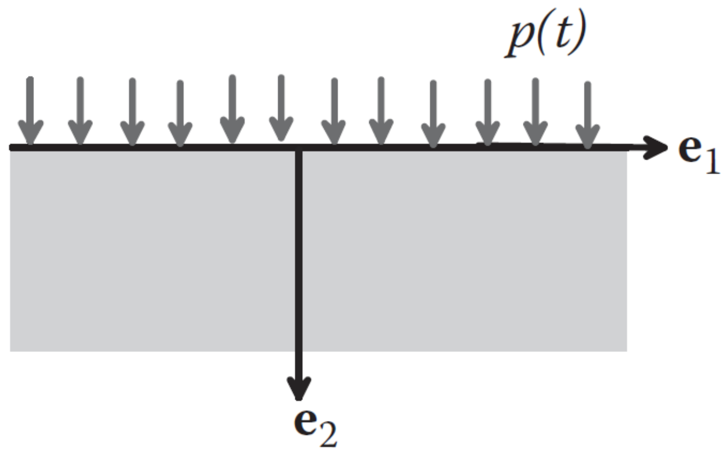
Simple Dynamic Solutions for Linear Elastic Solids

mi@seu.edu.cn

Outline

- Surface under time-varying normal pressure（自由表面时变压力载荷）
- Surface under constant normal pressure（自由表面常压）
- Surface under time-varying shear traction（自由表面时变剪应力载荷）
- One-dimensional bar under end loading（端部受载杆件）
- Plane waves in infinite solids（无限固体中的平面波）
- Wave speeds in isotropic solids(各向同性固体中的波速)
- Reflection of longitudinal waves at a free surface(反射波)
- Reflection and transmission of waves at a bimaterial interface（反射波与透射波）

Surface under Time-varying Normal Pressure



- The solid is at rest and stress free at time $t = 0$.
- Strain-displacement relation:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- Isotropic Hooke's Law:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}; \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

- Linear momentum balance equations: $\dfrac{\partial \sigma_{ij}}{\partial x_i} = \rho \dfrac{\partial^2 u_j}{\partial t^2}$
- Symmetry condition indicates

$$u_1 = u_3 = 0, \quad u_2 = u_2\left[x_2, t\right]$$

- The only nontrivial strain component: $\varepsilon_2 = \dfrac{\partial u_2}{\partial x_2}$

Surface under Time-varying Normal Pressure

- Nontrivial stress components:

$$\sigma_2[x_2,t]=\frac{E}{(1+\nu)}\left\{\varepsilon_2+\frac{\nu}{1-2\nu}\varepsilon_2\right\}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial u_2}{\partial x_2},\quad \sigma_1[x_2,t]=\sigma_3[x_2,t]=\frac{E\nu}{(1+\nu)(1-2\nu)}\frac{\partial u_2}{\partial x_2}.$$

- The only nonzero linear momentum balance equation:

$$\frac{\partial \sigma_2}{\partial x_2}=\rho\frac{\partial^2 u_2}{\partial t^2}\quad\Rightarrow\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial^2 u_2}{\partial x_2^2}=\rho\frac{\partial^2 u_2}{\partial t^2}$$

$$\Rightarrow\frac{\partial^2 u_2}{\partial x_2^2}=\frac{1}{c_L^2}\frac{\partial^2 u_2}{\partial t^2},\quad c_L^2=\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}$$

- The 1-D wave equation has the general solution:

$$u_2[x_2,t]=f[t-x_2/c_L]+g[t+x_2/c_L]$$

- Initial conditions:

$$\begin{cases}u_2[x_2,0]=0\\ \dfrac{\partial u_2[x_2,0]}{\partial t}=0\end{cases}\Rightarrow\begin{cases}f[-x_2/c_L]+g[x_2/c_L]=0\\ f'[-x_2/c_L]+g'[x_2/c_L]=0\end{cases}\Rightarrow f[-x_2/c_L]=-g[x_2/c_L]=A$$

Surface under Time-varying Normal Pressure

- The vertical displacement:

$$g\left[x_2/c_L\right]=-A \quad \Rightarrow g\left[t+x_2/c_L\right]=-A$$

$$u_2\left[x_2,t\right]=f\left[t-x_2/c_L\right]+g\left[t+x_2/c_L\right]=f\left[t-x_2/c_L\right]-A$$

$$\Rightarrow \sigma_2\left[x_2,t\right]=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial u_2}{\partial x_2}=-\frac{1}{c_L}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}f'\left[t-x_2/c_L\right]$$

- Boundary conditions at the plane surface:

$$\sigma_2\left[0,t\right]=-p\left[t\right] \quad \Rightarrow -\frac{1}{c_L}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}f'\left[t\right]=-p\left[t\right] \quad \Rightarrow f'\left[t\right]=\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}p\left[t\right]$$

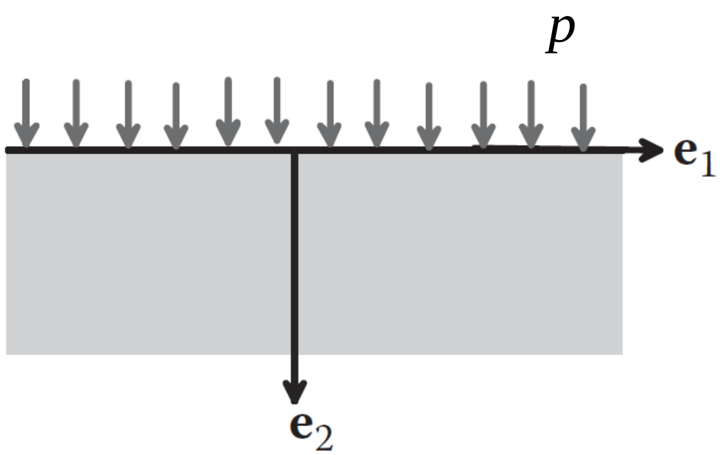
$$\Rightarrow f\left[t\right]=\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\int_0^t p\left[\tau\right]d\tau+B \quad \Rightarrow f\left[t-x_2/c_L\right]=\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\int_0^{t-x_2/c_L} p\left[\tau\right]d\tau+B$$

- Determine B: $t=0 \quad \Rightarrow A=f\left[-x_2/c_L\right]=B$
- Final solutions:

$$u_2\left[x_2,t\right]=\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\int_0^{t-x_2/c_L} p\left[\tau\right]d\tau, \quad \sigma_2\left[x_2,t\right]=-p\left[t-x_2/c_L\right], \quad t-x_2/c_L\geq 0$$

Surface under Constant Normal Pressure

- For the particular case of a constant pressure of magnitude



$$u_2[x_2,t]=\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}(t-x_2/c_L)\,p$$

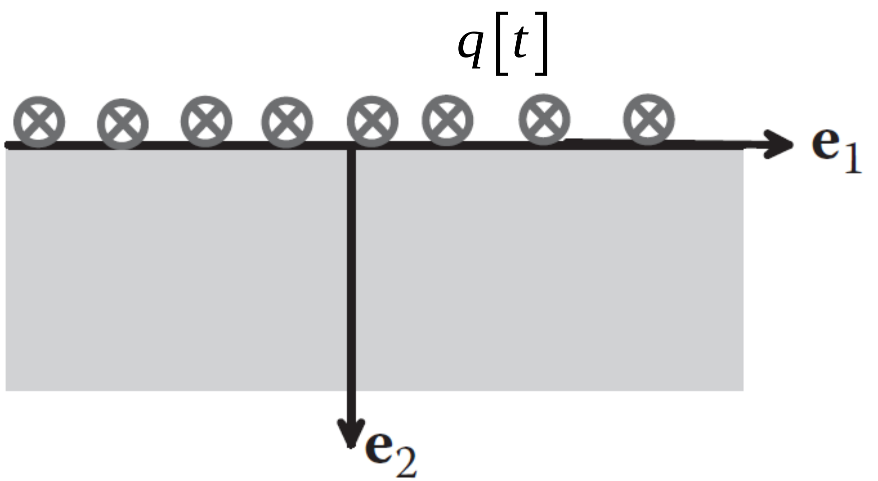
$$\sigma_2[x_2,t]=-p,\quad t-x_2/c_L\geq 0$$

- Evidently, a stress pulse equal in magnitude to the surface pressure propagates vertically through the half space with speed c_L.
- The velocity of the solid is constant in: $0 < x_2 < tc_L$.

$$v_2[x_2,t]=\frac{\partial u_2[x_2,t]}{\partial t}=\frac{c_L\,p}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}=\frac{p}{\rho c_L}$$

Surface under Time-varying Shear Traction

- The solid is at rest and stress free at time $t = 0$.



- Strain-displacement relation:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

- Isotropic Hooke's Law:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}; \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

- Linear momentum balance equations: $\dfrac{\partial \sigma_{ij}}{\partial x_i} = \rho \dfrac{\partial^2 u_j}{\partial t^2}$

- Symmetry condition indicates

$$u_1 = u_2 = 0, \quad u_3 = u_3\left[x_2, t\right]$$

- The only nontrivial strain component: $\varepsilon_{23} = \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_2}$

Surface under Time-varying Shear Traction

- The only nontrivial stress component:

$$\sigma_{23} = \frac{E}{(1+\nu)}\varepsilon_{23} = \frac{E}{2(1+\nu)}\frac{\partial u_3}{\partial x_2}$$

- The only nonzero linear momentum balance equation:

$$\frac{\partial \sigma_{23}}{\partial x_2} = \rho \frac{\partial^2 u_3}{\partial t^2} \quad \Rightarrow \frac{E}{2(1+\nu)}\frac{\partial^2 u_3}{\partial x_2^2} = \rho \frac{\partial^2 u_3}{\partial t^2} \quad \Rightarrow \frac{\partial^2 u_3}{\partial x_2^2} = \frac{1}{c_S^2}\frac{\partial^2 u_3}{\partial t^2}, \quad c_S^2 = \frac{E}{2(1+\nu)\rho}$$

- The 1-D wave equation has the general solution:

$$u_3[x_2, t] = f[t - x_2/c_S] + g[t + x_2/c_S]$$

- Initial conditions:

$$\begin{cases} u_3[x_2, 0] = 0 \\ \dfrac{\partial u_3[x_2, 0]}{\partial t} = 0 \end{cases} \Rightarrow \begin{cases} f[-x_2/c_S] + g[x_2/c_S] = 0 \\ f'[-x_2/c_S] + g'[x_2/c_S] = 0 \end{cases} \Rightarrow f[-x_2/c_S] = -g[x_2/c_S] = A$$

Surface under Time-varying Shear Traction

- The nonzero displacement:

$$g[x_2/c_S] = -A \quad \Rightarrow g[t + x_2/c_S] = -A$$

$$u_3[x_2,t] = f[t - x_2/c_S] + g[t + x_2/c_S] = f[t - x_2/c_S] - A$$

$$\Rightarrow \sigma_{23}[x_2,t] = \frac{E}{2(1+\nu)}\frac{\partial u_3}{\partial x_2} = -\frac{1}{c_S}\frac{E}{2(1+\nu)} f'[t - x_2/c_S]$$

- Boundary conditions at the plane surface:

$$\sigma_{23}[0,t] = -q[t] \quad \Rightarrow -\frac{1}{c_S}\frac{E}{2(1+\nu)} f'[t] = -q[t] \quad \Rightarrow f'[t] = \frac{2(1+\nu)c_S}{E} q[t]$$

$$\Rightarrow f[t] = \frac{2(1+\nu)c_S}{E}\int_0^t q[\tau]d\tau + B \quad \Rightarrow f[t - x_2/c_S] = \frac{2(1+\nu)c_S}{E}\int_0^{t-x_2/c_S} q[\tau]d\tau + B$$

- Determine B: $t = 0 \quad \Rightarrow A = f[-x_2/c_S] = B$

- Final solutions:

$$u_3[x_2,t] = \frac{2(1+\nu)c_S}{E}\int_0^{t-x_2/c_S} q[\tau]d\tau, \quad \sigma_{23}[x_2,t] = -q[t - x_2/c_S], \quad t - x_2/c_S \geq 0$$

One-dimensional Bar Subjected to End Loading

- The bar is at rest and stress free at time $t = 0$.
- We cheat by assuming that σ_{11} is the only nonzero stress.
- The strain-displacement relation: $\varepsilon_1 = \partial u_1 / \partial x_1$
- Hooke's Law: $\sigma_1 = E\varepsilon_1 = E\,\partial u_1 / \partial x_1$
- Linear momentum balance equations:

$$\frac{\partial \sigma_1}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2} \quad \Rightarrow E\frac{\partial^2 u_1}{\partial x_1^2} = \rho \frac{\partial^2 u_1}{\partial t^2} \quad \Rightarrow \frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_B^2}\frac{\partial^2 u_1}{\partial t^2}, \quad c_B^2 = \frac{E}{\rho}$$

- This equation is exact for $\nu = 0$ but cannot be correct in general, because transverse motion is neglected.
- The 1-D wave equation has the general solution:

$$u_1[x_1, t] = f\left[t - x_1/c_B\right] + g\left[t + x_1/c_B\right]$$

One-dimensional Bar Subjected to End Loading

- Initial conditions

$$\begin{cases} u_1[x_1,0]=0 \\ \dfrac{\partial u_1[x_1,0]}{\partial t}=0 \end{cases} \Rightarrow \begin{cases} f[-x_1/c_B]+g[x_1/c_B]=0 \\ f'[-x_1/c_B]+g'[x_1/c_B]=0 \end{cases} \Rightarrow f[-x_1/c_B]=-g[x_1/c_B]=A$$

$$g[x_1/c_B]=-A \quad \Rightarrow g[t+x_1/c_B]=-A$$

- The nonzero displacement:

$$u_1[x_1,t]=f[t-x_1/c_B]+g[t+x_1/c_B]=f[t-x_1/c_B]-A$$

$$\Rightarrow \sigma_1[x_1,t]=E\frac{\partial u_1}{\partial x_1}=-\frac{E}{c_B}f'[t-x_1/c_B]$$

- Boundary conditions at $x=0$:

$$\sigma_1[0,t]=-p[t] \quad \Rightarrow -\frac{E}{c_B}f'[t]=-p[t] \quad \Rightarrow f'[t]=\frac{c_B}{E}p[t]$$

$$\Rightarrow f[t]=\frac{c_B}{E}\int_0^t p[\tau]d\tau+B \quad \Rightarrow f[t-x_1/c_B]=\frac{c_B}{E}\int_0^{t-x_1/c_B}p[\tau]d\tau+B$$

- Determine B: $t=0 \quad \Rightarrow A=f[-x_1/c_B]=B$
- Final solutions:

$$\boxed{u_1[x_1,t]=\frac{c_B}{E}\int_0^{t-x_1/c_B}p[\tau]d\tau, \quad \sigma_1[x_1,t]=-p[t-x_1/c_B], \quad t-x_1/c_B\geq 0}$$

Plane Waves in an Infinite Solid

- A plane wave that travels in direction **p** at speed c

$$u_i\left[x_j,t\right]=a_i f\left[ct-x_k p_k\right]$$

- Linear momentum balance equations:

$$\begin{cases}\sigma_{ij}=C_{ijkl}\varepsilon_{kl}=\frac{1}{2}C_{ijkl}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)=C_{ijkl}\frac{\partial u_k}{\partial x_l}\\ \frac{\partial \sigma_{ij}}{\partial x_j}=\rho\frac{\partial^2 u_i}{\partial t^2}\end{cases}\Rightarrow C_{ijkl}\frac{\partial u_k}{\partial x_j\partial x_l}=\rho\frac{\partial^2 u_i}{\partial t^2}$$

- Substituting the plane wave solution for displacement

$$u_k=a_k f\left[ct-x_m p_m\right]\Rightarrow\frac{\partial^2 u_k}{\partial x_j\partial x_l}=\frac{\partial}{\partial x_j}\left\{-p_l a_k f'\left[ct-x_m p_m\right]\right\}=p_j p_l a_k f''\left[ct-x_m p_m\right]$$

$$\Rightarrow C_{ijkl}p_j p_l a_k f''\left[ct-x_m p_m\right]=\rho a_i c^2 f''\left[ct-x_m p_m\right]$$

$$\Rightarrow C_{ijkl}p_j p_l a_k=\rho a_i c^2 \quad \Rightarrow\left(\frac{C_{ijkl}p_j p_l}{\rho}-c^2\delta_{ik}\right)a_k=0$$

- For any **p**: three wave speeds and three displacement directions

Plane Waves in an Infinite Solid

- For the special case of an isotropic solid

$$C_{ijkl} = \frac{2\mu\nu}{(1-2\nu)}\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\Rightarrow C_{ijkl}p_j p_l a_k = \left(\frac{2\mu\nu}{(1-2\nu)}p_i p_k + \mu\left(\delta_{ik} + p_i p_k\right)\right)a_k = \frac{\mu}{(1-2\nu)}p_i p_k a_k + \mu a_i = \rho a_i c^2$$

$$\Rightarrow \boxed{\frac{\mu}{(1-2\nu)}p_i p_k a_k + \left(\mu - \rho c^2\right)a_i = 0}$$

- Only need to differentiate between P-wave and S-wave

$$p_k a_k = 0 \quad \Rightarrow c_S^2 = \mu/\rho$$

$$a_k = \eta p_k \quad \Rightarrow \left\{\frac{\mu}{(1-2\nu)} + \mu - \rho c^2\right\}\eta p_i = 0 \quad \Rightarrow c_L^2 = \frac{2\mu(1-\nu)}{\rho(1-2\nu)}$$

- The S-wave travels at speed c_S, and material particles are displaced perpendicular to the direction of motion of the wave.
- The P-wave travels at speed c_L, and material particles are displaced parallel to the direction of motion of the wave.

Summary of Wave Speeds in Isotropic Solids

- Longitudinal waves

$$c_L^2 = \frac{E\left(1-\nu\right)}{\rho\left(1+\nu\right)\left(1-2\nu\right)}$$

- Shear waves

$$c_S^2 = \frac{E}{2\left(1+\nu\right)\rho}$$

- Waves in an 1-D bar

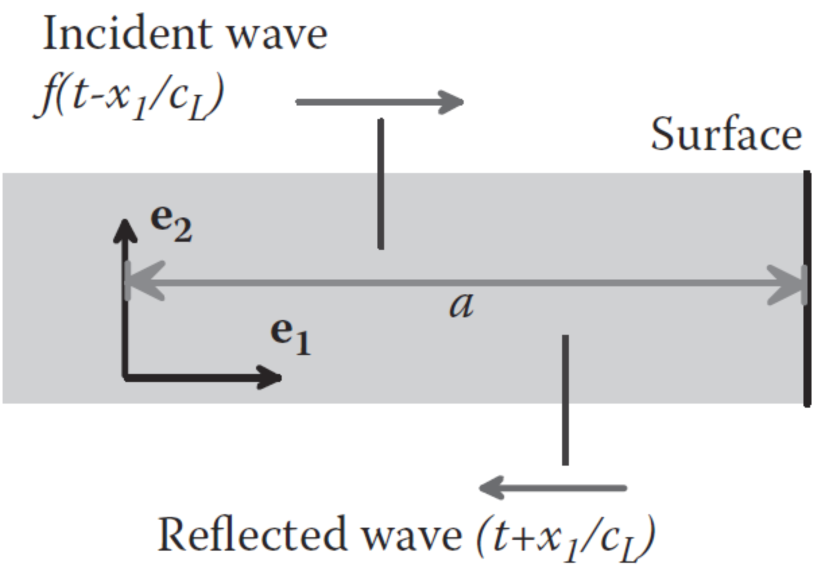
$$c_B^2 = \frac{E}{\rho}$$

- For most real materials, $c_L > c_B > c_S$.

Reflection of a Longitudinal Wave at a Free Surface

- Consider a longitudinal wave travelling towards a free surface at $x_1 = a$

$$u_1'[x_1,t] = -\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\int_0^{t-x_1/c_L} f[\tau]d\tau, \quad \sigma_1'[x_1,t] = f[t-x_1/c_L]$$



- The material ahead of the wave front is at rest and stress free, whereas behind the front, material has a constant stress and velocity.

- At time $t = a/c_L$, the front would reach the free surface and be reflected.

$$\sigma_1''[x_1,t] = g[t+x_1/c_L]$$

Reflection of a Longitudinal Wave at a Free Surface

- Total stress after reflection

$$\sigma_1[x_1,t]=\sigma_1'[x_1,t]+\sigma_1''[x_1,t]=f[t-x_1/c_L]+g[t+x_1/c_L]$$

- Stress free condition at $x_1 = a$

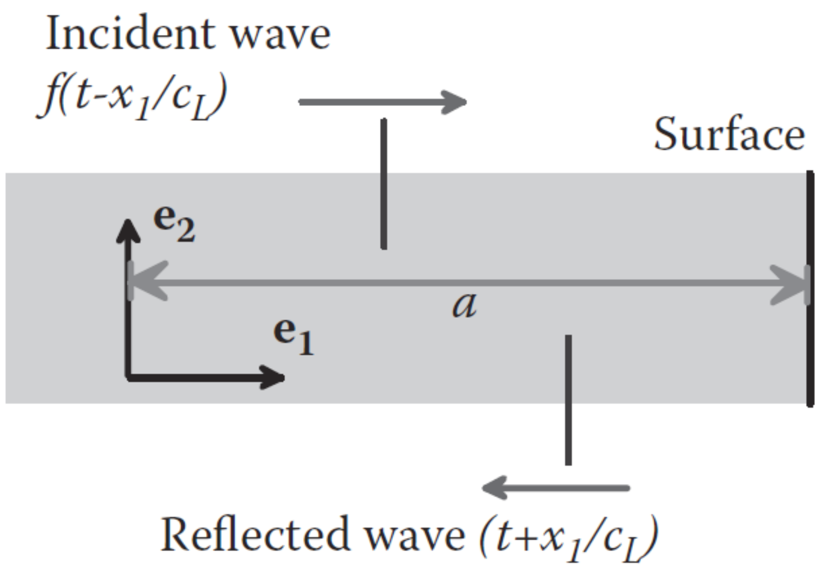
$$0=\sigma_1[a,t]=f[t-a/c_L]+g[t+a/c_L]$$

$$\Rightarrow g[t+a/c_L]=-f[t-a/c_L]$$

$$\Rightarrow g[t+x_1/c_L+a/c_L-a/c_L]$$

$$=-f[t+x_1/c_L-a/c_L-a/c_L]$$

$$\Rightarrow \boxed{\begin{aligned} &g[t+x_1/c_L] \\ &=-f[t-a/c_L+(x_1-a)/c_L] \end{aligned}}$$



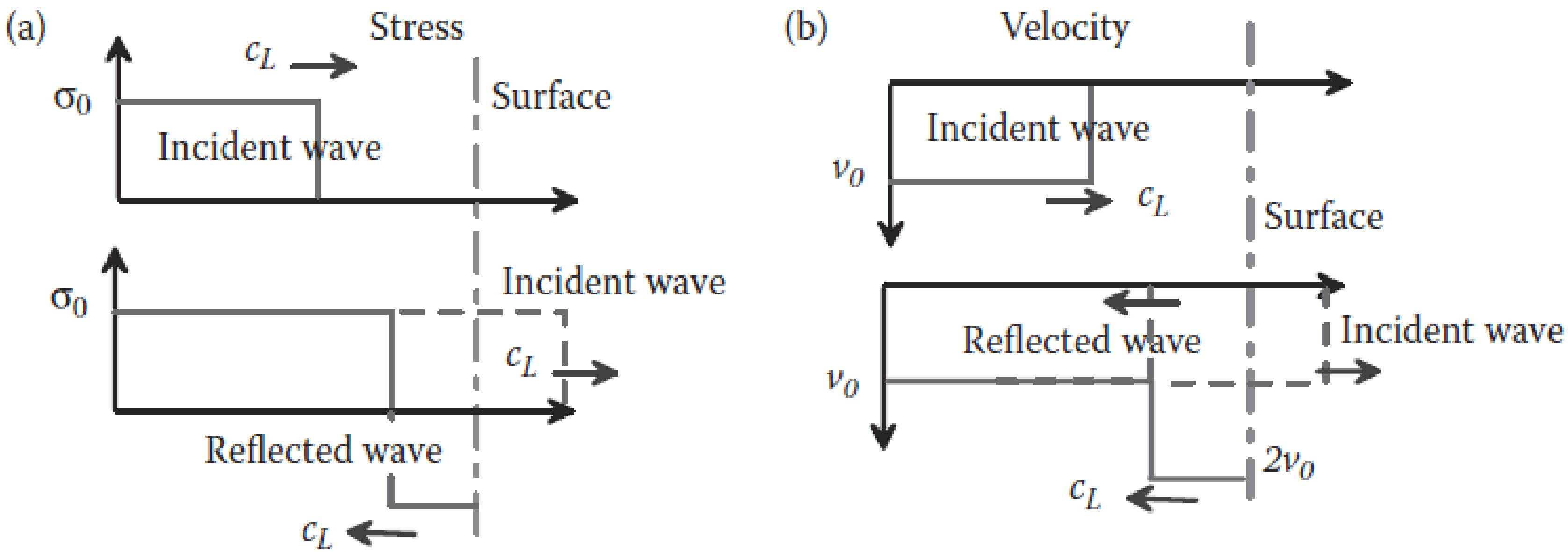
- Full solution

$$\sigma_1[x_1,t]=f[t-x_1/c_L]-f[t-a/c_L+(x_1-a)/c_L],$$

$$u_1[x_1,t]=-\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\left\{\int_0^{t-x_1/c_L} f[\tau]d\tau+\int_0^{t-a/c_L+(x_1-a)/c_L} f[\tau]d\tau\right\}$$

Reflection of a Longitudinal Wave at a Free Surface

- As a specific example, consider a plane, constant-stress wave that is incident on a free surface.



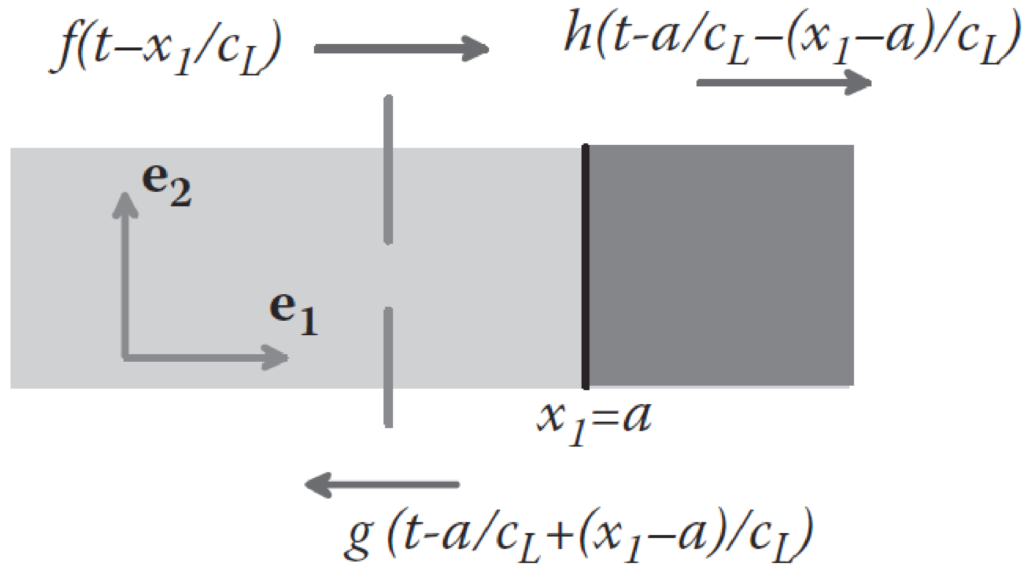
- Behind the incident wave, the stress and velocity are both constant.
- At time, $t = a/c_L$, the wave reaches the free surface. At this time, an equal and opposite stress pulse is reflected.
- Behind the reflected wave, the solid is stress free and has constant velocity: $u_1[x_1,t] = -2\rho\sigma_0/c_L$.

Reflection and Transmission of Waves

- Consider a longitudinal wave travelling towards a bimaterial interface at $x_1 = a$

$$u_1'[x_1,t] = -\frac{c_L}{E}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\int_0^{t-x_1/c_L} f[\tau]d\tau, \quad \sigma_1'[x_1,t] = f[t-x_1/c_L]$$

- Some of the energy will be reflected, whereas some will be transmitted into the adjacent solid.



- Reflection wave: $\sigma_1''[x_1,t] = g\left[t-a/c_L+(x_1-a)/c_L\right]$
- Transmission wave: $\bar{\sigma}_1[x_1,t] = h\left[t-a/c_L-(x_1-a)/\bar{c}_L\right]$

Reflection and Transmission of Waves

- Stress continuity at the interface, $x_1 = a$, requires

$$f\left[t-a/c_L\right]+g\left[t-a/c_L\right]=h\left[t-a/\bar{c}_L\right]$$

- Displacement continuity at the interface, $x_1 = a$, requires

$$\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial^2 \bar{u}_1}{\partial t^2} \quad \Rightarrow \frac{1}{\rho}\frac{\partial \sigma_1}{\partial x_1}=\frac{1}{\bar{\rho}}\frac{\partial \bar{\sigma}_1}{\partial x_1} \quad \Rightarrow -\frac{f'\left[t-a/c_L\right]}{\rho c_L}+\frac{g'\left[t-a/c_L\right]}{\rho c_L}=-\frac{h'\left[t-a/\bar{c}_L\right]}{\bar{\rho}\bar{c}_L}$$

$$\Rightarrow -\frac{f\left[t-a/c_L\right]}{\rho c_L}+\frac{g\left[t-a/c_L\right]}{\rho c_L}=-\frac{h\left[t-a/\bar{c}_L\right]}{\bar{\rho}\bar{c}_L}+C, \quad t=0, f=g=h=0 \quad \Rightarrow C=0$$

- The conditions of stress and displacement continuity yield

$$\sigma_1''\left[x_1,t\right]=\beta_r f\left[t-a/c_L+\left(x_1-a\right)/c_L\right], \quad \beta_r=\frac{\bar{\rho}\bar{c}_L-\rho c_L}{\bar{\rho}\bar{c}_L+\rho c_L}$$

$$\bar{\sigma}_1\left[x_1,t\right]=\beta_t f\left[t-a/c_L-\left(x_1-a\right)/\bar{c}_L\right], \quad \beta_t=\frac{2\bar{\rho}\bar{c}_L}{\bar{\rho}\bar{c}_L+\rho c_L}$$