Centroids & Centers of Gravity

Contents

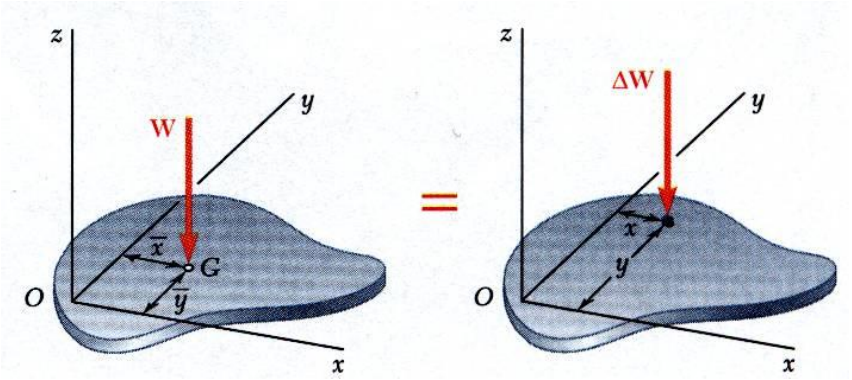
Introduction

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.

- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.

Center of Gravity of a 2D Body

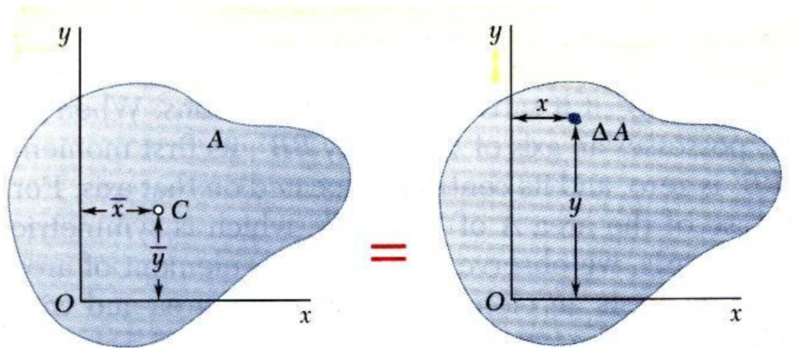
- Center of gravity of a plate



$$\sum M_y = \bar{x}W = \sum \bar{x}_i \Delta W_i$$
$$= \int x dW$$
$$\sum M_x = \bar{y}W = \sum \bar{y}_i \Delta W_i$$
$$= \int y dW$$

Centroids and First Moments of Areas

- Centroid of an area (assuming uniform thickness and density)



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\rho g A t) = \int x(\rho g t)\,dA$$

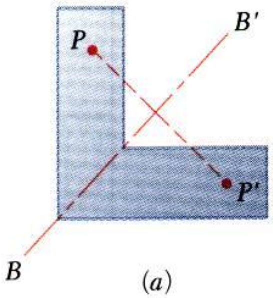
$$\bar{x}A = \int x\,dA = S_y$$

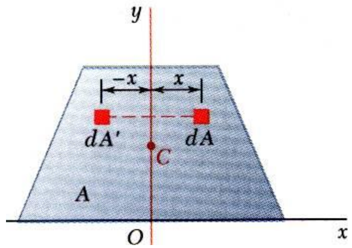
= first moment with respect to y

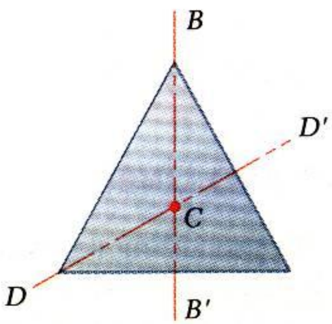
$$\bar{y}A = \int y\,dA = S_x$$

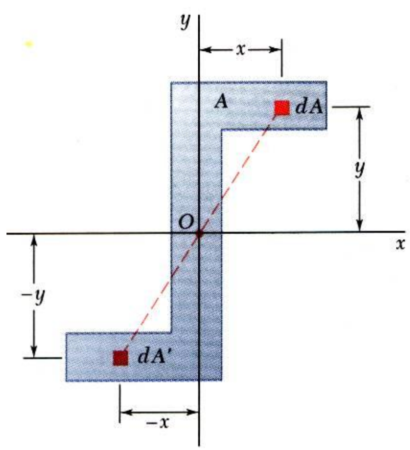
= first moment with respect to x

Centroids and First Moments of Areas



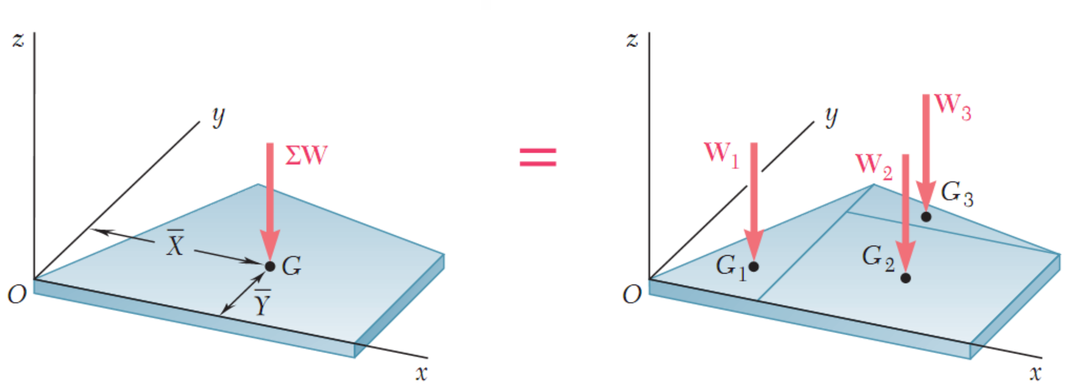






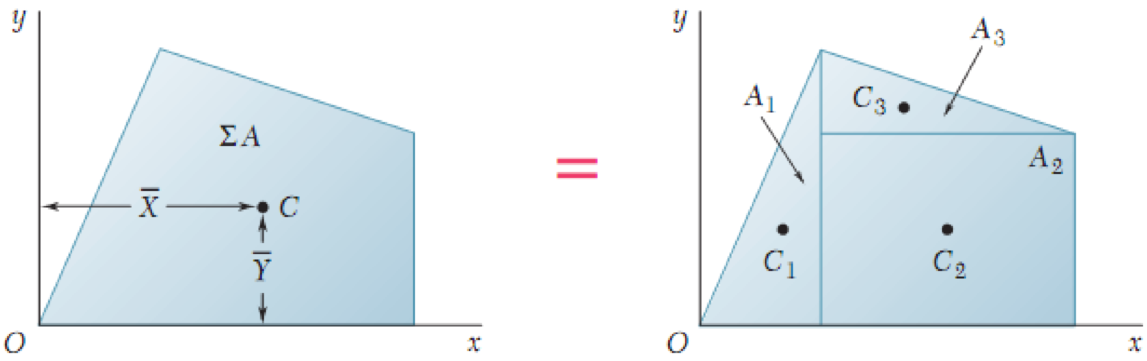
- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

Centroids of Composite Areas



- Composite plates

$$\bar{X}\sum W = \sum \bar{x}_i W_i$$
$$\bar{Y}\sum W = \sum \bar{y}_i W_i$$



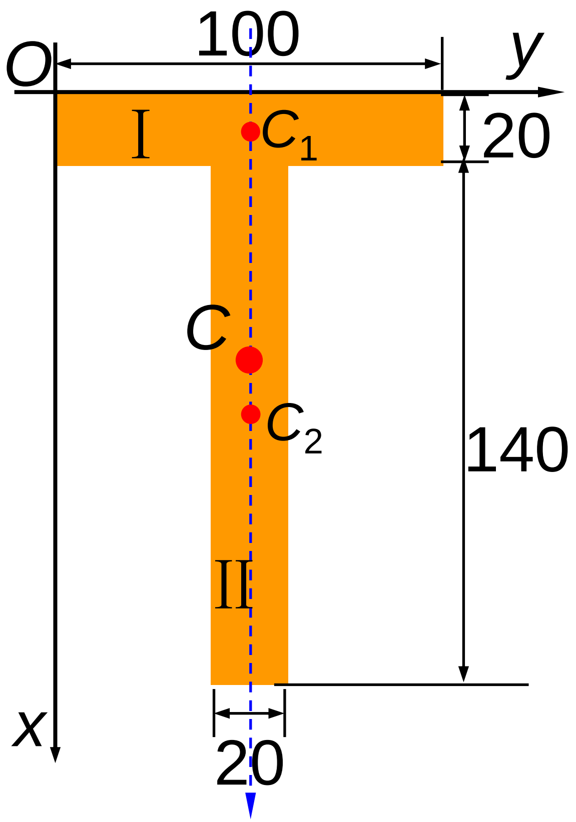
- Composite area

$$\bar{X}\sum A = \sum \bar{x}_i A_i$$
$$\bar{Y}\sum A = \sum \bar{y}_i A_i$$

Sample Problem

$$S_x = \int_A y dA = A\bar{y} \qquad S_y = \int_A x dA = A\bar{x}$$

$$S_x = \sum_1^n S_{xi} = \sum_{i=1}^n A_i \bar{y}_i \quad ; \quad S_y = \sum_{i=1}^n A_i \bar{x}_i$$

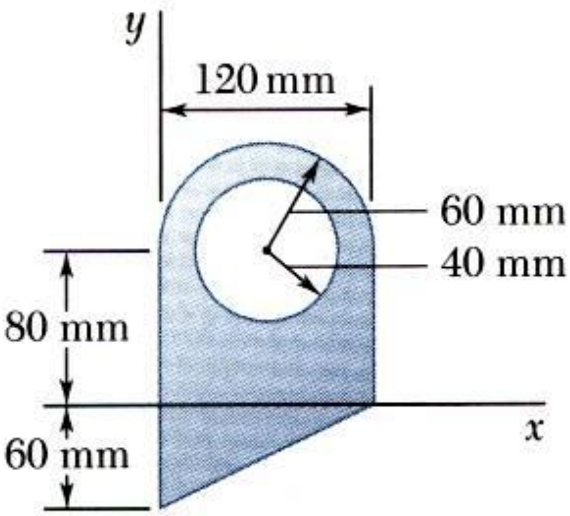


$$\bar{x} = \frac{S_y}{A} = \frac{\sum_{i=1}^n A_i \bar{x}_i}{A} = \frac{10 \times 2000 + (20+70) \times 2800}{2000 + 2800}$$

$$= 56.66$$

$$\bar{y} = \frac{S_x}{A} = \frac{\sum_{i=1}^n A_i \bar{y}_i}{A} = 50$$

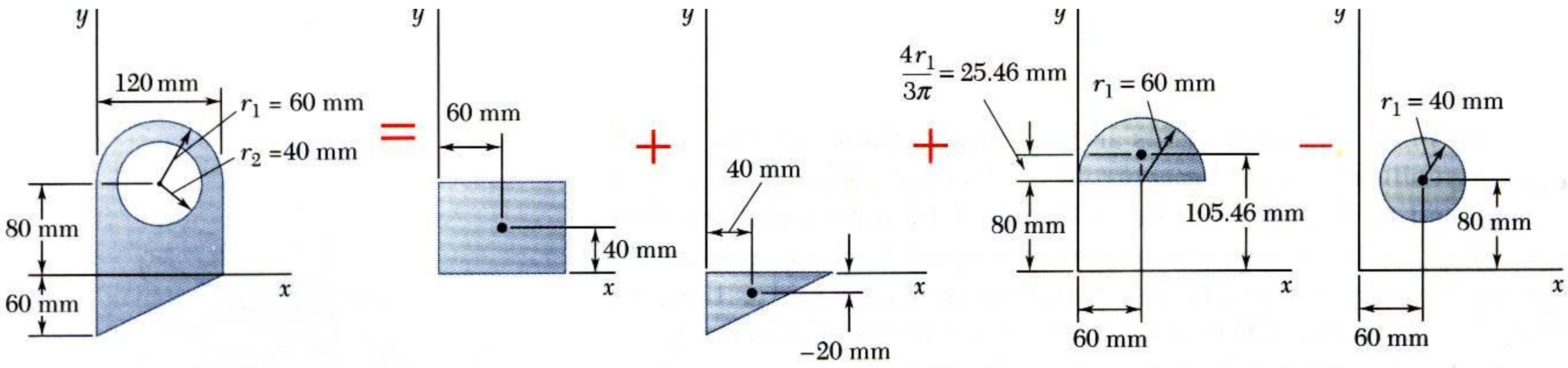
Sample Problem



For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.



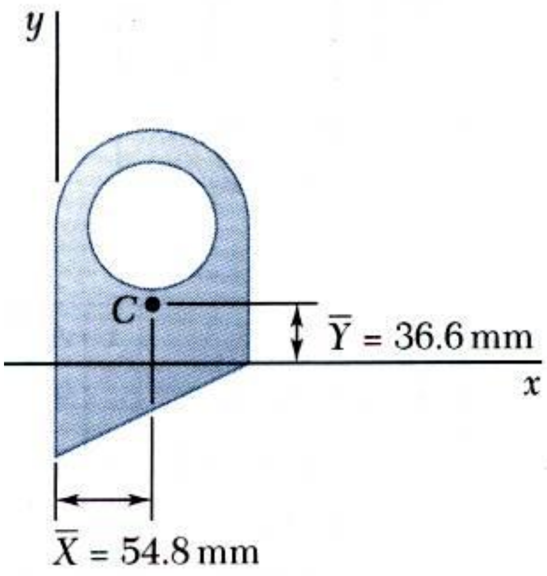
Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$S_x = +506.2 \times 10^3 \text{ mm}^3$$

$$S_y = +757.7 \times 10^3 \text{ mm}^3$$

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



$$\overline{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\boxed{\overline{X} = 54.8\,\text{mm}}$$

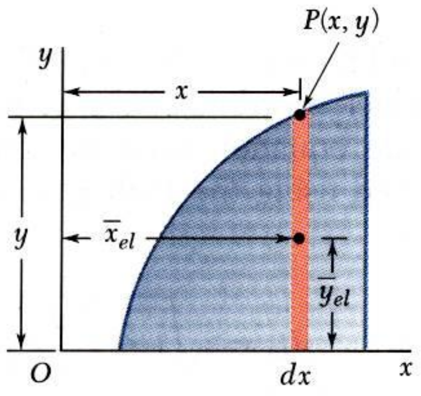
$$\overline{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\boxed{\overline{Y} = 36.6\,\text{mm}}$$

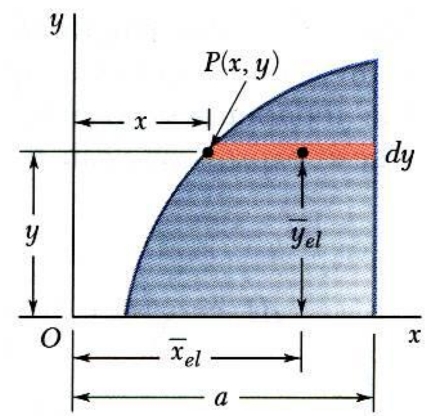
Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$
$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$

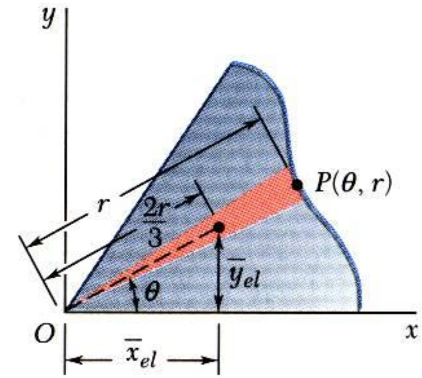
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int x\left(y\,dx\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{y}{2}\left(y\,dx\right)$$

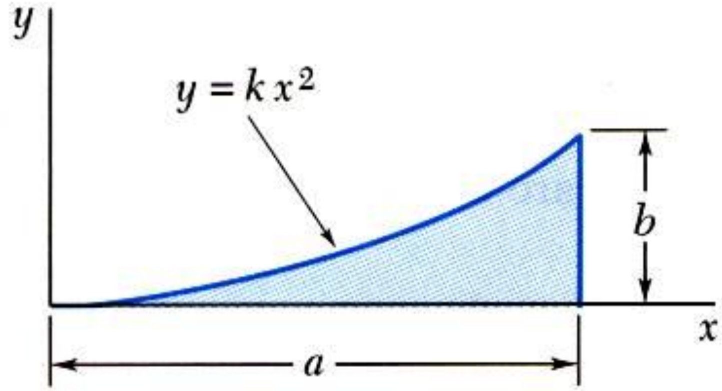


$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}\left[\left(a-x\right)dy\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int y\left[\left(a-x\right)dy\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2\,d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2\,d\theta\right)$$

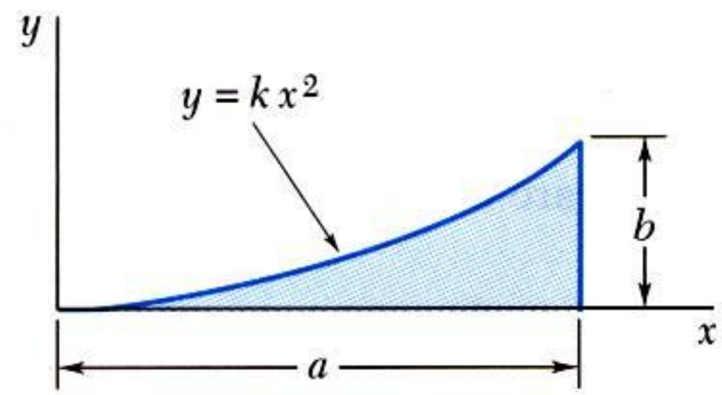
Sample Problem

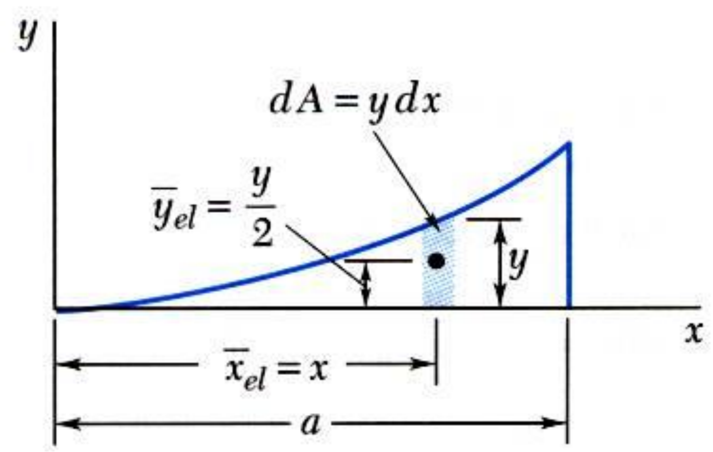


Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.





SOLUTION:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \;\Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

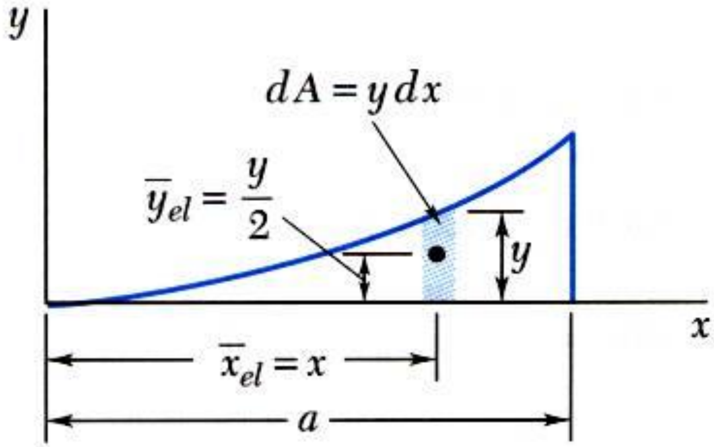
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

- Using vertical strips, perform a single integration to find the first moments.

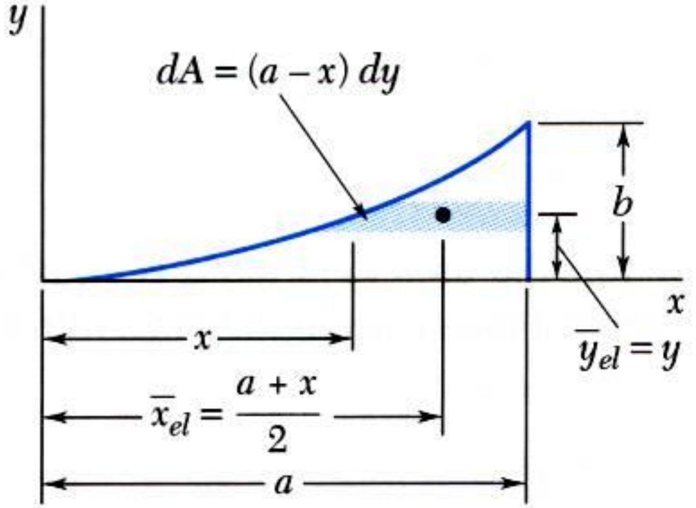


$$S_y = \int \bar{x}_{el} dA = \int xy dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$S_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$



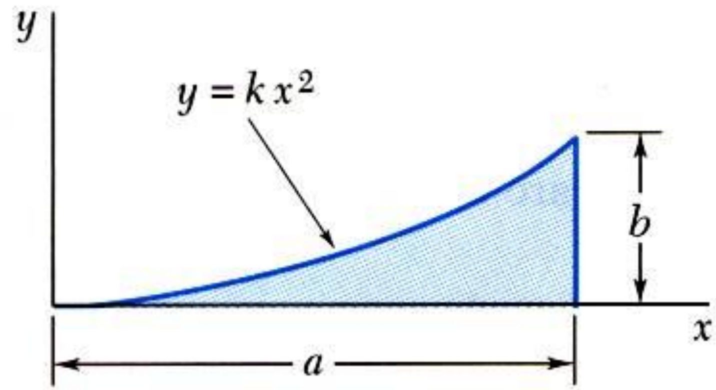
- Or, using horizontal strips, perform a single integration to find the first moments.

$$S_y = \int \overline{x}_{el} dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$S_x = \int \overline{y}_{el} dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$



- Evaluate the centroid coordinates.

$$\bar{x}A = S_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4}$$

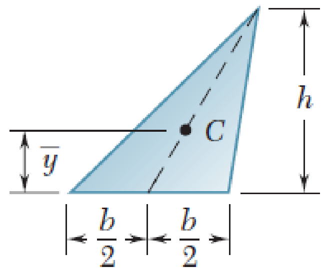
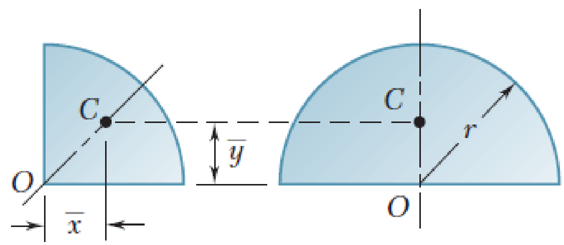
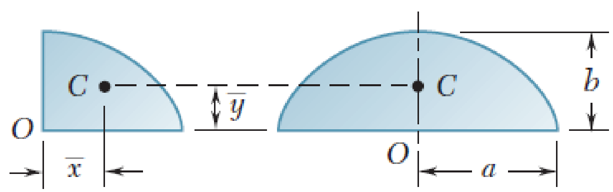
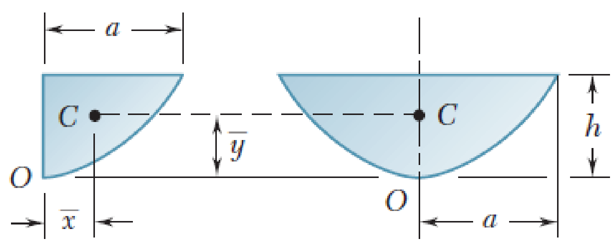
$$\boxed{\bar{x} = \frac{3}{4}a}$$

$$\bar{y}A = S_x$$

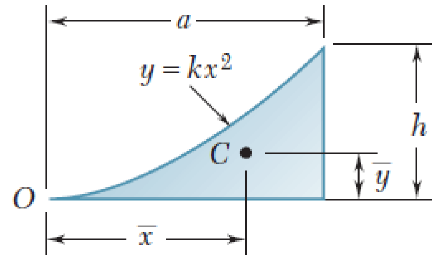
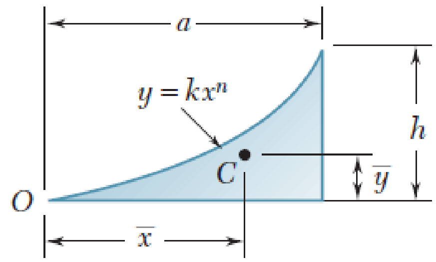
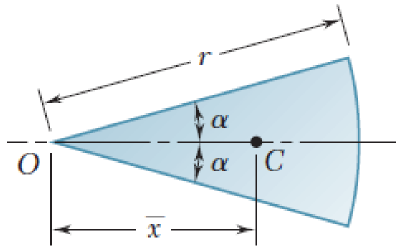
$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\boxed{\bar{y} = \frac{3}{10}b}$$

Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$

Centroids of Common Shapes of Areas

Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2