# Hyper-elastic Materials

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# Outline

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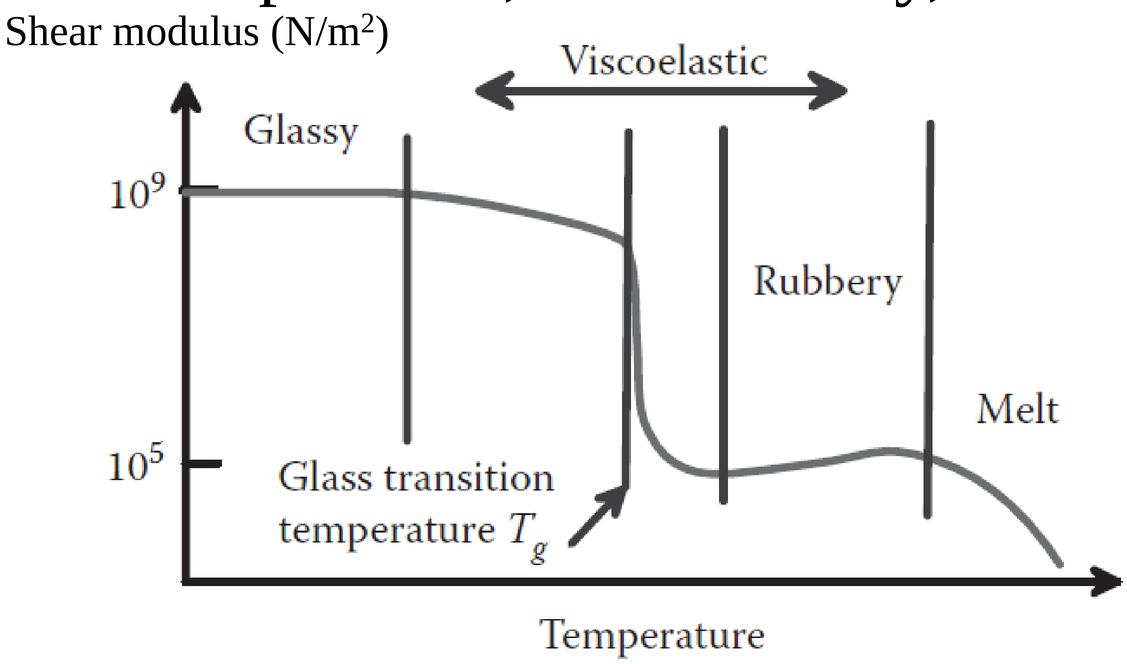
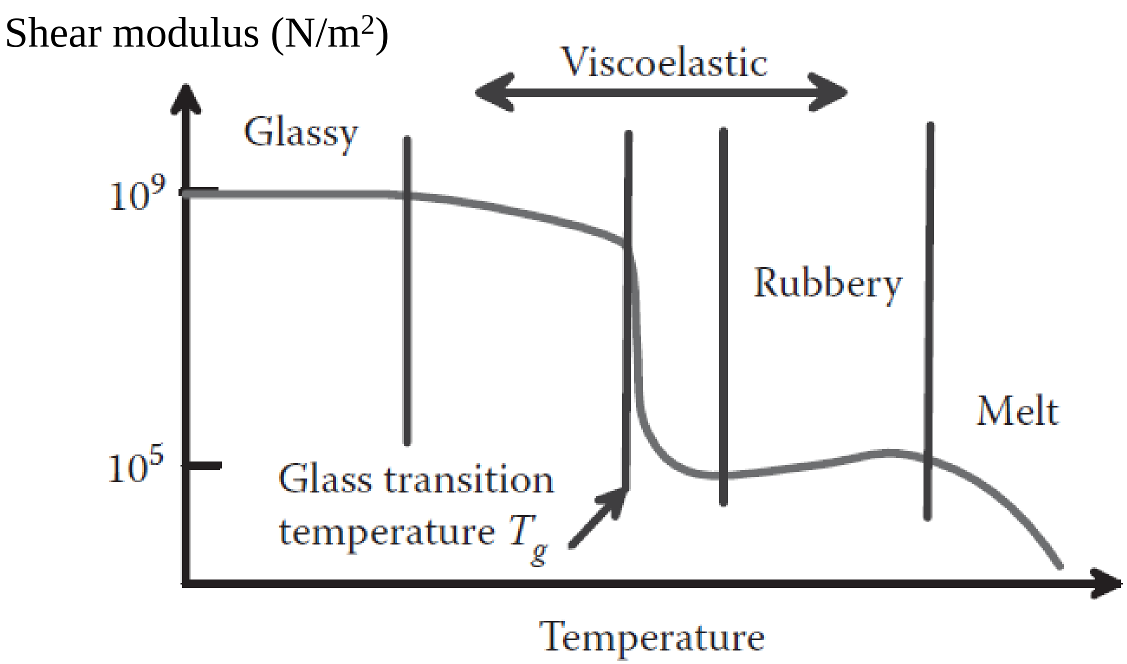
# Introduction

- Main applications of the theory are (1) to model the rubbery behavior of a polymeric material and (2) to model polymeric foams that can be subjected to large reversible shape changes (e.g., a sponge).
- In general, the response of a typical polymer is strongly dependent on temperature, strain history, and loading rate.



# Introduction

- Rubbery behavior: the response is elastic, the stress does not depend strongly on strain rate or strain history, and the modulus increases with temperature.
- Heavily cross-linked polymers (elastomers) are the most likely to show ideal rubbery behavior.
- Hyperelastic constitutive laws are intended to approximate this rubbery behavior.



# Mechanical Behavior of Rubbers

- Features of the behavior of a solid rubber:
  - The material is close to ideally elastic.
  - The material strongly resists volume changes. The bulk modulus is comparable with that of metals.
  - The material is very compliant in shear: shear modulus is of the order of $10^{-5}$ times that of most metals.
  - The material is isotropic.
  - The shear modulus is temperature dependent: the material becomes stiffer as it is heated, in sharp contrast to metals.

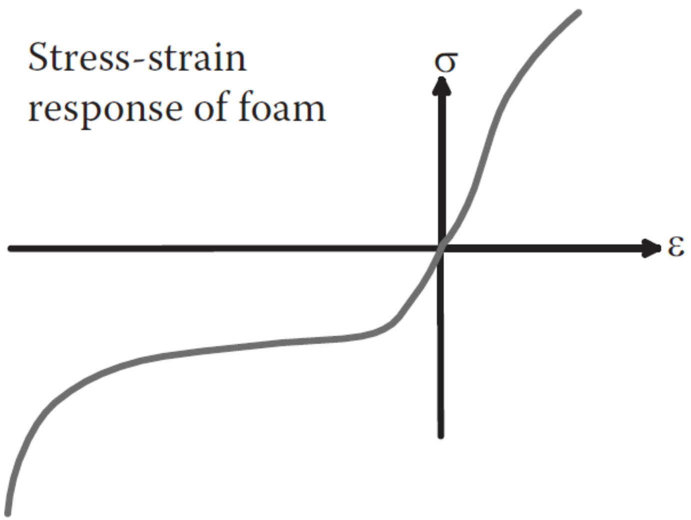
# Mechanical Behavior of Polymeric Foams

- Polymeric foams:

➢ Polymeric foams are close to reversible and show little rate or history dependence.

➢ In contrast to rubbers, most foams are highly compressible; bulk and shear moduli are comparable.

➢ Foams have a complicated true stress-true strain response. The finite strain response of the foam in compression is quite different from that in tension because of buckling in the cell walls.




➢ Foams can be anisotropic depending on their cell structure. Foams with a random cell structure are isotropic.

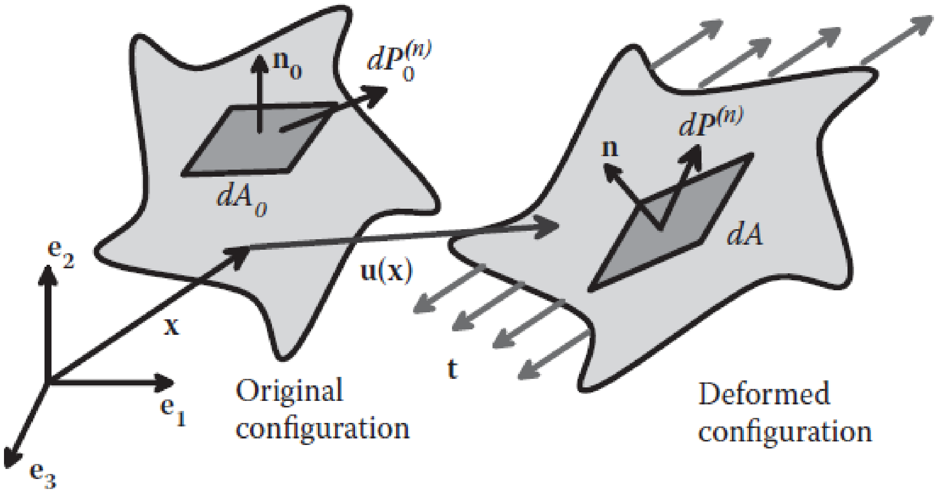
# Strain Measure

- Define the stress-strain relation for the solid by specifying its strain energy density as a function of deformation gradient tensor: $W = W(\mathbf{F})$. The general form of the strain energy density is guided by experiment.
- If $W$ is a function of the left Cauchy-Green deformation tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$, the constitutive equation is automatically isotropic.
- Invariants of **B**:

$$I_1 = B_{kk}$$

$$I_2 = \frac{1}{2}\left(B_{ii}B_{kk} - B_{ik}B_{ki}\right) = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right)$$

$$I_3 = \det\left[B_{ij}\right] = J^2$$



# Strain Measure

- An alternative set of invariants of **B** more convenient for models of nearly incompressible materials

$$\bar{I}_1 = \frac{I_1}{J^{2/3}} = \frac{B_{kk}}{J^{2/3}}$$

$$\bar{I}_2 = \frac{I_2}{J^{4/3}} = \frac{1}{2J^{4/3}}\left(I_1^2 - B_{ik}B_{ki}\right) = \frac{1}{2}\left(\bar{I}_1^2 - \frac{B_{ik}B_{ki}}{J^{4/3}}\right)$$

$$I_3 = \det\left[B_{ij}\right] = J^2$$

- Note that the first two invariants remain constant under a pure volume change.

$$\left[B_{ij}\right] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \Rightarrow I_1 = B_{kk} = 3\lambda^2,$$

$$I_2 = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right) = 3\lambda^4, \quad I_3 = \det\left[B_{ij}\right] = J^2 = \lambda^6$$

$$\Rightarrow \boxed{\bar{I}_1 = \frac{I_1}{J^{2/3}} = 3, \quad \bar{I}_2 = \frac{I_2}{J^{4/3}} = 3.}$$

- Principal stretches and principal directions

$$\mathbf{B} = \lambda_1^2 \mathbf{b}_1 \otimes \mathbf{b}_1 + \lambda_2^2 \mathbf{b}_2 \otimes \mathbf{b}_2 + \lambda_3^2 \mathbf{b}_3 \otimes \mathbf{b}_3, \quad B_{ij} = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix}$$

# Stress Measure and General Constitutive Law

- Stress measure: $dP_j^{(\mathbf{n})} = dA n_i \sigma_{ij}$

- Strain energy density:

$$W(\mathbf{F}) = U(\mathbf{B}) = U(I_1, I_2, I_3) = \bar{U}(\bar{I}_1, \bar{I}_2, \bar{I}_3) = \tilde{U}(\lambda_1, \lambda_2, \lambda_3)$$

- Cauchy stress in terms of deformation gradient **F**

$$\dot{W} = \iint_S T_i^n v_i dS + \iiint_V F_i v_i dV = \iiint_{V_0} S_{ij} \dot{F}_{ji} d V_0 + \frac{d}{dt} \iiint_{V_0} \frac{1}{2} \rho_0 v_i v_i d V_0$$

$$\boxed{\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \mathbf{S} \quad \Rightarrow \sigma_{ij} = \frac{1}{J} F_{ik} S_{kj} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial F_{jk}}}$$

# General Constitutive Law

- Cauchy stress in terms of invariants of **B**

$$\sigma_{ij} = \frac{1}{J} F_{ik} S_{kj} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial F_{jk}} = \frac{1}{\sqrt{I_3}} F_{ik} \left( \frac{\partial U}{\partial I_1} \frac{\partial I_1}{\partial F_{jk}} + \frac{\partial U}{\partial I_2} \frac{\partial I_2}{\partial F_{jk}} + \frac{\partial U}{\partial I_3} \frac{\partial I_3}{\partial F_{jk}} \right)$$

- Derivatives of **B** w.r.t. **F**:

$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T, B_{pk} = F_{pm} F_{km} \quad \Rightarrow B_{kk} = F_{km} F_{km} \quad \Rightarrow \frac{\partial B_{kk}}{\partial F_{ij}} = 2F_{km} \frac{\partial F_{km}}{\partial F_{ij}} = 2F_{ij}$$

$$\Rightarrow \frac{\partial B_{pk}}{\partial F_{ij}} = \frac{\partial F_{pm}}{\partial F_{ij}} F_{km} + F_{pm} \frac{\partial F_{km}}{\partial F_{ij}} = \delta_{pi} \delta_{mj} F_{km} + F_{pm} \delta_{ki} \delta_{mj} = \delta_{pi} F_{kj} + F_{pj} \delta_{ki}$$

$$(p \to i) \Rightarrow \frac{\partial B_{ik}}{\partial F_{ij}} = \underbrace{\delta_{ii}}_{=1} F_{kj} + F_{ij} \delta_{ki} = 2F_{kj}$$

# General Constitutive Law

- Derivatives of invariants w.r.t. **F**:

$$I_1 = B_{kk}, I_2 = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right) = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ik}\right), I_3 = \det\left[B_{ij}\right] = J^2$$

$$\Rightarrow \begin{cases} \dfrac{\partial I_1}{\partial F_{ij}} = \dfrac{\partial B_{kk}}{\partial F_{ij}} = 2F_{ij}, \qquad \dfrac{\partial I_3}{\partial F_{ij}} = \dfrac{\partial\left(J^2\right)}{\partial F_{ij}} = 2J\dfrac{\partial J}{\partial F_{ij}} = 2J\left(JF_{ji}^{-1}\right) = 2I_3F_{ji}^{-1} \\ \dfrac{\partial I_2}{\partial F_{ij}} = \dfrac{1}{2}\left\{2I_1\dfrac{\partial I_1}{\partial F_{ij}} - 2B_{ik}\dfrac{\partial B_{ik}}{\partial F_{ij}}\right\} = \dfrac{1}{2}\left\{2I_1\left(2F_{ij}\right) - 2B_{ik}\left(2F_{kj}\right)\right\} = 2\left\{I_1F_{ij} - B_{ik}F_{kj}\right\} \end{cases}$$

- Cauchy stress in terms of invariants of **B**

$$\sigma_{ij} = \frac{1}{J}F_{ik}S_{kj} = \frac{1}{J}F_{ik}\frac{\partial W}{\partial F_{jk}} = \frac{1}{\sqrt{I_3}}F_{ik}\left\{\frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial F_{jk}} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial F_{jk}} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial F_{jk}}\right\}$$

$$= \frac{1}{\sqrt{I_3}}F_{ik}\left\{\frac{\partial U}{\partial I_1}\left(2F_{jk}\right) + \frac{\partial U}{\partial I_2}\left(2\left\{I_1F_{jk} - B_{jm}F_{mk}\right\}\right) + \frac{\partial U}{\partial I_3}\left(2I_3F_{kj}^{-1}\right)\right\}$$

$$= \frac{2}{\sqrt{I_3}}\left\{\frac{\partial U}{\partial I_1}\left(F_{ik}F_{jk}\right) + \frac{\partial U}{\partial I_2}\left(I_1F_{ik}F_{jk} - B_{jm}F_{ik}F_{mk}\right) + \frac{\partial U}{\partial I_3}\left(I_3F_{ik}F_{kj}^{-1}\right)\right\}$$

$$\Rightarrow \boxed{\sigma_{ij} = \frac{2}{\sqrt{I_3}}\left\{B_{ij}\frac{\partial U}{\partial I_1} + \left(I_1B_{ij} - B_{im}B_{jm}\right)\frac{\partial U}{\partial I_2}\right\} + 2\sqrt{I_3}\delta_{ij}\frac{\partial U}{\partial I_3}}$$

# General Constitutive Law

- Cauchy stress in terms of alternative invariants of **B**

$$\bar{I}_1 = \frac{I_1}{J^{2/3}}, \bar{I}_2 = \frac{I_2}{J^{4/3}}, I_3 = J^2, \qquad \frac{\partial J}{\partial F_{ij}} = JF_{ji}^{-1}, \qquad \frac{\partial I_1}{\partial F_{ij}} = 2F_{ij}, \frac{\partial I_2}{\partial F_{ij}} = 2\left(I_1 F_{ij} - B_{ik}F_{kj}\right), \frac{\partial I_3}{\partial F_{ij}} = 2I_3 F_{ji}^{-1}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \bar{I}_1}{\partial F_{ij}} = \dfrac{1}{J^{2/3}}\dfrac{\partial I_1}{\partial F_{ij}} - \dfrac{2}{3J^{5/3}}\dfrac{\partial J}{\partial F_{ij}} I_1 = \dfrac{1}{J^{2/3}}\left(2F_{ij}\right) - \dfrac{2}{3J^{5/3}}\left(JF_{ji}^{-1}\right)I_1 = \dfrac{2}{J^{2/3}}F_{ij} - \dfrac{2}{3}\bar{I}_1 F_{ji}^{-1} \\ \dfrac{\partial \bar{I}_2}{\partial F_{ij}} = \dfrac{1}{J^{4/3}}\dfrac{\partial I_2}{\partial F_{ij}} - \dfrac{4}{3J^{7/3}}\dfrac{\partial J}{\partial F_{ij}} I_2 = \dfrac{1}{J^{4/3}}2\left(I_1 F_{ij} - B_{ik}F_{kj}\right) - \dfrac{4}{3J^{7/3}}JF_{ji}^{-1}I_2 = \dfrac{2}{J^{2/3}}\bar{I}_1 F_{ij} - \dfrac{2}{J^{4/3}}B_{ik}F_{kj} - \dfrac{4}{3}\bar{I}_2 F_{ji}^{-1} \end{cases}$$

$$\sigma_{ij} = \frac{1}{J}F_{ik}S_{kj} = \frac{1}{J}F_{ik}\frac{\partial W}{\partial F_{jk}} = \frac{1}{J}F_{ik}\left(\frac{\partial \bar{U}}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial F_{jk}} + \frac{\partial \bar{U}}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial F_{jk}} + \frac{\partial \bar{U}}{\partial J}\frac{\partial J}{\partial F_{jk}}\right)$$

$$= \frac{1}{J}F_{ik}\left\{\frac{\partial \bar{U}}{\partial \bar{I}_1}\left(\frac{2}{J^{2/3}}F_{jk} - \frac{2}{3}\bar{I}_1 F_{kj}^{-1}\right) + \frac{\partial \bar{U}}{\partial \bar{I}_2}\left(\frac{2}{J^{2/3}}\bar{I}_1 F_{jk} - \frac{2}{J^{4/3}}B_{jm}F_{mk} - \frac{4}{3}\bar{I}_2 F_{kj}^{-1}\right) + \frac{\partial \bar{U}}{\partial J}\left(JF_{kj}^{-1}\right)\right\}$$

$$\Rightarrow \boxed{\sigma_{ij} = \left\{\frac{\partial \bar{U}}{\partial \bar{I}_1}\left(\frac{2}{J^{5/3}}B_{ij} - \frac{2}{3J}\bar{I}_1\delta_{ij}\right) + \frac{\partial \bar{U}}{\partial \bar{I}_2}\left(\frac{2}{J^{5/3}}\bar{I}_1 B_{ij} - \frac{2}{J^{7/3}}B_{im}B_{jm} - \frac{4}{3J}\bar{I}_2\delta_{ij}\right) + \frac{\partial \bar{U}}{\partial J}\delta_{ij}\right\}}$$

$$\Rightarrow \boxed{\sigma_{ij} = \frac{2}{J^{5/3}}\frac{\partial \bar{U}}{\partial \bar{I}_1}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + \frac{2}{J^{7/3}}\frac{\partial \bar{U}}{\partial \bar{I}_2}\left(B_{kk}B_{ij} - B_{im}B_{jm} - \frac{1}{3}\left(I_1^2 - B_{mn}B_{mn}\right)\delta_{ij}\right) + \frac{\partial \bar{U}}{\partial J}\delta_{ij}}$$

# General Constitutive Law

- Cauchy stress in terms of principal values of **B**

$$\mathbf{B} = \lambda_1^2 \mathbf{b}_1 \otimes \mathbf{b}_1 + \lambda_2^2 \mathbf{b}_2 \otimes \mathbf{b}_2 + \lambda_3^2 \mathbf{b}_3 \otimes \mathbf{b}_3, \quad B_{ij} = \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} I_1 = B_{kk} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 = \dfrac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right) = \dfrac{1}{2}\left\{\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^2 - \left(\lambda_1^4 + \lambda_2^4 + \lambda_3^4\right)\right\} = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 = \det\left[B_{ij}\right] = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2 \end{cases}$$

$$\Rightarrow \frac{\partial I_1}{\partial \lambda_i} = 2\lambda_i, \quad \frac{\partial I_2}{\partial \lambda_i} = \frac{\partial I_2}{\partial \lambda_i^2}\frac{\partial \lambda_i^2}{\partial \lambda_i} = 2\lambda_i\left(I_1 - \lambda_i^2\right), \quad \frac{\partial I_3}{\partial \lambda_i} = \frac{\partial I_3}{\partial \lambda_i^2}\frac{\partial \lambda_i^2}{\partial \lambda_i} = 2\lambda_i \frac{I_3}{\lambda_i^2}$$

$$W\left(\mathbf{F}\right) = U\left(\mathbf{B}\right) = U\left(I_1, I_2, I_3\right) = \tilde{U}\left(\lambda_1, \lambda_2, \lambda_3\right)$$

$$\Rightarrow \frac{\partial \tilde{U}}{\partial \lambda_i} = \frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial \lambda_i} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial \lambda_i} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial \lambda_i} = 2\lambda_i\left\{\frac{\partial U}{\partial I_1} + \left(I_1 - \lambda_i^2\right)\frac{\partial U}{\partial I_2} + \frac{I_3}{\lambda_i^2}\frac{\partial U}{\partial I_3}\right\}$$

# General Constitutive Law

$$\frac{\partial \tilde{U}}{\partial \lambda_i} = \frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial \lambda_i} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial \lambda_i} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial \lambda_i} = 2\lambda_i \left\{ \frac{\partial U}{\partial I_1} + \left(I_1 - \lambda_i^2\right)\frac{\partial U}{\partial I_2} + \frac{I_3}{\lambda_i^2}\frac{\partial U}{\partial I_3} \right\}$$

$$\sigma_{ij} = \frac{1}{J}F_{ik}S_{kj} = \frac{1}{J}F_{ik}\frac{\partial W}{\partial F_{jk}} = \frac{2}{\sqrt{I_3}}\left\{ B_{ij}\frac{\partial U}{\partial I_1} + \left(I_1 B_{ij} - B_{im}B_{jm}\right)\frac{\partial U}{\partial I_2} + I_3\frac{\partial U}{\partial I_3}\delta_{ij} \right\}$$

$$\Rightarrow \sigma_{11} = \frac{2}{\sqrt{I_3}}\left\{ \lambda_1^2\frac{\partial U}{\partial I_1} + \left(I_1\lambda_1^2 - \lambda_1^4\right)\frac{\partial U}{\partial I_2} + I_3\frac{\partial U}{\partial I_3} \right\} = \frac{2\lambda_1^2}{\sqrt{I_3}}\left\{ \frac{\partial U}{\partial I_1} + \left(I_1 - \lambda_1^2\right)\frac{\partial U}{\partial I_2} + \frac{I_3}{\lambda_1^2}\frac{\partial U}{\partial I_3} \right\} = \frac{\lambda_1}{\sqrt{I_3}}\frac{\partial \tilde{U}}{\partial \lambda_i}$$

$$\Rightarrow \sigma_{11} = \frac{\lambda_1}{\sqrt{I_3}}\frac{\partial \tilde{U}}{\partial \lambda_1}, \quad \sigma_{22} = \frac{\lambda_2}{\sqrt{I_3}}\frac{\partial \tilde{U}}{\partial \lambda_2}, \quad \sigma_{33} = \frac{\lambda_3}{\sqrt{I_3}}\frac{\partial \tilde{U}}{\partial \lambda_3}$$

$$\boxed{\sigma_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\frac{\partial \tilde{U}}{\partial \lambda_i}, \quad \sigma_{ij} = \frac{1}{\lambda_1\lambda_2\lambda_3}\begin{bmatrix} \lambda_1\frac{\partial \tilde{U}}{\partial \lambda_1} & & \\ & \lambda_2\frac{\partial \tilde{U}}{\partial \lambda_2} & \\ & & \lambda_3\frac{\partial \tilde{U}}{\partial \lambda_3} \end{bmatrix}}$$

# Incompressibility

- The preceding formulas assume some compressibility.
- Most rubbers strongly resist volume changes, and, it is sometimes convenient to approximate them as perfectly incompressible.

$$J=1 \quad \Rightarrow \bar{I}_1=\frac{I_1}{J^{2/3}}=I_1, \quad \bar{I}_2=\frac{I_2}{J^{4/3}}=I_2=\frac{1}{2}\left(I_1^2-B_{ik}B_{ki}\right), \quad I_3=\det\left[B_{ij}\right]=J^2=1$$

$$W(\mathbf{F})=U(\mathbf{B})=U(I_1,I_2) \qquad \Rightarrow \sigma_{ij}=\frac{2}{\sqrt{I_3}}\left\{B_{ij}\frac{\partial U}{\partial I_1}+\left(I_1B_{ij}-B_{im}B_{jm}\right)\frac{\partial U}{\partial I_2}\right\}+\cancel{2\sqrt{I_3}\delta_{ij}\frac{\partial U}{\partial I_3}}$$

$$\Rightarrow \sigma_{kk}=2\left\{B_{kk}\frac{\partial U}{\partial I_1}+\left(I_1B_{kk}-B_{km}B_{km}\right)\frac{\partial U}{\partial I_2}\right\}=2\left\{I_1\frac{\partial U}{\partial I_1}+2I_2\frac{\partial U}{\partial I_2}\right\}$$

$$\sigma_{ij}=\sigma'_{ij}+\tilde{\sigma}_{ij}=\left(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}\right)+p\delta_{ij}=2\left\{\left(B_{ij}-\frac{\delta_{ij}}{3}I_1\right)\frac{\partial U}{\partial I_1}+\left(I_1B_{ij}-B_{im}B_{jm}-\frac{2I_2}{3}\delta_{ij}\right)\frac{\partial U}{\partial I_2}\right\}+p\delta_{ij}$$

$$=2\left\{\left(\frac{\partial U}{\partial I_1}+I_1\frac{\partial U}{\partial I_2}\right)B_{ij}-\left(I_1\frac{\partial U}{\partial I_1}+2I_2\frac{\partial U}{\partial I_2}\right)\frac{\delta_{ij}}{3}-B_{im}B_{jm}\frac{\partial U}{\partial I_2}\right\}+p\delta_{ij}$$

- The hydrostatic stress $p$ is an unknown variable, which must be calculated by solving the equilibrium equations and BCs.

# Polynomial Models for Rubber Elasticity

- Limit of alternative strain invariants at small deformations:

$$J = \lambda_1 \lambda_2 \lambda_3 \to 1 \quad \Rightarrow \bar{I}_1 = \frac{I_1}{J^{2/3}} = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{J^{2/3}} \to 3, \;\; \bar{I}_2 = \frac{I_2}{J^{4/3}} = \frac{\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2}{J^{4/3}} \to 3$$

- **Generalized polynomial rubber elasticity potential**

$$\bar{U}\left(\bar{I}_1, \bar{I}_2, J\right) = \sum_{i+j=1}^{N} C_{ij} \left(\bar{I}_1 - 3\right)^i \left(\bar{I}_2 - 3\right)^j + \sum_{i=1}^{N} \frac{K_i}{2} \left(J - 1\right)^{2i}$$

- **Generalized Mooney-Revlin model**

$$N = 1 \quad \Rightarrow \bar{U}\left(\bar{I}_1, \bar{I}_2, J\right) = C_{10}\left(\bar{I}_1 - 3\right) + C_{01}\left(\bar{I}_2 - 3\right) + \frac{K_1}{2}\left(J - 1\right)^2$$

$$\Rightarrow \frac{\partial \bar{U}}{\partial \bar{I}_1} = C_{10}, \qquad \frac{\partial \bar{U}}{\partial \bar{I}_2} = C_{01}, \qquad \frac{\partial \bar{U}}{\partial J} = K_1\left(J - 1\right)$$

$$\sigma_{ij} = \frac{2}{J^{5/3}} \frac{\partial \bar{U}}{\partial \bar{I}_1} \left( B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right) + \frac{2}{J^{7/3}} \frac{\partial \bar{U}}{\partial \bar{I}_2} \left( B_{kk} B_{ij} - B_{im} B_{jm} - \frac{1}{3} \left( I_1^2 - B_{mn} B_{mn} \right) \delta_{ij} \right) + \frac{\partial \bar{U}}{\partial J} \delta_{ij}$$

$$\Rightarrow \boxed{\sigma_{ij} = \frac{2C_{10}}{J^{5/3}} \left( B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right) + \frac{2C_{01}}{J^{7/3}} \left( B_{kk} B_{ij} - B_{im} B_{jm} - \frac{1}{3} \left( I_1^2 - B_{mn} B_{mn} \right) \delta_{ij} \right) + K_1 \left(J - 1\right) \delta_{ij}}$$

# Polynomial Models for Rubber Elasticity

- Physical interpretation $K_1$

$$[B]=\begin{bmatrix}\lambda^2 & 0 & 0\\ 0 & \lambda^2 & 0\\ 0 & 0 & \lambda^2\end{bmatrix}, \qquad \Rightarrow I_1=B_{kk}=3\lambda^2,\ \ I_2=\frac{1}{2}\left(I_1^2-B_{ik}B_{ki}\right)=\frac{1}{2}\left(\left(3\lambda^2\right)^2-3\lambda^4\right)=3\lambda^4,$$

$$I_3=\det\left[B_{ij}\right]=J^2=\lambda^6, \quad \Rightarrow J=\lambda^3,\ \ \bar{I}_1=\frac{I_1}{J^{2/3}}=3,\ \ \bar{I}_2=\frac{I_2}{J^{4/3}}=3.$$

$$\sigma_{ij}=C_{10}\left(\frac{2}{J^{5/3}}B_{ij}-\frac{2}{3J}\bar{I}_1\delta_{ij}\right)+C_{01}\left(\frac{2}{J^{5/3}}\bar{I}_1B_{ij}-\frac{2}{J^{7/3}}B_{im}B_{jm}-\frac{4}{3J}\bar{I}_2\delta_{ij}\right)+K_1\left(J-1\right)\delta_{ij}$$

$$\Rightarrow \sigma_{kk}=C_{10}\left(\frac{2}{J^{5/3}}B_{kk}-\frac{2}{J}\bar{I}_1\right)+C_{01}\left(\frac{2}{J^{5/3}}\bar{I}_1B_{kk}-\frac{2}{J^{7/3}}B_{km}B_{km}-\frac{4}{J}\bar{I}_2\right)+3K_1\left(J-1\right)$$

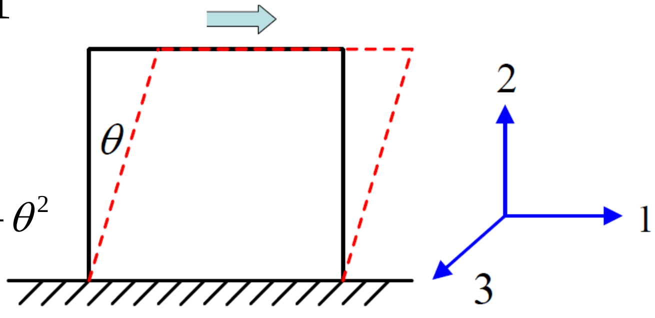
$$\Rightarrow \sigma_{kk}=C_{10}\left(\frac{2}{\lambda^5}3\lambda^2-\frac{2}{\lambda^3}3\right)+C_{01}\left(\frac{2}{\lambda^5}3\left(3\lambda^2\right)-\frac{2}{\lambda^7}3\lambda^4-\frac{4}{\lambda^3}3\right)+3K_1\left(\lambda^3-1\right)=3K_1\left(\lambda^3-1\right)$$

$$\boxed{J\to 1 \quad \Rightarrow p=\frac{\sigma_{kk}}{3}=K_1\left(\lambda^3-1\right)\to K_1\varepsilon_{kk}.}$$

- For small strains: bulk modulus = $K_1$.

# Polynomial Models for Rubber Elasticity

- Physical interpretation of $C_{10}$ and $C_{01}$




$$y_1 \approx x_1 + x_2\theta,\ y_2 = x_2,\ y_3 = x_3$$

$$\Rightarrow [B]=[F][F]^T = \begin{bmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+\theta^2 & \theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Rightarrow B_{kk} = 3+\theta^2$$

$$\sigma_{ij} = \frac{2C_{10}}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + \frac{2C_{01}}{J^{7/3}}\left(B_{kk}B_{ij} - B_{im}B_{jm} - \frac{1}{3}\left(I_1^2 - B_{mn}B_{mn}\right)\delta_{ij}\right) + K_1\left(J-1\right)\delta_{ij}$$

$$J \to 1 \quad \Rightarrow \sigma_{12} = 2C_{10}B_{12} + 2C_{01}\left(B_{kk}B_{12} - B_{1m}B_{2m}\right) = 2C_{10}\theta + 2C_{01}\left\{\left(3+\theta^2\right)\theta - \left(\left(1+\theta^2\right)\theta + \theta\right)\right\} = 2\left(C_{10}+C_{01}\right)\theta$$

$$\Rightarrow \boxed{\sigma_{12} = 2\left(C_{10}+C_{01}\right)\gamma_{12}, \qquad \Rightarrow G = 2\left(C_{10}+C_{01}\right)}$$

- For small strains: shear modulus = 2 ( $C_{10}$ + $C_{01}$).

- **Generalized neo-Hookean model**

$$\bar{U}\left(\bar{I}_1, \bar{I}_2, J\right) = C_{10}\left(\bar{I}_1 - 3\right) \cancel{+C_{01}\left(\bar{I}_2 - 3\right)} + \frac{K_1}{2}\left(J-1\right)^2 \Rightarrow \boxed{\sigma_{ij} = \frac{2C_{10}}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + K_1\left(J-1\right)\delta_{ij}}$$

# More Sophisticated Models for Rubber Elasticity

- Arruda–Boyce eight-chain rubber elasticity model

$$\bar{U} = \mu\left\{\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20\beta^2}(\bar{I}_1^2 - 9) + \frac{11}{1050\beta^4}(\bar{I}_1^3 - 27) + \ldots\right\} + \frac{K}{2}(J-1)^2$$

$$\sigma_{ij} = \frac{\mu}{J^{5/3}}\left(1 + \frac{B_{kk}}{5J^{2/3}\beta^2} + \frac{33(B_{kk})^2}{525\beta^4 J^{4/3}} + \ldots\right)\left(B_{ij} - \frac{B_{kk}}{3}\delta_{ij}\right) + K_1(J-1)\delta_{ij}$$

- Ogden rubber elasticity model

$$\tilde{U} = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3) + \frac{K_1}{2}(J-1)^2 \qquad \bar{\lambda}_i = \lambda_i / J^{1/3}$$

# Foam Elasticity Models

- Ogden–Storakers hyperelastic foam

$$\tilde{U} = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2}\left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \frac{1}{\beta_i}(J^{-\alpha_i\beta_i} - 1 \right)$$

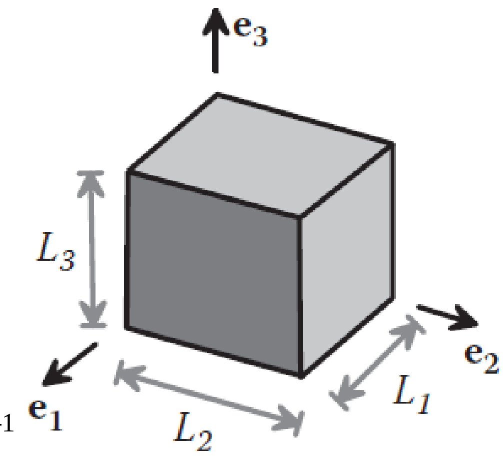
- Blatz–Ko foam rubber

$$U(I_1, I_2, I_3) = \frac{\mu}{2}\left( \frac{I_1}{I_2} + 2\sqrt{I_3} \right)$$

- In foam models, the shear and compression responses are coupled.

# Calibrating Nonlinear Elastic Models

- Stress-stretch relations predicted by Neo-Hookean model
- Uniaxial tension test

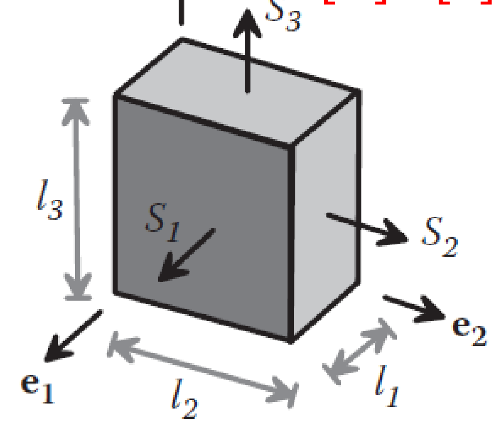


Undeformed

$$[U]=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda^{-1/2} & 0\\ 0 & 0 & \lambda^{-1/2}\end{bmatrix} \quad \Rightarrow [F]=[R][U]=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda^{-1/2} & 0\\ 0 & 0 & \lambda^{-1/2}\end{bmatrix}$$

$$\Rightarrow [B]=[F][F]^T=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda^{-1/2} & 0\\ 0 & 0 & \lambda^{-1/2}\end{bmatrix}^2=\begin{bmatrix}\lambda^2 & \theta & 0\\ \theta & \lambda^{-1} & 0\\ 0 & 0 & \lambda^{-1}\end{bmatrix}, \quad \Rightarrow B_{kk}=\lambda^2+2\lambda^{-1}$$

$[R]=[I]$



Deformed

$$\sigma'_{ij}=2C_{10}\left(B_{ij}-\frac{1}{3}B_{kk}\delta_{ij}\right) \quad \Rightarrow [\sigma']=\frac{2C_{10}}{3}\begin{bmatrix}2\lambda^2-2\lambda^{-1} & \theta & 0\\ \theta & \lambda^{-1}-\lambda^2 & 0\\ 0 & 0 & \lambda^{-1}-\lambda^2\end{bmatrix}$$

$$[\sigma]=\frac{1}{J}[F][S]=[F][S]=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda^{-1/2} & 0\\ 0 & 0 & \lambda^{-1/2}\end{bmatrix}\begin{bmatrix}S_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}\lambda S_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \sigma'_{11}=\frac{2}{3}\lambda S_{11}=\frac{2C_{10}}{3}\left(2\lambda^2-2\lambda^{-1}\right) \quad \Rightarrow \boxed{S_{11}=2C_{10}\left(\lambda-\lambda^{-2}\right)}$$

# Calibrating Nonlinear Elastic Models

- To use any of these constitutive relations, we need to determine values for the material constants.
- We can perform simple tension, pure shear, equibiaxial tension, or volumetric compression tests.
- The parameters can then be chosen to give the best (least-squared) fit to experimental behavior.

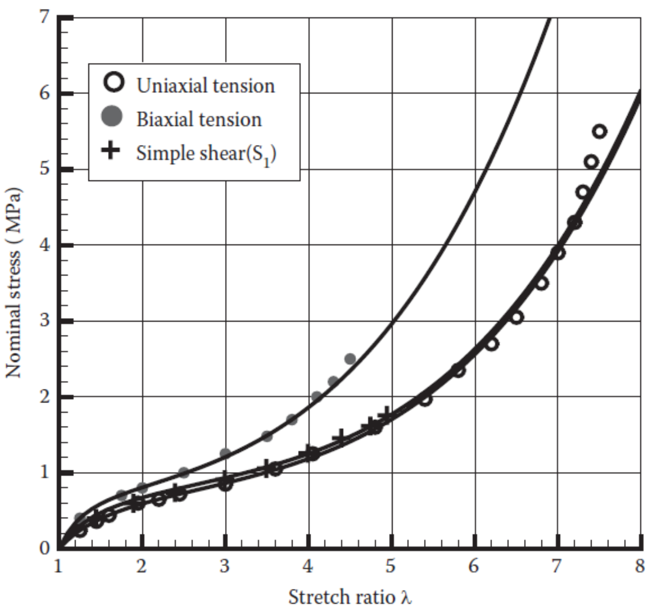



TABLE 3.7 Representative Material Properties for Vulcanized Rubber

| | |
|---|---|
| Neo-Hookean | $\mu_1 = 0.4$ MNm$^{-2}$ |
| Mooney-Rivlin | $\mu_1 = 0.39$ MNm$^{-2}$, $\mu_2 = 0.015$ MNm$^{-2}$ |
| Arruda-Boyce | $\mu_1 = 0.4$ MNm$^{-2}$, $\beta = 10$ |
| Ogden | $\mu_1 = 0.62$ MNm$^{-2}$, $\alpha_1 = 1.3$<br>$\mu_2 = 0.00118$ MNm$^{-2}$, $\alpha_2 = 5$<br>$\mu_3 = -0.00981$ MNm$^{-2}$, $\alpha_3 = -2$ |