{\partial z^2} = \frac{3z^3}{R^5} \tag{23.14}$$

and hence the surface $z = 0$ is free of traction except at the origin as required[5].

[4] The normal traction is zero everywhere except at the origin, where there is a delta function loading, i.e. $\sigma_{zz} = -F\delta(x)\delta(y)$.

[5] The condition $\sigma_{zr} = 0$ on the surface is guaranteed by the use of Solution F.

The force applied at the origin is

$$\begin{aligned} F &= -2\pi \int_0^\infty r\sigma_{zz}(r,h)dr \\ &= -6\pi h^3 \int_0^\infty \frac{rdr}{(r^2+h^2)^{5/2}} = -2\pi \end{aligned} \tag{23.15}$$

and hence the stress field due to a force F in the z-direction applied at the origin is obtained from the potential

$$\varphi = -\frac{F}{2\pi} \ln(R+z) \; . \tag{23.16}$$

The displacements at the surface $z=0$ are of particular interest, in view of the application to contact problems[6]. We have

$$2\mu u_r(r,0) = (1-2\nu)\frac{\partial \varphi}{\partial r} = -\frac{F(1-2\nu)}{2\pi r} \;\; ; \;\; z=0 \tag{23.17}$$

$$2\mu u_z(r,0) = -2(1-\nu)\frac{\partial \varphi}{\partial z} = \frac{F(1-\nu)}{\pi r} \;\; ; \;\; z=0 \; . \tag{23.18}$$

Thus, both displacements vary inversely with distance from the point of application of the force F, as indeed we could have deduced from dimensional considerations. The singularity in displacement at the origin is not serious, since in any practical application, the force will be distributed over a finite area. If the solution for such a distributed force is found by superposition using the above result, the singularity will be integrated out.

We note that the displacements are bounded at infinity, in contrast to the two-dimensional case (see Chapter 12). It is therefore possible to regard the point at infinity as a reference for rigid-body displacements, if appropriate.

The radial surface displacement u_r is negative, indicating that a force F directed into the half-space — i.e. a compressive force — causes the surrounding surface to move towards the origin[7]. We can define a *surface dilatation* e_s on the surface $z=0$, such that

$$e_s \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \; . \tag{23.19}$$

Substituting for the displacement from equation (23.17), we obtain

[6] cf. Chapter 12.

[7] Most engineers would say that this is what they would intuitively expect, but the validity of this intuition is suspect, since the corresponding result for a tensile force — for which the radial displacement is directed away from the origin — is counter-intuitive. Our expectation here is probably determined by the thought that forces of either direction will stretch the surface and hence draw points towards the origin, but this is a second order (non-linear) effect and cannot be admitted in the linear theory.

$$e_s = -\frac{F(1-2\nu)}{4\pi\mu}\left(-\frac{1}{r^2}+\frac{1}{r^2}\right) = 0 \; . \tag{23.20}$$

In other words, the normal force F does not cause any dilatation of the surface $z=0$, except at the origin. However, if we draw a circle of radius a, centre the origin on the surface, it is clear that the radius of this circle gets smaller by the amount of the inward radial displacement and hence the area of the circle A increases by

$$\Delta A = 2\pi a u_r = -\frac{F(1-2\nu)}{2\mu} \; . \tag{23.21}$$

This result is independent of a — i.e. all circles reduce in area by the same amount[8] and all of this reduction must therefore be concentrated at the origin, where in a sense a small amount of the surface is lost. We can generalize this result by superposition to state that, if there is an arbitrary distribution of purely normal (compressive) traction on the surface $z=0$ of the half-space, the area A enclosed by any closed curve will change by ΔA of equation (23.21), where F is now to be interpreted as the resultant of the tractions acting *within* the area A.

23.3 Other singular solutions

We have already shown how the singular solution $\ln(R+z)$ can be obtained from $1/R$ by partial integration, which of course is a form of superposition. A whole sequence of axially symmetric solutions can be obtained in the same way. Defining

$$\phi_0 = \frac{1}{R} \; , \tag{23.22}$$

we obtain

$$\begin{aligned} \phi_{-1} &= \ln(R+z) \;\; ; \;\; \phi_{-2} = z\ln(R+z) - R \\ \phi_{-3} &= \frac{1}{4}\left\{(2z^2-r^2)\ln(R+z) - 3Rz + r^2\right\} \\ \phi_{-4} &= \frac{1}{36}\left\{3(2z^3-3zr^2)\ln(R+z) + 9zr^2 - 11z^2R + 4r^2R\right\} \;\; ; \;\; \ldots \end{aligned} \tag{23.23}$$

by the operation[9]

$$\phi_{n-1}(r,z) = \int_{-\infty}^{0} \phi_n(r,(z-\zeta))d\zeta \; . \tag{23.24}$$

---

[8] as could be argued from the fact that the surface dilatation is zero.

[9] Some care needs to be taken with the behaviour of the integral at the upper limit to obtain bounded partial integrals by this method. A more systematic method of developing this sequence will be introduced in §24.6 below.

All of these functions are harmonic except at the origin and on the negative z-axis, where $R+z\to 0$. A corresponding set of harmonic potentials singular only on the *positive* z-axis can be obtained by setting $z\to -z$ in (23.23), which is equivalent to reflecting the potentials about the plane $z=0$. It is convenient to define the signs of these functions such that

$$\begin{aligned}
\phi_{-1} &= -\ln(R-z) \ ; \ \phi_{-2} = -z\ln(R-z) - R \\
\phi_{-3} &= -\frac{1}{4}\left\{(2z^2-r^2)\ln(R-z) + 3Rz + r^2\right\} \\
\phi_{-4} &= -\frac{1}{36}\left\{3(2z^3-3zr^2)\ln(R-z) + 9zr^2 + 11z^2R - 4r^2R\right\} \ ; \ \ldots ,
\end{aligned} \tag{23.25}$$

since with this convention, the recurrence relation between both sets of potentials is defined by

$$\phi_n = \frac{\partial \phi_{n-1}}{\partial z} \ . \tag{23.26}$$

The sequence can also be extended to functions with stronger singularities by applying (23.26) to (23.22), obtaining

$$\phi_1 = -\frac{z}{R^3} \ ; \ \phi_2 = \frac{3z^2}{R^5} - \frac{1}{R^3} \ ; \ \phi_3 = -\frac{15z^3}{R^7} + \frac{9z}{R^5} \ ; \ \ldots \tag{23.27}$$

These functions are harmonic and bounded everywhere except at the origin.

We can also generate non-axisymmetric potential functions by differentiation with respect to x or y. For example, the function

$$\frac{\partial}{\partial x}\left(-\frac{z}{R^3}\right) = \frac{3xz}{R^5} = \frac{3\sin(2\beta)\cos\theta}{2R^3} \tag{23.28}$$

in spherical coordinates, is a non-axisymmetric harmonic function. However, these solutions can be obtained more systematically in terms of Legendre polynomials, as will be shown in the next chapter.

The integrated solutions solutions defined by equation (23.24) can be generalized to permit an arbitrary distribution of sources along a line or surface. For example, a fairly general axisymmetric harmonic potential can be written in the form

$$\phi(r,z) = \int_a^b \frac{g(\zeta)dt}{\sqrt{r^2+(z-\zeta)^2}} \ , \tag{23.29}$$

which defines the potential due to a distribution of sources of strength $g(z)$ along the z-axis in the range $a<z<b$. Such a solution defines bounded stresses and displacements in any body which does not include this line segment. When the boundary conditions of the problem are expressed in terms of a representation such as (23.29), the problem is essentially reduced to the solution of an integral equation or a set of integral equations. This technique has a long history in the solution of axisymmetric flow problems around smooth bodies and it can be used for the related elasticity problem in which a uniform tensile stress is perturbed by an axisymmetric cavity. We shall also use an adaptation of the same method for solving crack and contact problems for the half-space in Chapters 30,31.

23.4 Image methods

As in the two dimensional case, singular solutions for the half space can be obtained by placing appropriate image singularities outside the body. We consider the case in which the half space $z>0$ with elastic constants μ_1, κ_1 is bonded to the half space $z<0$ with constants μ_2, κ_2. Suppose that a singularity such as a concentrated force exists somewhere in the half space $z>0$ and that the same singularity in an infinite body can be defined in terms of the complex Papkovich-Neuber solution of §21.6 through the potentials $\phi^{(0)}(\zeta, \bar{\zeta}, z), \psi^{(0)}(\zeta, \bar{\zeta}, z), \omega^{(0)}(\zeta, \bar{\zeta}, z)$. In other words these are the potentials that would apply if the entire space had the same elastic constants μ_1, κ_1.

Aderogba[10] shows that the stress field in the composite body can then be obtained from the potentials

$$\begin{aligned}
\phi^{(1)} &= \phi^{(0)} - A\kappa_1 \left[\phi^{(0)}\right]_{z\to -z} + \frac{(A\kappa_1^2 - B)}{2}\left[\int \omega^{(0)} dz\right]_{z\to -z} \\
&\quad +(H - A\kappa_1)\left[\mathcal{L}\left\{\int \psi^{(0)} dz\right\}\right]_{z\to -z} + C\left[\mathcal{J}\left\{\int\int \psi^{(0)} dz dz\right\}\right]_{z\to -z} \\
\psi^{(1)} &= \psi^{(0)} - H\left[\psi^{(0)}\right]_{z\to -z} \\
\omega^{(1)} &= \omega^{(0)} - A\kappa_1\left[\omega^{(0)}\right]_{z\to -z} + 2A\left[\frac{\partial \phi^{(0)}}{\partial z}\right]_{z\to -z} \\
&\quad -(H - A\kappa_1)\left[\mathcal{J}\left\{\int \psi^{(0)} dz\right\}\right]_{z\to -z} + 2A\left[\mathcal{L}\left\{\psi^{(0)}\right\}\right]_{z\to -z}
\end{aligned} \qquad (23.30)$$

for $z>0$ and

$$\begin{aligned}
\phi^{(2)} &= (A+1)\phi^{(0)} + \frac{(A\kappa_1^2 - B)\Gamma}{2}\int \omega^{(0)} dz + (1 + D - A\kappa_1)\Gamma\mathcal{L}\left\{\int \psi^{(0)} dz\right\} \\
&\quad +C\Gamma\mathcal{J}\left\{\int\int \psi^{(0)} dz dz\right\} \\
\psi^{(2)} &= -D\Gamma\psi^{(0)} \\
\omega^{(2)} &= -(B+1)\phi^{(0)} - (D\Gamma + B + 1)\mathcal{J}\left\{\int \psi^{(0)} dz\right\}
\end{aligned} \qquad (23.31)$$

for $z<0$, where

$$\mathcal{J}\{\psi\} = \frac{\partial \psi}{\partial \zeta} + \frac{\partial \overline{\psi}}{\partial \bar{\zeta}}\ ; \quad \mathcal{L}\{\psi\} = \frac{1}{2}\frac{\partial}{\partial z}\left(\zeta\overline{\psi} + \bar{\zeta}\psi\right) - z\mathcal{J}\{\psi\}\ , \qquad (23.32)$$

and the composite material properties are defined as

[10] K.Aderogba, On eigenstresses in dissimilar media, *Philosophical Magazine*, Vol. 35 (1977), pp.281–292.

$$\Gamma = \frac{\mu_2}{\mu_1} \; ; \quad H = \frac{(\Gamma-1)}{(\Gamma+1)} \; ; \quad A = \frac{(\Gamma-1)}{(\Gamma\kappa_1+1)} \; ; \quad B = \frac{(\Gamma\kappa_1-\kappa_2)}{(\Gamma+\kappa_2)}$$

$$C = \frac{1}{2}[2H(\kappa_1+1) - A\kappa_1(\kappa_1+2) - B] \; ; \quad D = (H-1)(\kappa_1+1)(\kappa_2+1) \; .$$

The arbitrary functions implied by the partial integrals in equations (23.30, 23.31) must be chosen so as to ensure that the resulting terms are harmonic and non-singular in the corresponding half space.

23.4.1 The traction-free half space

The special case of the traction-free half space is recovered by setting $\Gamma=0$. We then have

$$A = B = H = -1 \; ; \quad C = \frac{(\kappa^2-1)}{2}$$

and

$$\begin{aligned}
\phi &= \phi^{(0)} + \kappa \left[\phi^{(0)}\right]_{z\to -z} - \frac{(\kappa^2-1)}{2}\left[\int \omega^{(0)} dz\right]_{z\to -z} \\
&\quad +(\kappa-1)\left[\mathcal{L}\left\{\int \psi^{(0)} dz\right\}\right]_{z\to -z} + \frac{(\kappa^2-1)}{2}\left[\mathcal{J}\left\{\int\int \psi^{(0)} dz dz\right\}\right]_{z\to -z} \\
\psi &= \psi^{(0)} + \left[\psi^{(0)}\right]_{z\to -z} \\
\omega &= \omega^{(0)} + \kappa\left[\omega^{(0)}\right]_{z\to -z} - 2\left[\frac{\partial \phi^{(0)}}{\partial z}\right]_{z\to -z} \\
&\quad -(\kappa-1)\left[\mathcal{J}\left\{\int \psi^{(0)} dz\right\}\right]_{z\to -z} - 2\left[\mathcal{L}\left\{\psi^{(0)}\right\}\right]_{z\to -z}
\end{aligned} \tag{23.33}$$

Example

As an example, we consider the case of the traction-free half space with a concentrated force F acting in the z-direction at the point $(0,0,a)$, a distance a below the surface[11]. The unperturbed field for a concentrated force at the origin is given in §23.1.2 by the potential

$$\omega = -\frac{F}{8\pi(1-\nu)R} = -\frac{F}{8\pi(1-\nu)\sqrt{r^2+z^2}} \; ; \quad \phi = \psi = 0 \; . \tag{23.34}$$

For an equal force at the point $(0,0,a)$ in the infinite body, we need to make use of the results of §20.3.1 with $\boldsymbol{s}=\boldsymbol{k}a$, $\boldsymbol{\psi}=\boldsymbol{k}\omega$ and $z\to(z-a)$. We obtain

[11] This problem was first solved by R.D.Mindlin, Force at a point in the interior of a semi-infinite solid, *Physics*, Vol.7 (1936), pp.195–202. A more compendious list of solutions for various singularities in the interior of the half space was given by R.D.Mindlin and D.H.Cheng, Nuclei of strain in the semi-infinite solid, *Journal of Applied Physics*, Vol.21 (1950), pp.926–930.

$$\boldsymbol{s} \cdot \boldsymbol{\psi} = a\omega$$

and hence, using equation (20.21),

$$\omega^{(0)} = -\frac{F}{8\pi(1-\nu)\sqrt{r^2+(z-a)^2}} \; ; \quad \phi^{(0)} = \frac{Fa}{8\pi(1-\nu)\sqrt{r^2+(z-a)^2}} \tag{23.35}$$

and $\psi^{(0)} = 0$. In constructing the partial integral of $\omega^{(0)}$ for (23.33), we can ensure that the resulting term is harmonic by using the results of §23.3, but we must also ensure that the resulting potential has no singularities in the half space $z > 0$. We therefore modify the first of equations (23.25) by a change of origin and write

$$\int \omega^{(0)} dz = \frac{F}{8\pi(1-\nu)} \ln\left(\sqrt{r^2+(z-a)^2} - (z-a)\right) ,$$

which after the substitution $z \to -z$ yields

$$\frac{F}{8\pi(1-\nu)} \ln\left(\sqrt{r^2+(z+a)^2} + z + a\right) .$$

This function is singular on the half line $r=0, z<-a$, but this lies entirely outside the half space $z>0$ as required. Using this result and (23.35) in (23.33) and substituting $\kappa = (3-4\nu)$, we obtain

$$\begin{aligned} \phi = & \frac{Fa}{8\pi(1-\nu)\sqrt{r^2+(z-a)^2}} + \frac{F(3-4\nu)a}{8\pi(1-\nu)\sqrt{r^2+(z+a)^2}} \\ & -\frac{F(1-2\nu)}{2\pi} \ln\left(\sqrt{r^2+(z+a)^2} + z + a\right) \\ \omega = & -\frac{F}{8\pi(1-\nu)\sqrt{r^2+(z-a)^2}} - \frac{F(3-4\nu)}{8\pi(1-\nu)\sqrt{r^2+(z+a)^2}} \\ & -\frac{Fa(z+a)}{4\pi(1-\nu)\left(r^2+(z+a)^2\right)^{3/2}} . \end{aligned}$$

The full stress field is then obtained by substitution into equations (21.45–21.50) and it is easily verified in Maple or Mathematica that the traction components $\sigma_{zr}, \sigma_{z\theta}, \sigma_{zz}$ go to zero on $z=0$.

In the limit $a \to 0$, we recover the Boussinesq solution of §23.2.1 in which the force F is applied at the surface of the otherwise traction-free half space. The potentials then reduce to

$$\phi = -\frac{F(1-2\nu)}{2\pi} \ln(R+z) \; ; \quad \omega = -\frac{F}{2\pi R} ,$$

agreeing with (23.16) and (21.37).

PROBLEMS

1. Starting from the potential function solution of Problem 21.2, use dimensional arguments to determine a suitable *axisymmetric* harmonic singular potential to solve the problem of a concentrated *tangential* force F in the x-direction, applied to the surface of the half-space at the origin.

2. Find the stresses and displacements[12] corresponding to the use of $\Psi = 1/R$ in Solution E. Show that the surfaces of the cone, $r = z\tan\beta$ are free of traction and hence use the solution to determine the stresses in a conical shaft of semi-angle β_0, transmitting a torque, T.

Find the maximum shear stress at the section $z = c$, and compare it with the maximum shear stress in a cylindrical bar of radius $a = c\tan\beta_0$ transmitting the same torque. Note that (i) the *maximum* shear stress will not generally be either $\sigma_{z\theta}$ or $\sigma_{r\theta}$, but must be found by appropriate coördinate transformation, and (ii) it may not *necessarily* occur at the outer radius of the cone, $r = c\tan\beta_0$.

3. An otherwise uniform cylindrical bar of radius b has a small spherical hole of radius a on the axis.

Combine the function $\Psi = Az/R^3$ in solution E with the elementary torsion solution for the bar without a hole (see, for example, Problem 21.1) and show that, with a suitable choice of the arbitrary constant A, the surface of the hole, $R = a$ can be made traction-free. Hence deduce the stress field near the hole for this problem.

Assume that $b \gg a$, so that the perturbation due to the hole has a negligible influence on the stresses near the outer surface, $r = b$.

4. The area A on the surface of an elastic half-space is subjected to a purely normal pressure $p(x, y)$ corresponding to the total force, F. The loaded region is now expanded in a self-similar manner by multiplying all its linear dimensions by the same ratio, λ. Each point in the new contact area is subjected to a self-similar loading such that the pressure at the point $(\lambda x, \lambda y)$ is $Cp(x, y)$, where the constant C is chosen to ensure that the total force F is independent of λ.

Show that the deflection at corresponding points $(\lambda x, \lambda y, \lambda z)$ will then be proportional to F/λ.

5. The surface of the half space $z > 0$ is traction-free and a concentrated heat source Q is applied at the origin. Determine the stress and displacement field in the half space in the steady state.

The traction-free condition can be satisfied by using solution P of Table 22.1. You can then use dimensional arguments to determine the dependence of the heat flux on R and hence to choose the appropriate singular potential.

[12] This solution is known as the *centre of rotation*.

6. Use Aderogba's formula (23.33) to determine the potentials ϕ, ψ, ω defining the stress and displacement fields in the traction-free half space $z > 0$, subjected to a concentrated force in the x-direction acting at the point $(0, 0, a)$.

7. Determine the potentials ϕ, ψ, ω defining the stress and displacement field in the traction-free half space $z > 0$ due to a centre of dilatation of strength ΔV located at the point $(0, 0, a)$.

24

SPHERICAL HARMONICS

The singular solutions introduced in the last chapter are particular cases of a class of functions known as spherical harmonics. In this chapter, we shall develop these functions and some related harmonic potential functions in a more formal way. In particular, we shall identify:-

(i) Finite polynomial potentials expressible in the alternate forms $P_n(x,y,z)$, $P_n(r,z)\cos(m\theta)$ or $R^nP_n(\cos\beta)\cos(m\theta)$, where P_n represents a polynomial of degree n.
(ii) Potentials that are singular only at the origin.
(iii) Potentials including the factor $\ln(R+z)$ that are singular on the negative z-axis $(z<0, r=0)$.
(iv) Potentials including the factor $\ln\{(R+z)/(R-z)\}$ that are singular everywhere on the z-axis.
(v) Potentials including the factor $\ln(r)$ and/or negative powers of r that are singular everywhere on the z-axis.

All of these potentials can be obtained in axisymmetric and non-axisymmetric forms. When used in solutions A,B and E of Tables 21.1, 21.2, the bounded potentials (i) provide a complete set of functions for the sphere, cylinder or cone with prescribed surface tractions or displacements on the curved surfaces. These problems are three-dimensional counterparts of those considered in Chapters 5, 8 and 11. Problems for the hollow cylinder and cone can be solved by supplementing the bounded potentials with potentials (v) and (iv) respectively. *Axisymmetric* problems for the hollow sphere or the infinite body with a spherical hole can be solved using potentials (i,ii), but these potentials do not provide a complete solution to the non-axisymmetric problem[1]. Functions (ii) and (iii) are useful for problems of the half space, including crack and contact problems.

[1] See §26.1.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_24, © Springer Science+Business Media B.V. 2010

24.1 Fourier series solution

The Laplace equation in spherical polar coördinates takes the form

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial R^2} + \frac{2}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2\sin^2\beta}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial\beta^2} + \frac{\cot\beta}{R^2}\frac{\partial\phi}{\partial\beta} = 0\ . \tag{24.1}$$

We first perform a Fourier decomposition with respect to the variable θ, giving a series of terms involving $\sin(m\theta), \cos(m\theta), \quad m = 0, 1, 2, \ldots, \infty$. For the sake of brevity, we restrict attention to the cosine terms, since the sine terms will be of the same form and can be reintroduced at the end of the analysis. We therefore assume that a potential function ϕ can be written

$$\phi = \sum_{m=0}^{\infty} f_m(R, \beta)\cos(m\theta)\ . \tag{24.2}$$

Substituting this series into (24.1), we find that the functions f_m must satisfy the equation

$$\frac{\partial^2 f}{\partial R^2} + \frac{2}{R}\frac{\partial f}{\partial R} - \frac{m^2 f}{R^2\sin^2\beta} + \frac{1}{R^2}\frac{\partial^2 f}{\partial\beta^2} + \frac{\cot\beta}{R^2}\frac{\partial f}{\partial\beta} = 0\ . \tag{24.3}$$

24.2 Reduction to Legendre's equation

We now cast equation (24.3) in terms of the new variable

$$x = \cos\beta\ , \tag{24.4}$$

for which we need the relations

$$\frac{\partial}{\partial\beta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\beta} = -\sin\beta\frac{\partial}{\partial x} = -\sqrt{1-x^2}\frac{\partial}{\partial x} \tag{24.5}$$

$$\frac{\partial^2}{\partial\beta^2} = -\sqrt{1-x^2}\frac{\partial}{\partial x}\left(-\sqrt{1-x^2}\frac{\partial}{\partial x}\right) = (1-x^2)\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}\ . \tag{24.6}$$

Substituting into (24.3), we obtain

$$\frac{\partial^2 f}{\partial R^2} + \frac{2}{R}\frac{\partial f}{\partial R} - \frac{m^2 f}{R^2(1-x^2)} - \frac{2x}{R^2}\frac{\partial f}{\partial x} + \frac{(1-x^2)}{R^2}\frac{\partial^2 f}{\partial x^2} = 0\ . \tag{24.7}$$

Finally, noting that (24.7) is homogeneous in powers of R, we expand $f_m(R, x)$ as a power series — i.e.

$$f_m(R, x) = \sum_{n=-\infty}^{\infty} R^n g_{mn}(x)\ , \tag{24.8}$$

which on substitution into equation (24.7) requires that the functions $g_{mn}(x)$ satisfy the ordinary differential equation

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \left(n(n+1) - \frac{m^2}{(1-x^2)}\right)g = 0\ . \qquad (24.9)$$

This is the standard form of *Legendre's equation*[2].

24.3 Axisymmetric potentials and Legendre polynomials

In the special case where $m=0$, the functions (24.2) will be independent of θ and hence define axisymmetric potentials. Equation (24.9) then reduces to

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + n(n+1)g = 0\ , \qquad (24.10)$$

which has a polynomial solution of degree n, known as a *Legendre polynomial* $P_n(x)$. It can readily be determined by writing g in polynomial form, substituting into equation (24.10) and equating coefficients. The first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x = \cos\beta\\
P_2(x) &= \frac{1}{2}(3x^2-1) = \frac{1}{4}\{3\cos(2\beta)+1\}\\
P_3(x) &= \frac{1}{2}(5x^3-3x) = \frac{1}{8}\{5\cos(3\beta)+3\cos\beta\}\\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) = \frac{1}{64}\{35\cos(4\beta)+20\cos(2\beta)+9\}\\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x) = \frac{1}{128}\{63\cos(5\beta)+35\cos(3\beta)+30\cos\beta\}\ ,
\end{aligned} \qquad (24.11)$$

where we have also used the relation $x=\cos\beta$ to express the results in terms of the polar angle β.

The sequence can be extended to higher values of n by using the recurrence relation[3]

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (24.12)$$

or the differential definition

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\ . \qquad (24.13)$$

---

[2] See for example I.S.Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980, §8.70.

[3] See I.S.Gradshteyn and I.M.Ryzhik, *loc. cit.* §§8.81, 8.82, 8.91 for this and more results concerning Legendre functions and polynomials.

In view of the above derivations, it follows that the axisymmetric functions

$$R^n P_n(\cos\beta) \tag{24.14}$$

are harmonic for all n. These functions are known as *spherical harmonics.*

24.3.1 Singular spherical harmonics

Legendre polynomials are defined only for $n \geq 0$, so the corresponding spherical harmonics (24.14) will be bounded at the origin, $R=0$. However, if we set

$$n = -p - 1 \ , \tag{24.15}$$

where $p \geq 0$, we find that

$$n(n+1) = (-p-1)(-p) = p(p+1) \tag{24.16}$$

and hence equation (24.9) becomes

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \left(p(p+1) - \frac{m^2}{(1-x^2)}\right) g = 0 \ . \tag{24.17}$$

In other words, the equation is of the same form, with p replacing n. It follows that

$$R^{-n-1} P_n(\cos\beta) \tag{24.18}$$

are harmonic functions, which are singular at $R=0$ for $n \geq 0$. If n is allowed to take all non-negative values, the two expressions (24.14, 24.18) define axisymmetric harmonics for all integer powers of R, as indicated formally by the series (24.8). These functions are generated by the Maple and Mathematica files 'sp0', using the recurrence relation (24.12). In combination with solutions A,B and E of Chapter 21, they provide a general solution to the problem of a solid or hollow sphere loaded by arbitrary axisymmetric tractions.

24.3.2 Special cases

If we set $n=0$ in (24.18), we obtain

$$\phi = P_0(z/R)R^{-1} = \frac{1}{R} \ , \tag{24.19}$$

which we recognize as the source solution of the previous chapter.

Furthermore, if we next set $n=1$ in the same expression, we obtain

$$\phi = P_1(z/R)R^{-2} = \frac{z}{R^3} \ , \tag{24.20}$$

which is proportional to the function ϕ_1 of (23.27). Similar relations exist between the sequence $\phi_2, \phi_3, \ldots$, and the higher order singular spherical harmonics.

24.4 Non-axisymmetric harmonics

When $m \neq 0$, equation (24.2) will define harmonic functions that are not axisymmetric. Corresponding solutions of Legendre's equation (24.9) can be developed from the axisymmetric forms by the relations

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \tag{24.21}$$

for $m \leq n$, or

$$P_n^m(x) = (1-x^2)^{-m/2} \int_x^1 \dots \int_x^1 P_n(x)(dx)^m \; . \tag{24.22}$$

for $m > n$. The functions $P_n^m(x)$ are known as *Legendre functions.* For problems of the sphere, only the harmonics with $m \leq n$ are useful, since the others involve singularities on either the positive or negative z-axis.

The first few non-axisymmetric Legendre functions are

$$\begin{aligned}
P_1^1(x) &= -(1-x^2)^{\frac{1}{2}} = -\sin\beta \\
P_2^1(x) &= -3x(1-x^2)^{\frac{1}{2}} = -\frac{3}{2}\sin(2\beta) \\
P_2^2(x) &= 3(1-x^2) = \frac{3}{2}\{1-\cos(2\beta)\} \\
P_3^1(x) &= -\frac{3}{2}(5x^2-1)(1-x^2)^{\frac{1}{2}} = -\frac{3}{8}\{\sin\beta + 5\sin(3\beta)\} \\
P_3^2(x) &= 15x(1-x^2) = \frac{15}{4}\{\cos\beta - \cos(3\beta)\} \\
P_3^3(x) &= -15(1-x^2)^{\frac{3}{2}} = -\frac{15}{4}\{3\sin\beta - \sin(3\beta)\} \; .
\end{aligned} \tag{24.23}$$

The spherical harmonics

$$R^n P_n^m(\cos\beta)\cos(m\theta) \quad ; \quad R^n P_n^m(\cos\beta)\sin(m\theta) \tag{24.24}$$

$$R^{-n-1} P_n^m(\cos\beta)\cos(m\theta) \quad ; \quad R^{-n-1} P_n^m(\cos\beta)\sin(m\theta) \tag{24.25}$$

in combination with solutions A,B and E of Chapter 21 can be used for problems of a solid or hollow sphere loaded by non-axisymmetric tractions, but the solution is complete only for the solid sphere. The functions (24.24, 24.25) are generated by the Maple and Mathematica files 'spn'.

24.5 Cartesian and cylindrical polar coördinates

The bounded Legendre polynomial solutions (24.14, 24.24) also correspond to finite polynomial functions in cylindrical polar coördinates and hence can be used for the problem of the solid circular cylinder loaded by polynomial tractions on its curved boundaries. For example, the axisymmetric functions (24.14) take the form

$$R^n P_n(z/R) \tag{24.26}$$

and the first few can be expanded in cylindrical polar coördinates as

$$\begin{aligned}
R^0 P_0(z/R) &= 1 \\
R^1 P_1(z/R) &= z \\
R^2 P_2(z/R) &= \frac{1}{2}(3z^2 - R^2) = \frac{1}{2}(2z^2 - r^2) \\
R^3 P_3(z/R) &= \frac{1}{2}(5z^3 - 3zR^2) = \frac{1}{2}(2z^3 - 3zr^2) \\
R^4 P_4(z/R) &= \frac{1}{8}(35z^4 - 30z^2R^2 + 3R^4) = \frac{1}{8}(8z^4 - 24z^2r^2 + 3r^4) \\
R^5 P_5(z/R) &= \frac{1}{8}(63z^5 - 70z^3R^2 + 15zR^4) = \frac{1}{8}(8z^5 - 40z^3r^2 + 15zr^4) \; ,
\end{aligned} \tag{24.27}$$

where we have used the relation $R^2 = r^2 + z^2$. This sequence of functions is generated in the Maple and Mathematica files 'cyl0'.

These functions can also be expanded as finite polynomials in Cartesian coördinates, using the relations $x = r\cos\theta, y = r\sin\theta$. For example, the function $R^3 P_3^1(z/R)\sin\theta$ expands as

$$\begin{aligned}
R^3 P_3^1(z/R)\sin\theta &= -\frac{3}{2}(5z^2 - R^2)\sqrt{R^2 - z^2}\sin\theta = -\frac{3}{2}(4z^2 - r^2)r\sin\theta \\
&= -\frac{3}{2}(4z^2y - x^2y - y^3) \; .
\end{aligned} \tag{24.28}$$

Of course, these polynomial solutions can also be obtained by assuming a general polynomial form of the appropriate order and substituting into the Laplace equation to obtain constraint equations, by analogy with the procedure for biharmonic polynomials developed in §5.1.

24.6 Harmonic potentials with logarithmic terms

Equation (24.10), is a second order ordinary differential equation and as such must have two linearly independent solutions for each value of n, one of which is the Legendre polynomial $P_n(x)$. To find the other solution, we define a new function $h(x)$ through the relation

$$g(x) = P_n(x)h(x) \tag{24.29}$$

and substitute into (24.10), obtaining

$$\begin{aligned}
&(1 - x^2)P_n(x)h''(x) + 2\{(1 - x^2)P_n'(x) - xP_n(x)\}h'(x) \\
&\quad + \{(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n + 1)P_n(x)\}h(x) = 0 \; .
\end{aligned} \tag{24.30}$$

The last term in this equation must be zero, since $P_n(x)$ satisfies (24.10). We therefore have

$$(1-x^2)P_n(x)h''(x) + 2\{(1-x^2)P_n'(x) - xP_n(x)\}h'(x) = 0 \; , \qquad (24.31)$$

which is a homogeneous first order ordinary differential equation for h' and can be solved by separation of variables.

If the non-constant solution of (24.31) is substituted into (24.29), we obtain the logarithmic Legendre function $Q_n(x)$, the first few such functions being

$$\begin{aligned}
Q_0(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \\
Q_1(x) &= \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 \\
Q_2(x) &= \frac{1}{4}(3x^2-1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2} \\
Q_3(x) &= \frac{1}{4}(5x^3-3x)\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3} \\
Q_4(x) &= \frac{1}{16}(35x^4-30x^2+3)\ln\left(\frac{1+x}{1-x}\right) - \frac{35x^3}{8} + \frac{55x}{24} \\
Q_5(x) &= \frac{1}{16}(63x^5-70x^3+15x)\ln\left(\frac{1+x}{1-x}\right) - \frac{63x^4}{8} + \frac{49x^2}{8} - \frac{8}{15} \; .
\end{aligned} \qquad (24.32)$$

The recurrence relation

$$(n+1)Q_{n+1}(x) - (2n+1)xQ_n(x) + nQ_{n-1}(x) = 0 \qquad (24.33)$$

can be used to extend this sequence to higher values of n.

The corresponding harmonic functions are logarithmically singular at all points on the z-axis. For example

$$\begin{aligned}
R^0Q_0(z/R) &= \frac{1}{2}\ln\left(\frac{R+z}{R-z}\right) \\
R^1Q_1(z/R) &= \frac{z}{2}\ln\left(\frac{R+z}{R-z}\right) - R \\
R^2Q_2(z/R) &= \frac{1}{4}(3z^2-R^2)\ln\left(\frac{R+z}{R-z}\right) - \frac{3Rz}{2}
\end{aligned} \qquad (24.34)$$

and the factor $(R-z) \to 0$ on the positive z-axis $(r=0, z>0)$, where $R=\sqrt{r^2+z^2} \to z$, whilst $(R+z) \to 0$ on the negative z-axis $(r=0, z<0)$, where $R=\sqrt{r^2+z^2} \to -z$. These potentials in combination with the bounded potentials (24.14), permit a general solution of the problem of a hollow cone loaded by prescribed axisymmetric polynomial tractions on the curved surfaces. They are generated in the Maple and Mathematica files 'Qseries', using the recurrence relation (24.33).

The attentive reader will recognize a similarity between the logarithmic terms in this series and the functions $\phi_{-1}, \phi_{-2}, \ldots$ of equation (23.23), obtained from the source solution by successive partial integrations. However, the two sets of functions are not identical. In effect, the functions in (23.23) correspond to distributions of sources along the negative z-axis, whereas those in (24.34) involve distributions of sources along both the positive and negative z-axes. The functions (23.23) are actually more useful, since they are harmonic throughout the region $z>0$ and hence can be applied to problems of the half space with concentrated loading, as we discovered in §23.2.1.

The functions $Q_n(x)$ can all be written in the form

$$Q_n(x) = P_n(x) \ln\left(\frac{1+x}{1-x}\right) - W_{n-1}(x) \ , \tag{24.35}$$

where

$$W_{n-1}(x) = \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(x) P_{n-k}(x) \tag{24.36}$$

is a finite polynomial of degree of degree $(n-1)$. In other words, the multiplier on the logarithmic term is the Legendre polynomial $P_n(x)$ of the same order. This is also true for the functions of equation (23.23), as can be seen by comparing them with equations (24.34). A convenient way to extend the sequence of functions (23.23) is therefore to assume a solution of the form

$$\phi_{-n-1} = R^n P_n(z/R) \ln(R+z) + R_n(R, z) \ , \tag{24.37}$$

where R_n is a general polynomial of degree n in R, z, substitute into the Laplace equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{24.38}$$

and use the resulting equations to determine the coefficients in R_n. This method is used in the Maple and Mathematica files 'sing'.

24.6.1 Logarithmic functions for cylinder problems

The function $\ln(R+z)$ corresponds to a uniform distribution of sources along the negative z-axis. A similar distribution along the *positive* z-axis would lead to the related harmonic function $\ln(R-z)$. Alternatively, we could add these two functions, obtaining the solution for a uniform distribution of sources along the entire z-axis $(-\infty<z<\infty)$ with the result

$$\ln(R+z) + \ln(R-z) = \ln(R^2 - z^2) = 2\ln(r) \ . \tag{24.39}$$

Not surprisingly, this function is independent of z. It is actually the source solution for the two-dimensional Laplace equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{24.40}$$

and could have been obtained directly by seeking a singular solution of this equation that was independent of θ.

More importantly, a similar superposition can be applied to the functions (24.37) to obtain a new class of logarithmically singular harmonic functions of the form

$$\phi = R^n P_n(z/R)\ln(r) + S_n(r,z) \;, \tag{24.41}$$

where S_n is a polynomial of degree n in r, z. These functions, in combination with the bounded harmonics of equation (24.27) and solutions A,B and E, provide a general solution to the problem of a hollow circular cylinder loaded by axisymmetric polynomial tractions on the curved surfaces. As before, the sequence (24.41) is most conveniently obtained by assuming a solution of the given form with S_n a general polynomial of r, z, substituting into the Laplace equation (24.38), and using the resulting equations to determine the coefficients in S_n. This method is used in the Maple and Mathematica files 'hol0'. The first few functions in the sequence are

$$\begin{aligned}
\varphi_0 &= \ln(r)\\
\varphi_1 &= z\ln(r)\\
\varphi_2 &= \frac{1}{2}\{(2z^2 - r^2)\ln(r) + r^2\}\\
\varphi_3 &= \frac{1}{2}\{(2z^3 - 3zr^2)\ln(r) + 3zr^2\}\\
\varphi_4 &= \frac{1}{8}\left\{(8z^4 - 24z^2r^2 + 3r^4)\ln(r) - \frac{9r^4}{2} + 24z^2r^2\right\}\\
\varphi_5 &= \frac{1}{8}\left\{(8z^5 - 40z^3r^2 + 15zr^4)\ln(r) - \frac{45zr^4}{2} + 40z^3r^2\right\}\;.
\end{aligned} \tag{24.42}$$

Notice that only even powers of r can occur in these functions.

24.7 Non-axisymmetric cylindrical potentials

As in §24.4, we can develop non-axisymmetric harmonic potentials in cylindrical polar coördinates in the form

$$\phi_m = f_m(r,z)\begin{Bmatrix}\sin(m\theta)\\ \cos(m\theta)\end{Bmatrix} \;, \tag{24.43}$$

with $m \neq 0$. Substitution in (24.38) shows that the function f_m must satisfy the equation

$$\frac{\partial^2 f_m}{\partial r^2} + \frac{1}{r}\frac{\partial f_m}{\partial r} - \frac{m^2 f_m}{r^2} + \frac{\partial^2 f_m}{\partial z^2} = 0 \;. \tag{24.44}$$

A convenient way to obtain such functions is to note that if ϕ_m is harmonic, the function

$$\phi_{m+1} = f_{m+1}(r,z)\cos\big((m+1)\theta\big) \tag{24.45}$$

will also be harmonic if

$$f_{m+1} = \frac{\partial f_m}{\partial r} - \frac{m f_m}{r} \equiv r^m \frac{\partial}{\partial r}\left(\frac{f_m}{r^m}\right) , \tag{24.46}$$

which can also be written

$$f_m = r^m \mathcal{L}^m f_0 \qquad \text{where} \qquad \mathcal{L}\{\cdot\} = \frac{1}{r}\frac{\partial}{\partial r} . \tag{24.47}$$

To prove this result, we first note that

$$\frac{\partial^2 f_m}{\partial z^2} = -\frac{\partial^2 f_m}{\partial r^2} - \frac{1}{r}\frac{\partial f_m}{\partial r} + \frac{m^2 f_m}{r^2} , \tag{24.48}$$

from (24.44). We then substitute (24.46) into (24.45) and the resulting expression into (24.38), using (24.48) to eliminate the derivatives with respect to z. The coefficients of all the remaining derivatives will then be found to be identically zero, confirming that (24.45) is harmonic.

The relation (24.46) can be used recursively to generate a sequence of non-axisymmetric harmonic potentials, starting from $m = 0$ with any of the axisymmetric functions developed above. For example, starting with the axisymmetric function

$$\phi_0 = f_0 = R^4 P_4(z/R) = \frac{1}{8}(8z^4 - 24z^2r^2 + 3r^4) , \tag{24.49}$$

from equation (24.27), we can construct

$$f_1 = \frac{\partial f_0}{\partial r} = \frac{3}{2}(-4z^2 r + r^3) \;\; ; \;\; f_2 = \frac{\partial f_1}{\partial r} - \frac{f_1}{r} = 3r^2 ,$$

from which we obtain the harmonic potentials

$$\phi_1 = \frac{3}{2}(-4z^2 r + r^3)\cos\theta \;\; ; \;\; \phi_2 = 3r^2\cos(2\theta) . \tag{24.50}$$

Similarly, from the logarithmically singular function

$$\phi_0 = f_0 = \varphi_2 = \frac{1}{2}\{(2z^2 - r^2)\ln(r) + r^2\} \tag{24.51}$$

of equation (24.42), we obtain

$$f_1 = \frac{\partial f_0}{\partial r} = -r\ln(r) + \frac{z^2}{r} + \frac{r}{2} = \;\; ; \;\; f_2 = \frac{\partial f_1}{\partial r} - \frac{f_1}{r} = -1 - \frac{2z^2}{r^2} ,$$

defining the potentials

$$\phi_1 = \left(-r\ln(r) + \frac{z^2}{r} + \frac{r}{2}\right)\cos\theta \;\; ; \;\; \phi_2 = \left(-1 - \frac{2z^2}{r^2}\right)\cos(2\theta)\ . \tag{24.52}$$

This process is formalized in the Maple and Mathematica files ‘cyln’ and ‘holn’, which generate a sequence of functions F_{mn}, S_{mn} such that the potentials

$$F_{mn}\cos(m\theta)\ ; \quad F_{mn}\sin(m\theta)\ ; \quad S_{mn}\cos(m\theta)\ ; \quad S_{mn}\sin(m\theta) \tag{24.53}$$

are harmonic. These files are easily extended to larger values of m or n if required. For convenience, we list here the first few functions of this sequence for the case $m=1$.

$$\begin{aligned} F_{11} &= r \\ F_{12} &= 3zr \\ F_{13} &= \frac{3}{2}(4z^2r - r^3) \\ F_{14} &= \frac{5}{2}(4z^3r - 3zr^3) \\ F_{15} &= \frac{15}{8}(8z^4r - 12z^2r^3 + r^5)\ . \end{aligned} \tag{24.54}$$

$$\begin{aligned} S_{10} &= \frac{1}{r} \\ S_{11} &= \frac{z}{r} \\ S_{12} &= -r\ln(r) + \frac{z^2}{r} + \frac{r}{2} \\ S_{13} &= -3zr\ln(r) + \frac{z^3}{r} + \frac{3zr}{2} \\ S_{14} &= \frac{3}{2}(r^3 - 4z^2r)\ln(r) + \frac{z^4}{r} + 3z^2r - \frac{15r^3}{8} \\ S_{15} &= \frac{5}{2}(3zr^3 - 12z^3r)\ln(r) + \frac{z^5}{r} + 5z^3r - \frac{75zr^3}{8}\ . \end{aligned} \tag{24.55}$$

24.8 Spherical harmonics in complex notation

In §21.6 we developed a version of the Papkovich-Neuber solution in which the Cartesian coördinates (x, y, z) were replaced by $(\zeta, \bar{\zeta}, z)$, where $\zeta = x + \imath y$, $\bar{\zeta} = x - \imath y$. In this system, the Laplace equation takes the form

$$\nabla^2\psi \equiv 4\frac{\partial^2\psi}{\partial\zeta\partial\bar{\zeta}} + \frac{\partial^2\psi}{\partial z^2} = 0\ . \tag{24.56}$$

We consider solutions of the series form

$$\psi_{2k}(\zeta,\bar{\zeta},z)=\sum_{j=0}^{k}\frac{z^{2j}f_{k-j}(\zeta,\bar{\zeta})}{(2j)!}\ . \tag{24.57}$$

Substitution into (24.56) then shows that

$$\sum_{j=0}^{k}\frac{4z^{2j}}{(2j)!}\frac{\partial^2 f_{k-j}}{\partial\zeta\partial\bar{\zeta}}+\sum_{j=1}^{k}\frac{z^{2j-2}f_{k-j}}{(2j-2)!}=0 \tag{24.58}$$

and equating coefficients of $z^{2k-2i-2}$, we have

$$\frac{\partial^2 f_{i+1}}{\partial\zeta\partial\bar{\zeta}}=-\frac{f_i}{4}\ ;\quad i=(0,k-1)\ , \tag{24.59}$$

$$\frac{\partial^2 f_0}{\partial\zeta\partial\bar{\zeta}}=0\ , \tag{24.60}$$

which defines a recurrence relation for the functions f_i. Furthermore, equation (24.60) is the two-dimensional Laplace equation whose general solution can be written

$$f_0=g_1(\zeta)+\overline{g}_2(\bar{\zeta})\ , \tag{24.61}$$

where $g_1,\overline{g}_2$ are general holomorphic functions of the complex variable ζ and its conjugate $\bar{\zeta}$, respectively. Thus, the most general solution of the form (24.57) can be constructed by successive integrations of (24.59), starting from (24.61). This defines an even function of z, but it is easily verified that the odd function

$$\psi_{2k+1}(\zeta,\bar{\zeta},z)=\sum_{j=0}^{k}\frac{z^{2j+1}f_{k-j}(\zeta,\bar{\zeta})}{(2j+1)!} \tag{24.62}$$

also satisfies (24.56), where the functions f_i again satisfy the recurrence relations (24.59, 24.60).

24.8.1 Bounded cylindrical harmonics

The bounded harmonics of equations (24.26, 24.27, 24.54) can be obtained from equations (24.57, 24.62) by choosing

$$f_0(\zeta,\bar{\zeta})=g_1(\zeta)=\zeta^m\ . \tag{24.63}$$

Successive integrations of equation (24.59) then yield

$$f_1(\zeta,\bar{\zeta})=\frac{\zeta^{m+1}\bar{\zeta}}{(-4)(m+1)}\ ;\quad f_2(\zeta,\bar{\zeta})=\frac{\zeta^{m+2}\bar{\zeta}^2}{(-4)^2(m+1)(m+2)(2)}\ , \tag{24.64}$$

or in general

$$f_i(\zeta,\bar{\zeta})=\frac{m!(\zeta\bar{\zeta})^i\zeta^m}{(-4)^i(m+i)!i!}\ , \tag{24.65}$$

where we have omitted the arbitrary functions of integration. Substituting f_i into (24.57, 24.62), we can then construct the harmonic functions

$$\chi_{2k}^m = \sum_{j=0}^{k} \frac{m! z^{2j} (\zeta\bar{\zeta})^{k-j} \zeta^m}{(-4)^{k-j}(m+k-j)!(k-j)!(2j)!}$$
$$\chi_{2k+1}^m = \sum_{j=0}^{k} \frac{m! z^{2j+1} (\zeta\bar{\zeta})^{k-j} \zeta^m}{(-4)^{k-j}(m+k-j)!(k-j)!(2j+1)!} , \qquad (24.66)$$

the first few functions in the sequence being

$$\begin{aligned}
\chi_0^m(\zeta, \bar{\zeta}, z) &= \zeta^m \\
\chi_1^m(\zeta, \bar{\zeta}, z) &= z\zeta^m \\
\chi_2^m(\zeta, \bar{\zeta}, z) &= \frac{z^2\zeta^m}{2!} + \frac{(\zeta\bar{\zeta})\zeta^m}{(-4)(m+1)} \qquad (24.67) \\
\chi_3^m(\zeta, \bar{\zeta}, z) &= \frac{z^3\zeta^m}{3!} + \frac{z(\zeta\bar{\zeta})\zeta^m}{(-4)(m+1)} \\
\chi_4^m(\zeta, \bar{\zeta}, z) &= \frac{z^4\zeta^m}{4!} + \frac{z^2(\zeta\bar{\zeta})\zeta^m}{(-4)(m+1)(2!)} + \frac{(\zeta\bar{\zeta})^2\zeta^m}{(-4)^2(m+1)(m+2)(2)} .
\end{aligned}$$

The functions (24.66, 24.67) are three-dimensionally harmonic and satisfy the recurrence relations

$$\frac{\partial \chi_n^m}{\partial z} = \chi_{n-1}^m \; ; \quad \frac{\partial \chi_n^m}{\partial \zeta} = m\chi_n^{m-1} \; ; \quad \frac{\partial \chi_n^m}{\partial \bar{\zeta}} = -\frac{\chi_{n-2}^{m+1}}{4(m+1)} . \qquad (24.68)$$

or

$$\int \chi_n^m dz = \chi_{n+1}^m \; ; \quad \int \chi_n^m d\zeta = \frac{\chi_n^{m+1}}{(m+1)} \; ; \quad \int \chi_n^m d\bar{\zeta} = -4m\chi_{n+2}^{m-1} .$$

The factor

$$(\zeta\bar{\zeta}) = re^{\imath\theta}re^{-\imath\theta} = r^2 \quad \text{and} \quad \zeta^m = r^m(\cos(m\theta) + \imath\sin(m\theta)) \qquad (24.69)$$

and hence the functions χ_n^m can be expanded as functions of r, z with Fourier multipliers as in (24.53, 24.54). In fact they are related to the spherical harmonics (24.24) by the expressions

$$\chi_n^m(\zeta, \bar{\zeta}, z) = \frac{(-2)^m m!}{(n+2m)!} R^{m+n} P_{m+n}^m(z/R) \exp(\imath m\theta) , \qquad (24.70)$$

where

$$R = \sqrt{z^2 + \zeta\bar{\zeta}} = \sqrt{x^2 + y^2 + z^2} . \qquad (24.71)$$

A few low order potentials that we shall need later are

$$\chi_0^0 = 1 \; ; \quad \chi_1^0 = z \; ; \quad \chi_2^0 = \frac{z^2}{2} - \frac{\zeta\bar{\zeta}}{4} \; ; \quad \chi_3^0 = \frac{z^3}{6} - \frac{z\zeta\bar{\zeta}}{4} \; ,$$

$$\chi_0^1 = \zeta \; ; \quad \chi_1^1 = z\zeta \; ; \quad \chi_2^1 = \frac{z^2\zeta}{2} - \frac{\zeta^2\bar{\zeta}}{8} \; ; \quad \chi_3^1 = \frac{z^3\zeta}{6} - \frac{z\zeta^2\bar{\zeta}}{8} \; , \tag{24.72}$$

$$\chi_0^2 = \zeta^2 \; ; \quad \chi_1^2 = z\zeta^2 \; ; \quad \chi_2^2 = \frac{z^2\zeta^2}{2} - \frac{\zeta^3\bar{\zeta}}{12} \; ; \quad \chi_3^2 = \frac{z^3\zeta^2}{6} - \frac{z\zeta^3\bar{\zeta}}{12} \; .$$

24.8.2 Singular cylindrical harmonics

A similar procedure can be used to develop complex versions of the singular harmonics of equations (24.53, 24.55), starting with the function

$$\begin{aligned} f_0(\zeta, \bar{\zeta}) &= \zeta^{-m} \quad ; \quad m \geq 1 \\ &= \ln(\zeta\bar{\zeta}) \; ; \quad m = 0 \; . \end{aligned} \tag{24.73}$$

Notice that in the complex variable formulation of two-dimensional problems, $\ln(\zeta)$ is generally excluded as being multivalued and hence non-holomorphic. However,

$$\ln(\zeta\bar{\zeta}) = \ln(\zeta) + \ln(\bar{\zeta}) = 2\ln(r)$$

is clearly a real single-valued harmonic function (being of the form (24.61). Singular potentials with $m > 0$ will eventually integrate up to include logarithmic terms when following the procedure (24.59). We see this for example in the real stress function versions (24.55) for $m=1$.

PROBLEMS

1. Use equation (24.22) to evaluate the function $P_1^2(x)$ and verify that it satisfies Legendre's equation (24.9) with $m = 2$ and $n = 1$. Construct the appropriate harmonic potential function from equation (24.24) and verify that it is singular on the z-axis.

2. The functions (23.23) are singular only on the negative z-axis ($r=0, z<0$). Similar functions singular only on the positive z-axis are given in equations (23.25). Develop the first three of (24.42) by superposition, noting that $\ln(R+z)+\ln(R-z)=2\ln(r)$.

3. Use the methodology of §5.1 to construct the most general harmonic polynomial function of degree 3 in Cartesian coördinates x, y, z. Decompose the resulting polynomial into a set of spherical harmonics.

4. Express the recurrence relation of equations (24.43–24.46) in spherical polar coördinates R, θ, β. Check your result by deriving the first three functions starting from the source solution $\phi_0 = 1/R$ and compare the results with the expressions derived in the Maple or Mathematica file 'spn'.

25

CYLINDERS AND CIRCULAR PLATES

The bounded harmonic potentials of equations (24.27) in combination with solutions A, B and E provide a complete solution to the problem of a solid circular cylinder loaded by axisymmetric polynomial tractions on its curved surfaces. The corresponding problem for the hollow cylinder can be solved by including also the singular potentials of equation (24.42). The method can be extended to non-axisymmetric problems using the results of §24.7. If strong boundary conditions are imposed on the curved surfaces and weak conditions on the ends, the solutions are most appropriate to problems of 'long' cylinders in which $L \gg a$, where L is the length of the cylinder and a is its outer radius. At the other extreme, where $L \ll a$, the same harmonic functions can be used to obtain three-dimensional solutions for in-plane loading and bending of circular plates, by imposing strong boundary conditions on the plane surfaces and weak conditions on the curved surfaces. As in Chapter 5, some indication of the order of polynomial required can be obtained from elementary Mechanics of Materials arguments.

25.1 Axisymmetric problems for cylinders

If axisymmetric potentials are substituted into solutions A and B, we find that the circumferential displacement u_θ is zero everywhere, as are the stress components $\sigma_{\theta r}, \sigma_{\theta z}$. There is therefore no torque transmitted across any cross-section of the cylinder and no twist. The only force resultant on the cross-section is a force F in the z-direction given by

$$F = \int_a^b \int_0^{2\pi} \sigma_{zz} r d\theta dr = 2\pi \int_a^b r\sigma_{zz} dr \ , \qquad (25.1)$$

where a, b are the inner and outer radii of the cylinder respectively.

By contrast, substitution of an axisymmetric potential into solution E yields a stress and displacement state where the *only* non-zero displacement

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_25, © Springer Science+Business Media B.V. 2010

and stress components are $u_\theta, \sigma_{\theta r}, \sigma_{\theta z}$. In this case, the axial force $F=0$, but in general there will be a transmitted torque

$$T = \int_a^b \int_0^{2\pi} \sigma_{\theta z} r^2 d\theta dr = 2\pi \int_a^b r^2 \sigma_{\theta z} dr \; . \tag{25.2}$$

Notice that the additional power of r in equation (25.2) compared with (25.1) is the moment arm about the z-axis for the elemental force $\sigma_{\theta z} r dr d\theta$.

Axisymmetric problems are therefore conveniently partitioned into *irrotational* (torsionless) problems requiring solutions A and B only and *rotational* (torsional) problems requiring only solution E. There is a close similarity here with the partition of two-dimensional problems into *in-plane* problems (Chapters 5 to 14) and *antiplane* problems (Chapter 15). Notice that the rotational problem, like the antiplane problem, involves only one non-zero displacement (u_θ) and requires only a single harmonic potential function (in solution E). By contrast, the irrotational problem involves two non-zero displacements (u_r, u_z) and requires two independent harmonic potentials, one each in solutions A and B. In the same way, the in-plane two-dimensional solution involves two displacements (e.g. u_x, u_y) and requires a single biharmonic potential, which we know to be equivalent to two independent harmonic potentials.

25.1.1 The solid cylinder

Problems for the solid cylinder are solved by a technique very similar to that used in Chapter 5. We shall find that we can always find finite polynomials satisfying polynomial boundary conditions on the curved boundaries in the strong (pointwise) sense, but that the boundary conditions on the ends can then only be satisfied in the weak sense. Corrective solutions for these end effects can be developed from an eigenvalue problem, as in Chapter 6.

A torsional problem

As an example, we consider the solid cylinder $(0 \le r < a, 0 < z < L)$, built in at the end $z = L$ and subjected to the traction distribution

$$\sigma_{r\theta} = \frac{Sz}{L} \;\; ; \;\; \sigma_{rr} = \sigma_{rz} = 0 \;\; ; \;\; r = a \tag{25.3}$$

$$\sigma_{zr} = \sigma_{z\theta} = \sigma_{zz} = 0 \;\; ; \;\; z = 0 \; . \tag{25.4}$$

In other words, the end $z=0$ is traction-free and the curved surfaces are loaded by a linearly varying torsional traction. This is clearly a torsional problem and elementary equilibrium arguments show that the torque must increase with z^2 from the free end. This suggests a leading term proportional to $z^2 r$ in the stress component $\sigma_{z\theta}$ and we therefore use solution E with a 5th degree polynomial and below.

We use the trial function[1]

$$\psi = \frac{C_5}{8}(8z^5 - 40z^3r^2 + 15zr^4) + \frac{C_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{C_3}{2}(2z^3 - 3zr^2) + \frac{C_2}{2}(2z^2 - r^2) \qquad (25.5)$$

from equation (24.27) and substitute into solution E (Table 21.1), obtaining the non-zero stress and displacement components

$$2\mu u_\theta = 5C_5(4z^3r - 3zr^3) + 3C_4(4z^2r - r^3) + 6C_3zr + 2C_2r \qquad (25.6)$$

$$\sigma_{r\theta} = -15C_5zr^2 - 3C_4r^2 \qquad (25.7)$$

$$\sigma_{z\theta} = 15C_5\left(2z^2r - \frac{r^3}{2}\right) + 12C_4zr + 3C_3r \; . \qquad (25.8)$$

The boundary condition (25.3) must be satisfied for all z, giving

$$C_5 = -\frac{S}{15a^2L} \;\; ; \;\; C_4 = 0 \; . \qquad (25.9)$$

The traction-free condition (25.4) can only be satisfied in the weak sense. Substituting (25.8) into (25.2), and evaluating the integral at $z=0$, we obtain

$$T = \frac{\pi a^4(3C_3 - 5C_5a^2)}{2} \qquad (25.10)$$

and hence the weak condition $T = 0$ gives

$$C_3 = \frac{5C_5a^2}{3} = -\frac{S}{9L} \; , \qquad (25.11)$$

using (25.9). The complete stress field is therefore

$$\sigma_{r\theta} = \frac{Szr^2}{a^2L} \qquad (25.12)$$

$$\sigma_{z\theta} = \frac{Sr}{L}\left(\frac{r^2}{2a^2} - \frac{2z^2}{a^2} - \frac{1}{3}\right) \; . \qquad (25.13)$$

Notice that the constant C_2 is not determined in this solution, since it describes merely a rigid-body displacement and does not contribute to the stress field. In order to determine it, we need to generate a weak form for the built-in condition at $z=L$. An appropriate condition is[2]

$$\int_0^a \int_0^{2\pi} u_\theta r^2 d\theta dr = 0 \;\; ; \;\; z = L \; . \qquad (25.14)$$

[1] Notice that the first two functions from equation (24.27) give no stresses or displacements when substituted into solution E and hence are omitted here.

[2] See §9.1.1.

Substituting for u_θ from (25.6) and using (25.9, 25.11) we obtain

$$C_2 = \frac{2SL^2}{3a^2} \tag{25.15}$$

and hence

$$2\mu u_\theta = S\left(\frac{zr^3}{a^2L} - \frac{4z^3r}{3a^2L} - \frac{2zr}{3L} + \frac{4L^2r}{3a^2}\right) . \tag{25.16}$$

A thermoelastic problem

As a second example, we consider the traction-free solid cylinder $0 \le r < a, -L<z<L$, subjected to the steady-state thermal boundary conditions

$$q_r(a,z) = -q_0 \;\; ; \;\; T(r,\pm L) = 0 . \tag{25.17}$$

In other words, the curved surfaces $r=a$ are uniformly heated, whilst the ends $z=\pm L$ are maintained at zero temperature.

The problem is clearly symmetrical about $z=0$ and the temperature must therefore be described by even Legendre polynomial functions. Using solution T to describe the thermoelastic field, we note that temperature is proportional to $\partial\chi/\partial z$ and hence χ must be comprised of odd functions from equation (24.27) — in particular

$$\chi = \frac{C_3}{2}(2z^3 - 3zr^2) + C_1 z . \tag{25.18}$$

Substituting (25.18) into solution T (Table 22.1), yields

$$T = \frac{(1-\nu)}{\mu\alpha(1+\nu)}\left(\frac{C_3}{2}(6z^2 - 3r^2) + C_1\right) \tag{25.19}$$

$$q_r = -K\frac{\partial T}{\partial r} = \frac{3(1-\nu)KC_3 r}{\mu\alpha(1+\nu)} \tag{25.20}$$

and hence

$$C_3 = -\frac{\mu\alpha(1+\nu)q_0}{3Ka(1-\nu)} , \tag{25.21}$$

from (25.17, 25.21). The constant C_1 can be determined[3] from the weak form of $(25.17)_2$ — i.e.

[3] The constant C_1 serves merely to set the base level for temperature and since no stresses are generated in a traction-free body subject to a uniform temperature rise, we could set $C_1=0$ at this stage without affecting the final solution for the stresses. Notice however, that this choice would lead to different values for the constants A_4, A_2, B_3, B_1 in the following analysis. Also, a uniform temperature rise *does* cause dilatation and hence non-zero displacements, so it is essential to solve for C_1 if the displacement field is required.

$$2\pi \int_0^a T(r,\pm L) r dr = \frac{2\pi(1-\nu)}{\mu\alpha(1+\nu)} \left\{ \frac{C_3}{2}\left(3L^2a^2 - \frac{3a^4}{4}\right) + \frac{C_1 a^2}{2} \right\} = 0 \tag{25.22}$$

and hence

$$C_1 = -\frac{3C_3(4L^2-a^2)}{4} = \frac{\mu\alpha(1+\nu)q_0(4L^2-a^2)}{4Ka(1-\nu)} . \tag{25.23}$$

It follows that the complete temperature field is

$$T(r,z) = \frac{q_0(4(L^2-z^2)+2r^2-a^2)}{4Ka} , \tag{25.24}$$

from (25.19, 25.21, 25.23).

A particular solution for the thermal stress field is then obtained by substituting (25.21, 25.23, 25.18) into Table 22.1. To complete the solution, we must superpose the homogeneous solution which is here given by solutions A and B, since axisymmetric temperature fields give dilatational but irrotational stress and displacement fields. To determine the appropriate order of polynomials to include in ϕ and ω, we compare Tables 21.1 and 22.1 and notice that whilst solution B involves the same order of differentials as solution T, the corresponding components in solution A involve one further differentiation, indicating the need for a higher order polynomial function. We therefore try the functions

$$\phi = \frac{A_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{A_2}{2}(2z^2 - r^2) \;\; ; \;\; \omega = \frac{B_3}{2}(2z^3 - 3zr^2) + B_1 z . \tag{25.25}$$

Substitution into Tables 21.1, 22.1 then yields the stress components

$$\begin{aligned} \sigma_{rr} &= A_4\left(\frac{9r^2}{2} - 6z^2\right) - A_2 - 3B_3(z^2 + 2\nu z^2 - \nu r^2) - 2\nu B_1 \\ &\quad -3C_3(3z^2 - r^2) - 2C_1 \\ \sigma_{rz} &= -6(2A_4 + \nu B_3 + C_3)zr \\ \sigma_{zz} &= 6A_4(2z^2 - r^2) + 2A_2 + 3B_3(2^2 - \nu r^2 + 2\nu z^2) + 2\nu B_1 + 6C_3 z^2 . \end{aligned} \tag{25.26}$$

The strong traction-free boundary conditions

$$\sigma_{rr}(a,z) = 0 \;\; ; \;\; \sigma_{rz}(a,z) = 0 \tag{25.27}$$

yield the three equations

$$\begin{aligned} -6A_4 - 3B_3(1+2\nu) &= 9C_3 \\ \frac{9A_4a^2}{2} - A_2 + 3B_3\nu a^2 - 2\nu B_1 &= -3C_3a^2 + 2C_1 \\ 2A_4 + \nu B_3 &= -C_3 \end{aligned} \tag{25.28}$$

and another equation is obtained from the weak condition $F=0$ on the ends $z=\pm L$. Substituting for σ_{zz} into (25.1) and evaluating the integral, we obtain

$$F = \pi a^2 \left\{ 3A_4(4L^2 - a^2) + 3B_3 \left(\frac{(1-\nu)a^2}{2} + 2\nu L^2 \right) + 2A_2 \right.$$
$$\left. - B_1(1-\nu) + 6C_3L^2 \right\} = 0 \; . \qquad (25.29)$$

Finally, solving (25.28, 25.29) for A_4, A_2, B_3, B_1 and substituting the resulting expressions into (25.25) and Tables 21.1, 22.1 yields the complete stress field

$$\sigma_{rr} = \frac{q_0\mu\alpha(a^2 - r^2)}{4Ka} \;\; ; \;\; \sigma_{\theta\theta} = \frac{q_0\mu\alpha(a^2 - 3r^2)}{4Ka} \;\; ; \;\; \sigma_{zz} = \frac{q_0\mu\alpha(2r^2 - a^2)}{2Ka}$$
$$\sigma_{rz} = \sigma_{r\theta} = \sigma_{\theta z} = 0 \; . \qquad (25.30)$$

25.1.2 The hollow cylinder

Problems for the hollow cylinder $a < r < b$ can be solved by supplementing the bounded potentials (24.27) with the logarithmically singular potentials of equation (24.42). The bounded potentials required are the same as those needed for a solid cylinder with similar boundary conditions, but the logarithmic potentials are generally several orders lower, being typically no higher than the order of the most rapidly varying traction on the curved surfaces[4].

To illustrate the procedure, we consider the hollow cylinder $a < r < b$ subjected to the torsional tractions

$$\sigma_{r\theta} = \frac{Sz}{L} \;\; ; \;\; \sigma_{rr} = \sigma_{rz} = 0 \;\; ; \;\; r = b \qquad (25.31)$$

$$\sigma_{r\theta} = \sigma_{rr} = \sigma_{rz} = 0 \;\; ; \;\; r = a \qquad (25.32)$$

$$\sigma_{zr} = \sigma_{z\theta} = \sigma_{zz} = 0 \;\; ; \;\; z = 0 \; . \qquad (25.33)$$

This is the hollow cylinder equivalent of the torsional example considered in §25.1.1 above and hence the required bounded potentials are given by equation (25.5), though of course the values of the multiplying constants will be different. In the interests of brevity, we shall solve only for the stress field and hence the potential $C_2(2z^2 - r^2)/2$ can be omitted, since it defines only a rigid-body displacement.

The traction varies with z^1, so for the hollow cylinder, we need to add in the logarithmic potentials φ_1, φ_0, giving the potential function

$$\psi = \frac{C_5}{8}(8z^5 - 40z^3r^2 + 15zr^4) + \frac{C_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{C_3}{2}(2z^3 - 3zr^2)$$
$$+ A_1 z \ln(r) + A_0 \ln(r) \; . \qquad (25.34)$$

[4] This arises because when we differentiate the potentials by parts, some of the resulting expressions involve differentiation only of the logarithmic multiplier, leaving higher order polynomial expressions for the stress and displacement components on a given radial surface.

Substituting into solution E (Table 21.1) we obtain

$$\sigma_{r\theta} = -15C_5 z r^2 - 3C_4 r^2 + \frac{2A_1 z}{r^2} + \frac{2A_0}{r^2} \tag{25.35}$$

$$\sigma_{z\theta} = 15C_5\left(2z^2 r - \frac{r^3}{2}\right) + 12C_4 z r + 3C_3 r - \frac{A_1}{r}\,. \tag{25.36}$$

The boundary conditions (25.31, 25.32) must be satisfied for all z, giving the four equations

$$-15C_5 a^2 + \frac{2A_1}{a^2} = 0 \tag{25.37}$$

$$-3C_4 a^2 + \frac{2A_0}{a^2} = 0 \tag{25.38}$$

$$-15C_5 b^2 + \frac{2A_1}{b^2} = \frac{S}{L} \tag{25.39}$$

$$-3C_4 b^2 + \frac{2A_0}{b^2} = 0\,. \tag{25.40}$$

A fifth equation is obtained by substituting (25.36) into (25.2) and enforcing the weak condition $T=0$. Solving these equations for the five constants C_5, C_4, C_3, A_1, A_0 and back substitution into (25.35, 25.36) yields the final stress field

$$\sigma_{r\theta} = \frac{Sb^2 z(r^4 - a^4)}{Lr^2(b^4 - a^4)}$$

$$\sigma_{z\theta} = \frac{Sb^2(3(b^2+a^2)(a^4 - 4z^2r^2 + r^4) - 2(4a^4 + a^2b^2 + b^4)r^2)}{6Lr(b^2+a^2)(b^4-a^4)}\,.$$

25.2 Axisymmetric circular plates

Essentially similar methods can be applied to obtain three-dimensional solutions to axisymmetric problems of the circular plate $0 \le r < a$, $-h/2 < z < h/2$ loaded by polynomial tractions on the surfaces $z = \pm h/2$ (or with a polynomial temperature distribution), where $h \ll a$. In this case, we impose the strong boundary conditions on the plane surfaces $z = \pm h/2$ and weak conditions on $r = a$. As in the corresponding two-dimensional problems of Chapter 5, we generally need to start with a polynomial up to 3 orders higher than that suggested by the applied loading.

Weak boundary conditions at $r = a$ can be stated in terms of force and moment resultants per unit circumference or of averaged displacements. The force resultants are a membrane tensile force N and shear force V defined by

$$N = \int_{-h/2}^{h/2} \sigma_{rr} dz \ \ ; \ \ V = \int_{-h/2}^{h/2} \sigma_{rz} dz\,, \tag{25.41}$$

whilst the moment resultant is

$$M = \int_{-h/2}^{h/2} z\sigma_{rr} dz \; . \tag{25.42}$$

Weak displacement boundary conditions can be defined by analogy with equations (9.21).

25.2.1 Uniformly loaded plate on a simple support

As an example, we consider the plate $0 \le r < a$, $-h/2 < z < h/2$ loaded by a uniform compressive traction p_0 on the surface $z = h/2$, simply-supported at the edge $r = a$ and otherwise traction-free. Thus, the traction boundary conditions are

$$\sigma_{zz} = -p_0 \;\; ; \;\; \sigma_{zr} = 0 \;\; ; \;\; z = h/2 \tag{25.43}$$

$$\sigma_{zz} = \sigma_{zr} = 0 \;\; ; \;\; z = -h/2 \tag{25.44}$$

$$\int_{-h/2}^{h/2} \sigma_{rr} dz = 0 \;\; ; \;\; \int_{-h/2}^{h/2} z\sigma_{rr} dz = 0 \;\; ; \;\; r = a \; . \tag{25.45}$$

The loading is uniform suggesting potentials of degree 2 in ϕ and degree 1 in ω, but as in the plane stress problem of Figure 5.3, the loading will generate bending moments and bending stresses, requiring potentials three orders higher than this. We therefore start with the potential functions

$$\phi = \frac{A_5}{8}(8z^5 - 40z^3r^2 + 15zr^4) + \frac{A_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{A_3}{2}(2z^3 - 3zr^2) + \frac{A_2}{2}(2z^2 - r^2) \tag{25.46}$$

$$\omega = \frac{B_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{B_3}{2}(2z^3 - 3zr^2) + \frac{B_2}{2}(2z^2 - r^2) + B_1 z \; , \tag{25.47}$$

where we have omitted those terms leading only to rigid-body displacements. Substituting into Table 21.1, the stresses are obtained as

$$\sigma_{rr} = \frac{5A_5}{2}(9r^2z - 4z^3) + \frac{3A_4}{2}(3r^2 - 4z^2) - 3A_3 z - A_2 - 2B_4(3+4\nu)z^3 + \frac{3B_4}{2}(3+8\nu)r^2 z - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - B_2(1+4\nu)z - 2B_1\nu$$

$$\sigma_{\theta\theta} = \frac{5A_5}{2}(3r^2z - 4z^3) + \frac{3A_4}{2}(r^2 - 4z^2) - 3A_3 z - A_2 - 2B_4(3+4\nu)z^3 + \frac{3B_4}{2}(1+8\nu)r^2 z - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - B_2(1-4\nu)z - 2B_1\nu$$

$$\sigma_{rz} = \frac{15A_5}{2}(r^3 - 4rz^2) - 12A_4 rz - 3A_3 r - 6B_4(1+2\nu)z^2 r$$

$$-\frac{3B_4}{2}(1-2\nu)r^3 - 6B_3\nu zr + B_2(1-2\nu)r \qquad (25.48)$$

$$\begin{aligned}\sigma_{zz} &= 10A_5(2z^3-3r^2z)+6A_4(2z^2-r^2)+6A_3z+2A_2\\ &+4B_4(1+2\nu)z^3+6B_4(1-2\nu)r^2z+6B_3\nu z^2+3B_3(1-\nu)r^2\\ &-2B_2(1-2\nu)z-2B_1(1-\nu)\;.\end{aligned}$$

Notice that σ_{zz} is even in r and σ_{rz} is odd in r. The strong boundary conditions (25.43, 25.44) therefore require us to equate coefficients of r^2 and r^0 in σ_{zz} and of r^3 and r^1 in σ_{rz} on each of the boundaries $z=\pm h/2$, leading to the equations

$$\begin{aligned}\frac{15A_5}{2}-\frac{3B_4}{2}(1-2\nu) &= 0\\ -\frac{15A_5h^2}{2}-6A_4h-3A_3-\frac{3B_4h^2}{2}(1+2\nu)-3B_3\nu h+B_2(1-2\nu) &= 0\\ -\frac{15A_5h^2}{2}+6A_4h-3A_3-\frac{3B_4h^2}{2}(1+2\nu)+3B_3\nu h+B_2(1-2\nu) &= 0\\ -15A_5h-6A_4+3B_4(1-2\nu)h+3B_3(1-\nu) &= 0\\ 15A_5h-6A_4+3B_4(1-2\nu)h-3B_3(1-\nu) &= 0\\ \frac{5A_5h^3}{2}+3A_4h^2+3A_3h+2A_2+\frac{B_4(1+2\nu)h^3}{2} &\\ +\frac{3B_3\nu h^2}{2}-B_2(1-2\nu)h-2B_1(1-\nu) &= -p_0\\ -\frac{5A_5h^3}{2}+3A_4h^2-3A_3h+2A_2-\frac{B_4(1+2\nu)h^3}{2} &\\ +\frac{3B_3\nu h^2}{2}+B_2(1-2\nu)h-2B_1(1-\nu) &= 0\;.\end{aligned}$$

Two additional equations are obtained by substituting (25.48) into the weak conditions (25.45) and evaluating the integrals, with the result

$$\begin{aligned}\frac{A_4(9a^2h-h^3)}{2}-A_2h-\frac{B_3(1+2\nu)h^3}{4}+3B_3\nu a^2h-2B_1\nu h &= 0\\ \frac{A_5(15a^2h^3-h^5)}{8}-\frac{A_3h^3}{4}-\frac{B_4(3+4\nu)h^5}{40} &\\ +\frac{B_4(3+8\nu)a^2h^3}{8}-\frac{B_2(1+4\nu)h^3}{12} &= 0\;.\end{aligned}$$

Solving these equations for the constants $A_5, A_4, A_3, A_2, B_4, B_3, B_2, B_1$ and substituting the resulting expressions into (25.48), we obtain the final stress field as

$$\sigma_{rr} = \frac{p_0}{h^3}\left\{\frac{3(3+\nu)(r^2-a^2)z}{4}+(2+\nu)\left(\frac{h^2z}{20}-z^3\right)\right\}$$

$$\sigma_{\theta\theta} = \frac{p_0}{h^3}\left\{\frac{3[(1+3\nu)r^2-(3+\nu)a^2]z}{4}+(2+\nu)\left(\frac{h^2z}{20}-z^3\right)\right\}$$

$$\sigma_{rz} = \frac{3p_0(h^2 - 4z^2)r}{4h^3} \tag{25.49}$$

$$\sigma_{zz} = \frac{p_0(4z^3 - 3h^2z - h^3)}{2h^3} \; .$$

The maximum tensile stress occurs at the point $0, -h/2$ and is

$$\sigma_{\max} = \frac{3(3+\nu)p_0a^2}{8h^2} + \frac{(2+\nu)p_0}{10} \; . \tag{25.50}$$

The elementary plate theory predicts the stress field[5]

$$\sigma_{rr} = \frac{3(3+\nu)p_0(r^2 - a^2)z}{4h^3}$$

$$\sigma_{\theta\theta} = \frac{3p_0\{(1+3\nu)r^2 - (3+\nu)a^2\}z}{4h^3}$$

$$\sigma_{rz} = \sigma_{zz} = 0 \; ,$$

with a maximum tensile stress

$$\sigma_{\max} = \frac{3(3+\nu)p_0a^2}{8h^2} \; . \tag{25.51}$$

Thus, the elementary theory provides a good approximation to the more exact solution as long as $h \ll a$. For example, it is in error by only about 0.2% if $h = a/10$.

25.3 Non-axisymmetric problems

Non-axisymmetric problems can be solved by expanding the loading as a Fourier series in θ and using the potentials of §24.7 in solutions A,B and E. All three solutions are generally required, since there is no equivalent of the partition into irrotational and torsional problems encountered for the axisymmetric case. The method is in other respects similar to that used for axisymmetric problems, but solutions tend to become algebraically complicated, simply because of the number of functions involved. For this reason we shall consider only a very simple example here. More complex examples are readily treated using Maple or Mathematica.

25.3.1 Cylindrical cantilever with an end load

We consider the case of the solid cylinder $0 < r < a, -L < z < L$, built in at $z = L$ and loaded only by a shear force F in the x-direction at $z = 0$. Thus, the curved surfaces are traction-free

[5] S.Timoshenko and S.Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 2nd edn., 1959, §16.

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \quad ; \quad r = a \; , \tag{25.52}$$

and on the end $z=0$ we have $\sigma_{zz}=0$ and

$$\int_0^a \int_0^{2\pi} (\sigma_{zr} \cos\theta - \sigma_{z\theta} \sin\theta) r d\theta dr = -F \; , \tag{25.53}$$

from equilibrium considerations.

As in §13.1, loading in the x-direction implies normal stress components varying with $\cos\theta$ and these will be obtained from potentials ϕ, ω with a $\cos\theta$ multiplier and from ψ with a $\sin\theta$ multiplier[6].

From elementary bending theory, we anticipate bending stresses σ_{zz} varying with $zr\cos\theta$, suggesting fourth order polynomials in ϕ, ψ and third order in ω. We therefore try

$$\begin{aligned}
\phi &= \frac{5A_4}{2}(4z^3 r - 3zr^3)\cos\theta + 3A_2 zr\cos\theta \\
\omega &= \frac{3B_3}{2}(4z^2 r - r^3)\cos\theta \\
\psi &= \frac{5E_4}{2}(4z^3 r - 3zr^3)\sin\theta \; ,
\end{aligned} \tag{25.54}$$

from equations (24.54). Notice that the problem is antisymmetric in z, so only odd powers of z are included in ϕ, ψ and even powers in ω. Also, we have omitted the second order term in ψ and the first order term in ω, since these correspond only to rigid-body displacements.

Substituting (25.54) into Table 21.1, we obtain

$$\begin{aligned}
\sigma_{rr} &= -3(15A_4 + (3+8\nu)B_3 + 10E_4) rz\cos\theta \\
\sigma_{r\theta} &= 3(5A_4 + B_3 + 10E_4) rz\sin\theta \\
\sigma_{\theta\theta} &= -3(5A_4 + (1+8\nu)B_3 - 10E_4) rz\cos\theta \\
\sigma_{zr} &= \frac{3}{2}\{5A_4(4z^2 - 3r^2) + 2A_2 + B_3(4(1+2\nu)z^2 + 3(1-2\nu)r^2) \\
&\qquad + 5E_4(4z^2 - r^2)\}\cos\theta \\
\sigma_{z\theta} &= -\frac{3}{2}\{5A_4(4z^2 - r^2) + 2A_2 + B_3(4(1+2\nu)z^2 + (1-2\nu)r^2) \\
&\qquad + 5E_4(4z^2 - 3r^2)\}\sin\theta \\
\sigma_{zz} &= 12(5A_4 - (1-2\nu)B_3) rz\cos\theta \; .
\end{aligned} \tag{25.55}$$

Substituting these expressions into (25.53) and evaluating the integral, we obtain the condition

$$\frac{3\pi a^2}{2}(-5A_4 a^2 + 2A_2 + B_3(1-2\nu)a^2 - 5E_4 a^2) = -F \tag{25.56}$$

[6] Notice that the expressions for normal stresses in solution E involve a derivative with respect to θ, so generally if cosine terms are required in A and B, sine terms will be required in E and *vice versa*.

and the strong boundary conditions (25.52) provide the three additional equations

$$\begin{aligned} -3(15A_4 + (3+8\nu)B_3 + 10E_4) &= 0 \\ 3(5A_4 + B_3 + 10E_4) &= 0 \\ -3(5A_4 + (1+8\nu)B_3 - 10E_4) &= 0 \; . \end{aligned} \tag{25.57}$$

Solving (25.56, 25.57) for the four constants A_4, A_2, B_3, E_4 and substituting the resulting expressions into (25.55), we obtain the final stress field

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{\theta\theta} = 0$$

$$\begin{aligned} \sigma_{zr} &= -\frac{F(3+2\nu)\left(a^2 - r^2\right)\cos\theta}{2\pi a^4(1+\nu)} \\ \sigma_{z\theta} &= \frac{F\left[(3+2\nu)a^2 - (1-2\nu)r^2\right]\sin\theta}{2\pi a^4(1+\nu)} \\ \sigma_{zz} &= -\frac{4Frz\cos\theta}{\pi a^4} \; . \end{aligned} \tag{25.58}$$

PROBLEMS

1. The solid cylinder $0 \le r < a,\; 0 < z < L$ is supported at the end $z = L$ and subjected to a uniform shear traction $\sigma_{rz} = S$ on the curved boundary $r = a$. The end $z = 0$ is traction-free.

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

2. The hollow cylinder $b < r < a,\; 0 < z < L$ is supported at the end $z = L$ and subjected to a uniform shear traction $\sigma_{rz} = S$ on the outer boundary $r = a$, the inner boundary $r = b$ and the end $z = 0$ being traction-free.

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

3. The solid cylinder $(0 \le r < a, -L < z < L)$ is subjected to the traction distribution

$$\begin{aligned} &\sigma_{rr} = -p_0\left(L^2 - z^2\right) \;\; ; \quad \sigma_{zr} = 0 \;\; ; \;\; r = a \; ; \\ &\sigma_{zr} = \sigma_{zz} = 0 \;\; ; \quad z = \pm L \; . \end{aligned}$$

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

4. The hollow cylinder $b < r < a, 0 < z < L$ is subjected to the traction distribution

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \ ; \ \ r = a$$
$$\sigma_{rz} = S \ ; \ \ \sigma_{rr} = \sigma_{r\theta} = 0 \ ; \ \ r = b$$
$$\sigma_{zr} = \sigma_{zz} = 0 \ ; \ \ \ z = 0 \ .$$

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

5. A heat exchanger tube $b<r<a$, $-L<z<L$ is subjected to the steady-state temperature distribution

$$T = T_0 \left(1 + \frac{z}{10a}\right) \quad ; \quad r = b$$
$$= -\frac{T_0 z}{10a} \quad ; \quad r = a \ .$$

Find the complete stress field in the tube if all the surfaces are traction-free. Find the location and a general expression for the magnitude of (i) the maximum tensile stress and (ii) the maximum Von Mises equivalent stress for the case where $b=0.8a$ and $L=20a$. Find the magnitude of these quantities when $T_0=50^o$C and the material is copper for which $E=121$ GPa, $\nu=0.33$ and $\alpha=17\times 10^{-6}$ per ^{o}C.

6. The circular plate $0\leq r<a$, $-h/2<z<h/2$ is simply supported at $r=a$ and loaded by a shear traction $\sigma_{zr}=Sr$ on the top surface $z=h/2$. All the other tractions on the plane surfaces $z=\pm h/2$ are zero. Find the stresses in the plate, using strong boundary conditions on $z=\pm h/2$ and weak boundary conditions on $r=a$.

7. The circular plate $0\leq r<a$, $-h/2<z<h/2$ rotates at constant speed Ω about the z-axis, all the surfaces being traction-free. Find the stresses in the plate, using strong boundary conditions on $z=\pm h/2$ and weak boundary conditions on $r=a$.

8. The solid cylinder $0\leq r<a$, $0<z<L$ with its axis horizontal is built in at $z=L$ and loaded only by its own weight (density ρ). Find the complete stress field in the cylinder.

9. The hollow cylinder $b<r<a$, $0<z<L$ with its axis horizontal is built in at $z=L$ and loaded only by its own weight (density ρ). Find the complete stress field in the cylinder.

10. The hollow cylinder $b\leq r<a$, $-L<z<L$ rotates at constant speed Ω about the x-axis, all the surfaces being traction-free. Find the complete stress field in the cylinder.

11. The solid cylinder $0\leq r<a$, $-L<z<L$ is accelerated from rest by two equal and opposite forces F applied in the positive and negative y-directions

respectively at the ends $z=\pm L$. All the other surface tractions are zero. Find the complete stress field in the cylinder at the instant when the forces are applied[7] (i.e. when the angular velocity is zero).

12. The circular plate $0 \leq r < a$, $-h/2 < z < h/2$ rotates at constant speed Ω about the x-axis, all the surfaces being traction-free. Find the stresses in the plate, using strong boundary conditions on $z=\pm h/2$ and weak boundary conditions on $r=a$. **Note:** This problem is not axisymmetric. It is the three-dimensional version of Problem 8.3.

[7] This is a three-dimensional version of the problem considered in §7.4.2.

26

PROBLEMS IN SPHERICAL COOÖRDINATES

The spherical harmonics of Chapter 24 can be used in combination with Tables 21.2, 22.2 to treat problems involving bodies whose boundaries are surfaces of the spherical coördinate system, for example the spherical surface $R=a$ or the conical surface $\beta=\beta_0$, where a, β_0 are constants. Examples of interest include the perturbation of an otherwise uniform stress field by a spherical hole or inclusion, the stresses in a solid sphere due to rotation about an axis and problems for a conical beam or shaft.

26.1 Solid and hollow spheres

The general solution for a solid sphere with prescribed surface tractions can be obtained using the spherical harmonics of equation (24.24) in solutions A,B and E. The addition of the singular harmonics (24.25) permits a general solution to the axisymmetric problem of the hollow sphere, but the corresponding non-axisymmetric solution cannot be obtained from equations (24.24, 24.25). To understand this, we recall from §20.4 that the elimination of one of the components of the Papkovich-Neuber vector function $\boldsymbol{\psi}$ is only generally possible when all straight lines drawn in a given direction cut the boundary of the body at only two points. This is clearly not the case for a body containing a spherical hole, since lines in any direction can always be chosen that cut the surface of the hole in two points and the external boundaries of the body at two additional points.

A similar difficulty arises for a body containing a plane crack, such as the penny-shaped crack problem of Chapter 31. There are various ways of getting around it. For example,

(i) We could decompose the loading into symmetric and antisymmetric components with respect to the plane $z=0$. Each problem would then be articulated for the hollow hemisphere with symmetric or antisymmetric boundary conditions on the cut surface. Since the hollow hemisphere satisfies the conditions of §20.4, a general solution can be obtained using

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_26, © Springer Science+Business Media B.V. 2010

solutions A,B and E, but generally we shall need to supplement the potentials (24.24, 24.25) with those of equations (24.37) that are singular on that half of the z-axis that does not pass through the hemisphere being analyzed.

(ii) We could include all four terms of the Papkovich-Neuber solution[1] or the six terms of the Galerkin vector of section §20.2. The former option is equivalent to the use of solutions A and B in combination with two additional solutions obtained by permuting suffices x, y, z in solution B.

(iii) In certain cases, we may be able to obtain particular non-axisymmetric solutions by superposition of axisymmetric ones. For example, the spherical hole in an otherwise uniform shear field can be solved by superposing the stress fields for the same body subject to equal and opposite tension and compression respectively about orthogonal axes.

26.1.1 The solid sphere in torsion

As a simple example, we consider the solid sphere $0 \leq R < a$ subjected to the torsional tractions

$$\sigma_{R\theta} = S_0 \sin(2\beta) \tag{26.1}$$

on the surface $R=a$, the remaining tractions $\sigma_{RR}, \sigma_{R\beta}$ being zero.

This is clearly an axisymmetric torsion problem, for which we require only solution E and the axisymmetric bounded potentials (24.14). Substitution of the first few potentials into Table 21.2 shows that the problem can be solved using the potential

$$\psi = E_3 R^3 P_3(\cos\beta) = \frac{E_3}{2}(5\cos^3\beta + 3\cos\beta) , \tag{26.2}$$

where E_3 is an unknown constant. The corresponding non-zero stress components in spherical polar coördinates are

$$\sigma_{R\theta} = 3E_3 R \sin\beta \cos\beta \ \ ; \ \ \sigma_{\beta\theta} = -3E_3 R \sin^2\beta , \tag{26.3}$$

so by inspection we see that the boundary condition (26.1) is satisfied by choosing $E_3 = 3S_0/2a$.

26.1.2 Spherical hole in a tensile field

A more interesting problem is that in which a state of uniaxial tension $\sigma_{zz} = S_0$ is perturbed by the presence of a small, traction-free, spherical hole of radius a — i.e.

[1] See for example, A.I.Lur'e, *Three-Dimensional Problems of the Theory of Elasticity*, Interscience, New York, 1964, Chapter 8.

$$\sigma_{zz} \to S \;\; ; \;\; R \to \infty \tag{26.4}$$

$$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr} \to 0 \;\; ; \;\; R \to \infty \tag{26.5}$$

$$\sigma_{RR} = \sigma_{R\theta} = \sigma_{R\beta} = 0 \;\; ; \;\; R = a \; . \tag{26.6}$$

This problem is the axisymmetric equivalent of the two-dimensional problem of §8.4.1 and as in that case it is convenient to start by determining a stress function description of the *unperturbed* uniaxial stress field. This is clearly axisymmetric and involves no torsion, so we use solutions A and B with the polynomials (24.27) in cylindrical polar coördinates. The stress components involve two differentiations in solution A and one in solution B, so we need the functions $R^2P_2(z/R)$ in ϕ and $R^1P_1(z/R)$ in ω to get a uniform state of stress. Writing

$$\phi = \frac{A_1}{2}(2z^2 - r^2) \;\; ; \;\; \omega = B_1 z \tag{26.7}$$

and substituting into Table 21.1, we obtain

$$\sigma_{rr} = \sigma_{\theta\theta} = -(A_1 + \nu B_1) \;\; ; \;\; \sigma_{zz} = 2A_1 - 2B_1(1-\nu) \;\; ; \;\; \sigma_{rz} = \sigma_{z\theta} = \sigma_{\theta r} = 0 \; . \tag{26.8}$$

The conditions (26.4, 26.5) are therefore satisfied by the choice

$$A_1 = \frac{S\nu}{(1+\nu)} \;\; ; \;\; B_1 = -\frac{S}{2(1+\nu)} \; . \tag{26.9}$$

To satisfy the traction-free condition at the hole surface (26.6), we now change to spherical polar coördinates and superpose singular potentials from (24.18) with the same Legendre polynomial form as those in the unperturbed solution — i.e.

$$\phi = A_1R^2P_2(\cos\beta) + A_2R^{-3}P_2(\cos\beta) + A_3R^{-1}P_0(\cos\beta) \tag{26.10}$$

$$\omega = B_1R^1P_1(\cos\beta) + B_2R^{-2}P_1(\cos\beta) \; . \tag{26.11}$$

Notice that in contrast to the two-dimensional problem of §8.4.1, we need to include also any lower order Legendre polynomial terms that have the same symmetry (in this case even orders in ϕ and odd orders in ω).

Substituting (26.10, 26.11) into Table 21.2, we obtain

$$\begin{aligned}\sigma_{RR} = & \left(3A_1 + \frac{18A_2}{R^5} - 2(1-2\nu)B_1 + \frac{2(5-\nu)B_2}{R^3}\right)\cos^2\beta \\ & -A_1 - \frac{6A_2}{R^5} + \frac{2A_3}{R^3} - 2\nu B_1 - \frac{2\nu B_2}{R^3}\end{aligned} \tag{26.12}$$

$$\sigma_{R\beta} = \left(-3A_1 + \frac{12A_2}{R^5} - 2(1-2\nu)B_1 + \frac{2(1+\nu)B_2}{R^3}\right)\sin\beta\cos\beta \tag{26.13}$$

and hence the boundary conditions (26.6) require that

$$3A_1 + \frac{18A_2}{a^5} - 2(1-2\nu)B_1 + \frac{2(5-\nu)B_2}{a^3} = 0$$
$$-A_1 - \frac{6A_2}{a^5} + \frac{2A_3}{a^3} - 2\nu B_1 - \frac{2\nu B_2}{a^3} = 0$$
$$-3A_1 + \frac{12A_2}{a^5} - 2(1-2\nu)B_1 + \frac{2(1+\nu)B_2}{a^3} = 0 \; .$$

Solving these equations and using (26.9), we obtain

$$A_2 = \frac{Sa^5}{(7-5\nu)} \;\; ; \;\; A_3 = \frac{Sa^3(6-5\nu)}{2(7-5\nu)} \;\; ; \;\; B_2 = \frac{-5Sa^3}{2(7-5\nu)} \qquad (26.14)$$

and the final stress field is

$$\sigma_{RR} = S\cos^2\beta + \frac{S}{(7-5\nu)}\left(\frac{a^3}{R^3}(6-5(5-\nu)\cos^2\beta) + \frac{6a^5}{R^5}(3\cos^2\beta - 1)\right)$$
$$\sigma_{\theta\theta} = \frac{3S}{2(7-5\nu)}\left(\frac{a^3}{R^3}(5\nu - 2 + 5(1-2\nu)\cos^2\beta) + \frac{a^5}{R^5}(1-5\cos^2\beta)\right)$$
$$\sigma_{\beta\beta} = S\sin^2\beta + \frac{3S}{2(7-5\nu)}\left(\frac{a^3}{R^3}(4-5\nu+5(1-2\nu)\cos^2\beta) - \frac{9a^5}{R^5}(3\cos^2\beta - 1)\right)$$
$$\sigma_{R\beta} = S\left\{-1 + \frac{1}{(7-5\nu)}\left(-\frac{5a^3(1+\nu)}{R^3} + \frac{12a^5}{R^5}\right)\right\}\sin\beta\cos\beta \; . \qquad (26.15)$$

The maximum tensile stress is $\sigma_{\beta\beta}$ at $\beta = \pi/2$, $R = a$ and is

$$\sigma_{\max} = \frac{3S(9-5\nu)}{2(7-5\nu)} \; . \qquad (26.16)$$

The stress concentration factor is $13/6 = 2.17$ for materials with $\nu = 0.5$ and decreases only slightly for other values in the practical range.

26.2 Conical bars

The equation $\beta = \beta_0$, where β_0 is a constant, define conical surfaces in the spherical polar coördinate system and the potentials developed in Chapter 24 can therefore be used to solve various problems for the solid or hollow conical bar.

Substitution of a potential of the form $R^n f(\theta, \beta)$ into Table 21.2 will yield stress components varying with R^{n-2} in solutions A and E or with R^{n-1} in solution B. The bounded potentials of equations (24.14, 24.24) therefore provide a general solution of the problem of a solid conical bar with polynomial tractions on the curved surfaces.

If the curved surfaces are traction-free, but the bar transmits a force or moment resultant, the stresses will increase without limit as the vertex (the origin) is approached, so in these cases we should expect to need singular potentials even for the solid bar.

For the hollow conical bar, we need additional potentials with the same power law variation with R and these are provided by the Q-series of equations (24.34). Notice that the singularity of these solutions on the z-axis is acceptable for the hollow bar, since the axis is not then contained within the body.

26.2.1 Conical bar transmitting an axial force

We consider the solid conical bar $0 \leq \beta < \beta_0$ transmitting a tensile axial force F, the curved surfaces $\beta = \beta_0$ being traction-free. We shall impose the strong (homogeneous) conditions

$$\sigma_{\beta R} = \sigma_{\beta\theta} = \sigma_{\beta\beta} = 0 \quad ; \quad \beta = \beta_0 \tag{26.17}$$

on the curved surfaces and weak conditions on the ends. If the bar has plane ends and occupies the region $a < z < b$, we have

$$\int_0^{2\pi} \int_0^{b \tan \beta_0} \sigma_{zz}(r, \theta, b) r dr d\theta = F \tag{26.18}$$

and a similar condition[2] at $z = a$. Alternatively, we could consider the equivalent problem of a bar with spherical ends $R_a < R < R_b$, for which the weak boundary condition takes the form[3]

$$\int_0^{2\pi} \int_0^{\beta_0} \{\sigma_{RR}(R_b, \theta, \beta) \cos \beta - \sigma_{R\beta}(R_b, \theta, \beta) \sin \beta\} R_b^2 \sin \beta d\beta d\theta = F \; . \tag{26.19}$$

The cross-sectional area of the ends of the conical bar increase with R^2 (the radius increases with R), so the stresses must decay with R^{-2} as in the point force solutions of §23.1.2 and §23.2.1. This suggests the use of the source solution $1/R$ in solution B and the potential $\ln(R+z)$ in solution A. Notice that this problem is axisymmetric and there is no torsion, so solution E will not be required. Also, notice that the bar occupies only the region $z > 0$ in cylindrical polar coördinates, so the singularities on the negative z-axis implied by $\ln(R+z)$ are acceptable.

Writing these potentials in spherical polar coördinates and introducing arbitrary multipliers A, B, we have

---

[2] As in earlier problems, the weak condition need only be imposed at one end, since the condition at the other end will be guaranteed by the fact that the stress field satisfies the equilibrium condition everywhere.

[3] To derive this result, note that the element of area on a spherical surface is $(Rd\beta)(rd\theta) = R^2 \sin \beta d\beta d\theta$ and the forces on this element due to the stress components $\sigma_{RR}, \sigma_{R\beta}$ must be resolved into the axial direction.

$$\phi = A\{\ln(R) + \ln(1+\cos(\beta))\} \;\; ; \;\; \omega = \frac{B}{R} \tag{26.20}$$

and substitution into Table 21.2 gives the non-zero stress components

$$\sigma_{RR} = -\frac{A}{R^2} + \frac{2(2-\nu)B\cos\beta}{R^2} \tag{26.21}$$

$$\sigma_{\theta\theta} = \frac{A}{R^2(1+\cos\beta)} - \frac{(1-2\nu)B\cos\beta}{R^2} \tag{26.22}$$

$$\sigma_{\beta\beta} = \frac{A\cos\beta}{R^2(1+\cos\beta)} - \frac{(1-2\nu)B\cos\beta}{R^2} \tag{26.23}$$

$$\sigma_{\beta R} = \frac{A\sin\beta}{R^2(1+\cos\beta)} - \frac{(1-2\nu)B\sin\beta}{R^2} \; . \tag{26.24}$$

The strong conditions (26.17) will therefore be satisfied as long as

$$\frac{A}{(1+\cos\beta_0)} = (1-2\nu)B \tag{26.25}$$

and the weak condition (26.19) after substitution for the stress components and evaluation of the integral gives

$$2\pi(1-\cos\beta_0)\left\{A + \left(\cos^2\beta_0 + \cos\beta_0 + 2 - 2\nu\right)B\right\} = F \; . \tag{26.26}$$

Solving equations (26.25, 26.26) for A, B, we obtain

$$A = \frac{F(1-2\nu)(1+\cos\beta_0)}{2\pi(1-\cos\beta_0)\left(1+2\nu\cos\beta_0+\cos^2\beta_0\right)}$$

$$B = \frac{F}{2\pi(1-\cos\beta_0)\left(1+2\nu\cos\beta_0+\cos^2\beta_0\right)} \tag{26.27}$$

and the stress components are then recovered by substituting for A, B into equations (26.21–26.24).

Elementary Mechanics of Materials arguments predict that the axial stress σ_{zz} would be uniform and equal to $F/\pi b^2$ at $z = b$. We can assess the approximation involved in this prediction by recasting the present solution in cylindrical coordinates — i.e. writing

$$\phi = A\ln\left(\sqrt{r^2+z^2} + z\right) \;\; ; \;\; \omega = \frac{B}{\sqrt{r^2+z^2}} \tag{26.28}$$

with A, B given by (26.27) and substituting into the cylindrical polar form of solutions A and B (Table 21.1). This is done at the end of the Maple file 'conetension'. The normalized stresses on a cross-sectional plane are shown in Figure 26.1 for various cone angles β_0. The axial stress is largest at the axis $r = 0$, but remains fairly uniform across the cross-section for $\beta_0 < 10^o$. Significant non-uniformity occurs for larger cone angles. For example, even for a 30^o cone, the maximum stress exceeds the elementary prediction by 46%.

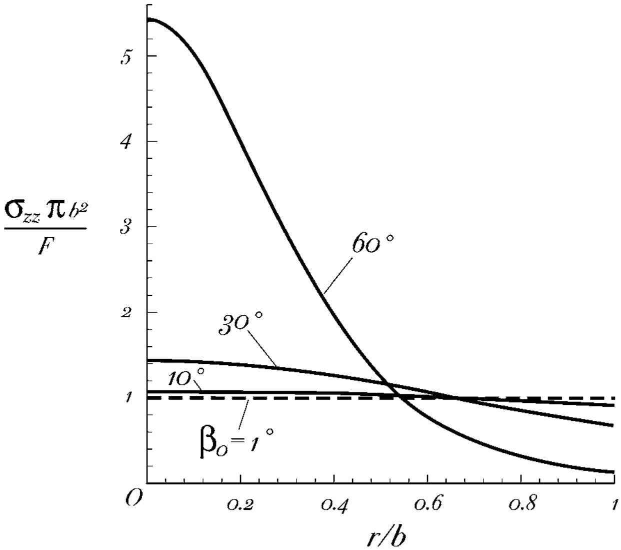

Figure 26.1: Distribution of the axial stress σ_{zz} for the cone loaded in tension.

In the special case where $\beta_0 = \pi/2$, the cone becomes a half space and we recover the Boussinesq solution of §23.2.1, albeit expressed in spherical polar coördinates. For values of β in the range $\pi/2 < \beta_0 < \pi$, the solution defines the problem of a large body with a conical notch loaded by a concentrated force at the vertex of the notch.

The related problem of the *hollow* conical bar $\beta_1 < \beta < \beta_2$ can be treated in the same way by supplementing ϕ in solution A by the term $A_2\{\ln(R) + \ln(1 - \cos(\beta))\}$. This is similar to the term in equation (26.20), except that the singularities are now on the positive z-axis, which of course is acceptable for the hollow bar, since this axis does not now lie inside the material of the bar[4].

26.2.2 Inhomogeneous problems

If polynomial tractions are applied to the curved surfaces of the cone, but there are no concentrated loads at the vertex, the stresses will vary with the same power of R as the tractions and the order of the corresponding potential is easily determined[5] by examining the derivatives in Table 21.2.

[4] See Problem 26.13.

[5] As in Chapter 8, some of the lower order bounded potentials correspond merely to rigid-body displacements and involve no stresses. In such cases, we need additional special potentials which are obtained from the logarithmic series of equations (24.37, 23.23), see for example Problem 26.17. Similar considerations apply to the cone loaded by low-order non-axisymmetric polynomial tractions. These special

Example

As a simple example, we consider the hollow cone $\beta_1 < \beta < \beta_2$ loaded by quadratic torsional shear tractions on the outer surface, the othere tractions being zero — i.e.

$$\sigma_{\beta\theta} = SR^2 \;\; ; \;\; \sigma_{\beta R} = \sigma_{\beta\beta} = 0 \;\; ; \;\; \beta = \beta_2$$
$$\sigma_{\beta\theta} = \sigma_{\beta R} = \sigma_{\beta\beta} = 0 \;\; ; \;\; \beta = \beta_1 \; . \tag{26.29}$$

This is clearly an axisymmetric torsion problem which requires solution E only. The tractions vary with R^2 and hence Ψ must vary with R^4, from Table 21.2. Since the cone is hollow, we require both P and Q series Legendre functions of this order, giving

$$\Psi = C_1 R^4 P_4(\cos\beta) + C_2 R^4 Q_4(\cos\beta) \; , \tag{26.30}$$

where P, Q are given by equations (24.11, 24.32) respectively. Substituting into Table 21.2 yields the non-zero stress components

$$\sigma_{\theta\beta} = -\frac{R^2}{2}\left\{30C_1 \sin^2\beta\cos\beta + 15C_2\sin^2\beta\cos\beta\ln\left(\frac{1-\cos\beta}{1+\cos\beta}\right) - \frac{C_2\left(30\cos^4\beta - 50\cos^2\beta + 16\right)}{\sin^2\beta}\right\} \tag{26.31}$$

$$\sigma_{R\theta} = \frac{R^2\sin\beta}{2}\left\{6C_1\left(5\cos^2\beta - 1\right) + 3C_2\left(5\cos^2\beta - 1\right)\ln\left(\frac{1-\cos\beta}{1+\cos\beta}\right) + \frac{C_2\left(30\cos^3\beta - 26\cos\beta\right)}{\sin^2\beta}\right\} \tag{26.32}$$

and the boundary conditions (26.29) will therefore be satisfied if

$$6C_1\left(5\cos^2\beta_1 - 1\right) + 3C_2\left(5\cos^2\beta_1 - 1\right)\ln\left(\frac{1-\cos\beta_1}{1+\cos\beta_1}\right) + \frac{C_2\left(30\cos^3\beta_1 - 26\cos\beta_1\right)}{\sin^2\beta_1} = 0 \tag{26.33}$$

$$6C_1\left(5\cos^2\beta_2 - 1\right) + 3C_2\left(5\cos^2\beta_2 - 1\right)\ln\left(\frac{1-\cos\beta_2}{1+\cos\beta_2}\right) + \frac{C_2\left(30\cos^3\beta_2 - 26\cos\beta_2\right)}{\sin^2\beta_2} = \frac{2S}{\sin\beta_2} \; . \tag{26.34}$$

Solution of these equations for C_1, C_2 and substitution of the result into (26.31, 26.32) completes the solution of the problem.

---

solutions are related to the two-dimensional corner fields described in §11.1.2, §11.1.3 and are needed because a state of uniform traction on the conical surface is not consistent with a locally homogeneous state of stress near the apex of the cone.

26.2.3 Non-axisymmetric problems

Non-axisymmetric problems for the solid or hollow cone can be solved using the potentials[6]

$$R^n P_n^m(\cos\beta)\left\{\begin{array}{l}\cos(m\theta)\\ \sin(m\theta)\end{array}\right. \;\; ; \;\; R^n Q_n^m(\cos\beta)\left\{\begin{array}{l}\cos(m\theta)\\ \sin(m\theta)\end{array}\right.$$

$$R^{-n-1} P_n^m(\cos\beta)\left\{\begin{array}{l}\cos(m\theta)\\ \sin(m\theta)\end{array}\right. \;\; ; \;\; R^{-n-1} Q_n^m(\cos\beta)\left\{\begin{array}{l}\cos(m\theta)\\ \sin(m\theta)\end{array}\right. .$$

As a simple example, we consider the solid cone $0 \leq \beta < \beta_0$ loaded only by a concentrated moment M about the x-axis at the vertex, the curved surfaces $\beta = \beta_0$ being traction free. There is no length scale, so the problem is self-similar and both the stresses and the potential functions must be of separated-variable form.

The applied moment leads to the transmission of a bending moment M along the cone and we anticipate stress components varying with y and hence with $\sin\theta$. The stress components on a given spherical surface will have moment arms proportional to the radius R and the area of this surface is proportional to R^2, so if the same moment is to be transmitted across all such surfaces, the stresses must decay with R^{-3}. This implies the use of a potential proportional to R^{-2} in solution B and to R^{-1} in solutions A and E. We therefore choose

$$\phi = AR^{-1}P_0^1(\cos\beta)\sin\theta = \frac{A\sin\beta\sin\theta}{R(1+\cos\beta)} \tag{26.35}$$

$$\omega = BR^{-2}P_1^1(\cos\beta)\sin\theta = \frac{B\sin\beta\sin\theta}{R^2} \tag{26.36}$$

$$\Psi = CR^{-1}P_0^1(\cos\beta)\cos\theta = \frac{C\sin\beta\cos\theta}{R(1+\cos\beta)} \; , \tag{26.37}$$

where we have used (24.22) to evaluate P_0^1.

Substituting these expressions into Table 21.2, we obtain the stress components

$$\begin{aligned}
\sigma_{RR} &= \frac{2\{A+2C+(5-\nu)B\cos\beta(1+\cos\beta)\}\sin\beta\sin\theta}{R^3(1+\cos\beta)} \\
\sigma_{\theta\theta} &= -\frac{\left\{(A+2C)(2+\cos\beta)+3(1-2\nu)B\cos\beta(1+\cos\beta)^2\right\}\sin\beta\sin\theta}{R^3(1+\cos\beta)^2} \\
\sigma_{\beta\beta} &= -\frac{\left\{A+2C+(1-2\nu)B(1+\cos\beta)^2\right\}\cos\beta\sin\beta\sin\theta}{R^3(1+\cos\beta)^2} \\
\sigma_{\theta\beta} &= \frac{\left\{A+2C+(1-2\nu)B(1+\cos\beta)^2\right\}\sin\beta\cos\theta}{R^3(1+\cos\beta)^2}
\end{aligned} \tag{26.38}$$

[6] but see footnote 5 on Page 411.

$$\sigma_{\beta R} = -\frac{\left\{2A + C(1-3\cos\beta) + 2B(1+\cos\beta)\left(1-2\nu + (1+\nu)\cos^2\beta\right)\right\}\sin\theta}{R^3(1+\cos\beta)}$$

$$\sigma_{R\theta} = -\frac{\left\{2A + C\left(4 - 3\cos\beta - 2\cos^2\beta\right) + 2(2-\nu)B\cos\beta(1+\cos\beta)\right\}\cos\theta}{R^3(1+\cos\beta)}$$

and hence the traction free boundary conditions $\sigma_{\beta R}=\sigma_{\beta\theta}=\sigma_{\beta\beta}=0$ on $\beta=\beta_0$ will be satisfied provided

$$\begin{aligned} 2A + C(1-3\cos\beta_0) + 2B(1+\cos\beta_0)\left(1-2\nu+(1+\nu)\cos^2\beta_0\right) &= 0 \\ A + 2C + (1-2\nu)B(1+\cos\beta_0)^2 &= 0 \\ A + 2C + (1-2\nu)B(1+\cos\beta_0)^2 &= 0\ . \end{aligned} \tag{26.39}$$

These three homogeneous equations have a non-trivial solution, since the last two equations are identical. The resulting free constant is determined from the equilibrium condition

$$\int_0^{\beta_0}\int_0^{2\pi} (\sigma_{R\theta}\cos\theta\cos\beta + \sigma_{R\beta}\sin\theta)R^3\sin\beta d\theta d\beta = M\ , \tag{26.40}$$

which states that the resultant of the tractions across any spherical surface of radius R is a moment M about the x-axis. Substituting for the stress components and evaluating the double integral, we obtain

$$\begin{aligned} &\pi\left(-2A(1-\cos\beta_0) + C\cos\beta_0\sin^2\beta_0\right. \\ &\qquad \left.+2B\left\{\cos^3\beta_0 + (1-2\nu)\cos\beta_0 - 2(1-\nu)\right\}\right) = M\ . \end{aligned} \tag{26.41}$$

The final stress field is obtained by solving (26.39, 26.41) for A, B, C and substituting into (26.38). These operations are performed in the Maple and Mathematica files ‘conebending’.

PROBLEMS

1. A traction-free solid sphere of radius a and density ρ rotates at constant angular velocity Ω about the z-axis[7]. Find the complete stress field in the sphere. Find the location and magnitude of (i) the maximum tensile stress and (ii) the maximum Von Mises stress and also (iii) the increase in the equatorial diameter and (iv) the decrease in the polar diameter due to the rotation.

2. A traction-free hollow sphere of inner radius b, outer radius a and density ρ rotates at constant angular velocity Ω about the z-axis. Find the complete

[7] This problem is of some interest in connection with the stresses and displacements in the earth's crust due to diurnal rotation.

stress field in the sphere. Find also the increase in the outer equatorial diameter and the decrease in the outer polar diameter due to the rotation.

3. A hollow sphere of inner radius b and outer radius a is filled with a gas at pressure p. Find the complete stress field in the sphere and the increase in the contained volume due to the internal pressure..

4. A state of uniaxial tension, $\sigma_{zz} = S$ in a large body is perturbed by the presence of a rigid spherical inclusion in the region $0 \le R < a$. The inclusion is perfectly bonded to the surrounding material, so that the boundary condition at the interface in a suitable frame of reference is one of zero displacement $(u_R = u_\theta = u_\beta = 0\ ;\ \ R = a)$.

Develop a solution for the stress field and find the stress concentration factor.

5. A rigid spherical inclusion of radius a in a large elastic body is subjected to a force F in the z-direction.

Find the stress field if the inclusion is perfectly bonded to the elastic body at $R = a$ and the stresses tend to zero as $R \to \infty$.

6. Investigate the steady-state thermal stress field due to the disturbance of an otherwise uniform heat flux, $q_z = Q$, by an insulated spherical hole of radius a.

You must first determine the perturbed temperature field, noting (i) that the temperature T is harmonic and hence can be represented in terms of spherical harmonics and (ii) that the insulated boundary condition implies that $\partial T / \partial R = 0$ at $R = a$.

The thermal stress field can now be obtained using Solution T supplemented by appropriate functions in Solutions A,B, chosen so as to satisfy the traction-free condition at the hole.

7. A state of uniaxial tension, $\sigma_{zz} = S$ in a large body (the matrix) of elastic properties μ_1, ν_1 is perturbed by the presence of a spherical inclusion in the region $0 \le R < a$ with elastic properties μ_2, ν_2. The inclusion is perfectly bonded to the matrix.

Develop a solution for the stress field and find the two stress concentration factors

$$K_1 = \frac{\sigma_1^{\max}}{S}\ \ ;\ \ K_2 = \frac{\sigma_2^{\max}}{S}\ ,$$

where $\sigma_1^{\max}, \sigma_2^{\max}$ are respectively the maximum tensile stresses in the matrix and in the inclusion.

Plot figures showing how K_1, K_2 vary with the modulus ratio μ_1/μ_2 and with the two Poisson's ratios.

8. A sphere of radius a and density ρ rests on a rigid plane horizontal surface. Assuming that the support reaction consists of a concentrated vertical force equal to the weight of the sphere, find the complete stress field in the sphere.

Hint: Start with the solution for a point force acting on the surface of a half-space (the Boussinesq solution of §23.2.1) and determine the tractions on an imaginary spherical surface passing through the point of application of the force. Comparison with the corresponding cylindrical problems 12.1, 12.2, 12.3, would lead one to expect that these tractions could be removed by a finite series of spherical harmonics, but this is not the case. In fact, the tractions are still weakly (logarithmically) singular, though the dominant $(1/R)$ singularity associated with the point force has been removed. A truncated series of spherical harmonics will give an approximate solution, but the series will be rather slowly convergent. A more rapidly convergent solution can be obtained by first removing the logarithmic singularity.

9. A sphere of radius a is loaded by two equal and opposite point forces F applied at the poles $\beta=0, 2\pi$). Find the stress field in the sphere. (Read the 'hint' for Problem 26.8).

10. A cylindrical bar of radius b transmits a torque T. It also contains a small spherical hole of radius a $(\ll b)$ on the axis. Find the perturbation in the elementary torsional stress field due to the hole.

11. A vertical conical tower, $0 < \beta < \beta_0$ of height h and density ρ is loaded only by its own weight. Find the stress field in the tower.

12. A vertical hollow conical tower, $\beta_1 < \beta < \beta_2$ of height h and density ρ is loaded only by its own weight. Find the stress field in the tower. Are there any conditions (values of β_1, β_2) for which tensile stresses are developed in the tower? If so, when and where?

13. The hollow conical bar $\beta_1 < \beta < \beta_2$ is loaded by a tensile axial force F, the curved surfaces $\beta = \beta_1, \beta_2$ being traction-free. Use the stress functions

$$\phi = A_1\{\ln(R) + \ln(1 + \cos(\beta))\} + A_2\{\ln(R) + \ln(1 - \cos(\beta))\} \; ; \; \omega = \frac{B}{R}$$

to obtain a solution for the stress field.

14. A *horizontal* solid conical bar, $0 < \beta < \beta_0$ of length L and density ρ is built in at $z = L$ and loaded only by its own weight. Find the stress field in the bar and the vertical displacement of the vertex.

15. Figure 26.2 shows the cross-section of a concrete dam which may be approximated as a sector of a truncated hollow cone. The dam is loaded by hydrostatic pressure (water density ρ) and by self weight (concrete density

$\rho_c = 2.3\rho$). Assuming the stress field is axisymmetric — i.e. that the tractions at the side support are the same as those that would occur on the equivalent plane in a complete truncated hollow cone loaded all around the circumference — determine the stress field in the dam using strong conditions on the curved surfaces and weak (traction-free) conditions on the truncated end.

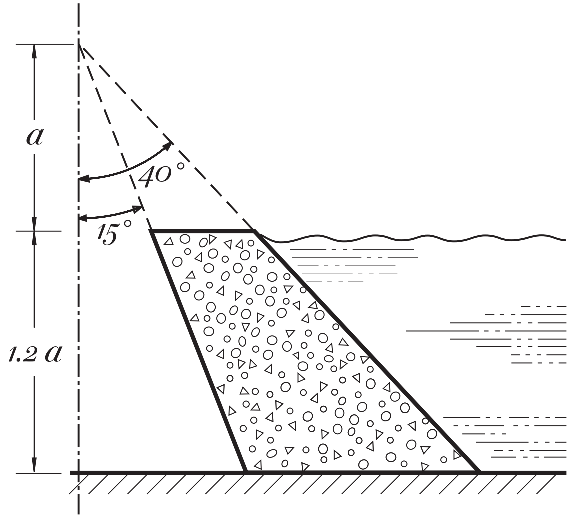

Figure 26.2: Truncated hollow conical dam.

16. Find the stress field in the solid conical shaft $0<\beta<\beta_0$, loaded only by a torque T at the vertex.

17. The solid conical bar $0 < \beta < \beta_0$ is loaded by uniform torsional shear tractions $\sigma_{\beta\theta} = S$ at $\beta = \beta_0$, all other traction components being zero. Find the stress field in the bar. **Note:** The 'obvious' stress function for this problem is $R^2 P_2(\cos\beta)$, but it corresponds only to rigid-body rotation and gives zero stress components. As in Chapter 8, we need a special stress function for this case, which is in fact ϕ_{-3} from equation (23.23). You will need to write $z = R\cos\beta$ to express this function in spherical polar coördinates. It will contain the term $R^2 \ln(R)$, but as in §10.3, the same degeneracy that causes the elementary potential to give zero stresses will cause there to be no logarithmic terms in the stress components.

18. Use spherical polar coördinates to find the stresses in the half-space $z>0$ due to a concentrated tangential force F in the x-direction, applied at the origin. **Note:** The half-space is equivalent to the cone $0 \leq \beta < \pi/2$. This problem is the spherical polar equivalent of Problem 23.1.

19. A solid sphere of radius a is loaded only by self-gravitation — i.e. each particle of the sphere is attracted to each other particle by a force $\gamma m_1 m_2/R^2$, where m_1, m_2 are the masses of the two particles, R is the distance betwen them and γ is the universal gravitational constant. Find the stress field in the sphere and in particular the location and magnitude of the maximum compressive stress and the maximum shear stress.

Assuming that the solution is to be applied to the earth, express the results in terms of the gravitational acceleration g at the surface.

20. The solid cone $0<\beta<\beta_0$ is loaded by a force in the x-direction applied at the vertex. The surface of the cone is otherwise traction-free. Find the stress field in the cone. Notice that Problem 18 is a special case of this solution for $\beta_0=\pi/2$.

21. The surface of a solid sphere of radius a is traction-free, but is subjected to the steady-state heat flux

$$q_R = q_0(3\cos(2\beta) + 1) \; .$$

In other words, it is heated near the equator and cooled near the poles.

Find the complete stress and displacement field in the sphere. Also, find the difference between the largest and smallest external diameter of the thermally distorted sphere.

27

AXISYMMETRIC TORSION

We have already remarked in §25.1 that the use of axisymmetric harmonic potential functions in Solution E provides a general solution of the problem of an axisymmetric body loaded in torsion. In this case, the only non-zero stress and displacement components are

$$\mu u_\theta = -\frac{\partial \Psi}{\partial r} \; ; \; \sigma_{\theta r} = \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{\partial^2 \Psi}{\partial r^2} \; ; \; \sigma_{\theta z} = -\frac{\partial^2 \Psi}{\partial r \partial z} \; . \tag{27.1}$$

A more convenient formulation of this problem can be obtained by considering the relationship between the stress components and the torque transmitted through the body. We first note that the two non-zero stress components (27.1) both act on the cross-sectional θ-plane, on which they constitute a vector field $\boldsymbol{\sigma_\theta}$ of magnitude

$$|\boldsymbol{\sigma_\theta}| = \sqrt{\sigma_{\theta r}^2 + \sigma_{\theta z}^2} \; . \tag{27.2}$$

This vector field is illustrated in Figure 27.1. It follows that the stress component $\sigma_{\theta n}$ normal to any of the 'flow lines' in this figure is zero and hence the complementary shear stress $\sigma_{n\theta} = 0$. Thus, the flow lines define traction-free surfaces in r, θ, z space.

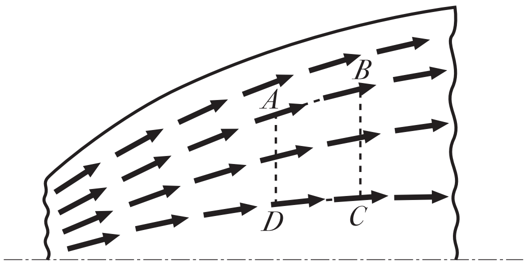

Figure 27.1: Vector representation of the shear stresses on the cross-sectional plane.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_27, © Springer Science+Business Media B.V. 2010

Consider now the equilibrium of the axisymmetric body defined by $ABCD$ in Figure 27.1. The surfaces AB and CD are flow lines and are therefore traction-free, so we must conclude that the resultant torque transmitted across the circular surface AD is equal to that transmitted across BC. Thus the flow lines define surfaces enclosing axisymmetric bodies that transmit constant torque along the axis. Conversely, if we use the torque $T(r,z)$ transmitted across the disk of radius r as our fundamental variable, a traction-free surface will then correspond to a line in the cross-section along which T is constant. This has many advantages, not the least of which is that we can make contour plots of candidate functions $T(r,z)$ and then adjust parameters in order to approximate the shape of the required body by one of the contours.

27.1 The transmitted torque

The torque transmitted across a disk of radius r is

$$T(r,z) = 2\pi \int_0^r r^2 \sigma_{z\theta} dr \; . \tag{27.3}$$

To avoid the factor of 2π, we shall define a stress function ϕ such that

$$T(r,z) = 2\pi\phi(r,z) \; . \tag{27.4}$$

Differentiation of (27.3) with respect to r then yields

$$\sigma_{z\theta} = \frac{1}{2\pi r^2}\frac{\partial T}{\partial r} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} \; . \tag{27.5}$$

The torque transmitted into a disk of thickness δz and radius r across the curved surface is

$$\delta T = -2\pi r^2 \sigma_{r\theta}\delta z \tag{27.6}$$

and hence

$$\sigma_{r\theta} = -\frac{1}{2\pi r^2}\frac{\partial T}{\partial z} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} \; . \tag{27.7}$$

27.2 The governing equation

Equations (27.5, 27.7) and (27.1) are alternative representations of the stress field and hence

$$-\frac{1}{r^2}\frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{\partial^2 \Psi}{\partial r^2} \;\; ; \;\; \frac{1}{r^2}\frac{\partial \phi}{\partial r} = -\frac{\partial^2 \Psi}{\partial r \partial z} \; . \tag{27.8}$$

Multiplying through by r^2 and differentiating these expressions, we obtain

$$\frac{\partial^2 \phi}{\partial z^2} = -r\frac{\partial^2 \Psi}{\partial r \partial z} + r^2\frac{\partial^3 \Psi}{\partial r^2 \partial z} \;\; ; \;\; \frac{\partial^2 \phi}{\partial r^2} = -2r\frac{\partial^2 \Psi}{\partial r \partial z} - r^2\frac{\partial^3 \Psi}{\partial r^2 \partial z} \tag{27.9}$$

and hence

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} = -3\frac{\partial^2 \Psi}{\partial r \partial z} = \frac{3}{r}\frac{\partial \phi}{\partial r} , \tag{27.10}$$

from (27.8). Thus, the stress function ϕ must satisfy the equation

$$\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 . \tag{27.11}$$

The axisymmetric torsion problem is therefore reduced to the determination of a function ϕ satisfying (27.11), such that ϕ is constant along any traction-free boundary. The stress field is then given by

$$\sigma_{r\theta} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} \;\; ; \;\; \sigma_{z\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} . \tag{27.12}$$

Recall also that the transmitted torque is related to ϕ through equation (27.4).

27.3 Solution of the governing equation

Equation (27.11) is similar in form to the Laplace equation (24.38) and solutions can be obtained in terms of Legendre functions, using a technique similar to that used in Chapter 24. A relationship with harmonic functions can be established by defining a new function f through the equation

$$\phi(r, z) = r^2 f(r, z) . \tag{27.13}$$

Substitution into (27.11) then shows that f must satisfy the equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2} + \frac{\partial^2 f}{\partial z^2} = 0 , \tag{27.14}$$

which is (24.44) with $m = 2$. It follows that (27.13) will define a function satisfying (27.11) if the function

$$f(r, z)\cos(2\theta)$$

is harmonic. In particular, we conclude that the functions

$$r^2 R^n P_n^2(\cos\beta) \;\; ; \;\; r^2 R^n Q_n^2(\cos\beta) \;\; ; \;\; r^2 R^{-n-1} P_n^2(\cos\beta) \;\; ; \;\; r^2 R^{-n-1} Q_n^2(\cos\beta) \tag{27.15}$$

satisfy (27.11). More generally, we can use equation (24.46) recursively to show that the function

$$f(r, z) = \frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial g}{\partial r}\right) \tag{27.16}$$

will satisfy (27.14) if $g(r, z)$ is an axisymmetric harmonic function and hence the function

$$\phi = r^2 \frac{\partial^2 g}{\partial r^2} - r\frac{\partial g}{\partial r} = r^3 \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial g}{\partial r}\right) \tag{27.17}$$

satisfies (27.11). This can also be verified directly by substitution (see Problem 27.1). Any axisymmetric harmonic function g can be used to generate solutions in this way, including for example those defined by equations (24.14, 24.18, 24.27, 24.34, 24.42).

The functions

$$R^n P_{n-2}^2(\cos\beta) \;\; ; \;\; R^n Q_{n-2}^2(\cos\beta) \;\; ; \;\; R^{3-n} P_{n-2}^2(\cos\beta) \;\; ; \;\; R^{3-n} Q_{n-2}^2(\cos\beta) \tag{27.18}$$

define solutions to equation (27.11) for $n \geq 4$. The P-series functions define the bounded polynomials

$$\begin{aligned}
\phi_4 &= R^4 \sin^4\beta = r^4 \\
\phi_5 &= R^5 \sin^4\beta\cos\beta = r^4 z \\
\phi_6 &= R^6\left(7\cos^2\beta - 1\right)\sin^4\beta = r^4\left(6z^2 - r^2\right) \\
\phi_7 &= R^7\left(3\cos^2\beta - 1\right)\sin^4\beta\cos\beta = r^4 z\left(2z^2 - r^2\right)
\end{aligned} \tag{27.19}$$

and a corresponding set of functions

$$\begin{aligned}
\phi_{-1} &= R^{-1}\sin^4\beta = \frac{r^4}{R^5} \\
\phi_{-2} &= R^{-2}\sin^4\beta\cos\beta = \frac{r^4 z}{R^7} \\
\phi_{-3} &= R^{-3} = \frac{r^4 z}{R^7}\left(7\cos^2\beta - 1\right)\sin^4\beta = \frac{r^4\left(6z^2 - r^2\right)}{R^9} \\
\phi_{-4} &= R^{-4}\left(3\cos^2\beta - 1\right)\sin^4\beta\cos\beta = \frac{r^4 z\left(2z^2 - r^2\right)}{R^{11}} \; .
\end{aligned} \tag{27.20}$$

that are singular only at the origin. The Q-series functions include logarithmic terms and are singular on the entire z-axis.

The two solutions of (27.11) of degree 0,1,2,3 in R are

$$R^n P_{n-2}^{-2}(\cos\beta) \;\; ; \;\; R^n Q_{n-2}^2(\cos\beta) \; , \tag{27.21}$$

which can be expanded as

$$\begin{aligned}
\phi_0 &= R^0(1-\cos\beta)^2(2+\cos\beta) = \frac{(R-z)^2(2R+z)}{R^3} \\
\phi_1 &= R^1(1-\cos\beta)^2 = \frac{(R-z)^2}{R} \\
\phi_2 &= R^2(1-\cos\beta)^2 = (R-z)^2 \\
\phi_3 &= R^3(1-\cos\beta)^2(2+\cos\beta) = (R-z)^2(2R+z)
\end{aligned} \tag{27.22}$$

$$\tag{27.23}$$

and

$$\begin{aligned} \varphi_0 &= 1 \\ \varphi_1 &= R\cos\beta = z \\ \varphi_2 &= R^2\cos\beta = Rz \\ \varphi_3 &= R^3 \end{aligned} \tag{27.24}$$

respectively.

27.4 The displacement field

Eliminating Ψ between equations (27.1), we have

$$\sigma_{\theta r} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \quad ; \quad \sigma_{\theta z} = \mu\frac{\partial u_\theta}{\partial z} \; . \tag{27.25}$$

Once the stress field is known, these equations can be solved for the only non-zero displacement component u_θ.

An alternative approach is to define a displacement function ψ through the equation

$$u_\theta = r\psi \; , \tag{27.26}$$

in which case

$$\sigma_{\theta r} = \mu r\frac{\partial \psi}{\partial r} \quad ; \quad \sigma_{\theta z} = \mu r\frac{\partial \psi}{\partial z} \; , \tag{27.27}$$

from (27.25, 27.26) and more generally the vector traction on the θ-plane is

$$\boldsymbol{\sigma_\theta} = \mu r \boldsymbol{\nabla}\psi \; . \tag{27.28}$$

It follows from (27.28) that on a traction free surface

$$\frac{\partial \psi}{\partial n} = 0 \; , \tag{27.29}$$

where n is the local normal to the surface. Thus lines of constant ϕ and lines of constant ψ are everywhere orthogonal to each other.

Equations (27.12) and (27.27) are alternative representations of the stress components and hence

$$\mu r\frac{\partial \psi}{\partial r} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} \quad ; \quad \mu r\frac{\partial \psi}{\partial z} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} \; . \tag{27.30}$$

Eliminating ϕ between these equations, we obtain

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \; , \tag{27.31}$$

which is the governing equation that must be satisfied by the displacement function ψ.

27.5 Cylindrical and conical bars

The formulation introduced in this chapter is particularly effective for determining the stresses in an axisymmetric bar loaded only by torques on the ends. In this case, the curved surfaces of the bar must be defined by lines of constant ϕ and the problem is reduced to the determination of a solution of (27.11) that is constant along a specified surface. It may be possible to determine this by inspection, using a suitable combination of functions such as (27.18–27.24). In other cases, an approximate solution may be obtained by making a contour plot of a candidate function and adjusting multiplying constants or other parameters until one of the contours approximates the required shape of the bar.

A simple case is the solid cylindrical bar of radius a, transmitting a torque T. Clearly the function

$$\phi = C\phi_4 = Cr^4 \tag{27.32}$$

is constant on $r = a$, so the traction-free condition is satisfied along this boundary. To determine the multiplying constant C, we use equation (27.4) to write

$$T = 2\pi\phi(a, z) - 2\pi\phi(0, z) = 2\pi Ca^4 \tag{27.33}$$

and hence

$$C = \frac{T}{2\pi a^4} \; . \tag{27.34}$$

The stress components are then recovered from (27.12) as

$$\sigma_{z\theta} = 4Cr = \frac{2Tr}{\pi a^4} \;\; ; \;\; \sigma_{r\theta} = 0 \; , \tag{27.35}$$

agreeing of course with the elementary torsion theory.

A more interesting example concerns the conical bar $0<\beta<\beta_0$ transmitting a torque T. The traction-free condition on the boundary $\beta=\beta_0$ will be satisfied if ϕ is independent of R and hence a function of β only in spherical polar coördinates. This condition is clearly satisfied by the functions ϕ_0, φ_0 of equations (27.22, 27.24), but the latter leads only to a null state of stress, so we take

$$\phi = C\phi_0 = C(1-\cos\beta)^2(2+\cos\beta) = \frac{C(R-z)^2(2R+z)}{R^3} \; . \tag{27.36}$$

For the constant C, we have

$$T = 2\pi\phi(R, \beta_0) - 2\pi\phi(R, 0) = 2\pi C(1-\cos\beta_0)^2(2+\cos\beta_0) \; , \tag{27.37}$$

from (27.4) and hence

$$C = \frac{T}{2\pi(1-\cos\beta_0)^2(2+\cos\beta_0)} \; . \tag{27.38}$$

The stress components are then obtained from (27.12) as

$$\sigma_{z\theta} = \frac{3Trz}{2\pi(1-\cos\beta_0)^2(2+\cos\beta_0)R^5} \;\; ; \;\; \sigma_{r\theta} = \frac{3Tr^2}{2\pi(1-\cos\beta_0)^2(2+\cos\beta_0)R^5} \; . \tag{27.39}$$

Simpler expressions for the stresses are obtained in spherical polar coördinates. Vector transformation of the tractions (27.12) on the θ-plane leads to the general expressions

$$\sigma_{\theta R} = \frac{1}{r^2}\frac{1}{R}\frac{\partial\phi}{\partial\beta} = \frac{1}{R^3\sin^2\beta}\frac{\partial\phi}{\partial\beta} \;\; ; \;\; \sigma_{\theta\beta} = -\frac{1}{r^2}\frac{\partial\phi}{\partial R} = -\frac{1}{R^2\sin^2\beta}\frac{\partial\phi}{\partial R} \; . \tag{27.40}$$

Substitution for ϕ from (27.36, 27.38) then yields

$$\sigma_{\theta R} = \frac{3T\sin\beta}{2\pi(1-\cos\beta_0)^2(2+\cos\beta_0)R^3} \;\; ; \;\; \sigma_{\theta\beta} = 0 \; , \tag{27.41}$$

showing that the traction on the θ-surface is purely radial. Of course, this follows immediately from the fact that ϕ is a function of β only and hence is constant on any line $\beta =$ constant.

The displacements for the conical bar can be obtained by equating (27.41) and (27.28), which in spherical polar coördinates takes the form

$$\sigma_{\theta R} = \mu R\sin\beta\frac{\partial\psi}{\partial R} \;\; ; \;\; \sigma_{\theta\beta} = \mu\sin\beta\frac{\partial\psi}{\partial\beta} \; . \tag{27.42}$$

Using (27.41), we therefore have

$$\frac{\partial\psi}{\partial R} = \frac{3T}{2\pi\mu(1-\cos\beta_0)^2(2+\cos\beta_0)R^4} \;\; ; \;\; \frac{\partial\psi}{\partial\beta} = 0 \; . \tag{27.43}$$

Thus ψ must be a function of R only and integrating the first of (27.43) we have

$$\psi = -\frac{T}{2\pi\mu(1-\cos\beta_0)^2(2+\cos\beta_0)R^3} \; . \tag{27.44}$$

The displacements are then recovered as

$$u_\theta = -\frac{Tr}{2\pi\mu(1-\cos\beta_0)^2(2+\cos\beta_0)R^3} = -\frac{T\sin\beta}{2\pi\mu(1-\cos\beta_0)^2(2+\cos\beta_0)R^2} \; , \tag{27.45}$$

from (27.26).

27.5.1 The centre of rotation

The stress function (27.36) can also be used to determine the stresses in an infinite body loaded only by a concentrated torque T about the z-axis acting at the origin. Using (27.40), we have

$$\sigma_{\theta R} = \frac{3C \sin \beta}{R^3} \quad ; \quad \sigma_{\theta\beta} = 0 \tag{27.46}$$

and the torque transmitted across the spherical surface $R\!=\!a$ is

$$T = 2\pi \int_0^{\pi} \sigma_{R\theta}(a, \beta) a^2 \sin^2 \beta d\beta = 8\pi C \; . \tag{27.47}$$

It follows that $C = T/8\pi$ and the stress field is

$$\sigma_{\theta R} = \frac{3T \sin \beta}{8\pi R^3} \quad ; \quad \sigma_{\theta\beta} = 0 \; . \tag{27.48}$$

The same result can be obtained from (27.41) by setting $\beta_0\!=\!\pi$, since a cone of angle π is really an infinite body with an infinitesimal hole along the negative z-axis[1]

27.6 The Saint Venant problem

As in Chapter 23, bounded polynomial solutions of (27.11) can be used to solve problems of the cylindrical bar with prescribed polynomial tractions on the curved surfaces, but the boundary conditions on the ends can generally only be satisfied only in the weak sense. The corrective solution needed to re-establish strong boundary conditions on the ends is a Saint Venant problem in the sense of Chapter 6 and can be treated by similar methods.

We first postulate the existence of solutions of (27.11) of the separated-variable form

$$\phi = e^{-\lambda z} f(r) \; . \tag{27.49}$$

Substitution in (27.11) then yields the ordinary differential equation

$$\frac{d^2 f}{dr^2} - \frac{3}{r}\frac{df}{dr} + \lambda^2 f = 0 \tag{27.50}$$

for the function $f(r)$. This equation has the general solution

$$f(r) = C_1 r^2 J_2(\lambda r) + C_2 r^2 Y_2(\lambda r) \; , \tag{27.51}$$

where J_2, Y_2 are Bessel functions of the first and second kind respectively of order 2 and C_1, C_2 are arbitrary constants. The boundaries $r = a, b$ of the hollow cylinder $a\!<\!r\!<\!b$ will be traction-free if $f(a)\!=\!f(b)\!=\!0$ and hence

$$\begin{aligned} C_1 J_2(\lambda a) + C_2 Y_2(\lambda a) &= 0 \\ C_1 J_2(\lambda b) + C_2 Y_2(\lambda b) &= 0 \; . \end{aligned} \tag{27.52}$$

These equations have a non-trivial solution for C_1, C_2 if and only if

[1] This hole does not alter the stress field, since the stresses are zero on the axis.

$$J_2(\lambda a)Y_2(\lambda b) - Y_2(\lambda a)J_2(\lambda b) = 0 \; , \tag{27.53}$$

which constitutes an eigenvalue equation for the exponential decay rate λ. The function Y_2 is unbounded at $r \to 0$ and hence must be excluded for the solid cylinder $0 \leq r < a$. In this case, the eigenvalue equation reduces to

$$J_2(\lambda a) = 0 \; . \tag{27.54}$$

As in Chapter 6, a more general solution can then be constructed for the corrective stress field in the form of an eigenfunction expansion.

In contrast to the plane problem, the eigenvalues of equations (27.53) are real. The first eigenvalue of (27.54) is $\lambda a = 5.136$, showing that self-equilibrated corrections to the boundary conditions on the ends decay quite rapidly with distance from the ends.

PROBLEMS

1. Verify that equation (27.17) defines a function satisfying equation (27.11) by direct substitution, using the result

$$\frac{\partial^2 g}{\partial z^2} = -\frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r}$$

(which is a restatement of the condition $\nabla^2 g = 0$) to eliminate the derivatives with respect to z.

2. The solid cylindrical bar $0 \leq r < a$, $0 < z < L$ is loaded by a uniform torsional shear traction $\sigma_{r\theta} = S$ on the surface $r = a$, the end $z = 0$ being traction-free. Find the stress field in the bar, using bounded polynomial solutions to equation (27.11).

3. Show that the function

$$\varphi_1 = z$$

satisfies equation (27.11) and defines a state of stress that is independent of z. Combine this solution with a bounded polynomial solution of (27.11) to determine the stress field in the hollow cylindrical bar $a < r < b$, $0 < z < L$ built in at $z = L$ and loaded by a uniform torsional shear traction $\sigma_{r\theta} = S$ on the inner surface $r = a$, the outer surface $r = b$ and the end $z = 0$ being traction-free.

4. A solid cylindrical bar of radius b transmitting a torque T, contains a small spherical hole of radius a ($\ll b$) centred on the axis. Find a bounded polynomial solution to equation (27.11) to describe the stress field in the bar in the absence of the hole and hence determine the perturbation in this field due to the hole by superposing a suitable singular solution.

5. Develop a contour plot in the r, z plane of a function comprising

(i) the function ϕ_4 of equation (27.19),
(ii) a centre of rotation, centred on the point $(0, a)$,
(iii) an equal negative centre of rotation, centred on the point $(0, -a)$.

By experimenting with different multiplying constants on these functions, show that the solution can be obtained for a cylindrical bar with an approximately elliptical central hole, transmitting a torque T. If the bar has outer radius b and the hole has radius $0.1b$ at the mid-plane and is of total axial length $0.4b$, estimate the stress at the point $(0.1b, 0)$.

6. Show that the lines $R - z = C$ define a set of parabolas, where C is a constant. Combine suitable functions from equations (27.18–27.24) to obtain a solution of (27.11) that is constant on these lines and hence solve the problem of a paraboloidal bar transmitting a torque T. Comment on the nature of the solution in the region $z \leq 0$ and in particular at the origin. Do your results place any restrictions on the applicability of the solution?

7. The solid cylindrical bar $0 \leq r < a$, $z > 0$ is bonded to a rigid body at the curved surface $r = a$ and loaded by torsional tractions on the plane end $z = 0$. Using a displacement function

$$\psi = e^{-\lambda z} f(r) ,$$

find suitable non-trivial solutions for the stress field and hence determine the slowest rate at which the stress field will decay along the bar.

28

THE PRISMATIC BAR

We have seen in Chapter 25 that three-dimensional solutions can be obtained to the problem of the solid or hollow cylindrical bar loaded on its curved surfaces, using the Papkovich-Neuber solution with spherical harmonics and related potentials. Here we shall show that similar solutions can be obtained for bars of more general cross-section, using the complex-variable form of the Papkovich-Neuber solution from §21.6.

We use a coördinate system in which the axis of the bar is aligned with the z-direction, one end being the plane $z=0$. The constant cross-section of the bar then comprises a domain Ω in the xy-plane, which may be the interior of a closed curve Γ, or that part of the region interior to a closed curve Γ_0 that is also *exterior* to one or more closed curves Γ_1, Γ_2, etc. The following derivations and examples will be restricted to the former, simply connected case, but it will be clear from the methods used that the additional complications asssociated with multiply connected cross-sections arise only in the solution of two-dimensional boundary-value problems, for which classical methods exist. As in Chapter 25, we shall apply weak boundary conditions at the ends of the bar, which implies that the solutions are appropriate only for relatively long bars in regions that are not too near the ends.

28.1 Power series solutions

The most general loading of the lateral surfaces Γ comprises the three traction components T_x, T_y, T_z, which we shall combine in the complex variable form as

$$T = T_x + \imath T_y \; , \quad T_z \; .$$

Consider the problem in which T, T_z can be written as power series in z — i.e.

$$T = \sum_{j=1}^{m-1} f_j(s) z^{j-1} \; ; \quad T_z = \sum_{j=1}^{m} g_j(s) z^{j-1} \; , \tag{28.1}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_28, © Springer Science+Business Media B.V. 2010

where f, g are arbitrary complex and real functions respectively of the curvilinear coordinate s defining position on Γ. We shall denote this system of tractions by the symbol $\boldsymbol{T}^{(m)}$. Notice that the in-plane tractions T are carried only up to the order z^{m-2} whereas the out-of-plane traction T_z includes a term proportional to z^{m-1}. Practical cases where the highest order term in all three tractions is the same (z^n say) can of course be treated by setting $m = n+2$ and $g_m(s) = 0$. We describe the problem (28.1) as $\mathcal{P}_m$ and denote the corresponding stress and displacement fields in the bar by

$$\boldsymbol{\sigma}^{(m)} = \left\{\Theta^{(m)},\ \Phi^{(m)},\ \Psi^{(m)},\ \sigma_{zz}^{(m)}\right\}\ ;\quad \boldsymbol{u}^{(m)} = \left\{u^{(m)},\ u_z^{(m)}\right\}\ , \tag{28.2}$$

respectively, using the complex-variable notation of §21.6.

The bar will transmit force resultants comprising an axial force F_z, shear forces F_x, F_y, a torque M_z and bending moments M_x, M_y, which are related to the stress components (28.2) by the equilibrium relations

$$F_z(z) = \iint_\Omega \sigma_{zz} d\Omega\ ;\quad F(z) \equiv F_x + \imath F_y = \iint_\Omega \Psi d\Omega$$

$$M_z(z) = \frac{\imath}{2}\iint_\Omega (\zeta\overline{\Psi} - \bar{\zeta}\Psi) d\Omega\ ;\quad M(z) \equiv M_x + \imath M_y = -\imath \iint_\Omega \zeta\sigma_{zz} d\Omega\ . \tag{28.3}$$

Consideration of the equilibrium of a segment of the bar shows that these resultants will also take the form of finite polynomials in z, with highest power z^m in F_z, M and z^{m-1} in F, M_z.

The complete definition of problem $\mathcal{P}_m$ requires that we also specify the force resultants $F(0), F_z(0), M(0), M_z(0)$ on the end $z = 0$. However, it is sufficient at this stage to obtain a particular solution with any convenient end condition. Correction of the end condition in the weak sense can then be completed using the techniques of Part III.

28.1.1 Superposition by differentiation

Suppose that the solution to a given problem $\mathcal{P}_m$ is known — i.e. that we have found stress and displacement components $\boldsymbol{\sigma}^{(m)}$, $\boldsymbol{u}^{(m)}$ that reduce to a particular set of polynomial tractions (28.1) on Γ and that satisfy the quasi-static equations of elasticity in the absence of body forces, which we here represent in the symbolic form

$$\mathcal{L}\left(\boldsymbol{\sigma}^{(m)}, \boldsymbol{u}^{(m)}\right) = 0\ , \tag{28.4}$$

where $\mathcal{L}$ is a set of linear differential operators. Differentiating (28.4) with respect to z, we have

$$\frac{\partial}{\partial z}\mathcal{L}\left(\boldsymbol{\sigma}^{(m)}, \boldsymbol{u}^{(m)}\right) = \mathcal{L}\left(\frac{\partial \boldsymbol{\sigma}^{(m)}}{\partial z}, \frac{\partial \boldsymbol{u}^{(m)}}{\partial z}\right) = 0\ . \tag{28.5}$$

It follows that the new set of stresses and displacements defined by differentiation as

$$\boldsymbol{\sigma}^{(m-1)} = \frac{\partial \boldsymbol{\sigma}^{(m)}}{\partial z} \; ; \quad \boldsymbol{u}^{(m-1)} = \frac{\partial \boldsymbol{u}^{(m)}}{\partial z} \tag{28.6}$$

will also satisfy the equations of elasticity and will correspond to the tractions

$$T = \sum_{j=2}^{m-1} (j-1) f_j(s) z^{j-2} \; ; \quad T_z = \sum_{j=2}^{m} (j-1) g_j(s) z^{j-2} \; . \tag{28.7}$$

This process of generating a new particular solution by differentiation with respect to a spatial coordinate can be seen as a form of linear superposition of the original solution on itself after an infinitesimal displacement in the z-direction and was already used in §23.3 to generate singular spherical harmonics. The resulting tractions (28.7) are polynomials in z with highest order terms z^{m-3}, z^{m-2} in T, T_z respectively and hence are of the form $\boldsymbol{T}^{(m-1)}$. Thus the stress field $\partial\boldsymbol{\sigma}^{(m)}/\partial z$ is the solution of a particular problem of the class $\mathcal{P}_{m-1}$.

28.1.2 The problems $\mathcal{P}_0$ and $\mathcal{P}_1$

Repeating this differential operation m times, we find that the stress and displacement fields

$$\boldsymbol{\sigma}^{(0)} = \frac{\partial^m \boldsymbol{\sigma}^{(m)}}{\partial z^m} \; ; \quad \boldsymbol{u}^{(0)} = \frac{\partial^m \boldsymbol{u}^{(m)}}{\partial z^m} \tag{28.8}$$

correspond to the physical problem $\mathcal{P}_0$ in which the tractions $\boldsymbol{T}=0$, and the only non-zero force resultants are the axial force F_z and the bending moment M, which are independent of z. This is a classical problem for which the only non-zero stress component is

$$\sigma_{zz}^{(0)} = A_0\zeta + \overline{A}_0\bar{\zeta} + B_0 \; , \tag{28.9}$$

where A_0 is a complex constant and B_0 a real constant.

The class $\mathcal{P}_1$ includes the problems of Chapters 16,17, where an otherwise traction-free bar is loaded by a lateral force $F(0)$ and a torque $T_z(0)$ at the end $z = 0$. However, it also permits z-independent antiplane tractions T_z on the lateral surfaces, which also generally lead to the bending moment M and/or the axial force F_z varying linearly with z. This generalization is described by Milne-Thomson[1] as the 'push' problem. In $\mathcal{P}_1$, the in-plane stress components $\Theta^{(1)}, \Phi^{(1)}$ remain zero, the antiplane stress components $\Psi^{(1)}$ are independent of z and $\sigma_{zz}^{(1)}$ has the form

$$\sigma_{zz}^{(1)} = (A_0\zeta + \overline{A}_0\bar{\zeta} + B_0)z + A_1\zeta + \overline{A}_1\bar{\zeta} + B_1 \; . \tag{28.10}$$

[1] L.M.Milne-Thomson, *Antiplane Elastic Systems*, Springer, Berlin, 1962.

28.1.3 Properties of the solution to $\mathcal{P}_m$

From these results, it is clear that the solution of problem $\mathcal{P}_m$ posesses the following properties:-

(i) The stress and displacement components, $\boldsymbol{\sigma}^{(m)}$, $\boldsymbol{u}^{(m)}$ can be expressed as power series in z.
(ii) The highest order terms in z in the in-plane stress components $\Theta^{(m)}, \Phi^{(m)}$ are of order z^{m-2}, since after $m-1$ differentiations with respect to z they are zero.
(iii) The highest order terms in z in the antiplane stress components $\Psi^{(m)}$ are of order z^{m-1}, since after $m-1$ differentiations with respect to z they are independent of z.
(iv) The highest order terms in the stress component $\sigma_{zz}^{(m)}$ are of the form

$$\sigma_{zz}^{(m)} = \frac{(A_0\zeta + \overline{A}_0\bar{\zeta} + B_0)z^m}{m!} + \frac{(A_1\zeta + \overline{A}_1\bar{\zeta} + B_1)z^{m-1}}{(m-1)!} \; , \qquad (28.11)$$

from (28.8, 28.10).

Conclusions (ii,iii) explain why we chose to terminate the traction series (28.1) at z^{m-2} in T and at z^{m-1} in T_z.

28.2 Solution of $\mathcal{P}_m$ by integration

The process of differentiation elaborated in §28.1.1 shows that for every problem $\mathcal{P}_m$, there exists a set of lower order problems $\mathcal{P}_j$, $j=(0, m-1)$ satisfying the recurrence relations

$$\boldsymbol{T}^{(j)} = \frac{\partial \boldsymbol{T}^{(j+1)}}{\partial z} \; ; \quad \boldsymbol{\sigma}^{(j)} = \frac{\partial \boldsymbol{\sigma}^{(j+1)}}{\partial z} \; ; \quad \boldsymbol{u}^{(j)} = \frac{\partial \boldsymbol{u}^{(j+1)}}{\partial z} \; . \qquad (28.12)$$

The lowest-order problems in this set, $\mathcal{P}_0$, $\mathcal{P}_1$ can be solved by the methods of Part III, so the more general problem $\mathcal{P}_m$ could be solved recursively if we could devise a method to generate the solution of $\mathcal{P}_{j+1}$ from that of $\mathcal{P}_j$. This process clearly involves partial integration of $\boldsymbol{\sigma}^{(j)}, \boldsymbol{u}^{(j)}$ with respect to z. In a formal sense, we can write

$$\boldsymbol{\sigma}^{(j+1)} = \int \boldsymbol{\sigma}^{(j)} dz + \boldsymbol{f}(x, y) \; , \qquad (28.13)$$

where $\boldsymbol{f}(x, y)$ is an arbitrary function of x, y only. Since $\boldsymbol{\sigma}^{(j)}$ is in the form of a finite power series in z, it is always possible to find a particular integral for the first term in (28.13). The function $\boldsymbol{f}$ must then be chosen to satisfy two conditions:-

(i) The complete stress field $\boldsymbol{\sigma}^{(j+1)}$ including $\boldsymbol{f}(x, y)$ must satisfy the equations of elasticity (28.4), and

(ii) the stresses must reduce to the known tractions $\boldsymbol{T}^{(j+1)}$ on Γ.

Condition (ii) involves only the coefficient of z^0 in $\boldsymbol{T}^{(j+1)}$, since the coefficients of higher order terms will have been taken care of at an earlier stage in the recursive process. Similarly, the conditions imposed by the equations of elasticity can only arise in the lowest order terms in the stress field. We shall show that the determination of $\boldsymbol{f}$ is a well-posed two-dimensional problem.

The concept of a recursive solution to problems $\mathcal{P}_m$ was first enunciated by Ieşan[2], who started from a state of arbitrary rigid-body displacement, rather than $\mathcal{P}_0$ of §28.1.2. A rigid body displacement lies in the class $\mathcal{P}_{-1}$ and since it involves no stresses, it is the same for all domains Ω. The present treatment will place emphasise on the analytical solution of specific problems rather than on general features of the solution method and is based on the paper J.R.Barber, Three-dimensional elasticity solutions for the prismatic bar, *Proceedings of the Royal Society of London*, Vol. A462 (2006), pp.1877–1896.

28.3 The integration process

The recursive procedure is most easily formulated in the context of the complex-variable Papkovich-Neuber (P-N) solution of §21.6 and equations (21.45–21.50). The prismatic bar satisfies the condition that all lines drawn in the z-direction cut the boundary of the body at only two points (the ends of the bar), which is the criterion established in §20.4.1 for the elimination of the component ψ_z in the P-N solution without loss of generality. We can therefore simplify the solution by omitting all the terms involving ω in equations (21.45–21.50), giving

$$2\mu u = 2\frac{\partial \phi}{\partial \bar{\zeta}} - (3-4\nu)\psi + \zeta\frac{\partial \overline{\psi}}{\partial \bar{\zeta}} + \bar{\zeta}\frac{\partial \psi}{\partial \bar{\zeta}} \tag{28.14}$$

$$2\mu u_z = \frac{\partial \phi}{\partial z} + \frac{1}{2}\left(\bar{\zeta}\frac{\partial \psi}{\partial z} + \zeta\frac{\partial \overline{\psi}}{\partial z}\right) \tag{28.15}$$

$$\Theta = -\frac{\partial^2 \phi}{\partial z^2} - 2\left(\frac{\partial \psi}{\partial \zeta} + \frac{\partial \overline{\psi}}{\partial \bar{\zeta}}\right) - \frac{1}{2}\left(\zeta\frac{\partial^2 \overline{\psi}}{\partial z^2} + \bar{\zeta}\frac{\partial^2 \psi}{\partial z^2}\right) \tag{28.16}$$

$$\Phi = 4\frac{\partial^2 \phi}{\partial \bar{\zeta}^2} - 4(1-2\nu)\frac{\partial \psi}{\partial \bar{\zeta}} + 2\zeta\frac{\partial^2 \overline{\psi}}{\partial \bar{\zeta}^2} + 2\bar{\zeta}\frac{\partial^2 \psi}{\partial \bar{\zeta}^2} \tag{28.17}$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} - 2\nu\left(\frac{\partial \psi}{\partial \zeta} + \frac{\partial \overline{\psi}}{\partial \bar{\zeta}}\right) + \frac{1}{2}\left(\bar{\zeta}\frac{\partial^2 \psi}{\partial z^2} + \zeta\frac{\partial^2 \overline{\psi}}{\partial z^2}\right) \tag{28.18}$$

$$\Psi = 2\frac{\partial^2 \phi}{\partial \bar{\zeta}\partial z} - (1-2\nu)\frac{\partial \psi}{\partial z} + \zeta\frac{\partial^2 \overline{\psi}}{\partial \bar{\zeta}\partial z} + \bar{\zeta}\frac{\partial^2 \psi}{\partial \bar{\zeta}\partial z}\ , \tag{28.19}$$

[2] D.Ieşan, On Saint-Venant's Problem, *Archive for Rational Mechanics and Analysis*, Vol. 91 (1986), pp.363–373.

where ϕ, ψ are respectively a real and a complex harmonic function of $\zeta, \bar{\zeta}, z$.

None of the expressions (28.14–28.19) include explicit z-multipliers and hence the integration (28.13) can be performed on the P-N potentials ϕ, ψ, rather than directly on the stress and displacement components. The requirement §28.2(i) that the solution satisfy the equations of elasticity (28.4) will then be met by ensuring that the integrated potentials are three-dimensional harmonic functions.

Suppose that the complex potential ψ in (28.14–28.19) for problem $\mathcal{P}_j$ is given by a known function ψ_j. This must be three-dimensionally harmonic, and hence

$$\nabla^2 \psi_j \equiv \frac{\partial^2 \psi_j}{\partial z^2} + 4\frac{\partial^2 \psi_j}{\partial \zeta \partial \bar{\zeta}} = 0 , \tag{28.20}$$

from (24.56). We next write

$$\psi_{j+1} = h_1(\zeta, \bar{\zeta}, z) + h_2(\zeta, \bar{\zeta}) , \tag{28.21}$$

where

$$h_1 = \int \psi_j(\zeta, \bar{\zeta}, z) dz \tag{28.22}$$

is any partial integral of ψ_j. The recurrence relations (28.12) then imply that h_2 is independent of z. Applying the Laplace operator (28.20) to the function ψ_{j+1} of equation (28.21), we obtain

$$4\frac{\partial^2 h_2}{\partial \zeta \partial \bar{\zeta}} = -\nabla^2 h_1 , \tag{28.23}$$

and the function h_2 can then be recovered by integration with respect to ζ and $\bar{\zeta}$. These integrations will introduce arbitrary holomorphic functions that will be used to satisfy the zeroth-order boundary conditions in problem $\mathcal{P}_{j+1}$.

We know from §28.1.3 that at any given stage in the recurrence procedure, ψ can be written as a finite power series in z and one way to write this series is as a set of terms of the form

$$\psi_{2n}(\zeta, \bar{\zeta}, z) = \sum_{i=0}^{n} \frac{z^{2i} f_{n-i}(\zeta, \bar{\zeta})}{(2i)!} \ ; \quad \psi_{2n+1}(\zeta, \bar{\zeta}, z) = \sum_{i=0}^{n} \frac{z^{2i+1} f_{n-i}(\zeta, \bar{\zeta})}{(2i+1)!}$$

from §24.8, where the functions f_k satisfy the recurrence relations (24.59, 24.60). It is easily verified that suitable z-integrals of these functions that satisfy the three-dimensional Laplace equation (28.20) are

$$\int \psi_{2n}(\zeta, \bar{\zeta}, z) dz = \psi_{2n+1}(\zeta, \bar{\zeta}, z) \ ; \quad \int \psi_{2n+1}(\zeta, \bar{\zeta}, z) dz = \psi_{2n+2}(\zeta, \bar{\zeta}, z) , \tag{28.24}$$

where ψ_{2n+2} is obtained from ψ_{2n} by replacing n by $n+1$.

The first term of this form is the two-dimensionally harmonic function

$$\psi_0 = f_0(\zeta, \bar{\zeta}) = g_1(\zeta) + \overline{g}_2(\bar{\zeta}) ,$$

from (24.61). Thus, if the above integration process is applied sequentially, the arbitrary functions of ζ and $\bar{\zeta}$ that are introduced at stage i can be expressed in the form of the function ψ_0 and this will integrate up to the corresponding function ψ_{j-i} at stage j.

In the special case where the functions f_k are finite polynomials in $\zeta, \bar{\zeta}$, the potentials can be expressed as a series of the bounded cylindrical harmonics χ_n^m defined in equations (24.66, 24.67), which can be integrated using the relation

$$\int \chi_n^m dz = \chi_{n+1}^m \ , \tag{28.25}$$

from (24.68).

Similar procedures can be used to produce harmonic partial integrals of the real potential, ϕ_j, in which case the the arbitrary holomorphic functions in h_2 must be chosen so as to ensure that the resulting potential is real.

28.4 The two-dimensional problem $\mathcal{P}_0$

Problem $\mathcal{P}_0$ corresponds to the elementary stress field in which the lateral surfaces of the bar are traction-free and the bar transmits a bending moment M and an axial force F_z. The only non-zero stress component is $\sigma_{zz}^{(0)}$ given by equation (28.9), but to start the recursive process, we need to express this in terms of the P-N solution of §28.3.

It is readily verified by substitution in (28.16–28.19) that this elementary stress field is generated by the choice of potentials

$$\phi_0 = \mathcal{G}_0^0 \ ; \quad \psi_0 = \mathcal{H}_0^0 \ , \tag{28.26}$$

where we introduce the notation

$$\begin{aligned} \mathcal{G}_m^n &\equiv \frac{(1-2\nu)}{2(1-\nu^2)} \left(A_m \chi_{n+2}^1 + \bar{A}_m \overline{\chi}_{n+2}^1 \right) + \frac{B_m \chi_{n+2}^0}{(1+\nu)} \\ \mathcal{H}_m^n &\equiv \frac{\bar{A}_m \chi_{n+2}^0}{(1-\nu^2)} - \frac{(1-2\nu) A_m \chi_n^2}{8(1-\nu^2)} - \frac{B_m \chi_n^1}{4(1+\nu)} \end{aligned} \tag{28.27}$$

and χ_n^m are the bounded cylindrical harmonics defined in §24.8.1 and equations (24.66, 24.67). The recurrence relation (28.25) implies that

$$\int \mathcal{G}_m^n dz = \mathcal{G}_m^{n+1} \ ; \quad \int \mathcal{H}_m^n dz = \mathcal{H}_m^{n+1} \tag{28.28}$$

and we shall find this useful in subsequent developments.

Since the stress components $\Theta^{(0)}, \Phi^{(0)}, \Psi^{(0)}$ are identically zero, no tractions are implied on any surface Γ and hence that the potentials (28.26) define the solution of problem $\mathcal{P}_0$ for a bar of any cross-section Ω.

28.5 Problem $\mathcal{P}_1$

Problem $\mathcal{P}_1$ could be solved using the methods of Part III, but here we shall develop the solution by integration using the P-N solution, since similar procedures will be needed at higher-order stages of the recursive process. The strategy is first to develop a particular solution describing the most general force resultants that can be transmitted along the bar, but without regard to the exact form of the antiplane tractions on Γ. We then superpose a corrective antiplane solution, in the spirit of Chapter 15, but within the P-N formalism, to satisfy these boundary conditions in the strong sense.

Starting with the solution ϕ_0, ψ_0 of equation (28.26), we first generate a trial solution for ϕ, ψ by integration with respect to z. Using the recurrence relations (28.28) to ensure that the resulting functions are harmonic, we obtain

$$\phi = \mathcal{G}_0^1 \; ; \quad \psi = \mathcal{H}_0^1 \; . \tag{28.29}$$

The constants A_0, B_0 in these functions provide the degrees of freedom to define the lateral force resultants F, but in problem $\mathcal{P}_1$, we also need to allow for the transmission of a torque M_z. Any particular solution of the torsion problem can be used for this purpose (remember we are going to correct the tractions on the lateral surfaces at the next stage) and the simplest solution is described by the potentials

$$\phi^{(t)} = 0 \; ; \quad \psi^{(t)} = -\frac{\imath C_0 \chi_1^1}{2(1-\nu)} \; , \tag{28.30}$$

where C_0 is a real constant.

Combining these results, we define the particular antiplane solution as

$$\phi_P = \mathcal{G}_0^1 \; ; \quad \psi_P = \mathcal{H}_0^1 - \frac{\imath C_0 \chi_1^1}{2(1-\nu)} \; , \tag{28.31}$$

for which the corresponding non-zero stress components are

$$\sigma_{zz} = A_0 z\zeta + \bar{A}_0 z\bar{\zeta} + B_0 z \tag{28.32}$$

$$\Psi = -\frac{A_0(1+2\nu)\zeta^2}{4(1+\nu)} - \frac{\bar{A}_0\zeta\bar{\zeta}}{2(1+\nu)} - \frac{B_0\zeta}{2} + \imath C_0 \zeta \; , \tag{28.33}$$

from (28.31, 28.18, 28.19).

Notice that in this solution, the real constant B_0 corresponds to an axial force F_z that varies linearly with z and this will arise only when there are non-self-equilibrating uniform antiplane tractions on the surface Γ. Thus, if this method is used to solve the problems of Chapters 16 or 17, B_0 can be set to zero.

28.5.1 The corrective antiplane solution

Equation (28.33) defines non-zero values of Ψ and hence generally implies non-zero tractions T_z on Γ, which however are independent of z. We may indeed have non-zero tractions T_z, but these will not generally correspond to those of the particular solution. It is therefore necessary to superpose a corrective solution in which the tractions are changed by some known function ΔT_z without affecting the force resultants implied by the constants A_0, B_0, C_0. Thus, the corrective solution satisfies the conditions

$$\Theta = \Phi = \sigma_{zz} = 0 \; ; \quad \frac{\partial \Psi}{\partial z} = 0 \; , \tag{28.34}$$

with self-equilibrated tractions ΔT_z, and it is an antiplane solution in the sense of Chapter 15 and §19.2.

A convenient representation of the corrective solution in P-N form can be written in the form

$$\phi = z(h + \overline{h}) - \frac{z(\zeta h' + \overline{\zeta h'})}{4(1-\nu)} \; ; \quad \psi = \frac{z\overline{h'}}{2(1-\nu)} \; , \tag{28.35}$$

where h is a holomorphic function of ζ. These potentials satisfy the Laplace equation (28.20) and substitution into (28.14–28.19) show that the homogeneous conditions (28.34) are satisfied identically. The non-zero displacement and stress components are obtained as

$$2\mu u_z = h + \overline{h} \; ; \quad \Psi = \overline{h'} \; , \tag{28.36}$$

agreeing with the notation of equations (19.16, 19.18).

It follows from (19.25) that

$$\Delta T_z = \frac{\partial h_y}{\partial s} \; , \tag{28.37}$$

where s is a real coördinate locally tangential to the boundary Γ, and hence h_y is identical to the real Prandtl stress function φ for the antiplane correction. Since the corrective tractions ΔT_z are known everywhere, we can integrate (28.37) around Γ to determine h_y at all points on the boundary. Interior values of the real two-dimensionally harmonic function h_y and hence the holomorphic function of which it is the imaginary part can then be obtained using the real stress function methods of Chapters 15–17, or the complex variable method of §19.2. It is important that the integration of (28.37) should yield a single-valued function of s. This is equivalent to the requirement that the tractions in the corrective antiplane problem should sum to a zero axial force F_z. If the applied tractions do not meet this condition, F_z will be a linear function of z and hence the constant B_0 in (28.33) will be non-zero. Thus, the single-valued condition on h_y serves to determine the constant B_0.

28.5.2 The circular bar

To illustrate the process and to record some useful results, we consider the antiplane problem $\mathcal{P}_1$ for the circular bar of radius a, with zero tractions T_z. The potentials ϕ_P, ψ_P define the stress field

$$\sigma_{zr} + \imath\sigma_{z\theta} = e^{-\imath\theta}\Psi(r,\theta) = -\frac{A_0 r^2(1+2\nu)e^{\imath\theta}}{4(1+\nu)} - \frac{\bar{A}_0 r^2 e^{-\imath\theta}}{2(1+\nu)} - \frac{B_0 r}{2} + \imath C_0 r \;, \tag{28.38}$$

from (28.33), where we have also used the coördinate transformation relation (19.13). On the boundary $r=a$, we have $\zeta = a\exp(\imath\theta)$, $\bar{\zeta} = a\exp(-\imath\theta)$ and hence the antiplane traction is

$$T_z = \sigma_{zr}(a,\theta) = -\frac{(3+2\nu)a^2\left(\Re\{A_0\}\cos\theta - \Im\{A_0\}\sin\theta\right)}{4(1+\nu)} - \frac{B_0 a}{2} + \frac{1}{a}\frac{\partial h_y}{\partial\theta} \;, \tag{28.39}$$

where we have added in the antiplane corrective term from (28.37). Equating T_z to zero, we obtain

$$\frac{\partial h_y}{\partial\theta} = \frac{(3+2\nu)a^3\left(\Re\{A_0\}\cos\theta - \Im\{A_0\}\sin\theta\right)}{4(1+\nu)} + \frac{B_0 a^2}{2}$$

and this integrates to a single-valued function of θ if and only if $B_0=0$, giving

$$h_y = \frac{(3+2\nu)a^3\left(\Re\{A_0\}\sin\theta + \Im\{A_0\}\cos\theta\right)}{4(1+\nu)} \;.$$

The holomorphic function h whose imaginary part takes this boundary value can be obtained using the direct method of §19.2.1, but it can also be seen by inspection that the appropriate function is

$$h = \frac{(3+2\nu)A_0 a^2\zeta}{4(1+\nu)} = \frac{(3+2\nu)A_0 a^2\chi_0^1}{4(1+\nu)} \;, \tag{28.40}$$

using (24.72).

Substituting this result into (28.35) and adding the results to (28.29, 28.30) with $B_0=0$, we obtain

$$\phi_1 = \mathcal{Q}_0^0 \;; \quad \psi_1 = \mathcal{V}_0^0 \;, \tag{28.41}$$

where we introduce the notation

$$\begin{aligned} \mathcal{Q}_m^n &= \frac{(1-2\nu)}{2(1-\nu^2)}\left(A_m\chi_{n+3}^1 + \bar{A}_m\overline{\chi}_{n+3}^1\right) \\ &\quad + \frac{(3-4\nu)(3+2\nu)a^2(A_m\chi_{n+1}^1 + \bar{A}_m\overline{\chi}_{n+1}^1)}{16(1-\nu^2)} \\ \mathcal{V}_m^n &= \frac{\bar{A}_m\chi_{n+3}^0}{(1-\nu^2)} - \frac{(1-2\nu)A_m\chi_{n+1}^2}{8(1-\nu^2)} + \frac{(3+2\nu)\bar{A}_m a^2\chi_{n+1}^0}{8(1-\nu^2)} \\ &\quad - \frac{\imath C_m\chi_{n+1}^1}{2(1-\nu)} \;. \end{aligned} \tag{28.42}$$

Equation (28.41) defines the solution to $\mathcal{P}_1$ for the traction-free circular cylinder in P-N form. Notice that we have chosen to describe the new polynomial terms arising from (28.40) in terms of the functions χ_n^m of equations (24.66), since this and the notation (28.42) facilitates subsequent integrations through the recurrence relation (28.25), which here implies that

$$\int \mathcal{Q}_m^n dz = \mathcal{Q}_m^{n+1} \; ; \quad \int \mathcal{V}_m^n dz = \mathcal{V}_m^{n+1} \; . \tag{28.43}$$

The stress functions (28.41), excluding the torsion term, can be used as an alternative solution to the problem of §17.4.1 and §25.3.1.

28.6 The corrective in-plane solution

In higher order problems $\mathcal{P}_j$, $j > 1$, we shall also need to make corrections for zeroth-order in-plane tractions T, using an appropriate form of the two-dimensional plane strain equations. If we use the complex form of the P-N solution (28.14–28.19) and define plane harmonic functions through

$$\phi = f + \overline{f} \; ; \quad \psi = g \; , \tag{28.44}$$

where f, g are holomorphic functions of ζ, we obtain

$$2\mu u = -(3-4\nu)g + \zeta\,\overline{g'} + 2\,\overline{f'} \; ; \quad 2\mu u_z = 0 \; . \tag{28.45}$$

This is identical to the classical form of the plane strain complex variable solution (19.42) under the notation relation

$$g \Rightarrow -\chi \; ; \quad 2f' \Rightarrow -\theta \; . \tag{28.46}$$

We recall from §19.3.2 that we can set $\theta(0)=0$ without loss of generality, so the implied integration in obtaining f from θ does not require a constant of integration. The corresponding stress components are given by

$$\Theta = -2(g' + \overline{g'}) \; ; \quad \Phi = 2(\zeta\,\overline{g''} + 2\,\overline{f''}) \; ; \quad \sigma_{zz} = -2\nu(g' + \overline{g'}) \; ; \quad \Psi = 0 \; . \tag{28.47}$$

28.7 Corrective solutions using real stress functions

We have seen in §28.3 that the integration process is greatly facilitated by the use of the complex-variable notation. However, for many geometries, the in-plane and antiplane corrections are easier to perform using the real Airy and Prandtl stress functions of Parts II and III. In Chapter 19 we established simple relations between the real and complex formulations for two-dimensional problems. Here we shall use these relations to convert real stress function expressions to complex P-N form.

28.7.1 Airy function

If we define a corrective field by the Airy stress function φ, we can write it in the form

$$\varphi = g_1 + \overline{g}_1 + \bar{\zeta} g_2 + \zeta \, \overline{g}_2 \; , \tag{28.48}$$

using the procedure defined in §18.4.1 or the Maple and Mathematica files 'rtocb', where g_1, g_2 are holomorphic functions. The relations (19.62, 19.64) and (28.46) then imply that the appropriate complex P-N potentials are

$$\phi = -g_1 - \overline{g}_1 \; ; \quad \psi = -2g_2 \; . \tag{28.49}$$

28.7.2 Prandtl function

If a solution of problem $\mathcal{P}_1$ is defined in terms of the Prandtl function φ, the first stage is to determine the real constants A, B, C by substituting into equation (17.10)

$$\nabla^2 \varphi = \frac{A\nu y}{(1+\nu)} - \frac{B\nu x}{(1+\nu)} + C$$

and equating coefficients. By expressing the antiplane stress Ψ from equation (28.33) in terms of σ_{zx}, σ_{zy} and comparing the results with (17.8), we can show that the complex potentials ϕ_P, ψ_P of equation (28.31) are equivalent to the Prandtl function

$$\varphi_P = -\frac{B[3(1+2\nu)y^2 x - (1-2\nu)x^3]}{24(1+\nu)} + \frac{A[3(1+2\nu)x^2 y - (1-2\nu)y^3]}{24(1+\nu)} + \frac{C(x^2+y^2)}{4} \; , \tag{28.50}$$

if the constants A_0, B_0, C_0 are replaced by

$$A_0 = \frac{A - \imath B}{2} \; ; \quad B_0 = 0 \; ; \quad C_0 = -\frac{C}{2} \; .$$

Next, we construct the real harmonic function

$$\varphi_H(x, y) = \varphi - \varphi_P \; ,$$

representing the difference between the actual Prandtl function and the particular solution (28.50). This can be converted to complex form as discussed in §19.2.1. We first find the holomorphic function g such that $\varphi_H = g + \overline{g}$ (for example, using the Maple or Mathematica procedure 'rtoch') and then define

$$h = 2\imath g \; . \tag{28.51}$$

It then follows from $(19.28)_2$ that $\varphi_H = \Im(h)$ and the corresponding complex potentials ϕ_H, ψ_H are obtained by substituting h into (28.35). Finally, the complex potentials appropriate to φ are recovered as

$$\phi = \phi_P + \phi_H \; ; \quad \psi = \psi_P + \psi_H \; . \tag{28.52}$$

This procedure is contained in the Maple and Mathematica files 'Prandtl-toPN'.

28.8 Solution procedure

We are now in a position to summarize the solution procedure for the three-dimensional problem $\mathcal{P}_m$. We first differentiate the tractions $\boldsymbol{T}$ m times with respect to z in order to define the sub-problems $\mathcal{P}_j$, $j=(1,m)$. We shall be particularly interested in the zeroth-order tractions in each sub-problem — i.e. the terms in $\boldsymbol{T}^{(j)}$ that are independent of z, since these are the only terms that are active in the incremental solution.

Suppose the potentials ϕ_j, ψ_j corresponding to problem $\mathcal{P}_j$ are known. In other words, if we substitute these potentials into equations (28.14–28.19), the resulting stresses satisfy the boundary conditions on the tractions $\boldsymbol{T}^{(j)}$ on the lateral surfaces in problem $\mathcal{P}_j$. This solution will contain two undetermined constants A_{j-1}, C_{j-1}, corresponding to a lateral force and a torque on the end of the bar, but we do not concern ourselves with end conditions at this stage.

To move up to the solution of problem $\mathcal{P}_{j+1}$, we proceed as follows:-

(i) Define new potentials by adding in the pure bending solution (28.26) with new constants A_j, B_j — i.e.

$$\phi = \phi_j + \mathcal{G}_j^0 \; ; \quad \psi = \psi_j + \mathcal{H}_j^0 \; . \tag{28.53}$$

(ii) Integrate ϕ, ψ with respect to z as in §28.3. Notice that any terms in these potentials of the form $\chi_n^m, \mathcal{G}_m^n, \mathcal{H}_m^n, \mathcal{Q}_m^n, \mathcal{V}_m^n$ can be integrated using the recurrence relations (28.25, 28.28, 28.43).

(iii) Add in the torsion solution (28.30) with a new constant C_j. At this stage, Θ, Φ will generally contain terms up to order $j-1$ in z and Ψ will contain terms up to order j. However, all except the zeroth order terms (those independent of z) will satisfy the boundary conditions on the lateral surfaces, since both the tractions and the stresses were obtained by one integration with respect to z from the given solution $\mathcal{P}_j$.

(iv) To satisfy the boundary conditions on the zeroth order terms in the in-plane tractions $T^{(j+1)}$, we add in the in-plane solution from §28.6 and solve an in-plane boundary value problem, using the methods of §19.3. Alternatively, we could solve the corrective in-plane problem using the real Airy stress function of Chapters 5–13 and then use the relations in §28.7.1 to convert the resulting biharmonic function to P-N form. Solvability of this in-plane problem requires that the corrective tractions be self-equilibrated in the plane (see for example §19.5.1 and §19.6.1) and this provides a condition for determining the constants A_{j-1}, C_{j-1}.

(v) To satisfy the boundary conditions on the zeroth order term in the antiplane tractions $T_z^{(j+1)}$, we add in the antiplane corrective solution from (28.35) and determine the function $h(\zeta)$, using the procedure outlined in §28.5.1. This solution is possible only if the tractions associated with h alone are self-equilibrating and hence the solution at this stage will also determine the constant B_j.

This completes the solution of problem $\mathcal{P}_{j+1}$, except for the constants A_j, C_j. The same procedure can be adapted to give the solution of problem $\mathcal{P}_1$ by setting $j = 0$ and replacing (28.53) by (28.26).

After repeating the procedure m times, a solution will be obtained that satisfies the traction conditions $\boldsymbol{T}$ completely and which contains two free constants A_{m-1}, C_{m-1}. To complete the solution, we once again add in the zeroth order solution, as in equations (28.53), using constants A_m, B_m. Finally, we determine the constants $A_{m-1}, C_{m-1}, A_m, B_m$ from the weak end conditions (three force and three moment resultants) at $z=0$ in $\mathcal{P}_m$. More specifically, the conditions on axial force and torque determine the real constants B_m, C_{m-1}, respectively, the conditions on lateral forces determine A_{m-1} (which is a complex constant and hence has two degrees of freedom) and the conditions on bending moments determine A_m.

28.9 Example

The solution procedure described involves a sequence of essentially elementary steps, but for a problem of significant order the number of such steps is large and the solution can become extremely complicated. For this reason, in most cases it is only really practical to undertake the solution using Maple or Mathematica. However, to illustrate the procedure we shall give the solution of a suitably low-order problem.

We consider the circular bar of radius a defined by $0 \le r < a$, $0 < z < L$, in which the end $z = 0$ is traction-free and the curved surfaces are subjected to the tractions

$$T = 0 \; ; \quad T_z = Sz\cos\theta \; ,$$

where S is a real constant. It is clear that the loading is symmetrical about the plane $\theta = 0, \pi$ ($y = 0$) and hence that the bar will not be subject to torsion, so the constants C_j in §28.8(iii) will all be zero and can be omitted.

Comparison with (28.1) shows that this is a problem of class $\mathcal{P}_2$. The sub-problem $\mathcal{P}_1$ corresponds to the tractions

$$T = 0 \; ; \quad T_z = S\cos\theta \; . \tag{28.54}$$

28.9.1 Problem $\mathcal{P}_1$

The solution of problem $\mathcal{P}_0$ is defined by the functions ϕ_0, ψ_0 of equation (28.26). Integrating with respect to z as in §28.5, but omitting the torsion term, we obtain

$$\phi = \mathcal{G}_0^1 \; ; \quad \psi = \mathcal{H}_0^1 \; . \tag{28.55}$$

There are no in-plane stresses or tractions in problem $\mathcal{P}_1$, so no in-plane correction is required.

Antiplane correction

The antiplane stress components Ψ are then obtained as

$$\Psi = -\frac{A_0(1+2\nu)\zeta^2}{4(1+\nu)} - \frac{\bar{A}_0\zeta\bar{\zeta}}{2(1+\nu)} - \frac{B_0\zeta}{2} \,, \tag{28.56}$$

from (28.19). The antiplane traction T_z is obtained exactly as in §28.5.2 as

$$T_z = \sigma_{zr}(a,\theta) = -\frac{(3+2\nu)a^2\left(\Re\{A_0\}\cos\theta - \Im\{A_0\}\sin\theta\right)}{4(1+\nu)} - \frac{B_0 a}{2} + \frac{1}{a}\frac{\partial h_y}{\partial\theta} \tag{28.57}$$

and equating this to the boundary value (28.54), we obtain

$$\frac{\partial h_y}{\partial\theta} = \frac{(3+2\nu)a^3\left(\Re\{A_0\}\cos\theta - \Im\{A_0\}\sin\theta\right)}{4(1+\nu)} + \frac{B_0 a^2}{2} + Sa\cos\theta \,.$$

This integrates to a single-valued function of θ if and only if $B_0=0$, giving

$$h_y = \frac{(3+2\nu)a^3\left(\Re\{A_0\}\sin\theta + \Im\{A_0\}\cos\theta\right)}{4(1+\nu)} + Sa\sin\theta$$

and the holomorphic function h whose imaginary part takes this boundary value is

$$h = \frac{(3+2\nu)A_0a^2\zeta}{4(1+\nu)} + S\zeta \,. \tag{28.58}$$

Substituting this expression into (28.35) and adding the results to (28.55), we obtain

$$\phi_1 = \mathcal{Q}_0^0 + \frac{(3-4\nu)S(\chi_1^1+\overline{\chi}_1^1)}{4(1-\nu)} \;; \quad \psi_1 = \mathcal{V}_0^0 + \frac{S\chi_1^0}{2(1-\nu)} \,, \tag{28.59}$$

which defines the solution to $\mathcal{P}_1$ in P-N form.

28.9.2 Problem $\mathcal{P}_2$

The next stage is to substitute ϕ_1, ψ_1 into (28.53) with $j=1$ and perform a further integration with respect to z, using equations (28.25,28.28, 28.43) to ensure that the potentials ϕ, ψ are three-dimensionally harmonic. We obtain

$$\phi = \mathcal{Q}_0^1 + \frac{(3-4\nu)S(\chi_2^1+\overline{\chi}_2^1)}{4(1-\nu)} + \mathcal{G}_1^1 \;; \quad \psi = \mathcal{V}_0^1 + \frac{S\chi_2^0}{2(1-\nu)} + \mathcal{H}_1^1 \,. \tag{28.60}$$

In-plane correction

There are no z-independent components in the in-plane tractions T for problem $\mathcal{P}_2$ and we anticipate requiring a corrective solution to satisfy this condition. Substituting the partial solution (28.60) into equations (28.16, 28.17), we obtain the in-plane stress components

$$\Theta = \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2\right](A_0\zeta + \bar{A}_0\bar{\zeta})}{16(1-\nu^2)} - \frac{S(3-4\nu)(\zeta+\bar{\zeta})}{4(1-\nu)}$$

$$\Phi = \frac{(1+4\nu)A_0\zeta^3}{24(1+\nu)} + \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2\right]\bar{A}_0\zeta}{16(1-\nu^2)} - \frac{S\zeta}{4(1-\nu)} . \quad (28.61)$$

From equation (19.70), we have

$$Tds = \frac{\imath}{2}\left(\Phi d\bar{\zeta} - \Theta d\zeta\right)$$

and on $r=a$,

$$\zeta = ae^{\imath\theta} \; ; \quad \bar{\zeta} = ae^{-\imath\theta} \; ; \quad d\zeta = \imath ae^{\imath\theta}d\theta \; ; \quad d\bar{\zeta} = -\imath ae^{-\imath\theta}d\theta \; , \quad (28.62)$$

so the integral around a portion of the boundary of the complex traction T is

$$\int Tds = \frac{a}{2}\int\left(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}\right)d\theta \; . \quad (28.63)$$

Substituting the boundary values (28.62) in (28.61), we obtain

$$\Phi e^{-\imath\theta} + \Theta e^{\imath\theta} = -\frac{(19-18\nu-16\nu^2)A_0a^3e^{2\imath\theta}}{48(1-\nu^2)} - \frac{Sa(3-4\nu)e^{2\imath\theta}}{4(1-\nu)}$$
$$-\frac{\bar{A}_0a^3}{2} - Sa \; . \quad (28.64)$$

The in-plane corrective problem is solvable if and only if the tractions on the boundary are self-equilibrating, which requires that the integral (28.63) return a single-valued function of θ. This condition requires that

$$-\frac{\bar{A}_0a^3}{2} - Sa = 0 \quad \text{and hence} \quad \bar{A}_0 = -\frac{2S}{a^2} = A_0 \; . \quad (28.65)$$

Notice that since the constant $\bar{A}_0$ is real it is also equal to A_0. Using this result, we then obtain

$$\Phi e^{-\imath\theta} + \Theta e^{\imath\theta} = \frac{Sa(1-12\nu+8\nu^2)e^{2\imath\theta}}{24(1-\nu^2)} \; . \quad (28.66)$$

To satisfy the homogeneous in-plane boundary conditions on Γ in $\mathcal{P}_2$, we superpose holomorphic functions f, g through equations (28.44, 28.47) and choose them so as to reduce (28.66) to zero for all θ. This can be done using the direct method of §19.6.1, but the form of (28.66) makes it clear that low order polynomial terms in f,g will give an expression of the appropriate form. Choosing

$$f = 0 \; ; \quad g = C\zeta^2 = C\chi_0^2 \qquad \text{and hence} \qquad \phi = 0 \; ; \quad \psi = C\chi_0^2 \tag{28.67}$$

from (28.44), where C is a real constant, we obtain the additional stress components

$$\Theta = -4C(\zeta + \bar{\zeta}) \; ; \quad \Phi = 4C\zeta \; ,$$

from (28.47) and on the surface $r=a$,

$$\Phi e^{-\imath\theta} + \Theta e^{\imath\theta} = -4Cae^{2\imath\theta} \; .$$

The boundary condition $T=0$ will therefore be satisfied for all θ if

$$C = \frac{S(1-12\nu+8\nu^2)}{96(1-\nu^2)} \; .$$

Antiplane correction

Since there are no zeroth-order antiplane tractions in $\mathcal{P}_2$, the antiplane correction is identical to that in §28.5.2. In particular, we have $B_1=0$ and the additional potentials are derived from (28.40) with A_0 replaced by A_1.

Applying this and the in-plane correction (28.67) to (28.60) and using (28.65) to eliminate A_0, we then have

$$\phi_2 = -\frac{S(1-2\nu)}{a^2(1-\nu^2)}\left(\chi_4^1 + \overline{\chi}_4^1\right) - \frac{S(3-4\nu)}{8(1-\nu^2)}\left(\chi_2^1 + \overline{\chi}_2^1\right) + \mathcal{Q}_1^0 \tag{28.68}$$

$$\psi_2 = -\frac{2S\chi_4^0}{a^2(1-\nu^2)} + \frac{S(1-2\nu)\chi_2^2}{4a^2(1-\nu^2)} - \frac{S\chi_2^0}{4(1-\nu^2)} + \frac{S(1-12\nu+8\nu^2)\chi_0^2}{96(1-\nu^2)} + \mathcal{V}_1^0 \; , \tag{28.69}$$

which completes the solution of problem $\mathcal{P}_2$.

28.9.3 End conditions

It remains to satisfy the weak traction-free conditions on the end $z=0$. We add the pure bending solution (28.26) with new constants A_2, B_2 into (28.68, 28.69), which is achieved by adding the terms $\mathcal{G}_2^0, \mathcal{H}_2^0$ into ϕ, ψ respectively. We then substitute the resulting potentials into (28.18, 28.19) to obtain the complex stress components σ_{zz}, Ψ. In particular, the tractions on the end of the bar $z=0$ are obtained as

$$\sigma_{zz}(0,\zeta,\bar{\zeta}) = \frac{S(2+\nu)\zeta\bar{\zeta}(\zeta+\bar{\zeta})}{4a^2(1+\nu)} - \frac{S(3-2\nu)(\zeta+\bar{\zeta})}{6} + A_2\zeta + \bar{A}_2\bar{\zeta} + B_2$$
$$\Psi(0,\zeta,\bar{\zeta}) = \frac{\bar{A}_1[(3+2\nu)a^2 - 2\zeta\bar{\zeta}] - A_1(1+2\nu)\zeta^2}{4(1+\nu)} \ . \qquad (28.70)$$

Clearly the shear tractions $\Psi = \sigma_{zx} + \imath\sigma_{zy}$ on $z=0$ can be set to zero in the strong sense by choosing $A_1 = 0$. For the normal tractions, we use (28.3) to calculate the force and moment resultants

$$F_z = \int_0^a \int_0^{2\pi} \sigma_{zz}(0,r,\theta) r d\theta dr = \pi B_2 a^2$$

$$M_x + \imath M_y = \int_0^a \int_0^{2\pi} \sigma_{zz}(0,r,\theta) r^2 e^{\imath\theta} dr d\theta = -\frac{\pi S(1-2\nu^2)a^4}{12(1+\nu)} + \frac{\pi \bar{A}_2 a^4}{2} \ .$$

Thus, the weak traction-free conditions are satisfied by setting

$$A_2 = \bar{A}_2 = \frac{S(1-2\nu^2)}{6(1+\nu)} \ ; \quad B_2 = 0 \ .$$

Substituting these constants into the potentials and simplifying, we obtain

$$\phi = -\frac{S(1-2\nu)}{a^2(1-\nu^2)}\left(\chi_4^1 + \overline{\chi}_4^1\right) - \frac{S(7+\nu-8\nu^2-8\nu^3)}{24(1-\nu^2)(1+\nu)}\left(\chi_2^1 + \overline{\chi}_2^1\right)$$
$$\psi = -\frac{2S\chi_4^0}{a^2(1-\nu^2)} + \frac{S(1-2\nu)\chi_2^2}{4a^2(1-\nu^2)} - \frac{S(1+3\nu+4\nu^2)\chi_2^0}{12(1-\nu^2)(1+\nu)} - \frac{S(1+7\nu)\chi_0^2}{96(1-\nu^2)(1+\nu)} \ ,$$

which defines the complete solution of the problem. The complex stress components can be recovered by substitution into equations (28.16–28.19) as

$$\Theta = \frac{S(\zeta+\bar{\zeta})}{3} - \frac{S\zeta\bar{\zeta}(\zeta+\bar{\zeta})}{4(1+\nu)a^2}$$
$$\Phi = \frac{S\zeta[2a^2 - 3\zeta\bar{\zeta} - (1+4\nu)\zeta^2 + 2(1-2\nu)a^2]}{12(1+\nu)a^2}$$
$$\sigma_{zz} = \frac{S(2+\nu)(3\zeta\bar{\zeta} - 2a^2)(\zeta+\bar{\zeta})}{12(1+\nu)a^2} - \frac{Sz^2(\zeta+\bar{\zeta})}{a^2}$$
$$\Psi = \frac{Sz\left[(1+2\nu)\zeta^2 + 2\zeta\bar{\zeta} - a^2\right]}{2(1+\nu)a^2} \ .$$

We have deliberately given a very detailed and hence lengthy solution of this example problem in order to clarify the steps involved, but the astute reader will realize that many steps might reasonably have been omitted as trivial. For example, the original problem is even in z, which immediately allows us to omit the terms involving the constants $A_1, \bar{A}_1, B_1$, causing the in-plane correction to be trivial in problems $\mathcal{P}_j$ where j is odd and the antiplane correction to be trivial when j is even.

PROBLEMS

1. A circular cylindrical bar of radius a is loaded by the tangential tractions $\sigma_{r\theta} = S\cos\theta$ on $r = a$. The end of the bar $z=0$ is traction-free. Find the complete stress field in the bar.

2. A circular cylindrical bar of radius a is loaded by a uniform radially-inward load F_0 per unit length along the line $\theta = 0$. The end of the bar $z = 0$ is traction-free. Find the complete stress field in the bar.

3. A circular cylindrical bar of radius a is loaded by a uniform tangential load F_0 per unit length along the line $\theta=0$. The load acts in the positive θ-direction and the end of the bar $z=0$ is traction-free. Find the complete stress field in the bar.

4. The lateral surfaces Γ of a bar of equilateral trianglular cross-section are defined by the equations $x = \pm\sqrt{3}y, x = a$. The bar is twisted by in-plane tractions T on Γ that are independent of z, the end $z = 0$ being traction-free. The tractions on the surface $x = a$ are defined by

$$\sigma_{xx} = 0\ ; \quad \sigma_{xy} = S$$

and those on the other two surfaces are similar under a rotation of $\pm 120°$.

This problem lies in class $\mathcal{P}_2$. Use the strategy suggested in Problem 16.3 to find a Prandtl stress function solution of the corresponding problem $\mathcal{P}_1$, convert it to P-N form as in §28.5.4 and then integrate with respect to z to obtain a particular solution of $\mathcal{P}_2$. Determine the tractions on the surface $x = a$ implied by this solution and discuss possible strategies for completing the in-plane correction, but do not attempt it.

5. The rectangular bar $-a<x<a, -b<y<b, 0<z<L$ is built in at $z=L$ and loaded only by the uniform shear tractions

$$\sigma_{xy}(a, y, z) = S\ ; \quad \sigma_{xy}(-a, y, z) = -S$$

on the surfaces $x = \pm a$. Find the complete stress field in the bar using weak conditions on the end $z = 0$. Assume that $b \gg a$ and use weak boundary conditions on the edges $y = \pm b$. **Hint:** The problem is even in x and odd in y, which implies that $\Re(A_0) = C_0 = 0$. The easiest way to impose these symmetries on the rest of the solution is to use an Airy function for the in-plane corrective solution.

6. A bar of square cross-section $-a<x<a, = -a<y<a, 0<z<L$ is built in at $z=L$ and loaded only by the uniform tractions

$$\sigma_{yy}(x, a, z) = -p_0$$

on the surface $y = a$. Find the complete stress field in the bar using only the closed-form terms from the solution of §17.4.2 for the problem $\mathcal{P}_1$ and an appropriate finite polynomial as an Airy stress function for the in-plane correction in $\mathcal{P}_2$. Plot appropriate stress components to estimate the errors in the elementary bending solution.

29

FRICTIONLESS CONTACT

As we noted in §21.5.1, Green and Zerna's Solution F is ideally suited to the solution of frictionless contact problems for the half-space, since it identically satisfies the condition that the shear tractions be zero at the surface $z=0$. In fact the *surface* tractions for this solution take the form

$$\sigma_{zz} = -\frac{\partial^2 \varphi}{\partial z^2} \;\; ; \;\; \sigma_{zx} = \sigma_{zy} = 0 \;\; ; \;\; z = 0 \; , \tag{29.1}$$

whilst the surface displacements are

$$u_x = \frac{(1-2\nu)}{2\mu}\frac{\partial \varphi}{\partial x} \;\; ; \;\; u_y = \frac{(1-2\nu)}{2\mu}\frac{\partial \varphi}{\partial y} \;\; ; \;\; u_z = -\frac{(1-\nu)}{\mu}\frac{\partial \varphi}{\partial z} \;\; ; \;\; z = 0 \; , \tag{29.2}$$

from Table 21.3.

29.1 Boundary conditions

Following §12.5, we can now formulate the general problem of indentation of the half-space by a frictionless rigid punch whose profile is described by a function $u_0(x,y)$ (see Figure 12.5). Notice that since the problem is three-dimensional, the punch profile is now a function of two variables x, y and the contact area will be an extended region of the plane $z=0$, which we denote by A.

Within the contact area, the normal displacement of the half-space must conform to the shape of the punch and hence

$$u_z = u_0(x,y) + C_0 + C_1 x + C_2 y \quad \text{in } A \; , \tag{29.3}$$

where the constants C_0, C_1, C_2 define an arbitrary rigid-body displacement of the punch. Outside the contact area, there must be no normal tractions. i.e.

$$\sigma_{zz} = 0 \quad \text{in } \bar{A} \; , \tag{29.4}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_29, © Springer Science+Business Media B.V. 2010

where we denote the complement of A — i.e. that part of the surface $z=0$ which is *not* in contact — by $\bar{A}$.

We now use equations (29.1, 29.2) to write these conditions in terms of the potential function φ with the result

$$\frac{\partial\varphi}{\partial z} = -\frac{\mu}{(1-\nu)}u_0(x,y) \;\; ; \;\; \text{in } A \tag{29.5}$$

$$\frac{\partial^2\varphi}{\partial z^2} = 0 \;\; ; \;\; \text{in } \bar{A} \, , \tag{29.6}$$

where for brevity we have omitted the rigid-body displacement terms in equation (29.3), since these can easily be reintroduced as required or subsumed under the function $u_0(x,y)$.

29.1.1 Mixed boundary-value problems

Equations (29.5, 29.6) define a *mixed boundary-value problem* for the potential function φ in the region $z>0$. The word 'mixed' here refers to the fact that different derivatives of the function are specified at different parts of the boundary. More specifically, it is a *two-part* mixed boundary-value problem[1], since the boundary conditions are specified over two complementary regions of the boundary, A, $\bar{A}$.

Similar mixed boundary-value problems arise in many fields of engineering mechanics, such as heat conduction, electrostatics and fluid mechanics. For example, in heat conduction, a mathematically similar problem arises if the region A is raised to a prescribed temperature whilst $\bar{A}$ is insulated. They can be reduced to integral equations using a Green's function formulation. Thus, considering the normal traction $\sigma_{zz}(x,y,0)\equiv -p(x,y)$ in A as comprising a set of point forces $F=p(x,y)dxdy$ and using equation (23.18) for the normal surface displacement due to a concentrated normal force at the surface, we can write

$$u_z(x,y,0) = \frac{(1-\nu)}{2\pi\mu}\int\!\!\int_A \frac{p(\xi,\eta)d\xi d\eta}{r(x,y,\xi,\eta)} \, , \tag{29.7}$$

where

$$r(x,y,\xi,\eta) = \sqrt{(x-\xi)^2+(y-\eta)^2} \tag{29.8}$$

is the distance between the points $(\xi,\eta,0)$ and $(x,y,0)$. Substitution in (29.3) then yields a double integral equation for the unknown contact pressure $p(x,y)$. This is of course a generalization of the method used in Chapter 12. However, in three-dimensional problems, the resulting double integral equation is generally of rather intractable form. We shall find that other methods of solution are more efficient (see Chapter 30 below).

[1] Strictly this terminology is restricted to cases where the two regions A, $\bar{A}$ are connected. A special case where this condition is not satisfied is the indentation by an annular punch for which $\bar{A}$ has two unconnected regions — one inside the annulus and one outside. This would be referred to as a three-part problem.

The fact that Solution F is described in terms of a single harmonic function enables us to use results from potential theory to prove some interesting results. For example, we note that the derivative $\partial\varphi/\partial z$ must also be harmonic in the half-space $z>0$. Suppose we wish to locate the point where $\partial\varphi/\partial z$ is a maximum. It cannot be inside the region $z>0$, since at a such a maximum we would need

$$\nabla^2 \frac{\partial\varphi}{\partial z} < 0 \tag{29.9}$$

and the left-hand side of this expression is zero everywhere. It follows that the maximum must occur somewhere on the boundary $z=0$ or else at infinity. Furthermore, if it occurs on the boundary, all immediately adjacent points must have lower values and in particular, we must have

$$\frac{\partial^2\varphi}{\partial z^2} < 0 \tag{29.10}$$

at the maximum point.

Recalling equations (29.1, 29.2), we conclude that the maximum value of the surface displacement u_z must occur either at infinity or at a point where the contact traction σ_{zz} is compressive. If we choose a frame of reference such that the displacement at infinity is zero, it follows that the maximum can only occur there if the surface displacement is negative throughout the finite domain, which can only arise if the net force on the half-space is tensile. Thus, when the half-space is loaded by a net compressive force, the maximum surface displacement must occur in a region where the surface traction is compressive, and must be positive.

29.2 Determining the contact area

In many contact problems, the contact area is not known *a priori*, but has to be determined as part of the solution[2]. As in Chapter 12, the contact area has to be chosen so as to satisfy the two inequalities

$$\sigma_{zz} < 0 \;\; ; \;\; \text{in } A \tag{29.11}$$

$$u_z > u_0(x,y) + C_0 + C_1 x + C_2 y \;\; ; \;\; \text{in } \bar{A} \; , \tag{29.12}$$

which state respectively that the contact traction should never be tensile and that the gap between the indenter and the half-space should always be positive. It is worth noting that physical considerations demand that these conditions be satisfied in *all* problems — not only those in which the contact area is initially unknown. However, if the indenter is rigid and has sharp corners, the contact area will usually be identical with the plan-form of the indenter and conditions (29.11, 29.12) need not be explicitly imposed. Notice

[2] See for example §12.5.3.

however that if there is also distortion due to thermoelastic effects or if there is a sufficiently large rigid-body rotation as in Problem 12.4, the contact area may not be coextensive with the plan-form of the indenter. In general, conditions (29.11, 29.12) define a *solution space* in which the mathematical solution will be physically meaningful. If any further restrictions are imposed in the interests of simplicity, it is always wise to check the solution at the end to ensure that the inequalities are satisfied.

In treating the Hertzian problem in §12.5.3, we were able to satisfy the inequalities by choosing the contact area in such a way that the contact traction tended to zero at the edge. A similar technique can be applied in three-dimensional problems provided that the profile of the indenter is smooth, but now we have to determine the equation of a boundary line in the plane $z=0$, rather than two points defining the ends of a strip. This is generally a formidable problem and closed form solutions are known only for the case where the contact area is either an ellipse or a circle.

Boundedness of the contact traction at the edge of the contact area is a necessary but not sufficient condition for the inequalities to be satisfied. Suppose for example the indenter has concave regions as shown in Figure 29.1. For some values of the load, there will be contact in two discrete regions near to A and B. However, we could construct a mathematical solution assuming contact near B only, in which the traction was bounded throughout the edge of the contact area and in which there would be interpenetration (i.e. violation of inequality (29.12)) near A. In general, the boundedness condition may be insufficient to determine a unique solution if the physically correct contact area is not simply connected.

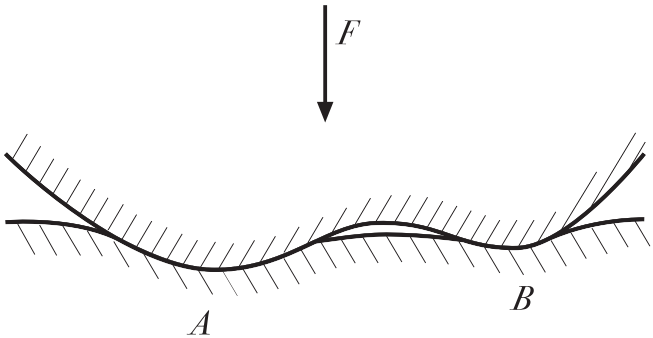

Figure 29.1: Contact problem which will have two unconnected multiply connected contact areas in an intermediate load range

In a numerical solution, this difficulty can be avoided by using the inequalities directly to determine the contact area by iteration. A simple approach is to guess the extent of the contact area A, solve the resulting contact problem numerically and then examine the solution to determine whether the inequalities (29.11, 29.12) are violated at any point. Any points in A at which the contact traction is tensile are then excluded from A in the next iteration, whereas any points in $\bar{A}$ at which the gap is negative are included in A.

This method is found to converge rapidly, but it involves the calculation of tractions or displacements at all surface nodes. An alternative method is to formulate a function which is minimized when the solution satisfies the inequalities and then treat the iterative process as an optimization problem. Kalker[3] and Kikuchi and Oden[4] have explored this technique at length and used it to develop finite element solutions to the contact problem.

A simpler method is available for the special problem of the indentation of an elastic half-space by a rigid indenter. Suppose we select a value at random for the contact area A, solve the resulting indentation problem, and then determine the corresponding total indentation force

$$F(A) = -\int\int_A \sigma_{zz}(x, y, 0)dxdy \ , \tag{29.13}$$

where σ_{zz} is the contact traction associated with the 'wrong' contact area, A. It can be shown that the maximum value of $F(A)$ occurs when A is chosen such that the inequalities (29.11, 29.12) are satisfied — i.e. when A takes its correct value. Thus, we can formulate the iteration process in terms of an optimization problem for $F(A)$. The proof of the theorem is as follows:-

Proof

Consider the effect of increasing A by a small element δA.

If $F(A)$ is thereby increased, the corresponding *differential* contact traction distribution — i.e. the difference between the final and the initial traction — amounts to a net compressive force on the half-space. Now we proved in §29.1.1 that in such cases, the maximum surface displacement must occur in a region where the traction is compressive and be positive in sign. Thus, the maximum differential surface displacement δu_z must occur somewhere in A or δA and be positive in sign. However, δu_z is zero throughout A, since the process of including a further region in the contact area does not affect the displacement boundary condition in A. It therefore follows that the differential displacement in δA is positive and that the contact traction there is compressive.

We know that the final value of u_z in δA is $f(x, y)+C_0+C_1x+C_2y$. The effect of the differential contact traction distribution has therefore been shown to be an *increase* of u_z at δA to the value $f(x, y)+C_0+C_1x+C_2y$ and it follows that δA must be an area for which initially

$$u_z < f(x, y) + C_0 + C_1x + C_2y \tag{29.14}$$

[3] J.J.Kalker, Variational principles of contact elastostatics, *Journal of the Institute for Mathematics and its Applications,* Vol. 20 (1977), pp.199–221.

[4] N.Kikuchi and J.T.Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

— i.e. for which the gap is negative.

Hence, an increase in $F(A)$ can only be achieved by increasing A if there are some areas δA outside A satisfying (29.14). By a similar argument, we can show that an increase in $F(A)$ can follow from a *reduction* in A only if there exist areas δA *inside* A for which

$$\sigma_{zz} > 0 \tag{29.15}$$

— i.e. for which the contact traction is tensile.

It follows that the maximum value of $F(A)$ must occur when A is chosen so that there are no regions satisfying (29.14, 29.15) — i.e. when the original inequalities (29.11, 29.12) are satisfied everywhere[5].

The above method has the formal advantage that it replaces the intractable inequality conditions by a variational statement, but in order to use it we need to have a way of determining the total load $F(A)$ on the indenter for an arbitrary contact area[6] A. We shall show in Chapter 34 how the reciprocal theorem can be used to simplify this problem[7].

29.3 Contact problems involving adhesive forces

At very small length scales, the attractive forces between molecules (van der Waals forces) become significant and must be taken into account in the solution of contact problems. This subject has increased in importance in recent years with the emphasis on micro- and nano-scale systems. One approach is to apply the same arguments as in the Griffith theory of brittle fracture (§13.3.1) and postulate that the contact area will adopt the value for which the total energy (comprising elastic strain energy, potential energy of external forces and surface energy) is at a minimum. Perturbing about this minimum energy state, we then obtain

$$G = \Delta\gamma \; , \tag{29.16}$$

where G is the energy release rate introduced in §13.3.3 and $\Delta\gamma$ is the *interface energy* of the two contacting materials — i.e. the work that must be done per unit area against interatomic forces at the interface to separate two bodies with atomically plane surfaces[8]. It follows from equation (13.54) that the contact tractions will exhibit a tensile square-root singularity at the edge of the contact area characterized by a stress intensity factor

[5] For more details of this argument see J.R.Barber, Determining the contact area in elastic contact problems, *Journal of Strain Analysis,* Vol. 9 (1974), pp.230–232.

[6] Even then, the variational problem is far from trivial.

[7] See also R.T.Shield, Load-displacement relations for elastic bodies, *Zeitschrift für angewandte Mathematik und Physik,* Vol. 18 (1967), pp.682–693.

[8] This will generally be less than the sum of the surface energies $\gamma_1 + \gamma_2$ of the two contacting materials, except in the case of similar materials.

$$K_I = \lim_{s \to 0} \sigma_{zz}(s)\sqrt{2\pi s} \ , \tag{29.17}$$

where $\sigma_{zz}(s)$ is the normal contact traction (tensile positive) at a distance s from the edge of the contact area. For the more general case of frictionless contact between dissimilar materials, equation (13.54) is modified to

$$G = \frac{K_I^2}{16}\left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_2+1}{\mu_2}\right)$$

and for three-dimensional problems, the local conditions at the edge of the contact are always those of plane strain ($\kappa = 3-4\nu$) giving

$$K_I = 2\sqrt{\Delta\gamma \Bigg/ \left(\frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}\right)} \ , \tag{29.18}$$

using (29.16). This approach to adhesive contact problems was first introduced by Johnson, Kendall and Roberts[9] and has come to be known as the JKR theory. Notice that it predicts that the stress intensity factor is constant around the edge of the contact region, regardless of its shape. Of course, in the special case where adhesive forces and hence surface energy are neglected, this stress intensity factor will be zero and the theory reduces to the classical condition that the contact tractions are square-root bounded at the edge of the contact area.

The JKR theory is approximate in that it neglects the attractive forces between the surfaces in the region where the gap is small but non-zero. More exact numerical solutions using realistic force separation laws show that the theory provides a good limiting solution in the range where the dimensionless parameter

$$\Psi \equiv \left[\frac{R(\Delta\gamma)^2}{4\varepsilon^3}\left(\frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}\right)^2\right]^{1/3} \gg 1 \ ,$$

where R is a representative radius of the contacting surfaces and ε is the equilibrium intermolecular distance[10].

---

[9] K.L.Johnson, K.Kendall and A.D.Roberts, Surface energy and the contact of elastic solids, *Proceedings of the Royal Society of London*, Vol. A324 (1971), pp.301–313.

[10] J.A.Greenwood, Adhesion of elastic spheres, *Proceedings of the Royal Society of London*, Vol. A453 (1997), pp.1277–1297.

PROBLEMS

1. Using the result of Problem 23.4 or otherwise, show that if the punch profile $u_0(x, y)$ has the form

$$u_0 = r^p f(\theta)$$

where $p > 0$ and $f(\theta)$ is any function of θ, the resulting frictionless contact problems at different indentation forces will be self-similar[11].

Show also that if l is a representative dimension of the contact area, the indentation force, F, and the rigid-body indentation, d, will vary according to

$$F \sim l^{p+1} \;\; ; \;\; d \sim l^p$$

and hence that the indentation has a stiffening load-displacement relation

$$F \sim d^{1+\frac{1}{p}} \; .$$

Verify that the Hertzian contact relations (§30.2.5 below) agree with this result.

2. A frictionless rigid body is pressed into an elastic body of shear modulus μ and Poisson's ratio ν by a normal force F, establishing a contact area A and causing the rigid body to move a distance δ. Use the results of §29.1 to define the boundary-value problem for the potential function φ corresponding to the *incremental problem*, in which an infinitesimal additional force increment ΔF produces an incremental rigid-body displacement $\Delta\delta$.

Suppose that the rigid body is a perfect electrical conductor at potential V_0 and the 'potential at infinity' in the elastic body is maintained at zero. Define the boundary-value problem for the potential V in the elastic body, noting that the current density vector $\boldsymbol{i}$ is given by Ohm's law

$$\boldsymbol{i} = -\frac{1}{\rho}\boldsymbol{\nabla} V \; ,$$

where ρ is the electrical resistivity of the material. Include in your statement an expression for the total current I transmitted through the contact interface.

By comparing the two potential problems or otherwise, show that the electrical contact resistance $R = V_0/I$ is related to the incremental contact stiffness $dF/d\delta$ by the equation

$$\frac{1}{R} = \frac{(1-\nu)}{\rho\mu}\frac{dF}{d\delta} \; .$$

[11] D.A.Spence, An eigenvalue problem for elastic contact with finite friction, *Proceedings of the Cambridge Philosophical Society,* Vol. 73 (1973), pp.249–268, has shown that this argument also extends to problems with Coulomb friction at the interface, in which case, the zones of stick and slip also remain self-similar with monotonically increasing indentation force.

3. Use equations (18.9, 18.11) to show that the function

$$f(x,z) = \Im\left\{\arcsin\left(\frac{x+\imath z}{a}\right)\right\}$$

satisfies Laplace's equation. Show that the function

$$\frac{\partial\varphi}{\partial z} = A + Bf(x,z)$$

can be used to solve the plane strain equivalent of the flat punch problem of §12.5.2 with suitable choices for the constants A, B. In particular, obtain expressions for the contact pressure distribution and for the normal surface displacement u_z outside the contact area.

Note: For complex arguments, we can write[12]

$$\arcsin(\zeta) = \frac{1}{\imath}\ln(\imath\zeta + \sqrt{1-\zeta^2}) \; .$$

4. Show that for frictionless contact problems for the half-space $z > 0$, the normal stress components near the surface $z = 0$ satisfy the relation

$$\sigma_{xx} + \sigma_{yy} = (1+2\nu)\sigma_{zz} \; ,$$

where σ_{zz} is the applied traction.

[12] See I.S.Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1980, §1.622.1.

30

THE BOUNDARY-VALUE PROBLEM

The simplest frictionless contact problem of the class defined by equations (29.3–29.6) is that in which the contact area A is the circle $0 < r < a$ and the indenter is axisymmetric, in which case we have to determine a harmonic function $\varphi(r, z)$ to satisfy the mixed boundary conditions

$$\frac{\partial\varphi}{\partial z} = -\frac{\mu}{(1-\nu)} u_0(r) \;\; ; \;\; 0 \le r < a \;, \; z = 0 \tag{30.1}$$

$$\frac{\partial^2\varphi}{\partial z^2} = 0 \;\; ; \;\; r > a \;, \; z = 0 \;. \tag{30.2}$$

30.1 Hankel transform methods

This is a classical problem and many solution methods have been proposed. The most popular is the Hankel transform method developed by Sneddon[1] and discussed in the application to contact problems by Gladwell[2]. The function

$$f(r, z) = \exp(-\xi z) J_0(\xi r) \;, \tag{30.3}$$

where ξ is a constant, is harmonic and hence a more general axisymmetric harmonic function can be written in the form

$$f(r, z) = \int_0^\infty A(\xi) \exp(-\xi z) J_0(\xi r) d\xi \;, \tag{30.4}$$

where $A(\xi)$ is an unknown function to be determined from the boundary conditions.

[1] I.N.Sneddon, Note on a boundary-value problem of Reissner and Sagoci, *Journal of Applied Physics,* Vol. 18 (1947), pp.130–132.

[2] G.M.L.Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_30,

Using this representation for φ and substituting into equations (30.1, 30.2), we find that

$$-\int_0^\infty \xi A(\xi) J_0(\xi r) d\xi = -\frac{\mu}{(1-\nu)} u_0(r) \ ; \ 0 \le r < a \tag{30.5}$$

$$\int_0^\infty \xi^2 A(\xi) J(\xi r) d\xi = 0 \ ; \ r > a \ . \tag{30.6}$$

These constitute a pair of *dual integral equations* for the function $A(\xi)$. Sneddon used the method of Titchmarsh[3] and Busbridge[4] to reduce equations of this type to a single equation, but a more recent solution by Sneddon[5] and formalized by Gladwell[6] effects this reduction more efficiently for the classes of equation considered here.

30.2 Collins' Method

A related method, which has the advantage of yielding a single integral equation in elementary functions directly, was introduced by Green and Zerna[7] and applied to a wide range of axisymmetric boundary-value problems by Collins[8].

30.2.1 Indentation by a flat punch

To introduce Collins' method, we first examine the simpler problem in which the punch is flat and hence $u_0(r)$ is a constant. This was first solved by Boussinesq in the 1880s.

A particularly elegant solution was developed by Love[9], using a series of complex harmonic potential functions generated from the real Legendre polynomial solutions of Chapter 24 by substituting $(z+\imath a)$ for z. This is tantamount to putting the origin at the 'imaginary' point $(0, -\imath a)$. The real and imaginary parts of the resulting functions are separately harmonic and have discontinuities at $r=a$ on the plane $z=0$.

---

3 E.C.Titchmarsh, *Introduction to the Theory of Fourier Integrals,* Clarendon Press, Oxford, 1937.

4 I.W.Busbridge, Dual integral equations, *Proceedings of the London Mathematical Society,* Ser.2 Vol. 44 (1938), pp.115–129

5 I.N.Sneddon, The elementary solution of dual integral equations, *Proceedings of the Glasgow Mathematical Association,* Vol. 4 (1960), pp.108–110.

6 G.M.L.Gladwell, *loc. cit.*, Chapters 5,10.

7 A.E.Green and W.Zerna, *loc. cit.*.

8 W.D.Collins, On the solution of some axisymmetric boundary-value problems by means of integral equations, II: Further problems for a circular disc and a spherical cap, *Mathematika,* Vol. 6 (1959), pp.120–133.

9 A.E.H.Love, Boussinesq's problem for a rigid cone, *Quarterly Journal of Mathematics,* Vol. 10 (1939), pp.161–175.

For example, if we start with the harmonic function $\ln(R+z)$ from equation (23.22) and replace z by $z+\imath a$, we can define the new harmonic function

$$\phi = \phi_1 + \imath\phi_2 = \ln(R^* + z + \imath a) \;, \tag{30.7}$$

where

$$R^{*2} = r^2 + (z + \imath a)^2 \;. \tag{30.8}$$

We also record the first derivative of ϕ which is

$$\frac{\partial \phi}{\partial z} = \frac{1}{R^*} \;. \tag{30.9}$$

On the plane $z=0$, $R^* \to \sqrt{r^2 - a^2}$ and hence

$$\phi(r,0) = \ln(\sqrt{r^2 - a^2} + \imath a) \tag{30.10}$$

$$\frac{\partial \phi}{\partial z}(r,0) = \frac{1}{\sqrt{r^2 - a^2}} \;. \tag{30.11}$$

Both the real and imaginary parts of these functions have discontinuities at $r=a$ on the plane $z=0$. For example

$$\phi_2(r,0) \equiv \Im\{\phi(r,0)\} = \frac{\pi}{2} \quad ; \quad 0 \le r < a \tag{30.12}$$

$$= \sin^{-1}\frac{a}{r} \quad ; \quad r > a \;, \tag{30.13}$$

whilst

$$\frac{\partial \phi_2}{\partial z}(r,0) = -\frac{1}{\sqrt{a^2 - r^2}} \quad ; \quad 0 \le r < a \tag{30.14}$$

$$= 0 \;\; ; \;\; r > a \;. \tag{30.15}$$

Comparing these results with the boundary conditions (30.1, 30.2), we see that the function

$$\frac{\partial \varphi}{\partial z} = -\frac{2\mu u_0}{\pi(1-\nu)}\Im\{\ln(R^* + z + \imath a)\} \tag{30.16}$$

identically satisfies both boundary conditions for the case where the function $u_0(r)$ is a constant — i.e. if the rigid indenter is flat and is pressed a distance u_0 into the half-space, as shown in Figure 30.1.

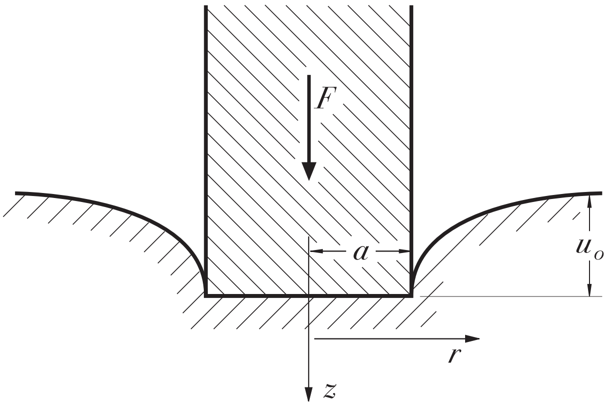

Figure 30.1: Indentation by a flat-ended cylindrical rigid punch.

The contact pressure distribution is then immediately obtained from equations (29.1, 30.14, 30.16) and is

$$p(r) = -\sigma_{zz}(r,0) = \frac{2\mu u_0}{\pi(1-\nu)\sqrt{a^2-r^2}} \quad ; \quad 0 \le r < a \; . \tag{30.17}$$

We also use (30.13) to record the surface displacement of the half-space *outside* the contact area, which is

$$u_z(r,0) = u_0 \sin^{-1}\frac{a}{r} \quad ; \quad r > a \tag{30.18}$$

and the total indenting force

$$F = 2\pi \int_0^a rp(r)dr = \frac{4\mu u_0}{(1-\nu)} \int_0^a \frac{rdr}{\sqrt{a^2-r^2}} = \frac{4\mu u_0 a}{(1-\nu)} \; . \tag{30.19}$$

The remaining stress and displacement components can be obtained by substituting (30.16) into the corresponding expressions for Solution F from Table 19.3.

This is Love's solution of the flat punch problem. To gain an appreciation of its elegance, it is really necessary to compare it with the original Hankel transform solution[10] which is extremely complicated.

30.2.2 Integral representation

Green and Zerna[11] extended the method of the last section to give a general solution of the boundary-value problem of equations (30.1, 30.2), using the integral representation

[10] I.N.Sneddon, Boussinesq's problem for a flat-ended cylinder, *Proceedings of the Cambridge Philosophical Society,* Vol. 42 (1946), pp.29–39.

[11] A.E.Green and W.Zerna, *loc.cit.*, §§5.8–5.10.

$$\frac{\partial \varphi}{\partial z} = \Re \int_0^a \frac{g(t)dt}{\sqrt{r^2 + (z + \imath t)^2}} = \frac{1}{2} \int_{-a}^a \frac{g(t)dt}{\sqrt{r^2 + (z + \imath t)^2}} \tag{30.20}$$

where $g(t)$ is an even function of t. It can be shown that equation (30.20) satisfies (30.2) identically and the remaining boundary condition (30.1) will yield an integral equation for the unknown function $g(t)$.

In effect, equation (30.20) is a superposition of solutions of the form $\Re(r^2+(z+\imath t)^2)^{-1/2}$ derived from the source solution $1/R$. We could therefore describe φ as the potential due to an arbitrary distribution $g(t)$ of point sources along the *imaginary* z-axis between (0,0) and $(0, \imath a)$. It is therefore a logical development of the classical method of obtaining axisymmetric potential functions by distributing singularities along the axis of symmetry[12]. A closely related solution is given by Segedin[13], who develops it as a convolution integral of an arbitrary kernel function with the Boussinesq solution of §30.2.1. He uses this method to obtain the solution for a power law punch ($u_0(r) \sim r^n$) and treats more general problems by superposition after expanding the punch profile as a power series in r. It should be noted that Segedin's solution is restricted to indentation by a punch of continuous profile[14] in which case the contact pressure tends to zero at $r = a$. The idea of representing a general solution by superposition of Boussinesq-type solutions for different values of a has also been used as a direct numerical method by Maw et al.[15].

Green's method was extensively developed by Collins, who used it to treat many interesting problems including the indentation problem for an annular punch[16] and a problem involving 'radiation' boundary conditions[17].

30.2.3 Basic forms and surface values

To represent harmonic potential functions we shall use suitable combinations of the four basic forms

$$\phi_1 = \Re \int_0^a g_1(t)\mathcal{F}(r, z, t)dt$$

$$\phi_2 = \Re \int_a^\infty g_2(t)\mathcal{F}(r, z, t)dt$$

---

[12] see §23.3 and particularly equation (23.29).

[13] C.M.Segedin, The relation between load and penetration for a spherical punch, *Mathematika,* Vol. 4 (1957), pp.156–161.

[14] see §29.2.

[15] N.Maw, J.R.Barber and J.N.Fawcett, The oblique impact of elastic spheres, *Wear,* Vol. 38 (1976), pp.101–114.

[16] W.D.Collins, On the solution of some axisymmetric boundary-value problems by means of integral equations, IV: Potential problems for a circular annulus, *Proceedings of the Edinburgh Mathematical Society,* Vol. 13 (1963), pp.235–246.

[17] W.D.Collins, On the solution of some axisymmetric boundary-value problems by means of integral equations, II: Further problems for a circular disc and a spherical cap, *Mathematika,* Vol. 6 (1959), pp.120–133.

$$\phi_3 = \Im \int_0^a g_3(t)\mathcal{F}(r,z,t)dt \tag{30.21}$$
$$\phi_4 = \Im \int_a^\infty g_4(t)\mathcal{F}(r,z,t)dt \; ,$$

where

$$\mathcal{F}(r,z,t) = \ln\left(\sqrt{r^2+(z+\imath t)^2}+z+\imath t\right) \; . \tag{30.22}$$

The square root in (30.22) is interpreted as

$$\sqrt{r^2+(z+\imath t)^2} = \rho e^{\imath\upsilon/2} \; , \tag{30.23}$$

where

$$\rho = \sqrt[4]{(r^2+z^2-t^2)^2+4z^2t^2} \;\; ; \;\; \upsilon = \tan^{-1}\left(\frac{2zt}{r^2+z^2-t^2}\right) \tag{30.24}$$

and $\rho \geq 0, \; 0 \leq \upsilon < \pi$.

Equations (30.21) can be written in two alternative forms which for ϕ_1 are

$$\phi_1 = \frac{1}{2}\int_0^a g_1(t)\{\mathcal{F}(r,z,t)+\mathcal{F}(r,z,-t)\}dt \tag{30.25}$$

and

$$\phi_1 = \frac{1}{2}\int_{-a}^a g_1(t)\mathcal{F}(r,z,t)dt \; . \tag{30.26}$$

We note that (30.25) is exactly equivalent to (30.26) if and only if g_1 is an even function of t. If the boundary values of $\phi, \partial\phi/\partial z$ etc. specified at $z=0$ are even in r, it will be found that g_1, g_2 are even and g_3, g_4 odd functions of t and forms like (30.26) can be used. The majority of problems fall into this category, but there are important exceptions such as the conical punch (where u_z is proportional to r) and problems with Coulomb friction for which σ_{zr} is proportional to σ_{zz} in some region. In these problems, forms like (30.26) can only be used if $g_i(t)$ is extended into $t<0$ by a definition with the required symmetry.

Expressions for the important derivatives of the functions ϕ_i at the surface $z=0$ are given in Table 30.1. In certain cases, higher derivatives are required — notably in thermoelastic problems, where heat flux is proportional to $\partial^3\phi/\partial z^3$ (see Solution P, Table 22.1). Higher derivatives are most easily obtained by differentiating *within* the plane $z = 0$, making use of the fact that for an axisymmetric harmonic function f,

$$\frac{\partial^2 f}{\partial z^2} = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r} \; . \tag{30.27}$$

The reader can verify that this result permits the expressions for $\partial^2\phi_i/\partial z^2$ in Table 30.1 to be obtained from those for $\partial\phi_i/\partial r$.

Table 30.1. Surface values of the derivatives of the functions ϕ_i (Equations (30.21)).

	$0<r<a$	$r>a$
$\frac{\partial\phi_1}{\partial r}$	$\frac{1}{r}\int_0^a g_1(t)dt - \frac{1}{r}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}}$	$\frac{1}{r}\int_0^a g_1(t)dt$
$\frac{\partial\phi_1}{\partial z}$	$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}}$	$\int_0^a \frac{g_1(t)dt}{\sqrt{r^2-t^2}}$
$\frac{\partial^2\phi_1}{\partial z^2}$	$\frac{1}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}}$	0
$\frac{\partial\phi_2}{\partial r}$	$\frac{1}{r}\int_a^\infty g_2(t)dt - \frac{1}{r}\int_a^\infty \frac{tg_2(t)dt}{\sqrt{t^2-r^2}}$	$\frac{1}{r}\int_a^\infty g_2(t)dt - \frac{1}{r}\int_r^\infty \frac{tg_2(t)dt}{\sqrt{t^2-r^2}}$
$\frac{\partial\phi_2}{\partial z}$	0	$\int_a^r \frac{g_2(t)dt}{\sqrt{r^2-t^2}}$
$\frac{\partial^2\phi_2}{\partial z^2}$	$\frac{1}{r}\frac{d}{dr}\int_a^\infty \frac{tg_2(t)dt}{\sqrt{t^2-r^2}}$	$\frac{1}{r}\frac{d}{dr}\int_r^\infty \frac{tg_2(t)dt}{\sqrt{t^2-r^2}}$
$\frac{\partial\phi_3}{\partial r}$	$-\frac{1}{r}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}}$	$-\frac{1}{r}\int_0^a \frac{tg_3(t)dt}{\sqrt{r^2-t^2}}$
$\frac{\partial\phi_3}{\partial z}$	$-\int_r^a \frac{g_3(t)dt}{\sqrt{t^2-r^2}}$	0
$\frac{\partial^2\phi_3}{\partial z^2}$	$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}}$	$\frac{1}{r}\frac{d}{dr}\int_0^a \frac{tg_3(t)dt}{\sqrt{r^2-t^2}}$
$\frac{\partial\phi_4}{\partial r}$	0	$-\frac{1}{r}\int_a^r \frac{tg_4(t)dt}{\sqrt{r^2-t^2}}$
$\frac{\partial\phi_4}{\partial z}$	$-\int_a^\infty \frac{g_4(t)dt}{\sqrt{t^2-r^2}}$	$-\int_r^\infty \frac{g_4(t)dt}{\sqrt{t^2-r^2}}$
$\frac{\partial^2\phi_4}{\partial z^2}$	0	$\frac{1}{r}\frac{d}{dr}\int_a^r \frac{tg_4(t)dt}{\sqrt{r^2-t^2}}$

30.2.4 Reduction to an Abel equation

Table 30.1 shows that we can satisfy the boundary condition (30.2) identically if we represent φ in the form of ϕ_1 of equations (30.21). Substitution in the remaining boundary condition (30.1) then gives the integral equation

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}} = -\frac{\mu}{(1-\nu)}u_0(r) \;\; ; \;\; 0 \le r < a \; , \tag{30.28}$$

for the unknown function $g_1(t)$. Notice that there is a one-to-one correspondence between the points $0 \le t < a$ in which $g_1(t)$ is defined and the domain $0 \le r < a$ in which the integral equation is to be satisfied. This is a necessary condition for the equation to be solvable[18].

Equation (30.28) is an *Abel integral equation.* Abel equations can be inverted explicitly. We operate on both sides of the equation with

[18] It is akin to the condition that the number of unknowns and the number of equations in a system of linear equations be equal.

$$\int_0^s \frac{rdr}{\sqrt{s^2 - r^2}} ,$$

obtaining the equation

$$\int_0^s \int_0^r \frac{rg_1(t)dtdr}{\sqrt{(s^2 - r^2)(r^2 - t^2)}} = -\frac{\mu}{(1-\nu)} \int_0^s \frac{ru_0(r)dr}{\sqrt{s^2 - r^2}} \quad ; \quad 0 \le r < a . \tag{30.29}$$

Since the unknown function, $g_1(t)$ does not depend upon r, it is advantageous to reverse the order of integration on the left-hand side of (30.29) and perform the inner integration. To obtain the limits on the new double integral, we consider r, t as coördinates defining the two-dimensional region of Figure 30.2, in which the domain of integration is shown shaded. It follows from this figure that when we reverse the order of integration the new limits are

$$\int_0^s \int_0^r \frac{rg_1(t)dtdr}{\sqrt{(s^2 - r^2)(r^2 - t^2)}} = \int_0^s g_1(t)dt \int_t^s \frac{rdr}{\sqrt{(s^2 - r^2)(r^2 - t^2)}} . \tag{30.30}$$

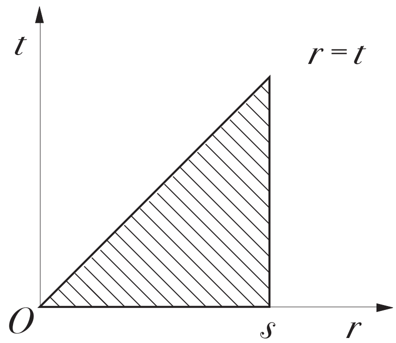

Figure 30.2: Domain of integration for equation (30.29)

The inner integral can now be evaluated as

$$\int_t^s \frac{rdr}{\sqrt{(s^2 - r^2)(r^2 - t^2)}} = \frac{\pi}{2} \tag{30.31}$$

and hence from equations (30.30, 30.31) we have

$$\int_0^s g_1(t)dt = -\frac{2\mu}{\pi(1-\nu)} \int_0^s \frac{ru_0(r)dr}{\sqrt{s^2 - r^2}} . \tag{30.32}$$

Differentiating with respect to s, we then have

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \frac{ru_0(r)dr}{\sqrt{s^2 - r^2}} , \tag{30.33}$$

which is the explicit solution of the Abel equation (30.29).

In general, we shall find that the use of the standard forms of equations (30.21) in two-part boundary-value problems leads to the four types of Abel equations whose solutions — obtained by a process similar to that given above — are given in Table 30.2.

Table 30.2. Inversions of Abel integral equations.

If $f(x) =$	$\int_0^x \frac{g(t)dt}{\sqrt{x^2-t^2}}$	$\int_x^a \frac{g(t)dt}{\sqrt{t^2-x^2}}$
then $g(x) =$	$\frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{tf(t)dt}{\sqrt{x^2-t^2}}$	$-\frac{2}{\pi}\frac{d}{dx}\int_x^a \frac{tf(t)dt}{\sqrt{t^2-x^2}}$
If $f(x) =$	$\int_a^x \frac{g(t)dt}{\sqrt{x^2-t^2}}$	$\int_x^\infty \frac{g(t)dt}{\sqrt{t^2-x^2}}$
then $g(x) =$	$\frac{2}{\pi}\frac{d}{dx}\int_a^x \frac{tf(t)dt}{\sqrt{x^2-t^2}}$	$-\frac{2}{\pi}\frac{d}{dx}\int_x^\infty \frac{tf(t)dt}{\sqrt{t^2-x^2}}$

Once g_1 is known, the complete stress field can be written down using equation (30.21) and Solution F of Table 21.3. In particular, the contact pressure

$$p(r) = -\sigma_{zz}(r,0) = \frac{\partial^2\varphi}{\partial z^2}(r,0) = \frac{1}{r}\frac{d}{dr}\int_r^b \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} \,, \tag{30.34}$$

from Table 30.1, and the total indenting force

$$F = 2\pi\int_0^a rp(r)dr = 2\pi\int_0^a\left(\frac{1}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}}\right)rdr = 2\pi\left[\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}}\right]_{r=0}^{r=a}$$
$$= -2\pi\int_0^a g_1(t)dt \,. \tag{30.35}$$

Example

As an example, we consider the problem in which a half-space is indented by a rigid cylindrical punch of radius b with a spherical end of radius $R \gg b$. The function u_0 is then defined by

$$u_0 = d - \frac{r^2}{2R} \,, \tag{30.36}$$

where d is the indentation at the centre of the punch. We suppose that the force is sufficient to ensure that the punch makes contact over the entire surface $0 \le r < b$.

Substituting (30.36) in (30.33) and performing the integration and differentiation, we obtain

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)}\left(d - \frac{s^2}{R}\right) \,. \tag{30.37}$$

Using this result in (30.34) the contact pressure distribution is obtained as

$$p(r) = \frac{2\mu}{\pi(1-\nu)R}\left(\frac{Rd+b^2-2r^2}{\sqrt{b^2-r^2}}\right) \tag{30.38}$$

and the total force is

$$F = \frac{4\mu}{(1-\nu)}\int_0^b \left(d-\frac{t^2}{R}\right)dt = \frac{4\mu b}{(1-\nu)}\left(d-\frac{b^2}{3R}\right) , \tag{30.39}$$

from (30.35, 30.37).

Equation (30.28) defines a positive contact pressure for all $r < b$ if and only if $d > b^2/R$ and hence

$$F \geq \frac{8\mu b^3}{3(1-\nu)R} . \tag{30.40}$$

For smaller values of F, contact will occur only in a smaller circle of radius $a < b$. In this case, the material of the punch outside $r = a$ plays no rôle in the problem and the contact pressure can be determined by replacing b by a in (30.38). Furthermore, the smooth contact condition requires that the $p(r)$ be bounded at $r = a$ and this is equivalent to enforcing the equality in (30.40) and solving for a, giving

$$a = \left(\frac{3(1-\nu)FR}{8\mu}\right)^{1/3} . \tag{30.41}$$

However, we shall introduce a more systematic approach to problems of this class in the next section.

30.2.5 Smooth contact problems

If the indenter is smooth, the extent of the contact area must be found as part of the solution by enforcing the unilateral inequalities (29.11, 29.12). One way to do this is first to solve the problem assuming a is an independent variable and then determine it from the condition that $p(r)$ be non-singular at the boundary of the contact area $r = a$, as in the two-dimensional Hertz problem of §12.5.3. Alternatively, we recall from Chapter 29 that the appropriate contact area is that which maximizes the indentation load F, so for a circular contact area of radius a,

$$\frac{dF}{da} = 0 \quad \text{and hence} \quad g_1(a) = 0 , \tag{30.42}$$

from (30.35).

However, more direct relationships can be established between the contact radius, the indenting force and the shape of the indenter, defined merely through its *slope* $u_0'(r)$. Starting with the right-hand side of equation (30.34), we integrate by parts, obtaining

$$\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} = g_1(a)\sqrt{a^2-r^2} - \int_r^a \sqrt{t^2-r^2}g_1'(t)dt \tag{30.43}$$

and hence

$$p(r) = -\frac{g_1(a)}{\sqrt{a^2-r^2}} + \int_r^a \frac{g_1'(t)dt}{\sqrt{t^2-r^2}} . \tag{30.44}$$

The second term is bounded at $r \to a$ and hence the tractions will be bounded there if and only if $g_1(a)=0$, as in (30.42). Using this result, we can then write a simpler expression for the contact pressure in smooth contact problems as

$$p(r) = \int_r^a \frac{g_1'(t)dt}{\sqrt{t^2-r^2}} . \tag{30.45}$$

Since the contact pressure depends only on the derivative g_1', we can also achieve some simplification by performing a similar integration by parts on the right-hand side of (30.33). We obtain

$$\int_0^s \frac{ru_0(r)dr}{\sqrt{s^2-r^2}} = u_0(0)s + \int_0^s \sqrt{s^2-r^2}u_0'(r)dr \tag{30.46}$$

and hence

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)}\left[d + s\int_0^s \frac{u_0'(r)dr}{\sqrt{s^2-r^2}}\right] , \tag{30.47}$$

where $d=u_0(0)$ represents the rigid-body indentation of the punch. We can determine d by setting $g_1(a)=0$ in (30.47), giving

$$d = -a\int_0^a \frac{u_0'(r)dr}{\sqrt{a^2-r^2}} \tag{30.48}$$

and hence

$$g_1(s) = \frac{2\mu}{\pi(1-\nu)}\left[a\int_0^a \frac{u_0'(r)dr}{\sqrt{a^2-r^2}} - s\int_0^s \frac{u_0'(r)dr}{\sqrt{s^2-r^2}}\right] . \tag{30.49}$$

Notice that this result depends only on the shape of the punch as defined by the derivative $u_0'(r)$. Also, differentiating (30.49), we obtain

$$g_1'(s) = -\frac{2\mu}{\pi(1-\nu)}\frac{d}{ds}\left[s\int_0^s \frac{u_0'(r)dr}{\sqrt{s^2-r^2}}\right] . \tag{30.50}$$

Substituting (30.49) into (30.35), we have

$$F = -\frac{4\mu}{(1-\nu)}\left[a^2\int_0^a \frac{u_0'(r)dr}{\sqrt{a^2-r^2}} - \int_0^a s\int_0^s \frac{u_0'(r)drds}{\sqrt{s^2-r^2}}\right] . \tag{30.51}$$

Changing the order of integration in the second term and performing the inner integral, we have

$$\int_0^a s \int_0^s \frac{u_0'(r)drds}{\sqrt{s^2-r^2}} = \int_0^a u_0'(r) \int_r^a \frac{sdsdr}{\sqrt{s^2-r^2}} = \int_0^a \sqrt{a^2-r^2}\, u_0'(r)dr \tag{30.52}$$

and hence, after some algebraic simplification,

$$F = -\frac{4\mu}{(1-\nu)} \int_0^a \frac{r^2 u_0'(r)dr}{\sqrt{a^2-r^2}} \,. \tag{30.53}$$

In a typical problem, the shape of the punch $u_0'(r)$ is a known function and either the force F or the indentation d will be prescribed. The contact radius a can then be determined by evaluating the integrals in (30.53) or (30.48) and solving the resulting equation. The function $g_1'(s)$ and the contact pressure can then be obtained from equations (30.50) and (30.45) respectively.

Example – The Hertz problem

As an example, we consider the problem in which a half-space is indented by a rigid sphere of radius R, in which case the slope of the indenter is defined by

$$u_0' = -\frac{r}{R} \,. \tag{30.54}$$

Substituting in equation (30.53) and performing the integration, we obtain

$$F = \frac{4\mu}{(1-\nu)R} \int_0^a \frac{r^3(r)dr}{\sqrt{a^2-r^2}} = \frac{8\mu a^3}{3(1-\nu)R} \,. \tag{30.55}$$

Also, the rigid-body indentation is obtained from (30.48) as

$$d = \frac{a}{R} \int_0^a \frac{rdr}{\sqrt{a^2-r^2}} = \frac{a^2}{R} \,. \tag{30.56}$$

Alternatively, we can eliminate a between equations (30.55, 30.56), obtaining

$$F = \frac{8\mu R^{1/2} d^{3/2}}{3(1-\nu)} \,, \tag{30.57}$$

which exhibits a stiffening characteristic with increasing indentation.

For the contact pressure distribution, we first note that

$$\int_0^s \frac{u_0'(r)dr}{\sqrt{s^2-r^2}} = -\frac{1}{R} \int_0^s \frac{rdr}{\sqrt{s^2-r^2}} = -\frac{s}{R} \,. \tag{30.58}$$

Substituting this result into the right-hand side of (30.50), we then obtain

$$g_1'(s) = \frac{4\mu s}{\pi(1-\nu)R} \,, \tag{30.59}$$

after which (30.45) gives

$$p(r) = \frac{4\mu}{\pi(1-\nu)R} \int_r^a \frac{tdt}{\sqrt{t^2-r^2}} = \frac{4\mu\sqrt{a^2-r^2}}{\pi(1-\nu)R} \,. \tag{30.60}$$

30.2.6 Choice of form

Each of the functions ϕ_i in Table 30.1 has a zero in $\partial\phi/\partial z$ or $\partial^2\phi/\partial z^2$ either in $0<r<a$ or in $r>a$. Thus, in the above examples it would have been possible to satisfy (30.2) by choosing $\partial\varphi/\partial z = \phi_3$ instead of $\varphi = \phi_1$.

The choice is best made by examining the requirements of continuity imposed at $r=a$, $z=0$ by the physical problem. It can be shown that, if $g_i(a)$ is bounded, the expressions in Table 30.1 will define continuous values of $\partial\phi_i/\partial z$, but $\partial^2\phi_1/\partial z^2$ will be discontinuous unless $g_i(a)=0$. Thus if, as in the present case, the function $\partial\varphi/\partial z$ represents a physical quantity like displacement or temperature which is required to be continuous, it is appropriate to choose $\partial\varphi/\partial z = \partial\phi_1/\partial z$. The alternative choice of $\partial\varphi/\partial z = \phi_3$ imposes too strong a continuity condition at $r=a$, since it precludes discontinuities in second derivatives and hence in the stress components.

Of course, in problems involving contact between smooth surfaces, the normal traction σ_{zz} must also be continuous at the edge of the contact region. The most straightforward treatment is that given in §30.2.5, in which the continuity condition furnishes an extra condition to determine the radius of the contact region, which is not known *a priori*. However, it would also be possible to force the required continuity through the formulation by using $\partial\varphi/\partial z = \phi_3$ in which case the contact radius is prescribed and the rigid-body indentation d of the punch must be allowed to float. This is essentially the technique used by Segedin[19].

30.3 Non-axisymmetric problems

The method described in this chapter can also be applied to problems in which non-axisymmetric boundary conditions are imposed interior to and exterior to the circle $r=a$ — for example, if the punch has a profile $u_0(r,\theta)$ that depends on both polar coördinates. The first step is to perform a Fourier decomposition of the profile, so that the boundary conditions can be expressed as a series of terms of the form $f_m(r)\cos(m\theta)$ or $f_m(r)\sin(m\theta)$, where m is an integer. These terms can then be treated separately using linear superposition.

Copson[20] shows that if a harmonic potential function $\phi(r,\theta,z)$ satisfies the mixed boundary conditions

$$\begin{aligned} \phi(r,\theta,0) &= f(r)\cos(m\theta)\ ; \quad 0\le r<a \\ \frac{\partial\phi}{\partial z}(r,\theta,0) &= 0\ ; \quad r>a\ , \end{aligned} \tag{30.61}$$

the surface value of the derivative in $0\le r<a$ can be written

[19] C.M.Segedin, The relation between load and penetration for a spherical punch, *Mathematika,* Vol. 4 (1957), pp.156–161.

[20] E.T.Copson, On the problem of the electrified disc, *Proceedings of the Edinburgh Mathematical Society,* Vol. 8 (1947), pp.14–19.

$$\frac{\partial\phi}{\partial z}(r,\theta,0) = r^{m-1}\cos(m\theta)\frac{d}{dr}\int_r^a \frac{tg(t)dt}{\sqrt{t^2-r^2}} \; , \tag{30.62}$$

where

$$g(t) = \frac{2}{\pi t^{2m}}\frac{d}{dt}\int_0^t \frac{s^{m+1}f(s)ds}{\sqrt{t^2-s^2}} \; . \tag{30.63}$$

This result can clearly be used to solve non-axisymmetric problems of the form (30.1, 30.2) by setting

$$\phi = \frac{\partial\varphi}{\partial z} \; . \tag{30.64}$$

With this notation, equations (30.61, 30.62) reduce to the equivalent expressions for ϕ_1 in Table 30.1 in the axisymmetric case $m=0$.

Example: The tilted flat punch

We suppose that a cylindrical rigid flat punch of radius a is pressed into the surface of the elastic half space $z>0$ by an offset force F, whose line of action passes through the point $(\epsilon,0,0)$ causing the punch to tilt through some small angle α. We restrict attention to the case where the offset is sufficiently small to ensure that the complete face of the punch makes contact with the half space. The boundary condition in $0\le r<a$ is then

$$u_z(r,\theta,0) = d + \alpha r\cos\theta \tag{30.65}$$

and since the problem is linear, we can decompose the solution into an axisymmetric and a non-axisymmetric term, the former being identical to that given in §30.2.1. To complete the solution, we therefore seek a harmonic function φ satisfying the boundary conditions

$$\frac{\partial\varphi}{\partial z} = -\frac{\mu\alpha r\cos\theta}{(1-\nu)} \quad ; \quad 0\le r<a \; , \; z=0$$
$$\frac{\partial^2\varphi}{\partial z^2} = 0 \quad ; \quad r>a \; , \; z=0 \; , \tag{30.66}$$

from (30.1, 30.2). Using (30.64), these conditions are equivalent to (30.61) with $m=1$ and

$$f(r) = -\frac{\mu\alpha r}{(1-\nu)} \; .$$

Equation (30.63) then gives

$$g(t) = -\frac{2\mu\alpha}{\pi(1-\nu)t^2}\frac{d}{dt}\int_0^t \frac{s^3 f(s)ds}{\sqrt{t^2-s^2}} = -\frac{4\mu\alpha}{\pi(1-\nu)}$$

and substituting into (30.62), we obtain

$$\frac{\partial^2\varphi}{\partial z^2}(r,\theta,0) = \cos\theta\frac{d}{dr}\int_r^a \frac{tg(t)dt}{\sqrt{t^2-r^2}} = \frac{4\mu\alpha r\cos\theta}{\pi(1-\nu)\sqrt{a^2-r^2}} \; .$$

The contact pressure distribution under the punch is then obtained as

$$p(r,\theta) = \frac{\partial^2 \varphi}{\partial z^2} = \frac{2\mu}{\pi(1-\nu)\sqrt{a^2-r^2}}\left(d + 2\alpha r\cos\theta\right) \;,$$

where we have superposed the axisymmetric contribution due to the term d in (30.65) using (30.17). This solution of the tilted punch problem was first given by A.E.Green[21]

Contact will occur over the complete face of the punch if and only if $p(r,\theta) > 0$ for all (r,θ) in $0 \le r < a$. This condition will fail at $(a, -\pi)$, implying local separation, if

$$\alpha > \frac{d}{2a} \;. \tag{30.67}$$

The rigid-body motion of the punch defined by the constants d, α can be related to the force F and its offset ϵ by equilibrium considerations. We obtain

$$F = \int_0^a \int_0^{2\pi} p(r,\theta) r d\theta dr = \frac{4\mu a d}{(1-\nu)}$$

$$F\epsilon = \int_0^a \int_0^{2\pi} p(r,\theta) r^2 \cos\theta d\theta dr = \frac{8\mu a^3 \alpha}{3(1-\nu)}$$

and using (30.67), we conclude that for full face contact, we require

$$\epsilon \le \frac{a}{3} \;.$$

In other words, the line of action of the indenting force must lie within a central circle of radius $a/3$.

30.3.1 The full stress field

In his solution of the tilted punch problem, Green used a semi-intuitive method similar to that introduced in §30.2.1 to deduce the form of the harmonic potential φ throughout the half space from the surface values obtained above. A more formal approach is to use the recurrence relation (24.45, 24.46) to relate the non-axisymmetric problem to an axisymmetric problem that can be solved using the method of §30.2.2 and Table 30.1.

For example, in the case of the tilted punch problem, the non-axisymmetric term in $\varphi(r,\theta,z)$ can be written $f_1(r,z)\cos\theta$, where f_1 satisfies the boundary conditions

$$\frac{\partial f_1}{\partial z} = -\frac{\mu\alpha r}{(1-\nu)} \;\; ; \;\; 0 \le r < a \;, \; z = 0$$

$$\frac{\partial^2 f_1}{\partial z^2} = 0 \;\; ; \;\; r > a \;, \; z = 0 \;, \tag{30.68}$$

[21] A.E.Green, On Boussinesq's problem and penny-shaped cracks, *Proceedings of the Cambridge Philosophical Society,* Vol. 45 (1949), pp.251–257.

from (30.66). Setting $m=0$ in (24.46) we can define an axisymmetric harmonic function $f_0(r,z)$ such that

$$f_1(r,z) = \frac{\partial f_0}{\partial r} \; . \tag{30.69}$$

Substituting this relation into $(30.68)_1$ and integrating with respect to r, we conclude that f_0 must satisfy the boundary condition

$$\frac{\partial f_0}{\partial z}(r,0) = \int \frac{\partial f_1}{\partial z}(r,0)dr = -\frac{\mu\alpha r^2}{2(1-\nu)} + C \;\; ; \;\; 0 \le r < a \; , \; z=0 \; , \tag{30.70}$$

where C is a constant of integration. Since the stress field must decay at large values of r, the corresponding integration of the outer boundary condition $(30.68)_2$ leads simply to

$$\frac{\partial^2 f_0}{\partial z^2} = 0 \;\; ; \;\; r > a \; , \; z=0 \; . \tag{30.71}$$

Equation (30.71) can be satisfied identically by representing the axisymmetric harmonic $f_0(r,z)$ in the form ϕ_1 of equation (30.21). Table 30.1 then gives

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}} = -\frac{\mu\alpha r^2}{2(1-\nu)} + C$$

and the Abel equation solution is obtained from Table 30.2 as

$$g_1(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \left(-\frac{\mu\alpha t^3}{2(1-\nu)} + Ct\right)\frac{dt}{\sqrt{x^2-t^2}} = \frac{2}{\pi}\left(-\frac{\mu\alpha x^2}{(1-\nu)} + C\right) \; .$$

The constant C must be chosen so as to satisfy the condition of continuity of displacement at $r=a$. As explained in §30.2.6, the representations (30.21) ensure continuity of the derivative $\partial\phi_i/\partial z$ at $r=a$ as long as the corresponding function $g_i(t)$ is bounded. In the present case, this means that $\partial f_0/\partial z$ will be continuous for all values of C. However, the differentiation in (30.69) will introduce a square-root singularity in $\partial f_1/\partial z$ unless the constant C is chosen so as to satisfy the stronger condition $g_1(a)=0$, giving

$$g_1(x) = \frac{2\mu\alpha(a^2-x^2)}{\pi(1-\nu)} \; .$$

The function $f_0(r,z)$ is then given by $(30.21)_1$ with this value of g_1 after which $f_1(r,z)$ is defined by (30.69).

For higher-order problems (larger values of m in $\varphi = f_m(r,z)\cos(m\theta)$), additional constants of integration will appear in the boundary conditions for $f_0(r,z)$ and stronger continuity conditions must be imposed at $r=a$. In fact $g_1(t)$ and its first $m-1$ derivatives must be set to zero at $t=a$.

PROBLEMS

1. A harmonic potential function $\omega(r,z)$ in the region $z>0$ is defined by the surface values $\omega(r,0)=A(a^2-r^2)$; $0\le r<a$; $\omega(r,0)=0$; $r>a$. Select a suitable form for the function using Table 30.1 and then use the boundary conditions to determine the corresponding unknown function $g_i(t)$.

2. Use equation (30.27) to derive the expressions for $\partial^2\phi_1/\partial z^2$ in Table 30.1 from those for $\partial\phi_1/\partial r$.

3. An elastic half space is indented by a rigid cylindrical punch of radius a with a *concave* spherical end of radius $R\gg a$. Find the contact pressure distribution $p(r)$ and hence determine the minimum force F_0 required to maintain contact over the entire punch surface. Do not attempt to solve the problem for $F<F_0$.

4. A rigid flat punch of radius a is bonded to the surface of the elastic half space $z>0$. A torque T is then applied to the punch. Use Solution E to describe the stress field in the half space and Collins' method to solve the resulting boundary-value problem for Ψ. In particular, find the distribution of traction between the punch and the half space and the rotation of the punch due to the torque.

5. A rigid sphere of radius R is pressed into an elastic half space of shear modulus μ and Poisson's ratio ν. Assuming that the interface energy for this material combination is $\Delta\gamma$, find a relation between the indenting force F and the radius of the contact area a. You should use the conditions (29.17, 29.18) to determine the unknown constant d in (30.36) and then (30.35) to determine F.

Hence determine the contact radius when the sphere is unloaded $F=0$ and the tensile force required to pull the sphere out of contact[22].

6.(i) An elastic half-space is indented by a conical frictionless rigid punch as shown in Figure 30.3, the penetration of the apex being d. Find the contact pressure distribution, the total load F and the contact radius a. The semi-angle of the cone is $\frac{\pi}{2}-\alpha$, where $\alpha\ll 1$.

---

[22] This solution was first given by K.L.Johnson, K.Kendall and A.D.Roberts, Surface energy and the contact of elastic solids, *Proceedings of the Royal Society of London*, Vol.A324 (1971), pp.301–313.

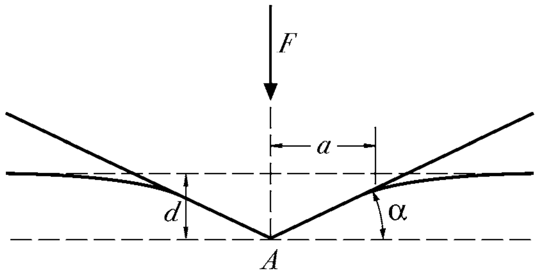

Figure 30.3: The conical punch.

(ii) What is the nature of the singularity in pressure at the vertex of the cone (point A).
(iii) Now suppose that the cone is truncated as shown in Figure 30.4. *Without finding the new contact pressure distribution*, find the new relationship between F and a. **Note:** Take the radius of the truncated end of the cone as b and refer the profile to the point A where the vertex would have been before truncation.
(iv) Would you expect a singularity in contact pressure at B and if so of what form. (The process of determining $p(r)$ is algebraically tedious. Try to find a simpler way to answer the question.)

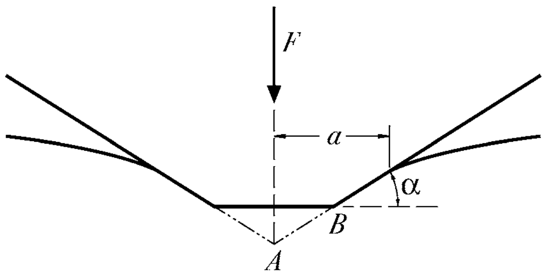

Figure 30.4: The truncated conical punch.

7. A rigid flat punch has rounded edges, as shown in Figure 30.5. The punch is pressed into an elastic half space by a force F. Assuming that the contact is frictionless, find the relation between F, the indentation u_0 and the radius a of the contact area.

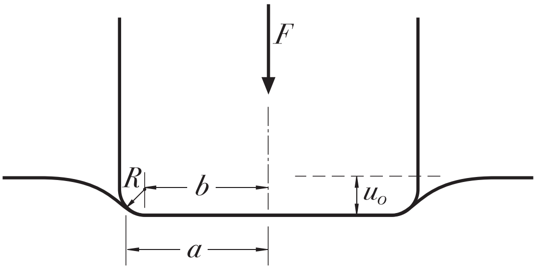

Figure 30.5: Flat punch with rounded corners.

8. An elastic half space is indented by a smooth axisymmetric rigid punch with the power law profile Cr^s, so the displacement in the contact area is

$$u_0(r) = d - Cr^s \ ,$$

where C, s are constants. Show that the indentation force F, the contact radius a and the indentation depth d are related by the equation

$$F = \frac{4\mu a d}{(1-\nu)} \left(\frac{s}{s+1} \right) \ .$$

9. The profile of a smooth axisymmetric frictionless rigid punch is described by the power law

$$u_0(r) = A_n r^{2n} \ ,$$

where n is an integer. The punch is pressed into an elastic half space by a force F. Find the indentation d, the radius of the contact area a and the contact pressure distribution $p(r)$. Check your results by comparison with the Hertz problem of §30.2.5 and give simplified expressions for the case of the fourth order punch

$$u_0(r) = A_2 r^4 \ .$$

10. For a smooth axisymmetric contact problem, the contact pressure must be continuous at $r=a$. Usually this condition is imposed explicitly and it serves to determine the radius a of the contact region. However, it can also be guaranteed by 'choice of form' as suggested in §30.2.6. Use this strategy to solve the Hertzian contact problem, for which

$$u_0'(r) = -\frac{r}{R} \ ,$$

where R is the radius of the indenter.

You will need to find a suitable representation of the form

$$\frac{\partial\varphi}{\partial z} = \phi_i \ ,$$

where ϕ_i is one of the four functions in equation (30.21), chosen so as to satisfy the homogeneous boundary condition (zero contact pressure in $r > a$). Then use Tables 30.1, 30.2 to set up and solve an Abel equation for the unknown function $g_i(t)$. In particular, find the contact pressure distribution $p(r)$ and the total load F needed to establish a circular contact area of radius a.

11. The flat end of a rigid cylindrical punch of radius a is pressed into the curved surface of an elastic cylinder of radius $R \gg a$ by a centric force F sufficient to ensure contact throughout the punch face. Use Copson's formula (30.61–30.63) to determine the contact pressure distribution and hence find the minimum value of F required for full contact.

12. Solve Problem 11 by relating the non-axisymmetric component of $\varphi(r, \theta, z)$ to a corresponding axisymmetric harmonic function $f_0(r, z)$, as in §30.3.1. Notice that the required function varies with $\cos(2\theta)$, so you will need to apply equation (24.46) twice recursively to relate $f_2(r, z)$ to $f_0(r, z)$. This will introduce two arbitrary constants in the boundary conditions for f_0 and these must be determined from the continuity conditions $g_1(a) = g_1'(a) = 0$.

31

THE PENNY-SHAPED CRACK

As in the two-dimensional case, we shall find considerable similarities in the formulation and solution of contact and crack problems. In particular, we shall find that problems for the plane crack can be reduced to boundary-value problems which in the case of axisymmetry can be solved using the method of Green and Collins developed in §30.2.

31.1 The penny-shaped crack in tension

The simplest axisymmetric crack problem is that in which a state of uniform tension $\sigma_{zz}=S$ in an infinite isotropic homogeneous solid is perturbed by a plane crack occupying the region $0\leq r<a,\ z=0$. Thus, the crack has the shape of a circular disk. This geometry has come to be known as the *penny-shaped* crack[1].

As in Chapter 13, we seek the solution in terms of an unperturbed uniform stress field $\sigma_{zz}=S$ (constant) and a corrective solution which tends to zero at infinity. The boundary conditions on the corrective solution are therefore

$$\sigma_{zz} = -S;\ \ \sigma_{zr} = 0\ \ ;\ \ 0\leq r<a\ ,\ \ z=0 \tag{31.1}$$

$$\sigma_{zz}, \sigma_{rz}, \sigma_{rr}, \sigma_{\theta\theta} \to 0\ \ ;\ \ R\to\infty\ . \tag{31.2}$$

The corrective solution corresponds to the problem in which the crack is opened by an internal pressure $-S$ and the body is not loaded at infinity.

The problem is symmetrical about the plane $z=0$ and it follows that on that plane there can be no shear stress σ_{zx}, σ_{zy} and no normal displacement u_z. We can therefore reduce the problem to a boundary-value problem for the half-space $z>0$ defined by the boundary conditions

[1] I.N.Sneddon, The distribution of stress in the neighbourhood of a crack in an elastic solid, *Proceedings of the Royal Society of London,* Vol. A187 (1946), pp.226–260.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_31, © Springer Science+Business Media B.V. 2010

$$\sigma_{zx} = \sigma_{zy} = 0 \; ; \; \text{all } r, \;\; z = 0 \tag{31.3}$$
$$\sigma_{zz} = -S \; ; \;\; 0 \le r < a, \;\; z = 0 \tag{31.4}$$
$$u_z = 0 \; ; \;\; r > a, \;\; z = 0 \; . \tag{31.5}$$

Equation (31.3) is a global condition and can be satisfied identically[2] by using Solution F of Table 21.3. The remaining conditions (31.4, 31.5) then define the mixed boundary-value problem

$$\frac{\partial^2 \varphi}{\partial z^2} = S \; ; \;\; 0 \le r < a, \;\; z = 0 \tag{31.6}$$
$$\frac{\partial \varphi}{\partial z} = 0 \; ; \;\; r > a, \;\; z = 0 \; . \tag{31.7}$$

This problem can be solved by the method of §30.2. We note from Table 30.1 that (31.7) can be satisfied identically by representing φ in the form of ϕ_3 of (30.21) — i.e.

$$\varphi = \Im \int_0^a g(t) F(r,z,t) dt \tag{31.8}$$

and the remaining boundary condition (31.6) then reduces to

$$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{t g(t) dt}{\sqrt{r^2 - t^2}} = S \; ; \;\; 0 \le r < a \; . \tag{31.9}$$

We now multiply both sides of this equation by r and integrate, obtaining

$$\int_0^r \frac{t g(t) dt}{\sqrt{r^2 - t^2}} = \frac{S r^2}{2} + C \; ; \;\; 0 \le r < a \; , \tag{31.10}$$

where C is an arbitrary constant.

Equation (31.10) is an Abel integral equation similar to (30.28) and can be inverted in the same way (see Table 30.2), with the result

$$x g(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \left(\frac{S r^2}{2} + C\right) \frac{r dr}{\sqrt{x^2 - r^2}} \tag{31.11}$$

and hence

$$g(x) = \frac{2 S x}{\pi} + \frac{2C}{\pi x} \; . \tag{31.12}$$

The second term in this expression is singular at $x \to 0$ and it is readily verified from Table 30.1 that $\partial\varphi/\partial z$ and hence u_z would be unbounded at the

[2] More generally, any problem involving a crack of arbitrary cross-section A on the plane $z = 0$ and loaded by a uniform tensile stress σ_{zz} at $z = \pm\infty$ has the same symmetry and can also be formulated using Solution F. In the general case, we obtain a two-part boundary-value problem for the half-space $z > 0$, in which conditions (31.6) and (31.7) are to be satisfied over A and $\bar{A}$ respectively, where $\bar{A}$ is the complement of A in the plane $z = 0$.

origin if $C \neq 0$. We therefore set $C = 0$ to retain continuity of displacement at the origin.

We can then recover the expression for the stress σ_{zz} in the region $r > a, z = 0$ which is

$$\sigma_{zz} = -\frac{\partial^2 \varphi}{\partial z^2} = -\frac{2S}{\pi r}\frac{d}{dr}\int_0^a \frac{x^2 dx}{\sqrt{r^2 - x^2}}$$
$$= -\frac{2S}{\pi}\left(\sin^{-1}\frac{a}{r} - \frac{a}{\sqrt{r^2 - a^2}}\right) . \qquad (31.13)$$

This of course is the corrective solution. To find the stress in the original crack problem, we must superpose the uniform tensile stress $\sigma_{zz} = S$, with the result

$$\sigma_{zz} = \frac{2S}{\pi}\left(\cos^{-1}\frac{a}{r} + \frac{a}{\sqrt{r^2 - a^2}}\right) . \qquad (31.14)$$

The stress intensity factor is defined as

$$K_I = \lim_{r \to a^+} \sigma_{zz}\sqrt{2\pi(r - a)} = 2S\sqrt{\frac{a}{\pi}} . \qquad (31.15)$$

We could have obtained this result directly from $g(x)$, without computing the complete stress distribution. We have

$$\sigma_{zz} = -\frac{1}{r}\frac{d}{dr}\int_0^a \frac{xg(x)dx}{\sqrt{r^2 - x^2}} \qquad (31.16)$$
$$= \frac{g(a)}{\sqrt{r^2 - a^2}} - \frac{g(0)}{r} - \int_0^a \frac{g'(x)dx}{\sqrt{r^2 - x^2}} . \qquad (31.17)$$

Now, unless $g'(x)$ is singular at $x = a$, the only singular term in (31.17) will be the first and the stress intensity factor is therefore

$$K_I = \lim_{r \to a^+} \sigma_{zz}\sqrt{2\pi(r - a)} = g(a)\sqrt{\frac{\pi}{a}} . \qquad (31.18)$$

In the present example, $g(a) = 2Sa/\pi$, leading to (31.15) as before.

We also compute the displacement u_z at $0 \le r < a$, $z = 0^+$, which is

$$u_z(r, 0^+) = -\frac{(1 - \nu)}{\mu}\frac{\partial \varphi}{\partial z} = \frac{(1 - \nu)}{\mu}\int_r^a \frac{g(x)dx}{\sqrt{x^2 - r^2}}$$
$$= \frac{2(1 - \nu)S\sqrt{a^2 - r^2}}{\pi\mu} . \qquad (31.19)$$

Since the crack opens symmetrically on each face, it follows that the crack-opening displacement is

$$\delta(r) = u_z(r, 0^+) - u_z(r, 0^-) = \frac{4(1 - \nu)S\sqrt{a^2 - r^2}}{\pi\mu} . \qquad (31.20)$$

31.2 Thermoelastic problems

In addition to acting as a stress concentration in an otherwise uniform stress field, a crack will obstruct the flow of heat and generate a perturbed temperature field in components of thermal machines. The simplest investigation of this effect assumes that the crack acts as a perfect insulator, so that the heat is forced to flow around it. For the penny-shaped crack[3], the appropriate thermal boundary conditions are then

$$q_z = -K\frac{\partial T}{\partial z} = 0 \;\; ; \;\; 0 \leq r < a, \;\; z = 0 \tag{31.21}$$

$$\to Q \;\; ; \;\; R \to \infty \; . \tag{31.22}$$

As in isothermal problems, we construct the temperature field as the sum of a uniform heat flux and a corrective solution, the boundary conditions on which become

$$q_z = -K\frac{\partial T}{\partial z} = -Q \;\; ; \;\; 0 \leq r < a, \;\; z = 0 \tag{31.23}$$

$$\to 0 \;\; ; \;\; R \to \infty \; . \tag{31.24}$$

This problem is antisymmetric about the plane $z=0$ and hence the temperature must be zero in the region $r>a, \; z=0$. We can therefore convert the heat conduction problem into a boundary-value problem for the half-space $z>0$ with boundary conditions

$$q_z = -Q \;\; ; \;\; 0 \leq r < a \tag{31.25}$$

$$T = 0 \;\; ; \;\; r > a \; . \tag{31.26}$$

The temperature field and a particular thermoelastic solution can be constructed using Solution P of Table 20.1, in terms of which (31.4, 31.5) define the mixed boundary-value problem

$$\frac{\partial^3 \psi}{\partial z^3} = -\frac{\mu\alpha(1+\nu)Q}{(1-\nu)K} \;\; ; \;\; 0 \leq r < a \tag{31.27}$$

$$\frac{\partial^2 \psi}{\partial z^2} = 0 \;\; ; \;\; r > a \; . \tag{31.28}$$

We require that the temperature and hence $\partial^2\psi/\partial z^2$ be continuous at $r=a, \; z=0$ and hence in view of the arguments of §30.3, we choose to satisfy (31.28) using the function ϕ_3 — i.e.

$$\frac{\partial \psi}{\partial z} = \Im \int_0^a g_3(t) F(r,z,t) dt \; . \tag{31.29}$$

[3] This problem was first solved by A.L.Florence and J.N.Goodier, The linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack, *International Journal of Engineering Science,* Vol. 1 (1963), pp.533–540.

Condition (31.27) then defines the Abel equation

$$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} = -\frac{\mu\alpha(1+\nu)Q}{(1-\nu)K} \;\; ; \;\; 0 \le r < a \; , \tag{31.30}$$

which has the solution

$$g_3(t) = -\frac{2\mu\alpha(1+\nu)Qt}{\pi(1-\nu)K} \; . \tag{31.31}$$

This defines a particular thermoelastic solution. We recall from Chapter 14 that the homogeneous solution corresponding to thermoelasticity is the general solution of the isothermal problem. Thus we now must superpose a suitable isothermal stress field to satisfy the mechanical boundary conditions of the problem.

The mechanical boundary conditions are that (i) the surfaces of the crack be traction-free and (ii) the stress field should decay to zero at infinity. However, the antisymmetry of the problem once again permits us to define boundary conditions on the half-space $z > 0$ which are

$$\sigma_{zz} = \sigma_{zr} \;=\; 0 \;\; ; \;\; 0 \le r < a, \;\; z = 0 \tag{31.32}$$

$$\sigma_{zz} = 0 \;\; ; \;\; u_r = 0 \;\; ; \;\; r > a, \;\; z = 0 \; . \tag{31.33}$$

Equations (31.32) state that the crack surfaces are traction-free and (31.33) are symmetry conditions.

We notice that taken together, these conditions imply that $\sigma_{zz} = 0$ throughout the plane $z = 0$. This is therefore a global condition and can be satisfied by the choice of form. It is already satisfied by the particular solution (see Table 20.1) and hence it must also be satisfied by the additional isothermal solution, for which we therefore use Solution G of Table 21.3.

The complete solution is therefore obtained by superposing Solutions P and G, the surface values of σ_{zr} and u_r being

$$\sigma_{zr} = \frac{\partial^2\chi}{\partial r\partial z} \tag{31.34}$$

$$2\mu u_r = 2(1-\nu)\left(\frac{\partial\psi}{\partial r} + \frac{\partial\chi}{\partial r}\right) \; . \tag{31.35}$$

The mixed conditions in (31.32, 31.33) then require

$$\frac{\partial^2\chi}{\partial r\partial z} = 0 \;\; ; \;\; 0 \le r < a, \;\; z = 0 \tag{31.36}$$

$$\frac{\partial\psi}{\partial r} + \frac{\partial\chi}{\partial r} = 0 \;\; ; \;\; r > a, \;\; z = 0 \; . \tag{31.37}$$

We can satisfy (31.37) by defining χ such that

$$\chi = -\psi + \Re\int_0^a g_1(t)F(r,z,t)dt \; , \tag{31.38}$$

with the auxiliary condition

$$\int_0^a g_1(t)dt = 0 \; , \tag{31.39}$$

(see Table 30.1).

The other boundary condition (31.36) can then be integrated to give

$$\frac{\partial \chi}{\partial z} = C \;\; ; \;\; 0 \le r < a, \;\; z = 0 \; , \tag{31.40}$$

where C is an arbitrary constant of integration, and hence

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2 - t^2}} = \frac{\partial \psi}{\partial z} + C \;\; ; \;\; 0 \le r < a \; , \tag{31.41}$$

using (31.38) and Table 30.1.

To solve for $g_1(t)$, we use the result

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left(\frac{\partial \psi}{\partial z}\right) = -\frac{\partial^3 \psi}{\partial z^3} = \frac{\mu\alpha(1+\nu)Q}{(1-\nu)K} \tag{31.42}$$

(see equation (30.27)), which can be integrated in the plane $z=0$ to give

$$\frac{\partial \psi}{\partial z} = \frac{\mu\alpha(1+\nu)Qr^2}{4(1-\nu)K} + B \; ; \quad 0 \le r < a, \; z = 0 \; , \tag{31.43}$$

where B is an arbitrary constant[4].

Substituting (31.43) into (31.41) and solving the resulting Abel integral equation using Table 30.2, we obtain

$$g_1(x) = \frac{2}{\pi}\left(B + C + \frac{\mu\alpha(1+\nu)Qx^2}{2(1-\nu)K}\right) \; . \tag{31.44}$$

Since B and C are two arbitrary constants and occur as a sum, there is no loss in generality in setting B to zero. The auxiliary condition (31.39) can then be used to determine C giving

$$C = -\frac{\mu\alpha(1+\nu)Qa^2}{6(1-\nu)K} \; . \tag{31.45}$$

Finally, we can recover the expression for the stress σ_{zr} on $r > a, \; z = 0$, which is

$$\sigma_{zr} = \frac{\partial^2 \chi}{\partial z \partial r} = \frac{d}{dr}\int_0^a \frac{g_1(t)dt}{\sqrt{r^2 - t^2}} + \frac{1}{r}\int_0^a \frac{tg_3(t)dt}{\sqrt{r^2 - t^2}} \tag{31.46}$$

$$= -\frac{2\mu\alpha(1+\nu)Qa^3}{3\pi(1-\nu)Kr\sqrt{r^2 - a^2}} \; , \tag{31.47}$$

[4] The other constant of integration leads to a term which is singular at the origin and has therefore been set to zero.

from Table 30.1 and equations (31.31, 31.44, 31.45).

Notice that the antisymmetry of the problem ensures that the crack tip is loaded in shear only — we argued earlier that σ_{zz} would be zero throughout the plane $z=0$. It also follows from a similar argument that the crack remains closed — there is a displacement u_z at the crack plane, but it is the same on both sides of the crack.

The stress intensity factor is of mode II (shear) form and is

$$K_{II} = \lim_{r\to a^+} \sigma_{zr}\sqrt{2\pi(r-a)} = -\frac{2\mu\alpha(1+\nu)Qa^{3/2}}{3\sqrt{\pi}(1-\nu)K} \; . \tag{31.48}$$

PROBLEMS

1. An infinite homogeneous body contains an axisymmetric *external* crack — i.e. the crack extends over the region $r>a$, $z=0$. Alternatively, the external crack can be considered as two half-spaces bonded together over the region $0\leq r<a$, $z=0$.

The body is loaded in tension, such that the total tensile force transmitted is F. Find an expression for the stress field and in particular determine the mode I stress intensity factor K_I.

2. An infinite homogeneous body containing a penny-shaped crack is subjected to torsional loading $\sigma_{z\theta}=Cr$, $R\to\infty$, where C is a constant[5]. Use Solution E (Table 21.1) to formulate the problem and solve the resulting boundary-value problem using the methods of §30.2.

3. The axisymmetric external crack of radius a (see Problem 31.1) is unloaded, but is subjected to the steady-state thermal conditions

$$\begin{aligned} T(r,z) &\to T_1 \;\; ; \;\; z\to\infty \\ &\to T_2 \;\; ; \;\; z\to-\infty \\ q_z(r,0) &= 0 \;\; ; \;\; r>a \; . \end{aligned}$$

In other words the extremities of the body are maintained at different temperatures T_1, T_2 causing heat to flow through the ligament $z=0$, $0\leq r<a$ and the crack faces are insulated.

Find the mode II stress intensity factor K_{II}.

4. The antisymmetry of the thermoelastic penny-shaped crack problem of §31.2 implies that there is no crack-opening displacement and hence the assumption of insulation (31.21) is arguably rather unrealistic. A more realistic assumption might be that the heat flux across the crack faces is proportional to the local temperature difference — i.e.

[5] This is known as the Reissner-Sagoci problem.

$$q_z(r,0) = h\left(T(r,0^-) - T(r,0^+)\right) \ ; \quad 0 \le r < a \ .$$

Use the methods of Chapter 30 to formulate the heat conduction problem. In particular, express the temperature in terms of one of the forms (30.21) and find the Abel integral equation which must be satisfied by the function $g_3(t)$. Do not attempt to solve this equation.

5. A long rectangular beam defined by the cross-section $-c<x<c$, $-d<y<d$ transmits a bending moment M about the positive x-axis. The beam contains a small penny-shaped crack of radius a in the cross-sectional plane $z=0$ with its centre at the point $(0,b,0)$ — i.e. the crack surface is defined by $0 \le \sqrt{x^2+(y-b)^2} < a \ ; \ z=0$. Assuming that the crack opens completely and that $c, d, d-b$ are all large compared with a, find the variation of K_I around the crack edge and hence determine the minimum value of b for the open-crack assumption to be valid. Find also the crack-opening displacement as a function of position within the crack.

Notice that the corrective problem involves both axisymmetric and non-axisymmetric terms. For the latter, you will need to apply the methods introduced in §30.3.

32

THE INTERFACE CRACK

The problem of a crack at the interface between dissimilar elastic media is of considerable contemporary importance in Elasticity, because of its relevance to the problem of debonding of composite materials and structures. Figure 32.1 shows the case where such a crack occurs at the plane interface between two elastic half-spaces. We shall use the suffices 1,2 to distinguish the stress functions and mechanical properties for the lower and upper half-spaces, $z > 0$, $z < 0$, respectively.

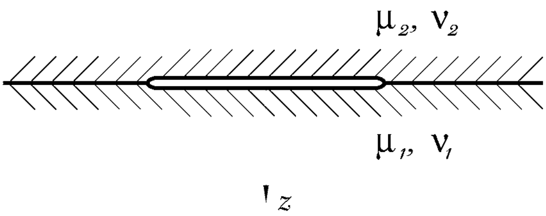

Figure 32.1: The plane interface crack.

The difference in material properties destroys the symmetry that we exploited in the previous chapter and we therefore anticipate shear stresses as well as normal stresses at the interface, even when the far field loading is a state of uniform tension. However, we shall show that we can still use the same methods to reduce the problem to a mixed boundary-value problem for the half-space.

32.1 The uncracked interface

The presence of the interface causes the problem to be non-trivial, even if there is no crack and the two bodies are perfectly bonded.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_32, © Springer Science+Business Media B.V. 2010

Consider the tensile loading of the composite bar of Figure 32.2(a).

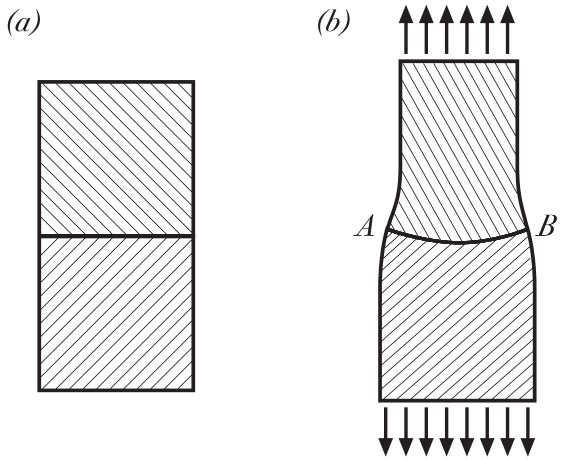

Figure 32.2: The composite bar in tension.

If we assume a state of uniform uniaxial tension $\sigma_{zz} = S$ exists throughout the bar, the Poisson's ratio strains, $e_{xx} = e_{yy} = -\nu_i S/E_i$ will imply an inadmissible discontinuity in displacements u_x, u_y at the interface, unless $\nu_1/E_1 = \nu_2/E_2$. In all other cases, a locally non-uniform stress field will be developed involving shear stresses on the interface and the bar will deform as shown in Figure 32.2(b). Indeed, for many material combinations, there will be a stress singularity at the edges A, B, which can be thought of locally as two bonded orthogonal wedges[1] and analyzed by the method of §11.2.

We shall not pursue the perfectly-bonded problem here — a solution for the corresponding two-dimensional problem is given by Bogy[2] — but we note that as long as the solution for the uncracked interface is known, the corresponding solution when a crack is present can be obtained as in §§13.3.2, 31.1, by superposing a corrective solution in which tractions are imposed on the crack face, equal and opposite to those transmitted in the uncracked state, and the distant boundaries of the body are traction-free. Furthermore, if the crack is small compared with the other linear dimensions of the body — notably the distance from the crack to the free boundary — it can be conceived as occurring at the interface between two bonded half-spaces, as in Figure 32.1.

[1] D.B.Bogy, Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading, *ASME Journal of Applied Mechanics,* Vol. 35 (1968), pp.460–466.

[2] D.B.Bogy, The plane solution for joined dissimilar elastic semistrips under tension, *ASME Journal of Applied Mechanics.,* Vol. 42 (1975), pp.93–98.

32.2 The corrective solution

If we assume that the loading conditions are such as to cause the crack to open fully and therefore to be traction-free, the corrective solution will be defined by the boundary conditions

$$\sigma_{xz1}(x,y,0^+) = \sigma_{xz2}(x,y,0^-) = -S_1(x,y) \tag{32.1}$$
$$\sigma_{yz1}(x,y,0^+) = \sigma_{yz2}(x,y,0^-) = -S_2(x,y) \tag{32.2}$$
$$\sigma_{zz1}(x,y,0^+) = \sigma_{zz2}(x,y,0^-) = -S_3(x,y)\ , \tag{32.3}$$

for $(x,y) \in A$ and

$$\sigma_{xxi}, \sigma_{yyi}, \sigma_{zzi}, \sigma_{xyi}, \sigma_{yzi}, \sigma_{zxi} \to 0\ \ ;\ \ R \to \infty\ , \tag{32.4}$$

where A is the region of the interfacial plane $z=0$ occupied by the crack and S_1, S_2, S_3 are the tractions that would be transmitted across the interface in the absence of the crack.

We shall seek a potential function solution of the problem in the half-space, so we supplement (32.1–32.4) by the continuity and equilibrium conditions

$$\sigma_{xz1}(x,y,0^+) = \sigma_{xz2}(x,y,0^-) \tag{32.5}$$
$$\sigma_{yz1}(x,y,0^+) = \sigma_{yz2}(x,y,0^-) \tag{32.6}$$
$$\sigma_{zz1}(x,y,0^+) = \sigma_{zz2}(x,y,0^-) \tag{32.7}$$
$$u_{x1}(x,y,0+) = u_{x2}(x,y,0-) \tag{32.8}$$
$$u_{y1}(x,y,0+) = u_{y2}(x,y,0-) \tag{32.9}$$
$$u_{z1}(x,y,0+) = u_{z2}(x,y,0-)\ , \tag{32.10}$$

in $\overline{A}$ — i.e. the uncracked part of the interface[3].

It is clear from (32.1–32.3) that (32.5–32.7) apply throughout the interface and are therefore global conditions. We can exploit this fact by expressing the fields in the two bonded half-spaces by separate sets of stress functions and then developing simple symmetric relations between the sets.

32.2.1 Global conditions

The global conditions involve the three stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$ at the surface $z=0$, so it is convenient to choose a formulation in which these components take simple forms at the surface and are as far as possible uncoupled. This can be achieved by superposing Solutions E,F,G of Chapter 21, for which

[3] Notice that only those stress components that act on the interface (and hence have a z-suffix) are continuous across the interface, because of the equilibrium requirement. The remaining three components will generally be discontinuous because of (32.8–32.10) and the dissimilar elastic properties.

$$\sigma_{zxi}(x,y,0) = \frac{\partial^2 \Psi_i}{\partial y \partial z} + \frac{\partial^2 \chi_i}{\partial x \partial z} \tag{32.11}$$

$$\sigma_{zyi}(x,y,0) = -\frac{\partial^2 \Psi_i}{\partial x \partial z} + \frac{\partial^2 \chi_i}{\partial y \partial z} \tag{32.12}$$

$$\sigma_{zzi} = -\frac{\partial^2 \varphi_i}{\partial z^2} \ , \tag{32.13}$$

from Tables 21.1, 21.3, where the suffix i takes the value 1,2 for bodies 1,2 respectively.

The global condition (32.7) now reduces to

$$\frac{\partial^2 \varphi_1}{\partial z^2}(x,y,0^+) = \frac{\partial^2 \varphi_2}{\partial z^2}(x,y,0^-) \tag{32.14}$$

and, remembering that φ_1 is defined only in $z>0$ and φ_2 only in $z<0$, we can satisfy it by imposing the symmetry relation

$$\varphi_1(x,y,z) = \varphi_2(x,y,-z) \equiv \varphi(x,y,z) \tag{32.15}$$

throughout the half-spaces. In the same way[4], the two remaining global conditions can be satisfied by demanding

$$\Psi_1(x,y,z) = -\Psi_2(x,y,-z) \equiv \Psi(x,y,z) \tag{32.16}$$

$$\chi_1(x,y,z) = -\chi_2(x,y,-z) \equiv \chi(x,y,z) \ , \tag{32.17}$$

where the negative sign in (32.16, 32.17) arises from the fact that symmetry of the derivatives $\partial\Psi/\partial z, \partial\chi/\partial z$ implies *antisymmetry* of Ψ, χ.

32.2.2 Mixed conditions

We can now use the remaining boundary conditions (32.1–32.3) and (32.8–32.10) to define a mixed boundary-value problem for the three potentials φ, Ψ, χ in the half-space $z>0$.

For example, using the expressions from Tables 21.1, 21.3 and the definitions (32.15–32.17), the boundary condition (32.8) reduces to

$$\frac{1}{\mu_1}\frac{\partial \Psi}{\partial y} + \frac{(1-2\nu_1)}{2\mu_1}\frac{\partial \varphi}{\partial x} + \frac{(1-\nu_1)}{\mu_1}\frac{\partial \chi}{\partial x} = -\frac{1}{\mu_2}\frac{\partial \Psi}{\partial y} + \frac{(1-2\nu_2)}{2\mu_2}\frac{\partial \varphi}{\partial x} - \frac{(1-\nu_2)}{\mu_2}\frac{\partial \chi}{\partial x} \tag{32.18}$$

in $\overline{A}$, $z=0$.

[4] This idea can also be extended to steady-state thermoelastic problems by adding in Solution P of Table 22.1, which leaves the expressions (32.11–32.13) for the interface stresses unchanged and requires a further symmetry relation between ψ_1, ψ_2 to satisfy the global condition of continuity of heat flux, $q_{z1}(x,y,0^+) = q_{z2}(x,y,0^-)$.

Rearranging the terms and dividing by the non-zero factor $\{(1-\nu_1)/\mu_1+(1-\nu_2)/\mu_2\}$ (which is $A/4$ in the terminology of equation (12.69) for plane strain), we obtain

$$\gamma\frac{\partial\Psi}{\partial y}+\beta\frac{\partial\varphi}{\partial x}+\frac{\partial\chi}{\partial x}=0\ ;\ \text{ in } \overline{A}\ , \tag{32.19}$$

where β is Dundurs' constant (see §§4.4.3, 12.7) and

$$\gamma=\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\Bigg/\left(\frac{(1-\nu_1)}{\mu_1}+\frac{(1-\nu_2)}{\mu_2}\right)\ . \tag{32.20}$$

A similar procedure applied to the boundary conditions (32.9, 32.10) yields

$$-\gamma\frac{\partial\Psi}{\partial x}+\beta\frac{\partial\varphi}{\partial y}+\frac{\partial\chi}{\partial y}=0 \tag{32.21}$$

$$\frac{\partial\varphi}{\partial z}+\beta\frac{\partial\chi}{\partial z}=0\ , \tag{32.22}$$

in $\overline{A}$.

In addition to (32.19, 32.21, 32.22), three further conditions are obtained from (32.1–32.3), using (32.11–32.13) and (32.15–32.17). These are

$$\frac{\partial^2\Psi}{\partial y\partial z}+\frac{\partial^2\chi}{\partial x\partial z}=-S_1(x,y) \tag{32.23}$$

$$-\frac{\partial^2\Psi}{\partial x\partial z}+\frac{\partial^2\chi}{\partial y\partial z}=-S_2(x,y) \tag{32.24}$$

$$-\frac{\partial^2\varphi}{\partial z^2}=-S_3(x,y)\ , \tag{32.25}$$

in A. The six conditions (32.19, 32.21–32.25) define a well-posed boundary-value problem for the three potentials φ,Ψ,χ, when supplemented by the requirements that (i) displacements should be continuous at the crack boundary (between A and $\bar{A}$) and (ii) the stress and displacement fields should decay as $R\to\infty$ in accordance with (32.4).

The two boundary conditions (32.19, 32.21) involve only two independent functions Ψ and $(\beta\varphi+\chi)$ and hence we can eliminate either by differentiation with the result

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-\frac{\partial^2\Psi}{\partial z^2}=0 \tag{32.26}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\beta\varphi+\chi)=-\beta\frac{\partial^2\varphi}{\partial z^2}-\frac{\partial^2\chi}{\partial z^2}=0 \tag{32.27}$$

in $\overline{A}$, where we have used the fact that the potential functions are all harmonic to express the boundary conditions in terms of a single derivative normal to the plane.

In the same way, (32.23, 32.24) yield

$$\frac{\partial^3 \Psi}{\partial z^3} = \frac{\partial S_1}{\partial y} - \frac{\partial S_2}{\partial x} \tag{32.28}$$

$$\frac{\partial^3 \chi}{\partial z^3} = \frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial y} , \tag{32.29}$$

in A.

Equations (32.26, 32.28) now define a two-part boundary-value problem for Ψ, similar to those solved in Chapter 30 for the axisymmetric geometry. It should be noted however that it is not necessary, nor is it generally possible, to satisfy the requirement of continuity of in-plane displacement u_x, u_y at the crack boundary in the *separate* contributions from the potential functions φ, Ψ, χ, as long as the *superposed* fields satisfy this requirement. Thus, the homogeneous problem for Ψ obtained by setting the right-hand side of (32.28) to zero has non-trivial solutions which are physically unacceptable in isolation, since they correspond to displacement fields which are discontinuous at the crack boundary, but they are an essential component of the solution of more general problems, where they serve to cancel similar discontinuities resulting from the fields due to φ, χ. This problem also arises for the crack in a homogeneous medium with shear loading[5] and in non-axisymmetric thermoelastic problems[6]

The remaining boundary conditions define a coupled two-part problem for φ, χ. As in the two-dimensional contact problems of Chapter 12, the coupling is proportional to the Dundurs' constant β. If the material properties are such that $\beta = 0$ (see §12.7), the problems of normal and shear loading are independent and (for example) the crack in a shear field has no tendency to open or close.

32.3 The penny-shaped crack in tension

We now consider the special case of the circular crack $0 \le r < a$, subjected to uniform tensile loading $S_3 = S,\ S_1 = S_2 = 0$. The boundary-value problem for φ, χ is then defined by the conditions[7]

---

5 See for example J.R.Barber, The penny-shaped crack in shear and related contact problems, *International Journal of Engineering Science,* Vol. 13, (1975), pp.815–832.

6 L.Rubenfeld, Non-axisymmetric thermoelastic stress distribution in a solid containing an external crack, *International Journal of Engineering Science,* Vol. 8 (1970), pp.499–509, J.R.Barber, Steady-state thermal stresses in an elastic solid containing an insulated penny-shaped crack, *Journal of Strain Analysis,* Vol. 10 (1975), pp.19–24.

7 This problem was first considered by V.I.Mossakovskii and M.T.Rybka, Generalization of the Griffith-Sneddon criterion for the case of a non-homogeneous body, *Journal of Applied Mathematics and Mechanics,* Vol. 28 (1964), 1277–1286, and by F.Erdogan, Stress distribution on bonded dissimilar materials containing circu-

$$\frac{\partial^3\chi}{\partial z^3} = 0 \;\; ; \;\; z=0,\; 0\leq r<a \tag{32.30}$$

$$\frac{\partial^2\varphi}{\partial z^2} = S \;\; ; \;\; z=0,\; 0\leq r<a \tag{32.31}$$

$$\frac{\partial\varphi}{\partial z}+\beta\frac{\partial\chi}{\partial z} = 0 \;\; ; \;\; z=0,\; r>a \tag{32.32}$$

$$\beta\frac{\partial^2\varphi}{\partial z^2}+\frac{\partial^2\chi}{\partial z^2} = 0 \;\; ; \;\; z=0,\; r>a\; . \tag{32.33}$$

From Table 30.1, we see that we can satisfy (32.32, 32.33) by writing[8]

$$\beta\varphi+\chi=\phi_1 \;\; , \;\; \varphi+\beta\chi=\phi_3 \tag{32.34}$$

— i.e.

$$\varphi=\frac{\phi_3-\beta\phi_1}{(1-\beta^2)} \;\; , \;\; \chi=\frac{\phi_1-\beta\phi_3}{(1-\beta^2)}\; , \tag{32.35}$$

where ϕ_1, ϕ_3 are defined by equation (30.21).

The remaining boundary conditions then define two coupled integral equations of Abel-type for the unknown functions $g_1(t), g_3(t)$ in the range $0\leq t<a$. To develop these equations, we first note that the condition $\nabla^2\chi=0$ and the fact that the solution is axisymmetric enable us to write (32.30) in the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{\partial\chi}{\partial z}(r,0) = 0 \;\; ; \;\; z=0,\; 0\leq r<a\; . \tag{32.36}$$

This can be integrated within the plane $z=0$ to give

$$\frac{\partial\chi}{\partial z} = C \;\; ; \;\; z=0,\; 0\leq r<a\; , \tag{32.37}$$

where C is a constant which will ultimately be chosen to ensure continuity of displacements at $r=a$ and we have eliminated a logarithmic term to preserve continuity of displacements at the origin.

Table 30.1 and (32.35, 32.37) now enable us to write

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}}+\beta\int_r^a \frac{g_3(t)dt}{\sqrt{t^2-r^2}} = (1-\beta^2)C \;\; ; \;\; 0\leq r<a\; , \tag{32.38}$$

whilst (32.31, 32.35) and Table 30.1 give

$$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}}-\frac{\beta}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} = (1-\beta^2)S \;\; ; \;\; 0\leq r<a\; , \tag{32.39}$$

---

lar or ring-shaped cavities, *ASME Journal of Applied Mechanics,* Vol. 32 (1965), pp.829–836. A formulation for the penny-shaped crack with more general loading was given by J.R.Willis, The penny-shaped crack on an interface, *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 25 (1972), pp.367–382.

[8] This method would fail for the special case $\beta=\pm 1$, but materials with positive Poisson's ratio are restricted by energy considerations to values in the range $-\frac{1}{2}\leq\beta\leq\frac{1}{2}$.

which can be integrated to give

$$\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} - \beta \int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} = \frac{(1-\beta^2)Sr^2}{2} + B \ ; \ 0 \le r < a \ , \qquad (32.40)$$

where B is an arbitrary constant.

We now treat (32.38) as an Abel equation for $g_1(t)$, carrying the g_3 integral onto the right-hand side and using the inversion rules (Table 30.2) to obtain

$$g_1(x) = \frac{2\beta}{\pi} \int_0^a \frac{tg_3(t)dt}{(x^2-t^2)} + \frac{2C(1-\beta^2)}{\pi} \ , \qquad (32.41)$$

where we have simplified the double integral term in g_3 by changing the order of integration and performing the resulting inner integral.

In the same way, treating (32.40) as an equation for $tg_3(t)$, we obtain

$$xg_3(x) = -\frac{2\beta}{\pi} \int_0^a \frac{t^2 g_1(t)dt}{(x^2-t^2)} + \frac{2B}{\pi} + \frac{2S(1-\beta^2)x^2}{\pi} \ . \qquad (32.42)$$

The function $g_3(x)$ must be bounded at $x=0$ and hence B must be chosen so that

$$B = -\beta \int_0^a g_1(t)dt \ . \qquad (32.43)$$

Thus, (32.42) reduces to

$$g_3(x) = -\frac{2\beta x}{\pi} \int_0^a \frac{g_1(t)dt}{(x^2-t^2)} + \frac{2S(1-\beta^2)x}{\pi} \ . \qquad (32.44)$$

32.3.1 Reduction to a single equation

We observe from equations (32.41, 32.44) that g_1 is an even function of x, whereas g_3 is odd. Using this result and expanding the integrands as partial fractions, we arrive at the simpler expressions[9]

$$g_1(x) = \frac{\beta}{\pi} \int_{-a}^a \frac{g_3(t)dt}{(x-t)} + \frac{2C(1-\beta^2)}{\pi} \ ; \ -a < x < a \qquad (32.45)$$

$$g_3(x) = -\frac{\beta}{\pi} \int_{-a}^a \frac{g_1(t)dt}{(x-t)} + \frac{2S(1-\beta^2)x}{\pi} \ ; \ -a < x < a \ . \qquad (32.46)$$

Two methods are available for reducing (32.45, 32.46) to a single equation. The most straightforward approach is to use (32.46) (for example) to substitute for g_3 in (32.45), resulting after some manipulations in the integral equation

[9] *cf.* equation (30.26) and the associated discussion.

$$g_1(x) + \frac{\beta^2}{\pi^2(1-\beta^2)} \int_{-a}^{a} \left(\ln\left|\frac{a-t}{a+t}\right| - \ln\left|\frac{a-x}{a+x}\right| \right) \frac{g_1(t)dt}{(t-x)}$$
$$= -\frac{2\beta S}{\pi^2}\left(2a + x\ln\left|\frac{a-x}{a+x}\right|\right) + \frac{2C}{\pi} \;\; ; \;\; -a < x < a \,. \qquad (32.47)$$

The kernel of this equation is bounded at $t = x$, the limiting form being obtainable by L'Hôpital's rule as $2a/(x^2 - a^2)$. Equations of this kind with definite integration limits and bounded kernels are known as *Fredholm integral equations*[10].

However, the identity of kernel and range of integration in equations (32.45, 32.46) permits a more direct approach. Multiplying (32.46) by $\imath$ and adding the result to (32.45), we find that both equations are contained in the complex equation

$$g(x) + \frac{\imath\beta}{\pi} \int_{-a}^{a} \frac{g(t)dt}{(x-t)} = \frac{2(1-\beta^2)}{\pi}(C + \imath Sx) \,, \qquad (32.48)$$

where $g(x) \equiv g_1(x) + \imath g_3(x)$. Furthermore, this equation can be inverted explicitly[11] in the form of an integral of the right-hand side, leading to a closed-form solution for $g(x)$. The tractions at the interface can then be recovered by separating $g(x)$ into its real and imaginary parts and substituting into the expressions

$$\sigma_{zr} = \frac{1}{(1-\beta^2)} \frac{d}{dr} \int_0^a \frac{g_1(t)dt}{\sqrt{r^2 - t^2}} \;\; ; \;\; z = 0, \; r > a \qquad (32.49)$$

$$\sigma_{zz} = -\frac{1}{(1-\beta^2)r} \frac{d}{dr} \int_0^a \frac{tg_3(t)dt}{\sqrt{r^2 - t^2}} \;\; ; \;\; z = 0, \; r > a \,. \qquad (32.50)$$

32.3.2 Oscillatory singularities

An asymptotic analysis of the integrals (32.49, 32.50) at $r = a + s$, $s \ll a$, shows that they have the form

$$\sigma_{zz} + \imath\sigma_{zr} = SK\left(\frac{s}{2a}\right)^{-\frac{1}{2}+\imath\epsilon} + O(1) \qquad (32.51)$$

$$= SK\left(\frac{s}{2a}\right)^{-\frac{1}{2}} \left[\cos\left(\epsilon \ln\frac{s}{2a}\right) + \imath \sin\left(\epsilon\ln\frac{s}{2a}\right)\right] + O(1) \,, \qquad (32.52)$$

where

$$\epsilon = \frac{1}{2\pi} \ln\left(\frac{1+\beta}{1-\beta}\right) \qquad (32.53)$$

[10] F.G.Tricomi, *Integral Equations*, Interscience, New York, 1957, Chapter 2.

[11] N.I.Muskhelishvili, *Singular Integral Equations*, (English translation by J.R.M.Radok, Noordhoff, Groningen, 1953)

and Kassir and Bregman[12] have found the complex constant K to be

$$K = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2+\imath\epsilon)}{\Gamma(\frac{1}{2}+\imath\epsilon)} \; . \tag{32.54}$$

Equation (32.52) shows that the stresses are square-root singular at the crack tip, but that they also oscillate with increasing frequency as r approaches a and $\ln(s/2a)\rightarrow-\infty$. This behaviour is common to all interface crack problems for which $\beta\neq 0$ and can be predicted from an asymptotic analysis similar to that of §11.2.2, from which a complex leading singular eigenvalue $\lambda=\frac{1}{2}\pm\imath\epsilon$ is obtained.

32.4 The contact solution

A more disturbing feature of this behaviour is that the crack opening displacement, $(u_{z1}(r,0)-u_{z2}(r,0))$, also oscillates as $r\rightarrow a^-$ and hence there are infinitely many regions of interpenetration of material near the crack tip.

This difficulty was first resolved by Comninou[13] for the case of the plane crack, by relaxing the superficially plausible assumption that the crack will be fully open and hence have traction-free faces in a tensile field, using instead the more rigorous requirements of frictionless unilateral contact — i.e. permitting contact to occur, the extent of the contact zones (if any) to be determined by the inequalities (29.11, 29.12). She found that a very small contact zone is established adjacent to each crack tip, the extent of which is of the same order of magnitude as the zone in which interpenetration is predicted for the original 'open crack' solution. At the closed crack tip, only the shear stresses in the bonded region are singular, leading to a 'mode II' (shear) stress intensity factor, by analogy with equation (13.45). However, the contact stresses show a proportional compressive singularity in $r\rightarrow a^-$. Both singularities are of the usual square-root form, without the oscillatory character of equations (32.52).

Solutions have since been found for other interface crack problems with contact zones, including the plane crack in a combined tensile and shear field[14]. These original solutions of the interface crack problem used numerical methods to solve the resulting integral equation, but more recently an analytical solution has been found by Gautesen and Dundurs[15].

[12] M.K.Kassir and A.M.Bregman, The stress-intensity factor for a penny-shaped crack between two dissimilar materials, *ASME Journal of Applied Mechanics,* Vol. 39 (1972), pp.308–310.

[13] M.Comninou, The interface crack, *ASME Journal of Applied Mechanics,* Vol. 44 (1977), pp.631–636.

[14] M.Comninou and D.Schmueser, The interface crack in a combined tension-compression and shear field, *ASME Journal of Applied Mechanics,* Vol. 46 (1979), pp.345–358.

[15] A.K.Gautesen and J.Dundurs, The interface crack in a tension field, *ASME Journal of Applied Mechanics,* Vol. 54 (1987), pp.93–98; A.K.Gautesen and J.Dundurs, The interface crack under combined loading, *ibid.*, Vol. 55 (1988), pp.580–586.

The problem of §32.3 — the penny-shaped crack in a tensile field — was solved in a unilateral contact formulation by Keer *et al.*[16]. They found a small annulus of contact $b<r<a$ adjacent to the crack tip. In fact, the formulation of §32.3 is easily adapted to this case. We note that the crack opening displacement must be zero in the contact zone, so the range of equation (32.32) is extended inwards to $b<r$, whilst (32.31) — setting the total normal tractions at the interface to zero — now only applies in the open region of the crack $0\le r<b$. This can be accommodated by replacing a by b in the definitions of ϕ_3, so that (32.38, 32.39) are replaced by

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}} + \beta \int_r^b \frac{g_3(t)dt}{\sqrt{t^2-r^2}} = (1-\beta^2)C\ ; \quad 0 \le r < a \quad (32.55)$$

$$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} - \frac{\beta}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} = (1-\beta^2)S\ ; \quad 0 \le r < b\ , \quad (32.56)$$

and (32.45, 32.46) by

$$g_1(x) = \frac{\beta}{\pi}\int_{-b}^b \frac{g_3(t)dt}{(x-t)} + \frac{2C(1-\beta^2)}{\pi}\ ; \quad -a < x < a \quad (32.57)$$

$$g_3(x) = -\frac{\beta}{\pi}\int_{-a}^a \frac{g_1(t)dt}{(x-t)} + \frac{2S(1-\beta^2)x}{\pi}\ ; \quad -b < x < b\ . \quad (32.58)$$

The difference in range in the integrals in (32.57, 32.58) now prevents us from combining them in a single complex equation, but it is still possible to develop a Fredholm equation in either function[17] by substitution, as in equation (32.47).

32.5 Implications for Fracture Mechanics

When an interface crack is loaded in tension, the predicted contact zones are very much smaller than the crack dimensions and it must therefore be possible to characterize all features of the local crack-tip fields, including the size of the contact zone, in terms of features of the elastic field further from the crack tip, where the open and contact solutions are essentially indistinguishable. Hills and Barber[18] have used this argument to derive the contact solution from the simpler 'open-crack' solution for a plane crack loaded in tension.

[16] L.M.Keer, S.H.Chen and M.Comninou, The interface penny-shaped crack reconsidered, *International Journal of Engineering Science,* Vol. 16 (1978), pp.765–772.

[17] The best choice here is to eliminate $g_1(x)$, since the opposite choice will lead to an integral equation on the range $-a<x<a$, which might have discontinuities in at the points $x=\pm b$. These could cause problems with convergence in the final numerical solution.

[18] D.A.Hills and J.R.Barber, Interface cracks, *International Journal of Mechanical Sciences,* Vol. 35 (1993), pp.27–37.

The contact zones may also be smaller than the fracture process zone defined in §13.3.1, in which case the conditions for fracture must also be capable of characterization in terms of the open-crack asymptotic field. If the Griffith fracture criterion applies, fracture will occur when the strain energy release rate G of §13.3.3 exceeds a critical value for the material. For a homogeneous material under in-plane loading, the energy release rate is given by equation (13.56) and is proportional to $K_1^2+K_{II}^2$. For the open solution of the interface crack, the oscillatory behaviour prevents the stress intensity factors being given their usual meaning, but the energy release rate depends on the asymptotic behaviour of $\sigma_{zr}^2+\sigma_{zz}^2$, which is non-oscillatory. The contact solution gives almost the same energy release rate, as indeed it must do, since the two solutions only differ in a very small region at the tip.

Experiments with homogeneous materials show that the energy release rate required for fracture varies significantly with the ratio K_{II}/K_I, which is referred to as the *mode mixity*[19]. This is probably more indicative of our over-idealization of the crack geometry than of any limitation on Griffith's theory, since real cracks are seldom plane and the attempt to shear them without significant opening will probably lead to crack face contact and consequent frictional forces.

It is difficult to extend the concept of mode mixity to the open interface crack solution, since, for example, the ratio σ_{zr}/σ_{zz} passes through all values, positive and negative, in each cycle of oscillation. However, the change in the ratio with s is very slow except in the immediate vicinity of the crack tip, leading Rice[20] to suggest that some specific distance from the tip be agreed by convention at which the mode mixity be defined.

An interesting feature of the asymptotics of the open solution is that, in contrast to the case $\beta=0$ (including the crack in a homogeneous material), all possible fields can be mapped into each other by a change in length scale and in one linear multiplier (e.g. the square root of the energy release rate). Thus, an alternative way of characterizing the conditions at the tip for fracture experiments would be in terms of the energy release rate and a characteristic length, which could be taken as the maximum distance from the tip at which the open solution predicts interpenetration. Conditions approach most closely to the classical mode I state when this distance is small.

It must be emphasised that these arguments are all conditional upon the characteristic length being small compared with all the other dimensions of the system, *including the anticipated process zone dimension.* If this condition is not satisfied, it is essential to use the unilateral contact formulation.

---

[19] See for example, H.A.Richard, Examination of brittle fracture criteria for overlapping mode I and mode II loading applied to cracks, **in** G.C.Sih, E.Sommer and W.Dahl, **eds.**, *Application of Fracture Mechanics to Structures*, Martinus Nijhoff, The Hague, 1984, pp.309–316.

[20] J.R.Rice, Elastic fracture mechanics concepts for interfacial cracks, *ASME Journal of Applied Mechanics,* Vol. 55 (1988), pp.98–103.

33

VARIATIONAL METHODS

Energy or variational methods have an important place in Solid Mechanics both as an alternative to the more direct method of solving the governing partial differential equations and as a means of developing convergent approximations to analytically intractable problems. They are particularly useful in situations where only a restricted set of results is required — for example, if we wish to determine the resultant force on a cross-section or the displacement of a particular point, but are not interested in the full stress and displacement fields. Indeed, such results can often be obtained in closed form for problems in which a solution for the complete fields would be intractable.

From an engineering perspective, it is natural to think of these methods as a consequence of the principle of conservation of energy or the first law of thermodynamics. However, conservation of energy is in some sense guaranteed by the use of Hooke's law and the equilibrium equations. Once these physical premises are accepted, the energy theorems we shall present here are purely mathematical consequences. Indeed, the finite element method, which is one of the more important developments of this kind, can be developed simply by applying arguments from approximation theory (such as a least-squares fit) to the governing equations introduced in previous chapters. For this reason, these techniques are now more often referred to as *Variational Methods*, meaning that instead of seeking to solve the governing partial differential equations directly, we seek to define a scalar function of the physical parameters which is stationary (generally maximum or minimum) in respect to infinitesimal variations about the solution.

In this chapter, we shall introduce only the principal theorems of this kind and indicate some of their applications[1]. We shall present most of the derivations in the context of general anisotropy, using the index notation of §1.1.2,

[1] For a more comprehensive review of variational methods in elasticity, the reader is referred to S.G.Mikhlin, *Variational Methods in Mathematical Physics*, Pergamon, New York, 1964 and S.P.Timoshenko and J.N.Goodier, *loc. cit.*, Chapter 8.

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_33, © Springer Science+Business Media B.V. 2010

since this leads to a more compact presentation as well as demonstrating the generality of the results.

33.1 Strain energy

When a body is deformed, the external forces move through a distance associated with the deformation and hence do work. The fundamental premise of linear elasticity is that the deformation is linearly proportional to the load, so if the load is now removed, the deformation must pass through the same sequence of states as it did during the loading phase. It follows that the work done during loading is exactly recovered during unloading and must therefore be stored in the deformed body as *strain energy.*

Suppose that the body Ω with boundary Γ is subjected to a body force distribution p_i and boundary tractions $t_i = \sigma_{ij}n_j$ (1.21), where n_j defines a unit vector in the direction of the local outward normal to Γ. If the loads are applied sufficiently slowly for inertia effects to be negligible[2], and if the final elastic displacements are u_i, the stored strain energy U in the body in the loaded state will be equal to the work W done by the external forces during loading, which is

$$U = W = \frac{1}{2}\iiint_{\Omega} p_i u_i d\Omega + \frac{1}{2}\iint_{\Gamma} t_i u_i d\Gamma \; . \tag{33.1}$$

33.1.1 Strain energy density

The same principle can be applied to determine the strain energy in an infinitesimal rectangular element of material subjected to a uniform state of stress σ_{ij}. We conclude that the *strain energy density* — i.e. the strain energy stored per unit volume — is given by

$$U_0 = \frac{1}{2}\sigma_{ij}e_{ij} \; . \tag{33.2}$$

Substituting for the stress or strain components from the generalized Hooke's law (1.55, 1.57), we obtain the alternative expressions

$$U_0 = \frac{1}{2}c_{ijkl}e_{ij}e_{kl} = \frac{1}{2}c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} = \frac{1}{2}s_{ijkl}\sigma_{ij}\sigma_{kl} \; . \tag{33.3}$$

Notice incidentally that U_0 must be positive for all possible states of stress or deformation and this places some inequality restrictions on the elasticity tensors c_{ijkl}, s_{ijkl}.

[2] This implies that the stress field is always quasi-static in the sense of Chapter 7 and hence that all the work done by the external loads appears as elastic strain energy in the body, rather than partially as kinetic energy.

If the material is isotropic, these expressions reduce to

$$U_0 = \frac{1}{2E}\Big[\left(\sigma_{xx}^2+\sigma_{yy}^2+\sigma_{zz}^2\right) - 2\nu(\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}+\sigma_{xx}\sigma_{yy})$$
$$+ 2(1+\nu)\left(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2\right)\Big]$$
$$= \mu\left[\frac{\nu e^2}{(1-2\nu)} + e_{xx}^2+e_{yy}^2+e_{zz}^2+2e_{yz}^2+2e_{zx}^2+2e_{xy}^2\right] . \qquad (33.4)$$

33.2 Conservation of energy

The strain energy U stored in the entire body Ω can be obtained by summing that stored in each of its individual particles, giving

$$U = \iiint_\Omega U_0 d\Omega . \qquad (33.5)$$

This expression and equation (33.1) must clearly yield the same result and hence

$$\frac{1}{2}\iiint_\Omega p_i u_i d\Omega + \frac{1}{2}\iint_\Gamma t_i u_i d\Gamma = \iiint_\Omega U_0 d\Omega . \qquad (33.6)$$

This argument appeals to the principle of conservation of energy, but this principle is implicit in Hooke's law, which guarantees that the loading is reversible. Thus, (33.6) can be derived from the governing equations of elasticity without explicitly invoking conservation of energy. To demonstrate this, we first substitute (1.21) into the second term on the left-hand side of (33.6) and apply the divergence theorem, obtaining

$$\frac{1}{2}\iint_\Gamma t_i u_i d\Gamma = \frac{1}{2}\iint_\Gamma n_j\sigma_{ji}u_i d\Gamma = \frac{1}{2}\iiint_\Omega \frac{\partial}{\partial x_j}(\sigma_{ji}u_i)d\Omega \qquad (33.7)$$
$$= \frac{1}{2}\iiint_\Omega \frac{\partial\sigma_{ji}}{\partial x_j}u_i d\Omega + \frac{1}{2}\iiint_\Omega \sigma_{ji}\frac{\partial u_i}{\partial x_j}d\Omega . \qquad (33.8)$$

Finally, we use the equilibrium equation (2.5) in the first term on the right-hand side of (33.8) and Hooke's law (1.57) in the second term to obtain

$$\frac{1}{2}\iint_\Gamma t_i u_i d\Gamma = -\frac{1}{2}\iiint_\Omega p_i u_i d\Omega + \frac{1}{2}\iiint_\Omega c_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_j}d\Omega , \qquad (33.9)$$

from which (33.6) follows after using (33.2) in the last term.

33.3 Potential energy of the external forces

We can also construct a *potential energy* of the external forces which we denote by V. The reader is no doubt familiar with the concept of potential energy as applied to gravitational forces, but the definition can be extended to more general force systems as *the work that the forces would do if they were allowed to return to their reference positions.* For a single concentrated force $\boldsymbol{F}$ displaced through $\boldsymbol{u}$ this is defined as

$$V = -\boldsymbol{F} \cdot \boldsymbol{u} = -F_i u_i \; . \tag{33.10}$$

It follows by superposition that the potential energy of the boundary tractions and body forces is given by

$$V = -\iint_{\Gamma_t} t_i u_i d\Gamma - \iiint_{\Omega} p_i u_i d\Omega \; , \tag{33.11}$$

where Γ_t is that part of the boundary over which the tractions are prescribed. We can then define the *total potential energy* Π as the sum of the stored strain energy and the potential energy of the external forces — i.e.

$$\Pi = U + V = \frac{1}{2}\iiint_{\Omega} c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega - \iint_{\Gamma_t} t_i u_i d\Gamma - \iiint_{\Omega} p_i u_i d\Omega \; . \tag{33.12}$$

33.4 Theorem of minimum total potential energy

Suppose that the displacement field u_i satisfies the equilibrium equations (2.13) for a given set of boundary conditions and that we then perturb this state by a small variation δu_i. The corresponding perturbation in Π is

$$\delta\Pi = \iiint_{\Omega} c_{ijkl}\frac{\partial \delta u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega - \iint_{\Gamma} t_i \delta u_i d\Gamma - \iiint_{\Omega} p_i \delta u_i d\Omega \; , \tag{33.13}$$

from (33.12). Notice that $\delta u_i = 0$ in any region Γ_u of Γ in which the displacement is prescribed and hence the domain of integration Γ_t in the second term on the right-hand side of (33.12) can be replaced by $\Gamma = \Gamma_u + \Gamma_t$.

Substituting for t_i from (1.21) into the second term on the right-hand side of (33.13) and then applying the divergence theorem, we have

$$\begin{aligned}\iint_{\Gamma} t_i \delta u_i d\Gamma = \iint_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma &= \iiint_{\Omega} \frac{\partial}{\partial x_j}\left(\sigma_{ij}\delta u_i\right) d\Omega \\ &= \iiint_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i d\Omega + \iiint_{\Omega} \frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} d\Omega \; .\end{aligned} \tag{33.14}$$

Finally, using the equilibrium equation (2.5) in the first term and Hooke's law (1.57) in the second, we obtain

$$\iint_{\Gamma} t_i \delta u_i d\Gamma = -\iiint_{\Omega} p_i \delta u_i d\Omega + \iiint_{\Omega} c_{ijkl}\frac{\partial \delta u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega \; , \tag{33.15}$$

and comparing this with (33.13), we see that $\delta\Pi = 0$. In other words, the equilibrium equation requires that the total potential energy must be stationary with regard to any kinematically admissible[3] small variation δu_i in the displacement field u_i, or

$$\frac{\partial \Pi}{\partial u_i} = 0 \, . \tag{33.16}$$

A more detailed second order analysis shows that the total potential energy must in fact be a minimum and this is intuitively reasonable, since if some variation δu_i from the equilibrium state could be found which reduced Π, the surplus energy would take the form of kinetic energy and the system would therefore generally accelerate away from the equilibrium state. However, we emphasise that the above derivation of equation (33.16) makes no appeal to the principle of conservation of energy.

33.5 Approximate solutions — the Rayleigh-Ritz method

We remarked in Chapter 2 that the problem of elasticity is to determine a stress field satisfying appropriate boundary conditions such that (i) the stresses satisfy the equilibrium equations (2.5) and (ii) the corresponding elastic strains satisfy the compatibility conditions (2.10). The compatibility conditions can be satisfied by defining the problem in terms of the displacements u_i and we have just shown that the equilibrium equations are equivalent to the principal of minimum total potential energy. Thus, an alternative formulation is to seek a kinematically admissible displacement field u_i such that the total potential energy Π is a minimum.

This approach is particularly useful as a method of generating approximate solutions to problems that might otherwise be analytically intractable. Suppose that we approximate the displacement field in the body by the finite series

$$u_i(x_1, x_2, x_3) = \sum_{k=1}^{n} A_k f_i^{(k)}(x_1, x_2, x_3) \, , \tag{33.17}$$

where the $f_i^{(k)}$ are a set of approximating functions and A_k are arbitrary constants constituting the *degrees of freedom* in the approximation. The $f_i^{(k)}$ must be continuous and single-valued and must also be chosen to satisfy any displacement boundary conditions of the problem, but are otherwise unrestricted as to form.

We can then substitute (33.17) into (33.12) and perform the integrals over Ω, obtaining the total potential energy as a quadratic function of the A_k. The theorem (33.16) demands that

[3] i.e. any perturbation that is a continuous function of position within the body and is consistent with the displacement boundary conditions.

$$\frac{\partial \Pi}{\partial A_k} = 0 \; ; \quad k = 1, n \; , \tag{33.18}$$

which defines n linear equations for the n unknown degrees of freedom A_k. The corresponding stress components can then be found by substituting (33.17) into Hooke's law (1.57).

If the approximating functions $f_i^{(k)}$ are defined over the entire body Ω, this typically leads to series solutions (e.g. power series or Fourier series) and the method is known as the *Rayleigh-Ritz method.* It is particularly useful in structural mechanics applications such as beam problems, where the displacement is a function of a single spatial variable, but it can also be applied (e.g.) to the problem of a rectangular plate, using double Fourier series or power series.

Example

To illustrate the method, we consider the problem of a rectangular plate $-a < x < a, -b < y < b$ which makes frictionless contact with fixed rigid planes at $y = \pm b$ and which is loaded by compressive tractions

$$\sigma_{xx}(\pm a, y) = -\frac{Sy^2}{b^2}$$

at $x = \pm a$. The problem is symmetric about both axes, so u_x must be odd in x and even in y, whilst u_y is even in x and odd in y. Also, the displacement boundary condition requires that $u_y = 0$ at $y = \pm b$. The most general third order polynomial approximation satisfying these conditions is

$$u_x = A_1 x^3 + A_2 x y^2 + A_3 x \; ; \quad u_y = A_4(b^2 - y^2)$$

and the corresponding strains are

$$\begin{aligned} e_{xx} &= \frac{\partial u_x}{\partial x} = 3A_1 x^2 + A_2 y^2 + A_3 \\ e_{yy} &= \frac{\partial u_y}{\partial y} = -2A_4 y \\ e_{xy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = A_2 x y \; . \end{aligned}$$

Substituting these results into (33.4) and then evaluating the integral (33.5), we obtain

$$\begin{aligned} U = \int_{-b}^{b}\int_{-a}^{a} U_0(x,y)dxdy &= \frac{4\mu ab(1-\nu)}{15(1-2\nu)}\Big[(27A_1^2a^4 + 3A_2^2b^4 + 15A_3^2 + 20A_4^2b^2 \\ &+ 10A_2A_3b^2 + 30A_1A_3a^2 + 10A_1A_2a^2b^2)\Big] + \frac{8\mu A_2^2 a^3 b^3}{9} \; . \end{aligned}$$

The potential energy of the tractions on the boundaries $x=\pm a$ is

$$V = 2\int_{-b}^{b}\frac{Sy^2(A_1a^3+A_2ay^2+A_3a)}{b^2}dy = 4Sab\left(\frac{A_1a^2}{3}+\frac{A_2b^2}{5}+\frac{A_3}{3}\right) ,$$

where the factor of 2 results from there being identical expressions for each boundary. Calculating $\Pi=U+V$ and imposing the condition (33.18) for each degree of freedom A_i, $i=1,4$, we obtain four linear equations whose solution is

$$A_1 = 0 \; ; \quad A_2 = -\frac{S(1-2\nu)}{\mu[5(1-2\nu)a^2+2(1-\nu)b^2]}$$

$$A_3 = -\frac{5Sa^2(1-2\nu)^2}{6\mu(1-\nu)[5(1-2\nu)a^2+2(1-\nu)b^2]} \; ; \quad A_4 = 0 \, .$$

Rayleigh-Ritz solutions are conceptually straightforward, but they tend to generate lengthy algebraic expressions, as illustrated in this simple example. However, this is no bar to their use provided the algebra is performed in Mathematica or Maple. Also, if Fourier series are used in place of power series, the orthogonality of the corresponding integrals will often lead to significant simplifications.

However, if high accuracy is required it is often more effective to use a set of piecewise continuous functions for the $f_i^{(k)}(x_1,x_2,x_3)$ of equation (33.17). The body is thereby divided into a set of *elements* and the displacement in each element is described by one or more *shape functions* multiplied by degrees of freedom A_k. Typically, the shape functions are defined such that the A_k represent the displacements at specified points or *nodes* within the body and the $f_i^{(k)}$ are zero except in those elements contiguous to node k. They must also satisfy the condition that the displacement be continuous between one element and the next for all A_k. Once the approximation is defined, equation (33.18) once again provides n linear equations for the n nodal displacements. This is the basis of the *finite element method*[4]. Since the theorem of minimum total potential energy is itself derivable from Hooke's law and the equilibrium equation, an alternative derivation of the finite element method can be obtained by applying approximation theory directly to these equations. To develop a set of n linear equations for the A_k, we substitute the approximate form (33.17) into the equilibrium equations, multiply by n *weight functions*, integrate over the domain Ω and set the resulting n linear functions of the A_k to zero. The resulting equations will be identical to (33.18) if the weight functions are chosen to be identical to the shape functions.

[4] For more detailed discussion of the finite element method, see O.C.Zienkiewicz, *The Finite Element Method*, McGraw-Hill, New York, 1977, K.J.Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1982, T.J.R.Hughes, *The Finite Element Method*, Prentice Hall, Englewood Cliffs, NJ, 1987.

33.6 Castigliano's second theorem

The strain energy U can be written as a function of the stress components, using the final expression in (33.3). We obtain

$$U = \frac{1}{2}\iiint_{\Omega} s_{ijkl}\sigma_{ij}\sigma_{kl}d\Omega \; . \tag{33.19}$$

We next consider the effect of perturbing the stress field by a small variation $\delta\sigma_{ij}$, chosen so that the perturbed field $\sigma_{ij}+\delta\sigma_{ij}$ still satisfies the equilibrium equation (2.5) — i.e.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + p_i = 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}(\sigma_{ij} + \delta\sigma_{ij}) + p_i = 0 \; . \tag{33.20}$$

The corresponding perturbation in U will be

$$\delta U = \iiint_{\Omega} s_{ijkl}\sigma_{kl}\delta\sigma_{ij}d\Omega = \iiint_{\Omega} \frac{\partial u_i}{\partial x_j}\delta\sigma_{ij}d\Omega \; . \tag{33.21}$$

The divergence theorem gives

$$\begin{aligned} \iint_{\Gamma} u_i\delta\sigma_{ij}n_j d\Gamma &= \iiint_{\Omega} \frac{\partial}{\partial x_j}\left(u_i\delta\sigma_{ij}\right)d\Omega \\ &= \iiint_{\Omega} \frac{\partial u_i}{\partial x_j}\delta\sigma_{ij}d\Omega + \iiint_{\Omega} \frac{\partial \delta\sigma_{ij}}{\partial x_j}u_i d\Omega \end{aligned} \tag{33.22}$$

and by taking the difference between the two equations (33.20), we see that the second term on the right-hand side must be zero. Using (33.21, 33.22) and $\delta\sigma_{ij}n_j = \delta t_i$, we then have

$$\delta U = \iint_{\Gamma_u} u_i\delta\sigma_{ij}n_j d\Gamma = \iint_{\Gamma_u} u_i\delta t_i d\Gamma \; , \tag{33.23}$$

where the integral is taken only over that part of the boundary Γ_u in which displacement u_i is prescribed, since no perturbation in traction is permitted in Γ_t where t_i is prescribed. It follows that the *complementary energy*

$$C = U - \iint_{\Gamma_u} u_i t_i d\Gamma \tag{33.24}$$

must be stationary with respect to all self-equilibrated variations of stress $\delta\sigma_{ij}$, or

$$\frac{\partial C}{\partial \sigma_{ij}} = 0 \; . \tag{33.25}$$

This is *Castigliano's second theorem*. It enables us to state a second alternative formulation of the elasticity problem as the search for a state of stress satisfying the equilibrium equation (2.5), such that the complementary energy C is stationary. It is interesting to note that the theorem of minimum potential energy and Castigliano's second theorem each substitutes for one of the two conditions equilibrium and compatibility identified in Chapter 2.

33.7 Approximations using Castigliano's second theorem

We recall that both the Airy stress function of Chapters 5–13 and the Prandtl stress function of Chapters 15–17 satisfy the equilibrium equations identically. In fact, these representations define the most general states of in-plane and antiplane stress respectively that satisfy the equilibrium equations without body force. Thus, if we describe the solution in terms of such a stress function, Castigliano's second theorem reduces the problem to the search for a stress function satisfying the boundary conditions and equation (33.25). This leads to a very effective approximate method for both in-plane and antiplane problems in which the stress function ϕ is represented in the form

$$\phi(x,y) = \sum_{k=1}^{n} A_k f_k(x,y) \;, \tag{33.26}$$

by analogy with (33.17). Also, the traction boundary conditions can generally be fairly easily satisfied by (33.26), making use of the mathematical description of the boundary line Γ. We shall start the discussion by considering the torsion problem of Chapter 16, for which this procedure is particularly straightforward. We shall then adapt it for the in-plane problem in §33.7.2.

33.7.1 The torsion problem

If the torsion problem is described in terms of Prandtl's stress function φ, the traction-free boundary condition for a simply connected region Ω can be satisfied by requiring that $\varphi = 0$ on the boundary Γ. Suppose this boundary is defined by a set of m line segments $f_i(x,y) = 0$, $i = 1, m$. Then the stress function

$$\varphi = f_1(x,y) f_2(x,y) ... f_m(x,y) g(x,y) \tag{33.27}$$

will go to zero at all points on Γ for any function $g(x,y)$. We can therefore choose a function $g(x,y)$ with an appropriate number of arbitrary constants A_k, $k = 1, n$ and then use Castigliano's second theorem to develop n equations for the n unknowns. For example, the equations $x = 0$ and $x^2 + y^2 - a^2 = 0$ define the boundaries of a semicircular bar of radius a whose centre is at the origin. Thus, the stress function

$$\varphi = x(x^2 + y^2 - a^2)(A_1 + A_2 x) \tag{33.28}$$

satisfies the traction-free condition at all points on the boundary and contains two degrees of freedom, A_1, A_2.

Once a function with an appropriate number of degrees of freedom has been chosen, we calculate the stress components

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} \; ; \quad \sigma_{zy} = -\frac{\partial \varphi}{\partial x} \;, \tag{33.29}$$

from (16.8) and hence the strain energy density from (33.4), which in this case reduces to

$$U_0 = \frac{\sigma_{zx}^2 + \sigma_{zy}^2}{2\mu} = \frac{1}{2\mu}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] . \tag{33.30}$$

The torque corresponding to this stress distribution is given by equation (16.14) as

$$T = 2\iint_\Omega \varphi d\Omega \tag{33.31}$$

and hence the complementary energy per unit length of bar is

$$\begin{aligned} C &= \iint_\Omega U_0 d\Omega - T\beta \\ &= \frac{1}{2\mu}\iint_\Omega \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 - 4\mu\beta\varphi\right] d\Omega , \end{aligned} \tag{33.32}$$

where β is the twist per unit length. Substitution in (33.25) then provides a set of equations for the unknown constants A_k.

Example

To illustrate the method, we consider a bar of square cross-section defined by $-a < x < a$, $-a < y < a$. The boundaries are defined by the four lines $x \pm a = 0$, $y \pm a = 0$ and the simplest function of the form (33.27) is

$$\varphi = A(x^2 - a^2)(y^2 - a^2) , \tag{33.33}$$

which has just one degree of freedom — the multiplying constant A. Substituting into equation (33.32) and evaluating the integrals, we have

$$C = \frac{1}{2\mu}\left[\frac{256A^2a^8}{45} - \frac{64\mu\beta A a^6}{9}\right] . \tag{33.34}$$

To determine A, we impose the condition

$$\frac{\partial C}{\partial A} = 0 \quad \text{giving} \quad A = \frac{5\mu\beta}{8a^2} , \tag{33.35}$$

after which the torque is recovered from (33.31) as

$$T = \frac{20\mu\beta a^4}{9} . \tag{33.36}$$

The exact result can be obtained by substituting $b = a$ in equation (16.50) and summing the series. The present one-term approximation underestimates the exact value by only about 1.3%. Closer approximations can be obtained by adding a few extra degrees of freedom, preserving the symmetry of the system with respect to x and y. For example, Timoshenko and Goodier[5] give a solution of the same problem using the two degree of freedom function

$$\varphi = (x^2 - a^2)(y^2 - a^2)[A_1 + A_2(x^2 + y^2)] \, ,$$

which gives a result for the torque within 0.15% of the exact value.

33.7.2 The in-plane problem

We showed in Chapter 4 that the most general in-plane stress field satisfying the equations of equilibrium (2.5) in the absence of body force can be expressed in terms of the scalar Airy stress function ϕ through the relations

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \, . \tag{33.37}$$

We also showed in §4.4.3 that the traction boundary-value problem — i.e. the problem of a two-dimensional body with prescribed tractions on the boundaries — has a solution in which the stress components are independent of Poisson's ratio. We can therefore simplify the following treatment by considering the special case of plane stress ($\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$) with $\nu = 0$ for which equation (33.4:i) reduces to

$$\begin{aligned} U_0 &= \frac{(\sigma_{xx}^2 + \sigma_{yy}^2)}{4\mu} + \frac{\sigma_{xy}^2}{2\mu} \\ &= \frac{1}{4\mu} \left[\left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] . \end{aligned} \tag{33.38}$$

We express the trial stress function ϕ in the form

$$\phi = \phi_P + \phi_H \, , \tag{33.39}$$

where ϕ_P is any particular function of x, y that satisfies the traction boundary conditions and ϕ_H is a function satisfying homogeneous (i.e. traction-free) boundary conditions and that contains one or more degrees of freedom A_k. The stress components (33.37) will define a traction-free boundary if

$$\phi = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial n} = 0 \, , \tag{33.40}$$

on the boundary, where n is the local normal. In order to satisfy these conditions, we note that if $f_i(x, y) = 0$ defines a line segment that is part of the boundary Γ, and if f_i is a continuous function in the vicinity of the line segment, then we can perform a Taylor expansion about any point on the

[5] *loc. cit.* Art.111.

boundary. It then follows that the function $[f_i(x,y)]^2$ will satisfy both conditions (33.40), since the first non-zero term in its Taylor expansion at the boundary will be quadratic. Thus, the conditions on ϕ_H can be satisfied by defining

$$\phi_H = [f_1(x,y)f_2(x,y)...f_m(x,y)]^2\, g(x,y) \tag{33.41}$$

where $f_i(x,y)=0 \; i=1,m$ define line segments comprising the boundary $\mathit{\Gamma}$ and $g(x,y)$ is any function containing one or more degrees of freedom.

Example

Consider the semicircular plate of Figure 33.1 subjected to the self-equilibrated tractions

$$\sigma_{xx} = S\left(1 - \frac{3y^2}{a^2}\right)$$

on the straight boundary $x=0$, the curved boundary $r^2-a^2=x^2+y^2-a^2=0$ being traction-free.

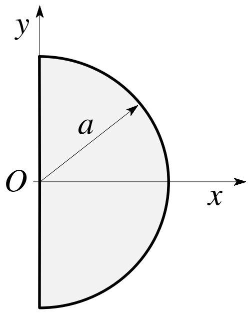

Figure 33.1: The semi-circular plate.

A particular solution maintaining the traction-free condition on the circle can be written

$$\phi_P = A(r^2-a^2)^2 = A(x^4+y^4+2x^2y^2-2a^2x^2-2a^2y^2+a^4) \; .$$

The stress components

$$\sigma_{xx} = \frac{\partial^2\phi_P}{\partial y^2} = A(12y^2+4x^2-4a^2) \; ; \quad \sigma_{xy} = -\frac{\partial^2\phi_P}{\partial x\partial y} = -8Axy$$

and on the boundary $x=0$,

$$\sigma_{xx} = 4A(3y^2-a^2) \; ; \quad \sigma_{xy} = 0 \; .$$

Thus, the boundary conditions can be satisfied by the choice

$$A = -\frac{S}{4a^2} \; ; \quad \phi_P = -\frac{S}{4a^2}(r^2 - a^2)^2 \; .$$

Notice that if the tractions on $x=0$ had been higher order polynomials in y, we could have found an appropriate particular solution satisfying the traction-free condition on the curved edge using the form $\phi_P = (r^2 - a^2)^2 g(x, y)$, where $g(x, y)$ is a polynomial in x, y of appropriate order.

The stress function ϕ_P provides already a first order approximation to the stress field, but to improve it we can add the homogeneous term

$$\phi_H = A_1 x^2 (r^2 - a^2)^2 = \frac{A_1 r^2 (r^2 - a^2)^2 (1 + \cos 2\theta)}{2} \; ,$$

which satisfies traction-free conditions on both straight and curved boundaries in view of (33.41). The stress components in polar coördinates are

$$\begin{aligned} \sigma_{rr} &= S\left(1 - \frac{r^2}{a^2}\right) + A_1(3r^4 - 4a^2r^2 + a^4) + A_1(r^4 - a^4)\cos 2\theta \\ \sigma_{\theta\theta} &= S\left(1 - \frac{3r^2}{a^2}\right) + A_1(15r^4 - 12a^2r^2 + a^4)(1 + \cos 2\theta) \\ \sigma_{r\theta} &= A_1(5r^4 - 6a^2r^2 + a^4)\sin 2\theta \end{aligned} \qquad (33.42)$$

from (8.12, 8.13) and the total strain energy is then calculated as

$$\begin{aligned} U &= \frac{1}{4\mu}\int_{-\pi/2}^{\pi/2}\int_0^a \left(\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + 2\sigma_{r\theta}^2\right) r dr d\theta \\ &= \frac{\pi a^2}{30\mu}\left(14a^8 A_1^2 - 5a^4 A_1 S + 5S^2\right) \end{aligned}$$

(using Mathematica or Maple). There are no displacement boundary conditions in this problem, so $C = U$ and the degree of freedom A_1 is determined from the condition

$$\frac{\partial C}{\partial A_1} = 0 \quad \text{or} \quad A_1 = \frac{5S}{28a^4} \; ,$$

after which the stress field is recovered by substitution in (33.42).

33.8 Uniqueness and existence of solution

A typical elasticity problem can be expressed as the search for a displacement field $\boldsymbol{u}$ satisfying the equilibrium equations (2.13), such that the stress components defined through equations (1.57) and the displacement components satisfy appropriate boundary conditions. Throughout this book, we have tacitly assumed that this is a well-posed problem — i.e. that if the problem is *physically* well defined in the sense that we could conceive of loading a body of the given geometry in the laboratory, then there exists one and only one solution.

To examine this question from a mathematical point of view, suppose provisionally that the solution is non-unique, so that there exist two distinct stress fields $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, both satisfying the field equations and the same boundary conditions. We can then construct a new stress field $\Delta\boldsymbol{\sigma}=\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$ by taking the difference between these fields, which is a form of linear superposition. The new field $\Delta\boldsymbol{\sigma}$ clearly involves no external loading, since the same external loads were included in each of the constituent solutions *ex hyp.* We therefore conclude from (33.1) that the corresponding total strain energy U associated with the field $\Delta\boldsymbol{\sigma}$ must be zero. However, U can also be written as a volume integral of the strain energy density U_0 as in (33.5), and U_0 must be everywhere positive or zero. The only way these two results can be reconciled is if U_0 is zero everywhere, implying that the stress is everywhere zero, from (33.3). Thus, the difference field $\Delta\boldsymbol{\sigma}$ is null, the two solutions must be identical *contra hyp.* and only one solution can exist to a given elasticity problem.

The question of existence of solution is much more challenging and will not be pursued here. A short list of early but seminal contributions to the subject is given by Sokolnikoff[6] who states that "the matter of existence of solutions has been satisfactorily resolved for domains of great generality." More recently, interest in more general continuum theories including non-linear elasticity has led to the development of new methodologies in the context of functional analysis[7].

33.8.1 Singularities

In §11.2.1, we argued that stress singularities are acceptable in the mathematical solution of an elasticity problem if and only if the strain energy in a small region surrounding the singularity is bounded. In the two-dimensional case (line singularity), this leads to equation (11.37) and hence to the conclusion that the stresses can vary with r^a as $r\rightarrow 0$ only if $a>-1$. We can now see the reason for this restriction, since if there were any points in the body where the strain energy was not integrable, the total strain energy (33.5) would be ill-defined and the above proof of uniqueness would fail.

If these restrictions on the permissible strength of singularities are not imposed, the uniqueness theorem fails and it is quite easy to generate examples in which a given set of boundary conditions permits multiple solutions. Consider the flat punch problem of §12.5.2 for which the contact pressure distribution is given by equation (12.47). If we differentiate the stress and displacement fields with respect to x, we shall generate a field in which the contact traction is

[6] I.S.Sokolnikoff, *Mathematical Theory of Elasticity,* McGraw-Hill, New York, 2nd.ed. 1956, §27.

[7] See for example, J.E.Marsden and T.J.R.Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliffs, 1983, Chapter 6.

$$p(x) = \frac{d}{dx}\left(-\frac{F}{\pi\sqrt{a^2-x^2}}\right) = -\frac{Fx}{\pi(a^2-x^2)^{3/2}}\ ; \quad -a<x<a$$

and the normal surface displacement is

$$u_y(x,0) = \frac{d}{dx}(C) = 0\ ,\ ; \quad -a<x<a$$

from (12.46). The traction-free condition is retained in $|x|>a, y=0$. We can now add this differentiated field to the solution of any frictionless contact problem over the contact region $-a<x<a$ without affecting the satisfaction of the displacement boundary condition (12.29), so solutions of such problems become non-unique. However, the superposed field clearly involves stresses varying with $r^{3/2}$ near the singular points $(\pm a, 0)$ and if (non-integrable) singularities of this strength are precluded, uniqueness is restored.

The three-dimensional equivalent of equation (11.37) would involve a stress field that tends to infinity with R^a as we approach the point $R=0$. In this case, the integral in spherical polar coordinates centred on the singular point would take the form

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^R \sigma_{ij}e_{ij}R^2\sin\beta dRd\beta d\theta = C\int_0^R R^{2a+2}dR\ ,$$

where C is a constant. This integral is bounded if and only if $a>-2$.

In both two and three-dimensional cases, the limiting singularity corresponds to that associated with a concentrated force F applied to the body, either at a point on the boundary or at an interior point. An 'engineering' argument can be made that the real loading in such cases consists of a distribution of pressure or body force over a small finite region $\mathcal{A}$, that is statically equivalent to F, in which case no singularity is involved. Furthermore, a unique solution is obtained for each member of a regular sequence of such problems in which $\mathcal{A}$ is progressively reduced. Thus, if we conceive of the point force as the limit of this set of distributions, the uniqueness theorem still applies. Saint-Venant's principle §3.1.2 implies that only the stresses close to $\mathcal{A}$ will be changed as $\mathcal{A}$ is reduced, so the concentrated force solution also represents an approximation to the stress field distant from the loaded region when $\mathcal{A}$ is finite. Sternberg and Eubanks[8] proposed an extension of the uniqueness theorem to cover these cases. Similar arguments can be applied to the stronger singularity due to a concentrated moment. However, there is now some ambiguity in the limit depending on the detailed sequence of finite states through which it is approached[9].

---

[8] E.Sternberg and R.A.Eubanks, On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity, *Journal of Rational Mechanics and Analysis*, Vol. 4 (1955), pp.135–168.

[9] E.Sternberg and V.Koiter, The wedge under a concentrated couple: A paradox in the two-dimensional theory of elasticity, *ASME Journal of Applied Mechanics*,

PROBLEMS

1. For an isotropic material, find a way to express the strain energy density U_0 of equation (33.4) in terms of the stress invariants of equations (1.28–1.30).

2. An annular ring $a<r<b$ is loaded by uniform (axisymmetric) shear tractions $\sigma_{r\theta}$ at $r=a$ that are statically equivalent to a torque T per unit thickness. By considering the equilibrium of the annular region $a<r<s$ where $s<b$, find the shear stress $\sigma_{r\theta}$ as a function of radius and hence find the total strain energy stored in the annulus. Use this result to determine the rotation of the outer boundary relative to the inner boundary, assuming the material has a shear modulus μ.

3. The isotropic semi-infinite strip $-a<y<a$, $0<x<\infty$ is attached to a rigid support at $y=\pm a$, whilst the end $x=0$ is subject to the antiplane displacement

$$u_z(0,y) = u_0\left(1-\frac{y^2}{a^2}\right) .$$

Since the resulting deformation will decay with x, approximate the displacement in the form

$$u_z(x,y) = u_0\left(1-\frac{y^2}{a^2}\right) e^{-\lambda x}$$

and use the minimum total potential energy theorem to determine the optimal value of the decay rate λ.

4. The rectangular plate $-a<x<a, -b<y<b$ is subject to the tensile tractions

$$\sigma_{xx} = S\left(1-\frac{y^2}{b^2}\right)$$

on the edges $x=\pm a$, all the other tractions being zero. Using appropriate third-order polynomials for the displacement components u_x, u_y, find an approximation for the displacement field and hence estimate the distribution of the stress component σ_{xx} on the plane $x=0$. Comment on the way this result is affected by the aspect ratio b/a.

5. Use the Maple or Mathematica file 'uxy' or the method of §9.1 to find the displacement field associated with the Airy stress function

$$\phi = \frac{G}{24}\left((y^2-b^2)^2-(x^2-a^2)^2\right) .$$

Vol.25 (1958), pp.575–581, X.Markenscoff, Some remarks on the wedge paradox and Saint Venant's principle, *ASME Journal of Applied Mechanics*, Vol.61 (1994), pp.519–523.

Using this displacement field, find the value of the single degree of freedom G that best approximates the stress field in the rectangular bar $-a<x<a$, $-b<y<b$ subjected only to the normal tractions

$$\sigma_{xx} = f(y) \; ; \quad x = \pm a \; .$$

6. A simple two-dimensional finite element mesh can be obtained by joining a series of nodes (x_i, y_i) $i=1, N$ by straight lines, forming a set of triangular elements. If the antiplane displacement u_z takes the values u_i at the three nodes $i=1,2,3$, develop an expression for a linear function of x, y that interpolates these nodal values. In other words determine the three constants A, B, C in

$$u_z(x, y) = Ax + By + C$$

such that $u_z(x_i, y_i) = u_i$ for $i=1,2,3$. Hence determine the strain energy U stored in the element as a function of u_i and use the principle of stationary potential energy to determine the element stiffness matrix k relating the nodal forces F_i at the nodes to the nodal displacements through the equation

$$F_i = k_{ij} u_j \; .$$

7. Use a one degree-of-freedom approximation and Castigliano's second theorem to estimate the twist per unit length β of a bar whose triangular cross-section is defined by the three corners $(0,0), (a,0), (0,a)$ and which is loaded by a torque T. Also, estimate the location and magnitude of the maximum shear stress.

8. Using the single degree of freedom stress function

$$\varphi = Ax(a^2 - x^2 - y^2) \; ,$$

estimate the torsional stiffness of the semicircular bar of radius a.

9. The rectangular plate $-a<x<a, -b<y<b$ is subject to the tractions

$$\sigma_{xx} = \frac{Sy^2}{b^2}$$

on the edges $x = \pm a$, all the other tractions being zero. Find an approximate solution for the stress field using Castigliano's second theorem. An appropriate particular solution is

$$\phi_P = \frac{Sy^4}{12b^2} \; .$$

Use a single degree of freedom approximation of the form (33.41) for the homogeneous solution.

10. A plate defined by the region $a<r<2a$, $-\alpha<\theta<\alpha$ is loaded only by the uniform tensile tractions

$$\sigma_{rr}(a,\theta) = \frac{F}{2a\alpha} \;; \quad \sigma_{rr}(2a,\theta) = \frac{F}{4a\alpha}$$

on the curved edges $r=a, 2a$. Find an approximate solution for the stress field using Castigliano's second theorem. Use a single degree of freedom approximation of the form (33.41) for the homogeneous solution ϕ_H.

34

THE RECIPROCAL THEOREM

In the previous chapter, we exploited the idea that the strain energy U stored in an elastic structure must be equal to the work done by the external loads, if these are applied sufficiently slowly for inertia effects to be negligible. Equation (33.1) is based on the premise that the loads are all increased at the same time and in proportion, but the final state of stress cannot depend on the exact history of loading and hence the work done W must also be history-independent. This conclusion enables us to establish an important result for elastic systems known as the reciprocal theorem.

34.1 Maxwell's Theorem

The simplest form of the theorem states that if a linear elastic body is kinematically supported and subjected to an external force $\boldsymbol{F}_1$ at the point P, which produces a displacement $\boldsymbol{u}_1(Q)$ at another point Q, then a force $\boldsymbol{F}_2$ at Q would produce a displacement $\boldsymbol{u}_2(P)$ at P where

$$\boldsymbol{F}_1 \cdot \boldsymbol{u}_2(P) = \boldsymbol{F}_2 \cdot \boldsymbol{u}_1(Q) . \tag{34.1}$$

To prove the theorem, we consider two scenarios — one in which the force $\boldsymbol{F}_1$ is applied slowly and then held constant whilst $\boldsymbol{F}_2$ is applied slowly and the other in which the order of application of the forces is reversed. Suppose that the forces $\boldsymbol{F}_1, \boldsymbol{F}_2$ produce the displacement fields $\boldsymbol{u}_1, \boldsymbol{u}_2$, respectively. If we apply $\boldsymbol{F}_1$ first, the work done will be

$$W_{11} = \frac{1}{2}\boldsymbol{F}_1 \cdot \boldsymbol{u}_1(P) . \tag{34.2}$$

If we now superpose the second force $\boldsymbol{F}_2$ causing additional displacements $\boldsymbol{u}_2$, the additional work done will be $W_{12}+W_{22}$, where

$$W_{12} = \boldsymbol{F}_1 \cdot \boldsymbol{u}_2(P) \tag{34.3}$$

J.R. Barber, *Elasticity*, Solid Mechanics and Its Applications 172,
DOI 10.1007/978-90-481-3809-8_34, © Springer Science+Business Media B.V. 2010

is the work done by the loads $\boldsymbol{F}_1$ 'involuntarily' moving through the additional displacement $\boldsymbol{u}_2(P)$ and

$$W_{22} = \frac{1}{2}\boldsymbol{F}_2 \cdot \boldsymbol{u}_2(Q) \tag{34.4}$$

is the work done by the force $\boldsymbol{F}_2$. Notice that there is no factor of $\frac{1}{2}$ in W_{12}, since the force $\boldsymbol{F}_1$ remains at its full value throughout the deformation. By contrast, $\boldsymbol{F}_2$ only increases gradually to its final value during the loading process so that the work it does W_{22} is the area under a load displacement curve, which introduces a factor of $\frac{1}{2}$. The total work done in this scenario is

$$W_1 = W_{11} + W_{12} + W_{22} = \frac{1}{2}\boldsymbol{F}_1 \cdot \boldsymbol{u}_1(P) + \boldsymbol{F}_1 \cdot \boldsymbol{u}_2(P) + \frac{1}{2}\boldsymbol{F}_2 \cdot \boldsymbol{u}_2(Q) \tag{34.5}$$

Consider now a second scenario in which the force $\boldsymbol{F}_2$ is applied first, followed by force $\boldsymbol{F}_1$. Clearly the total work done can be written down from equation (34.5) by interchanging the suffices 1,2 and the locations P, Q — i.e.

$$W_2 = W_{22} + W_{21} + W_{11} = \frac{1}{2}\boldsymbol{F}_2 \cdot \boldsymbol{u}_2(Q) + \boldsymbol{F}_2 \cdot \boldsymbol{u}_1(Q) + \frac{1}{2}\boldsymbol{F}_1 \cdot \boldsymbol{u}_1(P) \; . \tag{34.6}$$

In each case, since the forces were applied gradually, the work done will be stored as strain energy in the deformed body. But the final state in each case is the same, so we conclude that $W_1 = W_2$. Comparing the two expressions, we see that they will be equal if and only if $W_{12} = W_{21}$ and hence

$$\boldsymbol{F}_1 \cdot \boldsymbol{u}_2(P) = \boldsymbol{F}_2 \cdot \boldsymbol{u}_1(Q) \; , \tag{34.7}$$

proving the theorem. Another way of stating Maxwell's theorem is *"The work done by a force $\boldsymbol{F}_1$ moving through the displacements $\boldsymbol{u}_2$ due to a second force $\boldsymbol{F}_2$ is equal to the work done by the force $\boldsymbol{F}_2$ moving through the displacements $\boldsymbol{u}_1$ due to $\boldsymbol{F}_1$."*

34.2 Betti's Theorem

Maxwell's theorem is useful in Mechanics of Materials, where point forces are used to represent force resultants (e.g. on the cross-section of a beam), but in elasticity, the reciprocal theorem is more useful in the generalized form due to Betti, which relates the displacement fields due to two different distributions of surface traction and/or body force.

Suppose that the surface tractions $\boldsymbol{t}_i$ and body forces $\boldsymbol{p}_i$ produce the displacement fields $\boldsymbol{u}_i$, where i takes the values 1 and 2 respectively. Once again we consider two scenarios, one in which the loads $\boldsymbol{t}_1, \boldsymbol{p}_1$ are applied first, followed by $\boldsymbol{t}_2, \boldsymbol{p}_2$ and the other in which the order of loading is reversed. An argument exactly parallel to that in §34.1 then establishes that *"The work done by the loads $\boldsymbol{t}_1, \boldsymbol{p}_1$ moving through the displacements $\boldsymbol{u}_2$ due to $\boldsymbol{t}_2, \boldsymbol{p}_2$ is* *equal to the work done by the loads* $\boldsymbol{t}_2, \boldsymbol{p}_2$ *moving through the displacements* $\boldsymbol{u}_1$ *due to* $\boldsymbol{t}_1, \boldsymbol{p}_1$*."*

The mathematical statement of Betti's theorem for a body Ω with boundary Γ is

$$\iint_\Gamma \boldsymbol{t}_1 \cdot \boldsymbol{u}_2 d\Gamma + \iiint_\Omega \boldsymbol{p}_1 \cdot \boldsymbol{u}_2 d\Omega = \iint_\Gamma \boldsymbol{t}_2 \cdot \boldsymbol{u}_1 d\Gamma + \iiint_\Omega \boldsymbol{p}_2 \cdot \boldsymbol{u}_1 d\Omega \; . \quad (34.8)$$

34.3 Use of the theorem

Betti's theorem defines a relationship between two different stress and displacement states for the same body. In most applications, one of these states corresponds to the problem under investigation, whilst the second is an *auxiliary solution*, for which the tractions and surface displacements are known. Often the auxiliary solution will be a simple state of stress for which the required results can be written down by inspection.

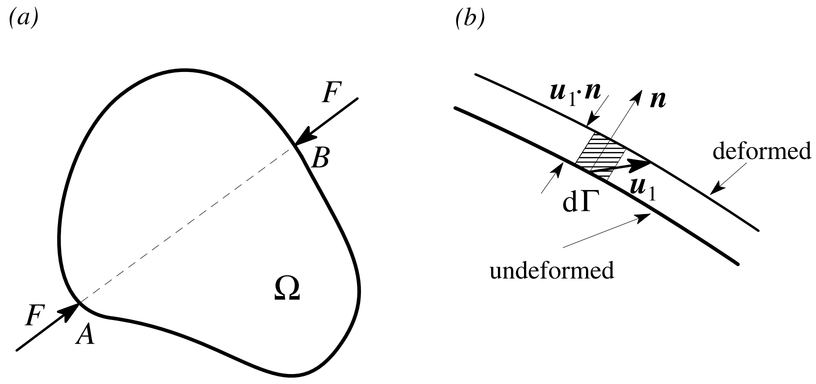

Figure 34.1: Increase in volume of a body due to two colinear forces.

The art of using the reciprocal theorem lies in choosing the auxiliary solution so that equation (34.8) yields the required result. For example, suppose we wish to determine the change in volume of a body Ω due to a pair of equal and opposite colinear concentrated forces, as shown in Figure 34.1(a). Figure 34.1(b) shows an enlarged view of a region of the boundary in the deformed and undeformed states, from which it is clear that the change in volume associated with some small region $d\Gamma$ of the boundary is $dV = \boldsymbol{u}_1 \cdot \boldsymbol{n} d\Gamma$, where $\boldsymbol{n}$ is the unit vector defining the outward normal to the surface at $d\Gamma$. Thus, the change in volume of the entire body is

$$\Delta V_1 = \iint_\Gamma \boldsymbol{u}_1 \cdot \boldsymbol{n} d\Gamma \; . \quad (34.9)$$

Now this expression can be made equal to the right-hand side of (34.8) if we choose $\boldsymbol{t}_2 = \boldsymbol{n}$ and $\boldsymbol{p}_2 = 0$. In other words, the auxiliary solution corresponds to the case where the body is subjected to a purely normal (tensile) traction of unit magnitude throughout its surface and there is no body force.

The solution of this auxiliary problem is straightforward, since the tractions are compatible with an admissible state of uniform hydrostatic tension

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 1 \;\; ; \;\; \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \tag{34.10}$$

throughout the body and the resulting strains are uniform and hence satisfy all the compatibility conditions. The stress-strain relations then give

$$e_{xx} = e_{yy} = e_{zz} = \frac{(1-2\nu)}{E} \;\; ; \;\; e_{xy} = e_{yz} = e_{zx} = 0 \tag{34.11}$$

and hence, in state 2, the body will simply deform into a larger, geometrically similar shape, any linear dimension L increasing to $L(1+(1-2\nu)/E)$.

It remains to calculate the *left-hand side* of (34.8), which we recall is the work done by the tractions $\boldsymbol{t}_1$ in moving through the surface displacements $\boldsymbol{u}_2$. In this case, $\boldsymbol{t}_1$ consists of the two concentrated forces, so the appropriate work is $F\delta_2$, where δ_2 is the amount that the length L of the line AB shrinks *in state 2*, permitting the points of application of the forces to approach each other.

From the above argument,

$$\delta_2 = -\frac{(1-2\nu)L}{E} \tag{34.12}$$

and hence we can collect results to obtain

$$\begin{aligned} \Delta V_1 &= \iint_\Gamma \boldsymbol{u}_1 \cdot \boldsymbol{n} d\Gamma = \iint_\Gamma \boldsymbol{u}_1 \cdot \boldsymbol{t}_2 d\Gamma \\ &= \iint_\Gamma \boldsymbol{u}_2 \cdot \boldsymbol{t}_1 d\Gamma = F\delta_2 = -\frac{F(1-2\nu)L}{E} \;, \end{aligned} \tag{34.13}$$

which is the required expression for the change in volume.

Clearly the same method would enable us to determine the change in volume due to any other self-equilibrated traction distribution $\boldsymbol{t}_1$, either by direct substitution in the integral $\iint_\Gamma \boldsymbol{u}_2 \cdot \boldsymbol{t}_1 d\Gamma$ or by using the force pair of Figure 34.1(a) as a Green's function to define a more general distribution through a convolution integral. It is typical of the reciprocal theorem that it leads to the proof of results of some generality.

34.3.1 A tilted punch problem

For any given auxiliary solution, Betti's theorem will yield only a single result, which can be the value of the displacement at a specific point, or more often, an integral quantity such as a force resultant or an average displacement. Indeed, when interest in the problem is restricted to such integrals, the reciprocal theorem should always be considered first as a possible method of solution.

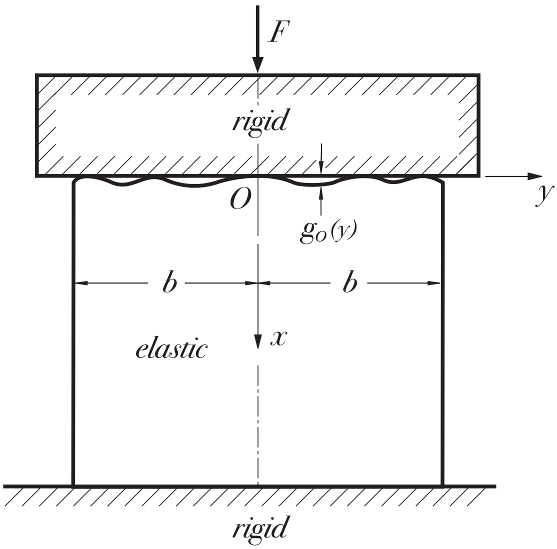

Figure 34.2: The tilted punch problem

As a second example, we shall consider the two-dimensional plane stress contact problem of Figure 34.2, in which a rectangular block with a slightly irregular surface is pressed down by a frictionless flat rigid punch against a frictionless rigid plane. We suppose that the force F is large enough to ensure that contact is established throughout the end of the block and that the line of action of the applied force passes through the mid-point of the block. The punch will therefore generally tilt through some small angle, α, and the problem is to determine α.

We suppose that the profile of the block is defined by an initial gap function $g_0(y)$ when the punch is horizontal and hence, if the punch moves downward a distance u_0 and rotates clockwise through α, the contact condition can be stated as

$$g(y) = g_0(y) + u_1(y) - u_0 - \alpha y = 0 \; , \tag{34.14}$$

where $u_1(y)$ is the downward normal displacement of the block surface, as shown in Figure 34.3(a). Since the punch is frictionless, the traction $\boldsymbol{t}_1$ is also purely normal, as shown.

The key to the solution lies in the fact that the line of action of $\boldsymbol{F}$, which is also the resultant of $\boldsymbol{t}_1$, passes through the centre of the block. Taking moments about this point, we therefore have

$$\int_{-b}^{b} y t_1(y) dy = 0 \; . \tag{34.15}$$

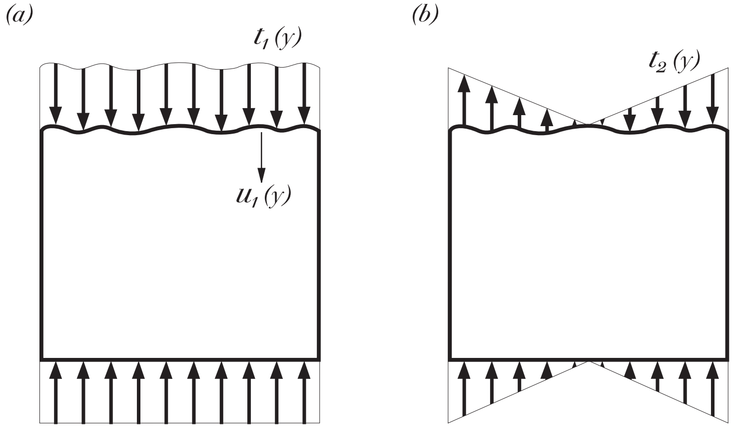

Figure 34.3: Tractions for the original and auxiliary problems for the tilted punch.

We can make this expression look like the left-hand side of (34.8) by choosing the auxiliary solution so that the normal displacement $u_2(y)=Cy$, which corresponds to a simple bending distribution with

$$t_2(y) = \frac{ECy}{a} , \tag{34.16}$$

as shown in Figure 34.3(b).

Betti's theorem now gives

$$\int_{-b}^{b} \frac{ECyu_1(y)dy}{a} = \int_{-b}^{b} Cyt_1(y)dy = 0 \tag{34.17}$$

and hence, treating (34.14) as an equation for $u_1(y)$ and substituting into (34.17), we obtain

$$\int_{-b}^{b} \{u_0 + \alpha y - g_0(y)\} y dy = 0 \tag{34.18}$$

— i.e.

$$\alpha = \frac{3}{b^3} \int_{-b}^{b} y g_0(y) dy , \tag{34.19}$$

which is the required result.

This result is relevant to the choice of end conditions in the problem of §9.1, since it shows that the end of the beam can be restored to a vertical plane by a self-equilibrated purely normal traction if and only if

$$\int_{-b}^{b} y g_0(y) dy = 0 , \tag{34.20}$$

which is equivalent to condition of $(9.21)_3$.

However, unfortunately we cannot rigorously justify the complete set of conditions (9.21) by this argument, since the full built-in boundary conditions (9.16) also involve constraints on the local extensional strain $e_{yy}=\partial u_y/\partial y$. It is clear from energy considerations that if these constraints were relaxed, any resulting motion would involve the end load doing positive work and hence the end deflection obtained with conditions (9.21) must be larger than that obtained with the strong conditions[1] (9.16).

In the above solution, we have tacitly assumed that the only contribution to the two integrals in (34.8) comes from the tractions and displacements at the top surface of the block. This is justified, since although there are also normal tractions $\boldsymbol{t}_1, \boldsymbol{t}_2$ at the lower surface, they make no contribution to the work integrals because the corresponding normal *displacements* ($\boldsymbol{u}_2, \boldsymbol{u}_1$ respectively) are zero. However, it is important to consider all surfaces of the body in the application of the theorem, including a suitably distant boundary in problems involving infinite or semi-infinite regions.

34.3.2 Indentation of a half-space

We demonstrated in §29.2 that the contact area A between a frictionless rigid punch and an elastic half-space is that value which maximizes the total load $F(A)$ on the punch. Shield[2] has shown that this can be determined without solving the complete contact problem, using Betti's theorem.

The total load on the punch is

$$F(A) = \iint_A p_1(x,y)dxdy \; , \tag{34.21}$$

where $p_1(x,y)$ is the contact pressure. We can make the left-hand side of (34.8) take this form by choosing the auxiliary solution such that the normal displacement $u_2(x,y)$ in A is uniform and of unit magnitude.

The simplest result is obtained by using as auxiliary solution that corresponding to the indentation of the half-space by a frictionless flat rigid punch of plan-form A, defined by the boundary conditions

$$\begin{aligned} u_2 \equiv u_z(x,y) = 1 \quad &; \quad (x,y) \in A \\ p_2 \equiv -\sigma_{zz}(x,y) = 0 \quad &; \quad (x,y) \in \overline{A} \\ \sigma_{xz} = \sigma_{yz} = 0 \quad &; \quad (x,y) \in A \cup \overline{A} \; . \end{aligned} \tag{34.22}$$

[1] J.D.Renton, Generalized beam theory applied to shear stiffness, *International Journal of Solids and Structures,* Vol. 27 (1991), pp.1955–1967, argues that the correct shear stiffness for a beam of any cross-section should be that which equates the strain energy associated with the shear stresses σ_{xz}, σ_{yz} and the work done by the shear force against the shear deflection. For the rectangular beam of §9.1, this is equivalent to replacing the multiplier on b^2/a^2 in (9.24b,c) by $2.4(1+\nu)$. This estimate differs from (9.24c) by at most 3%.

[2] R.T.Shield, Load-displacement relations for elastic bodies, *Zeitschrift für angewandte Mathematik und Physik,* Vol. 18 (1967), pp.682–693.

Since the only non-zero tractions in both solutions are the normal pressures in the contact area A, equation (34.8) reduces to

$$F(A) \equiv \iint_A p_1(x,y)dxdy = \iint_A p_2(x,y)u_1(x,y)dxdy \; , \qquad (34.23)$$

where $u_1(x,y)$ is the normal displacement under the punch, which is defined through the contact condition (29.3) in terms of the initial punch profile $u_0(x,y)$ and its rigid-body displacement.

In problems involving infinite domains, it is important to verify that no additional contribution to the work integrals in (34.8) is made on the 'infinite' boundaries. To do this, we consider a hemispherical region of radius b much greater than the linear dimensions of the loaded region A. The stress and displacement fields in both solutions will become self-similar at large R, approximating those of the point force solution (§23.2.1). In particular, the stresses will tend asymptotically to the form $R^{-2}f(\theta,\alpha)$ and the displacements to $R^{-1}g(\theta,\alpha)$.

If we now construct the integral $\iint \boldsymbol{t}_1 \cdot \boldsymbol{u}_2 d\Gamma$ over the hemisphere surface, it will therefore have the form

$$\int_0^{\pi/2}\int_0^{2\pi} b^{-2}f_1(\theta,\alpha)b^{-1}g_2(\theta,\alpha)b^2 d\theta d\alpha \; ,$$

which approaches zero with b^{-1} as $b \to \infty$, indicating that the distant surfaces make no contribution to the integrals in (34.8).

Of course, we can only use (34.23) to find $F(A)$ if we already know the solution $p_2(x,y)$ for the flat punch problem with plan-form A and exact solutions are only known for a limited number of geometries, such as the circle, the ellipse and the strip[3]. However, (34.23) does for example enable us to write down the indenting force for a circular punch of arbitrary profile[4] and hence, using the above theorem, to determine the contact area for the general axisymmetric contact problem.

34.4 Thermoelastic problems

We saw in §22.2 that the thermoelastic displacements due to a temperature distribution $T(x,y,z)$ can be completely suppressed if we superpose a body

[3] An *approximate* solution to this problem for a punch of fairly general plan-form is given by V.I.Fabrikant, Flat punch of arbitrary shape on an elastic half-space, *International Journal of Engineering Science,* Vol. 24 (1986), pp.1731–1740. Fabrikant's solution and the above reciprocal theorem argument are used as the bases of an approximate solution of the general smooth punch problem by J.R.Barber and D.A.Billings, An approximate solution for the contact area and elastic compliance of a smooth punch of arbitrary shape, *International Journal of Mechanical Sciences,* Vol. 32 (1990), pp.991–997.

[4] R.T.Shield, *loc. cit.*.

force $\boldsymbol{p}$ and tractions $\boldsymbol{t}$ defined by equations (22.51, 22.52) respectively. It follows that the displacements $\boldsymbol{u}_1$ in an unloaded body due to this temperature distribution are the same as those caused in a body at zero temperature by body forces and tractions equal and opposite to $\boldsymbol{p}$, $\boldsymbol{t}$ — i.e.

$$\boldsymbol{p}_1 = -\frac{2\mu(1+\nu)}{(1-2\nu)}\alpha\boldsymbol{\nabla}T \;\; ; \;\; \boldsymbol{t}_1 = \frac{2\mu(1+\nu)}{(1-2\nu)}\alpha\boldsymbol{n}T \; . \tag{34.24}$$

We can use this equivalence to derive a *thermoelastic reciprocal theorem*, relating the displacements $\boldsymbol{u}_1$ and the temperature T through an auxiliary isothermal solution $\boldsymbol{p}_2, \boldsymbol{t}_2, \boldsymbol{u}_2$. Substituting (34.24) into the left-hand side of (34.8), we obtain

$$\frac{2\mu(1+\nu)\alpha}{(1-2\nu)}\left(\iint_\Gamma T\boldsymbol{u}_2\cdot\boldsymbol{n}d\Gamma - \iiint_\Omega \boldsymbol{u}_2\cdot\boldsymbol{\nabla}Td\Omega\right)$$
$$= \iint_\Gamma \boldsymbol{t}_2\cdot\boldsymbol{u}_1 d\Gamma + \iiint_\Omega \boldsymbol{p}_2\cdot\boldsymbol{u}_1 d\Omega \; . \tag{34.25}$$

From the divergence theorem, we have

$$\iint_\Gamma T\boldsymbol{u}_2\cdot\boldsymbol{n}d\Gamma = \iiint_\Omega \text{div}(T\boldsymbol{u}_2)d\Omega$$
$$= \iiint_\Omega T\text{div}\;\boldsymbol{u}_2 d\Omega + \iiint_\Omega \boldsymbol{u}_2\cdot\boldsymbol{\nabla}Td\Omega \; , \tag{34.26}$$

after differentiating the integrand by parts. Using this result in (34.25), we obtain

$$\frac{2\mu(1+\nu)\alpha}{(1-2\nu)}\iiint_\Omega T\text{div}\;\boldsymbol{u}_2\,d\Omega = \iint_\Gamma \boldsymbol{t}_2\cdot\boldsymbol{u}_1 d\Gamma + \iiint_\Omega \boldsymbol{p}_2\cdot\boldsymbol{u}_1 d\Omega \; , \tag{34.27}$$

which is the required result[5].

The thermoelastic reciprocal theorem can be used in exactly the same way as Betti's theorem to develop general expressions for quantities that are expressible as integrals. For example, the change in the volume of a body due to an arbitrary temperature distribution can be found using the auxiliary solution (34.10, 34.11), for which $\boldsymbol{t}_2 = \boldsymbol{n}$ and $\boldsymbol{p}_2 = 0$. From (34.11) we have

$$\text{div}\;\boldsymbol{u}_2 = e_{xx} + e_{yy} + e_{zz} = \frac{3(1-2\nu)}{E} = \frac{3(1-2\nu)}{2\mu(1+\nu)} \tag{34.28}$$

and hence (34.27) reduces to

$$\delta V_1 = \iint_\Gamma \boldsymbol{u}_1\cdot\boldsymbol{n}d\Gamma = \frac{2\mu(1+\nu)\alpha}{(1-2\nu)}\iiint_\Omega T\text{div}\;\boldsymbol{u}_2\,d\Omega$$
$$= 3\alpha\iiint_\Omega Td\Omega \; . \tag{34.29}$$

[5] This proof was first given by J.N.Goodier, *Proceedings of the 3rd U.S. National Congress on Applied Mechanics*, (1958), pp.343–345.

In other words, the total change in volume of the body is the sum of that which would be produced by free thermal expansion of all its constituent particles if there were no thermal stresses. For additional applications of the thermoelastic reciprocal theorem, see J.N.Goodier, *loc. cit.* or S.P.Timoshenko and J.N.Goodier, *loc. cit.*, §155.

PROBLEMS

1. Figure 34.4 shows an elastic cube of side a which just fits between two parallel frictionless rigid walls. The block is now loaded by equal and opposite concentrated forces, F, at the mid-point of two opposite faces, as shown. 'Poisson's ratio' strains will cause the block to exert normal forces P on the walls. Find the magnitude of these forces.

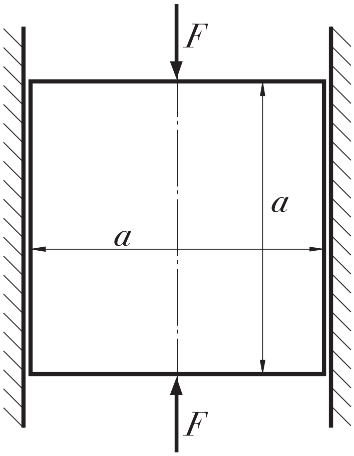

Figure 34.4: Loading of the constrained cube.

Would your result be changed if (i) the forces F were applied away from the mid-point or (ii) if their common line of action was inclined to the vertical?

2. Use the method of §29.2, §34.3.2 to find the relations between the indentation force, F, the contact radius, a and the maximum indentation depth, d, for a rigid frictionless punch with a fourth order profile $u_0 = -Cr^4$ indenting an elastic half-space.

3. The penny-shaped crack $0 \leq r < a$, $z = 0$ is opened by equal and opposite normal forces F applied to the two crack faces at a radius r_1. Find the volume of the opened crack.

4. A long elastic cylinder of constant but arbitrary cross-sectional area and length L rests horizontally on a plane rigid surface. Show that gravitational loading will cause the cylinder to increase in length by an amount

$$\delta = \frac{\nu \rho g L h}{E} ,$$

where ρ, E, ν are the density, Young's modulus and Poisson's ratio respectively of the material and h is the height of the centre of gravity of the cylinder above the plane.

5. A pressure vessel of arbitrary shape occupies a space of volume V_1 and encloses a hole of volume V_2, so that the volume of solid material is $V_M = V_1 - V_2$. If the vessel is now subjected to an external pressure p_1 and an internal pressure p_2, show that the total change in V_M is

$$\delta V_M = \frac{p_2 V_2 - p_1 V_1}{K_b} ,$$

where K_b is the bulk modulus of equation (1.76).

6. The long rectangular bar $0 < x < a, -b < y < b, a \gg b$ is believed to contain residual stresses σ_{xx} that are functions of y only, except near the ends, the other residual stress components being zero. To estimate the residual stress distribution, it is proposed to clamp the bar at $x = 0$, cut a groove as shown in Figure 34.5, using laser machining, and measure the vertical displacement u_y (if any) of the end $x = a$ due to the resulting stress relief.

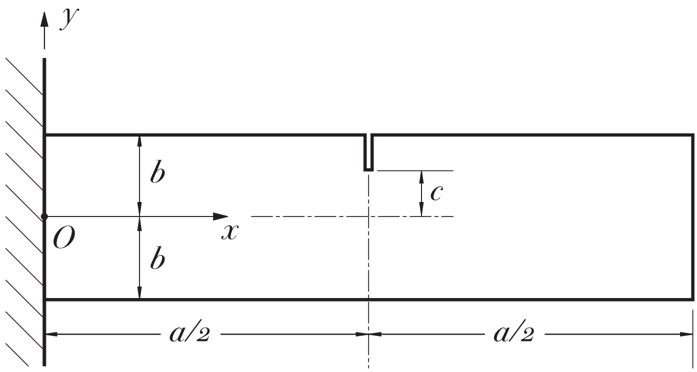

Figure 34.5: Groove cut in a bar with residual stresses.

What would be the auxiliary problem we would need to solve to establish an integral relation between the residual stress distribution and the displacement u_y, using the reciprocal theorem. (Do not attempt to solve the auxiliary problem!)

7. By differentiating the solution of the Hertz problem of §30.2.5 with respect to x, show that the contact pressure distribution

$$p(r, \theta) = \frac{Cr \cos\theta}{\sqrt{a^2 - r^2}} ; \quad 0 \le r < a$$

produces the normal surface displacement

$$u_z(r,\theta) = \frac{\pi C(1-\nu)r\cos\theta}{4\mu} \;; \quad 0 \le r < a,\; z=0\;.$$

Use this result and the reciprocal theorem to determine the angle through which the punch in §34.3.2 will tilt about the y-axis if the force F is applied along a line passing through the origin.

8. A long hollow cylinder of inner radius a and outer radius b has traction free surfaces, but it is subjected to a non-axisymmetric temperature increase $T(r,\theta)$. Find the change in the cross-sectional area of the hole due to thermoelastic distortion.

9. The long solid cylinder $0 \le r < a$, $0 < z < L$, $L \gg a$ is built in at $z = L$ and all the remaining surfaces are traction free. If the cylinder is now heated to an arbitrary non-uniform temperature T, find an integral expression for the displacement u_y at the free end.

10. A body Ω is raised to an arbitrary temperature field $T(x,y,z)$, but the boundaries are all free of traction. Show that the average value of the bulk stress $\bar{\sigma}$ of equation (1.75) is zero — i.e.

$$\iiint_\Omega \bar{\sigma} d\Omega = 0\;.$$

A

USING MAPLE AND MATHEMATICA

The algebraic manipulations involved in solving problems in elasticity are relatively routine, but they can become lengthy and complicated, particularly for three-dimensional problems. There are therefore significant advantages in using a symbolic programming language such as Maple or Mathematica. Before such programs were available, the algebraic complexity of manual calculations often made it quicker to use a finite element formulation, even when it was clear that an analytical solution could in principle be obtained. Apart from the undoubted saving in time, Maple or Mathematica also make it easy to plot stress distributions and perform parametric studies to determine optimum designs for components. From the engineering perspective, this leads to a significantly greater insight into the nature of the subject.

Using these languages efficiently necessitates a somewhat different approach from that used in conventional algebraic solutions. Anyone with more than a passing exposure to computer programs will know that it is almost always quicker to try a wide range of possible options, rather than trying to construct a logical series of steps leading to the required solution. This trial and error process also works well in the computer solution of elasticity problems. For example, if you are not quite sure which stress function to use in the problem, the easiest way to find out is to use your best guess, run the calculation and see what stresses are obtained. Often, the output from this run will make it clear that minor modifications or additional terms are required, but this is easily done. Remember that it is extremely easy and quick to make a small change in the problem formulation and then re-run the program. By contrast, in a conventional algebraic solution, it is essential to be very careful in the early stages because repeating a calculation with even a minor change in the initial formulation is very time consuming.

The best way to learn how to use these programs is also by trial and error. Don't waste time reading a compendious instruction manual or following a tutorial, since these will tell you how to do numerous things that you probably will never need. On the web site www.elasticity.org we provide the source code and the resulting output for the two-dimensional problem treated in §5.2.2.

This file has sufficient embedded explanation to show what is being calculated and you can easily copy the files and make minor changes to gain experience in the use of the method. The web site also includes an explanation of the commands you will need most often in the solution of elasticity problems and electronic versions of the Tables in the present Chapters 21 and 22 and of the recurrence relations for generating spherical harmonics.

INDEX

Abel integral equations, 463–465, 472, 478, 481, 482
  inversion, 465
adhesion, 452–453
Airy stress function, 46–49, 91–93, 110, 135, 219–220, 301, 349, 438, 507
  relation to complex potentials, 301–302
alternating tensor, 7, 19, 27
antiplane shear (antiplane strain), 227–232, 295–297, 435
approximate methods, 501–503, 505–509
auxiliary solution, 517
axisymmetric problems, 391–400, 406–412, 419–427
  plane thermoelastic, 349–350
  torsion, 419–427
asymptotic methods, 157–164
bending, 58–61, 240, 259, 433
Betti's theorem, 516–522
biharmonic equation, 49, 55, 79, 93, 111, 149, 220
body force, 25, 48, 91–105, 201, 220, 227, 329–330, 354–355
bonded interfaces, 163–164, 485–496
boundary conditions, 5, 228, 261
  global, 368, 478, 481, 487–488
  mixed, 448, 457, 478, 488–490
  weak, 40, 59, 61–63, 72, 105, 391, 397, 409
Boussinesq potentials, 333–336
Boussinesq problem, 367–370
bubble model, 204–205
bulk modulus, 23
Burgers' vector, 203
Carter's problem, 191–194
Cartesian tensors, 9
Castigliano's second theorem, 504–509
Cattaneo's problem, 188–191
Cauchy integral equations, 177, 210, 493
Cauchy integral theorem, 281–283, 306
Cauchy-Riemann relations, 274, 278, 295
centre of compression, 364
centre of dilatation, 364
centre of rotation, 425–426
change in volume, 517–518
circular cylinder, 263–264, 349–350, 391–397, 400–402
circular disk, 100, 112, 220
circular hole, 113–115, 119–120, 215, 351–354
circular plates, 397–400
circular sector, 166
closed sections, 249–251
closure conditions, 206, 207, 211
coefficient of friction, 187
Collins' method, 458–469
compatibility equations, 26–27, 93
compliance tensor, 20
completeness, 73, 82–83, 325
  of Papkovich-Neuber potentials, 325–329
complex variable formulation, 273 *et seq.*
  axisymmetric complex potentials, 459–463
concentrated force, 171–175, 199–201, 230, 365–370
conformal mapping, 288–290, 311–315
conical bar, 408–414, 424–425
conservative vector fields, 92
constraint equations, 56
contact problems, 93, 164, 171–194, 447–453, 518–522
  adhesive contact, 452–453
  frictionless contact, 177–183, 186, 339, 447–453, 518–522

smooth (incomplete) contact problems, 466–469
unilateral inequalities, 181, 449
continuity of displacement, 131, 135, 199–200, 469, 479, 491
contour integrals, 279 *et seq.*
corner fields, 88, 152
coördinate transformation, 8–12, 17–20, 228, 294
corrective solution, 39, 78–84, 114, 189, 192, 209–211, 352, 353, 407, 478–479, 487–490
Coulomb friction, 187, 462
crack opening displacement, 211, 212, 479, 494
cracks, 160, 162–163, 207–213, 314–315, 339, 477–483, 485–496
creep velocity, 192, 194
curved beams, 135–147
cylindrical cantilever, 263–264, 400-402, 436–437
D'Alembert's principle, 94
degeneracy, 116–118
special solutions, 116–118, 141–142, 145–147
dilatation, 22,23
centre of, 364
surface dilatation, 369
direction cosines, 11
dislocations, 202–206, 213, 222, 231
climb dislocation, 202–203, 210
dislocation derivatives, 207
glide dislocation, 203
screw dislocation, 231
displacement, 13
calculation from stress components, 123–129
notation for, 13
rigid-body, 14, 125–126, 128, 174, 175, 300
single-valued, 28, 131, 135, 199–200
displacement gradient, 176
Dundurs' constants, 50, 186, 489, 490
Dundurs' theorem, 223–224
eigenvalue problems, 80–84, 139–140, 157–165
eigenfunction expansion, 81, 84, 157, 166
elasticity tensor, 20
elastodynamics, 95
elliptical bar, 244
elliptical hole, 289, 313
end effects, 77–88, 426–427
energy release rate, 212–213, 496
equilibrium equations, 25, 26, 31, 101, 321, 322, 347, 354
existence of solution, 510
finite element method, 503
Flamant solution, 172–175, 199
flat punch indentation, 179–180
fracture mechanics, 207–208, 213, 495–496
fracture toughness, 208, 213
Fredholm integral equations, 493, 495
friction, 187
Fourier series methods, 68–72, 84–88, 111–113, 247–249, 264–266, 378
Fourier transform methods, 72–73, 165, 178–179
Galerkin vector, 322–324, 330
gap function, 184–186, 518
global boundary conditions, 368, 478, 481, 487–488
gravitational loading, 14–16, 91, 94,
Green-Collins representation, 460–463
Green's functions, 157, 174, 175, 184, 199, 206, 448, 518
Griffith fracture theory, 208, 495–496
gross slip, 190
half-plane, 154, 160, 173–176, 286–288
surface displacements, 173–175, 369
half-space, 339–341, 357–358, 367–370, 447, 487
coupling between normal and tangential effects, 186, 490
surface displacements, 369, 447
Hankel transform, 457–458
heat conduction, 221–222
Hertzian contact, 181–183, 186, 468
history-dependence, 187–194
holomorphic function, 274
Hooke's law, 3, 20–22, 220

hydrostatic stress, 10, 518
image methods, 213–215, 372–374
inclusions, 133, 207, 213
incompressible material, 23, 186
index notation, 6–8
  vector operators in, 7–8
inertia forces, 94
integral equations, 177, 210, 448, 458, 463, 493, 495
interface crack, 485–496
  contact solution, 494–495
irrotational vector field, 92, 322
isotropy, 20
Kelvin problem, 199–200, 365–366
Kronecker delta, 22
Lamé's constants, 22
Laurent series, 282
Legendre's equation, 378–379
Legendre functions, 381, 383, 421
  recurrence relations, 379, 381, 383, 386
Legendre polynomials, 371, 379, 384
Love's strain potential, 324, 335
Mathieu's method, 84–88
Maxwell's theorem, 515–516
Mellin transform, 165
membrane analogy, 245
Michell's solution, 118, 119, 129, 130, 276
microslip, 190, 193
Mindlin's problem, 188–191
mixed boundary-value problem, 448, 457, 478, 480, 488–490
mode mixity, 496
Mohr's circle, 9
multiply-connected bodies, 28–30, 131, 222, 249–251, 349
notches, 154–155, 157
oscillatory singularities, (*see* singular stress fields)
Papkovich-Neuber solution, 324–329, 334–335
  in complex variable notation, 342–343, 431
  change of coördinate system, 325
particular solution, 99–100
path-independent integrals, 28, 92, 279
penny-shaped crack, 477–483
  at an interface, 490–495
  obstructing heat conduction, 480–483
  in tension, 477–479
phase transformation, 364
plane crack in tension, 209–211
plane strain, 37–40
  relation to plane stress, 42, 220
  solution in complex variables, 298 *et seq.*
plane stress, 40–42
  generalized plane stress, 42
Poisson's ratio, 21
pole, 279–281
polynomial solutions, 55–68
potential energy, 499-500
  theorem of minimum total potential energy, 500–501
Prandtl's stress function, 242–243, 260, 295, 435, 438, 505
Prismatic bar, 427–444
process zone, 208, 213, 496
Rayleigh-Ritz method, 87, 501–503
reciprocal theorem, 515–524
rectangular bar, 229–230, 247–249, 264–266
rectangular beams, 55–73, 84–88, 97–99, 102–104, 123–127, 152
  end conditions, 77–88, 125–126, 520–521
  shear deflection, 127
residue, 279–281
residue theorem, 279, 288
rigid-body displacements, (*see* displacements)
rolling contact, 191–194
rotation of a line, 16, 17
rotation vector, 18, 19, 322
rotational acceleration, 100–104
Saint-Venant's principle, 40, 77, 239
self-equilibrated tractions, 39, 40, 77
self-similarity, 171–172, 199, 367
separated-variable solutions, 73, 79, 84, 172
shift, 187
simple radial distribution, 173
singular stress fields, 155–157, 162–163, 180, 205, 211, 364–370, 486, 510–511
  oscillatory singularities, 493–494
slip, 188–194
source solution, 363–366, 368, 371, 461

specific heat, 221
sphere, 405–406
spherical harmonics, 377–390
  in complex variable notation, 387–390
spherical hole, 406–408
spherical polar angle, 337
stick, 188–194
strain,
  notation for, 14
  shear strain, definition, 19
  tensile strain, 15–16
strain-displacement relations, 15–16, 19
  in polar coördinates, 111
strain energy, 156, 498–499
  strain energy density, 498
  strain energy release rate, 212–213, 496
strain potential, 322, 325, 334
strain suppression, 354–355
strain transformation relations, 20
stress,
  equivalent tensile stress, 13
  hydrostatic stress, 10, 518
  mean stress, 23
  notation for, 4
  principal stress, 9, 12–13
  shear stress, 5
stress concentrations, 113–115, 119–120, 137, 154, 406–408
stress concentration factors, 115, 120, 206, 408
stress intensity factors, 162, 163, 208, 211–213, 479, 483, 494, 496
stress-strain relations, 20–22, 220
stress transformation relations, 9, 47, 103
summation convention, 6
surface dilatation, 369
surface energy, 208, 452–453
symmetry, 66, 69, 80, 85, 101, 153, 158, 339, 394, 483
thermal conductivity, 221
thermal capacity, 221
thermal diffusivity, 221
thermal distortivity, 223, 358
thermal expansion coefficient, 219
thermoelastic displacement potential, 347–348, 364
thermoelasticity, 219–224, 347–358
  axisymmetric stress in the cylinder, 349–350, 394–396
  heat flow obstructed by a crack, 480–483
  plane problems, 219–224, 349–354
  steady-state temperature, 222–224, 350–354, 355–358
  Williams' solution, 357
  thick plate, 358
thermoelastic plane stress, 357–358
thermoelastic reciprocal theorem, 522–524
thin-walled sections, 245–246, 250–251
tilted punch, 470–472, 518–521
torsion, 241–251, 262, 419–427, 505–506
torsional rigidity, 244, 246, 247, 251
traction, 12, 303, 306
transformation of coördinates (*see* coördinate transformation)
transverse shear, 259–266
twist, 241–242
uniform rotation, 97–99
unilateral contact (*see* contact problems - unilateral inequalities)
uniqueness, 325–329, 509–511
variational methods, 497–511
vector notation, 6–8
vector operators in complex notation, 293–294
Von Mises stress, 13
wedge problems, 149–166
Williams' asymptotic method, 154–166
Young's modulus, 21