Mathematical Preliminaries

Outline

- Scalar, Vector and Matrix
- Indicial Notation and Summation Convention
- Kronecker Delta
- Levi-Civita symbol
- Coordinate Transformation
- Tensor
- Principal Values and Directions
- Tensor Algebra
- Tensor Calculus
- Integral Theorems
- Cylindrical and Spherical Coordinates

Scalar

- Scalar: representing a single magnitude at each point in space

- Material density
- Mass
- Energy
- Distance
- Volume
- Area
- Temperature

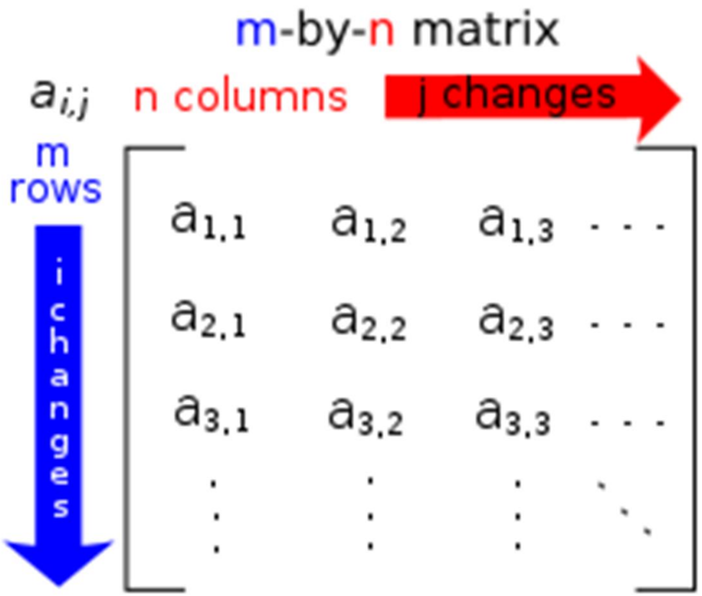
Vector

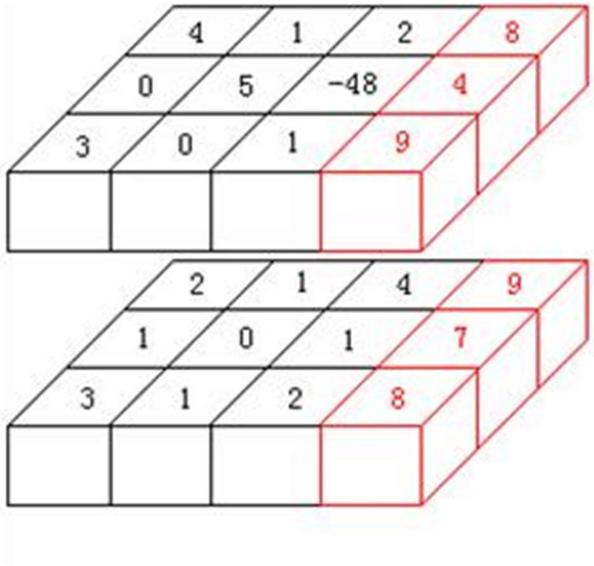
- Vector: representing physical quantities that have both magnitude and direction

- Electric field
- Force
- Displacement of material points
- Velocity
- Rotation of material points
- Force couple (Moment)

Matrix (Array)

- Matrix: a rectangular array of numbers





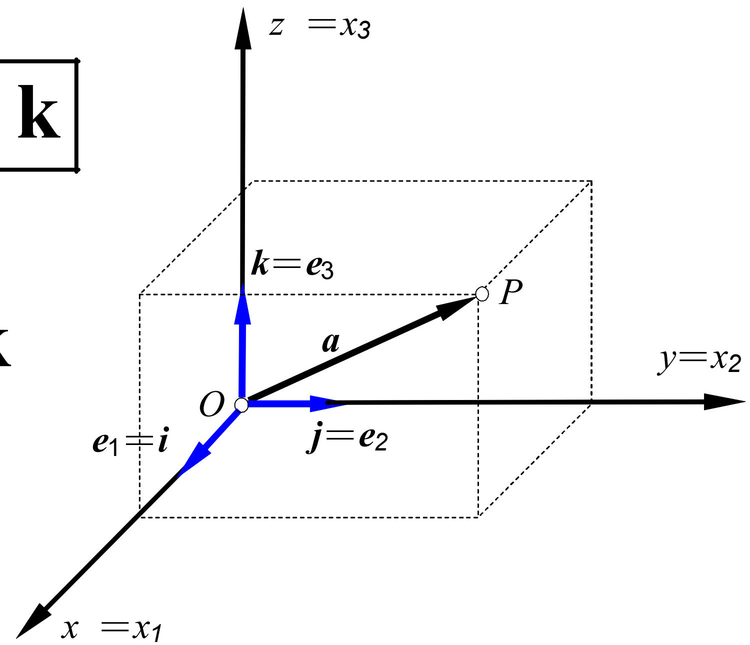
- Array: a data structure in which similar elements of data are arranged in a table

Indicial Notation

- Orthogonal unit vectors: $\boxed{\mathbf{i} \times \mathbf{j} = \mathbf{k}}$
- Vector decomposition:

$$\mathbf{a} = \mathbf{OP} = (\mathbf{a} \cdot \mathbf{i})\mathbf{i} + (\mathbf{a} \cdot \mathbf{j})\mathbf{j} + (\mathbf{a} \cdot \mathbf{k})\mathbf{k}$$

$$= a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} = \sum_{i=1}^{3} a_i\mathbf{e}_i$$



- Indicial notation: a shorthand scheme whereby a whole set of components is represented by a single symbol with subscripts

$$x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{e}_i = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \ a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_3 & a_{32} & a_{33} \end{bmatrix}$$

Indicial Notation

- Addition and subtraction

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \; a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

- Scalar multiplication

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \; \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix}$$

- Outer multiplication

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Indicial Notation

- Commutative, associative and distributive laws

$$a_i + b_i = b_i + a_i$$

$$a_{ij}b_k = b_k a_{ij}$$

$$a_i + (b_i + c_i) = (a_i + b_i) + c_i$$

$$a_i(b_{jk}c_l) = (a_i b_{jk})c_l$$

$$a_{ij}(b_k + c_k) = a_{ij}b_k + a_{ij}c_k$$

- Equality of two symbols

$$a_i = b_i \Rightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}, \qquad a_{ij} = b_{ij} \Rightarrow \begin{bmatrix} a_{11} = b_{11} & a_{12} = b_{12} & a_{13} = b_{13} \\ a_{21} = b_{21} & a_{22} = b_{22} & a_{23} = b_{23} \\ a_{31} = b_{31} & a_{32} = b_{32} & a_{33} = b_{33} \end{bmatrix}$$

- Avoid

$$a_i = b_j, \qquad a_{ij} = b_{kl} \cdots$$

Summation Convention

- Summation convention: if a subscript appears twice in the same term, then summation over that subscript from one to three is implied

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_{ij}b_j = \sum_{j=1}^{3} a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$

$$a_{ij}x_i x_j = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_i x_j$$

$$a_{ii} = a_{jj} = a_{kk} \cdots$$

- In a single term, no index can appear more than twice.
- Dummy (repeated) indices vs. free (distinct) indices
- Among terms, index property must match.

Contraction and Symmetry

- Contraction: for example, a_{ii} is obtained from a_{ij} by contraction on i and j
- Outer multiplication → contraction → inner product

$$a_i b_j \xrightarrow{\left(\text{contraction on } ij\right)} a_i b_i$$

$$a_{ij} b_{kl} \xrightarrow{\left(\text{contraction on } jk\right)} a_{ik} b_{kl}$$

- Symmetric vs. antisymmetric (skewsymmetric) w.r.t. two indices, i.e. m and n

$$a_{ij\ldots m\ldots n\ldots k} = a_{ij\ldots n\ldots m\ldots k} \qquad a_{ij\ldots m\ldots n\ldots k} = -a_{ij\ldots n\ldots m\ldots k}$$

- Useful identity: $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

Kronecker Delta

- Kronecker delta: a useful special symbol commonly used in index notational schemes
- Symmetric

$$\delta_{ij} = \begin{cases} 1, if \;\; i = j \; (no \;\; sum) \\ 0, if \;\; i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Replacement property

$$\delta_{ij} = \delta_{ji}$$

$$\delta_{ii} = 3, \delta_{i\underline{i}} = 1$$

$$\delta_{ij}a_j = a_i, \delta_{ij}a_i = a_j$$

$$\delta_{ij}a_{jk} = a_{ik}, \delta_{jk}a_{ik} = a_{ij}$$

$$\delta_{ij}a_{ij} = a_{ii}, \delta_{ij}\delta_{ij} = 3$$

Levi-Civita (Permutation) Symbol

- Antisymmetric w.r.t. any pair of its indices

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1, \varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \ldots = 0.$$

- Cross product of two vectors

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j (\varepsilon_{ijk} \mathbf{e}_k) = \varepsilon_{ijk} a_i b_j \mathbf{e}_k$$

- Scalar triple product of three vectors

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \varepsilon_{ijk} a_i b_j \mathbf{e}_k \cdot c_m \mathbf{e}_m = \varepsilon_{ijk} a_i b_j c_m (\mathbf{e}_k \cdot \mathbf{e}_m) = \varepsilon_{ijk} a_i b_j c_m \delta_{km} = \varepsilon_{ijk} a_i b_j c_k$$

Levi-Civita (Permutation) Symbol

- Determinant of a matrix (easily verifiable)

$$\det\left[a_{ij}\right] = \left|a_{ij}\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{3} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

- By the definition of Levi-Civita and note the following

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{rst} a_{ir} a_{js} a_{kt} &= \varepsilon_{ijk} a_{i1} a_{j2} a_{k3} - \varepsilon_{ijk} a_{i1} a_{j3} a_{k2} + \varepsilon_{ijk} a_{i2} a_{j3} a_{k1} \\ &\quad - \varepsilon_{ijk} a_{i2} a_{j1} a_{k3} + \varepsilon_{ijk} a_{i3} a_{j1} a_{k2} - \varepsilon_{ijk} a_{i3} a_{j2} a_{k1} \end{aligned}$$

$$\Rightarrow \boxed{\varepsilon_{ijk}\varepsilon_{rst} a_{ir} a_{js} a_{kt} = 6\det\left[a_{ij}\right], \qquad \varepsilon_{ijk} a_{il} a_{jm} a_{kn} = \varepsilon_{lmn} \det\left[a_{ij}\right]}$$

Levi-Civita (Permutation) Symbol

- ε-δ property

$$\varepsilon_{ijk} = \mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = \left(\delta_{ip}\mathbf{e}_p\right) \cdot \left(\delta_{jq}\mathbf{e}_q\right) \times \left(\delta_{kr}\mathbf{e}_r\right) \quad \Rightarrow \quad \boxed{\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}}$$

$$\varepsilon_{ijk}\varepsilon_{pqr} = \left|[A]\right|\left|[B]\right| = \left|[A]\right|\left|[B]^T\right| = \left|[A][B]^T\right|$$

$$= \left|\begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}\begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix}\right| = \begin{vmatrix} \delta_{im}\delta_{pm} & \delta_{im}\delta_{qm} & \delta_{im}\delta_{rm} \\ \delta_{jm}\delta_{pm} & \delta_{jm}\delta_{qm} & \delta_{jm}\delta_{rm} \\ \delta_{km}\delta_{pm} & \delta_{km}\delta_{qm} & \delta_{km}\delta_{rm} \end{vmatrix}$$

$$\Rightarrow \boxed{\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}} \quad \Rightarrow \quad \boxed{\varepsilon_{ijk}\varepsilon_{pqk} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{vmatrix} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}}$$

Sample Problem

- The matrix a_{ij} and vector b_i are specified by

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \; b_i = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

- Determine the following quantities, and indicate whether they are a scalar, vector or matrix.

$$a_{ii}\,, a_{ij}a_{ij}\,, a_{ij}a_{jk}\,, a_{ij}b_j\,, a_{ij}b_ib_j\,, b_ib_i\,, b_ib_j\,, a_{\text{symm.}}\,, a_{\text{anti.}}$$

- **Solution**:

$$a_{ii} = a_{11} + a_{22} + a_{33} = 7 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= 1 + 4 + 0 + 0 + 16 + 9 + 4 + 1 + 4 = 39 \text{ (scalar)}$$

Sample Problem - Solution

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = \begin{bmatrix} 1 & 10 & 6 \\ 6 & 19 & 18 \\ 6 & 10 & 7 \end{bmatrix} \text{(matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 10 \\ 16 \\ 8 \end{bmatrix} \text{(vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + \cdots = 84 \text{ (scalar)}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 16 + 0 = 20 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} 4 & 8 & 0 \\ 8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(matrix)}$$

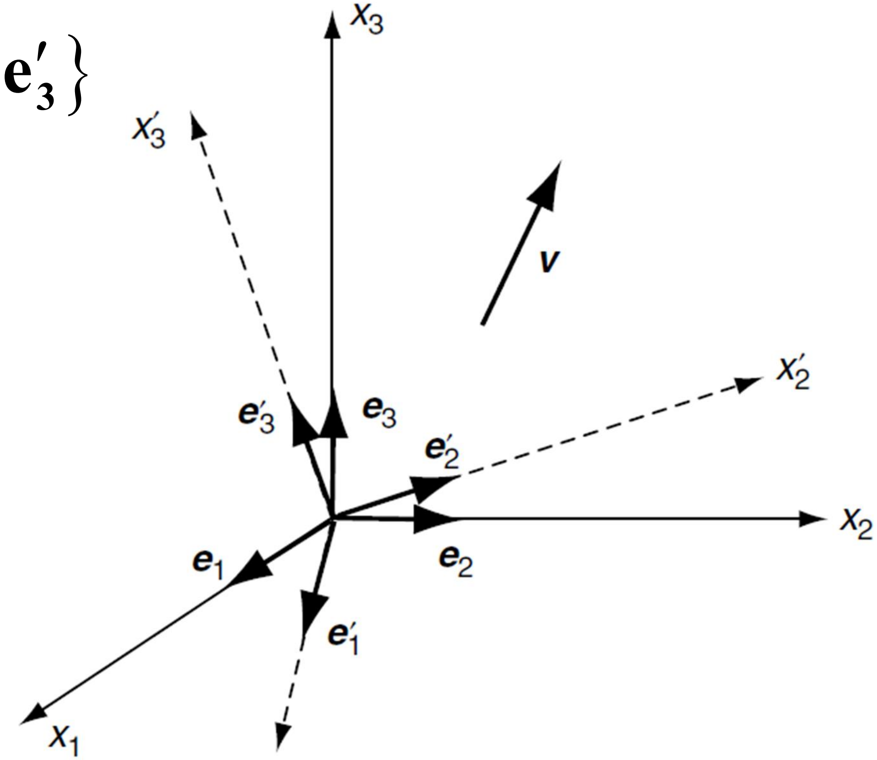
$$a_{\text{symm.}} = \frac{1}{2}\left(a_{ij} + a_{ji}\right) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} \text{(matrix)}$$

$$a_{\text{anti.}} = \frac{1}{2}\left(a_{ij} - a_{ji}\right) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \text{(matrix)}$$

Coordinate Transformation

- Unit base vectors for each frame

$$\{\mathbf{e_i}\} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}; \qquad \{\mathbf{e'_i}\} = \{\mathbf{e'_1}, \mathbf{e'_2}, \mathbf{e'_3}\}$$



- Transformation matrix

$$\boxed{Q_{ij} = \cos(x'_i, x_j)}$$

$$\mathbf{e}'_1 = Q_{11}\mathbf{e_1} + Q_{12}\mathbf{e_2} + Q_{13}\mathbf{e_3}$$

$$\mathbf{e}'_\mathbf{2} = Q_{21}e_1 + Q_{22}e_2 + Q_{23}\mathbf{e_3}$$

$$\mathbf{e}'_\mathbf{3} = Q_{31}\mathbf{e_1} + Q_{32}\mathbf{e_2} + Q_{33}\mathbf{e_3}$$

- Relations between base vectors

$$\mathbf{e'_i} = Q_{ij}\mathbf{e_j}; \qquad \mathbf{e_i} = Q_{ji}\mathbf{e'_j}$$

Coordinate Transformation

- Transformation between vectors

$$\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + v_3\mathbf{e_3} = v_i\mathbf{e_i} \boxed{= v_1{}'\mathbf{e}_1' + v_2'\mathbf{e}_2' + v_3'\mathbf{e}_3' = v_i'\mathbf{e}_i'}$$

$$v_i' = Q_{ij}v_j; \qquad v_i = Q_{ji}v_j'$$

- Property of transformation matrix

$$v_i = Q_{ji}v_j' = Q_{ji}Q_{jk}v_k \quad \Rightarrow \boxed{Q_{ji}Q_{jk} = \delta_{ik}} \boxed{\mathbf{Q}^T\mathbf{Q} = \mathbf{I}}$$

$$v_i' = Q_{ij}v_j = Q_{ij}Q_{kj}v_k'; \quad \Rightarrow \boxed{Q_{ij}Q_{kj} = \delta_{ik}} \boxed{\mathbf{Q}\mathbf{Q}^T = \mathbf{I}}$$

$$\boxed{\det\left[Q_{ij}\right] = \pm 1}$$

- For right-handed coordinates

$$\boxed{\det\left[Q_{ij}\right] = 1}$$

Tensor - Definition

- Transformation between matrices

$$\mathrm{A} = A'_{kl}\mathbf{e}'_{\mathbf{k}}\mathbf{e}'_{\mathbf{l}} = A'_{kl}\left(Q_{ki}\mathbf{e}_{\mathbf{i}}\right)\left(Q_{lj}\mathbf{e}_{\mathbf{j}}\right) = Q_{ki}Q_{lj}A'_{kl}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}} \boxed{= A_{ij}\mathbf{e}_{\mathbf{i}}\mathbf{e}_{\mathbf{j}}}$$

$$\Rightarrow \boxed{A'_{ij} = Q_{ik}Q_{jl}A_{kl}} \qquad \Rightarrow \boxed{A_{ij} = Q_{ki}Q_{lj}A'_{kl}}$$

- **Those vectors and matrices that satisfy the transformation rules are called tensors.**
- **Exceptional matrices do exist, i.e. spinors. Fortunately, this is not our concern.**

Tensor - Definition

- **A tensor can be any order and must satisfy**

$$a' = a, \text{ zero order (scalar)}$$

$$a'_i = Q_{ip} a_p, \text{ first order (vector)}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}, \text{ second order (matrix)}$$

$$a'_{ijk} = Q_{ip} Q_{jq} Q_{kr} a_{pqr}, \text{ third order}$$

$$a'_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} a_{pqrs}, \text{ fourth order}$$

- For general order

$$\boxed{A'_{ijkl\cdots} = Q_{im} Q_{jn} Q_{ko} Q_{lp} \cdots A_{mnop\cdots}}$$

$$\boxed{A_{ijkl\cdots} = Q_{mi} Q_{nj} Q_{ok} Q_{pl} \cdots A'_{mnop\cdots}}$$

Tensor Algebra

- The distinction between a tensor and its components

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i = v'_1\boldsymbol{e}'_1 + v'_2\boldsymbol{e}'_2 + v'_3\boldsymbol{e}'_3 = v'_i\boldsymbol{e}'_i$$

$$\begin{aligned}\mathbf{A} &= A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 \\ &+ A_{21}\boldsymbol{e}_2\boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 \\ &+ A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3 \\ &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j\end{aligned}$$

- Equality of two tensors: $\mathbf{A} = \mathbf{B} \Rightarrow A_i\mathbf{e_i} = B_i\mathbf{e_i} \Rightarrow A_i = B_i$
- Addition/subtraction
- Scalar multiplication: $\lambda\mathbf{A} = \lambda A_i\mathbf{e_i}$
- Cross product and triple product of first-order tensors:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk}a_jb_k\boldsymbol{e}_i$$

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk}a_ib_jc_k$$

Tensor Algebra

- Tensor product (outer multiplication)

$$\mathbf{A} \otimes \mathbf{B} = A_i \mathbf{e_i} \otimes B_j \mathbf{e_j} \Rightarrow A_i B_j \mathbf{e_i} \otimes \mathbf{e_j} = A_i B_j \mathbf{e_i} \mathbf{e_j}$$

- Contraction on a pair of indices

  (Tensor product →inner product)

$$\mathbf{A} \cdot \mathbf{B} = A_i \mathbf{e_i} \cdot B_j \mathbf{e_j} \Rightarrow A_i B_j \mathbf{e_i} \cdot \mathbf{e_j} = A_i B_i$$

- Generalized inner product (matrix product)

$$\boldsymbol{A}\boldsymbol{a} = [\mathbf{A}]\{\boldsymbol{a}\} = A_{ij}a_j = a_jA_{ij}$$

$$\boldsymbol{a}^{\boldsymbol{T}}\mathbf{A} = \{\boldsymbol{a}\}^T[\mathbf{A}] = a_iA_{ij} = A_{ij}a_i$$

$$\boldsymbol{A}\boldsymbol{B} = [\mathbf{A}][\boldsymbol{B}] = A_{ij}B_{jk}$$

$$\boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}} = A_{ij}B_{kj}$$

$$\mathbf{A}^{\boldsymbol{T}}\boldsymbol{B} = A_{ji}B_{jk}$$

$$tr(\mathbf{A}\boldsymbol{B}) = A_{ij}B_{ji}$$

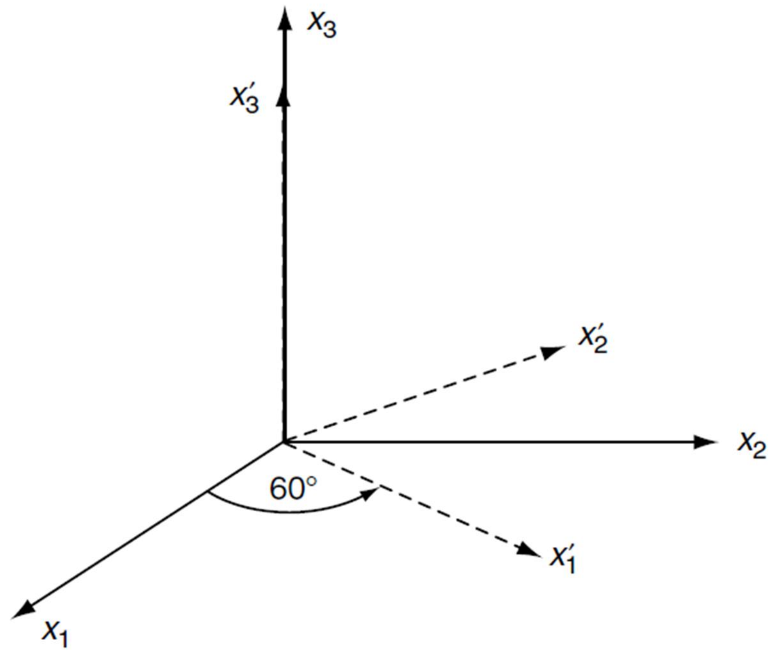
$$tr(\mathbf{A}\boldsymbol{B}^{\boldsymbol{T}}) = tr(\mathbf{A}^{\boldsymbol{T}}\boldsymbol{B}) = A_{ij}B_{ij}$$

Sample Problem

- The components of a first and a second-order tensor in a particular coordinate system are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, b_{ij} = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

- Determine the components of both tensors in a new coordinate system found through a rotation of 60º about the x_3-axis. Choose a counterclockwise rotation when viewing form the positive x_3-axis.



Sample Problem - Solution

$$Q_{ij} = \begin{bmatrix} \cos 60^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 150^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_i' = Q_{ij} a_j = \begin{vmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + 2\sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

$$b_{ij}' = Q_{ip} Q_{jq} b_{pq} = \begin{vmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2 + \sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1 - 3\sqrt{3}/2 \\ 3/2 + \sqrt{3} & 1 - 3\sqrt{3}/2 & 4 \end{bmatrix}$$

Isotropic Tensors

- Defined as those tensors whose components remain the same under all transformations
- All scalars (zero order) are isotropic
- No non-trivia isotropic tensor of the first order
- Kronecker Delta is the most general second order isotropic tensor

$$T_{ij} = \alpha\delta_{ij}$$

$$\Rightarrow T'_{ij} = Q_{ik}Q_{jl}T_{kl} = Q_{ik}Q_{jl}\alpha\delta_{kl} = \alpha Q_{ik}Q_{jk} = \alpha\delta_{ij} = T_{ij}$$

Isotropic Tensors

- Levi-Civita Symbol is the most general third order isotropic tensor

$$\det\left[\mathbf{a}\right] = a_{1m}a_{2n}a_{3o}\varepsilon_{mno} = a_{m1}a_{n2}a_{o3}\varepsilon_{mno}$$

$$\Rightarrow \varepsilon_{ijk}\det\left[\mathbf{a}\right] = a_{im}a_{jn}a_{ko}\varepsilon_{mno}$$

$$T_{ijk} = \alpha\varepsilon_{ijk}$$

$$\Rightarrow T'_{ijk} = Q_{im}Q_{jn}Q_{ko}T_{mno} = \alpha Q_{im}Q_{jn}Q_{ko}\varepsilon_{mno} = \alpha\varepsilon_{ijk}\det\left[\mathbf{Q}\right] = T_{ijk}$$

- The most general fourth order isotropic tensor

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$\Rightarrow T'_{ijkl} = Q_{im}Q_{jn}Q_{ko}Q_{lp}T_{mnop} = Q_{im}Q_{jn}Q_{ko}Q_{lp}\left(\alpha\delta_{mn}\delta_{op} + \beta\delta_{mo}\delta_{np} + \gamma\delta_{mp}\delta_{on}\right)$$

$$= \left(\alpha Q_{im}Q_{jm}Q_{ko}Q_{lo} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}\right) = T_{ijkl}$$

Tensor Field and Comma Notation

- The field concept of a tensor component

$$a = a(x_1, x_2, x_3) = a(x_i) = a(\boldsymbol{x})$$
$$a_i = a_i(x_1, x_2, x_3) = a_i(x_i) = a_i(\boldsymbol{x})$$
$$a_{ij} = a_{ij}(x_1, x_2, x_3) = a_{ij}(x_i) = a_{ij}(\boldsymbol{x})$$

- The Comma Notation for partial differentiation

Gradient operator: $\nabla = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3 = \frac{\partial}{\partial x_i}\mathbf{e}_i$

Laplacian operator: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial}{\partial x_i}\mathbf{e}_i \cdot \frac{\partial}{\partial x_j}\mathbf{e}_j = \frac{\partial^2}{\partial x_i \partial x_i}$

Gradient of a scalar: $\nabla\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i$

Laplacian of a scalar: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} = \phi_{,ii}$

Tensor Calculus

Directional derivative: $\frac{d\phi}{ds} = \nabla\phi \cdot \mathbf{n} = \left(\frac{\partial\phi}{\partial x_1}\mathbf{e_1} + \frac{\partial\phi}{\partial x_2}\mathbf{e_2} + \frac{\partial\phi}{\partial x_3}\mathbf{e_3} \right) \cdot \left(\frac{dx_1}{ds}\mathbf{e_1} + \frac{dx_2}{ds}\mathbf{e_2} + \frac{dx_3}{ds}\mathbf{e_3} \right)$

Gradient of a vector: $\boxed{\mathbf{u}\overleftarrow{\nabla} = \left(u_i\mathbf{e}_i\right)\left(\frac{\overleftarrow{\partial}}{\partial x_j}\mathbf{e}_j\right)} = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i\mathbf{e}_j = u_{i,j}\mathbf{e}_i\mathbf{e}_j$

Divergence of a vector: $\nabla \cdot \mathbf{u} = \left(\frac{\partial}{\partial x_j}\mathbf{e}_j\right) \cdot \left(u_i\mathbf{e}_i\right) = u_{i,j}\delta_{ij} = u_{i,i}$

Gradient of a tensor: $\boxed{\boldsymbol{\sigma}\overleftarrow{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\frac{\overleftarrow{\partial}}{\partial x_k}\mathbf{e}_k\right)} = \frac{\partial\sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$

Divergence of a tensor: $\nabla \cdot \boldsymbol{\sigma} = \left(\frac{\partial}{\partial x_k}\mathbf{e}_k\right) \cdot \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \frac{\partial\sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k \cdot \mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

Curl of a vector: $\nabla \times \mathbf{u} = \left(\frac{\partial}{\partial x_j}\mathbf{e}_j\right) \times \left(u_k\mathbf{e}_k\right) = \frac{\partial u_k}{\partial x_j}\left(\varepsilon_{ijk}\mathbf{e}_i\right) = \varepsilon_{ijk}u_{k,j}\mathbf{e}_i$

Laplacian of a vector: $\nabla^2\mathbf{u} = \frac{\partial^2}{\partial x_i \partial x_i}\left(u_k\mathbf{e}_k\right) = u_{k,ii}\mathbf{e}_k$

Useful Identities

$$\boldsymbol{\nabla}(\phi\psi) = (\boldsymbol{\nabla}\phi)\psi + \phi(\boldsymbol{\nabla}\psi)$$

$$\nabla^2(\phi\psi) = (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi$$

$$\boldsymbol{\nabla} \cdot (\phi\boldsymbol{u}) = \boldsymbol{\nabla}\phi \cdot \boldsymbol{u} + \phi(\boldsymbol{\nabla} \cdot \boldsymbol{u})$$

$$\boldsymbol{\nabla} \times (\phi\boldsymbol{u}) = \boldsymbol{\nabla}\phi \times \boldsymbol{u} + \phi(\boldsymbol{\nabla} \times \boldsymbol{u})$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2\phi$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} = 0$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \nabla^2\boldsymbol{u}$$

$$\boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}$$

- All can be proved with Indicial and Comma Notations.
- $\nabla \times \nabla\phi = \left(\frac{\partial}{\partial x_i}\mathbf{e}_i\right) \times \left(\frac{\partial\phi}{\partial x_j}\mathbf{e}_j\right) = \frac{\partial^2\phi}{\partial x_i \partial x_j}\varepsilon_{ijk}\mathbf{e}_k = \phi_{,ij}\varepsilon_{ijk}\mathbf{e}_k = \phi_{,ji}\varepsilon_{jik}\mathbf{e}_k = -\phi_{,ji}\varepsilon_{ijk}\mathbf{e}_k = -\phi_{,ij}\varepsilon_{ijk}\mathbf{e}_k = 0.$

Sample Problem

- Scalar and vector field functions are given by
  $\phi = x^2 - y^2$ and $\boldsymbol{u} = 2x\boldsymbol{e_1} + 3yz\boldsymbol{e_2} + xy\boldsymbol{e_3}$.
  Calculate the following expressions: $\nabla\phi, \nabla^2\phi, \nabla\cdot\mathbf{u}, \mathbf{u}\overleftarrow{\nabla}, \nabla\times\mathbf{u}$
- Solution

$$\nabla\phi = 2x\mathbf{e_1} - 2y\mathbf{e_2};\quad \nabla^2\phi = 2 - 2 = 0$$

$$\nabla\cdot\mathbf{u} = u_{i,i} = 2 + 3z + 0 = 2 + 3z$$

$$\mathbf{u}\overleftarrow{\nabla} = u_{i,j}\mathbf{e_i}\mathbf{e_j} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3z & 3y \\ y & x & 0 \end{bmatrix}$$

$$\nabla\times\mathbf{u} = \frac{\partial}{\partial x_i}\mathbf{e_i}\times u_j\mathbf{e_j} = \varepsilon_{kij}u_{j,i}\mathbf{e_k}$$

$$= \mathbf{e}_1\left(u_{3,2} - u_{2,3}\right) + \mathbf{e}_2\left(u_{1,3} - u_{3,1}\right) + \mathbf{e}_3\left(u_{2,1} - u_{1,2}\right)$$

$$= \mathbf{e}_1\left(x - 3y\right) + \mathbf{e}_2\left(-y\right)$$

Integral Theorems

- Gauss (Divergence) theorem

$$\iint_S \mathbf{u} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{u} dV;$$

$$\iint_S u_k n_k dS = \iiint_V u_{k,k} dV.$$

- Stokes theorem

$$\int_C \mathbf{u} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS;$$

$$\int_C u_k dx_k = \iint_S \varepsilon_{ijk} u_{k,j} n_i dS.$$

- Green's theorem on a plane

$$\int_C (u_1 dx_1 + u_2 dx_2) = \iint_S (u_{2,1} - u_{1,2}) dx_1 dx_2.$$

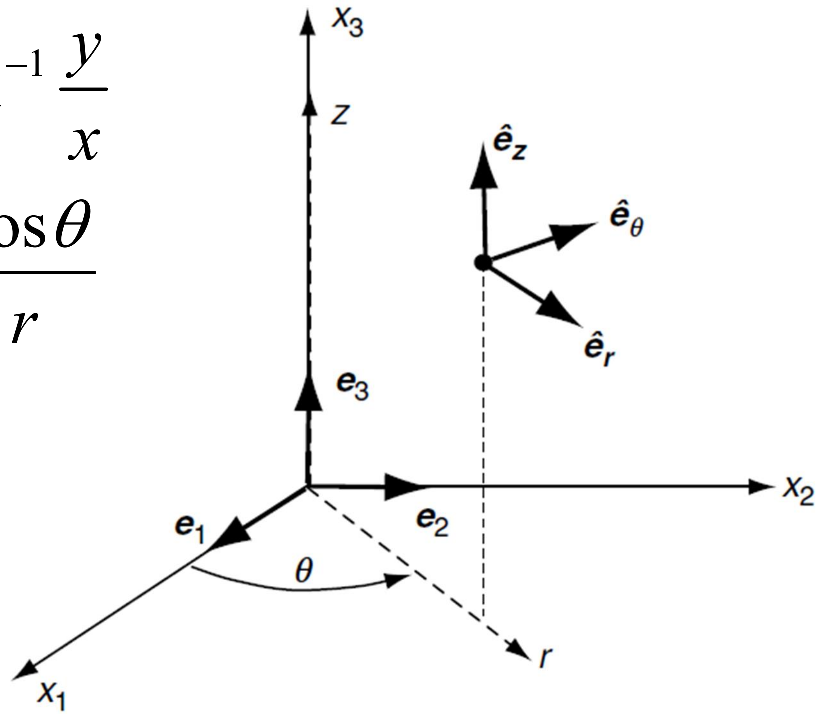
Cylindrical Coordinates

$$x = r\cos\theta, y = r\sin\theta, r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \cos\theta, \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \frac{\partial r}{\partial y} = \sin\theta, \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$



$$\begin{Bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{Bmatrix} = \begin{Bmatrix} \mathbf{e}_x\cos\theta + \mathbf{e}_y\sin\theta \\ -\mathbf{e}_x\sin\theta + \mathbf{e}_y\cos\theta \\ \mathbf{e}_z \end{Bmatrix}$$

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$$

Cylindrical Coordinates

$$\nabla = \mathbf{Q}\nabla = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

$$\boxed{\mathbf{u}\bar{\nabla} = \left(u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta + u_z\mathbf{e}_z\right)\left(\frac{\bar{\partial}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\bar{\partial}}{\partial \theta}\mathbf{e}_\theta + \frac{\bar{\partial}}{\partial z}\mathbf{e}_z\right)}$$

$$= \begin{bmatrix} \dfrac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right)\mathbf{e}_r\mathbf{e}_\theta + \dfrac{\partial u_r}{\partial z}\mathbf{e}_r\mathbf{e}_z \\ +\dfrac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\mathbf{e}_r + \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \dfrac{\partial u_\theta}{\partial z}\mathbf{e}_\theta\mathbf{e}_z \\ +\dfrac{\partial u_z}{\partial r}\mathbf{e}_z\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\mathbf{e}_z\mathbf{e}_\theta + \dfrac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z \end{bmatrix}$$

Cylindrical Coordinates

- Double gradient of a scalar (symmetric as expected)

$$\nabla\phi = \mathbf{e}_r \frac{\partial\phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{e}_z \frac{\partial\phi}{\partial z}$$

$$\Rightarrow \left(\nabla\phi\right)\overleftarrow{\nabla} = \begin{bmatrix} \dfrac{\partial^2\phi}{\partial r^2}\mathbf{e}_r\mathbf{e}_r + \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right)\mathbf{e}_r\mathbf{e}_\theta + \dfrac{\partial^2\phi}{\partial r\,\partial z}\mathbf{e}_r\mathbf{e}_z \\ + \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right)\mathbf{e}_\theta\mathbf{e}_r + \dfrac{1}{r}\left(\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2\phi}{\partial\theta^2}\right)\mathbf{e}_\theta\mathbf{e}_\theta + \dfrac{1}{r}\dfrac{\partial^2\phi}{\partial\theta\,\partial z}\mathbf{e}_\theta\mathbf{e}_z \\ + \dfrac{\partial^2\phi}{\partial r\,\partial z}\mathbf{e}_z\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial^2\phi}{\partial\theta\,\partial z}\mathbf{e}_z\mathbf{e}_\theta + \dfrac{\partial^2\phi}{\partial z^2}\mathbf{e}_z\mathbf{e}_z \end{bmatrix}$$

$$= \nabla\left(\nabla\phi\right)$$

Cylindrical Coordinates

$$\nabla \cdot \mathbf{u} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \right)$$

$$= \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla \times \mathbf{u} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \times \left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z \right)$$

$$= \mathbf{e}_r \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) + \mathbf{e}_\theta \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \mathbf{e}_z \left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{1}{r} u_\theta \right)$$

$$\nabla^2 = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical Coordinates

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_r}{\partial \theta^2} = \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\frac{\partial \mathbf{e}_r}{\partial \theta} = -\mathbf{e}_\theta$$

$$\frac{\partial}{\partial \theta}\left(u_r \mathbf{e}_r\right) = u_r \mathbf{e}_\theta + \mathbf{e}_r \frac{\partial u_r}{\partial \theta}, \frac{\partial^2}{\partial \theta^2}\left(u_r \mathbf{e}_r\right) = \mathbf{e}_r \frac{\partial^2 u_r}{\partial \theta^2} + 2\mathbf{e}_\theta \frac{\partial u_r}{\partial \theta} - u_r \mathbf{e}_r$$

$$\frac{\partial^2}{\partial \theta^2}\left(u_\theta \mathbf{e}_\theta\right) = \mathbf{e}_\theta \frac{\partial^2 u_\theta}{\partial \theta^2} + 2\frac{\partial u_\theta}{\partial \theta}\frac{\partial \mathbf{e}_\theta}{\partial \theta} + u_\theta \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = \mathbf{e}_\theta \frac{\partial^2 u_\theta}{\partial \theta^2} - 2\mathbf{e}_r \frac{\partial u_\theta}{\partial \theta} - u_\theta \mathbf{e}_\theta$$

$$\nabla^2 \mathbf{u} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\left(u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z\right)$$

$$= \left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right)\mathbf{e}_r + \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)\mathbf{e}_\theta + \nabla^2 u_z \mathbf{e}_z$$

Cylindrical Coordinates

$$\sigma\overleftarrow{\nabla} = \begin{pmatrix} \sigma_r \mathbf{e}_r\mathbf{e}_r + \tau_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + \tau_{rz}\mathbf{e}_r\mathbf{e}_z + \tau_{\theta r}\mathbf{e}_\theta\mathbf{e}_r + \sigma_\theta\mathbf{e}_\theta\mathbf{e}_\theta \\ +\tau_{\theta z}\mathbf{e}_\theta\mathbf{e}_z + \tau_{zr}\mathbf{e}_z\mathbf{e}_r + \tau_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + \sigma_z\mathbf{e}_z\mathbf{e}_z \end{pmatrix}\left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z\right)$$

$$\begin{aligned}
= \; & \frac{\partial\sigma_r}{\partial r}\mathbf{e}_r\mathbf{e}_r\mathbf{e}_r + \frac{\partial\tau_{r\theta}}{\partial r}\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_r + \frac{\partial\tau_{rz}}{\partial r}\mathbf{e}_r\mathbf{e}_z\mathbf{e}_r + \frac{\partial\tau_{\theta r}}{\partial r}\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_r + \frac{\partial\sigma_\theta}{\partial r}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_r \\
& + \frac{\partial\tau_{\theta z}}{\partial r}\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_r + \frac{\partial\tau_{zr}}{\partial r}\mathbf{e}_z\mathbf{e}_r\mathbf{e}_r + \frac{\partial\tau_{z\theta}}{\partial r}\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_r + \frac{\partial\sigma_z}{\partial r}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_r \\
& + \left(\frac{1}{r}\frac{\partial\sigma_r}{\partial\theta} - \frac{\tau_{r\theta}+\tau_{\theta r}}{r}\right)\mathbf{e}_r\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r}\right)\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{rz}}{\partial\theta} - \frac{\tau_{\theta z}}{r}\right)\mathbf{e}_r\mathbf{e}_z\mathbf{e}_\theta \\
& + \left(\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r}\right)\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\tau_{r\theta}+\tau_{\theta r}}{r}\right)\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_\theta \\
& + \left(\frac{1}{r}\frac{\partial\tau_{zr}}{\partial\theta} - \frac{\tau_{z\theta}}{r}\right)\mathbf{e}_z\mathbf{e}_r\mathbf{e}_\theta + \left(\frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\tau_{zr}}{r}\right)\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_\theta + \frac{1}{r}\frac{\partial\sigma_z}{\partial\theta}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_\theta \\
& + \frac{\partial\sigma_r}{\partial z}\mathbf{e}_r\mathbf{e}_r\mathbf{e}_z + \frac{\partial\tau_{r\theta}}{\partial z}\mathbf{e}_r\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial\tau_{rz}}{\partial z}\mathbf{e}_r\mathbf{e}_z\mathbf{e}_z + \frac{\partial\tau_{\theta r}}{\partial z}\mathbf{e}_\theta\mathbf{e}_r\mathbf{e}_z + \frac{\partial\sigma_\theta}{\partial z}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_z \\
& + \frac{\partial\tau_{\theta z}}{\partial z}\mathbf{e}_\theta\mathbf{e}_z\mathbf{e}_z + \frac{\partial\tau_{zr}}{\partial z}\mathbf{e}_z\mathbf{e}_r\mathbf{e}_z + \frac{\partial\tau_{z\theta}}{\partial z}\mathbf{e}_z\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial\sigma_z}{\partial z}\mathbf{e}_z\mathbf{e}_z\mathbf{e}_z
\end{aligned}$$

Cylindrical Coordinates

Gradient of a tensor: $\boxed{\sigma\overleftarrow{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\frac{\overleftarrow{\partial}}{\partial x_k}\mathbf{e}_k\right)} = \frac{\partial \sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$

Divergence of a tensor: $\nabla \cdot \sigma = \left(\frac{\partial}{\partial x_k}\mathbf{e}_k\right)\cdot\left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \frac{\partial \sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k \cdot \mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

$\nabla \cdot \sigma =$ contraction on the first and third index of $\sigma\overleftarrow{\nabla}$

$$\Rightarrow \boxed{\begin{aligned}\nabla \cdot \sigma = &\left(\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r}\right)\mathbf{e}_\theta \\ &+ \left(\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_z\end{aligned}}$$

Cylindrical Coordinates

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_r}{\partial \theta^2} = -\mathbf{e}_r, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\mathbf{e}_\theta$$

$$\nabla^2 \sigma = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) \begin{pmatrix} \sigma_r \mathbf{e}_r \mathbf{e}_r + \tau_{r\theta} \mathbf{e}_r \mathbf{e}_\theta + \tau_{rz} \mathbf{e}_r \mathbf{e}_z + \tau_{\theta r} \mathbf{e}_\theta \mathbf{e}_r + \sigma_\theta \mathbf{e}_\theta \mathbf{e}_\theta \\ + \tau_{\theta z} \mathbf{e}_\theta \mathbf{e}_z + \tau_{zr} \mathbf{e}_z \mathbf{e}_r + \tau_{z\theta} \mathbf{e}_z \mathbf{e}_\theta + \sigma_z \mathbf{e}_z \mathbf{e}_z \end{pmatrix}$$

$$= \left[\nabla^2 \sigma_r - \frac{2}{r^2}\left(\sigma_r - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} \right) \right] \mathbf{e}_r \mathbf{e}_r + \left[\nabla^2 \sigma_\theta + \frac{2}{r^2}\left(\sigma_r - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} \right) \right] \mathbf{e}_\theta \mathbf{e}_\theta$$

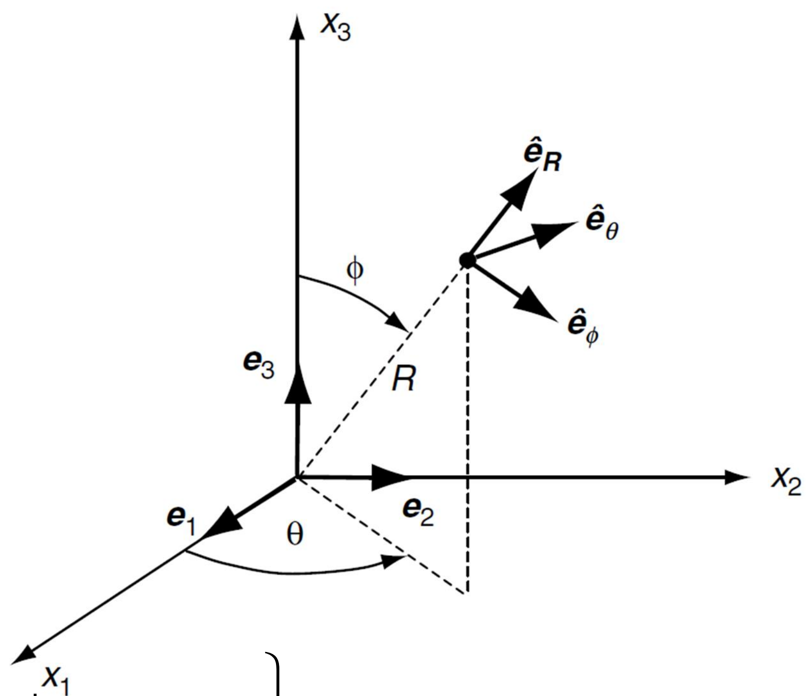
$$+ \left[\nabla^2 \sigma_z \right] \mathbf{e}_z \mathbf{e}_z + \left[\nabla^2 \tau_{r\theta} + \frac{2}{r^2}\left(\frac{\partial \sigma_r}{\partial \theta} - \frac{\partial \sigma_\theta}{\partial \theta} - \tau_{r\theta} - \tau_{\theta r} \right) \right] \mathbf{e}_r \mathbf{e}_\theta + \left[\nabla^2 \tau_{rz} - \frac{1}{r^2}\left(\tau_{rz} + 2\frac{\partial \tau_{\theta z}}{\partial \theta} \right) \right] \mathbf{e}_r \mathbf{e}_z$$

$$+ \left[\nabla^2 \tau_{\theta r} + \frac{2}{r^2}\left(\frac{\partial \sigma_r}{\partial \theta} - \frac{\partial \sigma_\theta}{\partial \theta} - \tau_{r\theta} - \tau_{\theta r} \right) \right] \mathbf{e}_\theta \mathbf{e}_r + \left[\nabla^2 \tau_{\theta z} + \frac{1}{r^2}\left(2\frac{\partial \tau_{rz}}{\partial \theta} - \tau_{\theta z} \right) \right] \mathbf{e}_\theta \mathbf{e}_z$$

$$+ \left[\nabla^2 \tau_{zr} - \frac{1}{r^2}\left(\tau_{zr} + 2\frac{\partial \tau_{z\theta}}{\partial \theta} \right) \right] \mathbf{e}_z \mathbf{e}_r + \left[\nabla^2 \tau_{z\theta} + \frac{1}{r^2}\left(2\frac{\partial \tau_{zr}}{\partial \theta} - \tau_{z\theta} \right) \right] \mathbf{e}_z \mathbf{e}_\theta$$

- The Laplacian is symmetric if the tensor is symmetric.

Spherical Coordinates



$$z = R\cos\varphi, r = R\sin\varphi, R = \sqrt{z^2 + r^2}, \varphi = \tan^{-1}\frac{r}{z}$$

$$\frac{\partial R}{\partial z} = \cos\varphi, \frac{\partial \varphi}{\partial z} = -\frac{\sin\varphi}{R}, \frac{\partial R}{\partial r} = \sin\varphi, \frac{\partial \varphi}{\partial r} = \frac{\cos\varphi}{R}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial R}\frac{\partial R}{\partial z} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial z} = \cos\varphi\frac{\partial}{\partial R} - \frac{\sin\varphi}{R}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial R}\frac{\partial R}{\partial r} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial r} = \sin\varphi\frac{\partial}{\partial R} + \frac{\cos\varphi}{R}\frac{\partial}{\partial \varphi}$$

$$\begin{Bmatrix} \mathbf{e}_R \\ \mathbf{e}_\varphi \\ \mathbf{e}_\theta \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_z \\ \mathbf{e}_r \\ \mathbf{e}_\theta \end{Bmatrix} = \begin{Bmatrix} \sin\varphi\mathbf{e}_r + \cos\varphi\mathbf{e}_z \\ \cos\varphi\mathbf{e}_r - \sin\varphi\mathbf{e}_z \\ \mathbf{e}_\theta \end{Bmatrix}$$

$$\frac{\partial \mathbf{e}_R}{\partial \varphi} = \mathbf{e}_\varphi, \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_R, \qquad \frac{\partial \mathbf{e}_R}{\partial \theta} = \sin\varphi\mathbf{e}_\theta, \frac{\partial \mathbf{e}_\varphi}{\partial \theta} = \cos\varphi\mathbf{e}_\theta,$$

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r = -\sin\varphi\mathbf{e}_R - \cos\varphi\mathbf{e}_\varphi \qquad \left(\mathbf{e}_r = \sin\varphi\mathbf{e}_R + \cos\varphi\mathbf{e}_\varphi\right)$$

Spherical Coordinates

$$\nabla = \mathbf{Q}\nabla = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial R} \\ \dfrac{1}{R}\dfrac{\partial}{\partial \varphi} \\ \dfrac{1}{R\sin\varphi}\dfrac{\partial}{\partial \theta} \end{Bmatrix}$$

$$\boxed{\mathbf{u}\overleftarrow{\nabla} = \left(u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi + u_\theta\mathbf{e}_\theta\right)\left(\frac{\overleftarrow{\partial}}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\overleftarrow{\partial}}{\partial \varphi}\mathbf{e}_\varphi + \frac{1}{R\sin\varphi}\frac{\overleftarrow{\partial}}{\partial \theta}\mathbf{e}_\theta\right)}$$

$$= \left[\begin{aligned} &\frac{\partial u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \left(\frac{1}{R}\frac{\partial u_R}{\partial \varphi} - \frac{u_\varphi}{R}\right)\mathbf{e}_R\mathbf{e}_\varphi + \left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R}\right)\mathbf{e}_R\mathbf{e}_\theta \\ &+\frac{\partial u_\varphi}{\partial R}\mathbf{e}_\varphi\mathbf{e}_R + \left(\frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial \varphi}\right)\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(-\frac{\cot\varphi u_\theta}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial \theta}\right)\mathbf{e}_\varphi\mathbf{e}_\theta \\ &+\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \frac{1}{R}\frac{\partial u_\theta}{\partial \varphi}\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\frac{u_R}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial \theta} + \frac{\cot\varphi u_\varphi}{R}\right)\mathbf{e}_\theta\mathbf{e}_\theta \end{aligned}\right]$$

Spherical Coordinates

$$\nabla\cdot\mathbf{u}=\left(\mathbf{e}_R\frac{\partial}{\partial R}+\mathbf{e}_\varphi\frac{1}{R}\frac{\partial}{\partial\varphi}+\mathbf{e}_\theta\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\right)\cdot\left(u_R\mathbf{e}_R+u_\varphi\mathbf{e}_\varphi+u_\theta\mathbf{e}_\theta\right)$$

$$=\frac{\partial u_R}{\partial R}+\frac{2u_R}{R}+\frac{\cot\varphi u_\varphi}{R}+\frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{u}=\left(\mathbf{e}_R\frac{\partial}{\partial R}+\mathbf{e}_\varphi\frac{1}{R}\frac{\partial}{\partial\varphi}+\mathbf{e}_\theta\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\right)\times\left(u_R\mathbf{e}_R+u_\varphi\mathbf{e}_\varphi+u_\theta\mathbf{e}_\theta\right)$$

$$=\left(-\frac{1}{R}\frac{\partial u_\theta}{\partial\varphi}-\frac{\cot\varphi u_\theta}{R}+\frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial\theta}\right)\mathbf{e}_R$$

$$+\left(-\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial\theta}+\frac{u_\theta}{R}+\frac{\partial u_\theta}{\partial R}\right)\mathbf{e}_\varphi+\left(\frac{1}{R}\frac{\partial u_R}{\partial\varphi}-\frac{u_\varphi}{R}-\frac{\partial u_\varphi}{\partial R}\right)\mathbf{e}_\theta$$

$$\nabla^2=\left(\mathbf{e}_R\frac{\partial}{\partial R}+\mathbf{e}_\varphi\frac{1}{R}\frac{\partial}{\partial\varphi}+\mathbf{e}_\theta\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\right)\cdot\left(\mathbf{e}_R\frac{\partial}{\partial R}+\mathbf{e}_\varphi\frac{1}{R}\frac{\partial}{\partial\varphi}+\mathbf{e}_\theta\frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}\right)$$

$$=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{\cot\varphi}{R^2}\frac{\partial}{\partial\varphi}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{R^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

Spherical Coordinates

$$\frac{\partial \mathbf{e}_R}{\partial \varphi} = \mathbf{e}_\varphi, \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_R, \frac{\partial \mathbf{e}_R}{\partial \theta} = \sin\varphi \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\varphi}{\partial \theta} = \cos\varphi \mathbf{e}_\theta, \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -sin\varphi \mathbf{e}_R - cos\varphi \mathbf{e}_\varphi$$

$$\frac{\partial^2 \mathbf{e}_R}{\partial \varphi^2} = -\mathbf{e}_R, \frac{\partial^2 \mathbf{e}_\varphi}{\partial \varphi^2} = -\mathbf{e}_\varphi$$

$$\frac{\partial^2 \mathbf{e}_R}{\partial \theta^2} = -\sin^2 \varphi \mathbf{e}_R - \sin\varphi \cos\varphi \mathbf{e}_\varphi, \frac{\partial^2 \mathbf{e}_\varphi}{\partial \theta^2} = -\sin\varphi \cos\varphi \mathbf{e}_R - \cos^2 \varphi \mathbf{e}_\varphi, \frac{\partial^2 \mathbf{e}_\theta}{\partial \theta^2} = -\mathbf{e}_\theta$$

$$\nabla^2 \mathbf{u} = \left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{\cot\varphi}{R^2}\frac{\partial}{\partial \varphi} + \frac{1}{R^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{R^2 \sin^2 \varphi}\frac{\partial^2}{\partial \theta^2} \right)\left(u_R \mathbf{e}_R + u_\varphi \mathbf{e}_\varphi + u_\theta \mathbf{e}_\theta \right)$$

$$= \left(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2\cot\varphi u_\varphi}{R^2} - \frac{2}{R^2}\frac{\partial u_\varphi}{\partial \varphi} - \frac{2}{R^2 \sin\varphi}\frac{\partial u_\theta}{\partial \theta} \right)\mathbf{e}_R$$

$$+ \left(\nabla^2 u_\varphi + \frac{2}{R^2}\frac{\partial u_R}{\partial \varphi} - \frac{u_\varphi}{R^2 \sin^2 \varphi} - \frac{2\cot\varphi}{R^2 \sin\varphi}\frac{\partial u_\theta}{\partial \theta} \right)\mathbf{e}_\varphi$$

$$+ \left(\nabla^2 u_\theta - \frac{u_\theta}{R^2 \sin^2 \varphi} + \frac{2}{R^2 \sin\varphi}\frac{\partial u_R}{\partial \theta} + \frac{2\cot\varphi}{R^2 \sin\varphi}\frac{\partial u_\varphi}{\partial \theta} \right)\mathbf{e}_\theta$$

Spherical Coordinates

$$\sigma\bar{\nabla} = \begin{pmatrix} \sigma_R \mathbf{e}_R \mathbf{e}_R + \tau_{R\varphi} \mathbf{e}_R \mathbf{e}_\varphi + \tau_{R\theta} \mathbf{e}_R \mathbf{e}_\theta + \tau_{\varphi R} \mathbf{e}_\varphi \mathbf{e}_R + \sigma_\varphi \mathbf{e}_\varphi \mathbf{e}_\varphi \\ +\tau_{\varphi\theta} \mathbf{e}_\varphi \mathbf{e}_\theta + \tau_{\theta R} \mathbf{e}_\theta \mathbf{e}_R + \tau_{\theta\varphi} \mathbf{e}_\theta \mathbf{e}_\varphi + \sigma_\theta \mathbf{e}_\theta \mathbf{e}_\theta \end{pmatrix} \left(\frac{\partial}{\partial R} \mathbf{e}_R + \frac{1}{R}\frac{\partial}{\partial \varphi} \mathbf{e}_\varphi + \frac{1}{R\sin\varphi}\frac{\partial}{\partial \theta} \mathbf{e}_\theta \right)$$

$$= \boxed{\begin{aligned} &\frac{\partial \sigma_R}{\partial R}\mathbf{e}_R\mathbf{e}_R\mathbf{e}_R + \frac{\partial \tau_{R\varphi}}{\partial R}\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial \tau_{R\theta}}{\partial R}\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial \tau_{\varphi R}}{\partial R}\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_R + \frac{\partial \sigma_\varphi}{\partial R}\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial \tau_{\varphi\theta}}{\partial R}\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial \tau_{\theta R}}{\partial R}\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_R \\ &+\frac{\partial \tau_{\theta\varphi}}{\partial R}\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_R + \frac{\partial \sigma_\theta}{\partial R}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_R + \left(\frac{1}{R}\frac{\partial \sigma_R}{\partial \varphi} - \frac{\tau_{R\varphi}+\tau_{\varphi R}}{R}\right)\mathbf{e}_R\mathbf{e}_R\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial \tau_{R\varphi}}{\partial \varphi} + \frac{\sigma_R-\sigma_\varphi}{R}\right)\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \varphi} - \frac{\tau_{\varphi\theta}}{R}\right)\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_\varphi \\ &+\left(\frac{1}{R}\frac{\partial \tau_{\varphi R}}{\partial \varphi} + \frac{\sigma_R-\sigma_\varphi}{R}\right)\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{\tau_{R\varphi}+\tau_{\varphi R}}{R}\right)\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial \tau_{\varphi\theta}}{\partial \varphi} + \frac{\tau_{R\theta}}{R}\right)\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\frac{1}{R}\frac{\partial \tau_{\theta R}}{\partial \varphi} - \frac{\tau_{\theta\varphi}}{R}\right)\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_\varphi \\ &+\left(\frac{1}{R}\frac{\partial \tau_{\theta\varphi}}{\partial \varphi} + \frac{\tau_{\theta R}}{R}\right)\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_\varphi + \frac{1}{R}\frac{\partial \sigma_\theta}{\partial \varphi}\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\frac{1}{R\sin\varphi}\frac{\partial \sigma_R}{\partial \theta} - \frac{\tau_{R\theta}+\tau_{\theta R}}{R}\right)\mathbf{e}_R\mathbf{e}_R\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{R\varphi}}{\partial \theta} - \frac{\cot\varphi\tau_{R\theta}+\tau_{\theta\varphi}}{R}\right)\mathbf{e}_R\mathbf{e}_\varphi\mathbf{e}_\theta \\ &+\left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{\sigma_R-\sigma_\theta+\cot\varphi\tau_{R\varphi}}{R}\right)\mathbf{e}_R\mathbf{e}_\theta\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{\varphi R}}{\partial \theta} - \frac{\cot\varphi\tau_{\theta R}+\tau_{\varphi\theta}}{R}\right)\mathbf{e}_\varphi\mathbf{e}_R\mathbf{e}_\theta \\ &+\left(\frac{1}{R\sin\varphi}\frac{\partial \sigma_\varphi}{\partial \theta} - \frac{\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R}\right)\mathbf{e}_\varphi\mathbf{e}_\varphi\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{\varphi\theta}}{\partial \theta} - \frac{\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)-\tau_{\varphi R}}{R}\right)\mathbf{e}_\varphi\mathbf{e}_\theta\mathbf{e}_\theta \\ &+\left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{\theta R}}{\partial \theta} + \frac{\sigma_R-\sigma_\theta}{R} + \frac{\cot\varphi\tau_{\varphi R}}{R}\right)\mathbf{e}_\theta\mathbf{e}_R\mathbf{e}_\theta + \left(\frac{1}{R\sin\varphi}\frac{\partial \tau_{\theta\varphi}}{\partial \theta} + \frac{\tau_{R\varphi}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R}\right)\mathbf{e}_\theta\mathbf{e}_\varphi\mathbf{e}_\theta \\ &+\left(\frac{1}{R\sin\varphi}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\tau_{R\theta}+\tau_{\theta R}+\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R}\right)\mathbf{e}_\theta\mathbf{e}_\theta\mathbf{e}_\theta \end{aligned}}$$

Spherical Coordinates

Gradient of a tensor: $\boxed{\sigma\overleftarrow{\nabla} = \left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right)\left(\frac{\overleftarrow{\partial}}{\partial x_k}\mathbf{e}_k\right)} = \frac{\partial \sigma_{ij}}{\partial x_k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \sigma_{ij,k}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$

Divergence of a tensor: $\nabla\cdot\sigma = \left(\frac{\partial}{\partial x_k}\mathbf{e}_k\right)\cdot\left(\sigma_{ij}\mathbf{e}_i\mathbf{e}_j\right) = \frac{\partial \sigma_{ij}}{\partial x_k}\left(\mathbf{e}_k\cdot\mathbf{e}_i\right)\mathbf{e}_j = \sigma_{ij,i}\mathbf{e}_j$

$\nabla\cdot\sigma =$ contraction on the first and third index of $\sigma\overleftarrow{\nabla}$

$$\Rightarrow \boxed{\begin{aligned}\nabla\cdot\sigma =& \left(\frac{\partial\sigma_R}{\partial R} + \frac{1}{R}\frac{\partial\tau_{\varphi R}}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta R}}{\partial\theta} + \frac{2\sigma_R - \sigma_\theta - \sigma_\varphi + \cot\varphi\,\tau_{\varphi R}}{R}\right)\mathbf{e}_R \\ &+\left(\frac{\partial\tau_{R\varphi}}{\partial R} + \frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta} + \frac{2\tau_{R\varphi} + \tau_{\varphi R} - \cot\varphi\left(\sigma_\theta - \sigma_\varphi\right)}{R}\right)\mathbf{e}_\varphi \\ &+\left(\frac{\partial\tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi} + \frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{R\theta} + \tau_{\theta R} + \cot\varphi\left(\tau_{\theta\varphi} + \tau_{\varphi\theta}\right)}{R}\right)\mathbf{e}_\theta\end{aligned}}$$

Outline

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- Kronecker Delta
- Levi-Civita symbol
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- Principal Values and Directions
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