Torsion

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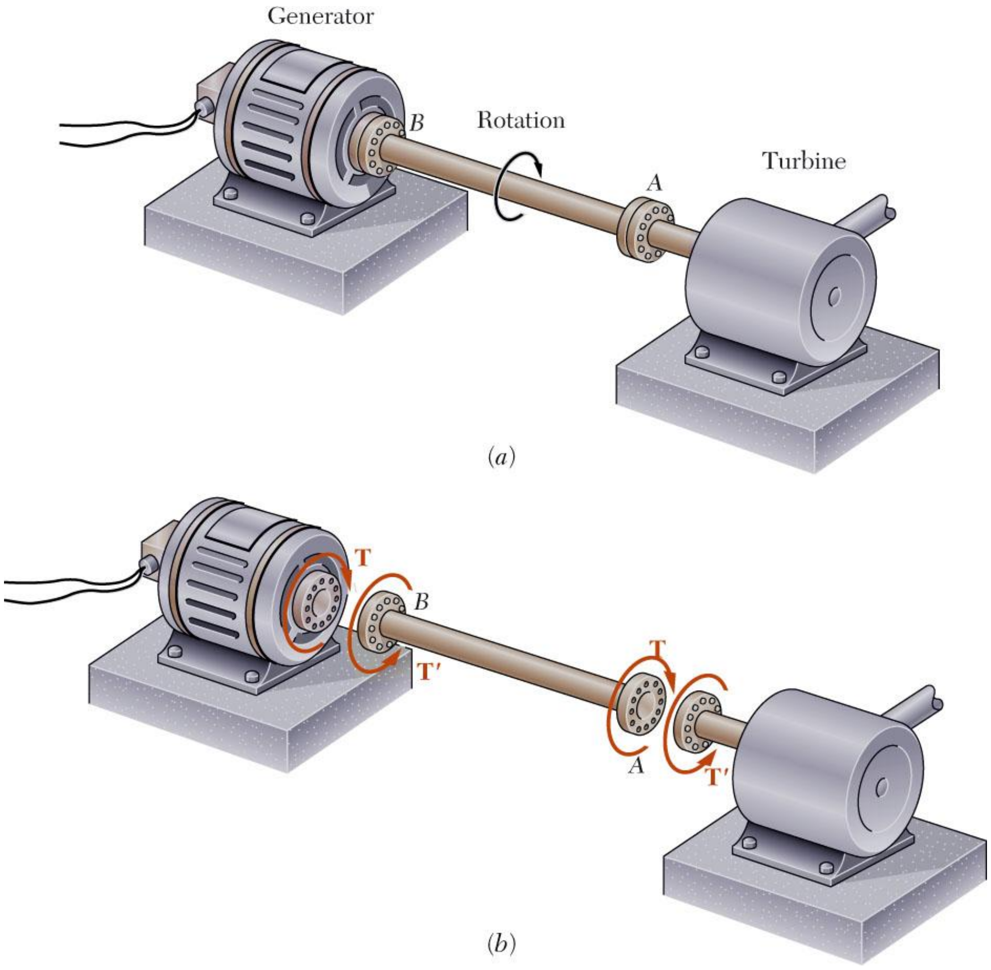
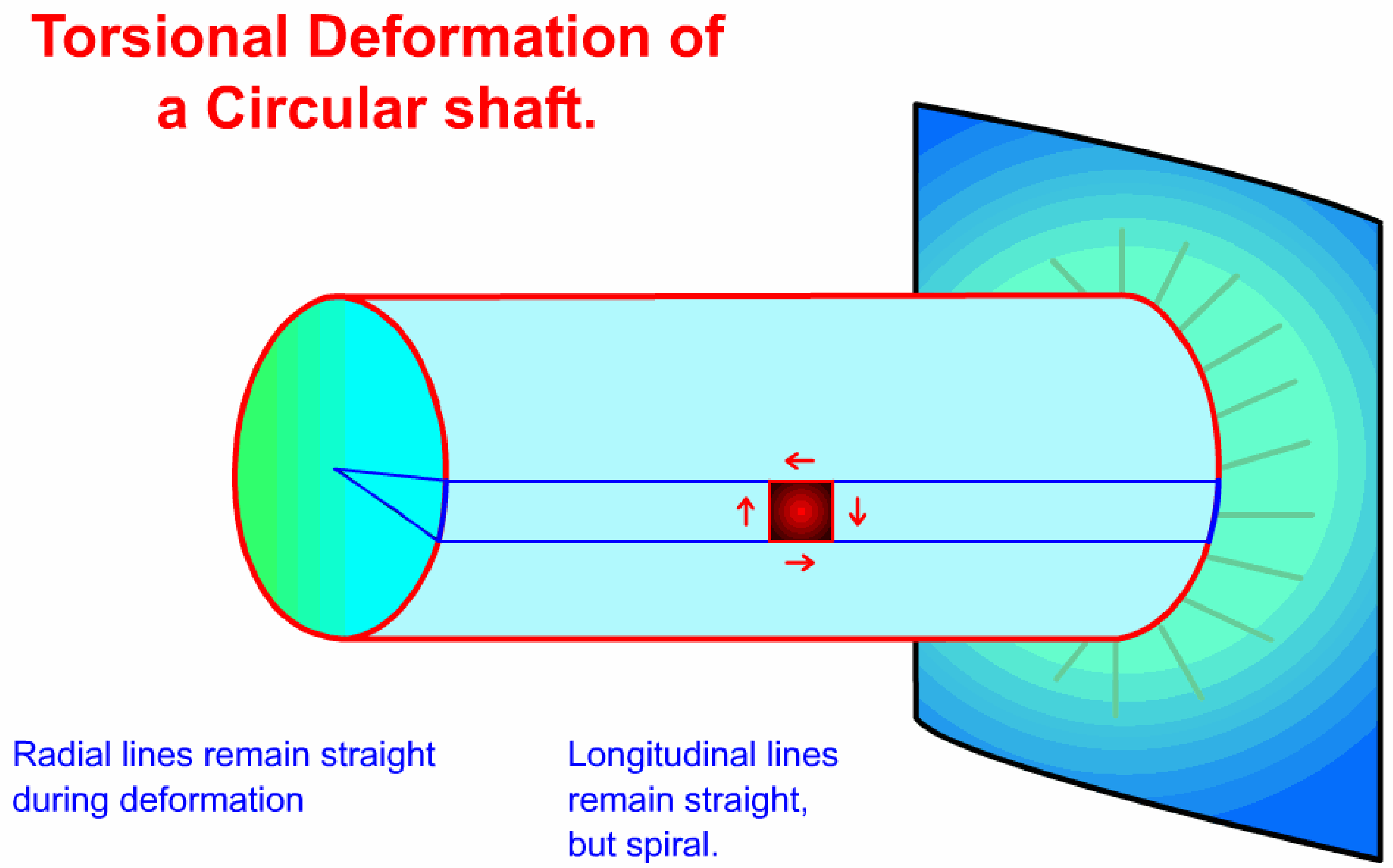
Introduction to Torsion



- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator

- Generator creates an equal and opposite torque *T*
- Cross section remains planar
- Interested in stresses and strains of circular shafts subjected to twisting couples or torques

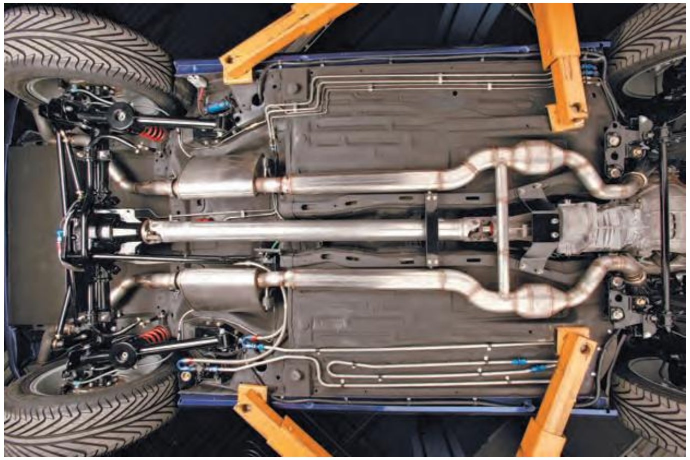
Introduction to Torsion



Notice the deformation of the rectangular element when it is subjected to a torque

Examples of Torsion Shafts

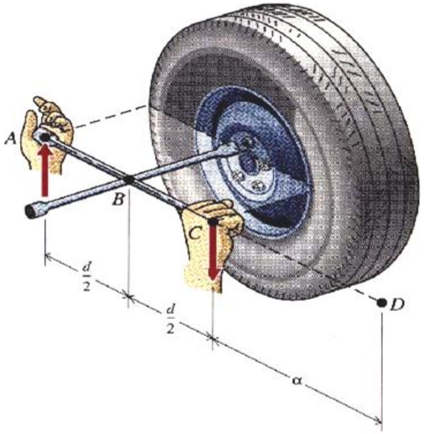
Power generation shaft



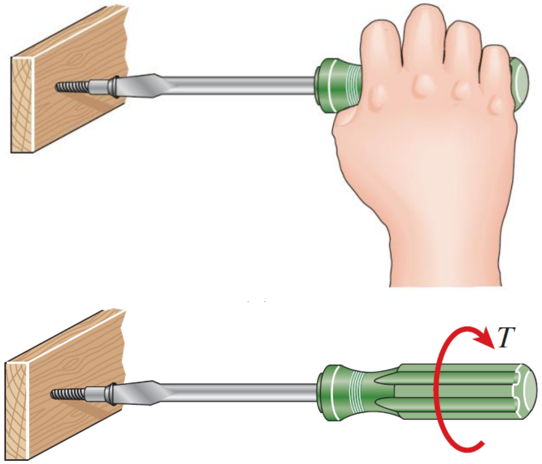
Automotive power train shaft

Ship drive shaft

Complex crank shaft

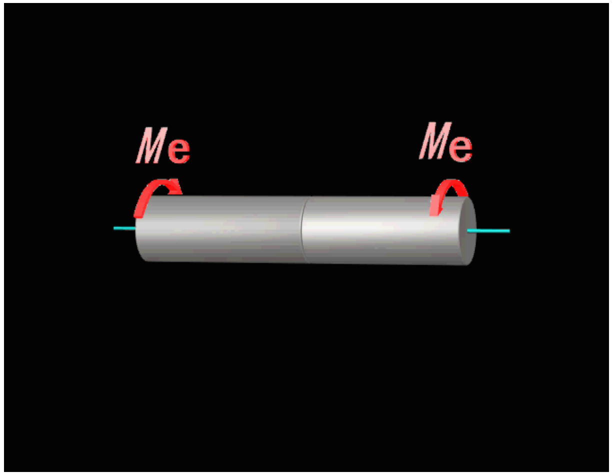


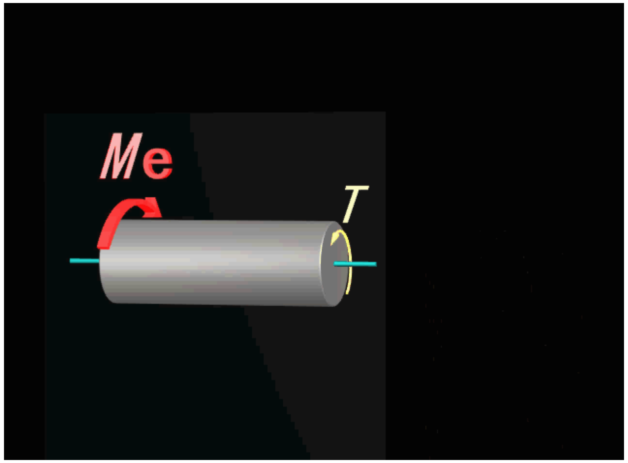
Tire shift drive

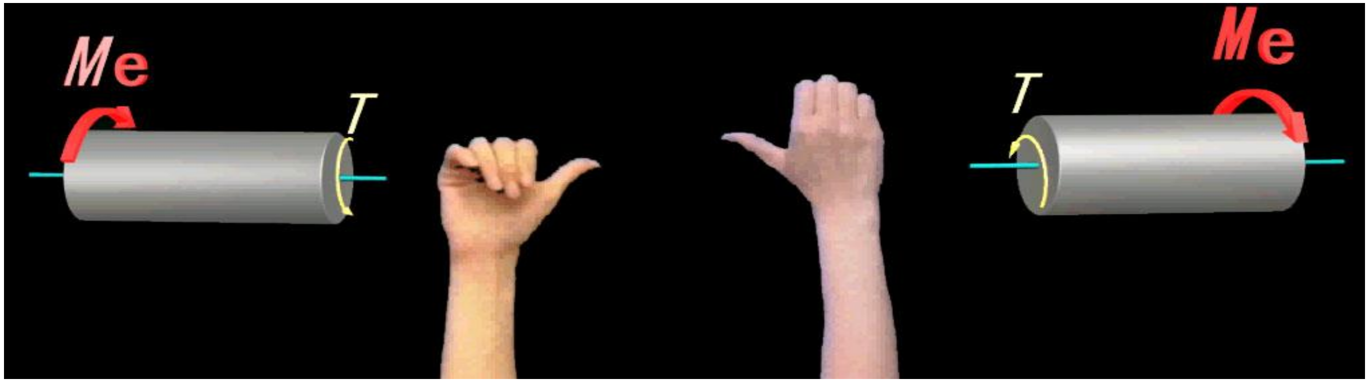


Screwdriver

Sign Convention of Torque



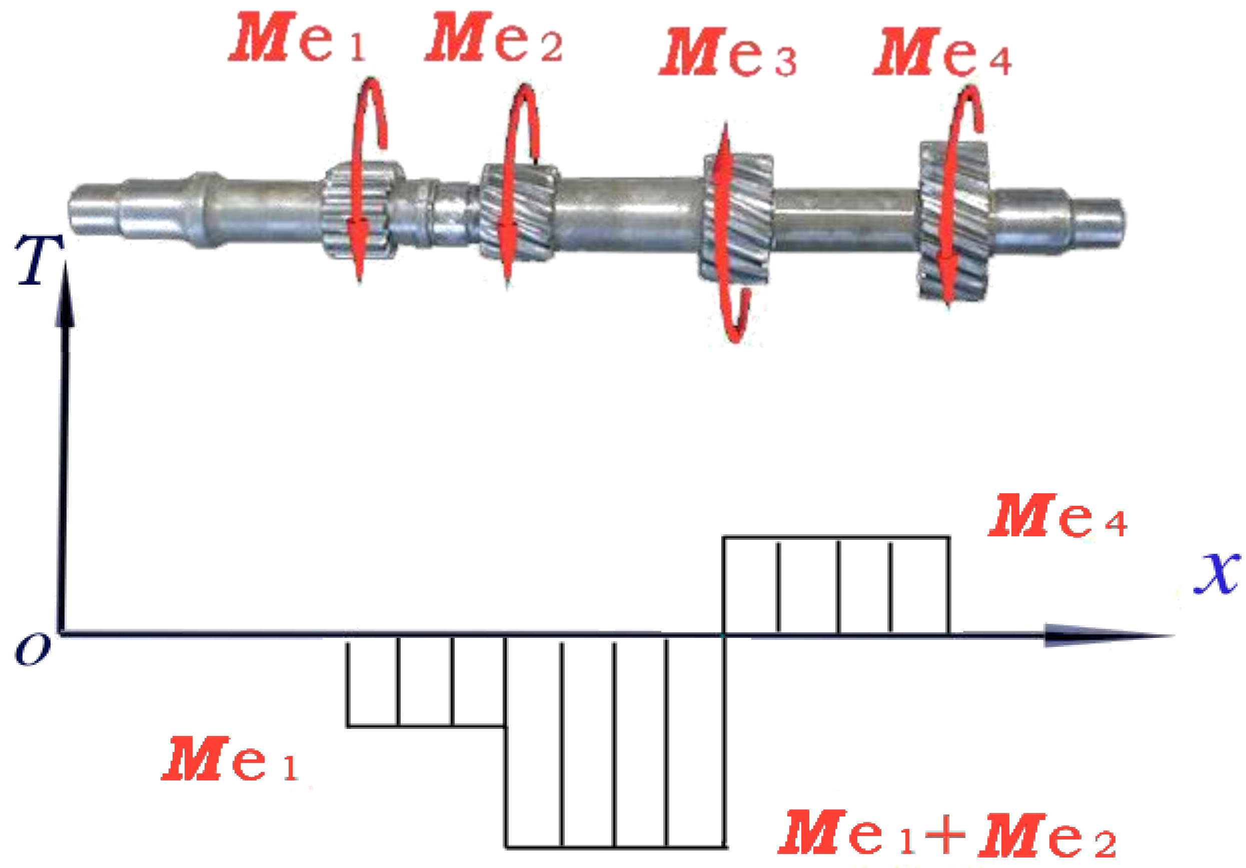




- Sign (Positive) convention (right-hand rule): thumb – cross-section normal; rest fingers – torque

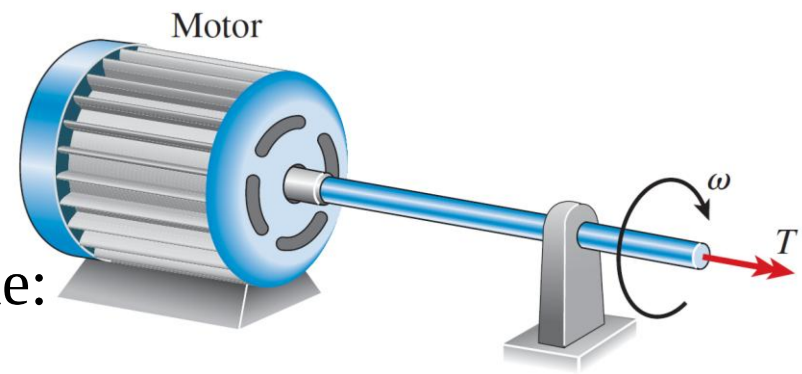
Torque Diagram

- Abscissa: cross-section position
- Ordinate: torque



Power & Torque

- Transformation from electric power to mechanical power



- motor power: P (kW); generated torque: M_e (N·m):

- n (rpm, revolutions per minute):

$$M_e \times 2\pi n = P \times 1000 \times 60 \Rightarrow M_e = \frac{1000 \times 60}{2\pi} \frac{P}{n} = 9549 \frac{P}{n}$$

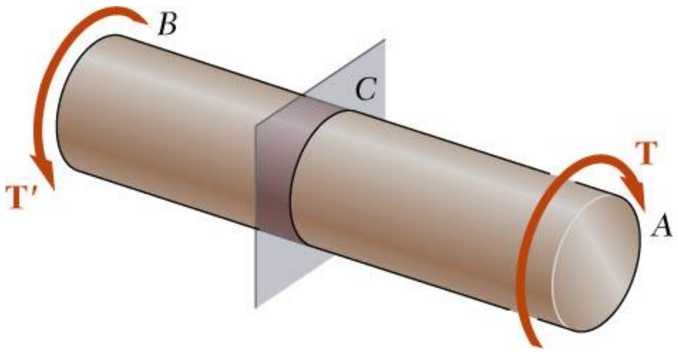
- n (rps, revolutions per second):

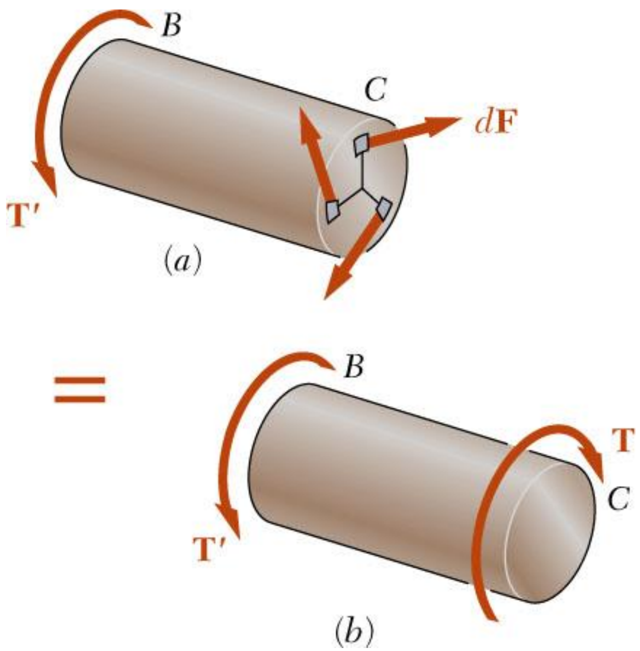
$$M_e \times 2\pi n = P \times 1000 \Rightarrow M_e = \frac{1000}{2\pi} \frac{P}{n} = 159 \frac{P}{n}$$

- ω (rad/s, radian per second):

$$M_e \times \omega = P \times 1000 \Rightarrow M_e = 1000 \frac{P}{\omega}$$

Internal Torque & Stress - Static Indeterminacy



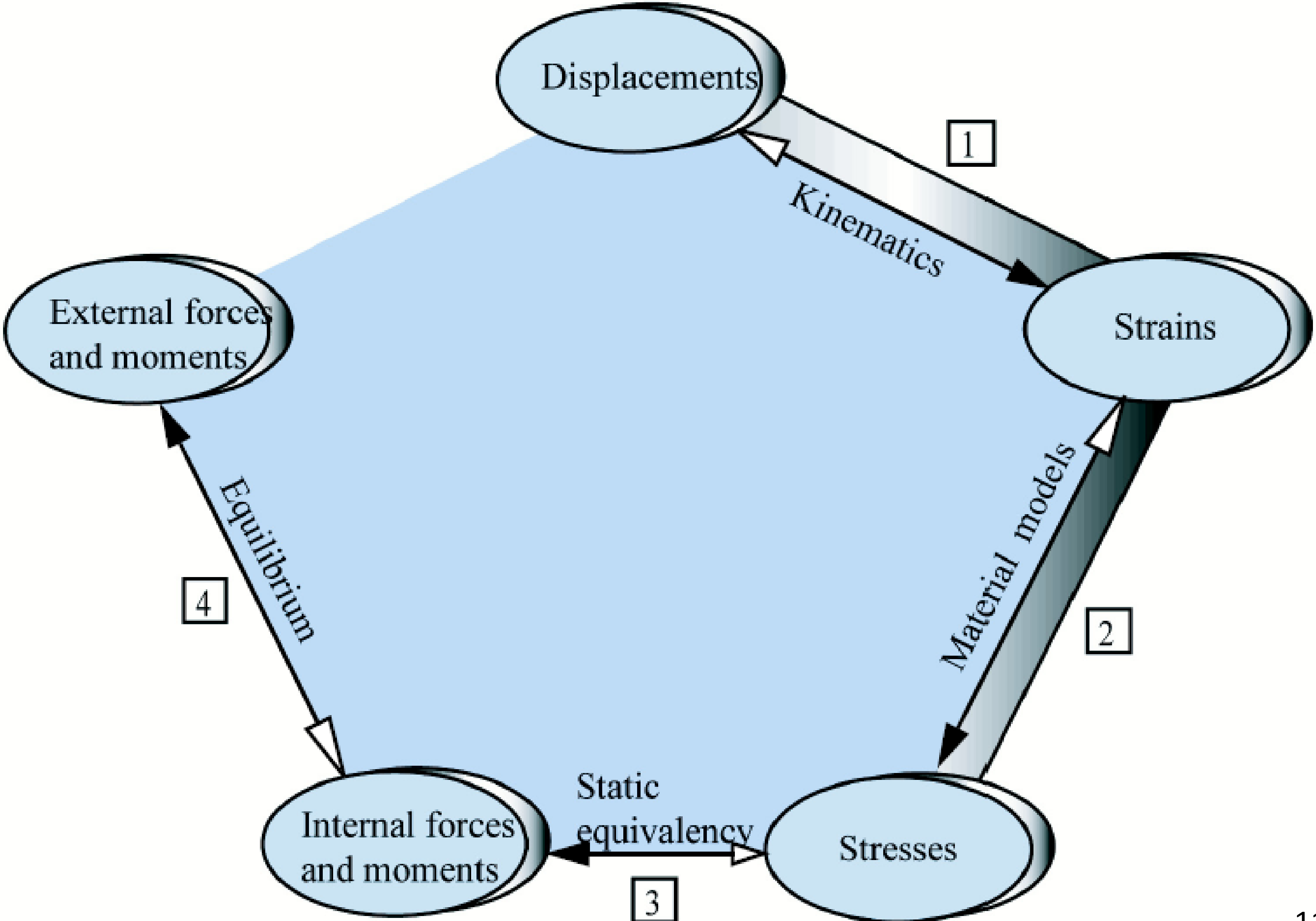


- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

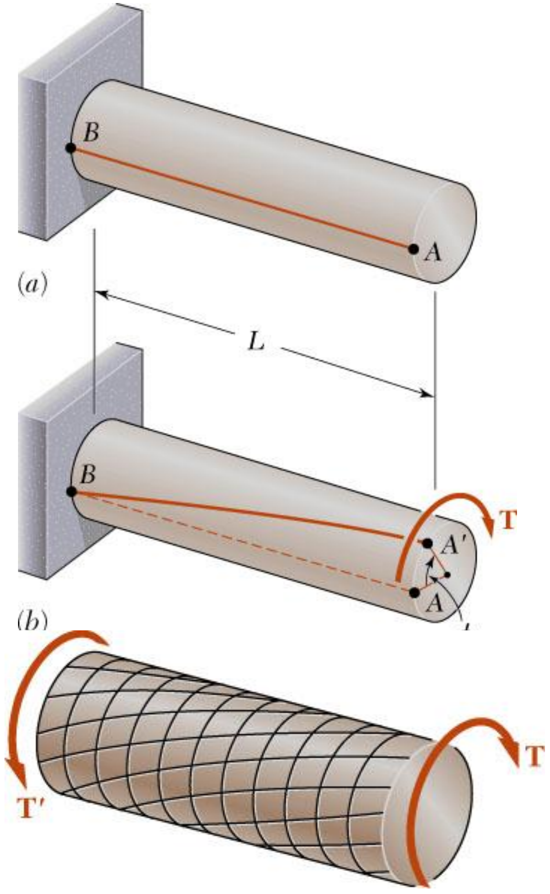
$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

General Relations Involved in Deformable Solids



Kinematics



- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

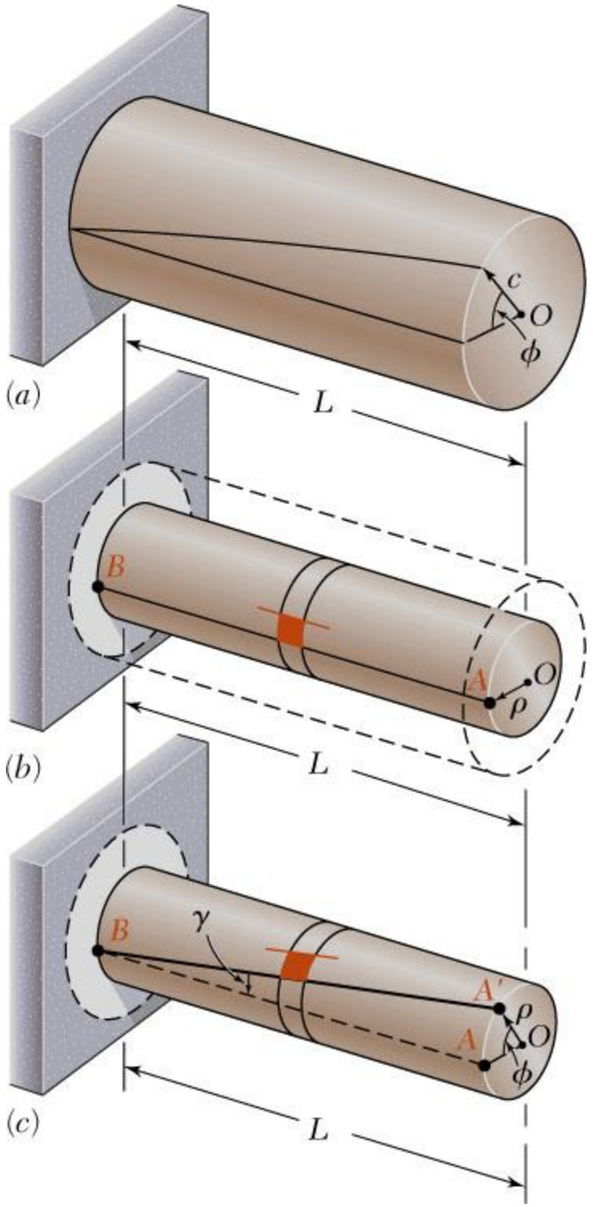
$$\phi \propto T, \qquad \phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.

- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.

- Since every cross section of the bar is identical, and since every cross section is subjected to the same internal torque, we say that the bar is in ***pure torsion***.

Kinematics



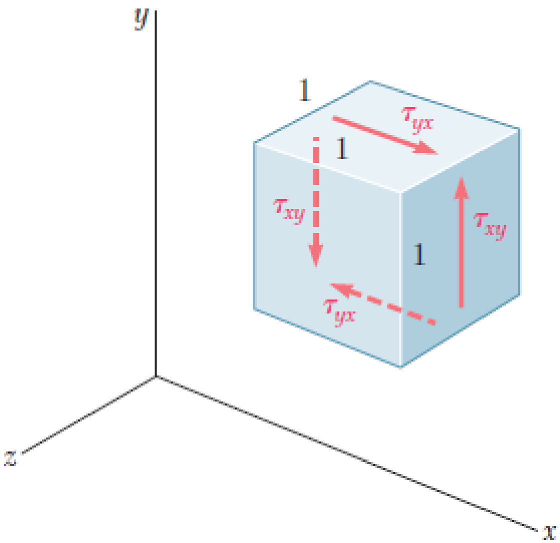
- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shearing strain may be related to the angle of twist.
- It follows that

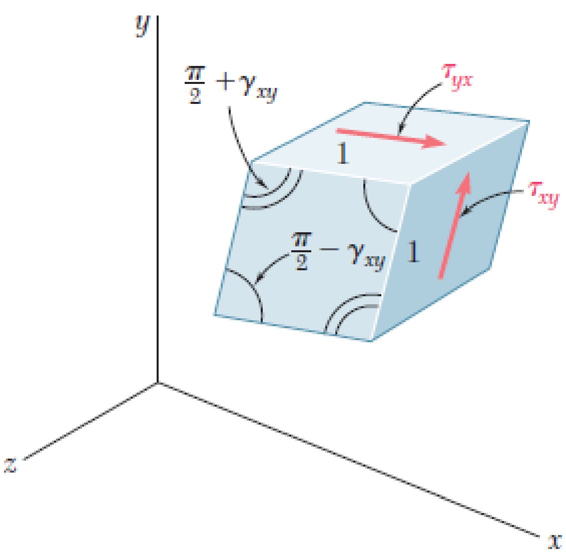
$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shearing strain is proportional to the angle of twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Hooke's Law for Shearing Deformation





- A cubic element subjected to a shearing stress will deform into a rhomboid. The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

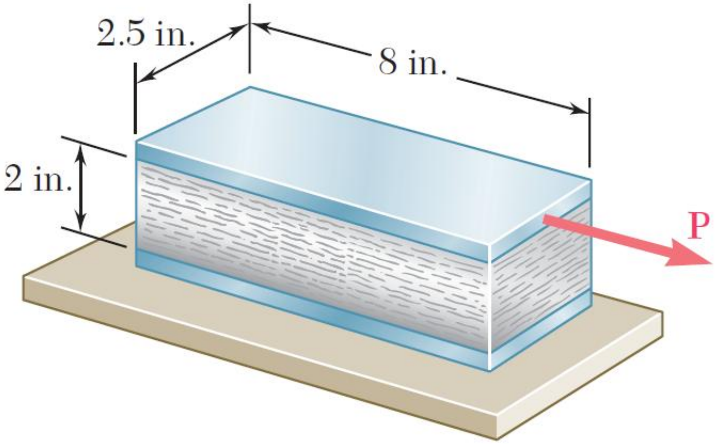
$$\tau_{xy} = f(\gamma_{xy})$$

- A plot of shearing stress vs. shear strain is similar the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where G is the modulus of rigidity or shear modulus.

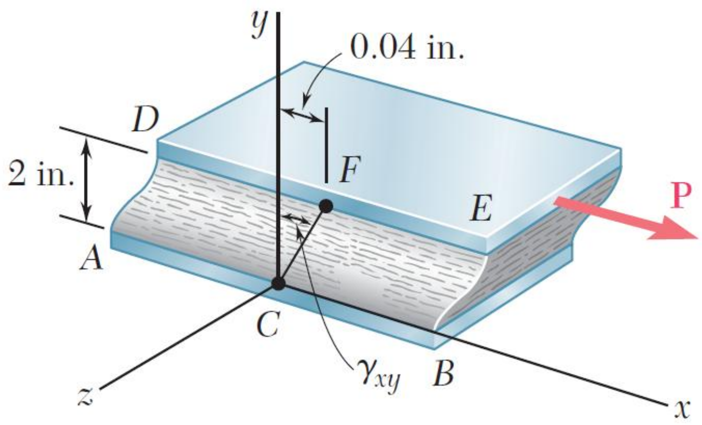
A Simple Example of Shearing Deformation



A rectangular block of material with modulus of rigidity G = 90 ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P.



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

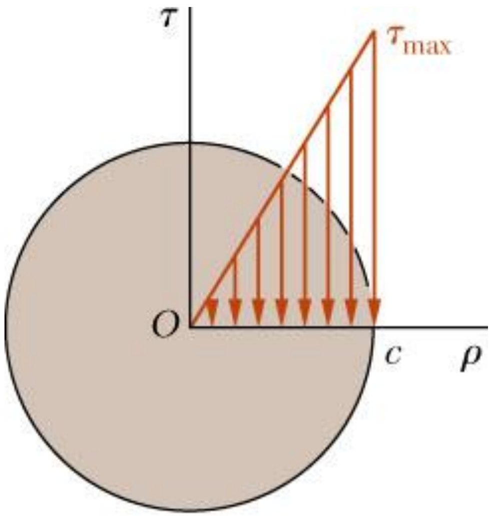
$$\tau_{xy} = G\gamma_{xy} = \left(90\times10^3\,\text{psi}\right)\left(0.020\,\text{rad}\right) = 1800\,\text{psi}$$

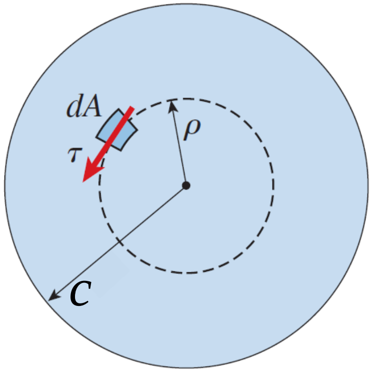
- Use the definition of shearing stress to find the force *P*.

$$P = \tau_{xy}A = \left(1800\,\text{psi}\right)\left(8\,\text{in.}\right)\left(2.5\,\text{in.}\right) = 36\times10^3\,\text{lb}$$

$$\boxed{P = 36.0\,\text{kips}}$$

Static Equivalency





- Multiplying the previous equation by the shear modulus, $G\gamma = \frac{\rho}{c} G\gamma_{max}$

  From Hooke's Law, $\tau = G\gamma$, so $\tau = \frac{\rho}{c}\tau_{max}$

  The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau \; dA = \frac{\tau_{max}}{c}\int \rho^2 \; dA = \frac{\tau_{max}}{c} I_p$$

- The results are known as the *elastic torsion formulas*, (***I*****$_p$** **– Polar moment of inertia**) $\tau_{max} = \frac{Tc}{I_p}$ and $\tau = \frac{T\rho}{I_p}$

Torsional Stress & Angle of Twist

- Torsional section modulus: [m³]: $W_p = I_P / c$.
- Torsional stress

$$\Rightarrow \tau_{\max} = \frac{Tc}{I_p} = \frac{T}{W_p}, \qquad \tau = \frac{T\rho}{I_p} = \frac{T}{W_p}\frac{\rho}{c}$$

- Angle of twist per unit length

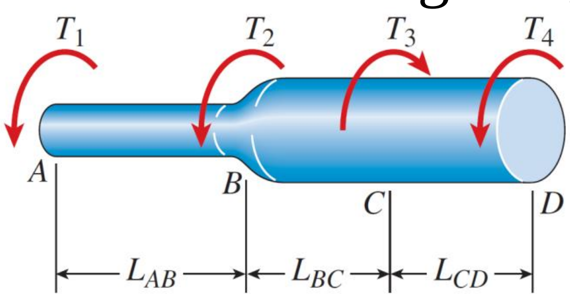
$$L\gamma = \rho\phi \quad \Rightarrow \quad \frac{\phi}{L} = \frac{\gamma}{\rho} = \frac{1}{\rho}\frac{\tau}{G} = \frac{1}{\rho}\frac{1}{G}\frac{T\rho}{I_p} = \frac{T}{GI_p} \quad \Rightarrow \quad \phi = \frac{TL}{GI_p}$$

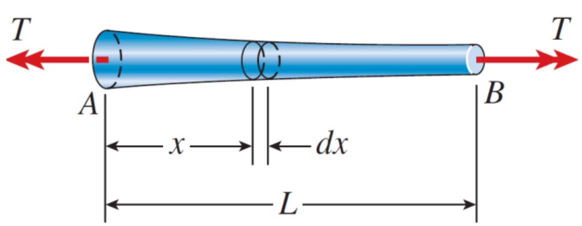
- GI_p: Torsional rigidity [N•m²]

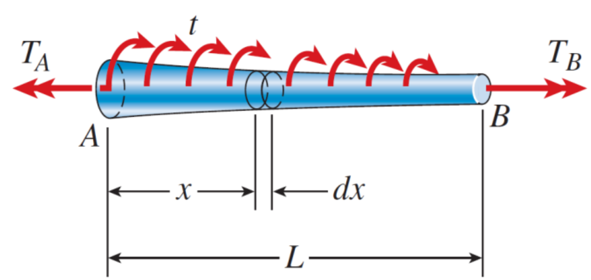
- The equations are limited to bars of circular cross section (either solid or hollow) that behave in a linearly elastic manner.

Nonuniform Torsion of Circular Shafts

- Pure torsion formula refers to torsion of a prismatic bar subjected to torques acting only at the ends.
- Nonuniform torsion differs from pure torsion in that the bar need not to be prismatic and the applied torques may act anywhere along the axis of the bar.
- Bars in nonuniform torsion can be analyzed by applying the formulas of pure torsion to finite segments of the bar and then adding the results, or by applying the formulas to differential elements of the bar and then integrating.







$$\phi = \sum_i \phi_i = \sum_i \frac{T_i L_i}{G_i (I_p)_i}$$

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)\,dx}{GI_p(x)}$$

Strength Analysis

- Strength check: $\tau_{\max} = (T / W_P)_{\max} \leq [\tau]$

- Cross-section design: $W_p \geq T_{\max} / [\tau]$

- Maximum allowable load: $T_{\max} \leq W_p [\tau]$

Stiffness Condition

- Angle of twist per unit length: $\phi' = \phi/L = T/GI_p$
- Stiffness condition: $\phi'_{\max} \leq [\phi']$

  $[\phi']$: allowable angle of twist per unit length

  [radian/m] or [degrees/m]

  ✓NC machine: $[\phi']$ = 0.15-0.3 degrees/m

  ✓Ordinary shaft: $[\phi']$ = 0.5-2.0 degrees/m

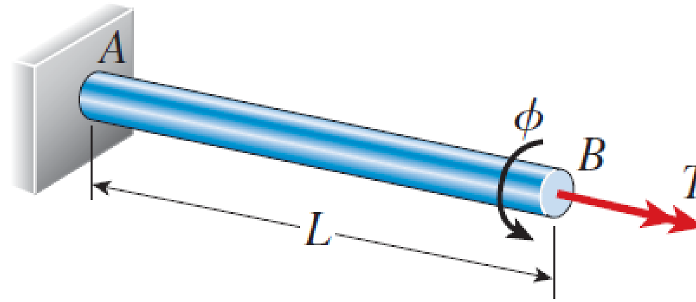
  ✓Shaft of drilling machine: $[\phi']$ = 2.0-4.0 degrees/m

- Stiffness check;
- Cross-section design;
- Allowable load.

Strain Energy

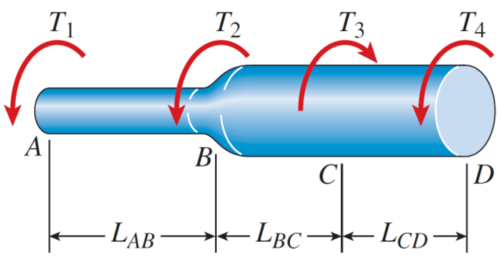
- Strain energy and work in pure torsion

$$U = W = \frac{1}{2}T\phi = \frac{T^2 L}{2GI_p} = \frac{GI_p}{2L}\phi^2 \qquad \left(\phi = \frac{TL}{GI_p}\right)$$

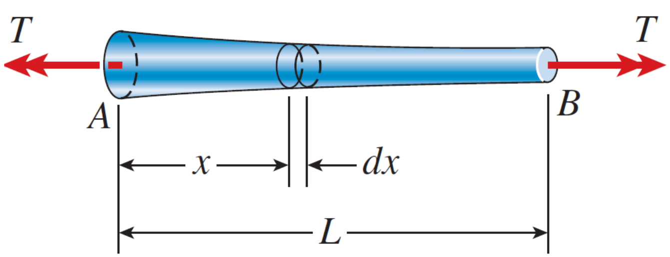


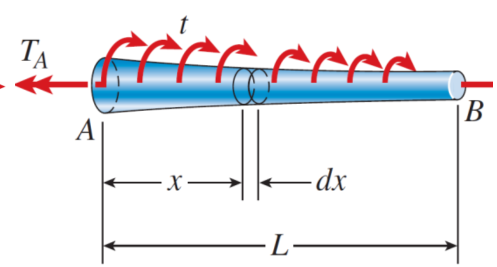
Prismatic bar in pure shear

- Nonuniform torsion

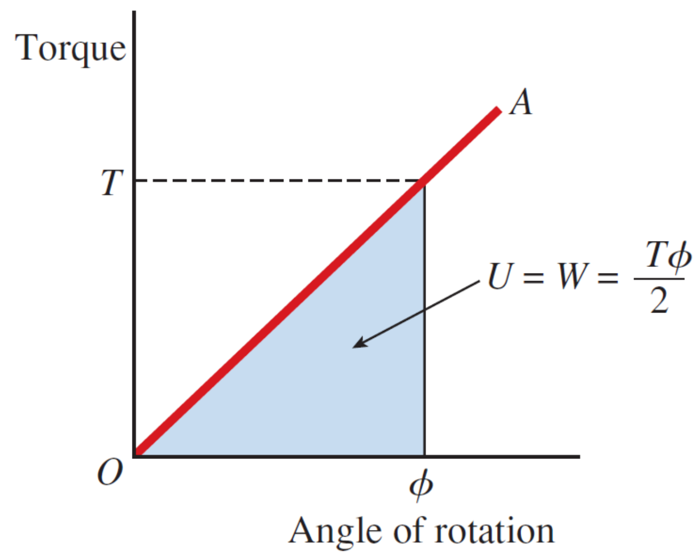


$$U = \sum_i U_i = \sum_i \frac{T_i^2 L_i}{2G_i \left(I_p\right)_i}$$





$$U = \int_0^L dU = \int_0^L \frac{\left[T(x)\right]^2 dx}{2GI_p(x)}$$



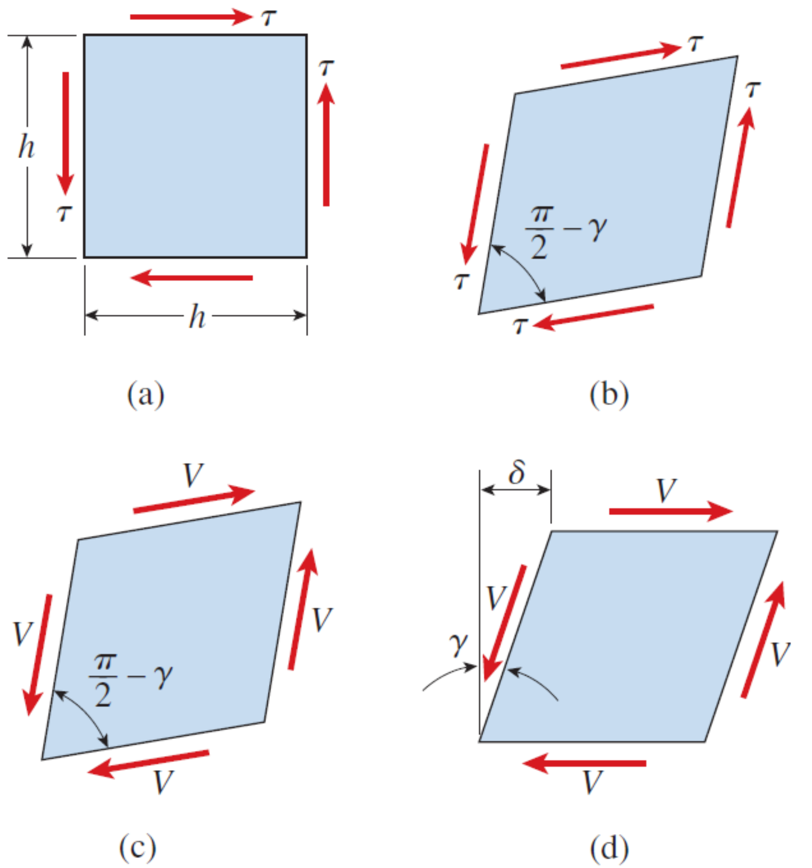
Torque-rotation diagram

Strain Energy Density

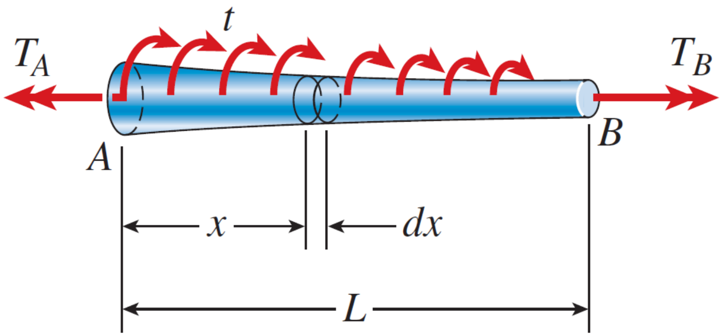
- Consider a differential cube of side h subjected to shearing stresses τ on its sides

$$U = W = \frac{1}{2}V\delta = \frac{1}{2}\left(\tau h^2\right)\left(\gamma h\right) = \frac{1}{2}\tau\gamma h^3$$

$$u = \frac{U}{h^3} = \frac{1}{2}\tau\gamma = \frac{1}{2}G\gamma^2 = \frac{1}{2G}\tau^2$$



- Total strain energy in torsion



$$U = \int_V u dV = \int_L \int_A u dA dx = \int_L \int_A \frac{\tau^2}{2G} dA dx$$

$$= \int_L \int_A \frac{1}{2G}\left(\frac{T\rho}{I_p}\right)^2 dA dx = \int_L \frac{T^2}{2GI_p^2}\left\{\int_A \rho^2 dA\right\} dx = \int_L \frac{T^2 dx}{2GI_p}$$

Polar Moment of Inertia & Section Modulus

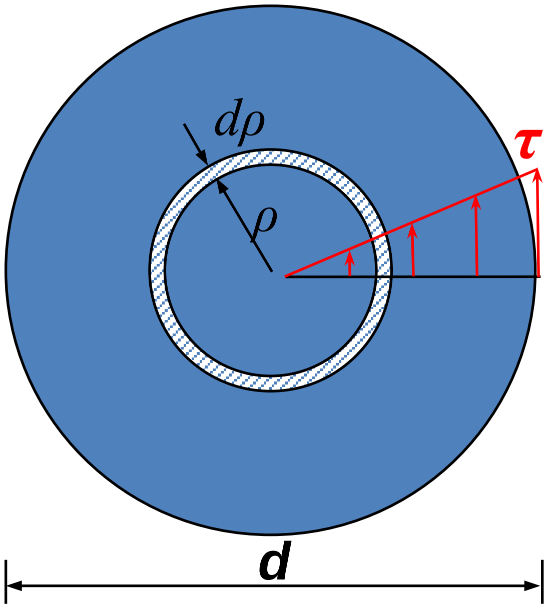
- Solid circular shafts

$$dA = 2\pi\rho d\rho$$

$$I_p = \int_A \rho^2 dA$$

$$= \int_0^{\frac{d}{2}} \rho^2 \cdot 2\pi\rho d\rho = \frac{\pi d^4}{32}$$

$$W_p = \frac{I_p}{r} = \frac{\pi d^4}{32} \times \frac{2}{d} = \frac{\pi d^3}{16}$$



Polar Moment of Inertia & Section Modulus

- Hollow circular shafts

$$\boldsymbol{I_p = \int_A \rho^2 dA}$$

$$= \int_{\frac{d}{2}}^{\frac{D}{2}} \rho^2 \cdot 2\pi\rho d\rho$$

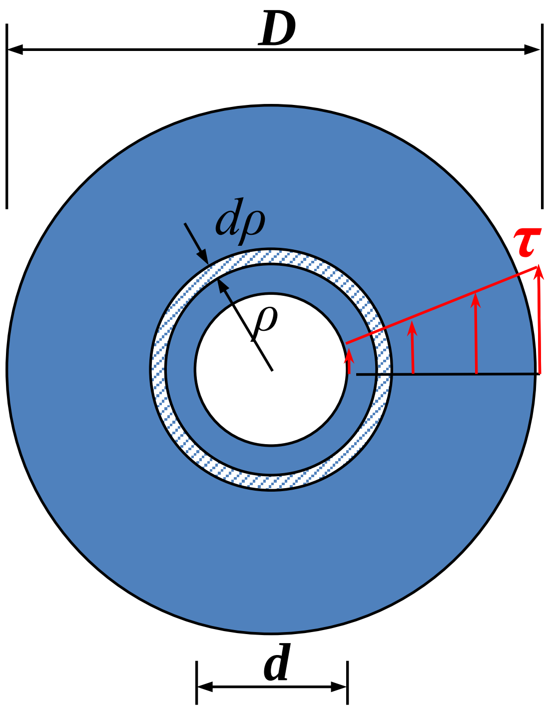
$$= \frac{\pi}{32}\left(D^4 - d^4\right)$$

$$= \frac{\pi D^4}{32}\left(1-\alpha^4\right)$$

$$\left(\alpha = \frac{d}{D}\right)$$

$$W_p = \frac{I_p}{D/2}$$

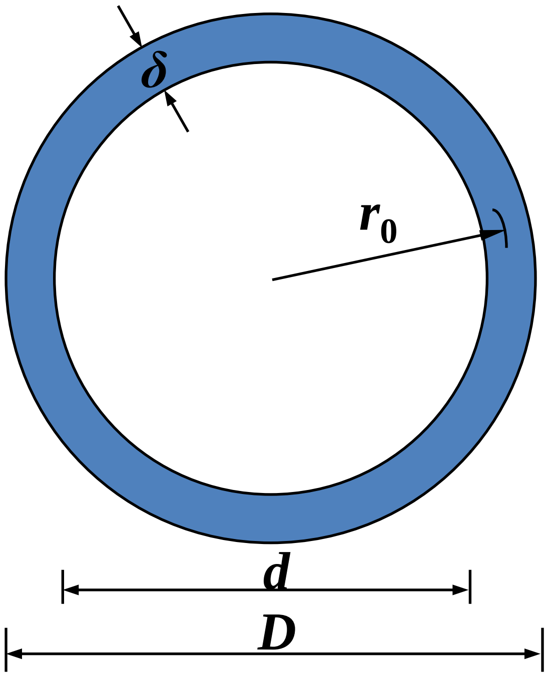
$$= \frac{\pi D^3}{16}\left(1-\alpha^4\right)$$



Polar Moment of Inertia & Section Modulus

- Thin-walled tubes: ($\delta << r_0$)

$$D = 2r_0 + \delta, \quad d = 2r_0 - \delta, \quad A \approx 2\pi r_0 \delta$$



$$I_p = \int_{\frac{d}{2}}^{\frac{D}{2}} \rho^2 \cdot dA \approx r_0^2 \int_{\frac{d}{2}}^{\frac{D}{2}} dA \approx 2\pi r_0^3 \delta$$

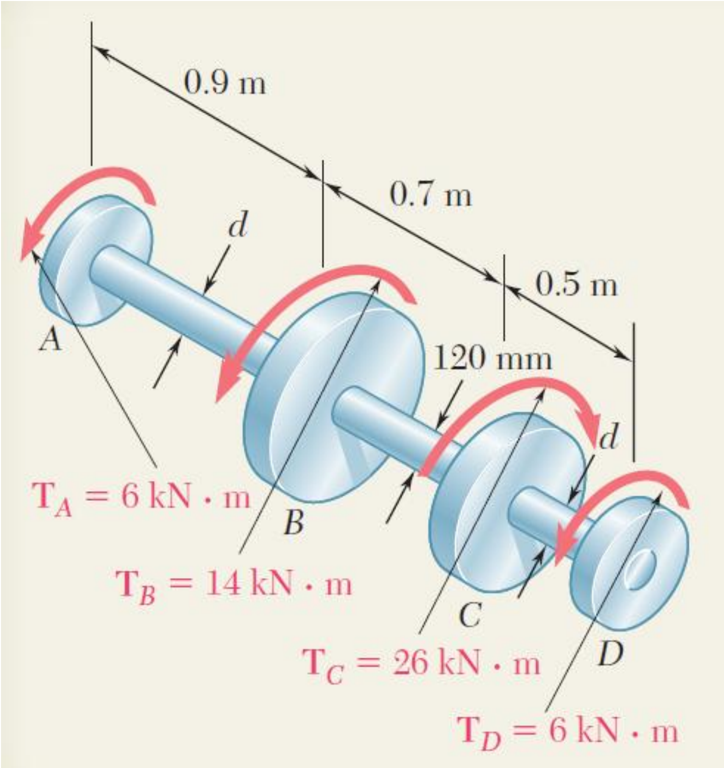
$$W_p = \frac{I_p}{r_0} \approx 2\pi r_0^2 \delta$$

- **Torsional stress in thin-walled tubes can be simplified to**

$$\tau = \frac{T}{W_p} = \frac{T}{2\pi r_0^2 \delta}$$

$$T = \int_A r(\tau dA) = r_0 \tau \int_A dA = r_0 \tau 2\pi r_0 \delta \quad \Rightarrow \quad \tau = \frac{T}{2\pi r_0^2 \delta}$$

Sample Problem



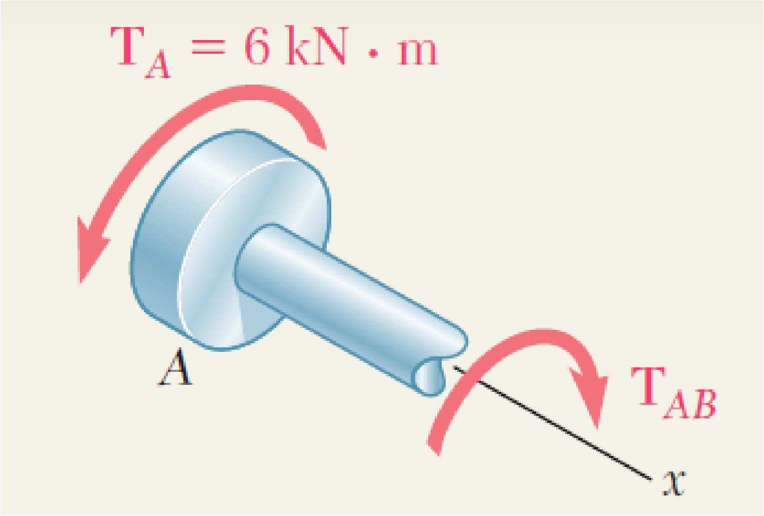
Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

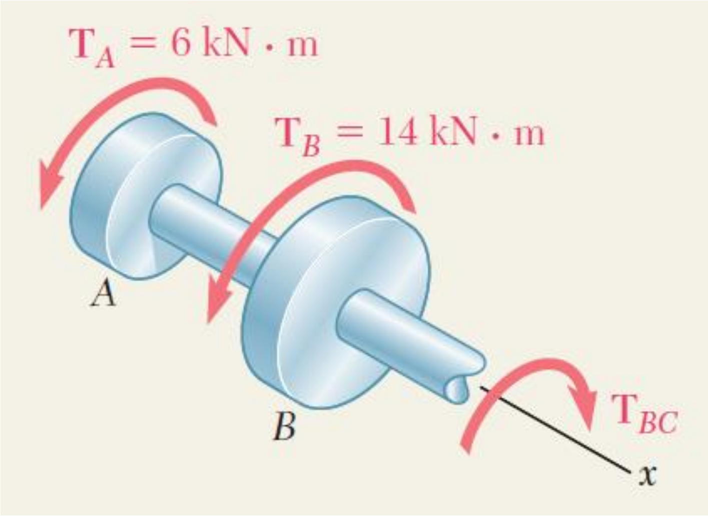
SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings

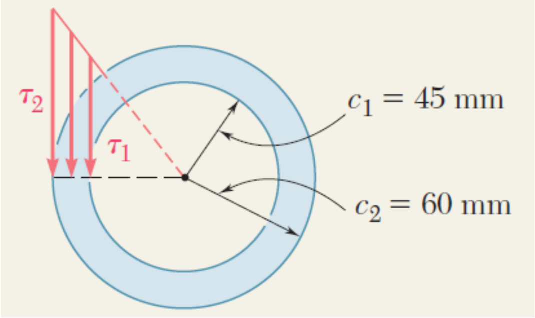




$$\sum T = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$
$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum T = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$
$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*



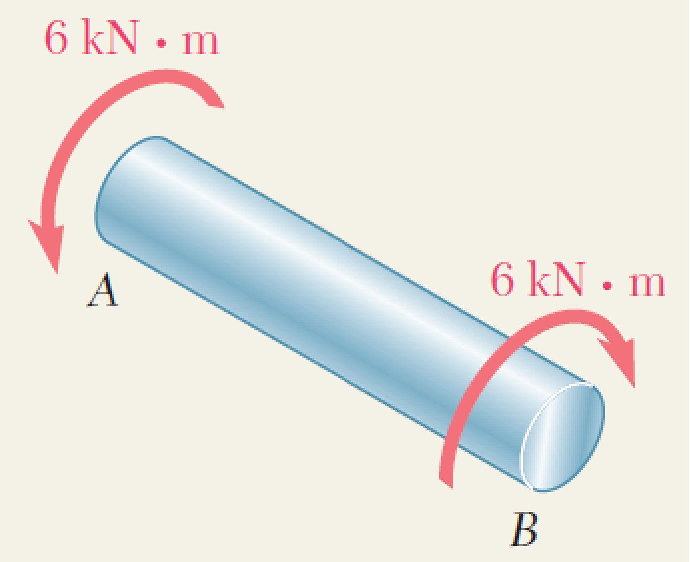
$$I_p = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$
$$= 13.92\times10^{-6}\,\text{m}^4$$

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{I_p} = \frac{(20\,\text{kN}\cdot\text{m})(0.060\,\text{m})}{13.92\times10^{-6}\,\text{m}^4}$$
$$= 86.2\,\text{MPa}$$

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2\,\text{MPa}} = \frac{45\,\text{mm}}{60\,\text{mm}}$$

$$\tau_{\min} = 64.7\,\text{MPa}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

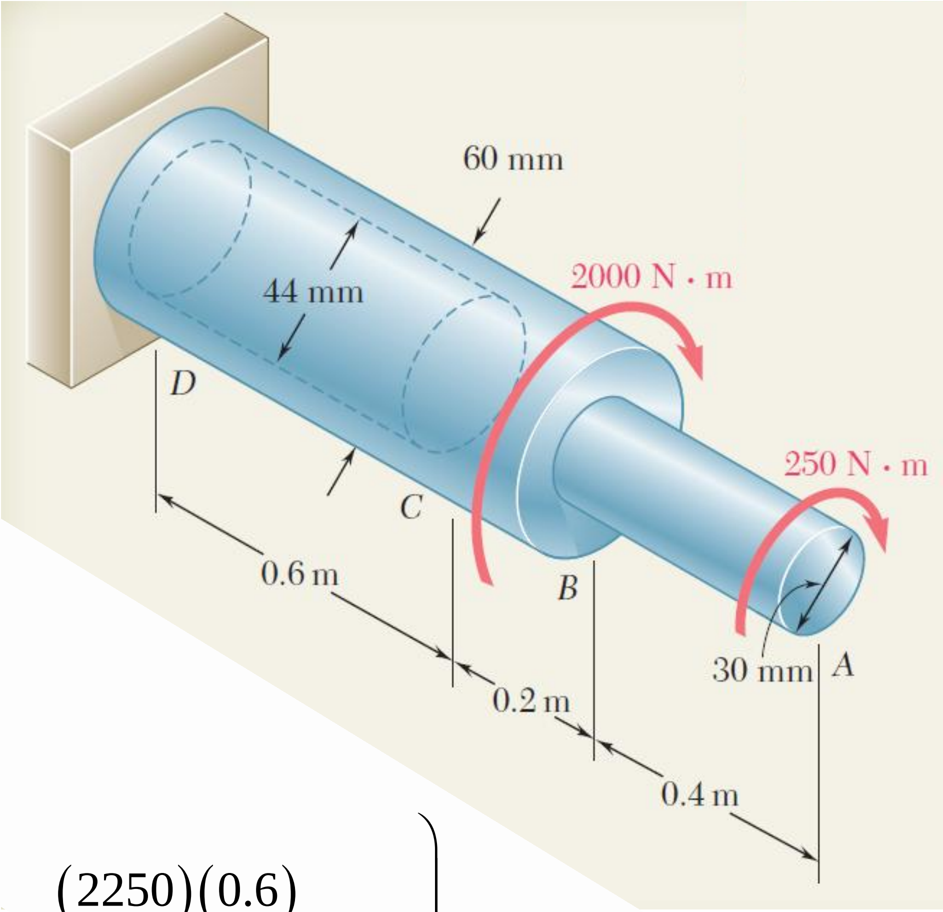


$$\tau_{\max} = \frac{Tc}{I_p} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\text{kN}\cdot\text{m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times10^{-3}\,\text{m}$$
$$d = 2c = 77.8\,\text{mm}$$

Sample Problem

- The horizontal shaft *AD* is attached to a fixed base at *D* and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion *CD* of the shaft. Knowing that the entire shaft is made of steel for which *G* = 77 GPa, determine the angle of twist at end *A*.



SOLUTION:

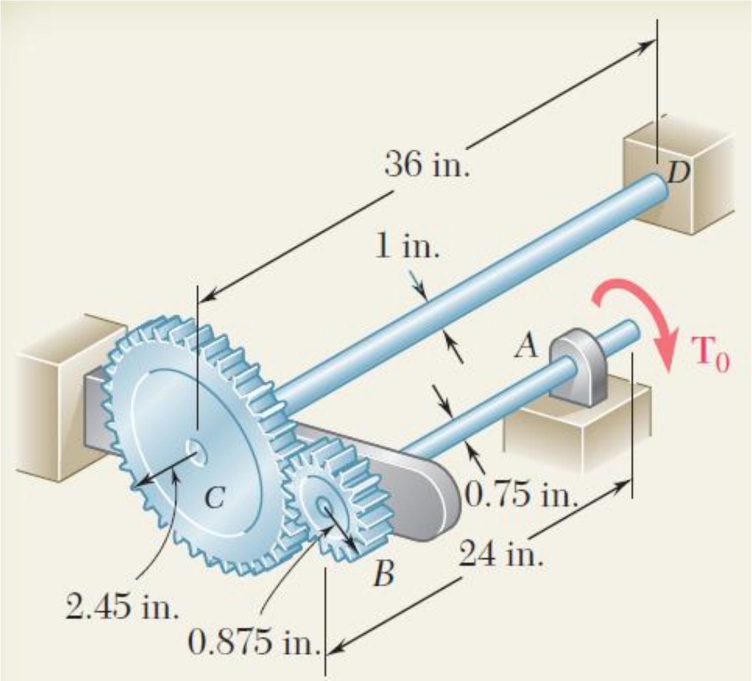
$$\phi = \frac{1}{G}\sum_i \left(\frac{T_{AB}L_{AB}}{I_{pAB}} + \frac{T_{BC}L_{BC}}{I_{pBC}} + \frac{T_{CD}L_{CD}}{I_{pCD}} \right)$$

$$= \frac{1}{G}\sum_i \left(\frac{(250)(0.4)}{\frac{\pi}{2}(0.015)^4} + \frac{(2250)(0.2)}{\frac{\pi}{2}(0.03)^4} + \frac{(2250)(0.6)}{\frac{\pi}{2}\left[(0.030)^4 - (0.022)^4\right]} \right)$$

$$= 0.01634 + 0.00459 + 0.01939$$

$$= 0.0403 \text{ rad} = 0.0403(360/2\pi) = 2.31^0$$

Sample Problem

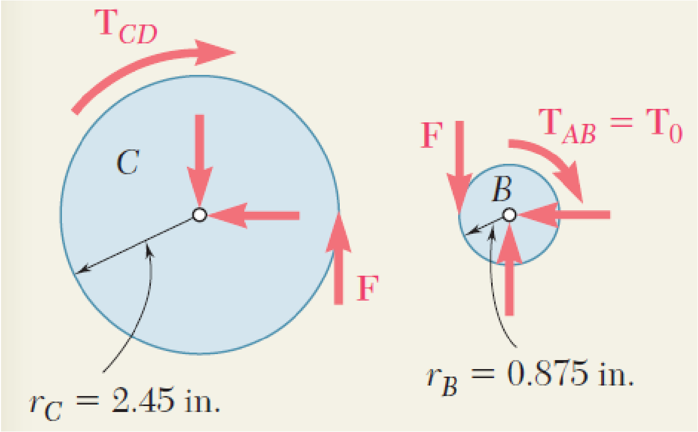


Two solid steel shafts are connected by gears. Knowing that for each shaft G = 11.2 x 10^6 psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0
- Apply a kinematic analysis to relate the angular rotations of the gears
- Find the maximum allowable torque on each shaft – choose the smallest
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0

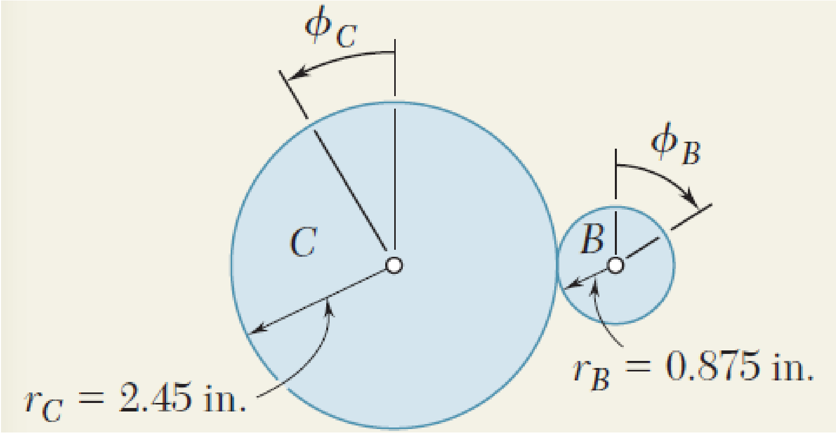


$$\sum M_B = 0 = F(0.875\text{in.}) - T_0$$

$$\sum M_C = 0 = F(2.45\text{in.}) - T_{CD}$$

$$T_{CD} = 2.8T_0$$

- Apply a kinematic analysis to relate the angular rotations of the gears

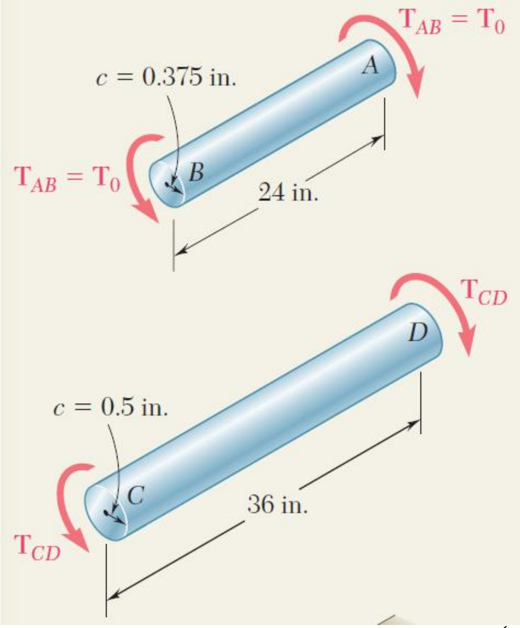


$$r_B\phi_B = r_C\phi_C$$

$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{2.45\text{in.}}{0.875\text{in.}}\phi_C$$

$$\phi_B = 2.8\phi_C$$

- Find the T_0 for the maximum allowable torque on each shaft



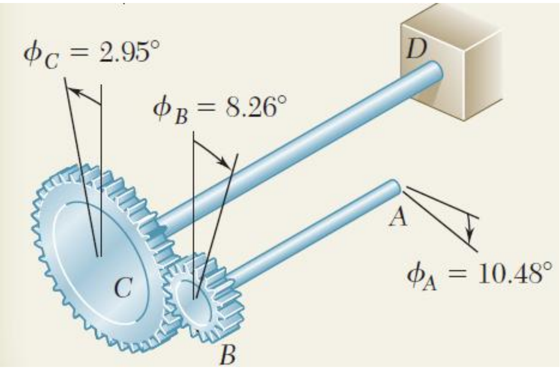
$$\tau_{\max} = \frac{T_{AB}c}{I_{pAB}} \qquad 8000\text{psi} = \frac{T_0(0.375\,\text{in.})}{\frac{\pi}{2}(0.375\,\text{in.})^4}$$

$$T_0 = 663\,\text{lb}\cdot\text{in.}$$

$$\tau_{\max} = \frac{T_{CD}c}{I_{pCD}} \qquad 8000\text{psi} = \frac{2.8T_0(0.5\,\text{in.})}{\frac{\pi}{2}(0.5\,\text{in.})^4}$$

$$T_0 = 561\,\text{lb}\cdot\text{in.}$$

- Find the corresponding angle of twist for each shaft and the net angular rotation of end A



$$\phi_{A/B} = \frac{T_{AB}L}{GI_{pAB}} = \frac{(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.375\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.387\,\text{rad} = 2.22^\text{o}$$

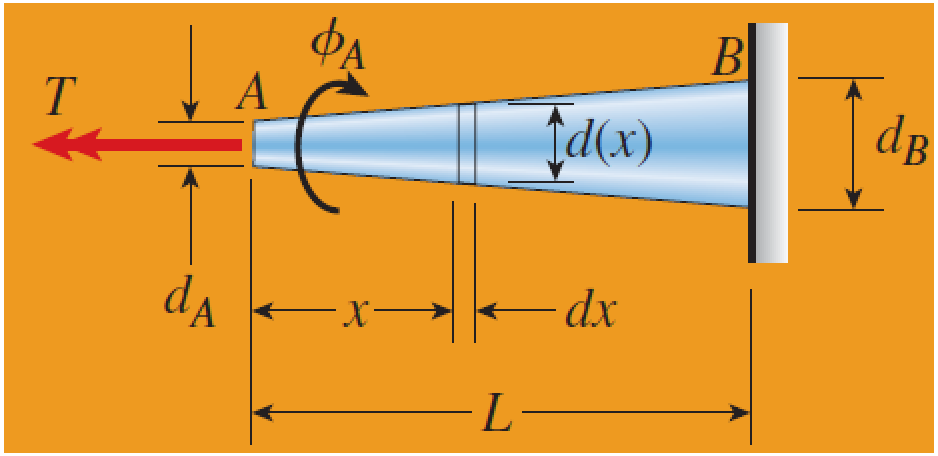
$$\phi_{C/D} = \frac{T_{CD}L}{GI_{pCD}} = \frac{2.8(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.5\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.514\,\text{rad} = 2.95^\text{o}$$

$$\phi_B = 2.8\phi_C = 2.8(2.95^\text{o}) = 8.26^\text{o}$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^\text{o} + 2.22^\text{o} = 10.48^\text{o}$$

Sample Problem



- Determine the angle of twist at end *A* of the shaft.

- Solution

$$d_x = d_A + \frac{x}{L}\left(d_B - d_A\right)$$

$$I_{px} = \frac{\pi d_x^4}{32} = \frac{\pi}{32}\left(d_A + \frac{x}{L}\left(d_B - d_A\right)\right)^4$$

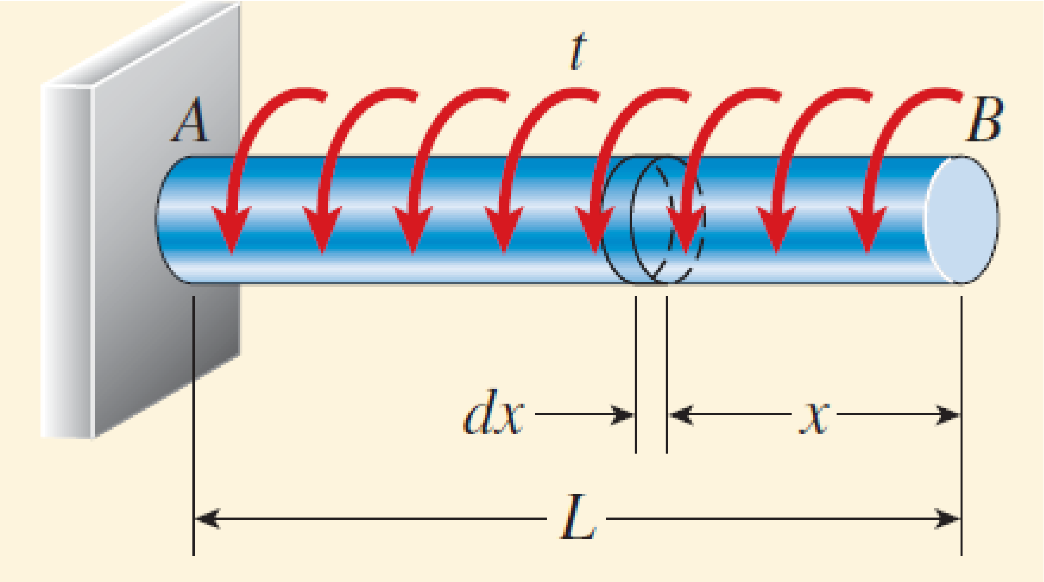
$$\phi_A = \int_0^L \frac{Tdx}{GI_{px}}$$

$$= \int_0^L \frac{Tdx}{G\frac{\pi}{32}\left(d_A + \frac{x}{L}\left(d_B - d_A\right)\right)^4}$$

$$= \frac{32T}{\pi G}\frac{L}{3\left(d_B - d_A\right)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

$$\phi_A = \frac{32TL}{3\pi G\left(d_B - d_A\right)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

Sample Problem



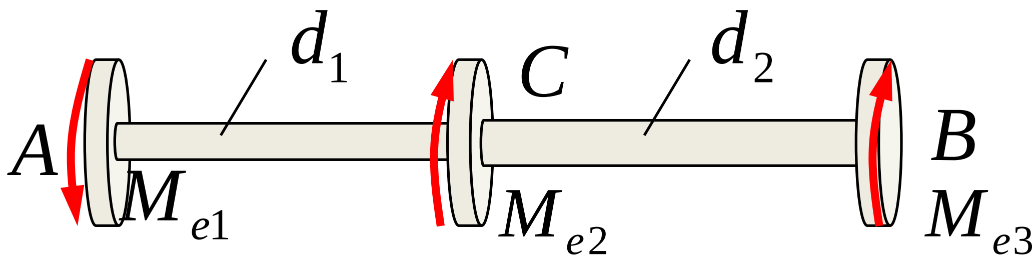
- Determine the angle of twist at cross-section *B* of the shaft.

- Solution

$$\phi_B = \int_0^L \frac{T}{GI_p}\,dx = \int_0^L \frac{tx}{GI_p}\,dx = \frac{tL^2}{2GI_p}$$

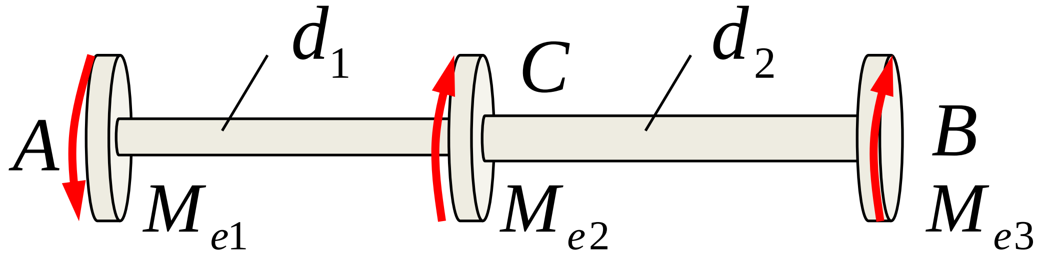
Sample Problem

- Given: n = 500 rpm, input gear A (P_1 = 400 kW), output gears C and B (P_2 = 160 kW, P_3 = 240 kW), $[\tau]$ = 70 MPa, $[\phi']$ = 1 degree/m, G = 80 GPa. Find (1) d_1 and d_2; (2) d if $d_{AC} = d_{BC}$ is required; (3) How to arrange the input and output gears to make the transmission more reasonable?



- Solution:

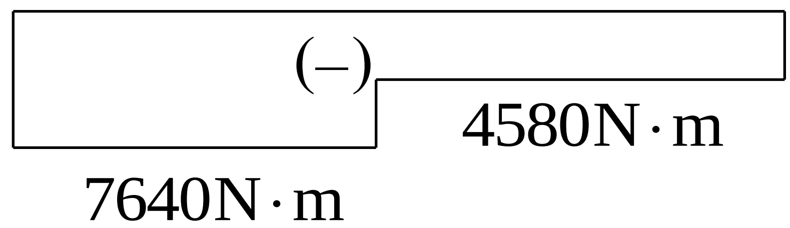
1. Torque diagram



$$M_{e1} = 9549\frac{P_1}{n} = 9549 \times \frac{400}{500}$$
$$= 7640\ \mathrm{N \cdot m}$$

$$M_{e2} = \frac{160}{400} M_{e1} = 3060\ \mathrm{N \cdot m}$$

$$M_{e3} = \frac{240}{400} M_{e1} = 4580\ \mathrm{N \cdot m}$$



2. Find d_1

- From strength condition:

$$\tau_{\max} = \frac{T}{\pi d_1^3 / 16} \le [\tau]$$

$$\Rightarrow d_1 \ge \sqrt[3]{\frac{16T}{\pi[\tau]}} = \sqrt[3]{\frac{16 \times 7640}{\pi \times 70 \times 10^6}} = 82.2 \times 10^{-3} \text{ m} = 82.2 \text{ mm}$$

- From stiffness condition:

$$\phi'_{\max} = \frac{T}{G \, \pi d_1^4 / 32} \times \frac{180^\circ}{\pi} \le [\phi']$$

$$\Rightarrow d_1 \ge \sqrt[4]{\frac{32T \times 180}{G\pi^2 \times [\phi']}} = \sqrt[4]{\frac{32 \times 7640 \times 180}{80 \times 10^9 \times \pi^2 \times 1}} = 86.4 \times 10^{-3} \text{ m} = 86.4 \text{ mm}$$

- Take d_1 = 86.4 mm

3. Find d_2

- From strength condition:

$$d_2 \geq \sqrt[3]{\frac{16T}{\pi[\tau]}} = \sqrt[3]{\frac{16\times 4580}{\pi\times 70\times 10^6}} = 69.3\times 10^{-3}\,\text{m} = 69.3\text{mm}$$

- From stiffness condition:

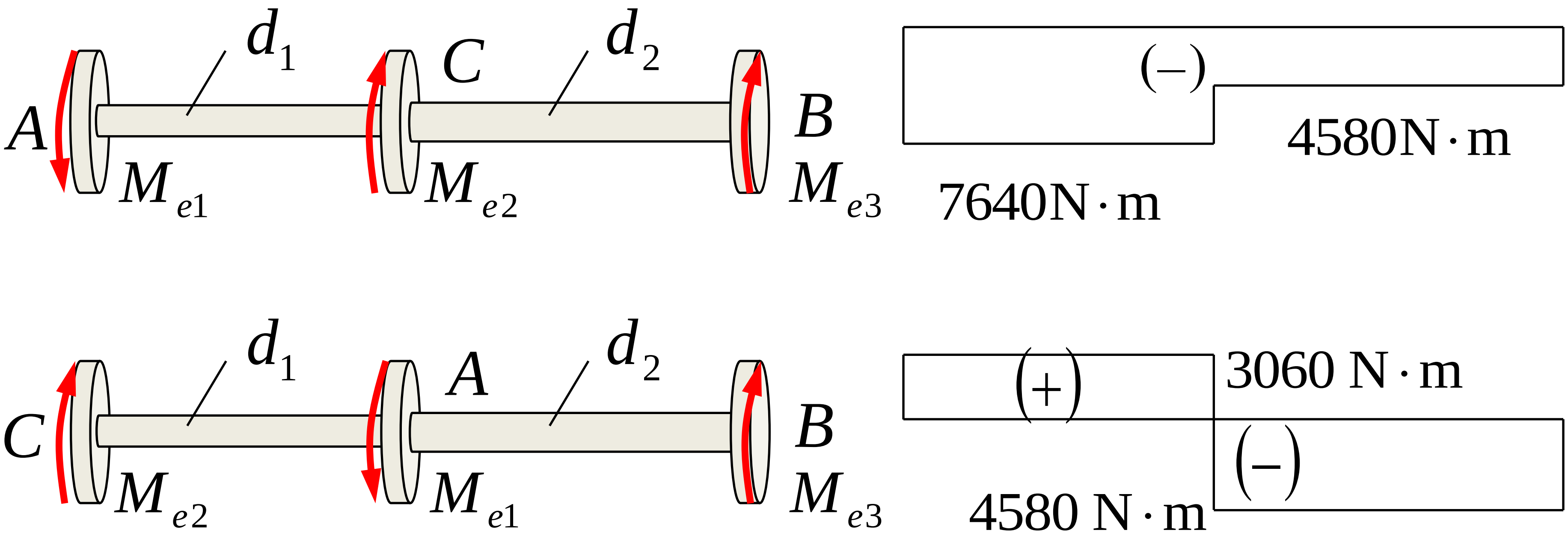
$$d_2 \geq \sqrt[4]{\frac{32T\times 180}{G\pi^2\times[\phi']}} = \sqrt[4]{\frac{32\times 4580\times 180}{80\times 10^9\times \pi^2\times 1}} = 76\times 10^{-3}\,\text{m} = 76\text{mm}$$

- Take d_2 = 76mm

4. If $d = d_1 = d_2$ is required, take the maximum value:

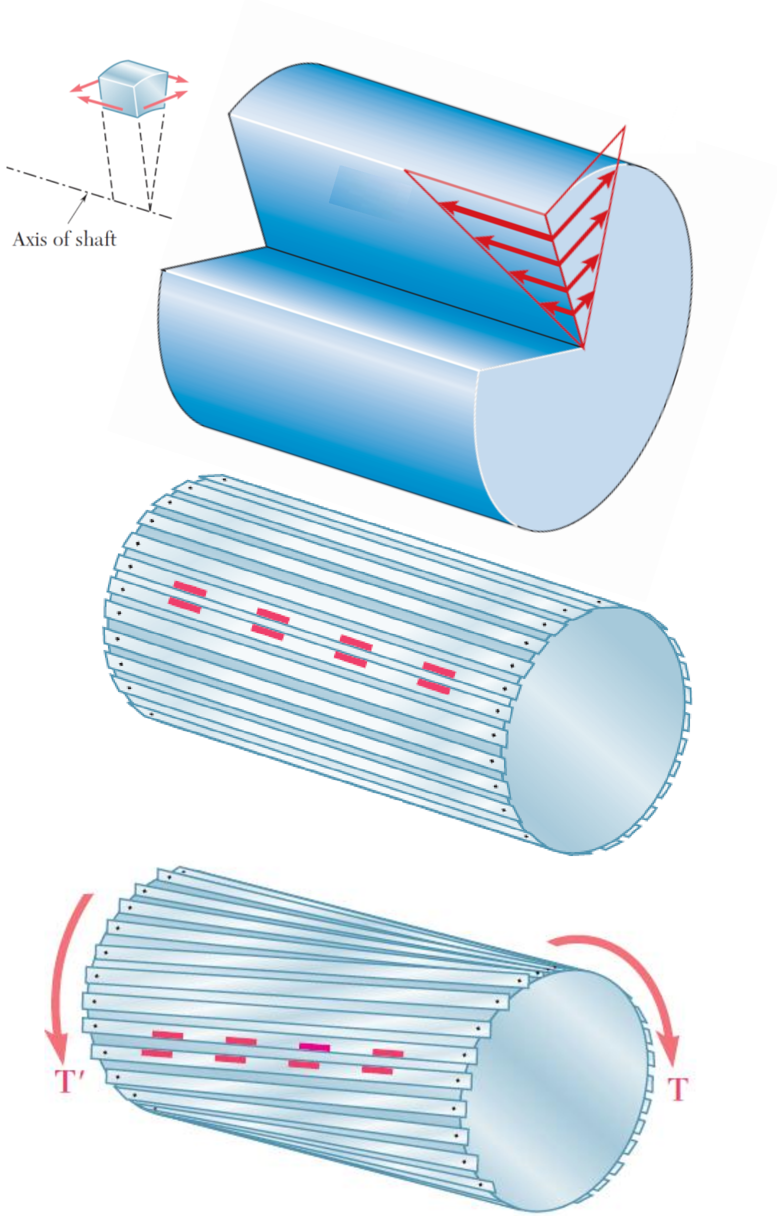
$$d = d_1 = 86.4\text{mm}$$

5. Rearrangement of gears to make the transmission more efficient. Put the input gear in between two output gears.



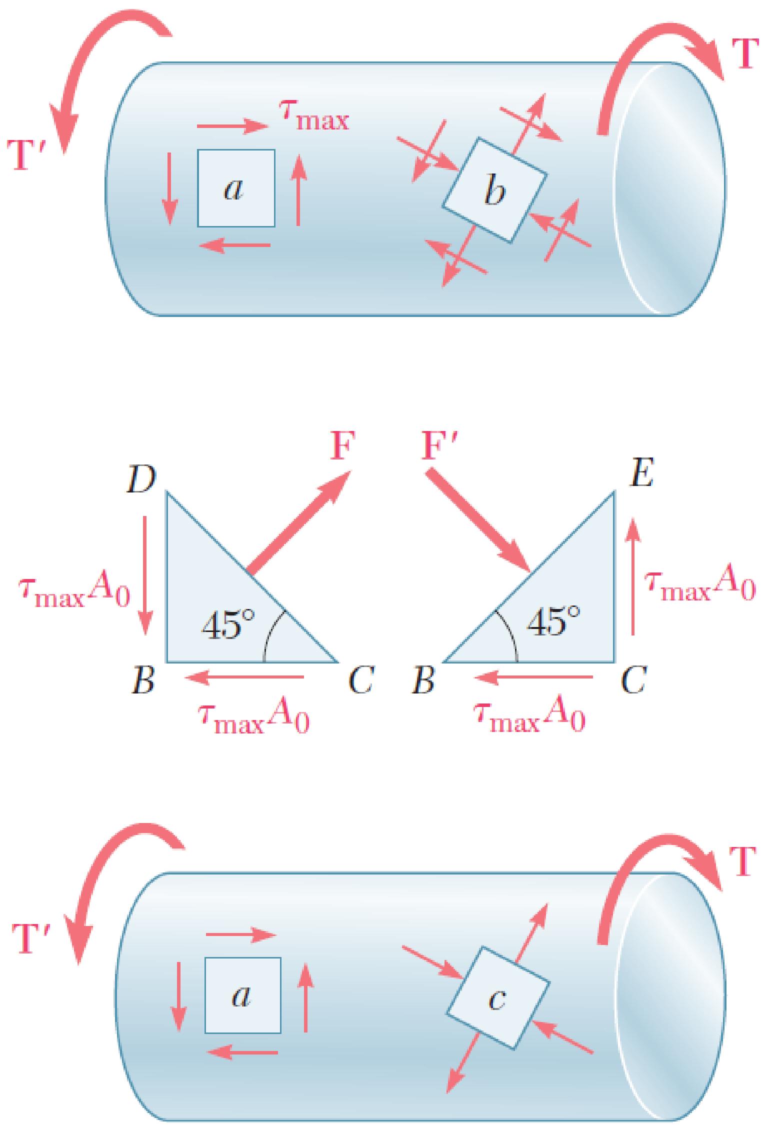
- Smaller maximum load carried by the system (4580/7640=60%)

Theorem of Conjugate Shearing Stress



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.
- The existence of the axial shear is demonstrated by considering a shaft made up of axial slats.
- The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Stresses on Oblique Cross Sections



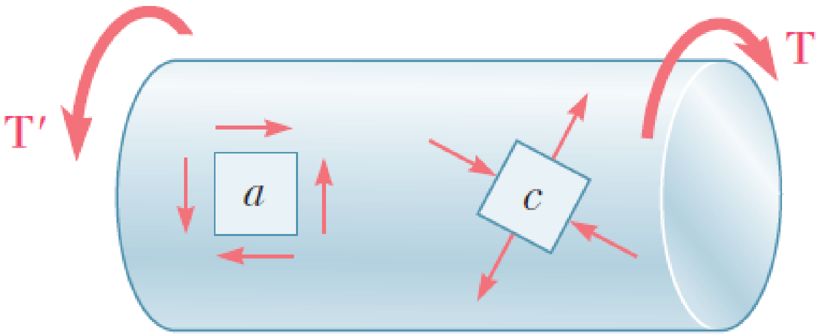
- Elements with faces parallel and perpendicular to the shaft axis are subjected to shearing stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.
- Consider an element at 45^0 to the shaft axis,

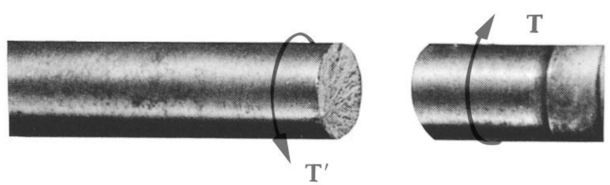
$$F = 2\left(\tau_{\max} A_0\right)\cos 45^\circ = \tau_{\max} A_0 \sqrt{2}$$

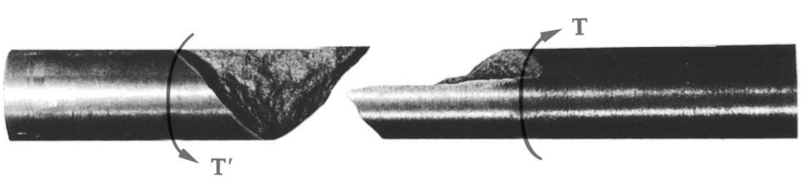
$$\sigma_{45^\circ} = \frac{F}{A} = \frac{\tau_{\max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{\max}$$

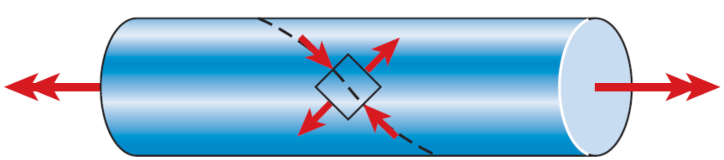
- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude
- What about the normal and shearing stress for an arbitrarily oriented element?

Failure Modes of Torsional Shafts





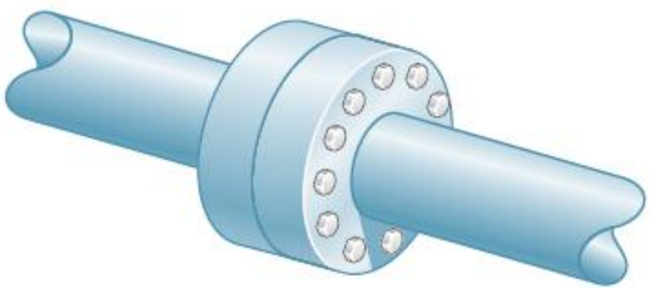


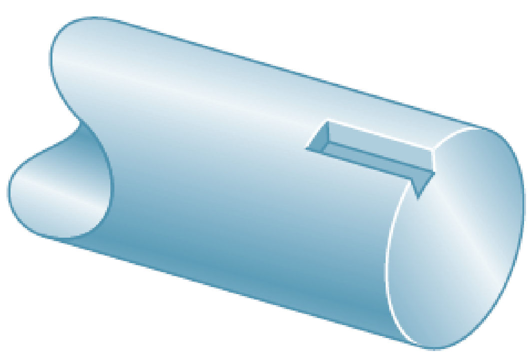


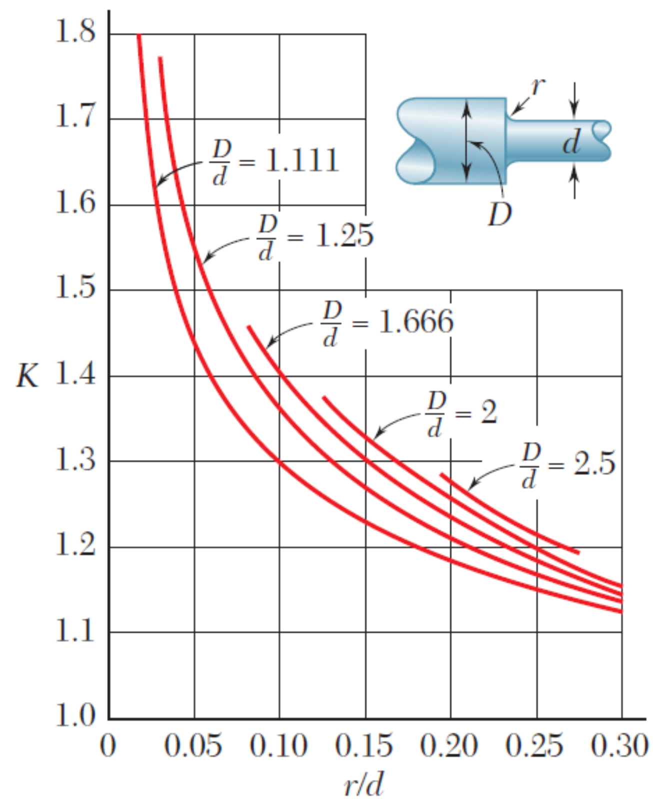
- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45^{o} to the shaft axis.

Stress Concentrations

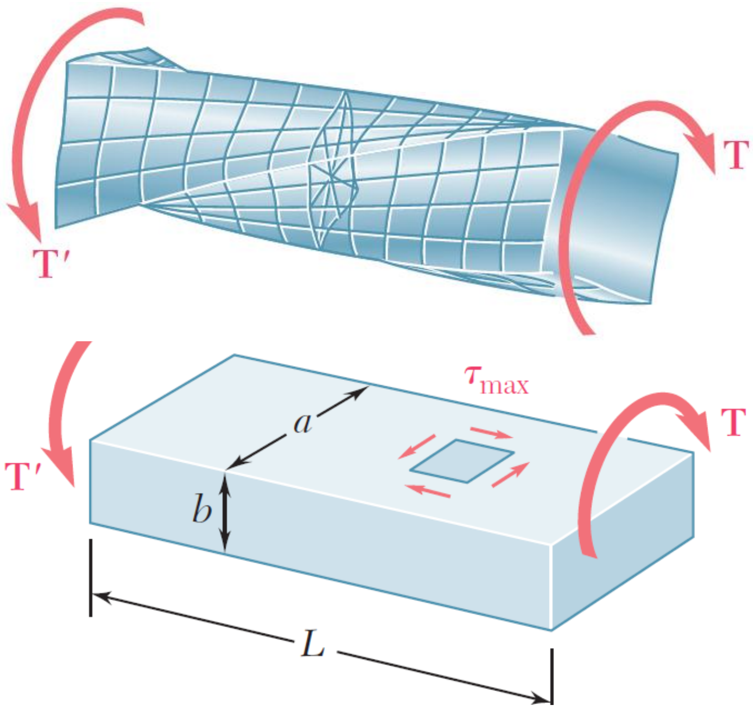
- The derivation of the torsion formula assumed a circular shaft with uniform cross-section loaded through rigid end plates.
- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations
- Experimental determined concentration factors are applied as $\tau_{\max} = K\left(Tc/I_p\right)$







Torsion of Noncircular Members

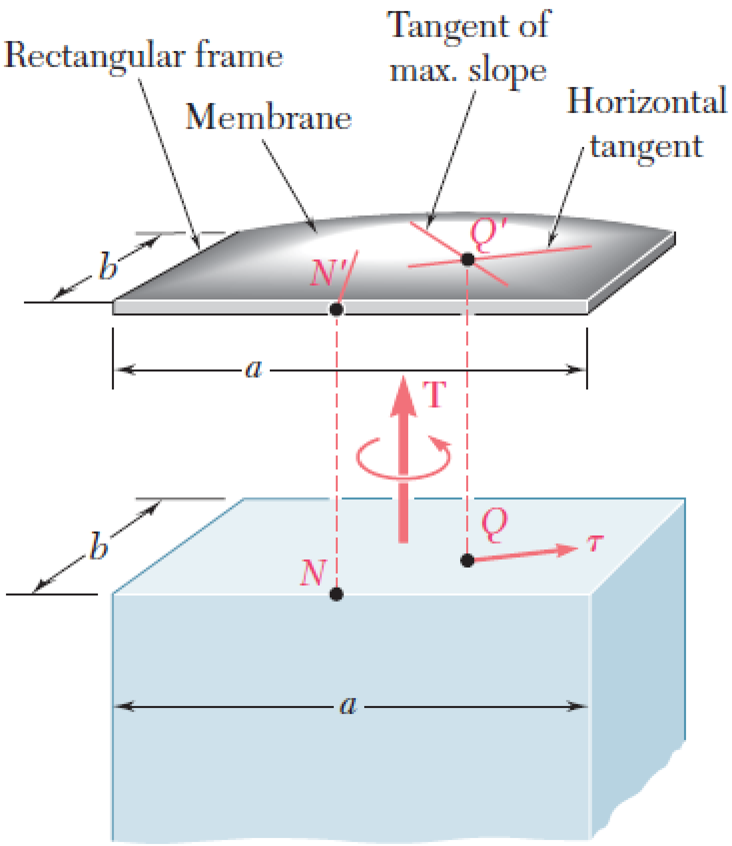


a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

- Previous torsion formulas are valid for axisymmetric or circular shafts
- Planar cross-sections of noncircular shafts do not remain planar and stress and strain distribution do not vary linearly
- The mathematical theory of elasticity for straight bars with a uniform rectangular cross section predicts that [S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, 3rd ed., McGraw-Hill, New York, 1969, Article 109]:

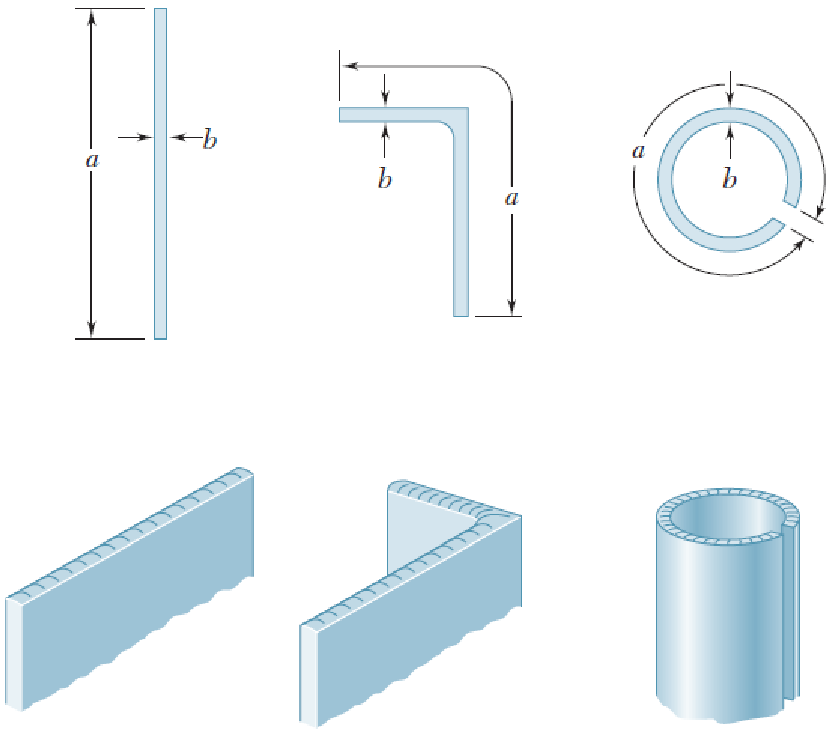
$$\tau_{max} = \frac{T}{c_1 a b^2} \qquad \phi = \frac{TL}{c_2 a b^3 G}$$

Membrane Analogy



- The determination of the deformation of the membrane depends upon the solution of the same partial differential equation as the determination of the shearing stresses in the bar.
- The shearing stress at Q will have the same direction as the horizontal tangent to the membrane at Q', and its magnitude will be proportional to the maximum slope of the membrane at Q'.
- The applied torque will be proportional to the volume between the membrane and the plane of the fixed frame.

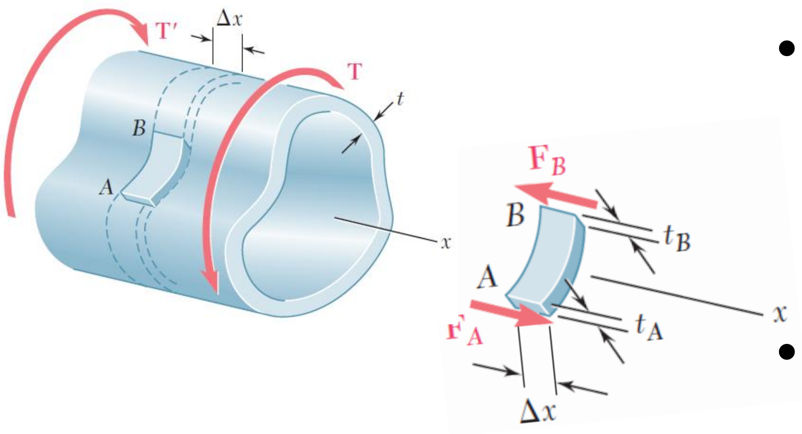
Torsion of Thin-Walled Open Shafts



- Using the membrane analogy to help us visualize the shearing stresses, we note that, if the same torque is applied to each member, the same volume will be located under each membrane, and the maximum slope will be about the same in each case.
- Thus, for a thin-walled member of uniform thickness and arbitrary shape, the shearing stress is the same as for a rectangular bar with a very large value of a/b and may be determined from

$$\tau_{\max} = \frac{T}{0.333ab^2} \qquad \phi = \frac{TL}{0.333ab^3G}$$

Torsion of Thin-Walled Hollow Shafts

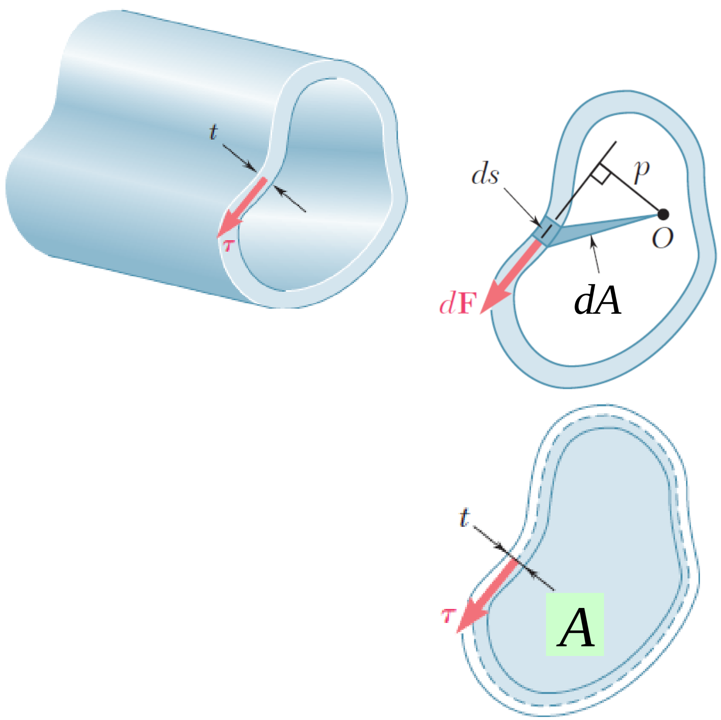


- Summing forces in the x-direction on AB

$$\sum F_x = 0 = \tau_A (t_A \Delta x) - \tau_B (t_B \Delta x)$$

$$\tau_A t_A = \tau_B t_B = \tau t = q = \text{shear flow}$$

- Shearing stress varies inversely with t.



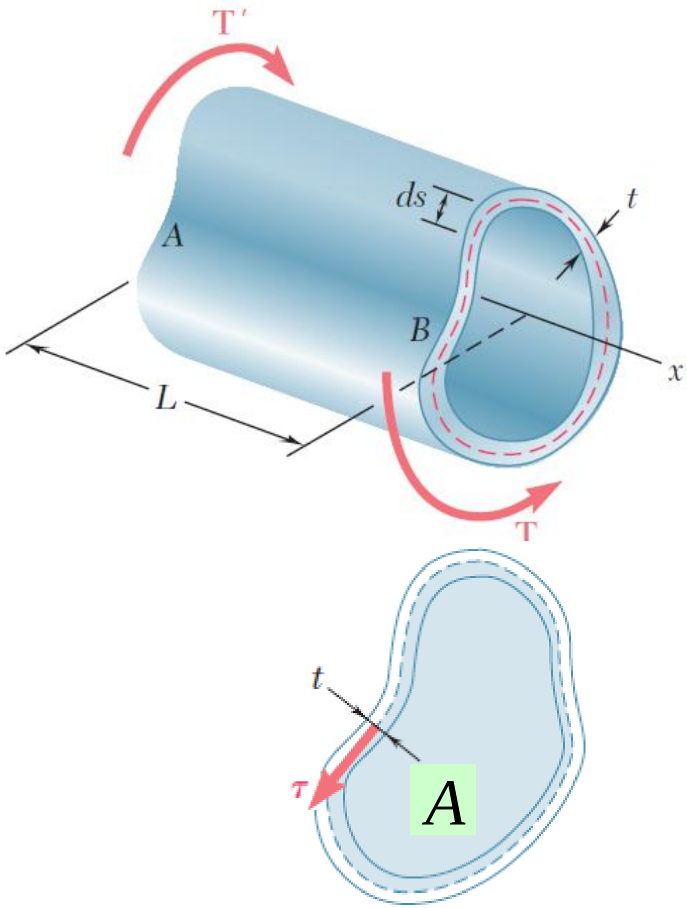
- Compute the shaft torque from the integral of the moments due to shearing stress

$$dM_0 = p\, dF = p\tau (t\, ds) = q(p ds) = 2q\, dA$$

$$T = \oint dM_0 = \oint 2q\, dA = 2qA \quad \Rightarrow \boxed{\tau = T/2tA}$$

where A is the area bounded by the center line of the wall cross section.

Torsion of Thin-Walled Hollow Shafts



- Strain energy density

$$\tau = T/2tA \quad \Rightarrow u = \frac{\tau^2}{2G} = \frac{T^2}{8Gt^2A^2}$$

- Strain energy and work done by the torque

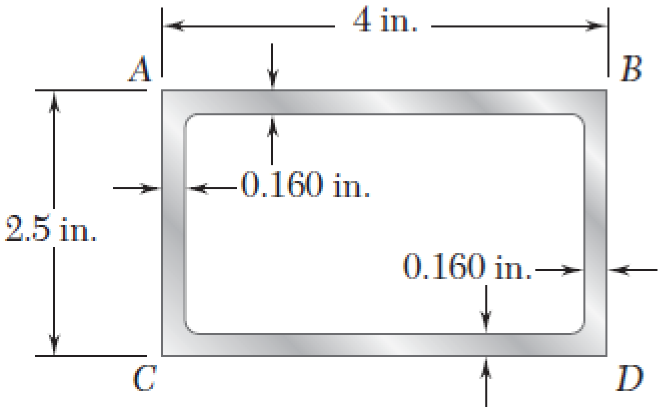
$$U = \int_0^L \oint u\left(tds\right)dx = \int_0^L \oint \frac{T^2}{8Gt^2A^2}\left(tds\right)dx$$

$$= \int_0^L \frac{T^2}{8GA^2}\left(\oint \frac{ds}{t}\right)dx = \frac{T^2L}{8GA^2}\left(\oint \frac{ds}{t}\right)$$

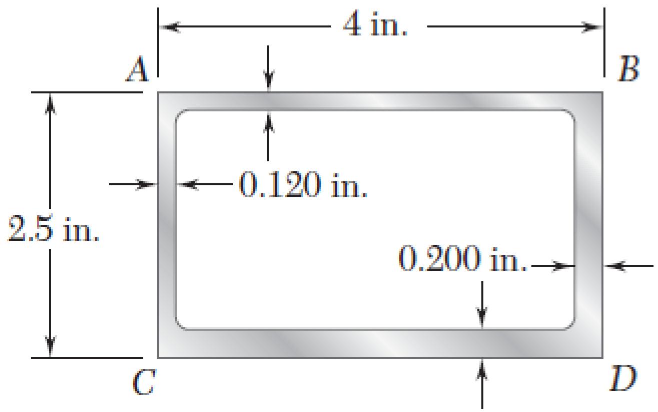
- Angle of twist by energy method

$$U = \frac{1}{2}T\phi \quad \Rightarrow \boxed{\phi = \frac{TL}{4GA^2}\oint \frac{ds}{t}}$$

Sample Problem



(a)

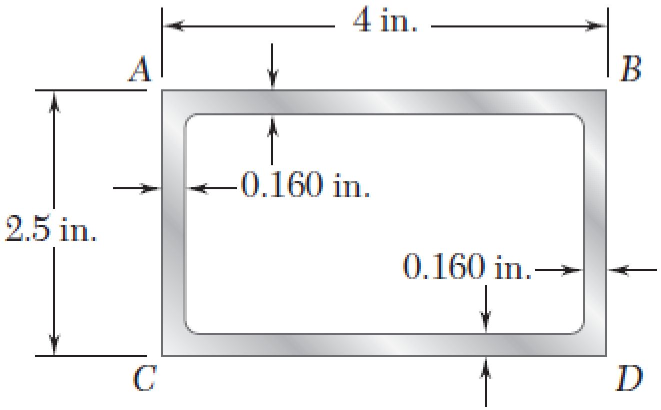


(b)

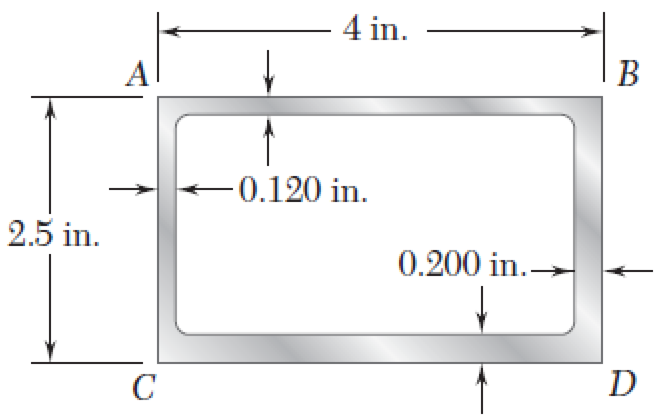
Extruded aluminum tubing with a rectangular cross-section has a torque loading of 24 kip-in. Determine the shearing stress in each of the four walls with (*a*) uniform wall thickness of 0.16 in. and wall thicknesses of (*b*) 0.12 in. on *AB* and *CD* and 0.2 in. on *CD* and *BD*.

SOLUTION:

- Determine the shear flow through the tubing walls
- Find the corresponding shearing stress with each wall thickness



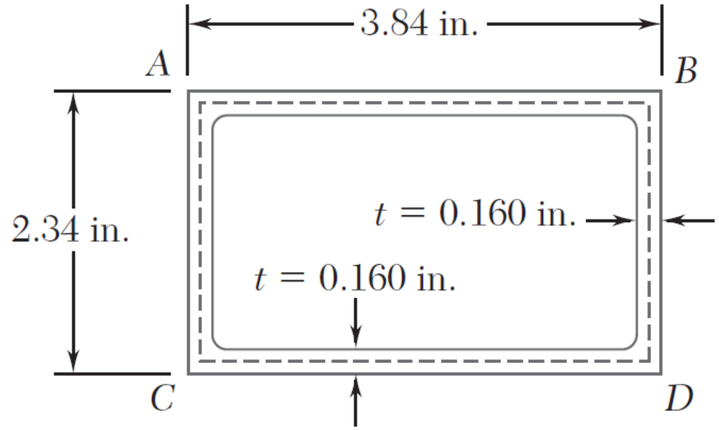
(a)



(b)

SOLUTION:

- Determine the shear flow



$$q = \frac{T}{2A}$$
$$= \frac{24\,\text{kip-in.}}{2(3.84\,\text{in.})(2.34\,\text{in.})}$$
$$= 1.335\,\frac{\text{kip}}{\text{in.}}$$

- Find the corresponding shearing stress with each wall thickness

(a) with a uniform wall thickness,

$$\tau = \frac{q}{t} = \frac{1.335\,\text{kip/in.}}{0.160\,\text{in.}} = 8.34\,\text{ksi}$$

(b) with a variable wall thickness

$$\tau_{AB} = \tau_{AC} = \frac{1.335\,\text{kip/in.}}{0.120\,\text{in.}} = 11.13\,\text{ksi}, \quad \tau_{BD} = \tau_{CD} = \frac{1.335\,\text{kip/in.}}{0.200\,\text{in.}} = 6.68\,\text{ksi}$$

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