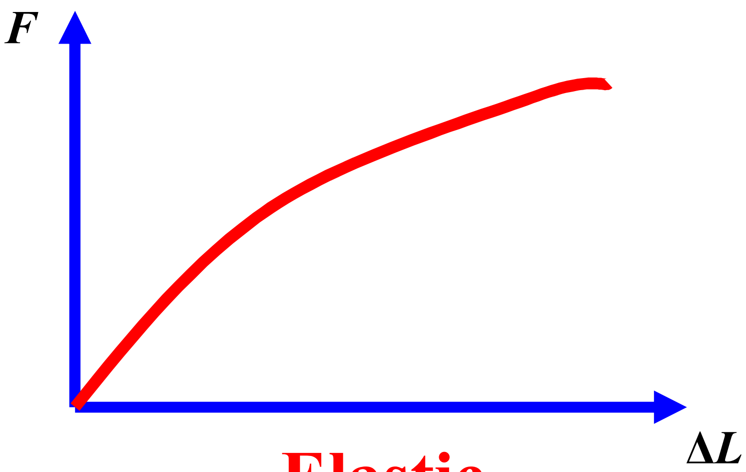
Introduction to Elasticity

Outline

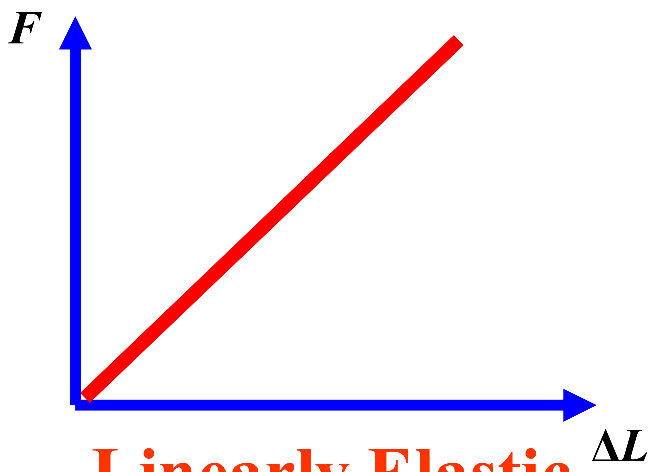
- What is Elasticity?
- A Brief History of Elasticity
- Tools of the Trade
- Engineering Applications of Elasticity
- Fundamental Concepts in Elasticity
- Assumptions of Elasticity Theory
- Geometry of Elastic Solids
- Topics That Will Be Covered
- Greek Alphabet

What is Elasticity?

- Concerned with determining stress, strain, and displacement distribution **in an elastic solid** under the influence of external forces.



Elastic



Linearly Elastic

- Using continuum mechanics formulation establishes a mathematical boundary value problem model - a set of governing partial differential field equations with particular boundary conditions.

History of Elasticity

- Started from early 19th century, primarily by **Navier, Cauchy, Saint-Venant**, Love, Muskhelisvili, Kirchhoff, Poisson, …
- Needed for understanding the deformation and damage mechanisms of bridges, roads, ships, military weapons, and etc.
- Understanding optical wave propagation requires elastic wave theory.

History of Elasticity

- **Claude-Louis Navier (1785−1836)**
- Unified the theory for beam bending (1826).

- Founder of Continuum Mechanics: submitted two monographs to French Academy of Science in 1821.
- Derived the equations of equilibrium and motion for elastic and isotropic solids.

$$C\left(\nabla^2 u_i + 2u_{j,ji}\right) + F_i = 0$$

- Promoted the method allowable stress for examining the strength condition.
- There is only one elastic constant in his formulations and no stress and strain concepts.

History of Elasticity

- **Augustin-Louis Cauchy (1789−1857)**
- **Founder of elasticity theory.** (800 papers and 7 books)
- Famous for his mathematical talent.
- Invented ε-δ in limit & continuity analysis.
- Founder of complex variable theory (Cauchy-Riemann conditions).
- Strain and strain-displacement equations.
- Stress, principal stress vs. principal strain » two elastic constants.
- Generalized Hooke's law: 36 elastic constants at most.
- Equations of equilibrium and BCs in terms of displacements.
- Studied the torsion of elastic rods.

History of Elasticity

- **Siméon Denis Poisson (1781−1840)**

- Solve the Poisson equation:
- Developed the Poisson distribution in probability theory.
- Longitudinal and transverse waves can propagate in elastic solids.
- Theoretically derived the Poisson's ratio (1/4).
- First derived the deflection equation of plates.

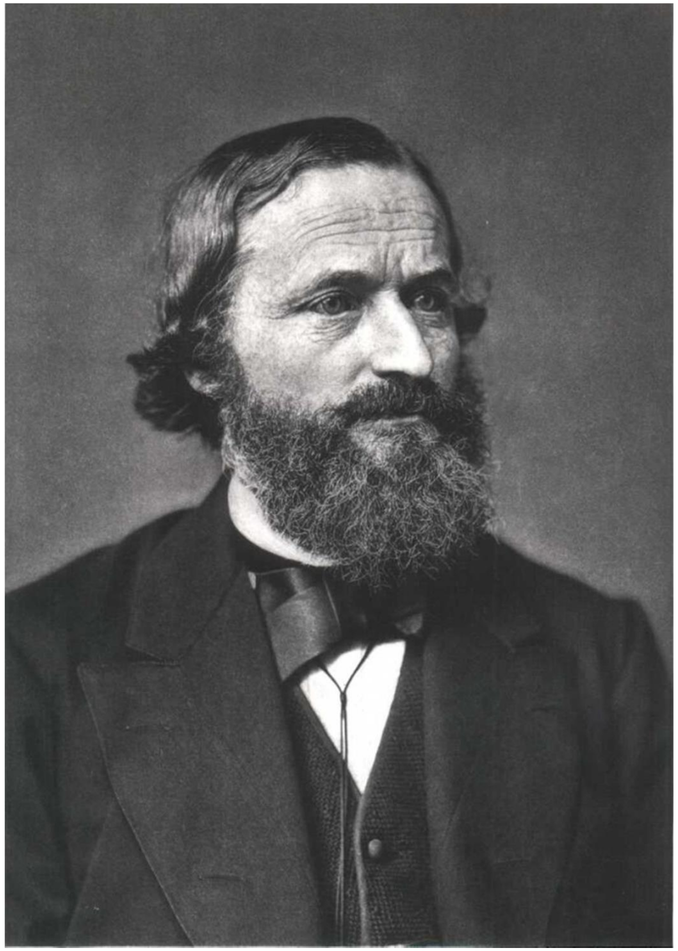
History of Elasticity

- **Adhémar Jean Claude Barré de Saint-Venant (1797−1886), a student of Navier**

- Developed the Saint-Venant Principle from pure bending of beams.
- First verified the precision of the two hypotheses assumed in beam bending.
- Semi-inverse solution for elasticity (1855)
- Improved the torsion of (non-circular) elastic rods by Cauchy.
- Derived solutions for a large number of elastic problems and promoted their use in engineering practice.

History of Elasticity

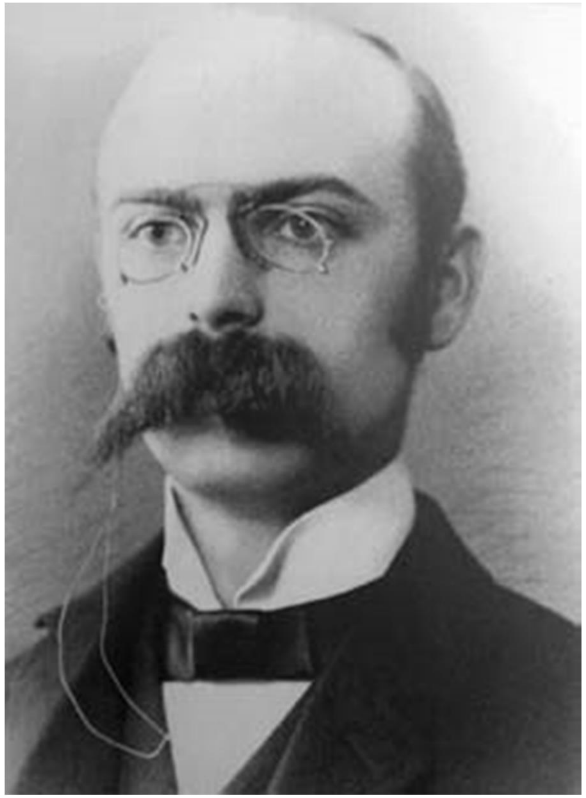
- **Gustav Robert Kirchhoff (1824−1887)**
- Contributed to the fundamental understanding of electrical circuits, spectroscopy, and the emission of black-body radiation by heated objects.



- Developed Kirchhoff plate theory by the principle of virtual displacement in 1850 (Straight normal line, two BCs only).
- Derived the equilibrium equations of elastic rods under large deflections, in analogy to the motion of a rigid body about a fixed point.

History of Elasticity

- **Augustus Edward Hough Love (1863−1940)**
- ***A treatise on the mathematical theory of elasticity***, 1892-1893, summarized all up-to-date achievements in elasticity
- The bending theory of thin-shells in 1888 (Kirchhoff-Love Hypothesis)
- Point-source theory in infinite elastic solids, later found its application in mathematical modeling of earthquake source.
- ***Some problems of geodynamics***: Love wave (earthquake) and Love number (elastic constants of earth).



History of Elasticity

- **Nikolay Ivanovich Muskhelisvili (1891−1976)**
- A Soviet Georgian mathematician
- Throughout his life, he solved many elastic problems using the complex variable formulation, including 2-D plane problems, the torsional problems, and some thermo-elastic problems.

- ***Some basic problems of the mathematical theory of elasticity***, 1963.
- ***Singular integral equations***, 1946.
- Provided the best method for solving practical engineering before FEM gains popular.

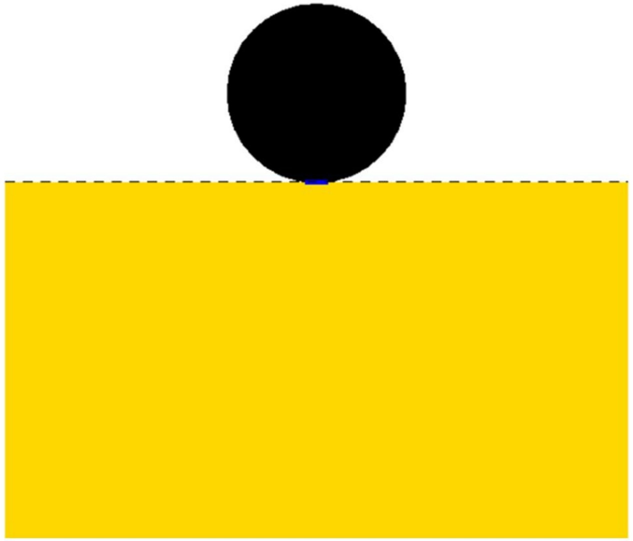
History of Elasticity

- **Solution to Some Important Problems in Elasticity**
- Sophie Germain (1776−1831): Vibrational modal shapes of elastic thin-plates, 1816 (A 3000 francs problem); also advanced Fermat's Last Theorem.
- Kelvin's problem: concentrated force acting in the interior of an infinite solid, 1848.
- Cerruti's Problem: concentrated force acting parallel to the free surface of a semi-infinite solid, 1882.
- Boussinesq's problem: concentrated force acting normal to the free surface of a semi-infinite solid, 1885.
- Mindlin's problem: concentrated force acting in the interior of a semi-infinite solid, 1936.

History of Elasticity

- **Solution to Some Important Problems in Elasticity**

- Contact problems of Hertz type: contact between a sphere and an elastic half-space.



- Number of elastic constants for the most anisotropic solids: 21, George Green (1793−1841); 15, Cauchy & Navier (wrong!).

- Measurement of Poisson's ratio: 1/3, Guillaume Wertheim (1815-1861).

- Equations of elasticity in curvilinear coordinates (Lamé problems): Lamé, 1859.

History of Elasticity

- **Energy Methods in Elasticity**
- Formulation of strain energy: Benoît Paul Émile Clapeyron (1799−1864).
- Variational calculus inspired by the Brachistochrone problem due to Johann Bernoulli (1667−1748) in 1696: Euler & Lagrange.
- Variational principles of elasticity: Hu, Washizu, Chien and Zienkiewicz.

History of Elasticity

- **Textbooks in Mechanics**

- Stephen Timoshenko (1878−1971): Timoshenko Beam, 20+ textbooks, still being used today…

- Textbooks in Mechanics: Encyclopedia style (before the 1950s); Timoshenko style (since the 1970s)

- Calling for concise, up-to-date and easily understandable textbooks.

Tools of the Trade

- **Mechanics of Materials:** Simplified analysis based upon the use of assumptions related to the geometry of the deformation, e.g., cross sections remain planar.
- **Theory of Elasticity:** General approach based upon the principles of continuum mechanics. Develops mathematical boundary-value problems for the solution to the stress, strain and displacement distributions in a given body.

Tools of the Trade

- **Computational Methods (Finite Elements, Boundary Elements, and Finite Differences):** Each of these approaches discretizes the body under study into many computational elements or cells. Computers are then used to calculate the stress and displacement in each element or cell.

- **Experimental Stress Analysis:** Numerous techniques such as photoelasticity, strain gages, brittle coatings, fiber optic sensors, Moiré holography, etc. have been developed to experimentally determine the stress, strain or displacements at specific locations in models or actual structures and machine parts.

Tools of the Trade

- **Numerical Methods in Elasticity**
- Rayleigh-Ritz Method: approximating a functional defined on a normed linear space by a linear combination of elements from that space.
- Method of Weighted Residuals (MWR): working with DEs directly to minimize solution error.
- Finite Element Method (FEM): Ray W. Clough (1960), O. C. Ziekiewitz, Kang Feng…
- Boundary Element Method: solving linear PDEs which have been formulated as integral equations (i.e. in boundary integral form).
- Finite Difference Method.

Engineering Applications of Elasticity

- Aeronautical/Aerospace Engineering - stress, fracture, and fatigue analysis in aero structures.
- Civil Engineering - stress and deflection analysis of structures including rods, beams, plates, and shells; geomechanics involving the stresses in soil, rock, concrete, and asphalt materials.
- Materials Engineering - to determine the stress fields in crystalline solids, around dislocations and in materials with microstructure.

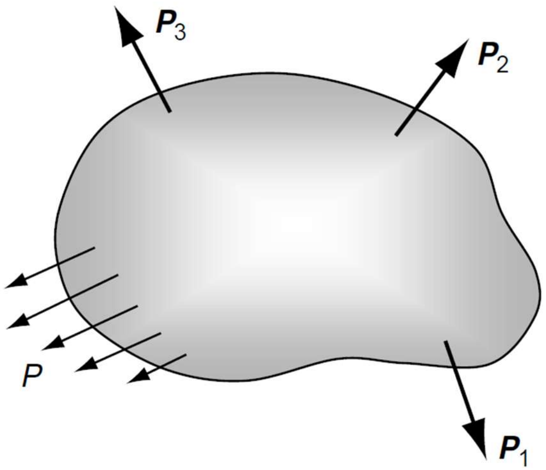
Engineering Applications of Elasticity

- Mechanical Engineering - analysis and design of machine elements, general stress analysis, contact stresses, thermal stress analysis, fracture mechanics, and fatigue.
- The subject also provides the basis for more advanced work in inelastic material behavior including plasticity and viscoelasticity, and to the study of computational stress analysis employing finite and boundary element methods.

Fundamental Concepts in Elasticity

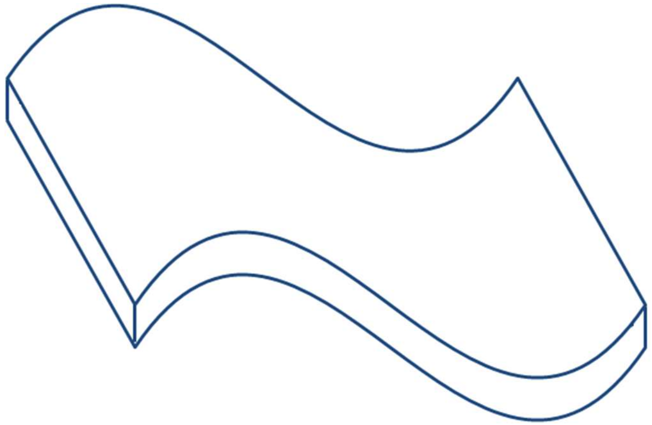
- Scalar, vector, tensor, coordinate transformation.
- Distributed vs. concentrated load.
- Body force, surface force and line force.
- Deformation: displacement and strain.
- Stress and equilibrium.
- Constitutive relation.
- Compatibility.
- Singularity.
- Boundary value problems.
- Solution Strategy.

Geometry of Elastic Solids

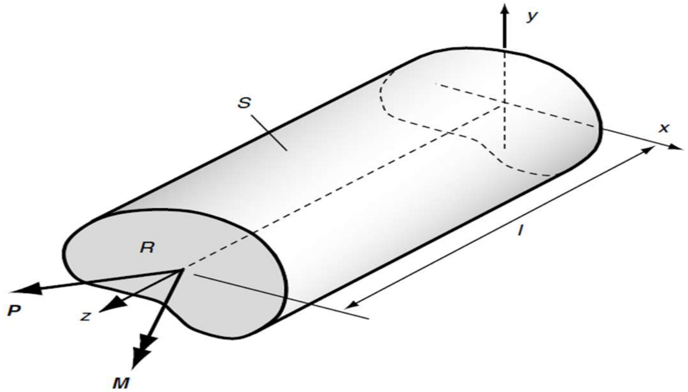


Bodies: three dimensions are equivalent.

Plates: two dimensions are far greater than the third. No curvature exists in the median plane of the plate.



Shells: two dimensions are far greater than the third. At least one principal curvature is nonzero in the median plane of the shell.



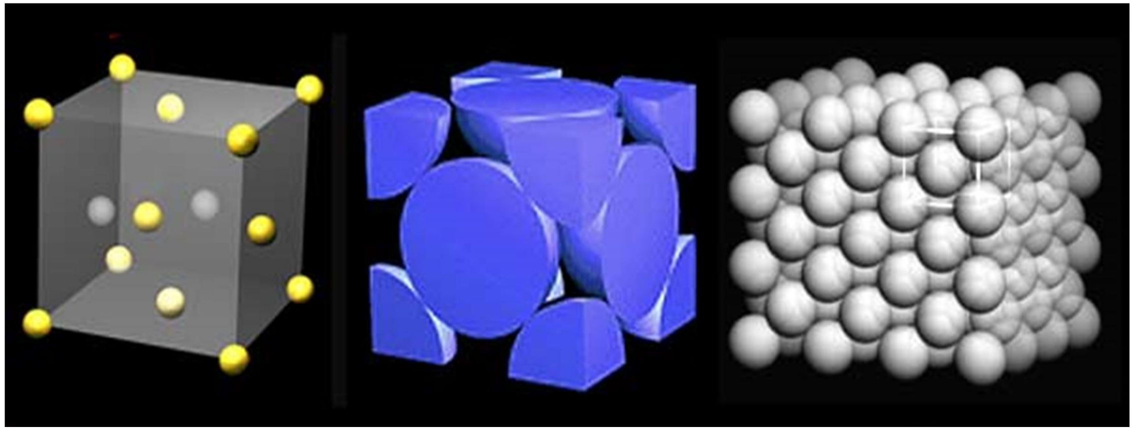
Rods: one dimension is far greater than the others. The bar axis may be either straight or curved.

Assumptions of Elasticity

- Continuity
- Homogeneity
- Isotropy
- Small Deformation & (Linear/Perfect) Elasticity

Continuity

- Matter fills up the whole space of a solid defined by its volume.
- Neither vacancies can be produced nor more materials can be added under normal working conditions.
- Arbitrary section or volume element can be extracted for force or deformation analysis.



Face-centered cubic lattice

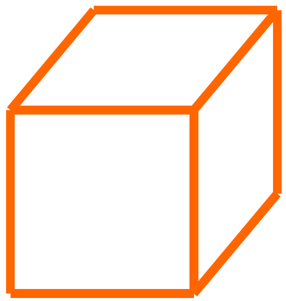
$$4r = a\sqrt{2} \Rightarrow r = \frac{\sqrt{2}}{4}a$$

$$V_{matter} = \frac{16}{3}\pi r^3 = \frac{16}{3}\pi\frac{2\sqrt{2}}{64}a^3 = \frac{\pi}{3\sqrt{2}}a^3$$

$$\Rightarrow \frac{V_{matter}}{V} = \frac{\pi}{3\sqrt{2}} \approx 74\%$$

Homogeneity

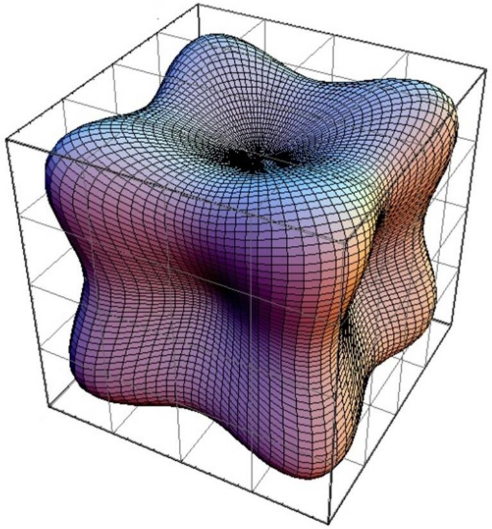
- Macroscopic material properties can be represented by those of any arbitrary representative volume element (RVE).
- The minimum size of a RVE depends on material types, i.e. $0.1 \times 0.1 \times 0.1$ mm for metals; $10 \times 10 \times 10$ mm for concrete
- A RVE should be composed of at least the number of "elementary entities" (usually atoms or molecules) in one mole, i.e. 6.022×10^{23}



A representative volume element in a solid.

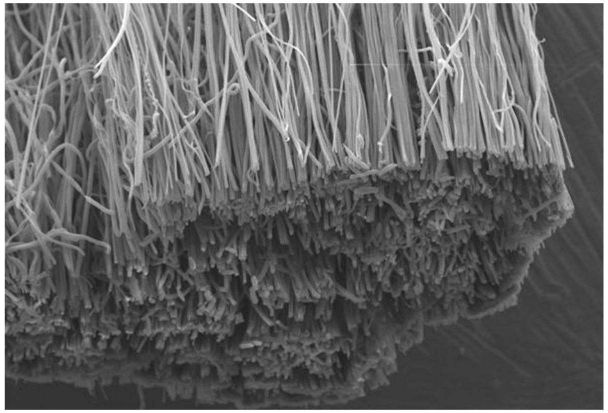
Isotropy

- Mechanical properties of materials are independent of directions.
- Mechanical properties of materials are taken as the statistical average of those along every direction.
- Strong transversely isotropic materials such as wood and fiber reinforced composites are still viewed as non-isotropic materials.



The directional dependence of Young's modulus

Cross section of woods.

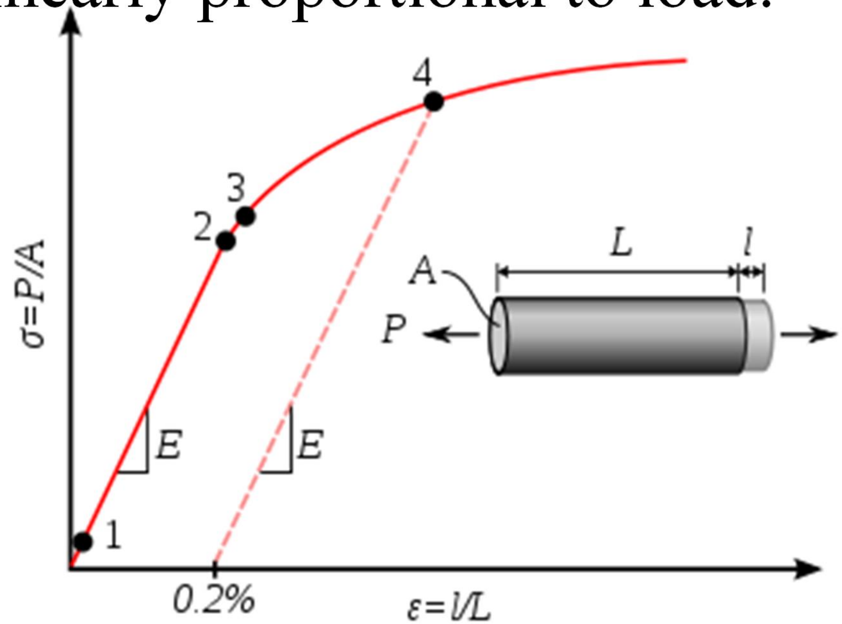


Carbon fiber reinforced composites

Small Deformation & Elasticity

- **Small Deformation:** deformation of structural elements under mechanical loads are negligible compared to their original size.
- Under small deformation, analysis of force and deformation can be based on a structure's size and shape prior to deformation.
- **Elasticity:** a structural element can restore to its original size and shape upon the removing of its external loading.
- **Linear Elasticity:** deformation is linearly proportional to load.

Stress-strain curve showing yield behavior. 1. A point within proportionality. 2. Proportionality limit. 3. Elastic limit (initial yield strength). 4. Subsequent yield strength.



Topics Being Covered

- Introduction to Elasticity
- Mathematical Preliminaries
- Deformation: Displacements and Strains
- Stress and Equilibrium
- Material Behavior - Linear Elastic Solids
- Formulation and Solution Strategies
- 2-D Formulation
- 2-D Problems in RCC
- 2-D Problems in Polar Coordinates
- Torsion of Non-circular Shafts

Advanced Topics

- 3-D Problems
- Bending of Thin Plates
- Thermo-elasticity
- Energy Methods and Variational Principles
- Introduction to Finite Element Method
- Other Numerical Methods

Greek Alphabet

Letter	Greek	English	Sounds As	Phonetic	Trsl.
Α α	ἄλφα	ālphā	ahl-fah	**a** in f**a**ther (long) **a** in d**a**d (short)	**a**
Β β	βῆτα	bētā	bay-tah	**b** in **b**all	**b**
Γ γ	γάμμα	gāmmā	gahm-mah	**g** in **g**ot	**g**
Δ δ	δέλτα	dĕltā	dell-tah	**d** in **d**og	**d**
Ε ε	ἒ ψιλόν	ĕpsīlŏn	eh-pseeh-lawn	**e** in n**e**t	**e**
Ζ ζ	ζῆτα	zētā	zay-tah	**z** in ga**z**e (initial) **dz** in a**dz** (medial)	**z**
Η η	ἦτα	ētā	ay-tah	**e** in ob**e**y	**ē**
Θ θ	θῆτα	thētā	thay-tah	**th** in **th**is	**th**
Ι ι	ἰῶτα	iōtā	yi-oh-tah	**i** in mach**i**ne (long) **i** in h**i**t (short)	**i**
Κ κ	κάππα	kăppā	kap-pah	**k** in **k**in	**k**
Λ λ	λάμβδα	lāmbdā	lahm-dah	**l** in **l**ot	**l**
Μ μ	μῦ	mū	mew	**m** in **m**an	**m**
Ν ν	νῦ	nū	new	**n** in **n**ew	**n**

Greek Alphabet

Letter	Greek	English	Sounds As	Phonetic	Trsl.
Ξ ξ	ξῖ	xsī	x-see	**x** in a**x**e	**x**
Ο ο	ὂ μικρόν	ŏmīkron	au-me-krahn	**ough** in **ough**t	**o**
Π π	πῖ	pī	peeh	**p** in **p**arty	**p**
Ρ ρ	ῥῶ	rhō	hrow	**r** in **r**ide **rh** in **rh**ino (aspirate)	**r**
Σ σ ς	σίγμα	sĭgmā	sig-mah	**s** in **s**it (unvoiced) **s** in i**s** (voiced)	**s**
Τ τ	ταῦ	tau	tau	**t** in **t**alk	**t**
Υ υ	ὖ ψιλόν	ūpsīlon	ew-pseeh-lawn	**u** in l**u**te (long) **u** in p**u**t (short	**y, u**
Φ φ	φῖ	phī	fee	**ph** in **ph**one	**ph**
Χ χ	χῖ	chī	khey	**ch** in **ch**emist	**ch**
Ψ ψ	ψῖ	psī	psee	**ps** in **ps**alm (initial) **ps** in li**ps** (medial)	**ps**
Ω ω	ὦ μέγα	ō mĕgā	oh-may-gah	**o** in n**o**te	**ō**

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