Simple Hyperelastic BVPs

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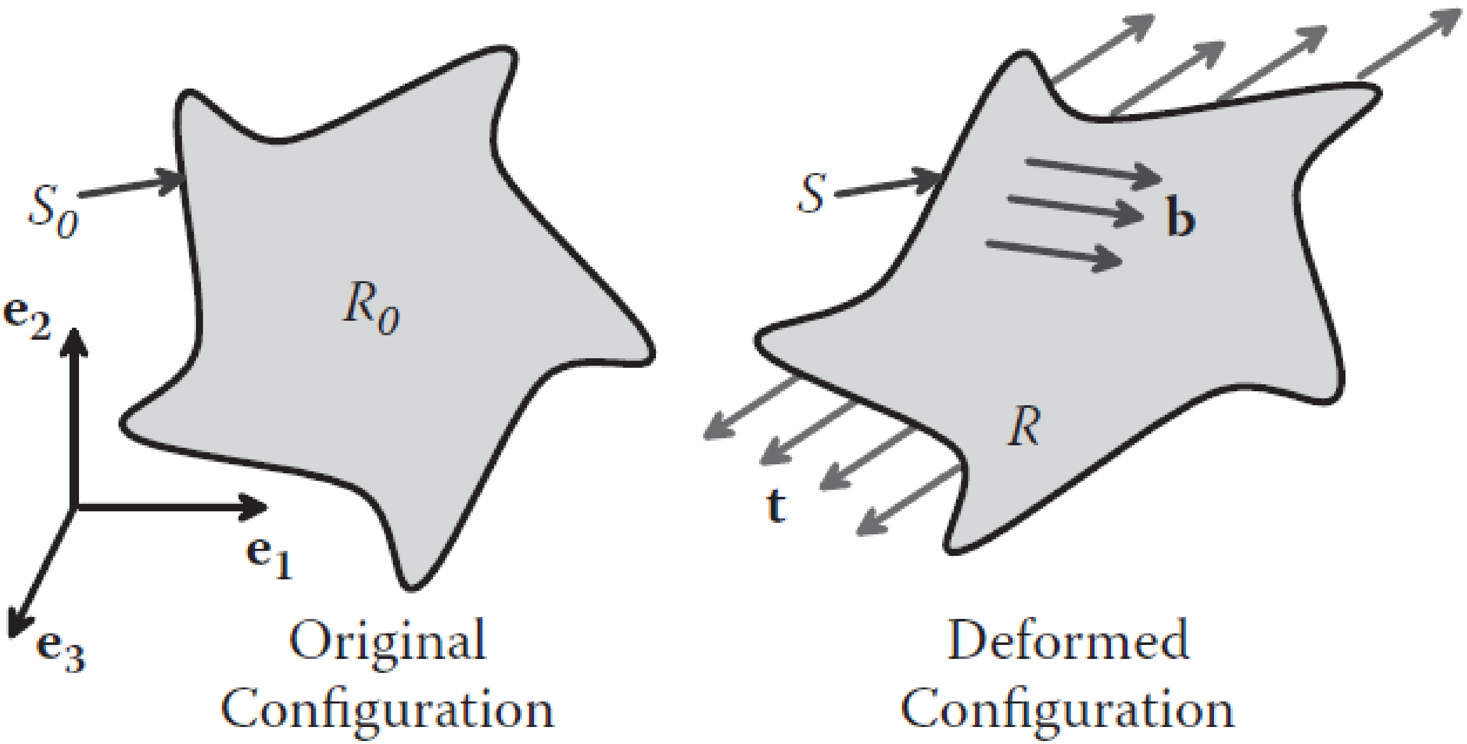
Outline

- Theory of hyperelasticity（超弹理论回顾）
- Incompressible spherically symmetric solids（不可压缩中心对称体）
- Pressurized hollow sphere（压力球腔）
  - Governing equations（控制方程）
  - Boundary condition（边界条件）
  - Displacement vs. Pressure（位移与压力函数关系）
  - Radial stress distribution（径向应力分布）
  - Hoop stress distribution（箍筋应力/周向应力）

Summary of the Theory of Hyperelasticity

- The solid is stress free in its undeformed configuration.
- Temperature changes during deformation are neglected.
- The solid is incompressible.



Summary of the Theory of Hyperelasticity

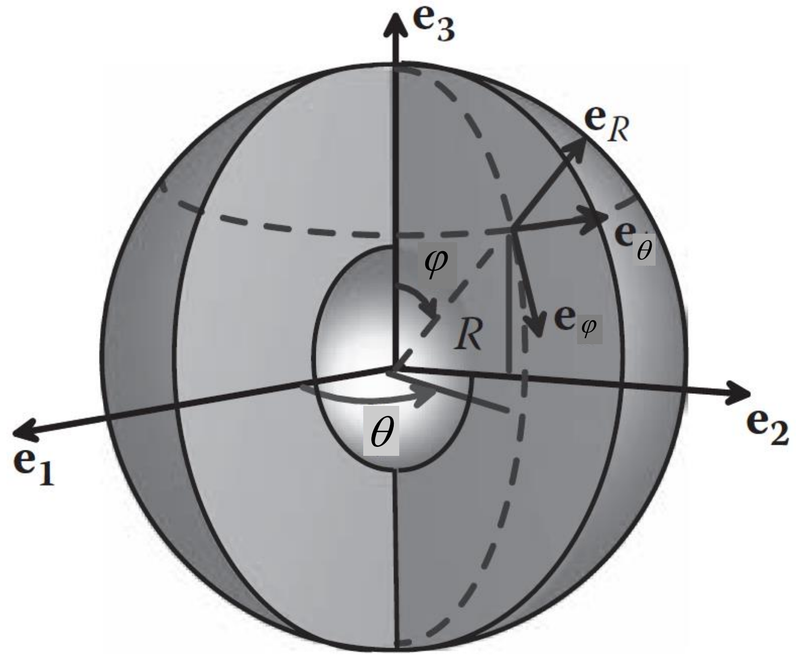
- Strain-displacement relations: $B_{ij} = F_{ik} F_{jk}, \quad F_{ij} = \delta_{ij} + u_{i,j}$
- Incompressibility: $J = \det\left[\mathbf{F}\right] = 1$
- Stress-strain relation $I_1 = B_{kk}, I_2 = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right),$

$$\sigma_{ij} = 2\left\{\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right) B_{ij} - \left(I_1 \frac{\partial U}{\partial I_1} + 2I_2 \frac{\partial U}{\partial I_2}\right)\frac{\delta_{ij}}{3} - B_{im}B_{jm}\frac{\partial U}{\partial I_2}\right\} + p\delta_{ij}.$$

- Equilibrium equations: $\frac{\partial \sigma_{ij}}{\partial y_i} + F_j = 0.$
- Traction BCs on S_t: $\sigma_{ij} n_i = t_j$
- Displacement BCs on S_u: $u_i = \bar{u}_i$

Incompressible Spherically Symmetric Solids

- Coordinates in **undeformed** configuration $\{R, \Phi, \Theta\}$
- Coordinates in **deformed** configuration $\{r, \varphi, \theta\}$
- Points only move radially, due to spherical symmetry



$$r = f(R) \qquad \theta = \Theta \qquad \varphi = \Phi.$$

- Position vector in the undeformed solid: $\mathbf{x} = R\mathbf{e}_r$
- Position vector in the deformed solid: $\mathbf{y} = r\mathbf{e}_r = f(R)\mathbf{e}_r$
- Displacement vector: $\mathbf{u} = \mathbf{y} - \mathbf{x} = r\mathbf{e}_r - R\mathbf{e}_r = (f(R) - R)\mathbf{e}_r$

Incompressible Spherically Symmetric Solids

- Cauchy stress: $\boldsymbol{\sigma} = \sigma_{rr}[r]\mathbf{e}_r\mathbf{e}_r + \sigma_{\varphi\varphi}[r]\mathbf{e}_\varphi\mathbf{e}_\varphi + \sigma_{\theta\theta}[r]\mathbf{e}_\theta\mathbf{e}_\theta, \quad \sigma_{\varphi\varphi} = \sigma_{\theta\theta}.$
- Deformation gradient:

$$F = F_{rr}[r,R]\mathbf{e}_r\mathbf{e}_R + F_{\varphi\varphi}[r,R]\mathbf{e}_\varphi\mathbf{e}_\Phi + F_{\theta\theta}[r,R]\mathbf{e}_\theta\mathbf{e}_\Theta, \quad F_{\varphi\varphi} = F_{\theta\theta}.$$

- Left C-G deformation tensor:

$$\mathbf{B} = B_{rr}[r]\mathbf{e}_r\mathbf{e}_r + B_{\varphi\varphi}[r]\mathbf{e}_\varphi\mathbf{e}_\varphi + B_{\theta\theta}[r]\mathbf{e}_\theta\mathbf{e}_\theta, \quad B_{\varphi\varphi} = B_{\theta\theta}.$$

- Strain-displacement relations:

$$F_{rr} = \frac{dr}{dR}, F_{\varphi\varphi} = F_{\theta\theta} = \frac{r}{R}, \quad \Rightarrow B_{rr} = \left(\frac{dr}{dR}\right)^2, B_{\varphi\varphi} = B_{\theta\theta} = \left(\frac{r}{R}\right)^2$$

- Incompressibility:

$$J = \det[\mathbf{F}] = \frac{dr}{dR}\left(\frac{r}{R}\right)^2 = 1.$$

Incompressible Spherically Symmetric Solids

- Stress-strain relation

$$\sigma_{rr} = 2\left\{\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right)B_{rr} - \frac{1}{3}\left(I_1 \frac{\partial U}{\partial I_1} + 2I_2 \frac{\partial U}{\partial I_2}\right) - B_{rr}^2 \frac{\partial U}{\partial I_2}\right\} + p,$$

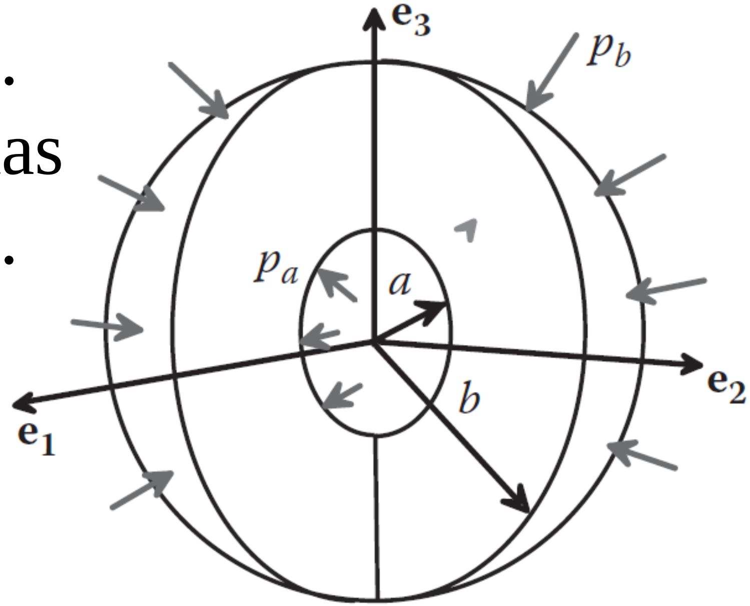
$$\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = 2\left\{\left(\frac{\partial U}{\partial I_1} + I_1 \frac{\partial U}{\partial I_2}\right)B_{\varphi\varphi} - \frac{1}{3}\left(I_1 \frac{\partial U}{\partial I_1} + 2I_2 \frac{\partial U}{\partial I_2}\right) - B_{\varphi\varphi}^2 \frac{\partial U}{\partial I_2}\right\} + p.$$

- Equilibrium equations: $\dfrac{d\sigma_{rr}}{dr} + \dfrac{2}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) + F_r = 0.$

- Traction BCs: $\sigma_{rr}\left[a\right] = \sigma_a, \sigma_{rr}\left[b\right] = \sigma_b.$
- Displacement BCs: $u_r\left[a\right] = u_a, u_r\left[b\right] = u_b.$

Pressurized Hollow Sphere: Incompressibility

- No body forces act on the sphere.
- Before deformation, the sphere has inner radius *A* and outer radius *B*.
- The solid is made from an incompressible Mooney-Rivlin solid.



- Solution:
- Incompressibility implies:

$$V = V_0 \quad \Rightarrow \int_a^r 4\pi r^2 dr = \int_A^R 4\pi R^2 dR \quad \Rightarrow r^3 - a^3 = R^3 - A^3$$

$$\Rightarrow r = \left(R^3 - A^3 + a^3\right)^{\frac{1}{3}}, \quad R = \left(r^3 - a^3 + A^3\right)^{\frac{1}{3}}$$

Pressurized Hollow Sphere: Governing Equations

- Deformation gradient and left C-G strain:

$$F_{rr} = \frac{dr}{dR} = \left(\frac{r}{R}\right)^2, F_{\varphi\varphi} = F_{\theta\theta} = \frac{r}{R}, \quad \Rightarrow B_{rr} = \left(\frac{r}{R}\right)^4, B_{\varphi\varphi} = B_{\theta\theta} = \left(\frac{r}{R}\right)^2$$

- Stress-strain relations:

$$I_1 = B_{kk} = B_{rr} + 2B_{\varphi\varphi}, \quad I_2 = \frac{1}{2}\left(I_1^2 - B_{ik}B_{ki}\right) = \frac{1}{2}\left\{\left(B_{rr} + 2B_{\varphi\varphi}\right)^2 - \left(B_{rr}^2 + 2B_{\varphi\varphi}^2\right)\right\} = \left(2B_{rr} + B_{\varphi\varphi}\right)B_{\varphi\varphi}$$

$$U = \frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right) \qquad \Rightarrow \frac{\partial U}{\partial I_1} = \frac{\mu_1}{2}, \quad \frac{\partial U}{\partial I_2} = \frac{\mu_2}{2}$$

$$\Rightarrow \boxed{\sigma_{rr} = \frac{2}{3}\left(\mu_1 + \mu_2 B_{\varphi\varphi}\right)\left(B_{rr} - B_{\varphi\varphi}\right) + p, \quad \sigma_{\varphi\varphi} = \sigma_{\theta\theta} = -\frac{1}{3}\left(\mu_1 + \mu_2 B_{\varphi\varphi}\right)\left(B_{rr} - B_{\varphi\varphi}\right) + p.}$$

- Equilibrium equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) + F_r = 0 \qquad \Rightarrow \sigma_{rr} = \mu_1\left(\frac{2R}{r} + \frac{R^4}{2r^4}\right) - \mu_2\left(\frac{2r}{R} - \frac{R^2}{r^2}\right) + C.$$

Pressurized Hollow Sphere: BCs

- Boundary conditions

$$\begin{cases} r=a; R=A: \sigma_{rr}=-p_a \\ r=b; R=B: \sigma_{rr}=-p_b \end{cases} \Rightarrow \begin{cases} -p_a=\mu_1\left(\frac{2A}{a}+\frac{A^4}{2a^4}\right)-\mu_2\left(\frac{2a}{A}-\frac{A^2}{a^2}\right)+C\equiv\mu_1\left(\frac{2}{\alpha}+\frac{1}{2\alpha^4}\right)-\mu_2\left(2\alpha-\frac{1}{\alpha^2}\right)+C \\ -p_b=\mu_1\left(\frac{2B}{b}+\frac{B^4}{2b^4}\right)-\mu_2\left(\frac{2b}{B}-\frac{B^2}{b^2}\right)+C\equiv\mu_1\left(\frac{2}{\beta}+\frac{1}{2\beta^4}\right)-\mu_2\left(2\beta-\frac{1}{\beta^2}\right)+C \end{cases}$$

- Adding the two equations gives the expression for *C*.

$$C=-\frac{\mu_1}{2}\left(\frac{2}{\alpha}+\frac{1}{2\alpha^4}+\frac{2}{\beta}+\frac{1}{2\beta^4}\right)+\frac{\mu_2}{2}\left(2\alpha-\frac{1}{\alpha^2}+2\beta-\frac{1}{\beta^2}\right)-\frac{p_a+p_b}{2}$$

- Although unnecessary, the Pressure *p* in the stress-strain relations can be determined by comparing expressions of the radial stress.

Pressurized Hollow Sphere: Displacement vs. Pressure

- Subtracting the first BC equation from the second results in the expression that relates α and β to the pressures.

$$\frac{p_a - p_b}{\mu_1} = 2\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) + \frac{1}{2}\left(\frac{1}{\beta^4} - \frac{1}{\alpha^4}\right) - \frac{2\mu_2}{\mu_1}(\beta - \alpha) + \frac{\mu_2}{\mu_1}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)$$

- Incompressibility yields

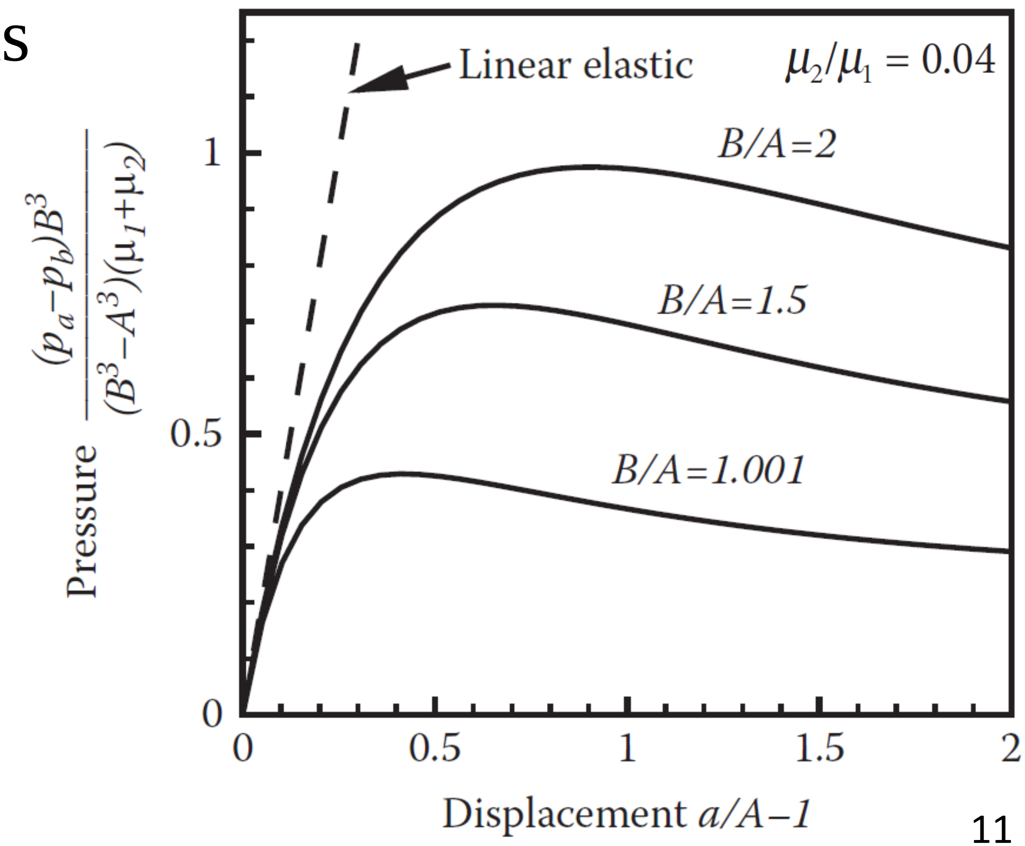
$$r^3 - a^3 = R^3 - A^3$$

$$\Rightarrow A^3 - a^3 = R^3 - r^3 = B^3 - b^3$$

$$\Rightarrow \frac{B^3}{A^3} = \frac{B^3}{A^3}\frac{A^3 - a^3}{B^3 - b^3} = \frac{1-\alpha^3}{1-\beta^3}$$

- The two equations can be solved for α and β.
- For graphing purpose, it is better to follow:

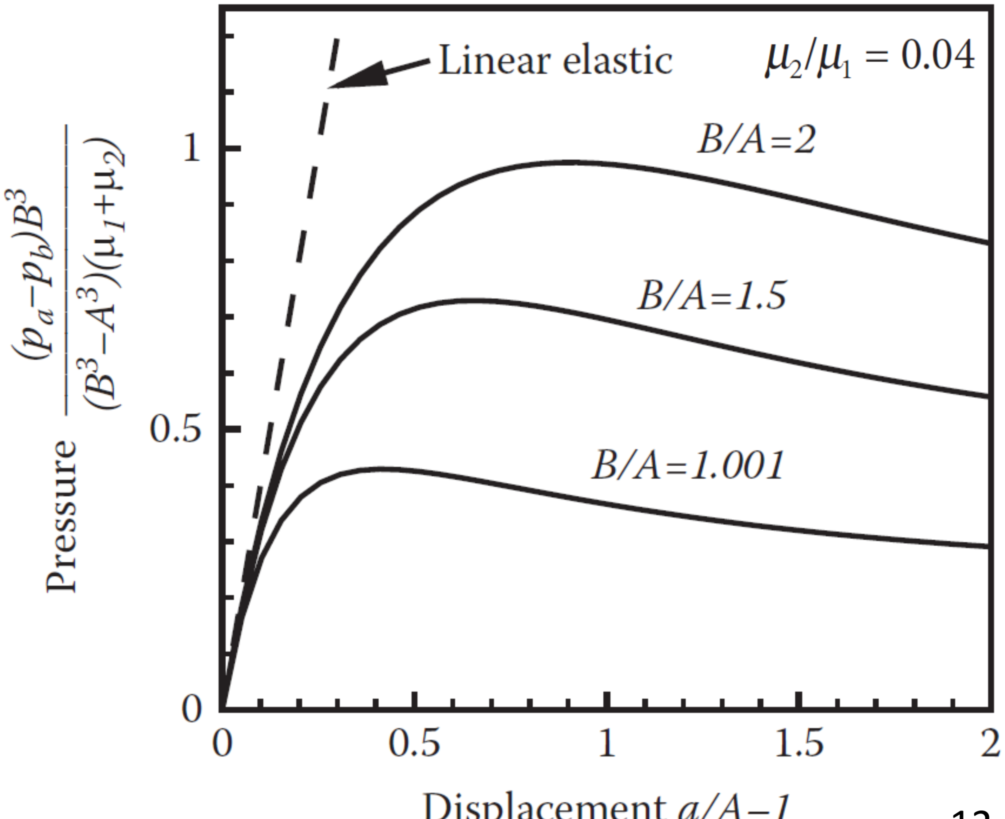
$$\alpha \to \beta \to p_a - p_b.$$



Pressurized Hollow Sphere: Displacement vs. Pressure

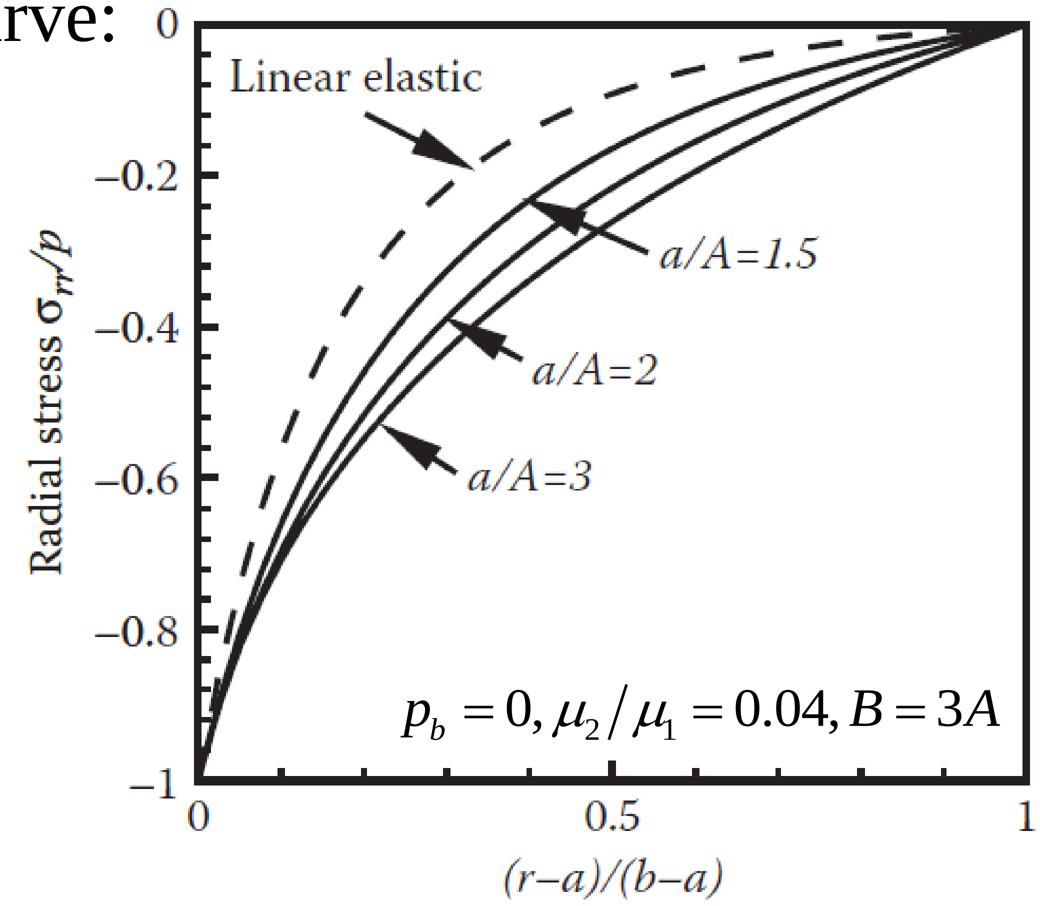
- For the linear elastic curve: $E = 3(\mu_1 + \mu_2), \nu = 0.5.$
- The linear elastic solution is accurate for $u/A < 0.05$.
- The relationship between pressure and displacement is nonlinear in the large deformation regimen.

- As the internal radius of the sphere increases, the pressure reaches a maximum and thereafter decreases because the wall thickness of the shell decreases as the sphere expands.



Pressurized Hollow Sphere: Radial Stress

- Radial stress distribution: $\sigma_{rr} = \mu_1\left(\frac{2R}{r}+\frac{R^4}{2r^4}\right)-\mu_2\left(\frac{2r}{R}-\frac{R^2}{r^2}\right)+C$
- For the linear elastic curve: $E = 3(\mu_1+\mu_2), \nu = 0.5.$
- The radial stress remains close to the linear elastic solution even in the large deformation regimen.



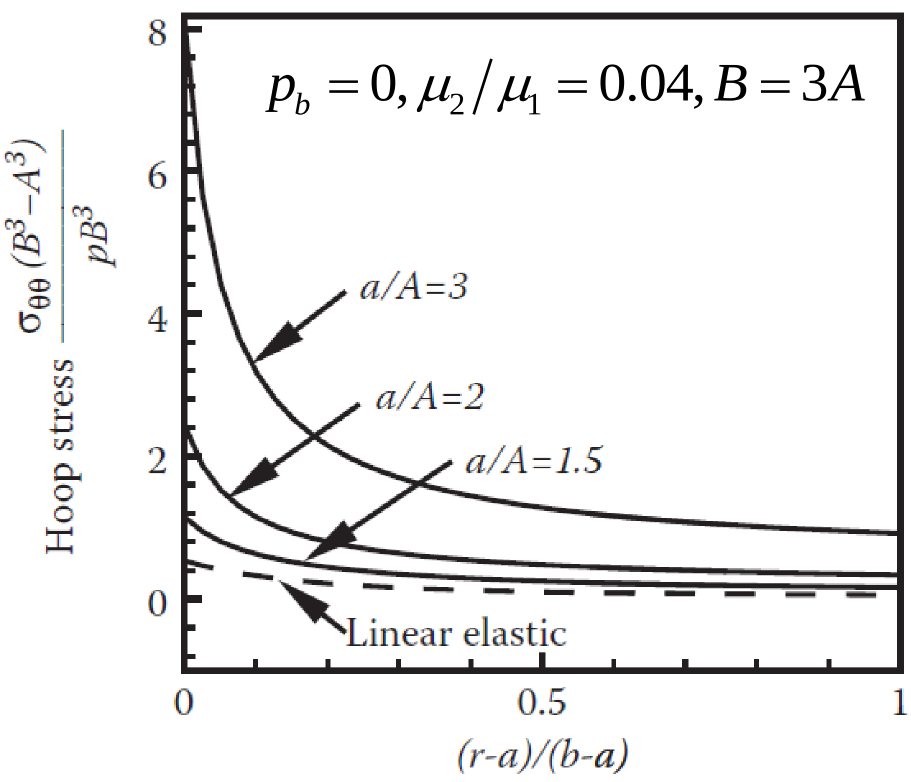
Pressurized Hollow Sphere: Hoop Stress

- Hoop stress

$$\sigma_{rr}=\frac{2}{3}\left(\mu_1+\mu_2 B_{\varphi\varphi}\right)\left(B_{rr}-B_{\varphi\varphi}\right)+p,\sigma_{\varphi\varphi}=\sigma_{\theta\theta}=-\frac{1}{3}\left(\mu_1+\mu_2 B_{\varphi\varphi}\right)\left(B_{rr}-B_{\varphi\varphi}\right)+p.$$

$$\Rightarrow \sigma_{\varphi\varphi}=\sigma_{rr}-\left(\mu_1+\mu_2 B_{\varphi\varphi}\right)\left(B_{rr}-B_{\varphi\varphi}\right)$$

$$=\mu_1\left(\frac{2R}{r}-\frac{R^4}{2r^4}+\frac{r^2}{R^2}\right)-\mu_2\left(\frac{2r}{R}-\frac{r^4}{R^4}\right)+C$$



- For the linear elastic curve: $E=3\left(\mu_1+\mu_2\right),\nu=0.5.$
- The hoop stress distribution is significantly altered as the deformation increases