FOUNDATIONS OF ENGINEERING MECHANICS

A. I. Lurie

Theory of Elasticity

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Translated by A. Belyaev

With 49 Figures

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Theory of Elasticity

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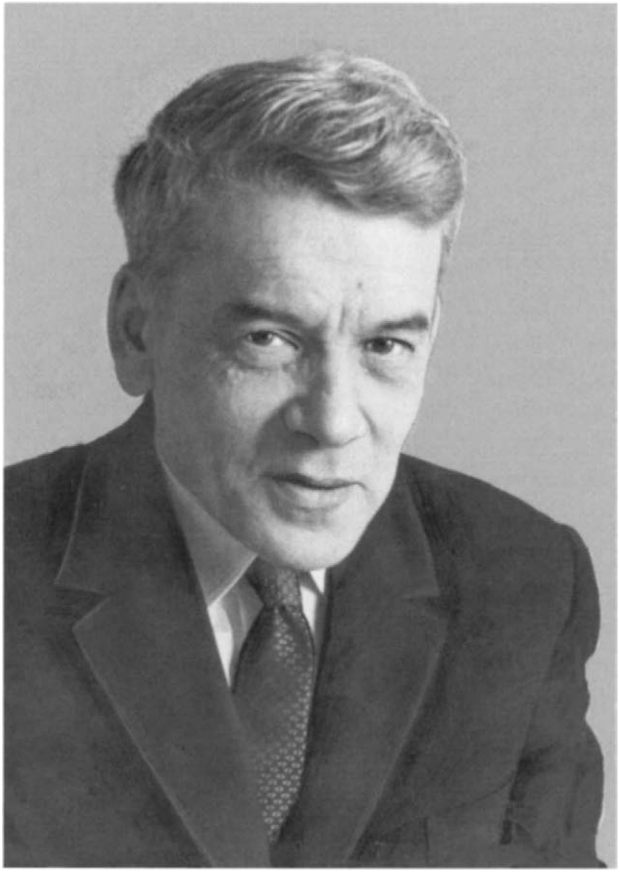

Anatolii I. Lurie

Anatolii Isakovich Lurie was born on the 19th of July, 1901, in Mogilev-on-the-Dnieper. In 1919 he graduated from gymnasium and was admitted to the School of Prospecting, the Ural Institute of Mines, in Ekaterinburg. Studies at this School did not satisfy him. The disciplines taught there were perceived more like the recipes mandatory for use. Mine survey was an exception, and he felt interest toward it. In 1923 he transferred himself to the Faculty of Physics and Mechanics of the Petrograd Polytechnic Institute (originally, St. Petersburg Polytechnic Institute named after Emperor Peter the Great), where he has been working since graduation in 1925. Academician A.F. Ioffe has founded this faculty in 1918; it had quite many outstanding scholars with European reputation who served as Professors. In 1939 A.I. Lurie was awarded the degree of Doctor of Science. He headed the Department of Theoretical Mechanics from 1936 to 1941, and through the period 1944 to 1977 he was the Head of the Department "Dynamics and Strength of Machines" (which in 1960 was renamed "Mechanics and Control Processes"). A.I. Lurie was elected a Corresponding Member of the Academy of Sciences of the USSR, Division of Mechanics and Control Processes. He was a member of the Presidium of the National Committee for Theoretical and Applied Mechanics and a member of the National Committee for Automatic Control. A.I. Lurie was a member of the Editorial Boards of the renowned Russian journals "Applied Mathematics and Mechanics" and "Mechanics of Solids'.

His scientific activities, lasting for more than half a century, brought about remarkable achievements. He wrote a number of magnificent books:

1. Nikolai E.L., and Lurie A.I. Vibrations of the Frame-Type Foundations. Leningrad and Moscow, Gosstroyizdat, 1933, 83 pp.
2. Loitsianskii L.G., and Lurie A.I. Theoretical Mechanics. Vols. 1 - 3. Leningrad and Moscow, GMTI, 1934.
3. Lurie A.I. Statics of Thin-Walled Elastic Shells. Moscow, Gostekhizdat, 1947, 252 pp.
4. Lurie A.I. Some Nonlinear Problems in the Theory of Automatic Control. Moscow, Gostekhizdat, 1951, 216 pp.
5. Lurie A.I. Operational Calculus and its Application to the Problems in Mechanics. Moscow, GITTL, 1951, 432 pp.
6. Lurie A.I. Three-dimensional Problems of the Theory of Elasticity. Moscow. GITTL, 1955, 492 pp.
7. Loitsianskii L.G., and Lurie A.I. A Course in Theoretical Mechanics. Vols. 1 and 2. Fifth edition. Moscow, GITTL, 1955, 380 pp., 596 pp.
8. Lurie A. I. Analytical Mechanics. Moscow, Nauka, 1961, 824 pp.
9. Lurie A.I. Theory of Elasticity. Moscow, Nauka, 1970, 940 pp.
10. Lurie A.I. Nonlinear Theory of Elasticity. Moscow, Nauka, 1980, 512 pp.

His books were translated into many languages (English, German, French, Chinese, Romanian, Bulgarian, and Armenian). The last book was written when A.I. Lurie was already seriously ill. He did not live to see both the proofs and the book out of print. This book was later translated into English by his son K.A. Lurie and published by North Holland Publishers in 1990.

The main features of his scientific style manifested itself in his early works; i.e. the ability to apply the achievements of classical mechanics to the needs of modern technology. His books are unparalleled by the number of practical applications. A. I. Lurie became an ardent promoter of the so-called direct or invariant vector and later tensor calculus. It is now difficult to imagine that once the relations in theoretical mechanics were expressed and written in the cumbersome coordinate form!

The works by A.I. Lurie in the field of application of operational calculus to the study of the stability of mechanical systems with distributed parameters brought him great fame. This study as well as his direct contacts with mathematicians stimulated research in the field of distribution of roots of quasi-polynomials.

The works by Lurie A.I. on the theory of absolute stability of control systems received much interest from the scientific community. The very statement of the problem and the application of the Lyapunov function method to its solution were pioneering and provoked a great flow of scientific publications.

Professor Lurie is also the author of a number of articles and books on the theory of elasticity. He devoted the last fifteen years of his life exclusively to the problems of the theory of elasticity. The typical feature of all these works was obtaining analytical results. He did not pay any attention to the numerical methods that are so popular nowadays.

Professor Lurie was an extraordinary person. He was always eager to share his ideas and listened with interest and respect to other people expressing their ideas and views. This especially attracted young scientists and lecturers to him. His study was always full of visitors seeking his advice, his reference on papers, or simply his support. He worked hard all his life: writing books, giving lectures, reviewing papers. He did not like and even disapproved of lazy though possibly talented people. I'd like to note for the sake of the Western reader that there is position of Professor Emeritus neither in the Soviet Union nor in modern Russia which ensures a decent salary and enables a Professor to work as much as he can or not to work at all if he is no longer able to do so. That is why Russian professors have to work until they die. The following episode is typical of him. In spring 1979 Professor Lurie underwent serious surgery. It took him the whole of summer to recover after it. In September he came back from Moscow. He looked fine. He said to me (I was already acting as the Head of the Chair): "I am going to read my favorite course "Theory of Elasticity". I tried to object to this and offered to read his lectures as well as mine. He reacted rather sharply and insisted on reading his own course. However he was able to do so only until October. In November he gave up saying that it was too difficult. He died on 12 February 1980. He was 78 yeas old.

Professor Vladimir A. Palmov

Foreword

The classical theory of elasticity maintains a place of honour in the science of the behaviour of solids. Its basic definitions are general for all branches of this science, whilst the methods for stating and solving these problems serve as examples of its application. The theories of plasticity, creep, viscoelasticity, and failure of solids do not adequately encompass the significance of the methods of the theory of elasticity for substantiating approaches for the calculation of stresses in structures and machines. These approaches constitute essential contributions in the sciences of material resistance and structural mechanics.

The first two chapters form Part I of this book and are devoted to the basic definitions of continuum mechanics; namely stress tensors (Chapter 1) and strain tensors (Chapter 2). The necessity to distinguish between initial and actual states in the nonlinear theory does not allow one to be content with considering a single strain measure. For this reason, it is expedient to introduce more rigorous tensors to describe the stress-strain state. These are considered in Section 1.3 for which the study of Sections 2.3-2.5 should precede. The mastering of the content of these sections can be postponed until the nonlinear theory is studied in Chapters 8 and 9.

Deriving closed systems of equations for the linear theory of elasticity and a description of the solution methods form the basic content of Part 2. In particular, Chapters 3 and 4 deal with constitutive laws and basic relationships. Part III (Chapters 5-7) is concerned with solving special problems. The subject of Chapter 5 coincides with that of the author's monograph "Three-dimensional problems of the theory of elasticity" Gostekhizdat, 1955. However, the statement of the problems considered is completely revised and some problems absent from the monograph are included here. These problems include: stresses due to a foreign inclusion, substantiation of the Saint-Venant principle, some problems of stress concentration (Neuber), elastostatic Robin's problem, etc.

Limitations on space gave rise to difficulties when selecting the material for Chapter 6 (the Saint-Venant problem) and Chapter 7 (plane problem). In Chapter 6 the statement of the Saint-Venant problem, the theorem on circulation, the question of the centre of rigidity, and variational methods are treated in some detail, whilst obtaining solutions for particular profiles is reduced to a minimum. In Chapter 7 applying the theory of a complex variable is limited to considering simple boundary-value problems. In addition, other methods of solution are demonstrated, which include: Mellin's transformation in the problem for a wedge, the operational solutions of the problems of a strip and a bar with a circular axis.

Part IV (Chapters 8 and 9) is devoted to the basics of the nonlinear theory of elasticity, namely statements of the constitutive law for the nonlinear elastic body, considering some simple problems, statement of problems relating to second order effects and bifurcation of the equilibrium.

The Appendices include descriptions of the methods of tensor calculus used in the book and some material on the theory of spherical and ellipsoidal functions.

Only "rigorous" statements of problems are considered in the book; that is, the solutions are not only statically admissible but they also satisfy the compatibility conditions. The original intention of including "technical" theories on thin rods, plates and shells was withdrawn because it would lead to an exorbitant increase in the volume of the book. For the same reason, only static problems are considered.

The literature referred to throughout the book does not constitute a comprehensive review of the principal investigations and solutions of the special problems of elasticity theory. To some extent, this shortcoming is compensated for by the review papers and monographs referred to, which contain an exhaustive bibliography on the special problems considered.

The book is addressed at readers who are interested in gaining deep understanding and knowledge of the theory of elasticity and having practice in solving problems. The book is also intended to be an aid for teaching a course in the mathematical theory of elasticity.

The first readers of the book were L.M. Zubov, who checked the formulae and calculations, and V.A. Palmov, who suggested a number of improvements and further explanations. It is a pleasant duty of the author to express a sincere gratitude for their valuable advice and critical suggestions.

Professor I.I. Vorovich and the staff involved in elasticity theory research at Rostov State University took the trouble of reviewing the manuscript. The author wishes to thank them for their fruitful and friendly critique.

Translator's preface

The book "Theory of Elasticity" by A.I. Lurie was printed in Russian with the edition of eleven thousand copies and immediately became a bibliographic rarity. In Russia, this monograph is deservedly considered to be a classical book in mechanics. Translation of this book is a great honour for me. Being a member of Lurie's Chair and one of his numerous pupils I consider this activity to be a debt of honour to perpetuate his memory in mechanics. Also from a professional perspective, the translation was a very interesting and cognitive experience.

The book can be viewed as an encyclopedia of closed-form solutions to a series of particular problems and analytical approaches to solving general problems. Despite the impressive progress of numerical methods, analytical methods in the theory of elasticity have not lost their practical importance. For example, new and rapidly developing areas of modern technology such as smart structures need analytical results for some types of structure in order to design the shape of sensors and actuators. It is important that all of the solutions obtained in the present book satisfy not only the necessary conditions for equilibrium (which is the case in the technical theories of rods, plates etc.) but also satisfy the sufficient conditions for equilibrium (i.e. the continuity or compatibility conditions). The latter conditions are of crucial importance for modern applications when structures are composed of several materials.

Although I tried to do my best while translating the book, some typing and other mistakes may have occurred in the translation, for which I would like to apologize. Also, I did not always succeed in finding the original references in English, German, French, Italian and other languages, and had to re-translate them from Russian into English. I apologise to the authors and the reader for possible inaccuracies in the titles of some references. While translating the book into English I tried to keep the author's nomenclature which does not always coincide with that adopted in Western books.

I am thankful to my son Nikita, from the State Polytechnical University of St. Petersburg, for the considerable technical support he gave during the translation. Also, I would like to express my sincere gratitude to Dr. Stewart McWilliam, from the University of Nottingham, UK who took the trouble of editing the manuscript which I translated into English. I am greatly obliged to him for his thorough correction of the galley-proofs.

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Part I

Basic concepts of continuum mechanics

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Stress tensor

1.1 Field of stresses in a continuum

1.1.1 Systems of coordinates in continuum mechanics

A continuum is characterised by a mass $dm = \rho d\tau$ contained in an elementary volume $d\tau$. The proportionality factor ρ, referred to as the mass density, is assumed to be a continuous function of the coordinates of the material particles of the medium.

A material continuum which was originally in equilibrium and occupied a volume v with a surface o reaches a new equilibrium state whose volume and surface are denoted by V and O respectively. The first state is referred to as the initial state (volume v or $v-$volume), whilst the second state is called the final state (volume V or $V-$volume). In what follows, the concept of the natural state will be of importance. The natural state is the state when the continuum is not stressed. Unless otherwise stated, the natural state is not identified with an initial state.

A Cartesian coordinate system $OX_1X_2X_3$ is introduced. The position M of a material particle in the initial state is given by its Cartesian coordinates a_1,a_2, a_3 in the system

$OX_1X_2X_3$, or by the position vector

$$\mathbf{r} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3 = a_s\mathbf{i}_s\,, \tag{1.1.1}$$

where $\mathbf{i}_s$ denotes the unit base vectors of the coordinate axes. The summation over a dummy index is omitted throughout the text as suggested in Appendices A-C.

The final position M' of the considered particle is described by coordinates x_1,x_2,x_3 in the same coordinate system, or by the position vector[1]

$$\mathbf{R} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = x_s\mathbf{i}_s\,. \tag{1.1.2}$$

The geometric difference $\mathbf{R}-\mathbf{r}$ determines the displacement vector of point M which is denoted by $\mathbf{u}$, i.e.

$$\mathbf{R} = \mathbf{r} + \mathbf{u} = (a_s + u_s)\,\mathbf{i}_s, \quad x_s = a_s + u_s\,. \tag{1.1.3}$$

The projections u_s of the displacement vector, referred to as the displacements, are viewed as being functions of the initial coordinates a_1,a_2, a_3 of the particle. The displacements and their derivatives with respect to variables a_1,a_2, a_3 are assumed to be continuous functions of the order that is necessary for further analysis. It is also assumed that equations (1.1.3) are uniquely resolvable for variables a_s, that is

$$\mathbf{r} = \mathbf{R} - \mathbf{u}, \quad a_s = x_s - u_s, \tag{1.1.4}$$

u_s now being considered as functions of coordinates x_s of the final state. The condition for unique solvability of the system of equations (1.1.3) is that the Jacobian

$$J(a_1, a_2, a_3) = \left|\frac{\partial x_s}{\partial a_k}\right| = \left|\delta_{sk} + \frac{\partial u_s}{\partial a_k}\right| = \begin{vmatrix} 1 + \dfrac{\partial u_1}{\partial a_1} & \dfrac{\partial u_1}{\partial a_2} & \dfrac{\partial u_1}{\partial a_3} \\ \dfrac{\partial u_2}{\partial a_1} & 1 + \dfrac{\partial u_2}{\partial a_2} & \dfrac{\partial u_2}{\partial a_3} \\ \dfrac{\partial u_3}{\partial a_1} & \dfrac{\partial u_3}{\partial a_2} & 1 + \dfrac{\partial u_3}{\partial a_3} \end{vmatrix} \tag{1.1.5}$$

does not vanish in the closed domain $v + o$. It is taken that $J > 0$. The Jacobian is known to be a ratio of the elementary volumes in the initial and final states

$$d\tau = J d\tau_0\,, \tag{1.1.6}$$

see also Subsection 2.5.5. According to the law of mass conservation

$$dm = \rho d\tau = \rho_0 d\tau_0\,, \tag{1.1.7}$$

[1] Translator'note. Equation numbering is as follows. The first number in parentheses indicates the Section, the second - the Subsection, whilst the third - the equation number in the Subsection. These three numbers appear when a cross-reference is made within the same chapter. When an equation from another chapter is referred to, they are completed by an indication of the number of this particular chapter. Reference to an Appendix is indicated using capital lettters.

so that

$$J = \frac{\rho_0}{\rho}. \tag{1.1.8}$$

The Cartesian coordinates a_s of the material particle in its initial state can be considered as variables related to this point and are thus retained in the final state, where these coordinates play the role of curvilinear coordinates. For example, points along a straight line $a_2 = a_2^0, a_3 = a_3^0$ which is parallel to axis OX_1 in volume v are located on the curve

$$x_s = x_s\left(a_1, a_2^0, a_3^0\right)$$

in volume V.

Due to the established terminology, a_s and x_s are referred to as Lagrangian and Eulerian coordinates, respectively. Better still, a_s are material coordinates which individualise a material particle and distinguish it from other particles, whereas x_s denote the coordinates of the particle in volume V.

The square of the linear element, which is the distance between two infinitesimally close points M and N in volume v,

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = da_1^2 + da_2^2 + da_3^2\,, \tag{1.1.9}$$

becomes

$$dS^2 = d\mathbf{R} \cdot d\mathbf{R} = dx_1^2 + dx_2^2 + dx_3^2 \tag{1.1.10}$$

when these points take the positions M' and N' respectively . In what follows, for the sake of brevity we will use terms $v-$metric and the $V-$metric depending upon what definition of the linear elements, eq. (1.1.9) or eq. (1.1.10), is accepted in each particular consideration. Clearly, both metrics are Euclidean (E_3).

Remark 1. A rigorous distinction between the initial and final states is necessary for considering finite deformation in continuous media. As a rule, there is no need in the framework of linear theory of elasticity to make such distinctions.

Remark 2. The material coordinates of a point are not necessarily the Cartesian coordinates a_s of the initial state. Presentation of the basics of continuous mechanics becomes more rigorous if any curvilinear coordinates q^1, q^2, q^3 are taken as the material coordinates. Then

$$a_s = a_s\left(q^1, q^2, q^3\right), \quad \mathbf{r} = \mathbf{r}\left(q^1, q^2, q^3\right), \tag{1.1.11}$$

as well as

$$x_s = x_s\left(q^1, q^2, q^3\right), \quad \mathbf{R} = \mathbf{R}\left(q^1, q^2, q^3\right) \tag{1.1.12}$$

should be considered as the coordinates of the particle and its position vector in volumes v and V, respectively.

1.1.2 External forces

In the present chapter a continuum is considered in its final state. The forces acting on the continuum are classified as external or internal forces. External forces represent actions exerted by bodies outside the considered volume V on the continuum particles. These may be surface and volume forces.

The force acting on each particle of the continuum is called a mass force. The vector of the mass force applied to a unit mass of the continuum is denoted by $\mathbf{K}$, then $\rho\mathbf{K}d\tau$ is the force acting on an elementary mass $\rho d\tau$ contained in volume $d\tau$ whilst $\rho\mathbf{K}$ is the force acting on a unit volume and is termed a volume force. The resultant vector and the resultant moment of the mass forces about the origin of the coordinate system are given by

$$\iiint\limits_V \rho\mathbf{K}d\tau, \quad \iiint\limits_V \mathbf{R}\times\rho\mathbf{K}d\tau. \tag{1.2.1}$$

A simple example of the mass force is the gravity force

$$\mathbf{K} = -\mathbf{k}g\,, \tag{1.2.2}$$

where $\mathbf{k}$ denotes the unit vector of the upward vertical and g is the acceleration due to gravity. When equilibrium of the continuum is considered relative to a moving coordinate system the inertia force of the translational motion

$$\mathbf{K} = -\mathbf{w}_e = -\left[\mathbf{w}_0 + \dot{\boldsymbol{\omega}}\times\mathbf{R} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})\right] \tag{1.2.3}$$

should be added to the mass forces. Here $\mathbf{w}_e$ denotes the vector of the translational acceleration which is equal to the geometric sum of the acceleration $\mathbf{w}_0$ of the origin of the coordinate system, rotational acceleration $\dot{\boldsymbol{\omega}}\times\mathbf{R}$ and centripetal acceleration $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$, with $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ denoting respectively the angular velocity and angular acceleration vectors. The Coriolis acceleration is not included in the right hand side of eq. (1.2.3) because the continuum does not move relative to the moving axes. In a particular case of the uniform rotation of the continuum about an immovable axis the centrifugal force

$$\mathbf{K} = -\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}) = \omega^2 h\mathbf{e}, \tag{1.2.4}$$

is a mass force. Here a particle is assumed to move along a circle of radius h and $\mathbf{e}$ denotes a unit vector directed along this radius from the centre of the circle. The origin of vector $\mathbf{R}$ lies on the rotation axis.

In the case of the potential mass forces we have

$$\mathbf{K} = -\operatorname{grad}\Pi, \tag{1.2.5}$$

where Π denotes the potential energy of a mass force field. For example, for the fields of gravity force and centrifugal force we have

$$\Pi = g\mathbf{k} \cdot \mathbf{R}, \quad \Pi = \frac{1}{2}\left[(\boldsymbol{\omega} \cdot \mathbf{R})^2 - \omega^2 R^2\right] = -\frac{1}{2}\left|\boldsymbol{\omega} \times \mathbf{R}\right|^2. \tag{1.2.6}$$

External surface forces are forces distributed over surface O of volume V. A surface force per unit area of this surface is denoted by $\mathbf{F}$. The resultant vector and the resultant moment of the surface forces are given by

$$\iint_O \mathbf{F} dO, \quad \iint_O \mathbf{R} \times \mathbf{F} dO. \tag{1.2.7}$$

Here dO denotes the area of surface O, which is in contrast to the area do of surface o bounding volume v of the continuum in the initial state. The unit vector of the normal to dO directed outwards from volume V is designated by $\mathbf{N}$, whereas $\mathbf{n}$ denotes the unit vector of the normal to do directed outwards from volume v. Moreover, $\mathbf{N}dO$ and $\mathbf{n}do$ are referred to as the vectors of the oriented surface on O and o, respectively. The normal component of force $\mathbf{F}$ and the component of $\mathbf{F}$ in the plane tangential to O are respectively given by

$$\mathbf{N} \cdot \mathbf{F}, \quad \mathbf{F} - \mathbf{NN} \cdot \mathbf{F} = (\mathbf{N} \times \mathbf{F}) \times \mathbf{N}. \tag{1.2.8}$$

An example of the surface force is the hydrostatic pressure in a fluid, in which the body is immersed

$$\mathbf{F} = -p\mathbf{N}. \tag{1.2.9}$$

Another example is a body resting on a foundation. In this case the reaction force is distributed over the contact area.

A surface force is a potential force provided that it retains its absolute value and direction during the deformation of the body from the initial to the final state. Then

$$\Pi = -\mathbf{F} \cdot \mathbf{R} = -\mathbf{F} \cdot (\mathbf{r} + \mathbf{u}) = \Pi_0 - \mathbf{F} \cdot \mathbf{u}. \tag{1.2.10}$$

1.1.3 *Internal forces in the continuum*

Consideration of the equilibrium of a continuum is based upon two statements: (i) when the whole continuum is in equilibrium, then any arbitrary part of this continuum is also in equilibrium (the free-body principle), and (ii) the equilibrium conditions for a rigid body are the necessary conditions of equilibrium of the considered part of the continuum (the principle of solidification).

Let us mentally divide volume V into two volumes V_1 and V_2. Let O' denote the surface separating two volumes and O_1 denote that part of

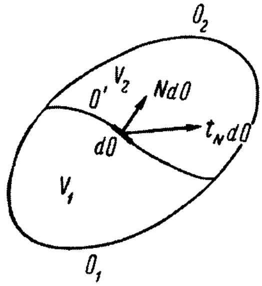

FIGURE 1.1.

O which bounds volume V_1. In addition to the external forces acting on the continuum in volume V_1 we should consider the reaction forces of the continuum in volume V_2 on volume V_1. If we do not include the latter forces, then the necessary conditions of equilibrium of the external forces, which are mass forces in V_1 and surface forces on O_1, are, in general, not satisfied. These forces should be equilibrated by the forces and moments of the interaction forces distributed over the separating surface O'. It is assumed that the distribution of these forces over surface dO of surface O' is statically equivalent to force $\mathbf{t}_N dO$, the orientation of surface dO being prescribed by a unit normal vector $\mathbf{N}$ directed outwards from V_1, see Fig. 1.1.

Therefore, for any oriented surface $\mathbf{N}dO$ at any location in the continuum, there exists a force $\mathbf{t}_N dO$ (a vector) which is the force exerted on $\mathbf{N}dO$ by the part "above" this surface. By virtue of the principle of action and reaction we have

$$\mathbf{t}_{-N} dO = -\mathbf{t}_N dO . \tag{1.3.1}$$

This interaction of the part of the continuum defines the field of internal forces, or in other words, the stress field in continuum. Not only the quantitative characteristics of the stress field vary from point to point, as in scalar fields, but it is also not possible to indicate a certain direction at any point, as in vector fields. The quantity prescribing the stress field must determine vector $\mathbf{t}_N dO$ at any point of the field and for any oriented surface $\mathbf{N}dO$ at this point (or vector $\mathbf{t}_N$ in terms of vector $\mathbf{N}$). This means that the physical state referred to as the stress field is determined by a quantity which relates vector $\mathbf{t}_N$ to vector $\mathbf{N}$. Adopting a linear relationship between these vectors (this question is considered in Subsection 1.4 in detail) means that this quantity is a tensor of second rank[2]. This tensor is referred to as the stress tensor and denoted as $\hat{T}$ whilst its components in a Cartesian coordinate system $OX_1X_2X_3$ are denoted as t_{ik}. Vector $\mathbf{t}_N$ is determined

[2] See the definition in Section A.3 (Appendix A).

by premultiplying $\hat{T}$ by $\mathbf{N}$

$$\mathbf{t}_N = \mathbf{N} \cdot \hat{T}. \tag{1.3.2}$$

Postmultiplying $\hat{T}$ by $\mathbf{N}$, i.e. $\hat{T} \cdot \mathbf{N}$, would only affect the notation of components of tensor $\hat{T}$.

Remark 1. It was assumed that the distribution of forces over the oriented surface $\mathbf{N}dO$ is statically equivalent to a single force $\mathbf{t}_N dO$. In other words, we assumed a zero principle moment of this force about a point on the line of action of this force. This assumption was omitted in the Cosserat continuum mechanics developed by at the beginning of the twentieth century. The reason for such a seemingly paradoxical statement that the moment has the order of smallness of the principle vector (order dO) is apparently due to the conditional character of the very concept of smallness in continuum mechanics. The so-called infinitesimally small volume comprises a complex object in itself and consists of a very large number of elementary particles, and the force transferred via an oriented surface should be treated as an integral effect of interaction of these particles. There is nothing logically inconsistent in that the influence of the moments can be comparable with that of the forces, at least at places of sharply changing stress state. In recent years the Cosserat ideas have been developed in numerous papers on non-symmetrical or moment theory of elasticity.

Remark 2. The adopted assumption that the reactive action of volume V_2 on V_1 can be replaced only by a system of forces distributed over surface O' is substantiated by the physical concept of the short-range interaction. In non-local theory of elasticity one also considers the mass forces of interaction of the "removed" part with the rest of the body.

1.1.4 Equilibrium of an elementary tetrahedron

Let us replace the assumption about a linear relationship between the vector of force $\mathbf{t}_N dO$ and the oriented surface $\mathbf{N}dO$ by assuming the general relationship

$$\mathbf{t}_N dO = \mathbf{f}\left(\mathbf{N}dO\right). \tag{1.4.1}$$

It is necessary to prove that $\mathbf{f}$ is a linear operation over vector $\mathbf{N}dO$. To this end, we consider equilibrium of the elementary tetrahedron with the vertex at point O and the edges $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ prescribed by vectors $\lambda\mathbf{e}_1$, $\lambda\mathbf{e}_2$, $\lambda\mathbf{e}_3$, with λ denoting a small scaling parameter, see Fig. 1.2. The outwards vectors of the oriented surfaces OAB, OBC, OCA are given by

$$\mathbf{N}_3 d\overset{3}{O} = \frac{1}{2}\lambda^2 \mathbf{e}_2 \times \mathbf{e}_1, \quad \mathbf{N}_1 d\overset{1}{O} = \frac{1}{2}\lambda^2 \mathbf{e}_3 \times \mathbf{e}_2, \quad \mathbf{N}_2 d\overset{2}{O} = \frac{1}{2}\lambda^2 \mathbf{e}_1 \times \mathbf{e}_3.$$

The right hand side of the easily proved identity

$$\mathbf{e}_3 \times \mathbf{e}_2 + \mathbf{e}_1 \times \mathbf{e}_3 + \mathbf{e}_2 \times \mathbf{e}_1 = (\mathbf{e}_3 - \mathbf{e}_1) \times (\mathbf{e}_2 - \mathbf{e}_1)$$

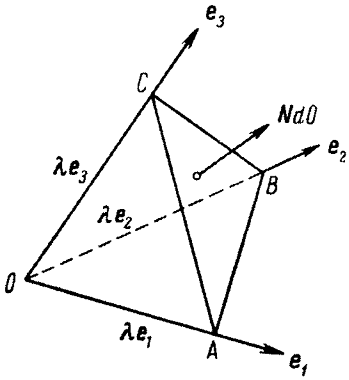

FIGURE 1.2.

is proportional to and directed in the opposite direction to vector $\mathbf{N}dO$ of the oriented surface ABC which is shown in Fig. 1.2. Hence

$$\mathbf{N}dO = \frac{1}{2}\lambda^2 (\mathbf{e}_2 - \mathbf{e}_1) \times (\mathbf{e}_3 - \mathbf{e}_1)$$

and thus

$$\mathbf{N}_1 d\overset{1}{O} + \mathbf{N}_2 d\overset{2}{O} + \mathbf{N}_3 d\overset{3}{O} = -\mathbf{N}dO. \tag{1.4.2}$$

The resultant vector of surface and mass forces acting on the tetrahedron vanishes, i.e.

$$\mathbf{t}_{N_1} d\overset{1}{O} + \mathbf{t}_{N_2} d\overset{2}{O} + \mathbf{t}_{N_3} d\overset{3}{O} + \mathbf{t}_N dO + \rho \mathbf{K} d\tau = 0.$$

The latter term is proportional to the elementary volume

$$d\tau = \frac{1}{6}\lambda^3 \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)$$

and should be neglected since the other terms are proportional to λ^2 as $\lambda \to 0$. Thus

$$-\mathbf{t}_N dO = \mathbf{t}_{N_1} d\overset{1}{O} + \mathbf{t}_{N_2} d\overset{2}{O} + \mathbf{t}_{N_3} d\overset{3}{O}. \tag{1.4.3}$$

Accounting for eqs. (1.4.2) and (1.4.3) and setting (1.3.1) in the following form

$$\mathbf{f}(-\mathbf{N}dO) = -\mathbf{f}(\mathbf{N}dO) \tag{1.4.4}$$

allows us to express eq. (1.4.1) as follows

$$\mathbf{f}\left(\mathbf{N}_1 d\overset{1}{O} + \mathbf{N}_2 d\overset{2}{O} + \mathbf{N}_3 d\overset{3}{O}\right) = \mathbf{f}\left(\mathbf{N}_1 d\overset{1}{O}\right) + \mathbf{f}\left(\mathbf{N}_2 d\overset{2}{O}\right) + \mathbf{f}\left(\mathbf{N}_3 d\overset{3}{O}\right).$$

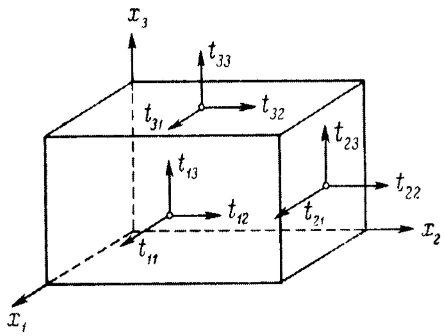

FIGURE 1.3.

This proves the linearity of the functional dependence (1.4.1). Relationship (1.3.2) defines the stress tensor $\hat{T}$ and, thus, is fundamental for constructing continuum mechanics.

By using a coordinate representation and eq. (1.4.2) we can write relationship (1.3.2) in the form

$$\left.\begin{aligned} t_{N_1} &= t_{11}N_1 + t_{21}N_2 + t_{31}N_3, \\ t_{N_2} &= t_{12}N_1 + t_{22}N_2 + t_{32}N_3, \\ t_{N_3} &= t_{13}N_1 + t_{23}N_2 + t_{33}N_3. \end{aligned}\right\} \tag{1.4.5}$$

Assuming $\mathbf{N} = \mathbf{i}_1$, so that $N_1 = 1, N_2 = N_3 = 0$ yields the vector of force acting on the unit area with the outward normal $\mathbf{i}_1$. Let us refer to this as stress vector $\mathbf{t}_1$. Its projections on the axes of system $OX_1X_2X_3$, i.e. t_{11}, t_{12}, t_{13} are termed stresses, where t_{11} is called the normal stress whilst t_{12} and t_{13} are called shear stresses. By analogy we introduce stress vectors $\mathbf{t}_2$ and $\mathbf{t}_3$ on the surfaces whose normals are the unit vectors of coordinate axes $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively. In the matrix of components of tensor $\hat{T}$

$$\begin{Vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{Vmatrix}, \tag{1.4.6}$$

the diagonal and non-diagonal elements present respectively normal and shear stresses. Figure 1.3 displays an elementary parallelepiped whose edges are parallel to the coordinate axes and the stresses on its three faces.

Remark 1. Relationships (1.4.5) were first obtained by Cauchy in 1827 by considering equilibrium of an elementary tetrahedron with edges parallel to the coordinate axes.

Remark 2. Stresses t_{sk} can only conditionally be called the projections of "vector" $\mathbf{t}_s$ since these quantities are transformed as tensor components rather than vector components under rotation of the coordinate system.

Quasivectors $\mathbf{t}_s$, see eq. (1.5.12), can be introduced in terms of a dyadic representation of the stress tensor

$$\hat{T} = \mathbf{i}_s \mathbf{i}_k t_{sk} = \mathbf{i}_s \mathbf{t}_s. \tag{1.4.7}$$

Remark 3. Figure 1.3 shows positive stresses t_{sk} ($t_{sk} > 0$) on faces with the outward normals whose directions are coincident with those of the coordinate axes. As $\mathbf{t}_{-s} = -\mathbf{t}_s$ the positive stresses t_{sk} have directions of $-\mathbf{i}_k$ on the faces with the normal $-\mathbf{i}_s$. This means that positive normal stresses are in tension whereas negative normal stresses are in compression. The moments of the positive normal stresses t_{sk} on faces $\mathbf{i}_s$ and $-\mathbf{i}_s$ about axis $\mathbf{i}_r$ have the sign of the corresponding Levi-Civita symbol e_{skr}, see eq. (A.1.2).

Remark 4. In the technical literature on the theory of elasticity the commonly accepted notation for normal and shear stresses are σ and τ with the corresponding indexes, so that the matrix of tensor $\hat{T}$ takes the form

$$\left\| \begin{array}{ccc} \sigma_x = \sigma_1 & \tau_{xy} = \tau_{12} & \tau_{xz} = \tau_{13} \\ \tau_{yx} = \tau_{21} & \sigma_y = \sigma_2 & \tau_{yz} = \tau_{23} \\ \tau_{zx} = \tau_{31} & \tau_{zy} = \tau_{32} & \sigma_z = \sigma_3 \end{array} \right\| . \tag{1.4.8}$$

This notation will be used in the present book along with notation (1.4.6). There are a number of other systems of notation, for example $\sigma_x = X_x$, $\tau_{xy} = X_y$ etc.

1.1.5 The necessary conditions for equilibrium of a continuum

Let us consider an arbitrary volume V_* bounded by a surface O_*. It is assumed that V_* lies completely within volume V, and O_* has no common points with surface O. The surface forces distributed over O_* which are internal for V and external for V_* are caused by the stress state described by tensor $\hat{T}$. They are given by the basic relationship (1.3.2) in which $\mathbf{N}$ denotes a unit vector of the external normal to O_*.

There are two groups of necessary conditions of equilibrium, namely equations of equilibrium in volume V and those on surface O.

The equations of equilibrium in the volume express the condition of vanishing principal vector and the principal moment of the mass and surface forces in an arbitrary volume V_* within volume V. Referring to eqs. (1.2.1) and (1.2.7) we have

$$\iiint\limits_{V_*} \rho \mathbf{K} d\tau + \iint\limits_{O_*} \mathbf{t}_N dO = 0, \quad \iiint\limits_{V_*} \mathbf{R} \times \rho \mathbf{K} d\tau + \iint\limits_{O_*} \mathbf{R} \times \mathbf{t}_N dO = 0 .$$

Replacing $\mathbf{t}_N$ by means of formula (1.3.2) yields

$$\left.\begin{aligned} \iiint\limits_{V_*} \rho \mathbf{K} d\tau + \iint\limits_{O_*} \mathbf{N} \cdot \hat{T} dO = 0, \\ \iiint\limits_{V_*} \mathbf{R} \times \rho \mathbf{K} d\tau + \iint\limits_{O_*} \mathbf{R} \times \mathbf{N} \cdot \hat{T} dO = 0. \end{aligned}\right\} \tag{1.5.1}$$

Transforming the surface integrals into volume integrals (see eqs. (B.5.5) and (B.5.6)) we obtain

$$\left.\begin{aligned} \iint\limits_{O_*} \mathbf{N} \cdot \hat{T} dO = \iiint\limits_{V_*} \operatorname{div} \hat{T} d\tau, \\ \iint\limits_{O_*} \mathbf{R} \times \mathbf{N} \cdot \hat{T} dO = \iiint\limits_{V_*} \left(\mathbf{R} \times \operatorname{div} \hat{T} - 2\boldsymbol{\omega}\right) d\tau, \end{aligned}\right\} \tag{1.5.2}$$

where $\boldsymbol{\omega}$ denotes a vector accompanying tensor $\hat{T}$. This vector is known to be determined by the skew-symmetric part of tensor $\hat{T}$. Then we arrive at the following equalities

$$\iiint\limits_{V_*} \left(\rho \mathbf{K} + \operatorname{div} \hat{T}\right) d\tau = 0, \quad \iiint\limits_{V_*} \left[\mathbf{R} \times \left(\rho \mathbf{K} + \operatorname{div} \hat{T}\right) - 2\boldsymbol{\omega}\right] d\tau = 0. \tag{1.5.3}$$

It follows from the equality

$$\iiint\limits_{V_*} f\left(x_1, x_2, x_3\right) d\tau = 0,$$

where V_* denotes an arbitrary volume and f is a continuous function of the coordinate, that $f \equiv 0$. Indeed, if one assumes that $f \neq 0$ at a point of volume V_*, then this function, as it is continuous, keeps the same sign in the vicinity of this point. This vicinity can be understood as a volume V_*, and the integral of a function with a constant sign can not vanish.

By virtue of the latter statement it follows from eq. (1.5.3) that

$$\operatorname{div} \hat{T} + \rho \mathbf{K} = 0. \tag{1.5.4}$$

This is the first equation of equilibrium for a continuum. Inserting it into the second equation in (1.5.3) yields $\boldsymbol{\omega} = 0$ which proves the symmetry of tensor $\hat{T}$

$$\hat{T} = \hat{T}^*. \tag{1.5.5}$$

Equilibrium equations for a continuum, eqs. (1.5.4) and (1.5.5), are written here in an invariant form. In Cartesian coordinates these equations have the form of three differential equations of statics of a continuum

$$\left.\begin{array}{l}\dfrac{\partial t_{11}}{\partial x_1}+\dfrac{\partial t_{21}}{\partial x_2}+\dfrac{\partial t_{31}}{\partial x_3}+\rho K_1=0,\\[2mm]\dfrac{\partial t_{12}}{\partial x_1}+\dfrac{\partial t_{22}}{\partial x_2}+\dfrac{\partial t_{32}}{\partial x_3}+\rho K_2=0,\\[2mm]\dfrac{\partial t_{13}}{\partial x_1}+\dfrac{\partial t_{23}}{\partial x_2}+\dfrac{\partial t_{33}}{\partial x_3}+\rho K_3=0\end{array}\right\} \tag{1.5.6}$$

and three equations expressing symmetry of the stress tensor

$$t_{23}=t_{32}, \quad t_{31}=t_{13}, \quad t_{12}=t_{21}. \tag{1.5.7}$$

By means of eqs. (1.5.6) and (1.5.7) a more general statement expressing the latter property can be set as follows

$$\mathbf{n}_1\cdot\hat{T}\cdot\mathbf{n}_2=\mathbf{n}_2\cdot\hat{T}\cdot\mathbf{n}_1\,, \tag{1.5.8}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are two arbitrary unit vectors. The latter equation says that the projection of the stress vector on the surface with normal $\mathbf{n}_1$ onto direction $\mathbf{n}_2$ is equal to the projection of the stress vector on the surface with normal $\mathbf{n}_2$ onto direction $\mathbf{n}_1$.

Equilibrium equations (1.5.6) and (1.5.7) can also be easily obtained by equating the principal vector and the principal moment of the surface and volume forces acting on an elementary parallelepiped to zero.

The surface forces of the faces perpendicular to axis $\mathbf{i}_1$ (the front and rear faces) are as follows

$$\begin{aligned}\mathbf{t}_1\left(x_1+\frac{1}{2}dx_1,x_2,x_3\right)dx_2dx_3&=\left(\mathbf{t}_1+\frac{1}{2}\frac{\partial\mathbf{t}_1}{\partial x_1}dx_1\right)dx_2dx_3\\ \mathbf{t}_{-1}\left(x_1-\frac{1}{2}dx_1,x_2,x_3\right)dx_2dx_3&=\left(\mathbf{t}_{-1}-\frac{1}{2}\frac{\partial\mathbf{t}_{-1}}{\partial x_1}dx_1\right)dx_2dx_3\,,\end{aligned}$$

where $\mathbf{t}_1\,(x_1,x_2,x_3)$ denotes the value of $\mathbf{t}_1$ at the center of the parallelepiped. Taking the origin at the vertex of the parallelepiped, the position radius of the points where these forces are applied can be considered as being equal to

$$\mathbf{i}_1dx_1+\frac{1}{2}\left(\mathbf{i}_2dx_2+\mathbf{i}_3dx_3\right)=\frac{1}{2}\mathbf{i}_1dx_1+\frac{1}{2}\mathbf{i}_sdx_s; \quad -\frac{1}{2}\mathbf{i}_1dx_1+\frac{1}{2}\mathbf{i}_sdx_s$$

where

$$\frac{1}{2}\mathbf{i}_sdx_s=\frac{1}{2}\left(\mathbf{i}_1dx_1+\mathbf{i}_2dx_2+\mathbf{i}_3dx_3\right)$$

is the position vector of the center of the parallelepiped. Similarly, we can construct expressions for the forces and the position vectors of the points where these forces are applied for the right and left faces which are perpendicular to $\mathbf{i}_2$

$$\begin{array}{cc} \left(\mathbf{t}_2 + \frac{1}{2}\frac{\partial \mathbf{t}_2}{\partial x_2} dx_2\right) dx_3 dx_1, & \left(\mathbf{t}_{-2} - \frac{1}{2}\frac{\partial \mathbf{t}_{-2}}{\partial x_2} dx_2\right) dx_3 dx_1, \\ \frac{1}{2}\mathbf{i}_2 dx_2 + \frac{1}{2}\mathbf{i}_s dx_s, & -\frac{1}{2}\mathbf{i}_2 dx_2 + \frac{1}{2}\mathbf{i}_s dx_s, \end{array}$$

as well as for the upper and lower faces which are perpendicular to $\mathbf{i}_3$

$$\begin{array}{cc} \left(\mathbf{t}_3 + \frac{1}{2}\frac{\partial \mathbf{t}_3}{\partial x_3} dx_3\right) dx_1 dx_2, & \left(\mathbf{t}_{-3} - \frac{1}{2}\frac{\partial \mathbf{t}_{-3}}{\partial x_2} dx_3\right) dx_1 dx_2, \\ \frac{1}{2}\mathbf{i}_3 dx_3 + \frac{1}{2}\mathbf{i}_s dx_s, & -\frac{1}{2}\mathbf{i}_3 dx_3 + \frac{1}{2}\mathbf{i}_s dx_s. \end{array}$$

The volume force $\rho \mathbf{K} dx_1 dx_2 dx_3$ is considered as being applied at the center of the parallelepiped. Equating the principal vector of the above forces and the principal moment of these forces about point O to zero and taking eq. (1.3.1) into account we arrive, after cancelling $dx_1 dx_2 dx_3$, at the following two vector equations

$$\frac{\partial \mathbf{t}_1}{\partial x_1} + \frac{\partial \mathbf{t}_2}{\partial x_2} + \frac{\partial \mathbf{t}_3}{\partial x_3} + \rho \mathbf{K} = 0, \tag{1.5.9}$$

$$\mathbf{i}_1 \times \mathbf{t}_1 + \mathbf{i}_2 \times \mathbf{t}_2 + \mathbf{i}_3 \times \mathbf{t}_3 + \frac{1}{2}\mathbf{i}_s dx_s \times \left(\frac{\partial \mathbf{t}_1}{\partial x_1} + \frac{\partial \mathbf{t}_2}{\partial x_2} + \frac{\partial \mathbf{t}_3}{\partial x_3} + \rho \mathbf{K}\right) = 0, \tag{1.5.10}$$

where the terms in parentheses in eq. (1.5.10) disappear due to eq. (1.5.9). Thus, we have derived relationships which are another form of equations (1.5.6) and (1.5.7)

$$\frac{\partial \mathbf{t}_s}{\partial x_s} + \rho \mathbf{K} = \left(\frac{\partial t_{st}}{\partial x_s} + \rho \mathbf{K}_t\right) \mathbf{i}_t = 0, \quad \frac{\partial t_{st}}{\partial x_s} + \rho K_t = 0, \tag{1.5.11}$$

$$\mathbf{i}_s \times \mathbf{t}_s = \mathbf{i}_s \times t_{st} \mathbf{i}_t = e_{rst} t_{st} \mathbf{i}_r = 0. \tag{1.5.12}$$

The three equilibrium equations (1.5.6) contain six components of the symmetric stress tensor. Clearly, equations (1.5.6) are only the necessary conditions for equilibrium, obtaining sufficient conditions inevitably requires consideration of a physical model of the continuum (elastic solid, viscous fluid etc.). The problem of equilibrium of a continuum is statically indeterminate.

Equilibrium equations on surface O bounding volume V are obtained from the basic relationship (1.3.2) where $\mathbf{t}_N$ is replaced by force $\mathbf{F}$ distributed over O

$$\mathbf{N} \cdot \hat{T} = \mathbf{F}. \tag{1.5.13}$$

Another form of this equality is

$$N_1 \mathbf{t}_1 + N_2 \mathbf{t}_2 + N_3 \mathbf{t}_3 = \mathbf{F} \tag{1.5.14}$$

or

$$\left.\begin{aligned} N_1 t_{11} + N_2 t_{21} + N_3 t_{31} &= F_1, \\ N_1 t_{12} + N_2 t_{22} + N_3 t_{32} &= F_2, \\ N_1 t_{13} + N_2 t_{23} + N_3 t_{33} &= F_3, \end{aligned}\right\} \tag{1.5.15}$$

where N_s denote projections of the unit vector $\mathbf{N}$ on the coordinate axes.

Let us agree to say that any particular solution of the equilibrium equations in the volume and on the surface determines a possible static state of the continuum. Such a state is described by a particular solution of the system of partial differential equations (1.5.6) with six unknown variables satisfying three boundary conditions (1.5.15). The goal of statics of a continuum is to determine a state corresponding to the adopted physical model.

1.1.6 Tensor of stress functions

Equilibrium equations for a continuum (1.5.4) are linear in components of the stress tensor, that is, the solution is the sum of a particular solution of the equation

$$\operatorname{div} \hat{T}^{(1)} + \rho \mathbf{K} = 0 \tag{1.6.1}$$

and the solution of the homogeneous equation

$$\operatorname{div} \hat{T}^{(2)} = 0. \tag{1.6.2}$$

The particular solution is assumed to be known; for $\rho = \text{const}$ and the mass forces encountered in practice (e.g. gravity force and centrifugal force) it can be found easily. For this reason we study a general representation of a tensor with zero divergence. In order to avoid notational complications we will denote it by $\hat{T}$ rather than $\hat{T}^{(2)}$. Such a tensor should be sought in the form

$$\hat{T} = \operatorname{rot} \hat{P}, \tag{1.6.3}$$

cf. eq. (B.4.16), where $\hat{P}$ is a tensor of second rank which, due to eq. (1.5.5), must obey the following condition

$$\operatorname{rot} \hat{P} = \left(\operatorname{rot} \hat{P}\right)^*. \tag{1.6.4}$$

Referring to eq. (B.4.13) we can satisfy this condition by taking

$$\hat{P} = \left(\operatorname{rot}\hat{\Phi}\right)^*, \tag{1.6.5}$$

where $\hat{\Phi}$ denotes any symmetric tensor of second rank. Thus, tensor

$$\hat{T} = \operatorname{rot}\left(\operatorname{rot}\hat{\Phi}\right)^* = \operatorname{inc}\hat{\Phi} \tag{1.6.6}$$

meets the required conditions because it is symmetric and its divergence is equal to zero. The symmetric tensor $\hat{\Phi}$ is referred to as the tensor of stress functions.

Adopting a diagonal form of $\hat{\Phi}$, i.e.

$$\hat{\Phi} = \mathbf{i}_1\mathbf{i}_1\Phi_{11} + \mathbf{i}_2\mathbf{i}_2\Phi_{22} + \mathbf{i}_3\mathbf{i}_3\Phi_{33}, \tag{1.6.7}$$

leads, by means of eq. (B.4.15), to the following representation of stresses in terms of Maxwell's stress functions

$$\left.\begin{aligned}
t_{11} &= \frac{\partial^2\Phi_{22}}{\partial x_3^2} + \frac{\partial^2\Phi_{33}}{\partial x_2^2}, & t_{12} = t_{21} &= -\frac{\partial^2\Phi_{33}}{\partial x_1\partial x_2},\\
t_{22} &= \frac{\partial^2\Phi_{33}}{\partial x_1^2} + \frac{\partial^2\Phi_{11}}{\partial x_3^2}, & t_{23} = t_{32} &= -\frac{\partial^2\Phi_{11}}{\partial x_2\partial x_3},\\
t_{33} &= \frac{\partial^2\Phi_{11}}{\partial x_2^2} + \frac{\partial^2\Phi_{22}}{\partial x_1^2}, & t_{31} = t_{13} &= -\frac{\partial^2\Phi_{22}}{\partial x_3\partial x_1}.
\end{aligned}\right\} \tag{1.6.8}$$

Representation of the stress tensor in terms of Morera's stress functions is obtained by assuming zero diagonal components

$$\hat{\Phi} = (\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)\,\Phi_{12} + (\mathbf{i}_2\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_2)\,\Phi_{23} + (\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_1\mathbf{i}_3)\,\Phi_{31}.$$

In this case

$$\left.\begin{aligned}
t_{11} &= -2\frac{\partial^2\Phi_{23}}{\partial x_2\partial x_3}, & t_{23} &= \frac{\partial}{\partial x_1}\left(\frac{\partial\Phi_{31}}{\partial x_2} + \frac{\partial\Phi_{12}}{\partial x_3} + \frac{\partial\Phi_{23}}{\partial x_1}\right),\\
t_{22} &= -2\frac{\partial^2\Phi_{31}}{\partial x_3\partial x_1}, & t_{31} &= \frac{\partial}{\partial x_2}\left(\frac{\partial\Phi_{12}}{\partial x_3} + \frac{\partial\Phi_{23}}{\partial x_1} + \frac{\partial\Phi_{31}}{\partial x_2}\right),\\
t_{33} &= -2\frac{\partial^2\Phi_{12}}{\partial x_1\partial x_2}, & t_{12} &= \frac{\partial}{\partial x_3}\left(\frac{\partial\Phi_{23}}{\partial x_1} + \frac{\partial\Phi_{31}}{\partial x_2} + \frac{\partial\Phi_{12}}{\partial x_3}\right).
\end{aligned}\right\} \tag{1.6.9}$$

Representation of the stress tensor in terms of Maxwell's functions is not invariant since any diagonal tensor is no longer diagonal under a coordinate transformation. Morera's representation is also not invariant. An invariant representation of the stress tensor in the form of eq. (1.6.6) was suggested independently by B. Finzi, Yu.A. Krutkov and V.I. Blokh.

In the plane problem of elasticity the stresses are independent of coordinate x_3 and components t_{23} and t_{31} of the stress tensor are absent.

An expression for the tensor of stress functions which is invariant about rotation about axis OX_3 can be taken in the following form

$$\hat{\Phi} = \left(\nu \hat{E}_2 + \mathbf{i}_3\mathbf{i}_3\right) U(x_1, x_2), \quad \hat{E}_2 = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2 = \hat{E} - \mathbf{i}_3\mathbf{i}_3, \tag{1.6.10}$$

where $\nu = \text{const}$. Then, by virtue of eq. (1.6.8)

$$t_{11} = \frac{\partial^2 U}{\partial x_2^2}, \; t_{22} = \frac{\partial^2 U}{\partial x_1^2}, \; t_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2}, \; t_{33} = \nu\left(\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2}\right). \tag{1.6.11}$$

Function $U(x_1, x_2)$ is the Airy stress function. It is easy to see that expressions (1.6.11) identically satisfy the homogeneous equations of equilibrium of the plane problem

$$\frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{21}}{\partial x_2} = 0, \quad \frac{\partial t_{12}}{\partial x_1} + \frac{\partial t_{22}}{\partial x_2} = 0.$$

Representation (1.6.6) suggests that for any stress tensor $\hat{T}$ the tensor of stress functions is determined up to an additive tensor $\hat{\Phi}^{(1)}$ which is symmetric and for which $\operatorname{inc}\hat{\Phi}^{(1)} = 0$. Thus, as we can see in Subsection 2.2.1, this tensor is a tensor of linear deformation of any vector $\mathbf{a}$

$$\hat{\Phi}^{(1)} = \operatorname{def}\mathbf{a}, \quad \operatorname{inc}\hat{\Phi}^{(1)} = \operatorname{inc}\operatorname{def}\mathbf{a} \equiv 0. \tag{1.6.12}$$

Hence, assuming

$$\hat{\Phi} = \hat{\Phi}_* + \operatorname{def}\mathbf{a}, \tag{1.6.13}$$

we have

$$\hat{T} = \operatorname{inc}\hat{\Phi} = \operatorname{inc}\hat{\Phi}_*. \tag{1.6.14}$$

Therefore, $\hat{\Phi}$ contains three arbitrarily prescribed functions a_s. This allows one to comprehend why six functions t_{sk} related by three differential equations (1.5.6) are expressed in terms of six rather than three stress functions Φ_{rt}.

1.2 The properties of the stress tensor

1.2.1 Component transformation, principal stresses and principal invariants

The stress tensor possesses all properties of the symmetric tensor as listed in Appendix A.

The law of transformation for the components of a stress tensor under rotation of the Cartesian coordinate system is given by formulae (A.3.6). These formulae can also be obtained from Cauchy's dependence (1.4.5). Let $\mathbf{N}$ be coincident with vector $\mathbf{i}'_k$ then $\alpha_{ks} = \mathbf{i}'_k \cdot \mathbf{i}_s = N_m$ and the projections of "quasivector" $\mathbf{t}'_k$ on the old axes, i.e. stresses on the surface with the normal $\mathbf{i}'_k$, are given, due to eq. (1.4.5), by the following equations

$$\mathbf{t}'_k \cdot \mathbf{i}_r = t_{mr} N_m = t_{mr} \alpha_{km},$$

whereas those on the new axes are

$$t'_{ks} = \alpha_{sr} \mathbf{t}'_k \cdot \mathbf{i}_r = \alpha_{km} \alpha_{sr} t_{mr}. \tag{2.1.1}$$

For example,

$$t'_{11} = \alpha_{11}^2 t_{11} + \alpha_{12}^2 t_{22} + \alpha_{13}^2 t_{33} + 2\left(\alpha_{11}\alpha_{12}t_{12} + \alpha_{12}\alpha_{13}t_{23} + \alpha_{13}\alpha_{11}t_{31}\right), \tag{2.1.2}$$

$$t'_{12} = \alpha_{11}\alpha_{21}t_{11} + \alpha_{12}\alpha_{22}t_{22} + \alpha_{13}\alpha_{23}t_{33} + \left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\right)t_{12} + \\ \left(\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{22}\right)t_{23} + \left(\alpha_{13}\alpha_{21} + \alpha_{11}\alpha_{23}\right)t_{31}, \tag{2.1.3}$$

It is easy to derive these formulae with the help of the following identity

$$\hat{T} = \hat{E} \cdot \hat{T} \cdot \hat{E},$$

where the unit tensor $\hat{E}$ is set as follows

$$\hat{E} = \mathbf{i}'_s \mathbf{i}'_s = \mathbf{i}'_k \mathbf{i}'_k.$$

Again we obtain eq. (2.1.1)

$$\hat{T} = t'_{ks} \mathbf{i}'_k \mathbf{i}'_s = \mathbf{i}'_k \mathbf{i}'_k \cdot t_{mr} \mathbf{i}_m \mathbf{i}_r \cdot \mathbf{i}'_s \mathbf{i}'_s = \alpha_{km} \alpha_{sr} t_{mr} \mathbf{i}'_k \mathbf{i}'_s. \tag{2.1.4}$$

The principal values of the stress tensor, referred to as the principal stresses, are equal to the roots t_1, t_2, t_3 of its characteristic equation

$$P_3(t) = |t_{sk} - \delta_{sk} t| = \begin{vmatrix} t_{11} - t & t_{12} & t_{13} \\ t_{21} & t_{22} - t & t_{23} \\ t_{31} & t_{32} & t_{33} - t \end{vmatrix} = 0. \tag{2.1.5}$$

The principal directions called principal axes form an orthogonal trihedron of the unit vectors $\overset{1}{\mathbf{e}}$, $\overset{2}{\mathbf{e}}$, $\overset{3}{\mathbf{e}}$ whose cosines of the angles to the coordinate axes $\overset{s}{e}_k = \overset{s}{\mathbf{e}} \cdot \mathbf{i}_k$ are determined by the following system of equations ($r = 1, 2, 3$; $s = 1, 2, 3$)

$$\left(t_{rm} - t_s \delta_{rm}\right) \overset{s}{e}_m = 0, \quad \overset{s}{e}{}_1^2 + \overset{s}{e}{}_2^2 + \overset{s}{e}{}_3^2 = 1, \quad (\not\Sigma_s) \tag{2.1.6}$$

where the symbol $\not\Sigma_s$ implies no summation over s. The diagonal representation of the stress tensor in terms of the principal axes is written in the form

$$\hat{T} = t_1 \overset{11}{\mathbf{ee}} + t_2 \overset{22}{\mathbf{ee}} + t_3 \overset{33}{\mathbf{ee}}, \tag{2.1.7}$$

that is, the principal stresses t_s on the surfaces with normal vectors $\overset{s}{\mathbf{e}}$ are normal whilst shear stresses are absent. Expressions for the components of the stress tensor in the system of axes $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ in terms of the principal stresses take the form

$$t_{sk} = \alpha_{s1}\alpha_{k1}t_1 + \alpha_{s2}\alpha_{k2}t_2 + \alpha_{s3}\alpha_{k3}t_3. \tag{2.1.8}$$

Here $\alpha_{sm} = \mathbf{i}_s \cdot \overset{m}{\mathbf{e}}$ since the principal axes play the role of the "old" axes. The notion of the Cauchy dependences (1.4.5) simplifies as well

$$t_{N1} = t_1 N_1, \quad t_{N2} = t_2 N_2, \quad t_{N3} = t_3 N_3, \quad \left(N_k = \mathbf{N} \cdot \overset{\mathbf{k}}{\mathbf{e}} \right). \tag{2.1.9}$$

Due to eq. (2.1.8) the normal stress on a surface with the normal $\mathbf{N}$ is expressed in terms of the principal stresses as follows

$$\sigma_N = \mathbf{N} \cdot \hat{T} \cdot \mathbf{N} = t_1 N_1^2 + t_2 N_2^2 + t_3 N_3^2, \tag{2.1.10}$$

which can be easily obtained from eqs. (2.1.7) and (2.1.9). At the same time, by virtue of eq. (2.1.9)

$$t_N^2 = \sigma_N^2 + \tau_N^2 = t_1^2 N_1^2 + t_2^2 N_2^2 + t_3^2 N_3^2, \tag{2.1.11}$$

which yields the square of the absolute value of quasivector t_N which is the total stress on the surface with normal $\mathbf{N}$. The total shear stress on this surface is denoted by τ_N, see Fig. 1.4.

The quantity σ_N represents the $\mathbf{NN}$−component of tensor $\hat{T}$ while t_N^2 is the square of the magnitude of vector $\mathbf{N} \cdot \hat{T}$. For this reason, in coordinate axes $\mathbf{i}_s$ we have

$$\left. \begin{aligned} \sigma_N &= \mathbf{N} \cdot \hat{T} \cdot \mathbf{N} = t_{st} N_s N_t, \\ t_N^2 &= \mathbf{N} \cdot \hat{T} \cdot \hat{T} \cdot \mathbf{N} = \mathbf{N} \cdot \hat{T}^2 \cdot \mathbf{N} = t_{sk} t_{kt} N_s N_t, \\ \tau_N^2 &= t_N^2 - \sigma_N^2 = N_s N_t \left(t_{sk} t_{kt} - N_k N_r t_{st} t_{kr} \right), \end{aligned} \right\} \tag{2.1.12}$$

where $N_s = \mathbf{N} \cdot \mathbf{i}_s$ and axes $\mathbf{i}_s$ are assumed not to be the principal axes.

In the plane problem of elasticity axis OX_3 is one of the principal axes since $t_{23} = t_{31} = 0$. Denoting the angles between axis $\mathbf{i}_1$ and the principal axes e_1 and e_2 as φ and $\frac{\pi}{2} - \varphi$, respectively, we have due to eq. (2.1.8)

$$\left. \begin{aligned} t_{11} &= t_1 \cos^2 \varphi + t_2 \sin^2 \varphi = \tfrac{1}{2}(t_1 + t_2) - \tfrac{1}{2}(t_2 - t_1) \cos 2\varphi, \\ t_{22} &= t_1 \sin^2 \varphi + t_2 \cos^2 \varphi = \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}(t_2 - t_1) \cos 2\varphi, \\ t_{12} &= (t_2 - t_1) \cos \varphi \sin \varphi = \tfrac{1}{2}(t_2 - t_1) \sin 2\varphi. \end{aligned} \right\} \tag{2.1.13}$$

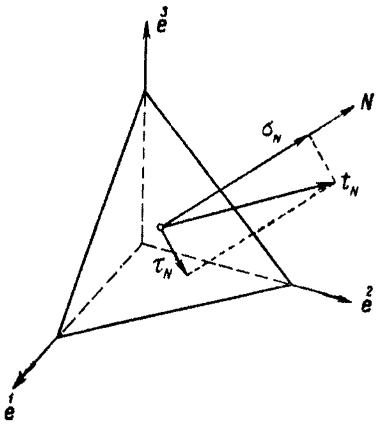

FIGURE 1.4.

Referring to eq. (1.6.11) it is easy to derive the basic relationships of the plane problem

$$\left.\begin{aligned} &t_1 + t_2 = t_{11} + t_{22} = \nabla^2 U, \\ &(t_2 - t_1)\, e^{2i\varphi} = t_{22} - t_{11} + 2it_{12} = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)^2 U. \end{aligned}\right\} \tag{2.1.14}$$

Using eqs. (A.10.4), (A.10.10) and (A.10.11) one can write the following equations for the principal invariants of tensor $\hat{T}$

$$\left.\begin{aligned} &I_1\left(\hat{T}\right) = t_1 + t_2 + t_3 = t_{11} + t_{22} + t_{33} = t_{ss}, \\ &I_2\left(\hat{T}\right) = \frac{1}{2}\left[(t_{ss})^2 - t_{sk}t_{ks}\right] = \frac{1}{2}\left[I_1^2\left(\hat{T}\right) - I_1\left(\hat{T}^2\right)\right], \\ &I_3\left(\hat{T}\right) = t_1t_2t_3 = |t_{sk}| = \frac{1}{6}\left[I_1^3\left(\hat{T}\right) - 3I_1\left(\hat{T}\right)I_1\left(\hat{T}^2\right) + 2I_1\left(\hat{T}^3\right)\right]. \end{aligned}\right\} \tag{2.1.15}$$

1.2.2 Mohr's circles of stress

We look for the surfaces on which the normal and shear stresses have *a priori* prescribed values σ_N and τ_N, respectively. The problem reduces to a search for three unknown values N_1^2, N_2^2, N_3^2 from eqs. (2.1.10) and (2.1.11) along with the following equation

$$N_1^2 + N_2^2 + N_3^2 = 1.$$

The sought-for solution is written down in the form

$$N_1^2 = \frac{f_1(\sigma_N, \tau_N)}{(t_1 - t_2)(t_1 - t_3)}, \quad N_2^2 = \frac{f_2(\sigma_N, \tau_N)}{(t_2 - t_1)(t_2 - t_3)}, \quad N_3^2 = \frac{f_3(\sigma_N, \tau_N)}{(t_3 - t_1)(t_3 - t_2)}, \tag{2.2.1}$$

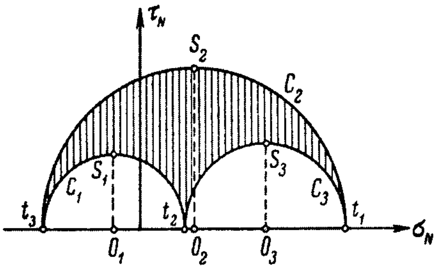

FIGURE 1.5.

where

$$\left.\begin{aligned} f_1(\sigma_N,\tau_N) &= \tau_N^2 + \left(\sigma_N - \frac{t_2+t_3}{2}\right)^2 - \left(\frac{t_2-t_3}{2}\right)^2, \\ f_2(\sigma_N,\tau_N) &= \tau_N^2 + \left(\sigma_N - \frac{t_3+t_1}{2}\right)^2 - \left(\frac{t_3-t_1}{2}\right)^2, \\ f_3(\sigma_N,\tau_N) &= \tau_N^2 + \left(\sigma_N - \frac{t_1+t_2}{2}\right)^2 - \left(\frac{t_1-t_2}{2}\right)^2. \end{aligned}\right\} \tag{2.2.2}$$

Let us agree to label the principal stresses in order of decreasing values, i.e. $t_1 > t_2 > t_3$. Clearly, only such σ_N, τ_N are possible, for which $N_s^2 > 0$. Hence, the following inequalities

$$f_1 > 0, \quad f_2 < 0, \quad f_3 > 0 \tag{2.2.3}$$

must hold true. Curves C_k, which lie in the half-plane $\tau_N > 0$ and for which $f_k = 0$, are the semicircles

$$\left.\begin{aligned} &C_1 \text{ with a centre at point } O_1 \left(\frac{t_2+t_3}{2}, 0\right) \text{ of radius } \frac{t_2-t_3}{2}, \\ &C_2 \text{ with a centre at point } O_2 \left(\frac{t_3+t_1}{2}, 0\right) \text{ of radius } \frac{t_1-t_3}{2}, \\ &C_2 \text{ with a centre at point } O_3 \left(\frac{t_1+t_2}{2}, 0\right) \text{ of radius } \frac{t_1-t_2}{2}. \end{aligned}\right\} \tag{2.2.4}$$

At the centres of these circles $f_k < 0$ $(k = 1,2,3)$, thus $f_k > 0$ in the parts of the half-plane which are outside of C_k. It follows from inequalities (2.2.3) that the region of possible σ_N, τ_N is located outside of C_3 and C_1 and inside of C_2, and is shown by the hatching in Fig. 1.5.

The point S_2 of semicircle C_2 corresponds to the maximum shear stress

$$(\tau_N)_{\max} = \tau_2 = \frac{t_1-t_3}{2} = (\tau_N)_{S_2}. \tag{2.2.5}$$

It is realised on the surfaces having the normal $\mathbf{N}^{(S_2)}$

$$N_1^{(S_2)} = \varepsilon \sqrt{\frac{f_1\left(\frac{t_1+t_3}{2}, \tau_2\right)}{(t_1-t_2)(t_1-t_3)}} = \varepsilon \sqrt{\frac{1}{2}},$$
$$N_2^{(S_2)} = 0, \; N_3^{(S_2)} = \varepsilon \sqrt{\frac{1}{2}}, \quad (\varepsilon = \pm 1).$$

The shear stresses corresponding to points S_3, S_1 of semicircles C_3 and C_1 are designated by

$$\tau_3 = \frac{t_1 - t_2}{2} = (\tau_N)_{S_3}, \quad \tau_1 = \frac{t_2 - t_3}{2} = (\tau_N)_{S_1}. \tag{2.2.6}$$

The orientation of the corresponding surfaces are given by their normals

$$N_1^{(S_3)} = \varepsilon \sqrt{\frac{1}{2}}, \quad N_2^{(S_3)} = \varepsilon \sqrt{\frac{1}{2}}, \quad N_3^{(S_3)} = 0.$$
$$N_1^{(S_3)} = 0, \qquad N_2^{(S_3)} = \varepsilon \sqrt{\frac{1}{2}}, \quad N_3^{(S_3)} = \varepsilon \sqrt{\frac{1}{2}}.$$

As one can see from these equations the shear stresses τ_k are observed on the planes which pass through the principal direction $\overset{k}{\mathbf{e}}$ and bisect the right angle between the principal coordinate planes intersecting in direction $\overset{k}{\mathbf{e}}$. One refers to τ_k as the principal shear stresses.

By using notation (2.2.5) and (2.2.6) we can easily obtain from eqs. (2.1.10) and (2.1.11) that

$$\tau_N^2 = 4\left(\tau_1^2 N_2^2 N_3^2 + \tau_2^2 N_3^2 N_1^2 + \tau_3^2 N_1^2 N_2^2\right). \tag{2.2.7}$$

In particular, on the octahedron plane, that is the plane equally inclined to the principal axes $\left(N_1^2 = N_2^2 = N_3^2 = 1/3\right)$ we have

$$\left.\begin{aligned} \sigma_N &= \frac{1}{3}(t_1 + t_2 + t_3) = \frac{1}{3} I_1\left(\hat{T}\right), \\ t_N^2 &= \frac{1}{3}\left(t_1^2 + t_2^2 + t_3^2\right) = \frac{1}{3} I_1\left(\hat{T}^2\right), \\ \tau_N^2 &= \frac{4}{9}\left(\tau_1^2 + \tau_2^2 + \tau_3^2\right). \end{aligned}\right\} \tag{2.2.8}$$

On the other hand, due to eqs. (A.11.6) and (A.10.10) we obtain

$$\tau_N^2 = t_N^2 - \sigma_N^2 = \frac{1}{3}\left[I_1\left(\hat{T}^2\right) - \frac{1}{3} I_1^2\left(\hat{T}\right)\right] = -\frac{2}{3} I_2\left(\operatorname{Dev}\hat{T}\right), \tag{2.2.9}$$

so that

$$I_2\left(\operatorname{Dev}\hat{T}\right) = -\frac{2}{3}\left(\tau_1^2 + \tau_2^2 + \tau_3^2\right). \tag{2.2.10}$$

It is worth noting that eq. (A.11.8) yields the same results. The value

$$\tau = \sqrt{-I_2\left(\operatorname{Dev}\hat{T}\right)} = \sqrt{\frac{2}{3}\left(\tau_1^2 + \tau_2^2 + \tau_3^2\right)} \tag{2.2.11}$$

is referred to as the intensity of shear stresses. The formulae obtained enable a mechanical interpretation of the invariants of the stress tensor.

The construction of the region for possible values of σ_N and τ_N was given by O. Mohr in 1882. Clearly, it is applicable for any symmetric tensor of second rank $\hat{Q}$, the role of σ_N and τ_N^2 being respectively played by $\mathbf{N}\cdot\hat{Q}\cdot\mathbf{N}$ and $\mathbf{N}\cdot\hat{Q}^2\cdot\mathbf{N}$.

1.2.3 Separating the stress tensor into a spherical tensor and a deviator

The stress tensor is represented in the form of eq. (A.11.1), i.e.

$$\hat{T} = \frac{1}{3}I_1\left(\hat{T}\right)\hat{E} + \operatorname{Dev}\hat{T} = \frac{1}{3}\sigma\hat{E} + \operatorname{Dev}\hat{T}. \tag{2.3.1}$$

Here $\frac{1}{3}\sigma$ is the mean value of the sum of three normal stresses on mutually orthogonal surfaces. Such a state of stress is realised in an ideal fluid or in a viscous fluid at rest with an equal stress $p = -\frac{1}{3}\sigma$ on any surface. Such a "hydrostatic" state of stress corresponds to the spherical part of the stress tensor whereas the deviation from the hydrostatic state is characterised by the deviator $\operatorname{Dev}\hat{T}$.

1.2.4 Examples of the states of stress

First example. In the state of stress referred to as pure shear, the stresses on the surfaces which are orthogonal to $\mathbf{i}_3$ as well as stresses t_{11}, t_{22} are absent. Tensor $\hat{T}$ is given by the equality

$$\hat{T} = (\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)\, t_{12} \tag{2.4.1}$$

and the characteristic equation (2.1.5) takes the form

$$\begin{vmatrix} -t & t_{12} & 0 \\ t_{21} & -t & 0 \\ 0 & 0 & -t \end{vmatrix} = -t\left(t^2 - t_{12}^2\right) = 0.$$

The principal stresses are

$$t_1 = t_{12}, \quad t_2 = 0, \quad t_3 = -t_{12}. \tag{2.4.2}$$

System of equations (2.1.6) determining the principal axis of stresses $\overset{1}{\mathbf{e}}$ is as follows

$$-t_{12}\overset{1}{e_1} + t_{12}\overset{1}{e_2} = 0, \; t_{21}\overset{1}{e_1} - t_{12}\overset{1}{e_2} = 0, \; -t_{12}\overset{1}{e_3} = 0, \; \overset{1}{e_1}{}^2 + \overset{1}{e_2}{}^2 + \overset{1}{e_3}{}^2 = 1.$$

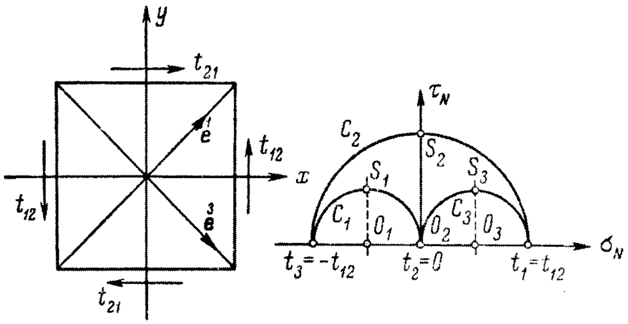

FIGURE 1.6.

One of these equations should be a consequence of the others. One can see that the second equation repeats the first equation. Hence

$$\overset{1}{e}_1=\overset{1}{e}_2=\pm\frac{1}{\sqrt{2}},\quad \overset{1}{e}_3=0.$$

By analogy we obtain

$$\overset{2}{e}_1=\overset{2}{e}_2=0,\quad \overset{2}{e}_3=\pm 1;\quad \overset{3}{e}_1=-\overset{3}{e}_2=\pm\frac{1}{\sqrt{2}},\quad \overset{3}{e}_3=0.$$

The principal axes $\overset{1}{\mathbf{e}}$ and $\overset{3}{\mathbf{e}}$ are directed along the diagonals of the square, see Fig. 1.6, and the principal axis $\overset{2}{\mathbf{e}}$ is coincident with $\mathbf{i}_3$ which follows directly from eq. (2.4.1) for tensor $\hat{T}$.

The location of Mohr's circles is shown in Fig. 1.6. The principal shear stresses and the intensity of shear stresses are given by

$$\tau_1=\tau_3=\frac{1}{2}t_{12},\quad \tau=t_{12},\quad \tau=\sqrt{\frac{2}{3}\left(\tau_1^2+\tau_2^2+\tau_3^2\right)}=t_{12}.$$

This explains the choice of multiplier 2/3 in definition (2.2.11) of quantity τ. In the case of pure shear the spherical part of $\hat{T}$ as well as the normal stresses on the octahedron surfaces are absent, the total shear stress on these surfaces being equal to $\sqrt{\frac{2}{3}}t_{12}$.

In an isotropic non-linear elastic medium the state of pure shear is not accompanied by pure shear strain, see Subsection 2.6.3 for details. The realisation of a pure shear strain requires the application of normal stresses.

Second example. Let us consider a linear elastic rod subjected to torsion about axis $\mathbf{i}_3$. The state of stress in the rod is described by the tensor

$$\hat{T}=t_{31}\left(\mathbf{i}_3\mathbf{i}_1+\mathbf{i}_1\mathbf{i}_3\right)+t_{23}\left(\mathbf{i}_2\mathbf{i}_3+\mathbf{i}_3\mathbf{i}_2\right). \tag{2.4.3}$$

The invariants of the tensor are

$$I_1\left(\hat{T}\right)=0, \quad I_2\left(\hat{T}\right)=-\left(t_{31}^2+t_{23}^2\right), \quad I_3\left(\hat{T}\right)=0, \tag{2.4.4}$$

and its characteristic equation, due to eq. (A.10.3), is as follows

$$-t^3+t\left(t_{31}^2+t_{23}^2\right)=0.$$

The principal stresses are equal to

$$t_1=\sqrt{t_{31}^2+t_{23}^2}, \quad t_2=0, \quad t_3=-\sqrt{t_{31}^2+t_{23}^2},$$

whilst the directions of the principal axes are given by the Table of the direction cosines

	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\overset{1}{\mathbf{e}}$	$\frac{\varepsilon}{\sqrt{2}}\cos\alpha$	$\frac{\varepsilon}{\sqrt{2}}\sin\alpha$	$\frac{\varepsilon}{\sqrt{2}}$
$\overset{2}{\mathbf{e}}$	$-\varepsilon\sin\alpha$	$\varepsilon\cos\alpha$	0
$\overset{3}{\mathbf{e}}$	$-\frac{\varepsilon}{\sqrt{2}}\cos\alpha$	$-\frac{\varepsilon}{\sqrt{2}}\sin\alpha$	$\frac{\varepsilon}{\sqrt{2}}$

Table 1.1 Table of the direction cosines

where $\cos\alpha=t_{31}/t_1$ and $\sin\alpha=t_{23}/t_1$. The state of stress on the faces of the parallelepiped with edges having the directions $\mathbf{i}_3$, the principal axis $\overset{2}{\mathbf{e}}$ and the normal $\mathbf{m}$, see Fig. 1.7, is a pure shear stress of intensity $\sqrt{t_{31}^2+t_{23}^2}$. In these axes the stress tensor is set as follows

$$\hat{T}=\sqrt{t_{31}^2+t_{23}^2}\left(\mathbf{m}\mathbf{i}_3+\mathbf{i}_3\mathbf{m}\right).$$

Third example. The tensor of equal shear stresses is given by the equalities

$$t_{ik}=\tau_0 \quad (i\neq k), \quad t_{11}=t_{22}=t_{33}=0. \tag{2.4.5}$$

Its invariants are

$$I_1\left(\hat{T}\right)=0, \quad I_2\left(\hat{T}\right)=-3\tau_0^2, \quad I_3\left(\hat{T}\right)=2\tau_0^3,$$

while the principal stresses are determined by the roots of the cubic equation

$$-t^3+3\tau_0^2t+2\tau_0^3=0$$

and are equal to

$$t_1=2\tau_0, \quad t_2=t_3=-\tau_0.$$

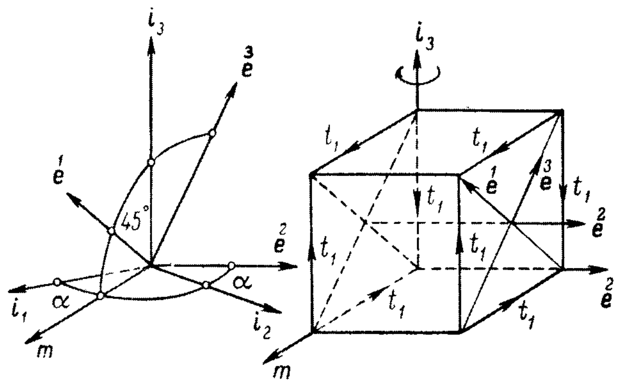

FIGURE 1.7.

The direction of the first principal axis is determined by the vector of the normal to the octahedron surface

$$\overset{1}{\mathbf{e}} = \pm\frac{1}{\sqrt{3}}\left(\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3\right),$$

whereas its principal axes $\overset{2}{\mathbf{e}}, \overset{3}{\mathbf{e}}$ lie in the plane orthogonal to $\overset{1}{\mathbf{e}}$ and are determined up to a rotation about this axis. Using eq. (A.9.14) we can put the stress tensor in the form

$$\hat{T} = \tau_0\left(3\,\overset{1}{\mathbf{e}}\overset{1}{\mathbf{e}} - \hat{E}\right).$$

A cylinder having axis of direction $\overset{1}{\mathbf{e}}$ is subjected to a tensile stress $2\tau_0$ along this axis and a compressive stress τ_0 on the side surface.

Fourth example. The electrostatic system of Maxwell stresses is given by the tensor

$$\hat{T} = \frac{kg}{4\pi}\left(\mathbf{EE} - \frac{1}{2}\hat{E}\mathbf{E}\cdot\mathbf{E}\right), \tag{2.4.6}$$

where g is the density of free charges, k is the dielectric constant which is assumed to be independent of g and $\mathbf{E}$ denotes the vector of the electric field strength. The electric field appears in the field of volume forces acting on a dielectric

$$\rho\mathbf{K} = -\operatorname{div}\hat{T} = -\frac{kg}{4\pi}\operatorname{div}\left(\mathbf{EE} - \frac{1}{2}\hat{E}\mathbf{E}\cdot\mathbf{E}\right).$$

Accounting for the relationships

$$\operatorname{div}\mathbf{EE} = \mathbf{E}\operatorname{div}\mathbf{E} - \mathbf{E}\times\operatorname{rot}\mathbf{E} + \frac{1}{2}\operatorname{grad}\mathbf{E}\cdot\mathbf{E}, \quad \mathbf{E} = \operatorname{grad}V, \quad \operatorname{rot}\mathbf{E} = 0,$$

where V denotes the electric potential, we have

$$\rho \mathbf{K} = -\frac{kg}{4\pi} \mathbf{E} \operatorname{div} \mathbf{E}. \tag{2.4.7}$$

One can immediately determine the principal axes and the principal stresses. Due to the definition of the principal axes

$$\frac{kg}{4\pi} \left(\mathbf{EE} - \frac{1}{2} \hat{E} \mathbf{E} \cdot \mathbf{E} \right) \cdot \mathbf{e} = t\mathbf{e},$$

and one can see that this equation is satisfied if one takes

$$\mathbf{e} = \overset{1}{\mathbf{e}} = \frac{\mathbf{E}}{|\mathbf{E}|}, \quad t = \frac{kg}{8\pi} \mathbf{E} \cdot \mathbf{E} = t_1. \tag{2.4.8}$$

The remaining solutions are obtained by assuming arbitrary directions of the unit vector $\mathbf{e}$ in the plane orthogonal to $\mathbf{E}$

$$\overset{2}{\mathbf{e}} \cdot \mathbf{E} = 0, \overset{3}{\mathbf{e}} \cdot \mathbf{E} = 0, t_2 = t_3 = -\frac{kg}{8\pi} \mathbf{E} \cdot \mathbf{E}. \tag{2.4.9}$$

A tensile stress is seen to act in the direction of the field whilst the compressive stresses of the same magnitude act in the transverse directions.

Fifth example. We consider a vessel under uniform pressure. The state of stress described by the spherical tensor

$$\hat{T} = -p\hat{E}, \tag{2.4.10}$$

is possible in a vessel subjected to an equal pressure inside and outside the vessel. Indeed, for the stress tensor (2.4.10) and absent volume forces the equilibrium equation (1.5.4) is satisfied in the volume whilst the condition

$$\mathbf{N} \cdot \hat{T} = -p\mathbf{N}$$

holds on any surface. This state is realised in a linearly elastic body.

1.3 Material coordinates

Study of this section requires familiarity with the contents of Sections 2.3-2.5.

1.3.1 Representation of the stress tensor

In Sections 1 and 2 of the present chapter the stress tensor $\hat{T}$ was prescribed by its components in the deformed medium (in volume V). In what follows these components are denoted as $t_{(sk)}$ in a Cartesian coordinate system $OX_1X_2X_3$. Proceeding to material coordinates q^s and a vector basis $\mathbf{R}_s$ we introduce the dyadic representations of the stress tensor

$$\hat{T} = \tilde{t}^{sk}\mathbf{R}_s\mathbf{R}_k = \tilde{t}_k^s\mathbf{R}_s\mathbf{R}^k \tag{3.1.1}$$

in terms of its contravariant $\tilde{t}^{sk}$ or mixed $\tilde{t}_k^s$ components (the covariant components of the stress tensor are normally not used). At the same time

$$\hat{T} = t_{(mn)}\mathbf{i}_m\mathbf{i}_n.$$

Comparing these expressions and referring to eqs. (3.1.6) and (3.1.8) of Chapter 2 we obtain the following relationships

$$t_{(mn)} = \tilde{t}^{rt}\frac{\partial x_m}{\partial q^r}\frac{\partial x_n}{\partial q^t} = \tilde{t}_t^r\frac{\partial x_m}{\partial q^r}\frac{\partial q^t}{\partial x_n}, \tag{3.1.2}$$

$$\tilde{t}^{rt} = t_{(mn)}\frac{\partial q^r}{\partial x_m}\frac{\partial q^t}{\partial x_n}, \quad \tilde{t}_t^r = t_{(mn)}\frac{\partial q^r}{\partial x_m}\frac{\partial x_n}{\partial q^t}. \tag{3.1.3}$$

1.3.2 Cauchy's dependences

The initial definition of the stress tensor (1.3.2) is now set in the form

$$\overset{(N)}{\mathbf{t}} = \mathbf{N}\cdot\hat{T} = \tilde{t}^{qs}\mathbf{R}_s\tilde{N}_q, \quad \tilde{N}_q = \mathbf{N}\cdot\mathbf{R}_q. \tag{3.2.1}$$

This yields

$$\overset{(N)}{\mathbf{t}}\cdot\mathbf{R}^k = \overset{(N)}{t^k} = \tilde{t}^{qk}\tilde{N}_q, \quad \overset{(N)}{\mathbf{t}}\cdot\mathbf{R}_k = \overset{(N)}{t_k} = \tilde{t}_k^q\tilde{N}_q. \tag{3.2.2}$$

These are Cauchy's dependences (1.4.5) expressing the contravariant and covariant components of stress vector $\overset{(N)}{\mathbf{t}}$ on the surface with the normal $\mathbf{N}$ in terms of the components (contravariant and mixed) of the stress tensor in the basis of volume V. By virtue of eq. (3.5.2) of Chapter 2, the force acting on this surface is presented by the expression

$$\overset{N}{\mathbf{t}}\,dO = \sqrt{\frac{G}{g}}\,\tilde{t}^{qs}\mathbf{R}_s\mathbf{r}_q\cdot\mathbf{n}do. \tag{3.2.3}$$

1.3.3 The necessary condition for equilibrium

An invariant notion of the equations of statics was presented in Subsection 1.1.5 by two relationships

$$\widetilde{\operatorname{div}}\hat{T} + \rho\mathbf{K} = 0, \quad \hat{T} = \hat{T}^*. \tag{3.3.1}$$

As suggested in Subsection 2.3.1, the tilde sign denotes that the operation of divergence is performed in the basis of volume V. By eq. (E.4.7) in this basis we have

$$\frac{1}{\sqrt{G}}\frac{\partial}{\partial q^s}\sqrt{G}\,\tilde{t}^{st}\mathbf{R}_t + \rho\mathbf{K} = 0. \tag{3.3.2}$$

The law of mass conservation and eq. (5.5.1) of Chapter 2 yield

$$\rho d\tau = \rho_0 d\tau_0, \quad \rho\sqrt{G} = \rho_0\sqrt{g},$$

hence, the equation of statics in volume V can be taken in another form

$$\frac{\partial}{\partial q^s}\sqrt{G}\;\tilde{t}^{st}\mathbf{R}_t + \rho_0\sqrt{g}\mathbf{K} = 0. \tag{3.3.3}$$

Referring to the rule of differentiation of the base vectors (E.2.2) we can represent the vector on the left hand side of this equation in terms of the contravariant components as follows

$$\frac{\partial}{\partial q^s}\sqrt{G}\,\tilde{t}^{sq} + \widetilde{\left\{ \begin{matrix} q \\ ts \end{matrix} \right\}}\sqrt{G}\,\tilde{t}^{st} + \rho_0\sqrt{g}\,\tilde{K}^q = 0. \tag{3.3.4}$$

The condition of symmetry of tensor $\hat{T}$ in terms of its contravariant and mixed components is written down in the standard form

$$\tilde{t}^{sk} = \tilde{t}^{ks}, \;\; \tilde{t}^{s}_{\cdot k} = \tilde{t}_k^{\cdot s} = \tilde{t}^{s}_{k\cdot}\,. \tag{3.3.5}$$

This follows from relationships (D.5.5) and, of course, is confirmed by the transformation formulae (3.1.3).

The equilibrium equations (3.3.3) can be easily derived by the following illustrative representation. Consider an elementary body (parallelepiped) bounded by surfaces q^s and $q^s + dq^s$ $(s = 1, 2, 3)$. Due to eqs. (3.2.3) and (3.5.3) of Chapter 2, the forces acting on these surfaces are presented by the expressions

$$\begin{aligned}
&-\sqrt{G}\,\tilde{t}^{1t}\mathbf{R}_t dq^2 dq^3, \quad \sqrt{G}\,\tilde{t}^{1t}\mathbf{R}_t dq^2 dq^3 + \frac{\partial}{\partial q^1}\sqrt{G}\,\tilde{t}^{1t}\mathbf{R}_t dq^1 dq^2 dq^3,\\
&-\sqrt{G}\,\tilde{t}^{2t}\mathbf{R}_t dq^3 dq^1, \quad \sqrt{G}\,\tilde{t}^{2t}\mathbf{R}_t dq^3 dq^1 + \frac{\partial}{\partial q^2}\sqrt{G}\,\tilde{t}^{2t}\mathbf{R}_t dq^2 dq^3 dq^1,\\
&-\sqrt{G}\,\tilde{t}^{3t}\mathbf{R}_t dq^1 dq^2, \quad \sqrt{G}\,\tilde{t}^{3t}\mathbf{R}_t dq^3 dq^1 + \frac{\partial}{\partial q^3}\sqrt{G}\,\tilde{t}^{3t}\mathbf{R}_t dq^2 dq^3 dq^1
\end{aligned}$$

since, for example, on the surface given by vectors $\mathbf{R}_2 dq^2$ and $\mathbf{R}_3 dq^3$ we have

$$\overset{1}{\mathbf{N}}\, d\overset{1}{O} = \mathbf{R}_2 \times \mathbf{R}_3 dq^2 dq^3 = \sqrt{G}\mathbf{R}^1 dq^2 dq^3, \;\; \overset{1}{\mathbf{N}}\, d\overset{1}{O}\cdot\hat{T} = \sqrt{G}\,\tilde{t}^{1t}\mathbf{R}_t dq^2 dq^3.$$

The mass force acting on the considered volume is equal to

$$\mathbf{K}dm = \mathbf{K}\rho d\tau = \mathbf{K}\rho_0 d\tau_0 = \mathbf{K}\rho_0\sqrt{g}dq^1dq^2dq^3.$$

Equation (3.3.3) indicates that the principal vector of the mentioned forces vanishes. The equilibrium equation on surface O bounding volume V suggests that vector $\overset{N}{t}$ determined via the stress tensor by eq. (3.2.1) is equal to the vector of the external surface force $\mathbf{F}$. Referring to eqs. (3.5.5) and (3.2.2) of Chapter 2 we have

$$\mathbf{F} = \mathbf{N}\cdot\hat{T} = \frac{1}{\sqrt{\mathbf{n}\cdot\hat{G}^{\times -1}\cdot\mathbf{n}}}\mathbf{R}^m n_m \cdot \tilde{t}^{st}\mathbf{R}_s\mathbf{R}_t,$$

or

$$\mathbf{F}\sqrt{G^{sk}n_sn_k} = \tilde{t}^{mt}n_m\mathbf{R}_t. \tag{3.3.6}$$

The latter equation written in terms of the contravariant components of the surface force has the form

$$\tilde{t}^{mt}n_m = \tilde{F}^t\sqrt{G^{sk}n_sn_k}, \tag{3.3.7}$$

which can also be set as follows

$$\tilde{F}^k = \tilde{t}^{qk}\tilde{N}_q,\ \tilde{F}^k = \tilde{t}^q_k\tilde{N}_q. \tag{3.3.8}$$

Notion (3.3.7) has an advantage in that it utilises the normal $\mathbf{n}$ to surface o of volume v which is given *a priori*, whilst surface O of volume V needs to be sought.

1.3.4 Another definition of the stress tensor

Trefftz, Hamel, Kappus and other authors propose that the stress tensor ${}_0\hat{T}$ is a tensor which is related to tensor $\hat{T}$ introduced earlier by the following formula

$${}_0\hat{T} = \sqrt{\frac{G}{g}}\hat{T} = \sqrt{\frac{G}{g}}\tilde{t}^{sq}\mathbf{R}_s\mathbf{R}_q = {}_0\tilde{t}^{sq}\mathbf{R}_s\mathbf{R}_q. \tag{3.4.1}$$

Then, by virtue of eq. (3.2.3),

$$\overset{(N)}{\mathbf{t}}\frac{dO}{do} = {}_0\tilde{t}^{sq}\mathbf{R}_sn_q = {}_0\overset{(N)}{\mathbf{t}}, \tag{3.4.2}$$

where ${}_0\overset{(N)}{\mathbf{t}}$ denotes the stress vector on the oriented surface $\mathbf{N}dO$ in volume V. However, this stress vector is related to a unit area of this surface in its initial state, i.e. in volume v.

The equilibrium equations in the volume and on the surface take the form

$$\frac{\partial}{\partial q^s}\sqrt{g}_0\tilde{t}^{st}\mathbf{R}_t + \rho_0\sqrt{g}\mathbf{K} = 0, \tag{3.4.3}$$

$$\tilde{P}^s = {}_0\tilde{t}^{sq}n_q. \tag{3.4.4}$$

Here $\mathbf{P}$ denotes a surface force per unit area of surface o in volume v

$$\mathbf{P}do = \mathbf{F}dO. \tag{3.4.5}$$

1.3.5 Elementary work of external forces

We consider equilibrium of a continuum in volume V bounded by surface O, the continuum being subjected to mass $\mathbf{K}$ and surface $\mathbf{F}$ forces. According to the principle of virtual work, the elementary work done by all external and internal forces due to virtual displacement of the continuum particles from their equilibrium position is equal to zero

$$\delta' a_{(e)} + \delta' a_{(i)} = 0. \tag{3.5.1}$$

The field of virtual displacement is prescribed by the vector

$$\delta\mathbf{u} = \delta(\mathbf{R} - \mathbf{r}) = \delta\mathbf{R}, \tag{3.5.2}$$

inasmuch as vector $\mathbf{r}$ identifying the considered particle in volume v remains unchanged ($\delta\mathbf{r} = 0$) under any virtual displacement from the equilibrium position in volume V.

The elementary work of the external forces is expressed in the form

$$\delta' a_{(e)} = \iiint\limits_V \rho\mathbf{K}\cdot\delta\mathbf{u}d\tau + \iint\limits_O \mathbf{F}\cdot\delta\mathbf{u}dO. \tag{3.5.3}$$

Replacing the surface force in the second integral by means of eq. (1.5.13), transforming the surface integral into a volume one due to eq. (B.5.5) and using eqs. (B.3.10), (B.3.10) and (1.5.5) yields

$$\begin{aligned}\iint\limits_O \mathbf{F}\cdot\delta\mathbf{u}dO &= \iint\limits_O \mathbf{N}\cdot\hat{T}\cdot\delta\mathbf{u}dO = \iiint\limits_V \tilde{\nabla}\cdot\hat{T}\cdot\delta\mathbf{u}d\tau \\ &= \iiint\limits_V \left[\left(\tilde{\nabla}\cdot\hat{T}\right)\cdot\delta\mathbf{u} + \hat{T}\cdot\cdot\tilde{\nabla}\delta\mathbf{u}\right]d\tau.\end{aligned} \tag{3.5.4}$$

Returning to eqs. (3.5.3) and (3.5.2) we obtain

$$\delta' a_{(e)} = \iiint\limits_V \left(\rho\mathbf{K} + \widetilde{\operatorname{div}}\hat{T}\right)\cdot\delta\mathbf{u}d\tau + \iiint\limits_V \hat{T}\cdot\cdot\tilde{\nabla}\delta\mathbf{R}d\tau.$$

In accordance with definition (D.4.3) of the nabla-operator in the metric of volume V

$$\tilde{\nabla}\delta\mathbf{R} = \mathbf{R}^s \frac{\partial}{\partial q^s}\delta\mathbf{R} = \mathbf{R}^s \delta\frac{\partial \mathbf{R}}{\partial q^s} = \mathbf{R}^s\delta\mathbf{R}_s,$$

and taking into account the symmetry of tensor $\hat{T}$ we have

$$\begin{aligned}\hat{T}\cdot\cdot\,\mathbf{R}^s\delta\mathbf{R}_s &= \tilde{t}^{qt}\mathbf{R}_q\mathbf{R}_t\cdot\cdot\,\mathbf{R}^s\delta\mathbf{R}_s = \tilde{t}^{qs}\mathbf{R}_q\cdot\delta\mathbf{R}_s \\ &= \frac{1}{2}\left(\tilde{t}^{qs}\mathbf{R}_q\cdot\delta\mathbf{R}_s + \tilde{t}^{sq}\mathbf{R}_s\cdot\delta\mathbf{R}_q\right) = \frac{1}{2}\tilde{t}^{qs}\delta\mathbf{R}_s\cdot\mathbf{R}_q.\end{aligned}$$

Now referring to eqs. (3.3.1) and (3.3.3) of Chapter 2 we arrive at the relationship

$$\delta' a_{(e)} = \frac{1}{2}\iiint\limits_V \tilde{t}^{sq}\delta G_{sq}d\tau = \frac{1}{2}\iiint\limits_v \sqrt{\frac{G}{g}}\tilde{t}^{sq}\delta G_{sq}d\tau_0. \tag{3.5.5}$$

By eqs. (3.6.3) of Chapter 2

$$\frac{1}{2}G_{sq} = \frac{1}{2}g_{sq} + \mathcal{E}_{sq},\; \delta\frac{1}{2}G_{sq} = \delta\mathcal{E}_{sq},$$

since tensor $\hat{g}$ remains unchanged under variation in volume V. This allows us to reset eqs. (3.5.5) and (3.5.1) in the form

$$\delta' a_{(e)} = -\delta' a_{(i)} = \iiint\limits_v \sqrt{\frac{G}{g}}\tilde{t}^{sq}\delta\mathcal{E}_{sq}d\tau_0. \tag{3.5.6}$$

We next consider the specific elementary work of external $\left(\delta' A_{(e)}\right)$ and internal $\left(\delta' A_{(i)}\right)$ forces

$$\delta' a_{(e)} = \iiint\limits_v \delta' A_{(e)}d\tau_0, \quad \delta' a_{(i)} = \iiint\limits_v \delta' A_{(i)}d\tau_0, \tag{3.5.7}$$

that is, the elementary work is related to a unit volume of the continuum in its initial state (volume v). Then, due to eq. (3.5.6) and the arbitrariness of volume V, we have

$$\delta' A_{(e)} = -\delta' A_{(i)} = \frac{1}{2}\sqrt{\frac{G}{g}}\tilde{t}^{sq}\delta G_{sq} = \sqrt{\frac{G}{g}}\tilde{t}^{sq}\delta\mathcal{E}_{sq}. \tag{3.5.8}$$

This expression can be derived in a simpler way by calculating the elementary work done by forces (listed in Subsection 1.3.3) acting in volume V on the elementary body bounded by surfaces q^s and $q^s + dq^s$. It is necessary to bear in mind that the virtual displacements of particles on the opposite faces differ in the following vector

$$\frac{\partial \delta \mathbf{u}}{\partial q^s} dq^s = \frac{\partial \delta \mathbf{R}}{\partial q^s} dq^s = \delta \mathbf{R}_s dq^s.$$

In particular, if $\hat{T}$ is a spherical tensor describing a hydrostatic state of stress (a uniform compression) of intensity $-p$ we have

$$\hat{T} = -p\hat{G}, \ \tilde{t}^{sq} = -pG^{sq} \tag{3.5.9}$$

and referring to eq. (A.7.9) we obtain

$$\delta' A_{(i)} = -\frac{1}{2} p \sqrt{\frac{G}{g}} G^{sq} \delta G_{sq} = -\frac{1}{2} p \frac{1}{\sqrt{gG}} \frac{\partial G}{\partial G_{sq}} \delta G_{sq} = -p \frac{\partial}{\partial G_{sq}} \sqrt{\frac{G}{g}} \delta G_{sq},$$

or

$$\delta' A_{(i)} = -p\delta \sqrt{\frac{G}{g}} = -\delta D. \tag{3.5.10}$$

Here D denotes the relative volume change, see eq. (5.5.1) of Chapter 2.

Remark 1. Virtual displacements in volume V and on surface O are those arbitrary infinitesimally small displacements which do not destroy the continuity of the continuum in V and are compatible with the imposed constraints on O, respectively. For this reason, on the part O_1 of the surface with the prescribed displacement we have

$$\delta \mathbf{u} = 0. \tag{3.5.11}$$

The surface forces $\mathbf{F}$ on O_1 are not known in advance. These are the reaction forces of those constraints which ensure prescribed displacements $\mathbf{u}$ at points on O_1. For instance, this is the reaction force of an immovable support ensuring $\mathbf{u} = 0$. However this does not prevent setting the elementary work of the surface forces from being expressed in the form

$$\iint_O \mathbf{F} \cdot \delta \mathbf{u} dO = \iint_O \mathbf{N} \cdot \hat{T} \cdot \delta \mathbf{u} dO,$$

as condition (3.5.11) holds on O_2.

Remark 2. A sign δ' in the expressions for the elementary work and specific elementary work is used to denote infinitesimally small quantities. A prime indicates that these quantities are not, in general, variations of some functions. There does not exist such a quantity $A_{(e)}$ (or $A_{(i)}$) whose variation is equal to the specific work of external (or internal) forces.

1.3.6 The energetic stress tensor

The strain measure is determined in the basis of volume v by eqs. (3.3.2) and (3.3.3) of Chapter 2

$$\hat{G}^{\times} = G_{sq}\mathbf{r}^s\mathbf{r}^q,$$

so that

$$\delta\hat{G}^{\times} = \mathbf{r}^s\mathbf{r}^q\delta G_{sq}. \tag{3.6.1}$$

The stress tensor $\hat{T}$ is presented in terms of the contravariant components in the basis of volume V by eq. (3.1.1). For this reason

$$\delta' A_{(e)} \neq \frac{1}{2}\sqrt{\frac{G}{g}}\hat{T}\cdot\cdot\,\delta\hat{G}^{\times}, \tag{3.6.2}$$

i.e. the specific elementary work of the external forces is not equal to the contraction (the first invariant of product) of stress tensor $\hat{T}$ and a variation of the first strain measure $\delta\hat{G}^{\times}$.

For this reason let us introduce the following tensor

$$\hat{Q} = t^{st}\mathbf{r}_s\mathbf{r}_t = \tilde{t}^{st}\mathbf{r}_s\mathbf{r}_t, \tag{3.6.3}$$

whose contravariant components in the basis of volume v are equal to the contravariant components $\tilde{t}^{st}$ of the stress tensor in the basis of volume V. Then

$$\hat{Q}\cdot\cdot\,\delta\hat{G}^{\times} = \tilde{t}^{sq}\mathbf{r}_s\mathbf{r}_q\cdot\cdot\,\mathbf{r}^m\mathbf{r}^n\delta G_{mn} = \tilde{t}^{sq}\delta G_{sq}$$

and

$$\delta' A_{(e)} = \frac{1}{2}\sqrt{\frac{G}{g}}\hat{Q}\cdot\cdot\,\delta\hat{G}^{\times} = \sqrt{\frac{G}{g}}\hat{Q}\cdot\cdot\,\delta\hat{\mathcal{E}}. \tag{3.6.4}$$

Here the specific elementary work is presented by the contraction of tensor $\hat{Q}$ and a variation of the first strain measure. For this reason tensor $\hat{Q}$ is referred to here as the energetic stress tensor.

The relation between tensors $\hat{T}$ and $\hat{Q}$ can also be represented in an invariant form. Referring to formula (3.2.3) of Chapter 2 we have

$$(\nabla\mathbf{R})^* = \mathbf{R}_s\mathbf{r}^s, \;\; \mathbf{R}_s = (\nabla\mathbf{R})^*\cdot\mathbf{r}_s = \mathbf{r}_s\cdot\nabla\mathbf{R}$$

and thus

$$\hat{T} = \mathbf{R}_s\tilde{t}^{sq}\mathbf{R}_q = (\nabla\mathbf{R})^*\cdot\mathbf{r}_s\tilde{t}^{sq}\mathbf{r}_q\cdot\nabla\mathbf{R}.$$

Using eq. (3.6.3) as well as eq. (3.2.3) of Chapter 2 yields

$$\hat{T} = (\nabla \mathbf{R})^* \cdot \hat{Q} \cdot \nabla \mathbf{R}, \; \hat{Q} = \left(\tilde{\nabla} \mathbf{r}\right)^* \cdot \hat{T} \cdot \tilde{\nabla} \mathbf{r}. \tag{3.6.5}$$

Let us notice that the equations of statics in the volume (3.3.3) and on the surface (3.3.8) contain components $\tilde{t}^{sq}$ of tensor $\hat{T}$ which are the components of tensor $\hat{Q}$ (but in another metric). Of course it would be an error to write these equations down in the following form

$$\widetilde{\operatorname{div}} \hat{Q} + \rho\sqrt{G}\mathbf{K} = 0, \;\; \mathbf{F} = \mathbf{N} \cdot \hat{Q}.$$

Let us present tensors $\hat{Q}$ and $\delta\hat{\mathcal{E}}$ by their spherical and deviatoric parts

$$\hat{Q} = \frac{1}{3}\hat{g} I_1\left(\hat{Q}\right) + \operatorname{Dev}\hat{Q},$$
$$\delta\hat{\mathcal{E}} = \frac{1}{3}\hat{g} I_1\left(\delta\hat{\mathcal{E}}\right) + \operatorname{Dev}\delta\hat{\mathcal{E}} = \frac{1}{3}\hat{g}\,\delta I_1\left(\hat{\mathcal{E}}\right) + \delta\operatorname{Dev}\hat{\mathcal{E}}.$$

Then, by virtue of eq. (3.6.4), we arrive at the expression for the specific elementary work

$$\delta' A_{(e)} = \sqrt{\frac{G}{g}}\left(\frac{1}{3}\hat{g} I_1\left(\hat{Q}\right) + \operatorname{Dev}\hat{Q}\right) \cdot\cdot \left(\frac{1}{3}\hat{g}\,\delta I_1\left(\hat{\mathcal{E}}\right) + \delta\operatorname{Dev}\hat{\mathcal{E}}\right).$$

Denoting the unit tensor in v–metric by $\hat{g}$ and keeping in mind the following relationships

$$\hat{g}\cdot\cdot\hat{g} = I_1\left(\hat{g}\cdot\hat{g}\right) = I_1\left(\hat{g}\right) = 3,$$
$$\hat{g}\cdot\cdot\,\delta\operatorname{Dev}\hat{\mathcal{E}} = I_1\left(\hat{g}\cdot\delta\operatorname{Dev}\hat{\mathcal{E}}\right) = \delta I_1\left(\operatorname{Dev}\hat{\mathcal{E}}\right) = 0,$$
$$\operatorname{Dev}\hat{Q}\cdot\cdot\hat{g} = 0,$$
$$\operatorname{Dev}\hat{Q}\cdot\cdot\,\delta\operatorname{Dev}\hat{\mathcal{E}} = I_1\left(\operatorname{Dev}\hat{Q}\cdot\delta\operatorname{Dev}\hat{\mathcal{E}}\right),$$

we obtain the following representation of the specific elementary work as a sum of two terms

$$\delta A_{(e)} = \sqrt{\frac{G}{g}}\left[\frac{1}{2} I_1\left(\hat{Q}\right)\delta I_1\left(\hat{\mathcal{E}}\right) + I_1\left(\operatorname{Dev}\hat{Q}\cdot\delta\operatorname{Dev}\hat{\mathcal{E}}\right)\right]. \tag{3.6.6}$$

In the linear theory of elasticity, the first and second terms are called the elementary work of change in volume and form, respectively. Such an interpretation does not take place in nonlinear theory.

1.3.7 Invariants of the stress tensor

Referring to eqs. (D.7.5) and (D.7.6) we have in volume V

$$I_1\left(\hat{T}\right) = G_{st}\tilde{t}^{st}, \quad I_3\left(\hat{T}\right) = G\left|\tilde{t}^{st}\right|. \tag{3.7.1}$$

Next, using first mixed and then contravariant components of $\hat{T}$ in formula (D.7.11) we obtain

$$\begin{aligned} I_2\left(\hat{T}\right) &= \frac{1}{2}\left(\tilde{t}_s^s\tilde{t}_r^r - \tilde{t}_r^s\tilde{t}_s^r\right) = \frac{1}{2}G_{sq}G_{rt}\left(\tilde{t}^{sq}\tilde{t}^{rt} - \tilde{t}^{rq}\tilde{t}^{st}\right) \\ &= \frac{1}{2}\tilde{t}^{sq}\tilde{t}^{rt}\left(G_{sq}G_{rt} - G_{rq}G_{st}\right). \end{aligned} \tag{3.7.2}$$

Also,

$$\begin{aligned} G_{sq}G_{rt} - G_{rq}G_{st} &= \left(\mathbf{R}_s \cdot \mathbf{R}_q\mathbf{R}_r - \mathbf{R}_r \cdot \mathbf{R}_q\mathbf{R}_s\right) \cdot \mathbf{R}_t \\ &= \left[\left(\mathbf{R}_s \times \mathbf{R}_r\right) \times \mathbf{R}_q\right] \cdot \mathbf{R}_t = \left(\mathbf{R}_s \times \mathbf{R}_r\right) \cdot \left(\mathbf{R}_q \times \mathbf{R}_t\right) \\ &= \in_{srm}\in_{qtn}\mathbf{R}^m \cdot \mathbf{R}^n = G^{mn}Ge_{srm}e_{qtn} \end{aligned}$$

which yields another representation for $I_2\left(\hat{T}\right)$

$$I_2\left(\hat{T}\right) = \frac{1}{2}GG^{mn}e_{srm}e_{qtn}\tilde{t}^{sq}\tilde{t}^{rt}. \tag{3.7.3}$$

1.4 Integral estimates for the state of stress

The content of this section is not related to the chapters that follow and can be omitted without sacrifice of understanding the following text.

1.4.1 Moments of a function

Let us agree to refer to the integrals

$$\iiint_V f\left(x_1, x_2, x_3\right) x_1^{s_1}x_2^{s_2}x_3^{s_3}\,d\tau, \quad d\tau = dx_1dx_2dx_3, \tag{4.1.1}$$

where

$$s_1 + s_2 + s_3 = n \tag{4.1.2}$$

as the $n-th$ order moments of a function $f\left(x_1, x_2, x_3\right)$ prescribed in volume V. For $s_1 = 0$ we have $n+1$ numbers $\left(s_2, s_3\right)$ whose sum is n, for $s_1 = 1$ we have n numbers $\left(s_2, s_3\right)$ whose sum is $n-1$ and so on. Hence, the total number of N moments of order n equals

$$N = (n+1) + n + \cdots + 1 = \frac{1}{2}(n+1)(n+2). \tag{4.1.3}$$

1.4.2 Moments of components of the stress tensor

The equilibrium equations in the volume, eq. (1.5.6), enable us to write the following $3N$ relationships

$$\iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} \left(\frac{\partial t_{1t}}{\partial x_1} + \frac{\partial t_{2t}}{\partial x_2} + \frac{\partial t_{3t}}{\partial x_3} \right) d\tau + \iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} \rho K_t d\tau = 0.$$

The first term is transformed by means of the Gauss-Ostrogradsky formula

$$\iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} \frac{\partial t_{kt}}{\partial x_k} d\tau = \iiint\limits_V \left[\frac{\partial \left(x_1^{s_1} x_2^{s_2} x_3^{s_3} t_{kt} \right)}{\partial x_k} - x_1^{s_1} x_2^{s_2} x_3^{s_3} \sum_{k=1}^{3} \frac{s_k}{x_k} t_{kt} \right] d\tau$$
$$= \iint\limits_O x_1^{s_1} x_2^{s_2} x_3^{s_3} F_t dO - \iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} \sum_{k=1}^{3} \frac{s_k}{x_k} t_{kt} d\tau,$$

where the equilibrium equation on the surface (1.5.15)

$$t_{kt} N_k = F_t$$

is used. Introducing the notion

$$\overset{t}{q}_{s_1 s_2 s_3} = \frac{1}{V} \left[\iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} \rho K_t d\tau + \iint\limits_O x_1^{s_1} x_2^{s_2} x_3^{s_3} F_t dO \right], \qquad (4.2.1)$$

$$\frac{1}{V} \iiint\limits_V x_1^{s_1} x_2^{s_2} x_3^{s_3} t_{kt} d\tau = \left(x_1^{s_1} x_2^{s_2} x_3^{s_3} t_{kt} \right)_m, \qquad (4.2.2)$$

we arrive at the following relationships

$$s_1 \left(x_1^{s_1 - 1} x_2^{s_2} x_3^{s_3} t_{1t} \right)_m + s_2 \left(x_1^{s_1} x_2^{s_2 - 1} x_3^{s_3} t_{2t} \right)_m +$$
$$s_3 \left(x_1^{s_1} x_2^{s_2} x_3^{s_3 - 1} t_{3t} \right)_m = \overset{t}{q}_{s_1 s_2 s_3}. \qquad (4.2.3)$$

Given the volume and surface forces on the entire surface O bounding volume V, the right hand sides of these equations are known. The total number of equations in (4.2.3) is equal to $\frac{3}{2}(n+1)(n+2)$ whereas the number of unknown quantities is equal to the number of moments of order $(n-1)$ for six functions t_{kt}, i.e. $3n(n+1)$.

1.4.3 The cases of $n = 0$ and $n = 1$

In this case $n = 1, s_1 = s_2 = s_3 = 0$ and the three following equations

$$\overset{t}{q}_{000} = \frac{1}{V} \left[\iiint\limits_V \rho K_t d\tau + \iint\limits_O F_t dO \right] = 0 \qquad (4.3.1)$$

express the condition of vanishing principal vector.

The case of $n = 1$ leads to nine equations in terms of six unknown values. From these equations one obtains the mean values of the six components of the stress tensor

$$(t_{1t})_m = \overset{t}{q}_{100}, \quad (t_{2t})_m = \overset{t}{q}_{010}, \quad (t_{3t})_m = \overset{t}{q}_{001}, \tag{4.3.2}$$

the condition of symmetry of the stress tensor

$$(t_{12})_m - (t_{21})_m = \overset{2}{q}_{100} - \overset{1}{q}_{010} = 0, \quad \overset{3}{q}_{010} - \overset{2}{q}_{001}, \quad \overset{1}{q}_{001} - \overset{3}{q}_{100} = 0 \tag{4.3.3}$$

expressing the requirement that three components of the principal moment of external forces are zero.

1.4.4 The first order moments for stresses

For $n = 2$ we arrive at 18 equations with the same number of unknown values

$$(t_{kt}x_r)_m = (t_{tk}x_r)_m \quad (r = 1, 2, 3). \tag{4.4.1}$$

The equations are split into two groups

$$2\,(t_{1t}x_1)_m = \overset{t}{q}_{200}, \quad 2\,(t_{2t}x_2)_m = \overset{t}{q}_{020}, \quad 2\,(t_{3t}x_3)_m = \overset{t}{q}_{002}, \tag{4.4.2}$$

$$\left.\begin{aligned} (t_{1t}x_2)_m + (t_{2t}x_1) &= \overset{t}{q}_{110}, \\ (t_{2t}x_3)_m + (t_{3t}x_2) &= \overset{t}{q}_{011}, \\ (t_{3t}x_1)_m + (t_{1t}x_3) &= \overset{t}{q}_{101}\,. \end{aligned}\right\} \tag{4.4.3}$$

They yield all the first order moments for six components of the stress tensor. For example, the first order moments of stresses t_{11}, t_{12} (divided by the volume) are

$$(t_{11}x_1)_m = \frac{1}{2}\overset{1}{q}_{200}, \quad (t_{11}x_2)_m = \overset{1}{q}_{110} - \frac{1}{2}\overset{2}{q}_{200}, \quad (t_{11}x_3)_m = \overset{1}{q}_{101} - \frac{1}{2}\overset{2}{q}_{200}, \tag{4.4.4}$$

$$(t_{12}x_1)_m = \frac{1}{2}\overset{2}{q}_{200}, \quad (t_{12}x_2)_m = \overset{1}{q}_{020}, \quad (t_{12}x_3)_m = \frac{1}{2}\left(\overset{2}{q}_{101} + \overset{1}{q}_{011} - \overset{3}{q}_{110}\right). \tag{4.4.5}$$

1.4.5 An example. A vessel under external and internal pressure

Let us denote the body volume and the volume of the internal space by V_e and V_i, respectively. The surfaces bounding these volumes are respectively denoted by O_e and O_i. The surface forces are an external pressure p_e uniformly distributed over O_e and internal pressure p_i uniformly distributed over O_i, that is

$$\mathbf{F} = \begin{cases} -p_e \mathbf{N}_e & \text{on } O_e \,, \\ -p_i \mathbf{N}_i & \text{on } O_i \,. \end{cases} \tag{4.5.1}$$

Here $\mathbf{N}_e$ and $\mathbf{N}_i$ designate the unit vectors of the outward normals to surfaces O_e and O_i, respectively. Clearly, vector $\mathbf{N}_i$ is directed into the internal space V_i. The origin of the coordinate system is taken at the centre of gravity of the body volume, so that

$$V_e x_s^e - V_i x_s^i = 0, \quad s = 1, 2, 3, \tag{4.5.2}$$

where x_s^e and x_s^i are respectively the coordinates of the centre of gravity of volumes V_e and V_i.

Neglecting mass forces and accounting for eq. (4.2.1) yields

$$\begin{aligned} \overset{t}{q}_{s_1 s_2 s_3} &= -\frac{1}{V}\left(p_e \iint\limits_{O_e} x_1^{s_1} x_2^{s_2} x_3^{s_3} N_{et} dO + p_i \iint\limits_{O_i} x_1^{s_1} x_2^{s_2} x_3^{s_3} N_{it} dO \right) \\ &= -\frac{1}{V}\left[p_e \iiint\limits_{V_e} \frac{\partial}{\partial x_t}\left(x_1^{s_1} x_2^{s_2} x_3^{s_3}\right) d\tau - p_i \iiint\limits_{V_i} \frac{\partial}{\partial x_t}\left(x_1^{s_1} x_2^{s_2} x_3^{s_3}\right) d\tau \right], \end{aligned}$$

where $V = V_e - V_i$. Of course, the equilibrium conditions (4.3.1) and (4.3.3) are satisfied. Taking into account eqs. (4.3.2), (4.4.2), (4.4.3) and (4.5.2), we obtain the mean values of the normal stresses and their first moments

$$(t_{qq})_m = -\frac{1}{V}\left(p_e V_e - p_i V_i\right)\left(\not{\Sigma}_q\right), \tag{4.5.3}$$

$$(t_{qq} x_s)_m = -\frac{1}{V}\left(p_e V_e x_s^e - p_i V_i x_s^i\right) = -\frac{V_e}{V}\left(p_e - p_i\right) x_s^e = -\frac{V_i}{V}\left(p_e - p_i\right) x_s^i. \tag{4.5.4}$$

The mean values and the first moments of shear stresses are equal to zero

$$(t_{qk})_m = 0, \quad (t_{qk} x_s)_m = 0 \quad (q \neq k). \tag{4.5.5}$$

1.4.6 *An example. Principal vector and principal moment of stresses in a plane cross-section of the body*

Let body V be loaded by mass and surface forces. We consider a part of the body which is cut by plane $x_3 = x_3^0$. Let us denote the volume of this part as T, it is bounded by surface $O_* + \Omega$, where O_* is the part of surface O of body V and Ω is the surface of the plane section. Taking $x_3 > x_3^0$ in volume T we have, due to eq. (4.2.1), that

$$\overset{r}{q}_{s_1 s_2 s_3} = \frac{1}{T}\left(\iiint\limits_T x_1^{s_1} x_2^{s_2} x_3^{s_3} \rho K_r d\tau + \iint\limits_{O_*} x_1^{s_1} x_2^{s_2} x_3^{s_3} F_r dO\right)$$
$$+\frac{1}{T}\iint\limits_{\Omega} x_1^{s_1} x_2^{s_2} x_3^{0 s_3} t_{rk} N_k dO = \overset{r*}{q}_{s_1 s_2 s_3} - \frac{1}{T}\iint\limits_{\Omega} x_1^{s_1} x_2^{s_2} x_3^{0 s_3} t_{r3} dO, \quad (4.6.1)$$

since $N_k = -\delta_{3k}$ for plane $x_3 = x_3^0$. The first terms in eq. (4.6.1) is determined by the prescribed external forces. The equilibrium equations (4.3.1) and (4.3.3) yield the following expressions for

a) the transverse force

$$\Omega\,(t_{13})_{m_*} = T\,\overset{1*}{q}_{000}, \quad \Omega\,(t_{23})_{m_*} = T\,\overset{2*}{q}_{000}, \quad (4.6.2)$$

b) the tensile force

$$\Omega\,(t_{33})_{m_*} = T\,\overset{3*}{q}_{000}, \quad (4.6.3)$$

c) the torque

$$\Omega\,(x_1 t_{23} - x_2 t_{31})_{m_*} = T\left(\overset{2*}{q}_{100} - \overset{1*}{q}_{010}\right), \quad (4.6.4)$$

d) the bending moments about axes Ox_1 and Ox_2

$$\left.\begin{aligned} \Omega\,(x_2 t_{33})_{m_*} &= T\left(\overset{3*}{q}_{010} - \overset{2*}{q}_{001} + x_3^0\,\overset{2*}{q}_{000}\right), \\ -\Omega\,(x_1 t_{33})_{m_*} &= T\left(\overset{1*}{q}_{001} - \overset{3*}{q}_{100} - x_3^0\,\overset{1*}{q}_{000}\right), \end{aligned}\right\} \quad (4.6.5)$$

Here m_* denotes a mean value over surface Ω

$$\frac{1}{\Omega}\iint\limits_{\Omega} f\left(x_1, x_2, x_3^0\right) dx_1 dx_2 = (f)_{m_*}.$$

1.4.7 An estimate of a mean value for a quadratic form of components of the stress tensor

In order to simplify the notion we introduce a single subscript notation

$$\tau_1 = t_{11}, \quad \tau_2 = t_{22}, \quad \tau_3 = t_{33}, \quad \tau_4 = t_{12}, \quad \tau_5 = t_{23}, \quad \tau_6 = t_{31}. \quad (4.7.1)$$

We introduce a positive 6×6 matrix $\|q_{rs}\|$. Let us note in passing that matrix $\|q_{rs}\|$ is termed positive if the quadratic form $q_{rs}x_s x_r$ is positive definite, i.e. this form vanishes only if all $x_s = 0$. Let us enter the following integral of the positive definite quadratic form over the body volume

$$\psi = \frac{1}{2} \iiint\limits_V q_{rs} \left[\tau_r - \left(a_0^r + \sum_{m=1}^{3} a_m^r x_m\right)\right] \left[\tau_s - \left(a_0^s + \sum_{k=1}^{3} a_k^r x_k\right)\right] d\tau, \quad (4.7.2)$$

the summation signs over r and s being omitted. The system of axes $Ox_1x_2x_3$ is taken to be coincident with the principal central axes of inertia of body V

$$\iiint\limits_V x_k d\tau = 0, \quad \iiint\limits_V x_k x_m d\tau = 0, \quad (k, m = 1, 2, 3;\ k \neq m). \quad (4.7.3)$$

We also introduce the notion

$$\iiint\limits_V x_k^2 d\tau = V j_k^2 \quad (k = 1, 2, 3), \quad (4.7.4)$$

so that

$$\rho_1^2 = j_2^2 + j_3^2, \quad \rho_2^2 = j_3^2 + j_1^2, \quad \rho_3^2 = j_1^2 + j_2^2$$

are the squares of the inertia radii about the central axes Ox_1, Ox_2 and Ox_3, respectively.

Quantity ψ considered as being a function of coefficients a_0^s and a_k^s has a minimum under the following conditions

$$\frac{\partial \psi}{\partial a_0^s} = -\iiint\limits_V q_{rs} \left[\tau_r - \left(a_0^r + \sum_{m=1}^{3} a_m^r x_m\right)\right] d\tau = -V q_{rs} \left[(\tau_r)_m - a_0^r\right],$$

$$\frac{\partial \psi}{\partial a_k^s} = -\iiint\limits_V q_{rs} x_k \left[\tau_r - \left(a_0^r + \sum_{m=1}^{3} a_m^r x_m\right)\right] d\tau$$
$$= -V q_{rs} \left[(\tau_r x_k)_m - a_k^r j_k^2\right].$$

As determinant $|q_{rs}| \neq 0$ it follows from these equations that a stationary value of ψ is a minimum due to the positive definiteness of ψ. This minimum occurs for the following values of a_0^r and a_k^r

$$a_0^r = (\tau_r)_m, \quad a_k^r = \frac{1}{j_k^2}(\tau_r x_k)_m$$

calculated by eqs. (4.3.2), (4.4.2) and (4.4.3). Returning to formulae (4.7.3) one can see that this minimum is as follows

$$\psi = \frac{1}{2}\iiint\limits_V q_{rs}\left\{\tau_r - \left[(\tau_r)_m + \sum_{q=1}^{3}\frac{(\tau_r x_q)_m}{j_q^2}x_q\right]\right\}\times$$
$$\left\{\tau_s - \left[(\tau_s)_m + \sum_{k=1}^{3}\frac{(\tau_s x_k)_m}{j_k^2}x_k\right]\right\}d\tau \tag{4.7.5}$$
$$= \frac{1}{2}\iiint\limits_V q_{rs}\tau_r\tau_s d\tau - \frac{1}{2}Vq_{rs}\left[(\tau_r)_m(\tau_s)_m + \sum_{k=1}^{3}\frac{1}{j_k^2}(\tau_r x_k)_m(\tau_s x_k)_m\right].$$

We arrive at the inequality

$$\frac{1}{V}\iiint\limits_V q_{rs}\tau_r\tau_s d\tau > q_{rs}\left[(\tau_r)_m(\tau_s)_m + \sum_{k=1}^{3}\frac{1}{j_k^2}(\tau_r x_k)_m(\tau_s x_k)_m\right], \tag{4.7.6}$$

which also holds with a sign $\geq$ for a positive semidefinite form $q_{rs}\tau_r\tau_s$, i.e. a form which retains its sign and vanishes not only for zero values of its variables. For example, let all q_{rs} but only a single q_{rr} be equal to zero, then

$$\left(\frac{1}{V}\iiint\limits_V \tau_r^2 d\tau\right)^{1/2} = \left(\tau_r^2\right)_m^{1/2} \geq \left[(\tau_r)_m^2 + \sum_{k=1}^{3}\frac{1}{j_k^2}(\tau_r x_k)_m^2\right]^{1/2}. \tag{4.7.7}$$

In the problem of the vessel, Subsection 1.4.5, the inequality

$$\iiint\limits_V q_{rs}\tau_r\tau_s d\tau \geq \frac{1}{V}\left[(p_eV_e - p_iV_i)^2 + V_e^2(p_e - p_i)^2\sum_{k=1}^{3}\frac{(x_k^e)^2}{j_k^2}\right]\sum_{r,s=1}^{3}q_{rs} \tag{4.7.8}$$

holds with an equality sign if $p_e = p_i$, see eqs. (4.5.3), (4.5.4) and (2.4.10). Assuming now $p_i = 0$ and a single nontrivial value $q_{ss} \neq 0$, we obtain the following inequality

$$|\tau_s|_{\max} > \left(\tau_s^2\right)_m^{1/2} \geq p_e\frac{V_e}{V}\left(1 + \sum_{k=1}^{3}\frac{(x_k^e)^2}{j_k^2}\right)^{1/2}, \tag{4.7.9}$$

indicating that the presence of an internal space is accompanied by an increase in stresses. This is because $V_e = V$ and $x_k^e = 0$ when no internal space is present and eq. (2.4.10) requires an equality sign in formula (4.7.9).

1.4.8 An estimate for the specific potential energy of the deformed linear-elastic body

As will be shown in Subsection 3.3.2, the specific potential energy is expressed in terms of the invariants of the stress tensor in the following way

$$A = \frac{1}{2E}\left[I_1^2\left(\hat{T}\right) - 2\left(1+\nu\right)I_2\left(\hat{T}\right)\right], \tag{4.8.1}$$

where E and ν denote respectively Young's modulus and Poisson's ratio of the material. Inequality (4.7.6) provides one with an estimate of this quantity in terms of the prescribed external forces. Let us denote

$$\overline{I_1\left(\hat{T}\right)} = \left(t_{rr}\right)_m, \quad \overline{\overline{I_1\left(\hat{T}^{(k)}\right)}} = \left(t_{rr}x_k\right)_m$$

and

$$\overline{I_2\left(\hat{T}\right)} = \left(t_{rr}\right)_m\left(t_{r+1,r+1}\right)_m - \left(t_{r,r+1}\right)_m^2,$$
$$\overline{I_2\left(\hat{T}^{(k)}\right)} = \left(t_{rr}x_k\right)_m\left(t_{r+1,r+1}x_k\right)_m - \left(t_{r,r+1}x_k\right)_m^2,$$

where the summation is carried out over subscript r from 1 to 3, with $r+1=4$ being replaced by 1. Then

$$\begin{aligned}(A)_m = \frac{1}{V}\iiint\limits_V A d\tau > \frac{1}{2E}&\left\{\left[\overline{I_1\left(\hat{T}\right)}\right]^2 - 2\left(1+\nu\right)\overline{I_2\left(\hat{T}\right)}\right\} \\ &+ \frac{1}{2E}\sum_{k=1}^{3}\frac{1}{j_k^2}\left\{\left[\overline{\overline{I_1\left(\hat{T}^{(k)}\right)}}\right]^2 - 2\left(1+\nu\right)\overline{\overline{I_2\left(\hat{T}^{(k)}\right)}}\right\}.\end{aligned} \tag{4.8.2}$$

1.4.9 An estimate of the specific intensity of shear stresses

Using eqs. (2.2.11), (A.11.8) and (A.10.5) we can write the square of the specific intensity of shear stresses, which is equal to the absolute value of the second invariant of the deviator of the stress tensor, as follows

$$\begin{aligned}\tau^2 &= \frac{1}{6}\left[\left(t_1-t_2\right)^2 + \left(t_2-t_3\right)^2 + \left(t_3-t_1\right)^2\right] \\ &= \frac{1}{6}\left[\left(t_{11}-t_{22}\right)^2 + \left(t_{22}-t_{33}\right)^2 + \left(t_{33}-t_{11}\right)^2 + 6\left(t_{12}^2+t_{23}^2+t_{31}^2\right)\right] \\ &= \frac{1}{6}\left[\left(t_{rr}-t_{r+1,r+1}\right)^2 + 6t_{r,r+1}^2\right].\end{aligned} \tag{4.9.1}$$

It is a positive semidefinite form as it vanishes for $t_{12} = t_{23} = t_{31} = 0, t_{11} = t_{22} = t_{33} \neq 0$. Due to eq. (4.7.6) we have

$$\left(\tau^2\right)_m = \frac{1}{V}\iiint\limits_V \tau^2 dx_1 dx_2 dx_3 \geq \frac{1}{6}\left[(t_{rr})_m - (t_{r+1,r+1})_m\right]^2 + (t_{r,r+1})_m^2$$

$$+\frac{1}{6}\sum_{k=1}^{3}\frac{1}{j_k^2}\left\{\left[(t_{rr}x_k)_m - (t_{r+1,r+1}x_k)_m\right]^2 + 6\left(t_{r,r+1}x_k\right)_m^2\right\} = \tau_*^2 . \quad (4.9.2)$$

According to the Mises yield criterion, inequality $\tau < \tau_T$ (τ_T denotes the yield stress of the material) at any point in the body ensures the absence of the plastic deformation zones. Inasmuch as $\left(\tau^2\right)_m^{1/2} < \tau_{\max}$, the condition $\tau_T > \left(\tau^2\right)_m^{1/2}$ presents a necessary, however not a sufficient, condition for unattainability of the yield stress. For this reason, the inequality

$$\tau_T \leq \tau_* \quad (4.9.3)$$

provides sufficient evidence for the presence of plastic zones, whereas the opposite inequality

$$\tau_T > \tau_* \quad (4.9.4)$$

is a necessary condition for their absence. As mentioned above, these criteria are expressed with the help of the formulae of Subsection 1.4.3 in terms of the external volume and surface forces, the latter being assumed to be prescribed over the entire surface O of the body.

1.4.10 Moments of stresses of second and higher order

When $n \geq 3$, then the number of equations (4.2.3) is less than the number of unknowns. For instance, for $n = 3$ we have 30 equations with 36 unknowns. However for $n \geq 3$ it is possible to determine 15 unknowns. These are the nine values

$$n\left(x_1^{n-1}t_{1s}\right)_m = \overset{s}{q}_{n00}, \quad n\left(x_2^{n-1}t_{2s}\right)_m = \overset{s}{q}_{0n0}, \quad n\left(x_3^{n-1}t_{3s}\right)_m = \overset{s}{q}_{00n} \quad (4.10.1)$$

as well as the six values of the following type

$$\begin{aligned}(n-1)\left(x_1^{n-2}x_2t_{11}\right)_m &= \overset{1}{q}_{n-1,10} - \frac{1}{n}\overset{2}{q}_{n00},\\ (n-1)\left(x_1^{n-2}x_3t_{11}\right)_m &= \overset{1}{q}_{n-1,01} - \frac{1}{n}\overset{3}{q}_{n00}\,.\end{aligned} \quad (4.10.2)$$

1.4.11 A lower bound for the maximum of the stress components

The derivation of formula (4.7.6) for a lower bound of the mean value of the quadratic form of the components of the stress tensor is based upon only on the properties of orthonormality, eqs. (4.7.3) and (4.7.4), of four polynomials of zeroth and first order

$$P_0 = \frac{1}{\sqrt{V}}, \quad P_s = \frac{x_s}{j_s\sqrt{V}} \quad (s = 1,2,3) \tag{4.11.1}$$

in volume V. This derivation can be carried out for a more general system of orthonormalised polynomials

$$P_0, P_1, P_2, P_3, \dots, P_r. \tag{4.11.2}$$

For example, one of the polynomials of second order

$$P_{3+q} = \frac{1}{j_{3+q}\sqrt{V}}\left[x_q^2 - j_q^2 - \frac{1}{V}\sum_{k=1}^{3}\frac{x_k}{j_k^2}\iiint\limits_V x_q^2 x_k d\tau\right], \tag{4.11.3}$$

where

$$j_{3+q}^2 = \frac{1}{V}\iiint\limits_V x_q^4 d\tau - j_q^4 - \frac{1}{V^2}\sum_{k=1}^{3}\frac{1}{j_k^2}\left(\iiint\limits_V x_q^2 x_k d\tau\right)^2 \tag{4.11.4}$$

can be added to the basic system (4.11.1). The calculation becomes more laborious when two polynomials, e.g. polynomial P_4 and polynomial P_5 containing x_2^2, are added to the basic system. In this case one needs to ensure the orthogonality of P_5 to the basic polynomials as well as to P_4.

Let a system of polynomials orthonormalised in V be constructed, that is

$$\iiint\limits_V P_t P_q d\tau = \begin{cases} 0, & q \neq t, \\ 1, & q = t, \end{cases} \quad t, q = 1, 2, \dots, s. \tag{4.11.5}$$

Instead of eq. (4.7.2) we introduce a more general form of the expression

$$\psi = \frac{1}{2}\iiint\limits_V q_{rq}\left(\tau_r - \sum_{t=0}^{s} a_t^r P_t\right)\left(\tau_q - \sum_{t=0}^{s} a_l^q P_l\right) d\tau.$$

Repeating the derivation of Subsection 1.4.7 we arrive at the inequality

$$\iiint\limits_V q_{rq}\tau_r\tau_q d\tau \geq V^2 q_{rq}\sum_{t=0}^{s}(\tau_r P_t)_m (\tau_q P_t)_m,$$

which is more general than (4.7.6). An equality sign can occur if $q_{rq}\tau_r\tau_q$ is a positive semidefinite form.

Let all q_{rq} but one with the coinciding subscripts be equal to zero. Taking into account that

$$\frac{1}{V}\iiint\limits_V \tau_q^2 d\tau \leq |\tau_q|^2_{\max} \tag{4.11.6}$$

leads to the following lower bound of the maximum of the absolute value for τ_q

$$|\tau_q|_{\max} \geq \left[V\sum_{t=0}^{s}(\tau_q P_t)^2_m\right]^{1/2}. \tag{4.11.7}$$

This lower bound appears to be very rough and it can be made more precise by adding new polynomials orthonormalised in V into the right hand side. However in choosing the polynomials one should allow for the possibility to express the values

$$(\tau_r P_t)_m = \frac{1}{V}\iiint\limits_V \tau_r P_t d\tau \tag{4.11.8}$$

in terms of the higher order moments (4.10.1) and (4.10.2) calculated by the external loads.

For instance, if we restrict ourselves to $n = 3$ in these formulae, then for $|\tau_1|_{\max}$ we can use, along with the basic polynomials (4.11.1), the above polynomial P_4. For a further refinement we can take one further polynomial with terms x_1x_2 or x_1x_3 orthonormalised with respect to the previous five polynomials. A further refinement requires constructing a system of seven orthonormalised polynomials (four basic and three quadratic ones containing x_1^2, x_1x_2, x_1x_3). This exhausts the possibility of further refining $|\tau_1|_{\max}$ for $n = 3$. Estimates for maxima for other components can be constructed by analogy. For example, when $n = 3$ the best lower bound for $|\tau_4|_{\max} = |t_{12}|_{\max}$ by formula (4.11.7) can be reached with the help of six polynomials, which are $P_0, \ldots, P_4$ and a polynomial with x_2^2 orthonormalised to the others.

1.4.12 A refined lower bound

We start from the equality

$$\iiint\limits_V \sum_{q=1}^{6}\beta_q\tau_q\sum_{t=0}^{s}a_t P_t d\tau = V\sum_{q=1}^{6}\beta_q\sum_{t=0}^{s}a_t(\tau_q P_t)_m, \tag{4.12.1}$$

where β_q $(q = 1, \dots, 6)$ and β_t $(t = 1, \dots, s)$ are some constants. Since the absolute value of the integral on the left hand side does not exceed the integral of absolute value of the integrand we have

$$\iiint_V \left|\sum_{q=1}^{6} \beta_q \tau_q\right| \left|\sum_{t=0}^{s} a_t P_t\right| d\tau \le \left|\sum_{q=1}^{6} \beta_q \tau_q\right|_{\max} \iiint_V \left|\sum_{t=0}^{s} a_t P_t\right| d\tau .$$

Returning to eq. (4.12.1) we arrive at the inequality

$$\left|\sum_{q=1}^{6} \beta_q \tau_q\right|_{\max} \ge \frac{\left|\sum_{q=1}^{6} \beta_q \sum_{t=0}^{s} a_t \left(\tau_q P_t\right)_m\right|}{\iiint_V \left|\sum_{t=0}^{s} a_t P_t\right| d\tau}. \qquad (4.12.2)$$

The presence of the integral of the absolute value on the right hand side complicates the calculation. However an advantage of this formula, in comparison with eq. (4.11.6), is the presence of a_t. Carefully choosing them we can increase the right hand side of the above inequality.

Assuming

$$\sum_{r,q=1}^{6} q_{rq} \tau_r \tau_q = \left(\sum_{q=1}^{6} \beta_q \tau_q\right)^2 ,$$

we have, due to eq. (4.11.6), that

$$\left|\sum_{q=1}^{6} \beta_q \tau_q\right|_{\max} \ge \left\{V \sum_{t=0}^{s} \left[\sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m\right]^2\right\}^{1/2} . \qquad (4.12.3)$$

Let us now take

$$a_t = \sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m \qquad (4.12.4)$$

in inequality (4.12.2) and denote its right hand side as

$$\gamma = V \frac{\sum_{t=0}^{s} \left[\sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m\right]^2}{\iiint_V \left|\sum_{t=0}^{s} \sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m P_t\right| d\tau}. \qquad (4.12.5)$$

Setting $\varphi = \operatorname{sgn} \psi$ in the Bunyakovsky-Schwarz inequality

$$\left[\iiint_V \varphi \psi d\tau\right]^2 \le \iiint_V \varphi^2 d\tau \iiint_V \psi^2 d\tau,$$

we have

$$\iiint\limits_V |\psi|\, d\tau \le \left(V \iiint\limits_V \psi^2 d\tau \right)^{1/2}.$$

This inequality, being applied to the right hand side of eq. (4.12.5), along with eq. (4.11.5) yields

$$\iiint\limits_V \left| \sum_{t=0}^{s} \sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m P_t \right| d\tau \le \left\{ V \iiint\limits_V \left[\sum_{t=0}^{s} \sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m P_t \right]^2 d\tau \right\}^{1/2}$$

$$= \left\{ V \sum_{t=0}^{s} \left[\sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m \right]^2 \right\}^{1/2},$$

which implies that

$$\gamma \ge \left\{ V \sum_{t=0}^{s} \left[\sum_{q=1}^{6} \beta_q \left(\tau_q P_t\right)_m \right]^2 \right\}^{1/2}.$$

Hence, the lower bound due to inequality (4.12.3) is worse than that given by inequality (4.12.2) if all the constants a_t are taken from eq. (4.12.4). For example, if a single β_q is not zero then, by virtue of eqs. (4.12.2) and (4.12.4), we have

$$|\tau_q|_{\max} \ge V \frac{\sum\limits_{t=0}^{s} \left(\tau_q P_t\right)_m^2}{\iiint\limits_V \left| \sum\limits_{t=0}^{s} \left(\tau_q P_t\right)_m P_t \right| d\tau} \tag{4.12.6}$$

and this estimate is better than eq. (4.11.7).

As an example, we derive a lower bound for the maximum of the absolute value of the temperature θ under an adiabatic loading. According to eq. (3.5.8) of Chapter 3

$$\theta = -\frac{\alpha \Theta_0}{c_p} \sigma,$$

where θ denotes the temperature difference from the natural state (i.e. the unloaded state), Θ_0 is the absolute temperature in the natural state, α is the coefficient of linear expansion, c_p is the heat capacity at constant pressure and $\sigma = t_{11} + t_{22} + t_{33}$ is the first invariant of the stress tensor. Taking only the basic polynomials (4.11.1) and using eq. (4.12.3) we obtain

$$|\theta|_{\max} \ge \frac{\alpha \Theta_0}{c_p} \left\{ V \sum_{t=0}^{3} \left(\sigma P_t\right)_m^2 \right\}^{1/2}. \tag{4.12.7}$$

A more refined estimate by eq. (4.12.5) yields

$$|\theta|_{\max} \geq \frac{\alpha\Theta_0}{c_p} V \frac{\sum\limits_{t=0}^{3} (\sigma P_t)_m^2}{\iiint\limits_V |(\sigma P_t)_m P_t| \, d\tau}. \tag{4.12.8}$$

2

Deformation of a continuum

2.1 Linear strain tensor

2.1.1 Outline of the chapter

As already mentioned in Subsection 1.1.1, the transition from the initial state (volume v) to the final state (volume V) is determined by prescribing the displacement vector $\mathbf{u}$ of the medium particles. Construction of a theory of continuum mechanics needs a mathematical means of describing change in the distance between the particles and angles between the chosen directions in terms of this vector field.

The problem is to follow the change in length and direction of an infinitesimally small linear element of volume v described by the vector

$$d\mathbf{r} = \mathbf{e}\,|d\mathbf{r}| = \mathbf{e}ds \tag{1.1.1}$$

as well as the change in the vector

$$d\mathbf{R} = \tilde{\mathbf{e}}\,|d\mathbf{R}| = \tilde{\mathbf{e}}dS \tag{1.1.2}$$

in volume V provided that this vector contains the same particles. The main issue is to relate vector $d\mathbf{R}$ to vector $d\mathbf{r}$. It is clear that the solution is bounded to introducing a second rank tensor. Indeed, let us consider the position vector $\mathbf{R}$ of a particle in volume V as a function of the material coordinates (these are, for example, the Cartesian coordinates a_1, a_2, a_3 of this particle in volume v), then by virtue of eq. (B.2.11) we have

$$d\mathbf{R} = d\mathbf{r} \cdot \nabla\mathbf{R} = (\nabla\mathbf{R})^* \cdot d\mathbf{r}, \tag{1.1.3}$$

where $\nabla \mathbf{R}$ denotes the gradient of vector $\mathbf{R}$ and is a second rank tensor. By means of this non-symmetrical tensor one can construct a symmetric tensor of the second rank which is referred to in what follows as the first measure of strain (Cauchy-Green). It allows one to obtain the solution to the above problem of the change in lengths and angles in volume v.

It does not exhaust the problem of determining the quantities characterising deformation since the inverse problem is also of importance. The inverse problem is concerned with determining vector $d\mathbf{r}$ in volume v given by vector $d\mathbf{R}$ in volume V and results in introducing the second measure of strain.

Given a prescribed oriented surface $\mathbf{n}do$ in volume v, another important geometric problem is the determination of the corresponding surface $\mathbf{N}dO$ in volume V. This problem and the inverse one, which is the determination of $\mathbf{n}do$ in terms of $\mathbf{N}dO$, are solved by introducing two additional strain measures defined by the second rank tensors which are inverse to the first and second measures, see Section A.7.

It follows from eq. (1.1.3) of Chapter 1 that

$$\nabla \mathbf{R} = \hat{E} + \nabla \mathbf{u} = \mathbf{i}_s \mathbf{i}_k \left(\delta_{sk} + \frac{\partial u_k}{\partial a_s} \right). \tag{1.1.4}$$

In the linear theory of elasticity, there is no need to deal with the above strain measures. The linear theory is based on the assumption that the elements of the matrix of tensor $\nabla \mathbf{u}$ are small

$$\left| \frac{\partial u_k}{\partial a_s} \right| \ll 1, \tag{1.1.5}$$

which is quite acceptable when analysing deformations of massive and slightly deformable bodies. Consequently, this assumption neglects the squares and the products of the components of tensor $\nabla \mathbf{u}$ compared with the first order terms. Under such an assumption, in order to describe the deformation it is sufficient to introduce a single second rank tensor which is referred to in what follows as the linear strain tensor.

2.1.2 Definition of the linear strain tensor

Let us consider two infinitesimally close points M and N in volume v

$$M \ldots \mathbf{r} = \mathbf{i}_s a_s, \; N \ldots \mathbf{r} + d\mathbf{r} = \mathbf{i}_s \left(a_s + da_s \right), \tag{1.2.1}$$

see Fig. 2.1. Their positions M' and N' in volume V are given by the following position vectors

$$\left. \begin{array}{l} M' \ldots \mathbf{R} = \mathbf{r} + \mathbf{u} = \mathbf{i}_s \left(a_s + u_s \right), \\ N' \ldots \mathbf{R} + d\mathbf{R} = \mathbf{r} + d\mathbf{r} + \mathbf{u} + d\mathbf{u} = \mathbf{i}_s \left(a_s + da_s + u_s + du_s \right). \end{array} \right\} \tag{1.2.2}$$

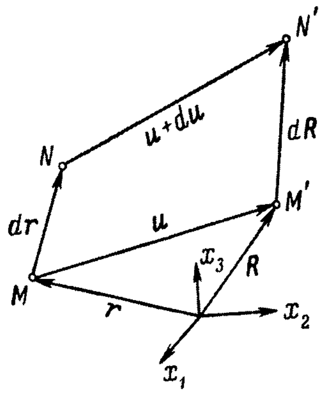

FIGURE 2.1.

Here $d\mathbf{u}$ represents the vector of the relative displacement of two infinitesimally close points in the medium. Then, due to eqs. (B.2.6) and (B.2.11) we have

$$d\mathbf{u} = \frac{d\mathbf{u}}{d\mathbf{r}} \cdot d\mathbf{r} = (\nabla\mathbf{u})^* \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla\mathbf{u}. \tag{1.2.3}$$

Tensor $\dfrac{d\mathbf{u}}{d\mathbf{r}}$, which is the derivative of vector $\mathbf{u}$ with respect to direction $\mathbf{r}$, can be set as the sum of its symmetric and skew-symmetric parts

$$\frac{d\mathbf{u}}{d\mathbf{r}} = \frac{1}{2}\left(\frac{d\mathbf{u}}{d\mathbf{r}} + \nabla\mathbf{u}\right) + \frac{1}{2}\left(\frac{d\mathbf{u}}{d\mathbf{r}} - \nabla\mathbf{u}\right) = \hat{\varepsilon} + \hat{\Omega}, \tag{1.2.4}$$

see eq. (A.4.8). The first component determines a symmetric tensor of second rank which is called the linear strain tensor

$$\hat{\varepsilon} = \operatorname{def}\mathbf{u} = \frac{1}{2}\left(\frac{d\mathbf{u}}{d\mathbf{r}} + \nabla\mathbf{u}\right) = \frac{1}{2}\left[(\nabla\mathbf{u})^* + \nabla\mathbf{u}\right]. \tag{1.2.5}$$

The matrix of the components of this tensor is set in the form

$$\left\| \begin{matrix} \varepsilon_{11} & \varepsilon_{12} = \frac{1}{2}\gamma_{12} & \varepsilon_{13} = \frac{1}{2}\gamma_{13} \\ \varepsilon_{21} = \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \varepsilon_{23} = \frac{1}{2}\gamma_{23} \\ \varepsilon_{31} = \frac{1}{2}\gamma_{31} & \varepsilon_{32} = \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{matrix} \right\|. \tag{1.2.6}$$

The expressions for components $\varepsilon_{ik} = \varepsilon_{ik}$ in terms of the derivatives of the displacement vector are, due to eq. (B.2.5), as follows

$$\varepsilon_{11} = \frac{\partial u_1}{\partial a_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial a_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial a_3}, \tag{1.2.7}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right), \ \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_3} + \frac{\partial u_3}{\partial a_2}\right), \ \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial a_1} + \frac{\partial u_1}{\partial a_3}\right). \tag{1.2.8}$$

In linear elasticity theory the diagonal components of matrix (1.2.6) relate to extensions, while the non-diagonal components γ_{ik} are referred to as shearing strains. The origin of this notion is explained in Subsection 2.3.6.

The second term in formula (1.2.4) is the skew-symmetric tensor of second rank

$$\hat{\Omega} = \frac{1}{2}\left(\frac{d\mathbf{u}}{d\mathbf{r}} - \nabla\mathbf{u}\right) = \frac{1}{2}\left[(\nabla\mathbf{u})^* - \nabla\mathbf{u}\right] \tag{1.2.9}$$

with the following matrix of the components

$$\begin{Vmatrix} 0 & \omega_{12} = -\omega_3 & \omega_{13} = \omega_2 \\ \omega_{21} = \omega_3 & 0 & \omega_{23} = -\omega_1 \\ \omega_{31} = -\omega_2 & \omega_{32} = \omega_1 & 0 \end{Vmatrix}. \tag{1.2.10}$$

Here the quantities

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial a_2} - \frac{\partial u_2}{\partial a_3}\right), \ \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_3} - \frac{\partial u_3}{\partial a_1}\right), \ \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} - \frac{\partial u_1}{\partial a_2}\right) \tag{1.2.11}$$

represent the projections of vector $\boldsymbol{\omega}$ referred to as the vector of rotation. This vector accompanies tensor $(\nabla\mathbf{u})^*$. Due to eq. (B.2.8) we have

$$\boldsymbol{\omega} = \frac{1}{2} e_{rts} \frac{\partial u_t}{\partial a_r} \mathbf{i}_s = \frac{1}{2} \mathbf{i}_r \times \mathbf{i}_t \frac{\partial u_t}{\partial a_r} = \frac{1}{2} \nabla \times \mathbf{u} = \frac{1}{2} \operatorname{rot} \mathbf{u}. \tag{1.2.12}$$

Rewriting formulae (1.2.5) and (1.2.9) in the form

$$\frac{d\mathbf{u}}{d\mathbf{r}} = \hat{\varepsilon} + \hat{\Omega}, \quad \nabla\mathbf{u} = \hat{\varepsilon} - \hat{\Omega} \tag{1.2.13}$$

and referring to eqs. (1.2.3) and (A.4.10) we obtain

$$d\mathbf{u} = \hat{\varepsilon} \cdot d\mathbf{r} + \hat{\Omega} \cdot d\mathbf{r} = \hat{\varepsilon} \cdot d\mathbf{r} + \boldsymbol{\omega} \times d\mathbf{r}. \tag{1.2.14}$$

The second term in this formula presents the displacement due to a rigid-body rotation of an infinitesimally small vicinity of point M whereas the first one determines the displacement of points of this vicinity due to deformation $\hat{\varepsilon}$.

The definition given here for displacement vector $\mathbf{u}$ can be generalised for any vector, see Section B.2. For instance, applying operation def to position vector $\mathbf{r}$ leads to the unit tensor

$$\operatorname{def} \mathbf{r} = \frac{1}{2}\left(\frac{d\mathbf{r}}{d\mathbf{r}} + \nabla\mathbf{r}\right) = \frac{1}{2}\left(\hat{E} + \hat{E}\right) = \hat{E}, \tag{1.2.15}$$

so that

$$\operatorname{def} \mathbf{R} = \hat{E} + \hat{\varepsilon}. \tag{1.2.16}$$

2.2 Determination of the displacement in terms of the linear strain tensor

2.2.1 *Compatibility of strains (Saint-Venant's dependences)*

We pose the problem of determining the displacement vector (or its projections u_s referred to, for the sake of brevity, as displacements) in terms of the prescribed linear strain tensor $\hat{\varepsilon}$. This involves an integration of the following system of six differential equations

$$\begin{gathered}\frac{\partial u_1}{\partial a_1} = \varepsilon_{11}, \qquad \frac{\partial u_2}{\partial a_2} = \varepsilon_{22}, \qquad \frac{\partial u_3}{\partial a_3} = \varepsilon_{33}, \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} + \frac{\partial u_1}{\partial a_2}\right) = \varepsilon_{12}, \ \frac{1}{2}\left(\frac{\partial u_3}{\partial a_2} + \frac{\partial u_2}{\partial a_3}\right) = \varepsilon_{23}, \ \frac{1}{2}\left(\frac{\partial u_1}{\partial a_3} + \frac{\partial u_3}{\partial a_1}\right) = \varepsilon_{31},\end{gathered} \tag{2.1.1}$$

where the right hand sides are assumed to be continuous together with the derivatives of the first and second order. The number of equations (six) exceeds the number of the unknowns (three), thus the problem will have a solution only when certain additional conditions are imposed on the components of tensor $\hat{\varepsilon}$. This can be illustrated by the following example. Let us assume that a medium is divided into elementary blocks. Let each block be subjected to a deformation in the form of small extensions and small shears of the original right-angled block. The obtained bodies can be a continuous (i.e. without gaps) medium only by properly matching the deformation of separate blocks. This occurs when a displacement vector $\mathbf{u}$ exists such that it is continuous along with the derivatives up to at least third order and the prescribed tensor $\hat{\varepsilon}$ is its deformation ($\hat{\varepsilon} = \operatorname{def} \mathbf{u}$). These are the conditions for integrability of the system of equations (2.1.1) and this explains why these conditions are termed the conditions for continuity or conditions for compatibility. It was Saint-Venant who pointed out the importance of these conditions in continuum mechanics, and it explains why the term "Saint-Venant's dependences" is often used.

We start from formula (1.2.14). Its right hand side contains an unknown skew-symmetric tensor $\hat{\Omega}$ which should be excluded from consideration. According to eq. (B.6.5), the condition for the integrability of relationship (1.2.14) (i.e. the condition for the existence of vector $\mathbf{u}$) is to have a vanishing rotor of tensor $\left(\hat{\varepsilon} + \hat{\Omega}\right)^*$, that is

$$\operatorname{rot}\left(\hat{\varepsilon} + \hat{\Omega}\right)^* = \operatorname{rot}\left(\hat{\varepsilon} - \hat{\Omega}\right),$$

as $\hat{\varepsilon}^* = \hat{\varepsilon}$ and $\hat{\Omega}^* = -\hat{\Omega}$. We arrive then at the condition

$$\operatorname{rot}\hat{\varepsilon} = \operatorname{rot}\hat{\Omega}. \tag{2.1.2}$$

The expression for the rotor of a skew-symmetric tensor is given by formula (B.3.9)

$$\operatorname{rot}\hat{\Omega} = (\nabla\boldsymbol{\omega})^* - \hat{E}\operatorname{div}\boldsymbol{\omega} = \frac{d\boldsymbol{\omega}}{d\mathbf{r}}, \tag{2.1.3}$$

since

$$\operatorname{div}\boldsymbol{\omega} = -\frac{1}{2}I_1(\operatorname{rot}\hat{\varepsilon}) = 0,$$

inasmuch as the first invariant of the rotor of a symmetric tensor is equal to zero.

Hence, we are led to the relationship

$$\operatorname{rot}\hat{\varepsilon} = \frac{d\boldsymbol{\omega}}{d\mathbf{r}}, \quad d\boldsymbol{\omega} = \operatorname{rot}\hat{\varepsilon}\cdot d\mathbf{r}. \tag{2.1.4}$$

Returning to eq. (B.6.5) we can reset the integrability condition in the form

$$\operatorname{rot}(\operatorname{rot}\hat{\varepsilon})^* = \operatorname{inc}\hat{\varepsilon} = 0. \tag{2.1.5}$$

When this condition is satisfied, then eq. (2.1.4) yields

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \int\limits_{M_0}^{M}\operatorname{rot}\hat{\varepsilon}\cdot d\mathbf{r}. \tag{2.1.6}$$

The right hand side in equation (1.2.14) for $d\mathbf{u}$ is now known and the condition for its integrability, eq. (2.1.3), is fulfilled. The latter equation yields vector $\boldsymbol{\omega}$ provided that condition (2.1.5) is satisfied. For this reason, eq. (2.1.5) represents the condition for integrability of relationship (1.2.14). In addition, this explains the nomenclature inc, namely condition $\operatorname{inc}\hat{P} \neq 0$ is incompatible with the existence of a vector whose deformation is the symmetric tensor $\hat{P}$, see Section B.4.

The notion of six conditions for tensor $\operatorname{inc}\hat{\varepsilon}$ to be zero follows from Table (B.4.15) of the components of tensor $\operatorname{inc}\hat{\varepsilon}$. The same conditions can be obtained by eliminating displacements u_1, u_2, u_3 from the system of equations (2.1.1). The elimination procedure is carried out as follows. Considering the three equations

$$\frac{\partial u_1}{\partial a_1} = \varepsilon_{11}, \quad \frac{\partial u_2}{\partial a_2} = \varepsilon_{22}, \quad \frac{\partial u_2}{\partial a_1} + \frac{\partial u_1}{\partial a_2} = \gamma_{12},$$

we notice that their left hand sides satisfy the identity

$$\frac{\partial^2}{\partial a_2^2}\frac{\partial u_1}{\partial a_1} + \frac{\partial^2}{\partial a_1^2}\frac{\partial u_2}{\partial a_2} = \frac{\partial^2}{\partial a_1 \partial a_2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right).$$

This immediately yields one of the three conditions for ensuring that the diagonal components of $\operatorname{inc}\hat{\varepsilon}$ are zero

$$\frac{\partial^2 \varepsilon_{11}}{\partial a_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial a_1^2} = \frac{\partial^2 \gamma_{12}}{\partial a_1 \partial a_2}. \tag{2.1.7}$$

One of the identities of the second group is

$$-2\frac{\partial^3 u_3}{\partial a_1 \partial a_2 \partial a_3} + \frac{\partial^2}{\partial a_2 \partial a_3}\left(\frac{\partial u_3}{\partial a_1} + \frac{\partial u_1}{\partial a_3}\right) + \frac{\partial^2}{\partial a_3 \partial a_1}\left(\frac{\partial u_3}{\partial a_2} + \frac{\partial u_2}{\partial a_3}\right) - \frac{\partial^2}{\partial a_3^2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right) = 0,$$

which can be used in the following relationship

$$-2\frac{\partial^2 \varepsilon_{33}}{\partial a_1 \partial a_2} + \frac{\partial}{\partial a_3}\left(\frac{\partial \gamma_{31}}{\partial a_2} + \frac{\partial \gamma_{23}}{\partial a_1} - \frac{\partial \gamma_{12}}{\partial a_3}\right) = 0. \tag{2.1.8}$$

The remaining conditions are obtained from eqs. (2.1.7) and (2.1.8) by means of a cyclic permutation of subscripts.

Let us also notice that formula (2.1.5) is applicable to any linear strain tensor $\operatorname{def}\mathbf{a}$ of any vector $\mathbf{a}$

$$\operatorname{inc}\operatorname{def}\mathbf{a} = 0 \tag{2.1.9}$$

and not just to the displacement vector. Any symmetric tensor with zero "incompatibility" represents the deformation of a certain vector. This statement was used in Subsection 1.1.6.

2.2.2 Displacement vector. The Cesaro formula

Using expression (2.1.6) for vector $\boldsymbol{\omega}$ we can rewrite relationship (1.2.14) as follows

$$d\mathbf{u} = \hat{\varepsilon} \cdot d\mathbf{r} + \boldsymbol{\omega}_0 \times d\mathbf{r} - d\mathbf{r} \times \int_{M_0}^{M} \operatorname{rot}\hat{\varepsilon}(\sigma) \cdot d\mathbf{r}(\sigma). \tag{2.2.1}$$

Let C denote the integration path, with M_0 and M being respectively the start and end points. Also let M' and M'' denote two point on this path. The position vectors of these points are denoted by $\mathbf{r}_0, \mathbf{r}(s), \mathbf{r}(\sigma)$ and $\mathbf{r}(\sigma')$, respectively. Then

$$\mathbf{u}(s) = \mathbf{u}_0 + \boldsymbol{\omega}_0 \times [\mathbf{r}(s) - \mathbf{r}_0] + \int_{M_0}^{M} \hat{\varepsilon}(\sigma) \cdot d\mathbf{r}(\sigma) - \int_{M_0}^{M} d\mathbf{r}(\sigma) \times \int_{M_0}^{M'} \operatorname{rot}\hat{\varepsilon}(\sigma') \cdot d\mathbf{r}(\sigma').$$

The double integral is transformed into a single integral in the following way

$$-\int_{M_0}^{M} d\mathbf{r}(\sigma) \times \int_{M_0}^{M'} \operatorname{rot}\hat{\varepsilon}(\sigma') \cdot d\mathbf{r}(\sigma') = \int_{M_0}^{M} \operatorname{rot}\hat{\varepsilon}(\sigma') \cdot d\mathbf{r}(\sigma') \times \int_{M''}^{M} d\mathbf{r}(\sigma)$$
$$= \int_{M_0}^{M} \operatorname{rot}\hat{\varepsilon}(\sigma') \cdot d\mathbf{r}(\sigma') \times [\mathbf{r}(s) - \mathbf{r}(\sigma')] .$$

Then we arrive at the Cesaro formula which determines the displacement vector in terms of the linear strain tensor

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) + \int_{M_0}^{M} \{\hat{\varepsilon}(\sigma) + [\mathbf{r}(\sigma) - \mathbf{r}(s)] \times \operatorname{rot}\hat{\varepsilon}(\sigma)\} \cdot d\mathbf{r}(\sigma)$$
$$= \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) + \int_{M_0}^{M} \hat{\Pi} \cdot d\mathbf{r}(\sigma) . \tag{2.2.2}$$

Here a non-symmetrical tensor of second rank

$$\hat{\Pi} = \hat{\varepsilon}' + (\mathbf{r}' - \mathbf{r}) \times \operatorname{rot}\hat{\varepsilon}' \tag{2.2.3}$$

is introduced. In this notion $\mathbf{r} = \mathbf{r}(s)$ and primes designate quantities at the integration points. Making use of the dyadic representation of this tensor

$$\hat{\Pi} = \left[\varepsilon'_{st} + (a'_q - a_q)\left(\frac{\partial \varepsilon'_{qt}}{\partial a'_s} - \frac{\partial \varepsilon'_{st}}{\partial a'_q}\right)\right] \mathbf{i}_s \mathbf{i}_t \tag{2.2.4}$$

we can put Cesaro's formula in the form

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) + \mathbf{i}_s \int_{M_0}^{M} \left[\varepsilon'_{st} + (a'_q - a_q)\left(\frac{\partial \varepsilon'_{qt}}{\partial a'_s} - \frac{\partial \varepsilon'_{st}}{\partial a'_q}\right)\right] da'_t. \tag{2.2.5}$$

Naturally, the displacement vector is determined correct to an additive vector

$$\mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) , \tag{2.2.6}$$

representing a small rigid-body displacement of the medium. This displacement is a geometric sum of displacement $\mathbf{u}_0$ of point M_0 and displacement $\boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0)$ due to a rotation about this point.

The integrability of expression (2.2.2) is proved directly. It is necessary to prove that condition (B.6.5) is met, that is

$$\operatorname{rot}\hat{\Pi}^* = \operatorname{rot}\left[\varepsilon' + (\mathbf{r}' - \mathbf{r}) \times \operatorname{rot}\hat{\varepsilon}'\right]^* = \operatorname{rot}\left[\hat{\varepsilon}' - \left(\operatorname{rot}\hat{\varepsilon}'\right)^* \times (\mathbf{r}' - \mathbf{r})\right] = 0. \tag{2.2.7}$$

Here we took into account identity (A.5.11)

$$\left[(\mathbf{r}' - \mathbf{r}) \times \operatorname{rot}\hat{\varepsilon}'\right]^* = -\left(\operatorname{rot}\hat{\varepsilon}'\right)^* \times (\mathbf{r}' - \mathbf{r}).$$

Assuming now $\hat{Q} = \left(\operatorname{rot}\hat{\varepsilon}'\right)^*$ in the identity

$$\operatorname{rot}\left(\hat{Q} \times \mathbf{r}\right) = \operatorname{rot}\hat{Q} \times \mathbf{r} + \hat{Q}^* - \hat{E}\operatorname{tr}\hat{Q} \tag{2.2.8}$$

yields the following result

$$\begin{aligned}\operatorname{rot}\left[\left(\operatorname{rot}\hat{\varepsilon}'\right)^* \times (\mathbf{r}' - \mathbf{r})\right] &= \operatorname{rot}\left(\operatorname{rot}\hat{\varepsilon}'\right)^* \times (\mathbf{r}' - \mathbf{r}) + \left(\operatorname{rot}\hat{\varepsilon}'\right)^{**} - \hat{E}\operatorname{tr}\left(\operatorname{rot}\hat{\varepsilon}'\right)^* \\ &= \operatorname{inc}\hat{\varepsilon}' \times (\mathbf{r}' - \mathbf{r}) + \operatorname{rot}\hat{\varepsilon}',\end{aligned}$$

since the trace of the rotor of a symmetric tensor (and its transpose) is zero. Inserting this result into eq. (2.2.7) we have

$$\operatorname{rot}\hat{\Pi}^* = \operatorname{rot}\varepsilon' - \operatorname{inc}\hat{\varepsilon}' \times (\mathbf{r}' - \mathbf{r}) - \operatorname{rot}\hat{\varepsilon}' = -\operatorname{inc}\hat{\varepsilon}' \times (\mathbf{r}' - \mathbf{r}) = 0, \tag{2.2.9}$$

and, due to the arbitrariness of vector $\mathbf{r}' - \mathbf{r}$ we arrive again at condition (2.1.5).

2.2.3 An example. The temperature field

Deformation of a single isotropic block in a temperature field $\theta(a_1, a_2, a_3)$ is given by the expression

$$\hat{\varepsilon} = \alpha\theta\hat{E} \quad (\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \alpha\theta,\ \gamma_{12} = \gamma_{23} = \gamma_{31} = 0), \tag{2.3.1}$$

where α denotes the coefficient of linear thermal expansion. In a solid such a deformation is possible under condition (2.1.5)

$$\begin{aligned}\operatorname{inc}\alpha\theta\hat{E} &= \operatorname{rot}\left(\operatorname{rot}\alpha\theta\hat{E}\right)^* = \mathbf{i}_s\frac{\partial}{\partial a_s} \times \left(\mathbf{i}_t\frac{\partial}{\partial a_t} \times \alpha\theta\mathbf{i}_q\mathbf{i}_q\right)^* \\ &= (\mathbf{i}_s \times \mathbf{i}_q)(\mathbf{i}_t \times \mathbf{i}_q)\frac{\partial^2\alpha\theta}{\partial a_s\partial a_t} = e_{sqr}e_{tqm}\mathbf{i}_r\mathbf{i}_m\frac{\partial^2\alpha\theta}{\partial a_s\partial a_t} \\ &= (\delta_{st}\delta_{rm} - \delta_{sm}\delta_{rt})\mathbf{i}_r\mathbf{i}_m\frac{\partial^2\alpha\theta}{\partial a_s\partial a_t} = 0.\end{aligned}$$

Thus

$$\operatorname{inc} \alpha\theta \hat{E} = \hat{E} \frac{\partial^2 \alpha\theta}{\partial a_s \partial a_s} - \frac{\partial^2 \alpha\theta}{\partial a_s \partial a_t} \mathbf{i}_s \mathbf{i}_t = \left(\hat{E} \nabla^2 - \nabla\nabla \right) \alpha\theta. \tag{2.3.2}$$

Taking into account that

$$\operatorname{tr} \left(\hat{E} \nabla^2 - \nabla\nabla \right) \alpha\theta = 0,$$

we have

$$\nabla\nabla \alpha\theta = 0, \quad \frac{\partial^2 \alpha\theta}{\partial a_s \partial a_t} = 0. \tag{2.3.3}$$

The condition for a feasible deformation, due to law (2.3.1), is the requirement that all second derivatives of θ with respect to coordinates vanish, so that θ is a linear function of the coordinates. Supposing that $\alpha = \text{const}$ we have

$$\theta = \theta_0 + q_1 a_1 + q_2 a_2 + q_3 a_3 = \theta_0 + \mathbf{q} \cdot \mathbf{r}, \tag{2.3.4}$$

where $\mathbf{q} = \operatorname{grad} \theta$ is a constant vector.

We find the displacement vector by substituting this expression into the Cesaro formula (2.2.5). Omitting the rigid-body displacement of the medium we obtain

$$\begin{aligned} \mathbf{u} &= \alpha \mathbf{i}_s \int\limits_{M_0}^{M} \left[\left(\theta_0 + q_k a_k' \right) \delta_{st} + \left(a_r' - a_r \right) \left(q_s \delta_{rt} - q_r \delta_{st} \right) \right] da_t' \\ &= \alpha \int\limits_{M_0}^{M} \left[\left(\theta_0 + \mathbf{q} \cdot \mathbf{r}' \right) d\mathbf{r}' + \frac{1}{2} \mathbf{q} d \left| \mathbf{r}' - \mathbf{r} \right|^2 - \mathbf{q} \cdot \left(\mathbf{r}' - \mathbf{r} \right) d\mathbf{r}' \right] \end{aligned}$$

which can be expressed in the final form

$$\mathbf{u} = \alpha\theta \left(\mathbf{r} - \mathbf{r}_0 \right) - \frac{1}{2} \alpha \left| \mathbf{r} - \mathbf{r}_0 \right|^2 \operatorname{grad} \theta. \tag{2.3.5}$$

Under any non-linear law of temperature distribution, a free thermal expansion due to law (2.3.1) can not take place. Tensor $\hat{E}\alpha\theta$ should be imposed by a compensating tensor $\hat{\varepsilon}^*$ such that, under the deformation

$$\hat{\varepsilon} = \hat{\varepsilon}^* + \hat{E} \alpha\theta, \tag{2.3.6}$$

the compatibility condition

$$\operatorname{inc} \hat{\varepsilon}^* = - \left(\hat{E} \nabla^2 - \nabla\nabla \right) \alpha\theta \tag{2.3.7}$$

is fulfilled. However it would be erroneous to think that the displacement vector $\mathbf{u}$ can be represented by a geometric sum of vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ determined by the conditions

$$\operatorname{def} \mathbf{u}_1 = \alpha\theta\hat{E}, \quad \operatorname{def} \mathbf{u}_2 = \hat{\varepsilon}^*,$$

since tensors $\alpha\theta\hat{E}$ and $\hat{\varepsilon}^*$, taken separately, do not satisfy the compatibility conditions and vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ do not exist.

2.2.4 The Volterra distortion

In a simply connected region the vectors of rotation $\boldsymbol{\omega}$ and displacement $\mathbf{u}$ given by integrals (2.1.6) and (2.2.2) are single-valued functions of coordinates a_k of point M which is the upper limit of the integral. In the case of a double connected region one needs to introduce the cyclic constant vectors

$$\oint_K \operatorname{rot} \hat{\varepsilon}' \cdot d\mathbf{r}' = \mathbf{b}, \quad \oint_K \left(\hat{\varepsilon}' + \mathbf{r}' \times \operatorname{rot} \hat{\varepsilon}'\right) \cdot d\mathbf{r}' = \mathbf{c}, \tag{2.4.1}$$

where K is a contour which can not be reduced to zero by a continuous transformation, see eq. (B.6.9). The values of $\boldsymbol{\omega}$ and $\mathbf{u}$ at point M can be written as follows

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \int_{M_0}^{M} \operatorname{rot} \hat{\varepsilon}' \cdot d\mathbf{r}' + n\mathbf{b}, \tag{2.4.2}$$

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{r} - \mathbf{r}_0) + \int_{M_0}^{M} \left(\hat{\varepsilon}' + (\mathbf{r}' - \mathbf{r}) \times \operatorname{rot} \hat{\varepsilon}'\right) \cdot d\mathbf{r}' + n\left(\mathbf{c} + \mathbf{b} \times \mathbf{r}\right), \tag{2.4.3}$$

where n denotes the number of revolutions along the integration path from M_0 to M on contour K. Vectors $\boldsymbol{\omega}$ and $\mathbf{u}$ are no longer uniquely defined. The uniqueness can be restored by introducing a barrier which transforms the double connected region into a simply connected region. However the transition through the barrier ensures that these vectors are no longer continuous.

Let σ denote the surface of the barrier while σ^- and σ^+ designate two surfaces which are congruent to σ and located immediately "above" and "below" the barrier. Let M, M^- and M^+ be some points that are close to each other on σ,σ^- and σ^+ respectively. Then, considering points M^- and M^+ as being the start and end points of the integration path, see Fig. 2.2, we have

$$\boldsymbol{\omega}^+ = \boldsymbol{\omega}^- + \int_{M^-}^{M^+} \operatorname{rot} \hat{\varepsilon}' \cdot d\mathbf{r}', \tag{2.4.4}$$

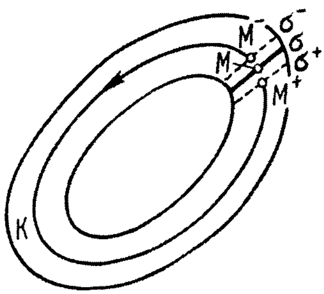

FIGURE 2.2.

$$\mathbf{u}^{+} = \mathbf{u}^{-} + \boldsymbol{\omega}^{-} \times \left(\mathbf{r}^{+} - \mathbf{r}^{-}\right) + \int\limits_{M^-}^{M^+} \left(\hat{\varepsilon}' + \left(\mathbf{r}' - \mathbf{r}^{+}\right) \times \operatorname{rot}\hat{\varepsilon}'\right) \cdot d\mathbf{r}'. \quad (2.4.5)$$

When we impose surfaces σ^- and σ^+ on barrier σ

$$\mathbf{r}^{+} \longrightarrow \mathbf{r}, \quad \mathbf{r}^{-} \longrightarrow \mathbf{r}, \quad \mathbf{r}^{+} - \mathbf{r}^{-} \longrightarrow 0$$

then integrals (2.4.4) and (2.4.5) are equal to the corresponding values along the closed contour K. We arrive at the Weingarten formulae (1901) yielding the jump in vectors $\boldsymbol{\omega}$ and $\mathbf{u}$ on the barrier

$$\boldsymbol{\omega}^{+} - \boldsymbol{\omega}^{-} = \mathbf{b}, \quad \mathbf{u}^{+} - \mathbf{u}^{-} = \mathbf{c} + \mathbf{b} \times \mathbf{r}. \quad (2.4.6)$$

These formulae indicate that the material on one side of the barrier experiences a small displacement relative to the material on the other side of the barrier. This displacement is a rigid-body displacement prescribed by the vectors of rotation $\mathbf{b}$ and translation $\mathbf{c}$. It can be explained as follows: a thin layer of material is removed after a double connected body (for instance, a torus) is cut along surface σ and then the congruent ends σ^- and σ^+ of the simply connected body obtained are connected together (into a torus), these ends being subjected to a small translation $\mathbf{c}$ and a small rotation described by $\mathbf{b}$. Volterra referred to this operation of creating a new body from the old one as a distortion whereas Love called it a dislocation. Nowadays a more general meaning is associated in the literature with the term "dislocation". Stresses appear in the elastic body subjected to a distortion. These stresses can be calculated theoretically for prescribed cyclic constant vectors $\mathbf{b}$ and $\mathbf{c}$. The latter can be determined experimentally by means of measuring displacements and rotations of the ends of the cut ring-like body.

The distortion in a simply connected body is impossible since after removing, say, a thin wedge-shaped body and matching the free ends, the strain tensor $\hat{\varepsilon}$ and thus the stress tensor become discontinuous. As mentioned above, it follows that the displacements in a simply connected body can not be multi-valued if tensor $\hat{\varepsilon}$ is continuous. This also explains why we required removal of a layer with necessarily congruent edges when considering the distortion. The reason is that the jump in vector $\mathbf{u}$ on the barrier, which is compatible with the assumption of continuity of tensor $\hat{\varepsilon}$, is a rigid-body displacement. For any jump of a more complicated nature this tensor is no longer continuous.

2.3 The first measure and the first tensor of finite strain

Reading of this section assumes a knowledge of Appendixes D and E.

2.3.1 Vector basis of volumes v and V

A particle in a continuum is prescribed by the material coordinates q^1, q^2 and q^3. The position of the particle in volume v is given by the position vector

$$\mathbf{r} = \mathbf{r}\left(q^1, q^2, q^3\right) = \mathbf{i}_s a_s \left(q^1, q^2, q^3\right). \tag{3.1.1}$$

In volume V the position of this particle can be expressed by the position vector

$$\mathbf{R} = \mathbf{R}\left(q^1, q^2, q^3\right) = \mathbf{i}_s x_s \left(q^1, q^2, q^3\right) = \mathbf{i}_s \left(a_s + u_s\right). \tag{3.1.2}$$

In particular, the Cartesian coordinates in volume v

$$a_s = q^s \tag{3.1.3}$$

can be taken as being the material coordinates. However we can equally well adopt the Cartesian coordinates in volume V

$$x_s = q^s \tag{3.1.4}$$

as the material coordinates. The vector basis in volume v is given by the three vectors

$$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial q^s} = \mathbf{i}_k \frac{\partial a_k}{\partial q^s}, \tag{3.1.5}$$

while the basis in volume V is given by other vectors

$$\mathbf{R}_s = \frac{\partial \mathbf{R}}{\partial q^s} = \mathbf{i}_k \frac{\partial x_k}{\partial q^s}. \tag{3.1.6}$$

The vectors of the cobasis in volumes v and V are constructed according to eq. (E.1.5)

$$\mathbf{r}^s = \frac{1}{2}\epsilon^{skt}\mathbf{r}_k \times \mathbf{r}_t, \tag{3.1.7}$$

$$\mathbf{R}^s = \frac{1}{2}\in^{skt}\mathbf{R}_k \times \mathbf{R}_t. \tag{3.1.8}$$

Here

$$\epsilon^{skt} = \frac{1}{\sqrt{g}}e^{skt}, \ \in^{skt} = \frac{1}{\sqrt{G}}e^{skt}, \tag{3.1.9}$$

with g and G denoting determinants of the matrices of the corresponding covariant components

$$g_{sk} = \mathbf{r}_s \cdot \mathbf{r}_k, \quad G_{sk} = \mathbf{R}_s \cdot \mathbf{R}_k \tag{3.1.10}$$

of the metric tensors $\hat{g}$ and $\hat{G}$ of volumes v and V, so that

$$g = |g_{sk}| = [\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)]^2, \quad G = |G_{sk}| = [\mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3)]^2. \tag{3.1.11}$$

These tensors play the role of unit tensors in volumes v and V and have the following dyadic notion

$$\hat{g} = g_{sk}\mathbf{r}^s\mathbf{r}^k = g^{sk}\mathbf{r}_s\mathbf{r}_k = \mathbf{r}^k\mathbf{r}_k, \tag{3.1.12}$$

$$\hat{G} = G_{sk}\mathbf{R}^s\mathbf{R}^k = G^{sk}\mathbf{R}_s\mathbf{R}_k = \mathbf{R}^k\mathbf{R}_k. \tag{3.1.13}$$

Here the contravariant components of the metric tensors $\hat{g}$ and $\hat{G}$ are introduced in the standard way

$$g^{sk} = \mathbf{r}^s \cdot \mathbf{r}^k, \quad G^{sk} = \mathbf{R}^s \cdot \mathbf{R}^k. \tag{3.1.14}$$

As tensors $\hat{g}$ and $\hat{G}$ are the unit tensors in volumes v and V, respectively, each of these tensors is equal to its inverse tensor

$$\hat{g} = \hat{g}^{-1}, \quad \hat{G} = \hat{G}^{-1}. \tag{3.1.15}$$

Definitions (3.1.7) and (3.1.8) yield the well known relationships

$$\mathbf{r}^s \cdot \mathbf{r}_k = g^s_k = \begin{cases} 1, & s = k, \\ 0, & s \neq k, \end{cases} \quad \mathbf{R}^s \cdot \mathbf{R}_k = G^s_k = \begin{cases} 1, & s = k, \\ 0, & s \neq k. \end{cases} \tag{3.1.16}$$

Inserting $\mathbf{r}_k$ and $\mathbf{R}_k$ from eqs. (3.1.5) and (3.1.6) yields

$$\mathbf{r}^s \cdot \mathbf{i}_t \frac{\partial a_t}{\partial q^k} = g^s_k, \quad \mathbf{R}^s \cdot \mathbf{i}_t \frac{\partial x_t}{\partial q^k} = G^s_k.$$

Then we have

$$\mathbf{r}^s \cdot \mathbf{i}_t \frac{\partial a_t}{\partial q^k} \frac{\partial q^k}{\partial a_r} = \mathbf{r}^s \cdot \mathbf{i}_t \frac{\partial a_t}{\partial a_r} = \mathbf{r}^s \cdot \mathbf{i}_r = g_k^s \frac{\partial q^k}{\partial a_r} = \frac{\partial q^s}{\partial a_r}.$$

Therefore

$$\mathbf{r}^s \cdot \mathbf{i}_r = \frac{\partial q^s}{\partial a_r}, \quad \mathbf{r}^s = \frac{\partial q^s}{\partial a_r} \mathbf{i}_r \tag{3.1.17$_1$}$$

and by analogy

$$\mathbf{R}^s \cdot \mathbf{i}_r = \frac{\partial q^s}{\partial x_r}, \quad \mathbf{R}^s = \frac{\partial q^s}{\partial x_r} \mathbf{i}_r. \tag{3.1.17$_2$}$$

As a sequence we obtain the above mentioned relationships

$$\mathbf{r}_s \mathbf{r}^s = \frac{\partial a_k}{\partial q^s} \frac{\partial q^s}{\partial a_r} \mathbf{i}_k \mathbf{i}_r = \hat{E}, \quad \mathbf{R}_s \mathbf{R}^s = \frac{\partial x_k}{\partial q^s} \frac{\partial q^s}{\partial x_r} \mathbf{i}_k \mathbf{i}_r = \hat{E}, \quad \hat{E} = \hat{g} = \hat{G}. \tag{3.1.18}$$

With the help of eqs. (3.1.10) and (3.1.14), we arrive at the formulae

$$g_{sk} = \frac{\partial a_t}{\partial q^s} \frac{\partial a_t}{\partial q^k}, \quad G_{sk} = \frac{\partial x_t}{\partial q^s} \frac{\partial x_t}{\partial q^k}, \tag{3.1.19}$$

$$g^{sk} = \frac{\partial q^s}{\partial a_t} \frac{\partial q^k}{\partial a_t}, \quad G^{sk} = \frac{\partial q^s}{\partial x_t} \frac{\partial q^k}{\partial x_t}. \tag{3.1.20}$$

The nabla-operator in volume v is determined by the following symbolic vector

$$\nabla = \mathbf{r}^s \frac{\partial}{\partial q^s}, \tag{3.1.21}$$

see eq. (E.4.3). In volume V vectors $\mathbf{R}^s$ play the role of $\mathbf{r}^s$ and the nabla-operator is denoted as follows

$$\tilde{\nabla} = \mathbf{R}^s \frac{\partial}{\partial q^s}. \tag{3.1.22}$$

In what follows, the operations in volumes v and V need to be clearly distinguished from each other. To this end, operations and quantities relating to volume V are marked by a tilde sign $(\sim)$. For example, a vector can be prescribed by its components in the basis of volume v as well as in the basis of volume V. Its covariant and contravariant components in the vector basis of volume v are denoted by a_s and a^s, whilst those in the vector basis of volume V are denoted by $\tilde{a}_s$ and $\tilde{a}^s$, so that

$$\mathbf{a} = a_s \mathbf{r}^s = a^s \mathbf{r}_s = \tilde{a}_s \mathbf{R}^s = \tilde{a}^s \mathbf{R}_s. \tag{3.1.23}$$

According to the above, the tensor equal to the gradient of vector **a** is written in the metric of volume v in the following form

$$\nabla \mathbf{a} = \mathbf{r}^s \frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{r}^s \mathbf{r}_k \nabla_s a^k = \mathbf{r}^s \mathbf{r}^k \nabla_s a_k. \tag{3.1.24}$$

However in the metric of volume V, we have another tensor

$$\tilde{\nabla} \mathbf{a} = \mathbf{R}^s \frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{R}^s \mathbf{R}_k \tilde{\nabla}_s \tilde{a}^k = \mathbf{R}^s \mathbf{R}^k \tilde{\nabla}_s \tilde{a}_k. \tag{3.1.25}$$

The difference is due to the fact that the transposed tensor $(\nabla \mathbf{a})^*$ is defined as the derivative of **a** with respect to direction **r**, while $\left(\tilde{\nabla}\mathbf{a}\right)^*$ is the derivative of **a** with respect to **R**

$$(\nabla \mathbf{a})^* = \frac{d\mathbf{a}}{d\mathbf{r}}, \quad \left(\tilde{\nabla}\mathbf{a}\right)^* = \frac{d\mathbf{a}}{d\mathbf{R}}. \tag{3.1.26}$$

Performing operations of the covariant differentiation in $v-$ and $V-$volumes, it is necessary to distinguish between the Christoffel symbols, namely

$$\nabla_s a^k = \frac{\partial a^k}{\partial q^s} + \left\{ {k \atop st} \right\} a^t, \quad \tilde{\nabla}_s \tilde{a}^k = \frac{\partial \tilde{a}^k}{\partial q^s} + \widetilde{\left\{ {k \atop st} \right\}} \tilde{a}^t. \tag{3.1.27}$$

The symbols with a tilde are calculated by means of components of tensor $\hat{G}$, and when the tilde is absent, the components of $\hat{G}$ should be taken as

$$\left. \begin{aligned} \left\{ {k \atop st} \right\} &= g^{kr} [st, r] = \frac{1}{2} g^{kr} \left(\frac{\partial g_{sr}}{\partial q^t} + \frac{\partial g_{tr}}{\partial q^s} - \frac{\partial g_{st}}{\partial q^r} \right), \\ \widetilde{\left\{ {k \atop st} \right\}} &= G^{kr} \widetilde{[st, r]} = \frac{1}{2} G^{kr} \left(\frac{\partial G_{sr}}{\partial q^t} + \frac{\partial G_{tr}}{\partial q^s} - \frac{\partial G_{st}}{\partial q^r} \right). \end{aligned} \right\} \tag{3.1.28}$$

Let us also notice the formulae

$$\mathbf{r}_s = \frac{1}{2} \epsilon_{stk} \mathbf{r}^t \times \mathbf{r}^k, \quad \mathbf{R}_s = \frac{1}{2} \in_{stk} \mathbf{R}^t \times \mathbf{R}^k, \tag{3.1.29}$$

which are inverse to eqs. (3.1.7) and (3.18). Here

$$\epsilon_{stk} = \mathbf{r}_s \cdot (\mathbf{r}_t \times \mathbf{r}_k) = \sqrt{g} e_{stk}, \; \in_{stk} = \mathbf{R}_s \cdot (\mathbf{R}_t \times \mathbf{R}_k) = \sqrt{G} e_{stk}. \tag{3.1.30}$$

2.3.2 Tensorial gradients $\nabla \mathbf{R}$ *and* $\tilde{\nabla} \mathbf{r}$

Using eqs. (3.1.21) and (3.1.22) and taking into account the definitions of the basis vectors (3.1.5) and (3.1.6) yields the dyadic representations of these tensors

$$\nabla \mathbf{R} = \mathbf{r}^s \frac{\partial \mathbf{R}}{\partial q^s} = \mathbf{r}^s \mathbf{R}_s, \tag{3.2.1}$$

$$\tilde{\nabla} \mathbf{r} = \mathbf{R}^s \frac{\partial \mathbf{r}}{\partial q^s} = \mathbf{R}^s \mathbf{r}_s. \tag{3.2.2}$$

The transposed tensors, which are the derivative of $\mathbf{R}$ with respect to $\mathbf{r}$ and the derivative of $\mathbf{r}$ with respect to $\mathbf{R}$, are given by the equalities

$$(\nabla\mathbf{R})^* = \frac{d\mathbf{R}}{d\mathbf{r}} = \mathbf{R}_s\mathbf{r}^s, \quad \left(\tilde{\nabla}\mathbf{r}\right)^* = \frac{d\mathbf{r}}{d\mathbf{R}} = \mathbf{r}_s\mathbf{R}^s. \tag{3.2.3}$$

These definitions yield the formulae

$$d\mathbf{R} = (\nabla\mathbf{R})^* \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla\mathbf{R}, \tag{3.2.4}$$

$$d\mathbf{r} = \left(\tilde{\nabla}\mathbf{r}\right)^* \cdot d\mathbf{R} = d\mathbf{R} \cdot \tilde{\nabla}\mathbf{r}, \tag{3.2.5}$$

which are crucial for further analysis. One can immediately see that tensors $\nabla\mathbf{R}$ and $\tilde{\nabla}\mathbf{r}$ are mutually inverse to each other

$$\left.\begin{array}{ll} \nabla\mathbf{R}\cdot\tilde{\nabla}\mathbf{r} = \mathbf{r}^s\mathbf{R}_s\cdot\mathbf{R}^k\mathbf{r}_k = \mathbf{r}^s\mathbf{r}_s = \hat{g}, & \nabla\mathbf{R} = \left(\tilde{\nabla}\mathbf{r}\right)^{-1}, \\ \tilde{\nabla}\mathbf{r}\cdot\nabla\mathbf{R} = \mathbf{R}^s\mathbf{r}_s\cdot\mathbf{r}^k\mathbf{R}_k = \mathbf{R}^s\mathbf{R}_s = \hat{G}, & \tilde{\nabla}\mathbf{r} = (\nabla\mathbf{R})^{-1}. \end{array}\right\} \tag{3.2.6}$$

Let us also notice that, in contrast to eqs. (3.2.1) and (3.2.2), tensors $\tilde{\nabla}\mathbf{R}$ and $\nabla\mathbf{r}$ are the unit tensors in the basis of volumes v and V respectively

$$\tilde{\nabla}\mathbf{R} = \mathbf{R}^s\mathbf{R}_s = \hat{G}, \quad \nabla\mathbf{r} = \mathbf{r}^s\mathbf{r}_s = \hat{g}. \tag{3.2.7}$$

2.3.3 The first measure of strain (Cauchy-Green)

As already mentioned in Subsection 2.1.2 vector $\overrightarrow{MN} = d\mathbf{r}$, determined by two infinitesimally close points M and N in volume v, becomes equal to vector $\overrightarrow{M'N'} = d\mathbf{R}$ in volume V. The relation between these vectors is given by eq. (3.2.4). It allows one to find the formula for the square of the linear element dS in volume V

$$d\mathbf{R}\cdot d\mathbf{R} = dS^2 = d\mathbf{r}\cdot\nabla\mathbf{R}\cdot(\nabla\mathbf{R})^*\cdot d\mathbf{r} = d\mathbf{r}\cdot\hat{G}^\times\cdot d\mathbf{r}. \tag{3.3.1}$$

Here we introduced tensor $\hat{G}^\times$referred to in what follows as the first strain measure or the Cauchy strain measure. According to eq. (3.2.1) this tensor is equal to

$$\hat{G}^\times = \nabla\mathbf{R}\cdot(\nabla\mathbf{R})^* = \mathbf{r}^s\mathbf{R}_s\cdot\mathbf{R}_k\mathbf{r}^k = G^\times_{sk}\mathbf{r}^s\mathbf{r}^k. \tag{3.3.2}$$

As follows from the latter equation, this tensor is determined in the vector basis of volume v by the covariant components $G^\times_{sk}$ equal to the covariant components of the unit tensor $\hat{G}$ in volume V, see the dyadic representation of $\hat{G}$, eq. (3.1.13),

$$G^\times_{sk} = G_{sk} = \mathbf{R}_s\cdot\mathbf{R}_k. \tag{3.3.3}$$

It is, however, erroneous to identify tensors $\hat{G}^{\times}$ and $\hat{G}$ relying on this formula. The contravariant components $G^{\times sk}$ of the strain measure tensor are determined according to the general rule (D.5.4) of transformation from covariant to contravariant components

$$G^{\times st} = g^{sr} g^{tq} G_{rq}, \tag{3.3.4}$$

and do not equal the contravariant components G^{sk} of tensor $\hat{G}$. The latter are given by formulae (3.1.14) and are the elements of the matrix inverse to $\|G_{sk}\|$.

We now return to the square of the linear element (3.3.1). Accounting for eqs. (3.3.2) and (3.3.3) we arrive at the familiar representation of dS^2 by a quadratic form of differentials dq^s obtained with the help of the covariant components of tensor $\hat{G}$

$$dS^2 = d\mathbf{r} \cdot \mathbf{r}^s G_{sk} \mathbf{r}^k \cdot d\mathbf{r} = dq^t \mathbf{r}_t \cdot \mathbf{r}^s G_{sk} \mathbf{r}^k \cdot \mathbf{r}_m dq^m = G_{sk} dq^s dq^k . \tag{3.3.5}$$

Formulae (3.1.19) are used for calculating the covariant components of the first strain measure

$$G_{sk} = \mathbf{R}_s \cdot \mathbf{R}_k = \mathbf{i}_t \frac{\partial x_t}{\partial q^s} \cdot \mathbf{i}_m \frac{\partial x_m}{\partial q^k} = \frac{\partial x_t}{\partial q^s} \frac{\partial x_t}{\partial q^k}. \tag{3.3.6}$$

Let us introduce now tensor $G^{\times^{-1}}$ which is the inverse of $G^{\times}$. Referring to eq. (1.7.14) we have

$$G^{\times^{-1}} = (\nabla \mathbf{R} \cdot \nabla \mathbf{R}^*)^{-1} = (\nabla \mathbf{R})^{*-1} \cdot (\nabla \mathbf{R})^{-1} = (\nabla \mathbf{R})^{-1*} \cdot (\nabla \mathbf{R})^{-1}$$

and making use of eqs. (3.2.6), (3.2.2), (3.2.3) and (3.1.14) we obtain

$$G^{\times^{-1}} = \left(\tilde{\nabla} \mathbf{r}\right)^* \cdot \tilde{\nabla} \mathbf{r} = \mathbf{r}_s \mathbf{R}^s \cdot \mathbf{R}^k \mathbf{r}_k = G^{sk} \mathbf{r}_s \mathbf{r}_k. \tag{3.3.7}$$

In the vector basis of volume v the contravariant components of this tensor, denoted by $\hat{m}$ in the following, are equal to the contravariant components of the unit tensor $\hat{G}$ of volume V.

Calculation of components G^{sk} involves the inversion of matrix G_{sk}. Alternatively let us assume for the time being that the Cartesian coordinate of points in volume V are the material coordinates

$$x_s = q^s, \quad \mathbf{R}_s = \mathbf{i}_s = \mathbf{R}^s, \quad \tilde{\nabla} \mathbf{r} = \mathbf{i}_s \frac{\partial \mathbf{r}}{\partial x_s}, \quad \left(\tilde{\nabla} \mathbf{r}\right)^* = \frac{\partial \mathbf{r}}{\partial x_s} \mathbf{i}_k .$$

Then, using eqs. (3.3.7) and (3.1.15) and returning to the material coordinates q^s we have

$$G^{\times^{-1}} = \hat{m} = \frac{\partial \mathbf{r}}{\partial x_s} \frac{\partial \mathbf{r}}{\partial x_s} = \mathbf{r}_t \mathbf{r}_m \frac{\partial q^t}{\partial x_s} \frac{\partial q^m}{\partial x_s}. \tag{3.3.8}$$

Hence, we obtain again formulae (3.1.20)

$$G^{tm} = \frac{\partial q^t}{\partial x_s} \frac{\partial q^m}{\partial x_s}, \tag{3.3.9}$$

and it can be easily proved that matrices $\|G_{sk}\|$ and $\|G^{st}\|$ are the inverse of each other. Indeed, by virtue of eq. (3.3.7),

$$G_{st}G^{tr} = \frac{\partial x_m}{\partial q^s} \frac{\partial x_m}{\partial q^t} \frac{\partial q^t}{\partial x_l} \frac{\partial q^r}{\partial x_l} = \frac{\partial x_m}{\partial q^s} \frac{\partial x_m}{\partial x_l} \frac{\partial q^r}{\partial x_l} = \frac{\partial x_m}{\partial q^s} \frac{\partial q^r}{\partial x^m} = \frac{\partial q^r}{\partial q^s} = \delta^r_s,$$

which is required. Calculation using formula (3.3.9) requires knowledge of the transformation that is the inverse of eq. (3.1.12), i.e. the expression for the material coordinates in terms of the Cartesian coordinates of volume V.

2.3.4 Geometric interpretation of the components of the first strain measure

Let us represent an infinitesimally small vector $d\mathbf{r}$ in eq. (3.3.1) as a product of its absolute value $|d\mathbf{r}| = ds$ and the unit vector representing its direction $\mathbf{e}$. Then we arrive at the equality

$$dS^2 = \mathbf{e} \cdot \hat{G}^\times \cdot \mathbf{e}\, ds^2, \quad \frac{dS}{ds} = \left(\mathbf{e} \cdot \hat{G}^\times \cdot \mathbf{e}\right)^{1/2}. \tag{3.4.1}$$

In particular, directing $d\mathbf{r}$ along the base vector $\mathbf{r}_t$ we have

$$\mathbf{e} = \frac{\mathbf{r}_t}{|\mathbf{r}_t|} = \frac{\mathbf{r}_t}{\sqrt{g_{tt}}} \qquad (\not{\Sigma}_t) \tag{3.4.2}$$

and referring to eq. (3.3.2) we obtain

$$dS^2 = \frac{1}{g_{tt}} \mathbf{r}_t \cdot G_{qm} \mathbf{r}^q \mathbf{r}^m \cdot \mathbf{r}_t ds^2 = \frac{G_{tt}}{g_{tt}} ds^2, \quad \frac{dS}{ds} = \sqrt{\frac{G_{tt}}{g_{tt}}}. \tag{3.4.3}$$

This result presents a geometric interpretation of the diagonal components of matrix $\|G_{sk}\|$. Let δ_t denote the extension of an elementary linear element directed along the basis vector $\mathbf{r}_t$ in volume v, then we have

$$\delta_t = \sqrt{\frac{G_{tt}}{g_{tt}}} - 1, \quad G_{tt} = g_{tt} (1 + \delta_t)^2 \quad (\not{\Sigma}_t). \tag{3.4.4}$$

By eq. (3.2.4), a unit vector $\tilde{\mathbf{e}}$ having direction $d\mathbf{R}$ in volume V is determined by the equality

$$d\mathbf{R} = \tilde{\mathbf{e}} dS = \mathbf{e} ds \cdot \nabla \mathbf{R} = (\nabla \mathbf{R})^* \cdot \mathbf{e} ds,$$

so that, according to eq. (3.4.1), we obtain

$$\tilde{\mathbf{e}} = \frac{\mathbf{e}\cdot\nabla\mathbf{R}}{\sqrt{\mathbf{e}\cdot\hat{G}^{\times}\cdot\mathbf{e}}} = \frac{(\nabla\mathbf{R})^{*}\cdot\mathbf{e}}{\sqrt{\mathbf{e}\cdot\hat{G}^{\times}\cdot\mathbf{e}}}. \tag{3.4.5}$$

Considering now two directions $\mathbf{e}$ and $\mathbf{e}'$ at a point M in volume v with the angle β between them and denoting the corresponding directions in volume V as $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{e}}'$ we obtain

$$\cos\tilde{\beta} = \tilde{\mathbf{e}}\cdot\tilde{\mathbf{e}}' = \frac{\mathbf{e}\cdot\nabla\mathbf{R}\cdot(\nabla\mathbf{R})^{*}\cdot\mathbf{e}'}{\sqrt{\mathbf{e}\cdot\hat{G}^{\times}\cdot\mathbf{e}\mathbf{e}'\cdot\hat{G}^{\times}\cdot\mathbf{e}'}}$$

or, referring to eq. (3.3.2),

$$\cos\tilde{\beta} = \frac{\mathbf{e}\cdot\hat{G}^{\times}\cdot\mathbf{e}'}{\sqrt{\mathbf{e}\cdot\hat{G}^{\times}\cdot\mathbf{e}\mathbf{e}'\cdot\hat{G}^{\times}\cdot\mathbf{e}'}}. \tag{3.4.6}$$

In particular, choosing $\mathbf{e}$ and $\mathbf{e}'$ along the directions of the basis vectors $\mathbf{r}_s$ and $\mathbf{r}_t$ we arrive at the formula

$$\cos\tilde{\beta}_{st} = \frac{G_{st}}{\sqrt{G_{ss}G_{tt}}} = \frac{G_{st}}{\sqrt{g_{ss}g_{tt}}}\frac{1}{(1+\delta_s)(1+\delta_t)} \quad \left(\not\Sigma_{s,t}\right), \tag{3.4.7}$$

highlighting the geometric meaning of the non-diagonal components of matrix $\|G_{st}\|$.

Defining angle φ_{st}, referred to as the angle of shear, by the equality

$$\tilde{\beta}_{st} = \beta_{st} - \varphi_{st},$$

we have

$$\begin{aligned}\cos\tilde{\beta}_{st} &= \cos\beta_{st}\cos\varphi_{st} + \sin\beta_{st}\sin\varphi_{st} \\ &= \frac{1}{\sqrt{g_{ss}g_{tt}}}\left(g_{st}\cos\varphi_{st} + \sqrt{g_{ss}g_{tt} - g_{st}^2}\sin\varphi_{st}\right).\end{aligned}$$

This enables one to put eq. (3.4.7) in the following form

$$g_{st}\cos\varphi_{st} + \sqrt{g_{ss}g_{tt} - g_{st}^2}\sin\varphi_{st} = \frac{G_{st}}{(1+\delta_s)(1+\delta_t)}. \tag{3.4.8}$$

2.3.5 *Change in the oriented surface*

The vector of the oriented surface $\mathbf{n}do$ in volume v can be represented as follows

$$\mathbf{n}do = \frac{1}{2}\mathbf{e}'\times\mathbf{e}''ds'ds'',$$

where $\mathbf{e}'$ and $\mathbf{e}''$ are the unit vectors in the plane of surface do. In volume V this vector transforms into the vector

$$\mathbf{N}dO = \frac{1}{2}\tilde{\mathbf{e}}' \times \tilde{\mathbf{e}}'' dS' dS'',$$

and, due to eqs. (3.4.5), (3.4.1), (3,2,1) and (3.2.3), we have

$$\mathbf{N}dO = \frac{1}{2} ds' ds'' \left[(\nabla \mathbf{R})^* \cdot \mathbf{e}' \right] \times (\mathbf{e}'' \cdot \nabla \mathbf{R}) = \frac{1}{2} ds' ds'' \mathbf{R}_s \times \mathbf{R}_q e'^s e''^q, \tag{3.5.1}$$

where e'^s and e''^q denote respectively the contravariant components of $\mathbf{e}'$ and $\mathbf{e}''$ in the basis of volume v. Referring to eqs. (3.1.20) and (3.1.30) we have

$$\begin{aligned} \mathbf{N}dO &= \frac{1}{2} ds' ds'' \in_{sqt} \mathbf{R}^t e'^s e''^q = \frac{1}{2} ds' ds'' \sqrt{\frac{G}{g}} (\mathbf{r}_s \times \mathbf{r}_q) \cdot \mathbf{r}_t e'^s e''^q \mathbf{R}^t \\ &= \frac{1}{2} \sqrt{\frac{G}{g}} ds' ds'' (\mathbf{e}' \times \mathbf{e}'') \cdot \mathbf{r}_t \mathbf{R}^t = \sqrt{\frac{G}{g}} \mathbf{n} \cdot \mathbf{r}_t \mathbf{R}^t do \end{aligned} \tag{3.5.2}$$

and by eq. (3.2.3)

$$\mathbf{N}dO = \sqrt{\frac{G}{g}} \mathbf{n} \cdot \left(\tilde{\nabla} \mathbf{r} \right)^* do = \sqrt{\frac{G}{g}} \tilde{\nabla} \mathbf{r} \cdot \mathbf{n} do. \tag{3.5.3}$$

It follows from this equation and eq. (3.3.7) that

$$\frac{dO}{do} = \left\{ \frac{G}{g} \left[\mathbf{n} \cdot \left(\tilde{\nabla} \mathbf{r} \right)^* \right] \cdot \left(\tilde{\nabla} \mathbf{r} \cdot \mathbf{n} \right) \right\}^{1/2} = \left[\frac{G}{g} \mathbf{n} \cdot \hat{G}^{\times^{-1}} \cdot \mathbf{n} \right]^{1/2} \tag{3.5.4}$$

provides a geometric interpretation of tensor $\hat{G}^{\times^{-1}}$. It determines the ratio of the areas of the oriented surfaces in volumes V and v in the same way that the deformation measure $\hat{G}^{\times}$ determines the ratio of the lengths of the line elements, see eq. (3.4.1).

Now by virtue of eq. (3.5.3) we have

$$\mathbf{N} = \frac{\tilde{\nabla} \mathbf{r} \cdot \mathbf{n}}{\sqrt{\mathbf{n} \cdot \hat{G}^{\times^{-1}} \cdot \mathbf{n}}} = \frac{(\nabla \mathbf{R})^{-1} \cdot \mathbf{n}}{\sqrt{\mathbf{n} \cdot \hat{G}^{\times^{-1}} \cdot \mathbf{n}}}. \tag{3.5.5}$$

This formula is analogous to eq. (3.4.5).

2.3.6 The first tensor of finite strain

Expressing the position vector $\mathbf{R}$ of a particle in volume V in the equation for the first strain measure in terms of the displacement vector $\mathbf{u}$ introduces a symmetric tensor of second rank referred to as the first tensor of finite strain or the Cauchy-Green tensor. It is denoted by

$$\hat{\mathcal{E}} = \operatorname{Def} \mathbf{u},$$

in contrast to the linear strain tensor $\hat{\varepsilon} = \operatorname{def} \mathbf{u}$. Referring to eqs. (1.1.4) and (3.3.2) and replacing the unit tensor $\hat{E}$ by $\hat{g}$ in the basis of volume v according to eq. (3.2.7), we obtain

$$\hat{G}^{\times} = (\hat{g} + \nabla \mathbf{u}) \cdot \left[\hat{g} + (\nabla \mathbf{u})^*\right] = \hat{g} + \nabla \mathbf{u} + (\nabla \mathbf{u})^* + \nabla \mathbf{u} \cdot (\nabla \mathbf{u})^* . \quad (3.6.1)$$

Introducing the definition

$$\hat{\mathcal{E}} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^* + \nabla \mathbf{u} \cdot (\nabla \mathbf{u})^*\right] = \hat{\varepsilon} + \frac{1}{2} \nabla \mathbf{u} \cdot (\nabla \mathbf{u})^*, \quad (3.6.2)$$

we obtain

$$\hat{G}^{\times} = \hat{g} + 2\hat{\mathcal{E}}, \quad \hat{\mathcal{E}} = \frac{1}{2}\left(\hat{G}^{\times} - \hat{g}\right). \quad (3.6.3)$$

If the Cartesian coordinates a_s in the system of axes $OX_1X_2X_3$ are taken as being the material coordinates and $u_{(k)}$ denotes projections of $\mathbf{u}$ on these axes, then the components of tensor $\hat{\mathcal{E}}$ in these axes have the following form

$$\mathcal{E}_{(ss)} = \varepsilon_{(ss)} + \frac{1}{2}\left[\left(\frac{\partial u_{(1)}}{\partial a_s}\right)^2 + \left(\frac{\partial u_{(2)}}{\partial a_s}\right)^2 + \left(\frac{\partial u_{(3)}}{\partial a_s}\right)^2\right], \quad (3.6.4)$$

$$\mathcal{E}_{(sk)} = \frac{1}{2}\Gamma_{(sk)} = \varepsilon_{(sk)} + \frac{1}{2}\left(\frac{\partial u_{(1)}}{\partial a_s}\frac{\partial u_{(1)}}{\partial a_k} + \frac{\partial u_{(2)}}{\partial a_s}\frac{\partial u_{(2)}}{\partial a_k} + \frac{\partial u_{(3)}}{\partial a_s}\frac{\partial u_{(3)}}{\partial a_k}\right), \quad (3.6.5)$$

where $\varepsilon_{(ss)}$ and $\varepsilon_{(sk)}$ are given by formulae (1.2.7) and (1.2.8) in which u_s is replaced by $u_{(s)}$ in order to distinguish the projections of $\mathbf{u}$ on the axes of the Cartesian system (i.e. the displacements) from the covariant components u_s of this vector in the vector basis $\mathbf{r}_s$.

Expressions for the covariant components $\hat{\mathcal{E}}_{sk}$ of the Cauchy deformation tensor, in terms of the covariant components of the displacement vector, are written, due to eqs. (E.4.5) and (E.4.6) in the form

$$\hat{\mathcal{E}}_{sk} = \frac{1}{2}\Gamma_{sk} = \varepsilon_{sk} + \frac{1}{2} g^{mt} \nabla_s u_m \nabla_k u_t, \quad (3.6.6)$$

where

$$\varepsilon_{sk} = \frac{1}{2}\left(\nabla_s u_k + \nabla_k u_s\right) = \frac{1}{2}\left(\frac{\partial u_s}{\partial q^k} + \frac{\partial u_k}{\partial q^s}\right) - \left\{ \begin{matrix} r \\ sk \end{matrix} \right\} u_r. \quad (3.6.7)$$

The formulae relating components of the tensor of finite strain to extensions δ_t of the elementary lines directed along the basis vectors $\mathbf{r}_s$ in volume v and the angles of shear φ_{st} are obtained directly from eqs. (3.4.4) and (3.4.8) by replacing G_{tt} and G_{st} by $g_{tt} + 2\varepsilon_{tt}$ and $g_{st} + 2\varepsilon_{st}$ respectively. They are set as follows

$$\delta_t = \sqrt{1 + 2\varepsilon_{tt}} - 1, \quad \sin\varphi_{st} = \frac{\Gamma_{st}}{(1+\delta_s)(1+\delta_t)}, \tag{3.6.8}$$

provided that the Cartesian coordinates a_s in volume v are considered as being the material coordinates.

As already mentioned in Subsection 2.1.1 the linear theory of elasticity assumes that the components of tensor $\nabla\mathbf{u}$ are small and neglects the squares of these values compared with the linear terms. Under this condition the tensor of finite strain is replaced by the linear strain tensor

$$\hat{\mathcal{E}} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^*\right] = \hat{\varepsilon}, \quad \mathcal{E}_{st} = \varepsilon_{st} \tag{3.6.9}$$

and, by virtue of formulae (3.6.8),

$$\delta_t = \varepsilon_{tt}, \quad \sin\varphi_{st} \approx \varphi_{st} = \gamma_{st}. \tag{3.6.10}$$

The latter equation explains the terminology of the linear theory of elasticity, namely that the diagonal components of tensor $\hat{\varepsilon}$ are termed extensions while the non-diagonal ones are called shears. The latter represent changes in the right angles between the lines which originally were parallel to the coordinate axes.

More often than not, extensions δ_k and shears φ_{st} turn out to be sufficiently small, which gives grounds to replace formulae (3.6.8) by approximate equalities (3.6.10). However, the smallness of extensions and shears alone can not substantiate replacement of $\hat{\mathcal{E}}$ by $\hat{\varepsilon}$. As already noted, smallness of all the components of the gradient of displacement $\nabla\mathbf{u}$ is required. An example of a rigid-body rotation of a medium is given in Subsection 2.3.8. It will be shown that $\hat{\mathcal{E}} = 0$ whereas $\hat{\varepsilon} \neq 0$ and, moreover, the components of the latter tensor can be arbitrarily large. Evidently, tensors $\hat{\mathcal{E}}$ and $\hat{\varepsilon}$ can not be identified in this particular case. Further analysis is carried out in Subsection 2.3.9.

2.3.7 The principal strains and principal axes of strain

As $\hat{G}^{\times}$ and $\hat{\mathcal{E}}$ are symmetric tensors of second rank, they possess all of the properties listed in Subsections 1.2.1 and 1.2.2. The principal strains denoted as E_s are determined from the characteristic equation for tensor $\hat{\mathcal{E}}$

$$\left|\mathcal{E}_{(st)} - E\delta_{st}\right| = 0 \tag{3.7.1}$$

or in the notation of eq. (D.7.4)

$$\left| g^{tr}\mathcal{E}_{sr} - Eg_s^t \right| = 0. \tag{3.7.2}$$

Analogous to the principal shear stresses in Subsection 1.2.2 we introduce the principal shear strains

$$E_2 - E_3 = \Gamma_1,\ E_1 - E_3 = \Gamma_2,\ E_1 - E_2 = \Gamma_3 \quad (E_1 > E_2 > E_3). \tag{3.7.3}$$

The second invariant of the deviator of tensor $\hat{\mathcal{E}}$ is expressed in terms of the principal shears

$$-I_2\left(\operatorname{Dev}\hat{\mathcal{E}}\right) = \frac{1}{6}\left(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2\right) = \frac{\Gamma^2}{4}. \tag{3.7.4}$$

Here, similar to eq. (2.2.11) of Chapter 1 we introduced the intensity of shear strains

$$\Gamma = \sqrt{\frac{2}{3}\left(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2\right)} = 2\sqrt{-I_2\left(\operatorname{Dev}\hat{\mathcal{E}}\right)}. \tag{3.7.5}$$

2.3.8 Finite rotation of the medium as a rigid body

Under such a displacement, the position vector $\mathbf{r}$ does not change its length and orientation with respect to the coordinate frame rotated together with the medium and becomes equal to, see eq. (A.8.3),

$$\mathbf{r}' = \mathbf{R} = \mathbf{r}\cdot\hat{A} = \mathbf{r}\cdot\mathbf{i}_s\mathbf{i}'_s = a_s\mathbf{i}'_s,$$

where $\hat{A}$ denotes the rotation tensor and $\mathbf{i}'_s$ are the unit base vectors of the rotated axes. In this case $\mathbf{r}_s = \mathbf{i}_s, \mathbf{R}_s = \mathbf{i}'_s$ so that, by virtue of eq. (3.2.1)

$$\nabla\mathbf{R} = \hat{A}, \quad (\nabla\mathbf{R})^* = \hat{A}^*, \quad \hat{G}^\times = \nabla\mathbf{R}\cdot(\nabla\mathbf{R})^* = \hat{E} = \hat{g}$$

and due to eq. (3.6.3)

$$\hat{\mathcal{E}} = 0. \tag{3.8.1}$$

This is what to be expected since a rigid-body displacement of the medium is not accompanied by changes in the elements' length and the angles between them. However the linear strain tensor is not equal to zero

$$\hat{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^*\right] = \frac{1}{2}\left[\nabla\mathbf{R} + (\nabla\mathbf{R})^* - 2\hat{E}\right] = \frac{1}{2}\left(\mathbf{i}_s\mathbf{i}'_s + \mathbf{i}'_s\mathbf{i}_s - 2\mathbf{i}_s\mathbf{i}_s\right). \tag{3.8.2}$$

For example, under a rotation through an angle of 90° about axis $\mathbf{i}_3$ we have

$$\mathbf{i}'_1 = \mathbf{i}_2, \quad \mathbf{i}'_2 = -\mathbf{i}_1, \quad \mathbf{i}'_3 = \mathbf{i}_3, \quad \hat{\varepsilon} = -\left(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2\right)$$

which means that $\varepsilon_{11} = \varepsilon_{22} = -1$.

The linear vector of rotation is given by eq. (1.2.12)

$$\boldsymbol{\omega} = \frac{1}{2}\operatorname{rot}\mathbf{u} = \frac{1}{2}\nabla\times(\mathbf{R}-\mathbf{r}) = \frac{1}{2}\mathbf{i}_s\times\frac{\partial}{\partial a_s}\left(\mathbf{i}'_k - \mathbf{i}_k\right)a_k = \frac{1}{2}\mathbf{i}_s\times\mathbf{i}'_s. \tag{3.8.3}$$

2.3.9 *Expression for the tensor of finite strain in terms of the linear strain tensor and the linear vector of rotation*

Turning to eqs. (1.2.13) and (3.6.2) we have

$$\hat{\mathcal{E}} = \hat{\varepsilon} + \frac{1}{2}(\hat{\varepsilon} - \Omega) \cdot (\hat{\varepsilon} + \Omega) = \hat{\varepsilon} + \frac{1}{2}\left(\hat{\varepsilon}^2 + \hat{\varepsilon} \cdot \hat{\Omega} - \hat{\Omega} \cdot \hat{\varepsilon} - \hat{\Omega}^2\right)$$

and, referring to eq. (A.6.12), we arrive at the formula

$$\hat{\mathcal{E}} = \hat{\varepsilon} + \frac{1}{2}\left[\hat{\varepsilon}^2 + \hat{g}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} - \boldsymbol{\omega} \times \hat{\varepsilon} - (\boldsymbol{\omega} \times \hat{\varepsilon})^*\right] \tag{3.9.1}$$

as $\hat{E} = \hat{g}$ in the basis of volume v. It follows from this equation that conditions

$$|\varepsilon_{sk}| \ll 1 \tag{3.9.2}$$

are not sufficient for identifying tensors $\hat{\mathcal{E}}$ and $\hat{\varepsilon}$ even if the components of tensor $\hat{\mathcal{E}}$ are small. According to eq. (3.9.2) vector $\boldsymbol{\omega}$ is small but it is not inconceivable that values of ε_{sk} have a higher order of smallness than ω_k. In eq. (3.9.1) one needs to retain the terms that are quadratic in $\boldsymbol{\omega}$, that is

$$\hat{\mathcal{E}} = \hat{\varepsilon} + \frac{1}{2}(\hat{g}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega}), \tag{3.9.3}$$

and for $|\varepsilon_{sk}| \ll |\omega_k|$ one can take formula (3.9.1) in the following form

$$\hat{\mathcal{E}} = \frac{1}{2}(\hat{g}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega}). \tag{3.9.4}$$

This formula can be applicable in problems of deforming bodies with dramatic change in dimension in the some directions, for example a thin rod or a thin plate, under certain loading conditions.

At the same time, it follows from eq. (3.9.1) that replacement of tensor $\hat{\mathcal{E}}$ by a linear strain tensor requires the same order of smallness not only of the components ε_{sk} but also the components of vector $\boldsymbol{\omega}$

$$|\varepsilon_{sk}| \ll 1, \quad |\omega_k| \ll 1. \tag{3.9.5}$$

Using eq. (1.1.5) these conditions are equivalent to

$$\left|\frac{\partial u_k}{\partial a_s}\right| \ll 1. \tag{3.9.6}$$

2.4 The second measure and the second tensor of finite strain

2.4.1 *The second measure of finite strain*

Introducing the first measure of strain $\hat{G}^{\times}$ and the inverse tensor $\hat{G}^{\times^{-1}}$ allows us to show ways of determining geometric quantities such as line lengths, angles and oriented surfaces, in volume V in terms of those in volume v. The inverse problem is considered here, namely determining the quantities in volume v in terms of those in volume V. Clearly, the solution reduces to replacing vector $\mathbf{r}$ by $\mathbf{R}$ and vice versa in the constructions of Section 2.3. We will consider both vectors as being functions of the material coordinates q^s.

Instead of eq. (3.2.4) the starting relationship is now eq. (3.2.5)

$$d\mathbf{r} = d\mathbf{R} \cdot \tilde{\nabla}\mathbf{r} = \left(\tilde{\nabla}\mathbf{r}\right)^* \cdot d\mathbf{R}.$$

Assuming $d\mathbf{R} = \tilde{\mathbf{e}}\,|d\mathbf{R}| = \tilde{\mathbf{e}}dS$ and $d\mathbf{r} = \mathbf{e}\,|d\mathbf{r}| = \mathbf{e}ds$ we obtain

$$d\mathbf{r} \cdot d\mathbf{r} = ds^2 = \tilde{\mathbf{e}} \cdot \tilde{\nabla}\mathbf{r} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^* \cdot \tilde{\mathbf{e}}dS^2 = \tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}dS^2. \tag{4.1.1}$$

The introduced symmetric tensor of second rank

$$\hat{g}^{\times} = \tilde{\nabla}\mathbf{r} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^* = \mathbf{R}^s\mathbf{r}_s \cdot \mathbf{r}_k\mathbf{R}^k = g_{sk}\mathbf{R}^s\mathbf{R}^k \tag{4.1.2}$$

is referred to as the second measure of strain. Its components g_{sk} in the vector basis of volume V are equal to the covariant components of the unit tensor $\hat{g}$ in volume v, however these tensor can not be identified. The contravariant components of the strain measure $\hat{g}^{\times}$ are given by

$$g^{\times^{sk}} = G^{sr}G^{kq} \cdot g_{rq}\,. \tag{4.1.3}$$

In order to find the covariant components of $\hat{g}^{\times}$ we use the following formulae

$$g_{sk} = \mathbf{i}_t \frac{\partial a_t}{\partial q^s} \cdot \mathbf{i}_q \frac{\partial a_q}{\partial q^k} = \frac{\partial a_t}{\partial q^s} \frac{\partial a_t}{\partial q^k}. \tag{4.1.4}$$

Tensor $\hat{g}^{\times^{-1}} = \hat{M}$, which is the inverse of $\hat{g}^{\times}$, is as follows

$$\hat{g}^{\times^{-1}} = \left[\tilde{\nabla}\mathbf{r} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^*\right]^{-1} = \left(\tilde{\nabla}\mathbf{r}\right)^{*-1} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^{-1} = (\nabla\mathbf{R})^* \cdot \nabla\mathbf{R}, \tag{4.1.5}$$

where equalities (3.2.6) were used. By eq. (3.2.3) we have

$$\hat{g}^{\times^{-1}} = \mathbf{R}_s\mathbf{r}^s \cdot \mathbf{r}^k\mathbf{R}_k = g^{sk}\mathbf{R}_s\mathbf{R}_k\,. \tag{4.1.6}$$

The components of this tensor are elements of the matrix inverse to g_{sk}. Repeating the calculation similar to eq. (3.3.9) we obtain

$$g^{sk} = \frac{\partial q^s}{\partial a_r} \frac{\partial q^k}{\partial a_r}. \tag{4.1.7}$$

2.4.2 The geometric meaning of the component of the second measure of strain

By eq. (4.1.1)

$$\frac{dS}{ds} = \left(\tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}\right)^{-1/2}. \tag{4.2.1}$$

In particular, considering the direction of basic vector $\mathbf{R}_k$ in volume V we return to formulae (3.4.3) and (3.4.4)

$$\frac{dS}{ds} = \left(\frac{g_{kk}}{G_{kk}}\right)^{-1/2} = 1 + \delta_k, \quad \frac{g_{kk}}{G_{kk}} = \frac{1}{\left(1 + \delta_k\right)^2} \quad (\not{\Sigma}_k). \tag{4.2.2}$$

By analogy with eq. (3.4.5), we obtain the following relationship

$$\mathbf{e} = \frac{\tilde{\mathbf{e}} \cdot \tilde{\nabla}\mathbf{r}}{\sqrt{\tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}}} = \frac{\left(\tilde{\nabla}\mathbf{r}\right)^{*} \cdot \tilde{\mathbf{e}}}{\sqrt{\tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}}} \tag{4.2.3}$$

determining the unit vector in volume v which has direction $\tilde{\mathbf{e}}$ in volume V. Then we have

$$\mathbf{e} \cdot \mathbf{e}' = \cos\beta = \frac{\tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}'}{\sqrt{\tilde{\mathbf{e}} \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}\tilde{\mathbf{e}}' \cdot \hat{g}^{\times} \cdot \tilde{\mathbf{e}}'}} \tag{4.2.4}$$

and directing $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{e}}'$ along the basis vectors $\mathbf{R}_s$ and $\mathbf{R}_t$ in volume V we arrive at the formula

$$\cos\beta_{st} = \frac{g_{st}}{\sqrt{g_{ss}g_{tt}}} = \frac{\mathbf{r}_s \cdot \mathbf{r}_t}{|\mathbf{r}_s|\,|\mathbf{r}_t|} \quad (\not{\Sigma}_{st}). \tag{4.2.5}$$

Let us also notice the following equations

$$\frac{dO}{do} = \sqrt{\frac{G}{g}}\left(\mathbf{N} \cdot \hat{g}^{\times^{-1}} \cdot \mathbf{N}\right)^{-1/2}, \quad \mathbf{n} = \frac{\nabla\mathbf{R} \cdot \mathbf{N}}{\sqrt{\mathbf{N} \cdot \hat{g}^{\times^{-1}} \cdot \mathbf{N}}} = \frac{\left(\tilde{\nabla}\mathbf{r}\right)^{-1} \cdot \mathbf{N}}{\sqrt{\mathbf{N} \cdot \hat{g}^{\times^{-1}} \cdot \mathbf{N}}}, \tag{4.2.6}$$

which are analogous to formulae (3.5.4) and (3.5.5).

2.4.3 The second tensor of finite strain (Almansi-Hamel)

Introducing into consideration the displacement vector and referring to eq. (3.2.7), we have

$$\tilde{\nabla}\mathbf{r} = \tilde{\nabla}\left(\mathbf{R} - \mathbf{u}\right) = \hat{G} - \tilde{\nabla}\mathbf{u}, \tag{4.3.1}$$

so that, due to eq. (3.2.7)

$$\hat{g}^{\times} = \tilde{\nabla}\mathbf{r} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^{*} = \hat{G} - \left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{*}\right] + \nabla\mathbf{u} \cdot (\nabla\mathbf{u})^{*} = \hat{G} - 2\widehat{\tilde{\mathcal{E}}}. \quad (4.3.2)$$

Here we introduced the Almansi-Hamel strain tensor

$$\widehat{\tilde{\mathcal{E}}} = \frac{1}{2}\left(\hat{G} - \hat{g}^{\times}\right) = \widehat{\tilde{\varepsilon}} - \frac{1}{2}\tilde{\nabla}\mathbf{u} \cdot \left(\tilde{\nabla}\mathbf{u}\right)^{*}, \quad (4.3.3)$$

where $\widehat{\tilde{\varepsilon}}$ designates the linear strain tensor

$$\widehat{\tilde{\varepsilon}} = \frac{1}{2}\left[\tilde{\nabla}\mathbf{u} + \left(\tilde{\nabla}\mathbf{u}\right)^{*}\right] = \frac{1}{2}\left(\frac{\partial \tilde{u}_s}{\partial q^t} + \frac{\partial \tilde{u}_t}{\partial q^s} - \widetilde{\left\{ \begin{matrix} r \\ st \end{matrix} \right\}}\tilde{u}_r\right)\mathbf{R}^s\mathbf{R}^t \quad (4.3.4)$$

calculated in the basis of volume V. Considering the Cartesian coordinates x_s in volume V as material ones we also have

$$\tilde{\mathcal{E}}_{sk} = \frac{1}{2}\left[\left(\frac{\partial u_{(s)}}{\partial x_k} + \frac{\partial u_{(k)}}{\partial x_s}\right) - \frac{\partial u_{(t)}}{\partial x_s}\frac{\partial u_{(t)}}{\partial x_k}\right]. \quad (4.3.5)$$

The Almansi-Hamel strain tensor is expressed in terms of the linear strain tensor and the linear vector of rotation by the formula

$$\widehat{\tilde{\mathcal{E}}} = \widehat{\tilde{\varepsilon}} - \frac{1}{2}\left[\widehat{\tilde{\varepsilon}}^{2} + \hat{G}\tilde{\omega} \cdot \tilde{\omega} - \tilde{\omega}\tilde{\omega} - \tilde{\omega} \times \widehat{\tilde{\varepsilon}} - \left(\tilde{\omega} \times \widehat{\tilde{\varepsilon}}\right)^{*}\right], \quad (4.3.6)$$

which is analogous to eq. (3.9.1).

Comparing equalities (3.6.3) and (4.3.2)

$$\left.\begin{aligned} \hat{\mathcal{E}} &= \mathcal{E}_{sk}\mathbf{r}^s\mathbf{r}^k = \frac{1}{2}\left(\hat{G}^{\times} - \hat{g}\right) = \frac{1}{2}\left(G_{sk} - g_{sk}\right)\mathbf{r}^s\mathbf{r}^k, \\ \widehat{\tilde{\mathcal{E}}} &= \tilde{\mathcal{E}}_{sk}\mathbf{R}^s\mathbf{R}^k = \frac{1}{2}\left(\hat{G} - \hat{g}^{\times}\right) = \frac{1}{2}\left(G_{sk} - g_{sk}\right)\mathbf{R}^s\mathbf{R}^k, \end{aligned}\right\} \quad (4.3.7)$$

yields the following

$$\mathcal{E}_{sk} = \tilde{\mathcal{E}}_{sk}. \quad (4.3.8)$$

The covariant components of tensor $\hat{\mathcal{E}}$ in the basis of volume v and those of $\widehat{\tilde{\mathcal{E}}}$ in the basis of volume V are coincident. However it would be erroneous to set these tensors equal, i.e. $\hat{\mathcal{E}} \neq \widehat{\tilde{\mathcal{E}}}$.

2.5 Relation between the strain measures

2.5.1 Strain measures and the inverse tensors

In Sections 3 and 4 of the present chapter we introduced into consideration four strain measures: the Cauchy measure $\hat{G}^{\times}$ and the tensor $\hat{M} = \hat{g}^{\times^{-1}}$

which is the inverse of measure $\hat{g}^{\times}$

$$\hat{G}^{\times} = \nabla\mathbf{R} \cdot (\nabla\mathbf{R})^{*}, \quad \hat{g}^{\times^{-1}} = \left(\tilde{\nabla}\mathbf{r}\right)^{*^{-1}} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^{-1} = (\nabla\mathbf{R})^{*} \cdot (\nabla\mathbf{R}) = \hat{M}, \tag{5.1.1}$$

as well as the second measure $\hat{g}^{\times}$ and the tensor which is the inverse of the Cauchy measure $\hat{m} = \hat{G}^{\times^{-1}}$

$$\hat{g}^{\times} = \tilde{\nabla}\mathbf{r} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^{*}, \quad \hat{G}^{\times^{-1}} = (\nabla\mathbf{R})^{*^{-1}} \cdot (\nabla\mathbf{R}) = \left(\tilde{\nabla}\mathbf{r}\right)^{*} \cdot \tilde{\nabla}\mathbf{r} = \hat{m}. \tag{5.1.2}$$

Tensors $\hat{G}^{\times}$ and $\hat{m}$ are determined in the basis of volume v

$$\hat{G}^{\times} = G_{sk}\mathbf{r}^{s}\mathbf{r}^{k}, \qquad \hat{m} = \hat{G}^{\times^{-1}} = G^{sk}\mathbf{r}_{s}\mathbf{r}_{k}, \tag{5.1.3}$$

where G_{sk} and G^{sk} denote the covariant and contravariant components of the unit tensor of volume V, respectively.

Representations of $\hat{g}^{\times}$ and $\hat{M}$ in the basis of volume V have the form

$$\hat{g}^{\times} = g_{sk}\mathbf{R}^{s}\mathbf{R}^{k}, \qquad \hat{M} = g^{sk}\mathbf{R}_{s}\mathbf{R}_{k}, \tag{5.1.4}$$

with g_{sk} and g^{sk} denoting respectively the covariant and contravariant components of the unit tensor of volume v.

The formulae for the components of the introduced tensors in the Cartesian system of axes $OX_1X_2X_3$ are written as follows

$$G_{(sk)} = \frac{\partial x_t}{\partial a_s}\frac{\partial x_t}{\partial a_k}, \quad M_{(sk)} = \frac{\partial x_s}{\partial a_t}\frac{\partial x_k}{\partial a_t}, \tag{5.1.5}$$

$$g_{(sk)} = \frac{\partial a_t}{\partial x_s}\frac{\partial a_t}{\partial x_k}, \quad m_{(sk)} = \frac{\partial a_s}{\partial x_t}\frac{\partial a_k}{\partial x_t}. \tag{5.1.6}$$

2.5.2 Relationships between the invariants

It is known, see eq. (A.9.16), that the tensors $\hat{Q}\cdot\hat{Q}^{*}$ and $\hat{Q}^{*}\cdot\hat{Q}$ have the same principal values. For this reason, denoting the principal values of tensors (5.1.1) and (5.1.2) as G_s, M_s, g_s, m_s we have

$$G_s = M_s, \quad g_s = m_s. \tag{5.2.1}$$

Along with this, the principal values of tensor $\hat{Q}^{-1}$ are equal to the inverse of the principal values of $\hat{Q}$. Thus

$$G_s = \frac{1}{m_s} = \frac{1}{g_s}, \quad g_s = \frac{1}{M_s} = \frac{1}{G_s}. \tag{5.2.2}$$

This yields the following relationships, see eqs. (A.10.15) and (A.10.16),

$$I_s\left(\hat{G}^\times\right) = I_s\left(\hat{M}\right), \quad I_s\left(\hat{g}^\times\right) = I_s\left(\hat{m}\right), \tag{5.2.3}$$

$$I_1\left(\hat{G}^\times\right) = \frac{I_2\left(\hat{g}^\times\right)}{I_3\left(\hat{g}^\times\right)}, \quad I_2\left(\hat{G}^\times\right) = \frac{I_1\left(\hat{g}^\times\right)}{I_3\left(\hat{g}^\times\right)}, \quad I_3\left(\hat{G}^\times\right) = \frac{1}{I_3\left(\hat{g}^\times\right)} \tag{5.2.4}$$

and, of course, the formulae for the inverses

$$I_1\left(\hat{g}^\times\right) = \frac{I_2\left(\hat{G}^\times\right)}{I_3\left(\hat{G}^\times\right)}, \quad I_2\left(\hat{g}^\times\right) = \frac{I_1\left(\hat{G}^\times\right)}{I_3\left(\hat{G}^\times\right)}, \quad I_3\left(\hat{g}^\times\right) = \frac{1}{I_3\left(\hat{G}^\times\right)}. \tag{5.2.5}$$

Formulae determining the principal invariants of the strain measures $\hat{G}^\times$ and $\hat{g}^\times$ should be added to the above relations. By eqs. (5.1.3) and (5.1.4) we have

$$I_1\left(\hat{G}^\times\right) = G_{sk}\mathbf{r}^s \cdot \mathbf{r}^k = g^{sk}G_{sk}, \quad I_1\left(\hat{g}^\times\right) = g_{sk}\mathbf{R}^s \cdot \mathbf{R}^k = G^{sk}g_{sk}, \tag{5.2.6}$$

which is in agreement with definition (D.7.5). Furthermore, due to eq. (D.7.6)

$$I_3\left(\hat{G}^\times\right) = \frac{|G_{sk}|}{g} = \frac{G}{g}, \quad I_3\left(\hat{g}^\times\right) = \frac{|g_{sk}|}{G} = \frac{g}{G}. \tag{5.2.7}$$

Now, by virtue of eqs. (5.2.4) and (5.2.5), we obtain

$$I_2\left(\hat{G}^\times\right) = \frac{G}{g}g_{sk}G^{sk}, \quad I_2\left(\hat{g}^\times\right) = \frac{g}{G}g^{sk}G_{sk}, \tag{5.2.8}$$

which corresponds to eq. (D.7.10).

2.5.3 Representation of the strain measures in terms of the principal axes

Let $\overset{q}{\mathbf{e}}$ denote the unit vectors of the principal axes of the strain measures $\hat{G}^\times$ and $\hat{G}^{\times^{-1}}$

$$\hat{G}^\times = \sum_{s=1}^{3} G_s \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}, \quad \hat{G}^{\times^{-1}} = \sum_{s=1}^{3} \frac{\overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}}{G_s}. \tag{5.3.1}$$

Taking $\mathbf{e} = \overset{q}{\mathbf{e}}$ in eq. (3.4.5) yields

$$\mathbf{e}\cdot\hat{G}^\times\cdot\mathbf{e} = \overset{s}{\mathbf{e}}\cdot\hat{G}^\times\cdot\overset{s}{\mathbf{e}} = G_s, \quad \overset{s}{\tilde{\mathbf{e}}} = \frac{\overset{s}{\mathbf{e}}\cdot\nabla\mathbf{R}}{\sqrt{G_s}} = \frac{\nabla\mathbf{R}^*\cdot\overset{s}{\mathbf{e}}}{\sqrt{G_s}}, \tag{5.3.2}$$

so that

$$\hat{A} = \overset{s}{\mathbf{e}}\overset{s}{\tilde{\mathbf{e}}} = \sum_{s=1}^{3} \frac{\overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}}{G_s} \cdot \nabla\mathbf{R} = \hat{G}^{\times^{-1/2}} \cdot \nabla\mathbf{R}. \tag{5.3.3}$$

This determines the tensor of rotation of the principal axes of tensor $\hat{G}^{\times}$ under deformation of volume v. The trihedrons of the principal axes of the tensors $\hat{G}^{\times}$ and $\hat{M}$ having coincident principal values are related by a transformation of rotation , see eq. (A.9.17), and the rotation of tensor $\hat{G}^{\times}$ is carried out by the rotation tensor (5.3.3). For this reason, $\hat{M}$ is just a "rotated tensor $\hat{G}^{\times}$" and due to eq. (A.9.17)

$$\hat{M} = \hat{A}^* \cdot \hat{G}^{\times} \cdot \hat{A} = \sum_{s=1}^{3} \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\mathbf{e}} \cdot \hat{G}^{\times} \cdot \sum_{q=1}^{3} \overset{q}{\mathbf{e}}\overset{q}{\tilde{\mathbf{e}}} = \sum_{s=1}^{3} G_s \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}, \tag{5.3.4}$$

which also implies that

$$\hat{g}^{\times} = \hat{M}^{-1} = \sum_{s=1}^{3} \frac{\overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}}{G_s}. \tag{5.3.5}$$

Let us also notice that relation (5.3.3), reset in the form

$$\nabla\mathbf{R} = \hat{G}^{\times^{1/2}} \cdot \hat{A}, \quad \nabla\mathbf{R}^* = \hat{A}^* \cdot \hat{G}^{\times^{1/2}}, \tag{5.3.6}$$

is in agreement with representation (A.10.17) for the non-symmetric tensor in the form of a symmetric tensor premultiplied or postmultiplied by a rotation tensor. Equation (5.3.6) yields relationship (5.3.4)

$$\nabla\mathbf{R}^* \cdot \nabla\mathbf{R} = \hat{M} = \hat{A}^* \cdot \hat{G}^{\times} \cdot \hat{A}.$$

A sequence of formulae (5.3.6) and (3.2.6) are representations of tensors $\tilde{\nabla}\mathbf{r}$ and $\tilde{\nabla}\mathbf{r}^*$ in the form

$$\tilde{\nabla}\mathbf{r} = \hat{A}^* \cdot \hat{G}^{\times^{-1/2}}, \quad \tilde{\nabla}\mathbf{r}^* = \hat{G}^{\times^{-1/2}} \cdot \hat{A}. \tag{5.3.7}$$

Given the original transformations (3.1.1) and (3.1.2) of volume v into volume V, determination of the rotation tensor requires knowledge of tensor $\hat{G}^{\times^{-1/2}}$. To this end, the principal directions and principal values of tensor $\hat{G}^{\times}$ need to be found. Another approach is based upon searching the components of tensor $\hat{G}^{\times^{-1/2}}$. By eq. (A.6.9) this reduces to the following system of equations

$$G^{-1/2}_{(st)} G^{-1/2}_{(tq)} = G^{-1}_{(sq)}. \tag{5.3.8}$$

The procedure for solving this system involves searching the principal directions and principal values of tensor $\hat{G}^{\times}$. This procedure is simplified considerably when a plane field of displacement is studied, see Subsection 2.6.2.

2.5.4 The invariants of the tensors of finite strain

These are calculated in terms of the invariants of strain measures $\hat{G}^\times$ and $\hat{g}^\times$ by means of the following relationships

$$\left.\begin{aligned}\hat{\mathcal{E}} = \frac{1}{2}\left(\hat{G}^\times - \hat{E}\right), \hat{\mathcal{E}}^2 = \frac{1}{4}\left(\hat{G}^{\times^2} - 2\hat{G}^\times + \hat{E}\right), \\ \hat{\mathcal{E}}^3 = \frac{1}{8}\left(\hat{G}^{\times^3} - 3\hat{G}^{\times^2} + 3\hat{G}^\times - \hat{E}\right), \hat{\tilde{\mathcal{E}}} = \frac{1}{2}\left(\hat{E} - \hat{g}^\times\right), \\ \hat{\tilde{\mathcal{E}}}^2 = \frac{1}{4}\left(\hat{g}^{\times^2} - 2\hat{g}^\times + \hat{E}\right), \hat{\tilde{\mathcal{E}}}^3 = \frac{1}{8}\left(-\hat{g}^{\times^3} + 3\hat{g}^{\times^2} - 3\hat{g}^\times + \hat{E}\right)\end{aligned}\right\} \tag{5.4.1}$$

and formulae

$$\left.\begin{aligned}I_2\left(\hat{Q}\right) = \frac{1}{2}\left[I_1^2\left(\hat{Q}\right) - I_1\left(\hat{Q}^2\right)\right], \\ I_3\left(\hat{Q}\right) = \frac{1}{3}\left[I_1\left(\hat{Q}^3\right) - I_1\left(\hat{Q}\right) I_1\left(\hat{Q}^2\right) + I_1\left(\hat{Q}\right) I_2\left(\hat{Q}\right)\right],\end{aligned}\right\} \tag{5.4.2}$$

see eqs. (A.10.10) and (A.10.11). The result is

$$I_1\left(\hat{\mathcal{E}}\right) = \frac{1}{2}\left[I_1\left(\hat{G}^\times\right) - 3\right], \quad I_2\left(\hat{\mathcal{E}}\right) = \frac{1}{4}\left[I_2\left(\hat{G}^\times\right) - 2I_1\left(\hat{G}^\times\right) + 3\right],$$
$$I_3\left(\hat{\mathcal{E}}\right) = \frac{1}{8}\left[I_3\left(\hat{G}^\times\right) - I_2\left(\hat{G}^\times\right) + I_1\left(\hat{G}^\times\right) - 1\right], \tag{5.4.3}$$

$$I_1\left(\hat{\tilde{\mathcal{E}}}\right) = \frac{1}{2}\left[3 - I_1\left(\hat{g}^\times\right)\right], \quad I_2\left(\hat{\tilde{\mathcal{E}}}\right) = \frac{1}{4}\left[I_2\left(\hat{g}^\times\right) - 2I_1\left(\hat{g}^\times\right) + 3\right],$$
$$I_3\left(\hat{\tilde{\mathcal{E}}}\right) = \frac{1}{8}\left[-I_3\left(\hat{g}^\times\right) + I_2\left(\hat{g}^\times\right) - I_1\left(\hat{g}^\times\right) + 1\right], \tag{5.4.4}$$

The inverse relationships are as follows

$$I_1\left(\hat{G}^\times\right) = 2I_1\left(\hat{\mathcal{E}}\right) + 3, \quad I_2\left(\hat{G}^\times\right) = 4I_2\left(\hat{\mathcal{E}}\right) + 4I_1\left(\hat{\mathcal{E}}\right) + 3,$$
$$I_3\left(\hat{G}^\times\right) = 1 + 2I_1\left(\hat{\mathcal{E}}\right) + 4I_2\left(\hat{\mathcal{E}}\right) + 8I_3\left(\hat{\mathcal{E}}\right), \tag{5.4.5}$$

$$I_1\left(\hat{g}^\times\right) = 3 - 2I_1\left(\hat{\tilde{\mathcal{E}}}\right), \quad I_2\left(\hat{g}^\times\right) = 3 - 4I_1\left(\hat{\tilde{\mathcal{E}}}\right) + 4I_2\left(\hat{\tilde{\mathcal{E}}}\right),$$
$$I_3\left(\hat{g}^\times\right) = 1 - 2I_1\left(\hat{\tilde{\mathcal{E}}}\right) + 4I_2\left(\hat{\tilde{\mathcal{E}}}\right) - 8I_3\left(\hat{\tilde{\mathcal{E}}}\right). \tag{5.4.6}$$

The dependences between the principal invariants of the strain tensors $\hat{\mathcal{E}}$ and $\hat{\tilde{\mathcal{E}}}$ are more complicated and can be obtained by means of formulae (5.2.4). For example,

$$I_1\left(\hat{g}^\times\right) = 3 - 2I_1\left(\hat{\tilde{\mathcal{E}}}\right) = \frac{I_2\left(\hat{G}^\times\right)}{I_3\left(\hat{G}^\times\right)}$$

and due to eq. (5.4.5)

$$I_1\left(\hat{\tilde{\mathcal{E}}}\right) = \frac{I_1\left(\hat{\mathcal{E}}\right) + 4I_2\left(\hat{\mathcal{E}}\right) + 12I_3\left(\hat{\mathcal{E}}\right)}{1 + 2I_1\left(\hat{\mathcal{E}}\right) + 4I_2\left(\hat{\mathcal{E}}\right) + 8I_3\left(\hat{\mathcal{E}}\right)}. \tag{5.4.7}$$

2.5.5 Dilatation

The elementary volumes of the medium in the initial and actual state are as follows

$$d\tau_0 = \mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)\, dq^1 dq^2 dq^3 = \sqrt{g}\mathbf{r}_1 \cdot \mathbf{r}^1 dq^1 dq^2 dq^3 = \sqrt{g} dq^1 dq^2 dq^3,$$
$$d\tau = \mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3)\, dq^1 dq^2 dq^3 = \sqrt{G}\mathbf{R}_1 \cdot \mathbf{R}^1 dq^1 dq^2 dq^3 = \sqrt{G} dq^1 dq^2 dq^3.$$

This yields

$$\frac{d\tau}{d\tau_0} = \sqrt{\frac{G}{g}}, \quad D = \frac{d\tau - d\tau_0}{d\tau_0} = \sqrt{\frac{G}{g}} - 1. \tag{5.5.1}$$

The quantity D which is a relative change of the elementary volume under deformation is called dilatation.

Referring to definition (D.7.6) of the third invariant in a non-orthogonal basis, eq. (D.7.6), we have

$$I_3\left(\hat{G}^\times\right) = \frac{1}{g}\left|G_{sk}\right| = \frac{G}{g} = (1 + D)^2 \tag{5.5.2}$$

or by virtue of eq. (5.4.5)

$$D = \left[1 + 2I_1\left(\hat{\mathcal{E}}\right) + 4I_2\left(\hat{\mathcal{E}}\right) + 8I_3\left(\hat{\mathcal{E}}\right)\right]^{1/2} - 1. \tag{5.5.3}$$

Taking into account eqs. (5.2.5) and (5.4.6) we can reset this equation in the following form

$$D = \left[1 - 2I_1\left(\hat{\tilde{\mathcal{E}}}\right) + 4I_2\left(\hat{\tilde{\mathcal{E}}}\right) - 8I_3\left(\hat{\tilde{\mathcal{E}}}\right)\right]^{-1/2} - 1. \tag{5.5.4}$$

A derivation of formulae (5.5.3) relies on the fact that the volume of a unit cube with edges, directed along the principal axes $\overset{q}{\mathbf{e}}$ of tensor $\hat{G}^\times$ in volume v, is equal to

$$(1 + \delta_1)(1 + \delta_2)(1 + \delta_3) = \sqrt{(1 + 2\mathcal{E}_1)(1 + 2\mathcal{E}_2)(1 + 2\mathcal{E}_3)} = D + 1$$

in volume V. Here δ_s denotes the principal extensions, whilst $\mathcal{E}_s$ denotes principal values of tensor $\hat{\mathcal{E}}$, see also eq. (3.6.8). It only remains to refer to formulae (A.10.4)-(A.10.6) which relate the principal invariants to the principal values of the tensor.

In a linear approximation, in which tensor $\hat{\mathcal{E}}$ is identified with a linear strain tensor $\hat{\varepsilon}$, the dilatation usually denoted as ϑ (instead of D) is set in the form

$$\vartheta = I_1(\hat{\varepsilon}) = g^{sk}\varepsilon_{sk}, \tag{5.5.5}$$

see eq. (5.5.3) as well as (D.7.5). Another form is due to eqs. (E.4.4) and (E.4.6)

$$\vartheta = g^{sk}\frac{1}{2}(\nabla_s u_k + \nabla_k u_s) = \nabla_s u^s = \operatorname{div}\mathbf{u} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}u^r}{\partial q^r}. \tag{5.5.6}$$

2.5.6 Similarity transformation

We consider two initial states of the medium: volume v and volume v_*, the latter being obtained from the first state by the similarity transformation

$$\mathbf{r}^* = K\mathbf{r}, \quad \mathbf{a}_{s*} = K\mathbf{a}_s. \tag{5.6.1}$$

Then the strain measures corresponding to the first and second initial states are related as follows

$$\hat{G}^\times_* = \frac{1}{K^2}\hat{G}^\times, \quad \hat{g}^\times_* = K^2\hat{g}^\times, \tag{5.6.2}$$

which follows directly from eq. (3.3.6). In order to prove this, it is sufficient to introduce Cartesian coordinates a_s in volume v.

The relation between the principal invariants of the strain measures I_k and I_k^* is given by the evident formulae

$$\begin{gathered}I_1\left(\hat{G}^\times_*\right) = \frac{1}{K^2}I_1\left(\hat{G}^\times\right), \; I_2\left(\hat{G}^\times_*\right) = \frac{1}{K^4}I_2\left(\hat{G}^\times\right), \; I_3\left(\hat{G}^\times_*\right) = \frac{1}{K^6}I_3\left(\hat{G}^\times\right) \\ I_1\left(\hat{g}^\times_*\right) = K^2 I_1\left(\hat{g}^\times\right), \; I_2\left(\hat{g}^\times_*\right) = K^4 I_2\left(\hat{g}^\times\right), \; I_3\left(\hat{g}^\times_*\right) = K^6 I_3\left(\hat{g}^\times\right).\end{gathered} \tag{5.6.3}$$

The formulae relating the invariants of the tensors of finite strain have a more complicated appearance. Using eqs. (5.6.3) and (5.4.3) we have

$$\begin{gathered}I_1\left(\hat{\mathcal{E}}_*\right) = \frac{1}{K^2}\left[I_1\left(\hat{\mathcal{E}}\right) + \frac{3}{2}\beta\right], \; I_2\left(\hat{\mathcal{E}}_*\right) = \frac{1}{K^4}\left[I_2\left(\hat{\mathcal{E}}\right) + \beta I_1\left(\hat{\mathcal{E}}\right) + \frac{3}{4}\beta^2\right], \\ I_3\left(\hat{\mathcal{E}}_*\right) = \frac{1}{K^6}\left[I_3\left(\hat{\mathcal{E}}\right) + \frac{1}{2}\beta I_2\left(\hat{\mathcal{E}}\right) + \frac{1}{4}\beta^2 I_1\left(\hat{\mathcal{E}}\right) + \frac{1}{8}\beta^3\right],\end{gathered} \tag{5.6.4}$$

where

$$\beta = 1 - K^2. \tag{5.6.5}$$

Similar equations in which K^2 and β are replaced respectively by K^{-2} and $K^{-2} - 1$ relate the invariants of the strain tensors $\widehat{\widehat{\mathcal{E}}}_*$ and $\widehat{\widehat{\mathcal{E}}}$.

Dealing with the theory of finite rotation, strain measures are preferred to strain tensors. As Kirchhoff said, "Introducing displacements instead of the coordinates one wins nothing. Quite the contrary, the brevity and clarity of equations are lost".

2.5.7 Determination of the displacement vector in terms of the strain measures

The presentation of this particular material is due to discussions with M.A. Zak.

Given strain measure $\hat{G}^\times$ (the matrix of components G_{sk}), tensor $\hat{G}^{\times -1}$ (the inverse matrix) can be found with the help of the Christoffel symbols of the second kind

$$\widetilde{\left\{ \begin{matrix} s \\ kq \end{matrix} \right\}} = G^{st} \left[\widetilde{k, q; t}\right] = \frac{1}{2} G^{st} \left(\frac{\partial G_{kt}}{\partial q^q} + \frac{\partial G_{qt}}{\partial q^k} - \frac{\partial G_{kq}}{\partial q^t} \right). \tag{5.7.1}$$

Clearly, it is assumed that the metric tensor $\hat{g}$ of volume v and the Christoffel symbols corresponding to this tensor

$$\left\{ \begin{matrix} s \\ kq \end{matrix} \right\} = g^{st} [k, q; t] = \frac{1}{2} g^{st} \left(\frac{\partial g_{kt}}{\partial q^q} + \frac{\partial g_{qt}}{\partial q^k} - \frac{\partial g_{kq}}{\partial q^t} \right). \tag{5.7.2}$$

are given.

Proceeding from the equalities

$$\mathbf{R}_q = \mathbf{r}_q \cdot \nabla \mathbf{R}, \quad \nabla \mathbf{R} = \mathbf{r}^s \mathbf{R}_s, \tag{5.7.3}$$

we have

$$\begin{aligned} \frac{\partial \nabla \mathbf{R}}{\partial q^k} &= \frac{\partial \mathbf{r}^s}{\partial q^k} \mathbf{R}_s + \mathbf{r}^s \frac{\partial \mathbf{R}_s}{\partial q^k} = - \left\{ \begin{matrix} s \\ kq \end{matrix} \right\} \mathbf{r}^q \mathbf{R}_s + \widetilde{\left\{ \begin{matrix} q \\ ks \end{matrix} \right\}} \mathbf{r}^s \mathbf{R}_q \\ &= \left(\widetilde{\left\{ \begin{matrix} q \\ ks \end{matrix} \right\}} - \left\{ \begin{matrix} q \\ ks \end{matrix} \right\} \right) \mathbf{r}^s \mathbf{R}_q = \left(\widetilde{\left\{ \begin{matrix} q \\ ks \end{matrix} \right\}} - \left\{ \begin{matrix} q \\ ks \end{matrix} \right\} \right) \mathbf{r}^s \mathbf{r}_q \cdot \nabla \mathbf{R}. \end{aligned}$$

Introducing the notion

$$\hat{\Gamma}_{[k]} = \left(\widetilde{\left\{ \begin{matrix} q \\ ks \end{matrix} \right\}} - \left\{ \begin{matrix} q \\ ks \end{matrix} \right\} \right) \mathbf{r}^s \mathbf{r}_q, \tag{5.7.4}$$

we arrive at the system of linear differential equations for tensor $\nabla \mathbf{R}$

$$\frac{\partial \nabla \mathbf{R}}{\partial q^k} = \hat{\Gamma}_{[k]} \cdot \nabla \mathbf{R}. \tag{5.7.5}$$

The integrability conditions for this system follow from the relationships

$$\frac{\partial \nabla \mathbf{R}}{\partial q^r \partial q^k} = \left(\frac{\partial \hat{\Gamma}_{[k]}}{\partial q^r} + \hat{\Gamma}_{[k]} \cdot \hat{\Gamma}_{[r]} \right) \cdot \nabla \mathbf{R}$$
$$= \left(\frac{\partial \hat{\Gamma}_{[r]}}{\partial q^k} + \hat{\Gamma}_{[r]} \cdot \hat{\Gamma}_{[k]} \right) \cdot \nabla \mathbf{R} = \frac{\partial^2 \mathbf{R}}{\partial q^k \partial q^r}$$

and reduce to the following form

$$\frac{\partial \hat{\Gamma}_{[k]}}{\partial q^r} - \frac{\partial \hat{\Gamma}_{[r]}}{\partial q^k} = \hat{\Gamma}_{[r]} \cdot \hat{\Gamma}_{[k]} - \hat{\Gamma}_{[k]} \cdot \hat{\Gamma}_{[r]}. \tag{5.7.6}$$

Performing the differentiation (by means of the formulae for differentiating basis vectors $\mathbf{r}^s$ and $\mathbf{r}_q$) and replacing $\hat{\Gamma}_{[k]}$ and $\hat{\Gamma}_{[r]}$ by eq. (5.7.4) we come to the relationships which are equivalent to the requirement that the components of the Ricci tensor $\tilde{A}^{mn}$are zero, cf. (E.6.14).

Assuming the solution of the system of equations (5.7.5) to be known and using eq. (5.7.3), we find $\mathbf{R}$ by means of the complete differential

$$d\mathbf{R} = \mathbf{R}_m dq^m = \mathbf{r}_m \cdot \nabla \mathbf{R} dq^m = d\mathbf{r} \cdot \nabla \mathbf{R}. \tag{5.7.7}$$

Given strain measure $\hat{g}^\times$ (and thus the inverse tensor $\hat{g}^{\times^{-1}} = \hat{M}$), vector $\mathbf{r}$, determining the position of a particle in volume v, is sought whereas its position vector $\mathbf{R}$ and the metric tensor $\hat{G}$ in volume V are known. For example, they are given by the Cartesian coordinates x_s and $\hat{G} = \hat{E} = \mathbf{i}_s \mathbf{i}_s$.

From the relationships

$$\tilde{\nabla} \mathbf{r} = \mathbf{R}^s \mathbf{r}_s, \ \mathbf{r}_q = \mathbf{R}_q \cdot \tilde{\nabla} \mathbf{r} \tag{5.7.8}$$

we obtain

$$\frac{\partial \tilde{\nabla} \mathbf{r}}{\partial q^k} = \left(\left\{ {q \atop ks} \right\} - \widetilde{\left\{ {q \atop ks} \right\}} \right) \mathbf{R}^s \mathbf{R}_q \cdot \tilde{\nabla} \mathbf{r}$$

and the system of differential equations (5.7.5) is replaced by the following system

$$\frac{\partial \tilde{\nabla} \mathbf{r}}{\partial q^k} = \hat{\tilde{\Gamma}}_{[k]} \cdot \tilde{\nabla} \mathbf{r}, \ \hat{\tilde{\Gamma}}_{[k]} = \left(\left\{ {q \atop ks} \right\} - \widetilde{\left\{ {q \atop ks} \right\}} \right) \mathbf{R}^s \mathbf{R}_q . \tag{5.7.9}$$

Determining tensor $\tilde{\nabla} \mathbf{r}$ from this system we find $\mathbf{r}$ by means of the complete differential

$$d\mathbf{r} = d\mathbf{R} \cdot \tilde{\nabla} \mathbf{r}. \tag{5.7.10}$$

The displacement vector $\mathbf{u}$ is then obtained as follows

$$\mathbf{u} = \mathbf{R} - \mathbf{r}.$$

2.6 Examples of deformations

2.6.1 Affine transformation

The affine transformation is defined by the equality

$$\mathbf{R} = \hat{\Lambda} \cdot \mathbf{r}, \tag{6.1.1}$$

where $\hat{\Lambda}$ is a constant tensor of second rank. From the relationship

$$d\mathbf{R} = \hat{\Lambda} \cdot d\mathbf{r} = d\mathbf{r} \cdot \hat{\Lambda}^*$$

we obtain, by eqs. (3.2.4), (3.3.2) and (3.3.7), that

$$\nabla \mathbf{R} = \hat{\Lambda}^*, \quad (\nabla \mathbf{R})^* = \hat{\Lambda}, \quad \hat{G}^{\times} = \hat{\Lambda}^* \cdot \hat{\Lambda}, \quad \hat{G}^{\times^{-1}} = \hat{m} = \left(\hat{\Lambda}^* \cdot \hat{\Lambda}\right)^{-1}, \tag{6.1.2}$$

and by eqs. (3.2.5), (4.1.2), (4.1.5) and (A.7.14) that

$$\tilde{\nabla} \mathbf{r} = \hat{\Lambda}^{*^{-1}}, \quad \left(\tilde{\nabla} \mathbf{r}\right)^* = \hat{\Lambda}^{-1}, \quad \hat{g}^{\times} = \left(\hat{\Lambda}^* \cdot \hat{\Lambda}\right)^{-1}, \quad \hat{M} = \hat{\Lambda} \cdot \hat{\Lambda}^*. \tag{6.1.3}$$

These formulae highlight the difference between the strain measures introduced. In what follows, the Cartesian coordinates are used in the components' indices. For this reason, in violating the general rules of tensor algebra (Appendix D) the dummy indices on the left and right hand sides of the formula occupy various positions, and summation is carried out only over the indices on the same level.

The components of the first measure of strain and strain tensor $\hat{\mathcal{E}} = \frac{1}{2}\left(\hat{G}^{\times} - \hat{g}\right)$ are represented in the form

$$G_{qt} = \lambda_{sq}\lambda_{st}, \quad \mathcal{E}_{qt} = \frac{1}{2}\left(\lambda_{sq}\lambda_{st} - \delta_{qt}\right), \tag{6.1.4}$$

and, due to eqs. (3.4.4) and (3.4.8), the formulae for the extensions and shears are given by

$$\delta_t = \left(\lambda_{1t}^2 + \lambda_{2t}^2 + \lambda_{3t}^2\right)^{1/2} - 1, \quad \sin \tilde{\varphi}_{qt} = \frac{\lambda_{sq}\lambda_{st}}{(1+\delta_q)(1+\delta_t)}. \tag{6.1.5}$$

Here λ^{sr} denotes the algebraic adjunct of element λ_{sr} of matrix $\|\lambda_{sr}\|$ divided by determinant $\lambda = |\lambda_{sr}|$ of this matrix. Then

$$g_{rt} = \lambda^{sr}\lambda^{st}, \quad \tilde{\mathcal{E}}_{rt} = \frac{1}{2}\left(\delta_{rt} - \lambda^{sr}\lambda^{st}\right) \tag{6.1.6}$$

and furthermore

$$\tilde{\delta}_t = \left(\lambda^{1t^2} + \lambda^{2t^2} + \lambda^{3t^2}\right)^{-1/2} - 1, \quad \sin \varphi_{qt} = \frac{\lambda^{st}\lambda^{sq}}{\left(1+\tilde{\delta}_t\right)\left(1+\tilde{\delta}_q\right)}. \tag{6.1.7}$$

Let us notice that $\delta_t \neq \tilde{\delta}_t$ for the following reason. In the first case, δ_t denotes an extension of a linear element of unit length which was parallel to axis $\mathbf{i}_t$ in volume v, and extends to length $1 + \delta_t$ in volume V. In the second case we speak about a linear element of length $1 + \tilde{\delta}_t$ in volume V which becomes parallel to axis $\mathbf{i}_t$ in the same volume. In eq. (6.1.5) the material coordinates are a_s (the Cartesian coordinates of volume v) whereas in eq. (6.1.6) the material coordinates are x_s (the Cartesian coordinates of volume V).

By eqs. (A.6.7) and (5.2.4) the invariants of the strain measure $\hat{G}^\times$ are represented in the form

$$I_1\left(\hat{G}^\times\right) = \hat{\Lambda}^* \cdot\cdot \hat{\Lambda} = \lambda_{st}\lambda_{st}, \quad I_2\left(\hat{G}^\times\right) = \frac{I_1\left(\hat{g}^\times\right)}{I_3\left(\hat{g}^\times\right)} = \lambda^2 \lambda^{st} \lambda^{st},$$
$$I_3\left(\hat{G}^\times\right) = \lambda^2. \tag{6.1.8}$$

2.6.2 A plane field of displacement

The mapping of volume v into volume V is prescribed by the relationship

$$x_1 = x_1(a_1, a_2), \quad x_2 = x_2(a_1, a_2), \quad x_3 = a_3. \tag{6.2.1}$$

In order to simplify the notion we introduce

$$\frac{\partial x_s}{\partial a_k} = \lambda_{sk} \ (s, k = 1, 2), \quad \lambda_{3s} = \lambda_{s3} = 0, \quad \lambda_{33} = 1. \tag{6.2.2}$$

Then, by analogy with eq. (6.1.2), we have

$$d\mathbf{R} = \hat{\Lambda} \cdot d\mathbf{r}$$

and, referring to eq. (6.1.4), we obtain

$$\left.\begin{array}{c} G_{11} = \lambda_{11}^2 + \lambda_{21}^2, \ G_{12} = \lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22}, \ G_{22} = \lambda_{12}^2 + \lambda_{22}^2, \\ G_{23} = 0, \qquad G_{31} = 0, \qquad G_{33} = 1, \end{array}\right\} \tag{6.2.3}$$

$$G = |G_{st}| = G_{11}G_{22} - G_{12}^2 = \left(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}\right)^2 = \lambda^2. \tag{6.2.4}$$

System of equations (5.3.8) determining the components of tensor $G^{\times^{1/2}}$, is set in the form

$$\left.\begin{array}{r} \left(G_{11}^{\times^{1/2}}\right)^2 + \left(G_{12}^{\times^{1/2}}\right)^2 = G_{11}, \\ G_{12}^{\times^{1/2}}\left(G_{11}^{\times^{1/2}} + G_{12}^{\times^{1/2}}\right) = G_{12}, \\ \left(G_{12}^{\times^{1/2}}\right)^2 + \left(G_{22}^{\times^{1/2}}\right)^2 = G_{22}. \end{array}\right\} \tag{6.2.5}$$

The square of the second invariant of this tensor is expressed in terms of the principal values and, in turn, in terms of the invariants of tensor $G^{\times}$

$$\left(G_{11}^{\times^{1/2}} + G_{22}^{\times^{1/2}}\right)^2 = I_1^2\left(G^{\times^{1/2}}\right) = \left(\sqrt{G_1^{\times}} + \sqrt{G_2^{\times}}\right)^2$$
$$= G_1^{\times} + G_2^{\times} + 2\sqrt{G_1^{\times} G_2^{\times}} = G_1^{\times} + G_2^{\times} + 2\sqrt{G} = \Delta.$$

Hence,

$$G_{11}^{\times^{1/2}} + G_{22}^{\times^{1/2}} = \sqrt{\Delta} = \sqrt{(\lambda_{11} + \lambda_{22})^2 + (\lambda_{12} - \lambda_{21})^2}. \qquad (6.2.6)$$

From the second equation in (6.2.5) we find $G_{12}^{\times^{1/2}}$ and then the diagonal elements $G_{11}^{\times^{1/2}}$ and $G_{22}^{\times^{1/2}}$

$$\left.\begin{aligned} G_{11}^{\times^{1/2}} &= \frac{1}{\sqrt{\Delta}}\left[\lambda_{11}(\lambda_{11} + \lambda_{22}) - \lambda_{21}(\lambda_{12} - \lambda_{21})\right], \\ G_{12}^{\times^{1/2}} &= \frac{1}{\sqrt{\Delta}}(\lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22}), \\ G_{22}^{\times^{1/2}} &= \frac{1}{\sqrt{\Delta}}\left[\lambda_{22}(\lambda_{11} + \lambda_{22}) + \lambda_{12}(\lambda_{12} - \lambda_{21})\right]. \end{aligned}\right\} \qquad (6.2.7)$$

Denoting

$$\cos\chi = \frac{1}{\sqrt{\Delta}}(\lambda_{11} + \lambda_{22}), \quad \sin\chi = \frac{1}{\sqrt{\Delta}}(\lambda_{21} - \lambda_{12}) \qquad (6.2.8)$$

we can set eq. (6.2.7) in the following form

$$\left.\begin{aligned} G_{11}^{\times^{1/2}} &= \lambda_{11}\cos\chi + \lambda_{21}\sin\chi, \\ G_{12}^{\times^{1/2}} &= \lambda_{12}\cos\chi + \lambda_{22}\sin\chi = -\lambda_{11}\sin\chi + \lambda_{21}\cos\chi, \\ G_{22}^{\times^{1/2}} &= -\lambda_{11}\sin\chi + \lambda_{22}\cos\chi. \end{aligned}\right\} \qquad (6.2.9)$$

The particular sign in equation (6.2.8) for χ is chosen such that the linear transformation

$$x_1 = \lambda_{11}a_1 + \lambda_{12}a_2, \quad x_2 = \lambda_{21}a_1 + \lambda_{22}a_2$$

is a rotation through angle χ about axis $\mathbf{i}_3$. In this case, the choice

$$\lambda_{11} = \lambda_{22} = \cos\chi, \quad \lambda_{21} = -\lambda_{12} = \sin\chi, \quad \sqrt{\Delta} = 2$$

satisfies definition (6.2.8) and moreover

$$x_1 = a_1\cos\chi - a_2\sin\chi, \quad x_2 = a_1\sin\chi + a_2\cos\chi, \quad \mathbf{R} = \mathbf{r}^{\backslash} = \mathbf{r}\cdot\hat{A} = \mathbf{i}'_s a_s, \qquad (6.2.10)$$

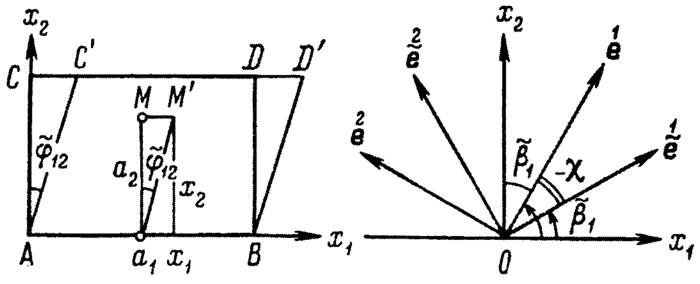

FIGURE 2.3.

which is required.

Representation of tensor $G^{\times^{1/2}}$ is now set as follows

$$G^{\times^{1/2}} = G_{sk}^{\times^{1/2}} \mathbf{i}_s \mathbf{i}_k + \mathbf{i}_3 \mathbf{i}_3 = (\mathbf{i}_s \mathbf{i}_k \cos\chi - \mathbf{i}_3 \times \mathbf{i}_s \mathbf{i}_k \sin\chi)\, \lambda_{sk} + \mathbf{i}_3 \mathbf{i}_3. \tag{6.2.11}$$

An expression for the rotation matrix is composed by means of eq. (5.3.6)

$$\hat{A} = G^{\times^{1/2}} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^* = (\mathbf{i}_s \mathbf{i}_k \cos\chi - \mathbf{i}_3 \times \mathbf{i}_s \mathbf{i}_k \sin\chi) \frac{\partial x_s}{\partial a_k} \cdot \mathbf{i}_q \mathbf{i}_r \frac{\partial a_q}{\partial x_r} + \mathbf{i}_3 \mathbf{i}_3$$
$$= (\mathbf{i}_s \mathbf{i}_r \cos\chi - \mathbf{i}_3 \times \mathbf{i}_s \mathbf{i}_r \sin\chi) \frac{\partial x_s}{\partial a_q} \frac{\partial a_q}{\partial x_r} + \mathbf{i}_3 \mathbf{i}_3$$

and furthermore

$$\hat{A} = \hat{E}_2 \cos\chi - \mathbf{i}_3 \times \hat{E}_2 \sin\chi + \mathbf{i}_3 \mathbf{i}_3 \qquad \left(\hat{E}_2 = \mathbf{i}_1 \mathbf{i}_1 + \mathbf{i}_2 \mathbf{i}_2\right). \tag{6.2.12}$$

The structure of this equation is coincident with that of eq. (A.8.8).

2.6.3 Simple shear

Simple shear should not be mixed up with pure shear, cf. Subsection 1.2.4. Simple shear is a special case of the plane affine transformation prescribed by the formulae

$$x_1 = a_1 + s a_2, \quad x_2 = a_2, \quad x_3 = a_3, \tag{6.3.1}$$

where a is a shear constant. Under a simple shear, a rectangle $ABCD$ becomes a parallelogram $A'B'C'D'$, see Fig. 2.3. Matrices $\hat{\Lambda}$ and $\hat{\Lambda}^*$ are written down in the form

$$\hat{\Lambda} = \hat{E} + s\mathbf{i}_1 \mathbf{i}_2, \quad \hat{\Lambda}^* = \hat{E} + s\mathbf{i}_2 \mathbf{i}_1, \tag{6.3.2}$$

and, by virtue of eqs. (6.1.3) and (6.1.4), we obtain tensor $\hat{G}^{\times}$ and the following non-vanishing components of tensor $\hat{\mathcal{E}}$

$$\hat{G}^{\times} = \hat{E} + s\left(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1\right) + s^2\mathbf{i}_2\mathbf{i}_2, \quad \mathcal{E}_{12} = \mathcal{E}_{21} = \frac{s}{2}, \quad \mathcal{E}_{22} = \frac{1}{2}s^2. \quad (6.3.3)$$

By eq. (6.1.5) we have

$$\delta_3 = \delta_1 = 0, \quad \delta_2 = \sqrt{1+s^2} - 1, \quad \sin\tilde{\varphi}_{12} = \frac{s}{\sqrt{1+s^2}}, \quad (6.3.4)$$

which can be easily seen in Fig. 2.3. From the characteristic equation for tensor $\hat{G}^{\times}$

$$\begin{vmatrix} 1-G & s & 0 \\ s & 1+s^2-G & 0 \\ 0 & 0 & 1-G \end{vmatrix} = (1-G)\left[(1-G)^2 - s^2G\right] = 0$$

we find the principal values

$$G_1 = \frac{1}{2}\left(2+s^2+s\sqrt{s^2+4}\right), \quad G_2 = \frac{1}{2}\left(2+s^2-s\sqrt{s^2+4}\right), \quad G_3 = 1. \quad (6.3.5)$$

The system of equations for determining the principal directions of $\hat{G}^{\times}$ is

$$(1-G_k)\cos\beta_k + s\sin\beta_k = 0, \quad s\cos\beta_k + \left(1+s^2-G_k\right)\sin\beta_k = 0, \quad (6.3.6)$$

where $\cos\beta_k = \overset{k}{\mathbf{e}} \cdot \mathbf{i}_1$, $\sin\beta_k = \overset{k}{\mathbf{e}} \cdot \mathbf{i}_2$. This yields

$$\tan\beta_1 = \frac{G_1-1}{s} = \frac{1}{2}\left(s+\sqrt{s^2+4}\right), \ \tan\beta_2 = \frac{G_2-1}{s} = \frac{1}{2}\left(s-\sqrt{s^2+4}\right) \quad (6.3.7)$$

implying that $\frac{\pi}{4} \le \beta_1 \le \frac{\pi}{2}$ and $\frac{3\pi}{4} \le \beta_2 \le \pi$ if $s > 0$.

The principal values of tensor $\hat{g}^{\times}$, by eqs. (5.2.2) and (6.3.6), equal

$$g_1 = \frac{1}{G_1} = G_2, \quad g_2 = \frac{1}{G_2} = G_1,$$

and the system of equations determining the principal values differs from (6.3.6) in replacing s by $-s$. For this reason

$$\tan\tilde{\beta}_1 = -\tan\beta_2, \quad \tan\tilde{\beta}_2 = -\tan\beta_1, \quad \beta_1 + \tilde{\beta}_1 = \frac{\pi}{2}.$$

The principal axes $\overset{s}{\mathbf{e}}$ and $\overset{s}{\tilde{\mathbf{e}}}$ of tensors $\hat{G}^{\times}$ and $\hat{g}^{\times}$ are shown in Fig. 2.3. The angle through which axes $\overset{s}{\mathbf{e}}$ need to be rotated about $\mathbf{i}_3$ in order to make these axes coincident with axes $\overset{s}{\tilde{\mathbf{e}}}$ is given by

$$\chi = \tilde{\beta}_1 - \beta_1 = \frac{\pi}{2} - 2\beta_1, \ \tan\chi = \cot 2\beta_1 = -\frac{1}{2}s. \quad (6.3.8)$$

On the other hand, in eqs. (6.2.6), (6.2.8) $\lambda_{11} = \lambda_{22} = 1$, $\lambda_{12} = s, \lambda_{21} = 0$ so that

$$\sqrt{\Delta} = \sqrt{s^2+4}, \quad \cos\chi = \frac{2}{\sqrt{s^2+4}}, \quad \sin\chi = -\frac{s}{\sqrt{s^2+4}}, \quad \tan\chi = -\frac{1}{2}s,$$

which is confirmed by eq. (6.3.8). That is, in the problem of simple shear, rotation tensor $\hat{A}$ takes the following form

$$\hat{A} = \overset{q}{\mathbf{e}}\overset{q}{\mathbf{e}} = \frac{1}{\sqrt{s^2+4}}\left(2\hat{E}_2 + s\mathbf{i}_3 \times \hat{E}_2\right) + \mathbf{i}_3\mathbf{i}_3. \tag{6.3.9}$$

2.6.4 *Torsion of a circular cylinder*

The coordinate transformation corresponding to this deformation can be described as a finite rotation of the medium about the cylinder axis $\mathbf{i}_3$. The rotation angle χ is a linear function of the abscissa measured along this axis

$$\chi = \chi_0 = \psi a_3\,, \tag{6.4.1}$$

where ψ denotes the torsion angle. The rotation tensor is given by formula (6.2.12), so that

$$\mathbf{R} = \mathbf{r}\cdot\hat{A}, \quad d\mathbf{R} = d\mathbf{r}\cdot\hat{A} + \mathbf{r}\cdot\hat{A}'da_3 = \hat{A}^*\cdot d\mathbf{r} + \hat{A}^{*'}\cdot\mathbf{r}da_3, \tag{6.4.2}$$

with a prime denoting differentiation with respect to a_3. Now we have

$$dS^2 = d\mathbf{R}\cdot d\mathbf{R} = d\mathbf{r}\cdot\hat{A}\cdot\hat{A}^*\cdot d\mathbf{r} + d\mathbf{r}\cdot\hat{A}\cdot\hat{A}^{*'}\cdot\mathbf{r}da_3 + \\ \mathbf{r}\cdot\hat{A}'\cdot\hat{A}^*\cdot d\mathbf{r}da_3 + \mathbf{r}\cdot\hat{A}'\cdot\hat{A}^{*'}\cdot\mathbf{r}da_3^2. \tag{6.4.3}$$

Utilising the relationships

$$\left(\mathbf{i}_3\times\hat{E}_2\right)\cdot\hat{E}_2 = \mathbf{i}_3\times\hat{E}_2, \quad \left(\mathbf{i}_3\times\hat{E}_2\right)^* = -\mathbf{i}_3\times\hat{E}_2, \\ \left(\mathbf{i}_3\times\hat{E}_2\right)\cdot\left(\mathbf{i}_3\times\hat{E}_2\right)^* = \hat{E}_2$$

and taking into account the equalities

$$\hat{A}\cdot\hat{A}^* = \hat{E}, \quad \left(\hat{A}\cdot\hat{A}^*\right)' = \hat{A}'\cdot\hat{A}^* + \hat{A}\cdot\hat{A}^{*'} = 0, \\ \hat{A}' = -\psi\left(\hat{E}_2\sin\chi + \mathbf{i}_3\times\hat{E}_2\cos\chi\right)$$

we have

$$\hat{A}'\cdot\hat{A}^* = -\psi\mathbf{i}_3\times\hat{E}_2, \quad \hat{A}'\cdot\hat{A}^{*'} = \psi^2\hat{E}_2$$

and, after substituting this result into eq. (6.4.3), we obtain

$$dS^2 = G_{st} da_s da_t = \delta_{st} da_s da_t + \psi \left[d\mathbf{r} \cdot \left(\mathbf{i}_3 \times \hat{E}_2 \right) \cdot \mathbf{r} - \mathbf{r} \cdot \left(\mathbf{i}_3 \times \hat{E}_2 \right) \cdot d\mathbf{r} \right] da_3 + \psi^2 \mathbf{r} \cdot \hat{E}_2 \cdot \mathbf{r} da_3^2 .$$

Therefore

$$dS^2 = \delta_{st} da_s da_t + 2\psi \left(a_1 da_2 - a_2 da_1 \right) da_3 + \psi^2 \left(a_1^2 + a_2^2 \right) da_3^2 \qquad (6.4.4)$$

and, by virtue of eq. (3.3.5), the components of the strain measure $\hat{G}^{\times}$ are as follows

$$\begin{aligned} &G_{11} = G_{22} = 1, \quad G_{33} = \psi^2 \left(a_1^2 + a_2^2 \right) + 1, \\ &G_{12} = 0, \quad G_{23} = \psi a_1, \quad G_{31} = -\psi a_2. \end{aligned} \qquad (6.4.5)$$

The non-vanishing components of the strain tensor are

$$\mathcal{E}_{23} = \frac{1}{2} \psi a_1, \quad \mathcal{E}_{31} = -\frac{1}{2} \psi a_2, \quad \mathcal{E}_{33} = \frac{1}{2} \psi^2 \left(a_1^2 + a_2^2 \right). \qquad (6.4.6)$$

The presence of component $\mathcal{E}_{33}$ caused by "the desire of the cylinder" to change its length indicates the necessity of an axial force. This force is required since u_3 is assumed to vanish, that is the length of the cylinder does not change. Then

$$\mathbf{u} = \mathbf{R} - \mathbf{r} = \mathbf{r} \cdot \left(\hat{A} - \hat{E} \right).$$

This is a manifestation of the effects which were found experimentally by Poynting in 1909. This particular effect can not be explained in terms of the linear theory of deformation.

The volume of the cylinder is retained under the deformation considered, indeed

$$G = |G_{st}| = \begin{vmatrix} 1 & 0 & -\psi a_2 \\ 0 & 1 & \psi a_1 \\ -\psi a_2 & \psi a_1 & 1 + \psi^2 \left(a_1^2 + a_2^2 \right) \end{vmatrix} = 1$$

and dilatation $D = 0$ due to eq. (5.5.1).

2.6.5 *Cylindrical bending of a rectangular plate*

Let us consider the following transformation

$$x_1 = C(a_1) \cos \frac{\alpha a_2}{b}, \quad x_2 = C(a_1) \cos \frac{\alpha a_2}{b}, \quad x_3 = e a_3, \qquad (6.5.1)$$

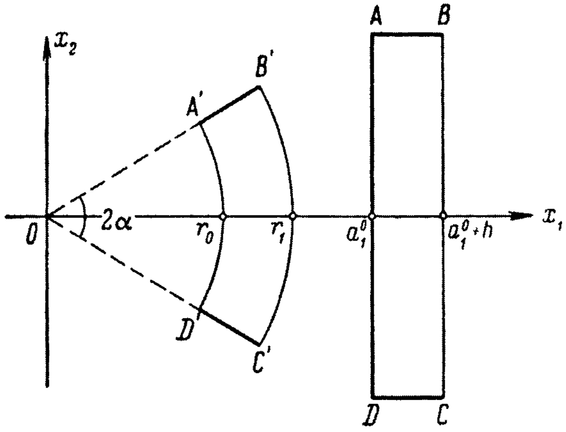

FIGURE 2.4.

which deforms the region of the parallelepiped

$$a_1^0 \leq a_1 \leq a_1^0 + h, \quad -b \leq a_2 \leq b \quad -l \leq a_3 \leq l \tag{6.5.2}$$

modelling a rectangular plate of thickness h, width $2b$ and height $2l$ into a cylindrical panel. The latter is a region bounded by the surfaces of two coaxial cylinders of radii

$$r_0 = C\left(a_1^0\right), r_1 = C\left(a_1^0 + h\right), \tag{6.5.3}$$

planes

$$x_2 = \pm x_1 \tan\alpha \tag{6.5.4}$$

and planes $x_3 = \pm el$, see Fig. 2.4. This deformation is assumed to occur under the conservation of the volume of the material.

The components of the strain measure $\hat{G}^\times$ are determined by means of eq. (3.3.6), the Cartesian coordinates of volume v being taken as material ones. The only non-vanishing components are the diagonal components of tensor $\hat{G}^\times$

$$G_{11} = C'^2\left(a_1\right), \quad G_{22} = \frac{\alpha^2}{b^2} C^2\left(a_1\right), \quad G_{33} = e^2. \tag{6.5.5}$$

From the condition of volume conservation, we have

$$\sqrt{G} = C'C\frac{\alpha e}{b} = 1, \quad C'C = \frac{b}{\alpha e}. \tag{6.5.6}$$

Integrating this equation and accounting for eq. (6.5.3) we find

$$C = \sqrt{\frac{2b}{\alpha e}\left(a_1 - a_1^0\right) + r_0^2}, \quad \frac{2b}{\alpha e}h = r_1^2 - r_0^2, \tag{6.5.7}$$

so that

$$C=\sqrt{\left(r_1^2-r_0^2\right)\frac{a_1-a_1^0}{h}+r_0^2}. \tag{6.5.8}$$

Let us assume that the deformed plate has such a plane $a_1 = a_1^*$ that the line segments $-b \le a_2 \le b$ on this plane which were parallel to axis i_2 in volume v retain their length in volume V. Then, due to eqs. (6.5.5) and (6.5.8), we obtain

$$G_{22}\left(a_1^*\right)=\left[\left(r_1^2-r_0^2\right)\frac{a_1^*-a_1^0}{h}+r_0^2\right]\frac{\alpha^2}{b^2}=1 \tag{6.5.9}$$

and the components of the strain measure are set in the form

$$G_{11}=\frac{1}{e^2}\frac{1+\dfrac{r_1^2-r_0^2}{r_0^2}\dfrac{a_1^*-a_1^0}{h}}{1+\dfrac{r_1^2-r_0^2}{r_0^2}\dfrac{a_1-a_1^0}{h}},\quad G_{22}=\frac{1}{e^2G_{11}},\quad G_{33}=e^2. \tag{6.5.10}$$

Taking into account eq. (6.5.7) we can put equality (6.5.9) in the form of a quadratic equation in $\left(r_1^2-r_0^2\right)/r_0^2$, whose positive solution is given by the formula

$$\frac{r_1^2-r_0^2}{r_0^2}=\frac{2h}{r_0e}\left[\sqrt{1+\left(\frac{a_1^*-a_1^0}{r_0e}\right)^2}+\frac{a_1^*-a_1^0}{r_0e}\right]. \tag{6.5.11}$$

From the latter, as well as from eq. (6.5.7), we obtain the ratio of the height of the rectangular strip to the length of the arc of the cross-section of the inner cylinder

$$\frac{b}{r_0\alpha}=\sqrt{1+\left(\frac{a_1^*-a_1^0}{r_0e}\right)^2}+\frac{a_1^*-a_1^0}{r_0e}. \tag{6.5.12}$$

It follows from the formulae derived, that for $e \approx 1$ the components of the strain measure differ from unity in terms which are of the order h/b, i.e. they are small for a thin plate. The components of the strain tensor have the same order whereas the displacements are by no means small.

2.6.6 Radial-symmetric deformation of a hollow sphere

Let the spherical coordinates of volume v

$$q^1=r,\quad q^2=\vartheta,\quad q^3=\lambda.$$

be taken as the material coordinates. Then, referring to Section C.8 we have

$$\mathbf{r} = r\mathbf{e}_R, \quad \mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial q^1} = \mathbf{e}_R, \quad \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial q^2} = r\mathbf{e}_\vartheta, \quad \mathbf{r}_3 = \frac{\partial \mathbf{r}}{\partial q^3} = \mathbf{e}_\lambda r \sin\vartheta$$

and the non-trivial covariant component of the unit tensor $\hat{g}$ of volume v are

$$g_{11} = \mathbf{r}_1 \cdot \mathbf{r}_1 = 1, \quad g_{22} = \mathbf{r}_2 \cdot \mathbf{r}_2 = \mathbf{r}^2, \quad g_{33} = \mathbf{r}_3 \cdot \mathbf{r}_3 = r^2 \sin^2\vartheta. \tag{6.6.1}$$

Next, we have

$$g = g_{11} g_{22} g_{33} = r^4 \sin^2\vartheta, \tag{6.6.2}$$

and the contravariant components of this tensor, due to eq. (E.5.7), are

$$g^{11} = \frac{1}{g}\frac{\partial g}{\partial g_{11}} = \frac{1}{g_{11}} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2 \sin^2\vartheta}. \tag{6.6.3}$$

Under a radial-symmetric deformation of the sphere bounded in volume v by the surfaces of the concentric spheres $r = r_0$ and $r = r_1$, we have in volume V

$$\mathbf{R} = R(r)\,\mathbf{e}_R, \; \mathbf{R}_1 = R'(r)\,\mathbf{e}_R, \; \mathbf{R}_2 = R(r)\,\mathbf{e}_\vartheta, \; \mathbf{R}_3 = R(r)\,\mathbf{e}_\lambda \sin\vartheta, \tag{6.6.4}$$

so that

$$G_{11} = R'^2, \quad G_{22} = R^2, \quad G_{33} = R^2 \sin\vartheta; \quad G = R^4 R'^2 \sin\vartheta.$$

For an incompressible material, i.e. for a material retaining the volume, we have, by eq. (5.5.1) that

$$\frac{G}{g} = 1, \quad R'^2 = \frac{r^4}{R^4}. \tag{6.6.5}$$

Integrating this equation, we obtain, as one would expect

$$R^3(r) - r^3 = \mathrm{const} = R_1^3 - r_1^3 = R_0^3 - r_0^3,$$

where R_1 and R_0 denote the sphere radii in the final state, that is in volume V. The non-trivial components of the unit tensor $\hat{G}$ are equal to

$$\left.\begin{aligned} G_{11} &= \frac{r^4}{R^4}, \quad G_{22} = R^2, \quad G_{33} = R^2 \sin^2\vartheta, \\ G^{11} &= \frac{R^4}{r^4}, \quad G^{22} = \frac{1}{R^2}, \quad G^{33} = \frac{1}{R^2 \sin^2\vartheta}. \end{aligned}\right\} \tag{6.6.6}$$

The principal invariants of the strain measure $\hat{G}^\times$ are found with the help of eq. (5.2.6) and (5.2.8)

$$I_1\left(\hat{G}^\times\right) = \frac{r^4}{R^4} + 2\frac{R^2}{r^2}, \; I_2\left(\hat{G}^\times\right) = \frac{R^4}{r^4} + 2\frac{r^2}{R^2}, \; I_3\left(\hat{G}^\times\right) = 1. \tag{6.6.7}$$

2.6.7 Axisymmetric deformation of a hollow cylinder

The calculation is fully analogous to that carried out in the previous subsection. The material coordinates are the cylindric coordinates

$$q^1 = r, \quad q^2 = \varphi, \quad q^3 = z.$$

Referring to formulae of Section C.7 we have

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{k}, \quad \mathbf{r}_1 = \mathbf{e}_r, \quad \mathbf{r}_2 = r\mathbf{e}_\varphi, \quad \mathbf{r}_3 = \mathbf{k}, \tag{6.7.1}$$

so that according to eqs. (C.7.1) and (E.5.7)

$$\left.\begin{array}{llll} g_{11} = 1, & g_{22} = r^2, & g_{33} = 1, & g = r^2 \\ g^{11} = 1, & g^{22} = \dfrac{1}{r^2}, & g^{33} = 1. & \end{array}\right\} \tag{6.7.2}$$

Under the axisymmetric deformation the position radius $\mathbf{R}$ in volume v is given by the equality

$$\mathbf{R} = R(r)\,\mathbf{e}_r + \alpha z\mathbf{k},$$

so that

$$\mathbf{R}_1 = R'(r)\,\mathbf{e}_r, \quad \mathbf{R}_2 = R(r)\,\mathbf{e}_\varphi, \quad \mathbf{R}_3 = \alpha\mathbf{k} \tag{6.7.3}$$

and

$$G_{11} = R'^2, \quad G_{22} = R^2, \quad G_{23} = \alpha^2; \quad G = R'^2 R^2 \alpha^2.$$

For an incompressible material

$$\frac{G}{g} = 1, \; R'^2 R^2 \alpha^2 = r^2, \; \alpha R^2 - r^2 = \text{const} = \alpha R_0^2 - r_0^2 = \alpha R_1^2 - r_1^2, \tag{6.7.4}$$

where r_0 and r_1 denote the radii of the concentric cylinders in volume V, while R_0 and R_1 are those in volume V. The covariant and contravariant components of tensor $\hat{G}$ are equal to

$$\left.\begin{array}{lll} G_{11} = \dfrac{r^2}{R^2\alpha^2}, & G_{22} = R^2, & G_{33} = \alpha^2, \\ G^{11} = \dfrac{R^2\alpha^2}{r^2}, & G^{22} = \dfrac{1}{R^2}, & G^{33} = \dfrac{1}{\alpha^2}, \end{array}\right\} \tag{6.7.5}$$

while the principal invariants of the tensor of strain measure $\hat{G}^\times$ are as follows

$$I_1\left(\hat{G}^\times\right) = \frac{r^2}{R^2\alpha^2} + \frac{R^2}{r^2} + \alpha^2, \; I_2\left(\hat{G}^\times\right) = \frac{\alpha^2 R^2}{r^2} + \frac{r^2}{R^2} + \frac{1}{\alpha^2}, \; I_3\left(\hat{G}^\times\right) = 1. \tag{6.7.6}$$

Part II

Governing equations of the linear theory of elasticity

3

The constitutive law in the linear theory of elasticity

3.1 Isotropic medium

3.1.1 Statement of the problem of the linear theory of elasticity

As repeatedly mentioned earlier, see Subsections 2.3.6 and 2.3.9, the tensors of finite strain can be replaced by a linear strain tensor $\hat{\varepsilon}$ provided that the components of the gradient of the displacement vector $\nabla \mathbf{u}$ are small. The latter is equivalent to the components of tensor $\hat{\varepsilon}$ and the rotation vector $\boldsymbol{\omega}$ being small

$$\left| \frac{\partial u_s}{\partial a_k} \right| \ll 1, \quad |\varepsilon_{sk}| \ll 1, \quad |\omega_s| \ll 1. \tag{1.1.1}$$

Under these conditions there is no necessity to distinguish between the derivatives with respect to the coordinates of the initial a_s and final x_s states. Indeed, for a function f we have the following derivatives

$$\frac{\partial f}{\partial a_k} = \frac{\partial f}{\partial x_s} \frac{\partial x_s}{\partial a_k} = \frac{\partial f}{\partial x_s} \left(\delta_{sk} + \frac{\partial u_s}{\partial a_k} \right) = \frac{\partial f}{\partial x_k} + \frac{\partial f}{\partial x_s} \frac{\partial u_s}{\partial a_k}$$

and, under the above assumption,

$$\frac{\partial f}{\partial a_k} = \frac{\partial f}{\partial x_k}. \tag{1.1.2}$$

Unless stated otherwise, in the linear theory of elasticity by the initial state we mean the natural state of the medium, i.e. the state without stresses.

The Cartesian coordinates of a particle in a stressed state are denoted by x_1, x_2, x_3 and those in the initial state as a_1, a_2, a_3, see Subsection 1.1.1

$$x_s = a_s + u_s \,. \tag{1.1.3}$$

There is, however, no need to introduce these coordinates explicitly. The body is assumed to have the same form in the initial and actual (loaded) states. Thus, there is no necessity to distinguish between volumes v and V as well as between surfaces o and O.

In contrast to the basic relationship (1.3.2) of Chapter 1, the stress tensor is introduced as follows

$$\mathbf{t}_n = \mathbf{n} \cdot \hat{T}. \tag{1.1.4}$$

Here $\mathbf{t}_n do$ denotes the force vector acting on the oriented surface $\mathbf{n}do$, with $\mathbf{n}$ and do denoting respectively the unit normal vector of this surface and the area of this surface, both in the initial state. The equilibrium equations in the volume are set, as before, in the form of eqs. (1.5.4) or (1.5.6) of Chapter 1. Relating mass to the initial volume, the mass density in the expression $\rho\mathbf{K}$ for the volume force is set equal to the initial value, i.e. $\rho = \rho_0$. According to eq. (1.1.4), the equilibrium equations on the surface are written in the form

$$\mathbf{F} = \mathbf{n} \cdot \hat{T}, \tag{1.1.5}$$

where $\mathbf{F}$ denotes the surface force related to the unit of the initial area of surface o and $\mathbf{n}$ is the unit vector of the normal to this surface.

In the linear theory of equilibrium of solids there is no need to discriminate between the strain tensors of Cauchy-Green $\hat{\mathcal{E}}$ and Almansi-Hamel $\hat{\tilde{\mathcal{E}}}$. As follows from eqs. (3.6.5) and (4.3.5) of Chapter 2, both tensors should be replaced, under conditions (1.1.1) and (1.1.2), by the linear strain tensor

$$\hat{\mathcal{E}} = \hat{\tilde{\mathcal{E}}} = \hat{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^*\right] \tag{1.1.6}$$

regardless of what independent variables (x_s or a_s) are used for the nabla-operator. Further, x_s is used to denote all these variables and V and O denote the volume and the bounding surface of the body, respectively. While utilising curvilinear coordinates q^s we adopt $\hat{g}$ and $g_{sk}, g^{sk}, g_s^k = \delta_s^k$ to denote the metric tensor and its components respectively. By eq. (5.5.5) of Chapter 2 the first invariant of the linear strain tensor, that is the dilatation in the linear approximation, is designated as follows

$$\vartheta = I_1(\hat{\varepsilon}) = \operatorname{div}\mathbf{u}. \tag{1.1.7}$$

It was stated in Subsection 1.1.5 that the objective of the static analysis of a continuum is to search for that state, among all feasible states of stress (satisfying the equations of statics throughout the volume and on the surface), which is actually realised for the adopted physical model of a particular medium. This model is determined by the constitutive law, namely, for a large number of media it consists of prescribing relations between the stress tensor and the strain tensor. In the linear elasticity theory it is a linear relationship between the stress tensor and the linear strain tensor. For a linear elastic body this relationship presents a system of linear equations relating the components of these tensors and expresses the generalised Hooke law. (It will be shown below that linearity of the relationships between the tensors is not equivalent to a linear relation between their components.) Also the temperature appears in the expression for the constitutive law.

Prescribing the constitutive law leads to a closed system of differential equations which allows one to determine the state of stress in the body and the displacement vector of the particles of the medium. Therefore, in a linear statement the problem as to determine the form and size of an elastic body in the final state is not important. These geometric characteristics are found after the problem has been solved under the assumption that the initial form of the body remains unchanged. This approach enables an inherent difficulty of the nonlinear elasticity theory to be avoided, namely, that the state of stress needs to be found in volume V which is a body with an unknown boundary O. The validity of the approach is substantiated using the following observation. While solving the problems of the nonlinear elasticity theory by the method of successive approximation, for example in the form of a power series in terms of a small parameter characterising the smallness of the gradient of the displacement vector, the initial approximation is the solution of the problem for a linear elastic body, with the constitutive equations being related to the initial volume and the initial form of the boundary.

In what follows, the study is carried out mostly in the Cartesian coordinate system $OX_1X_2X_3$. However the resulting relationships are formulated in an invariant form of dependences between vectors, tensors and tensor invariants. For this reason, a transition to curvilinear coordinates is straightforward.

3.1.2 Elementary work

Using the formulae of Subsections 1.3.5 and 1.3.6 one obtains an expression for the specific elementary work of external forces $\delta' A_{(e)}$ (and the equal in value, but opposite in sign, specific elementary work of internal forces $\delta' A_{(i)}$) by replacing ratio G/g and the strain tensor respectively by unity and the linear strain tensor. In the linear theory there is no necessity to distinguish between the metrics of volumes v and V and thus the energetic stress tensor is identical to stress tensor $\hat{T}$. Hence, by virtue of eq. (3.6.4) of Chapter 1 we have

$$\delta' A_{(e)} = -\delta' A_{(i)} = \hat{T} \cdot\cdot\, \delta\hat{\varepsilon} = I_1 \left(\hat{T} \cdot \delta\hat{\varepsilon}\right) = t_{sk}\delta\varepsilon_{sk}, \tag{1.2.1}$$

t_{sk} and $\hat{\varepsilon}_{sk}$ denoting respectively the components of tensors $\hat{T}$ and $\hat{\varepsilon}$ in the Cartesian coordinate system.

The first invariant of the stress tensor in further denoted by σ. It is a sum of three principal stresses or three normal stresses on the orthogonal surfaces

$$\sigma = t_{11} + t_{22} + t_{22} = t_1 + t_2 + t_3. \tag{1.2.2}$$

Using notion (1.1.7) we can present the specific elementary work, eq. (3.6.6) of Chapter 1, in terms of the spherical and deviatoric parts of tensor $\hat{T}$ and $\hat{\varepsilon}$ in the following form

$$\delta' A_{(e)} = \frac{1}{3}\sigma\delta\vartheta + I_1 \left(\operatorname{Dev}\hat{T} \cdot \delta \operatorname{Dev}\hat{\varepsilon}\right). \tag{1.2.3}$$

A link between the specific elementary work and the elementary work of the entire volume of the body is performed by the following integration

$$\delta' a_{(e)} = \iiint\limits_V \delta' A_{(e)} d\tau, \; \delta' a_{(i)} = \iiint\limits_V \delta' A_{(i)} d\tau. \tag{1.2.4}$$

3.1.3 Isotropic homogeneous medium of Hencky

The forthcoming consideration is restricted to media in which the stress tensor is defined by prescribing the strain tensor and temperature difference θ from the initial state. The components of these tensors are related by equations of the following type

$$t_{sk} = f_{sk}\left(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}; x_1, x_2, x_3, \theta\right) \tag{1.3.1}$$

along with certain invariance requirements, implying the validity of these relations under coordinate transformation. This requirement excludes the physical models with the stress tensor depending upon the tensors of strain rate or the time-history of deformation or "age" of the material etc. We also do not consider heterogeneous media for which coordinates x_1, x_2, x_3 appear explicitly in dependence (1.3.1).

By isotropic elastic media we mean media with tensors of strain and stress, that are aligned, cf. Subsection A.12. An elementary cube cut from such a medium deforms equally for any orientation of its edges. It follows from the Cayley-Hamilton theorem that two aligned tensors are related by a quadratic dependence of type (A.12.4). One of the difficulties of nonlinear elasticity theory is the necessity to find that strain measure which corresponds to the stress tensor. In the linear elasticity theory this difficulty does not exist and the quadratic dependence is replaced by a linear one of the type

$$\hat{T} = a\hat{E} + b\hat{\varepsilon}, \tag{1.3.2}$$

where a and b depend on the invariants of tensor $\hat{\varepsilon}$ and possibly on temperature. The unit tensor is denoted as $\hat{E}$.

An isotropic elastic medium obeying the constitutive law (1.3.2) is referred to as the Hencky medium. In terms of the components of tensors $\hat{T}$ and $\hat{\varepsilon}$, the latter equation is set as follows

$$t_{sk} = a\delta_{sk} + b\varepsilon_{sk}, \tag{1.3.3}$$

and, as a and b depend on the invariants of $\hat{\varepsilon}$, these relationships are nonlinear. The Hencky medium is geometrically linear but it is physically nonlinear. A particular case of this medium is the Hookean medium and a description of its behaviour comprises the major portion of the present book.[1]

Using eq. (1.3.2) we have

$$\sigma_1 = I_1\left(\hat{T}\right) = 3a + b\vartheta \tag{1.3.4}$$

so that

$$\operatorname{Dev}\hat{T} = \hat{T} - \frac{1}{3}\sigma\hat{E} = b\left(\hat{\varepsilon} - \frac{1}{3}\vartheta\hat{E}\right) = b\operatorname{Dev}\hat{\varepsilon} \tag{1.3.5}$$

and the dependence between the second (quadratic) invariants of the deviators of $\hat{T}$ and $\hat{\varepsilon}$ is written in the form

$$I_2\left(\operatorname{Dev}\hat{T}\right) = b^2 I_2\left(\operatorname{Dev}\hat{\varepsilon}\right).$$

Referring to formulae (2.2.11) of Chapter 1 and (3.7.4) of Chapter 2

$$\tau^2 = -I_2\left(\operatorname{Dev}\hat{T}\right), \quad \frac{\Gamma^2}{4} = -I_2\left(\operatorname{Dev}\hat{\varepsilon}\right) \tag{1.3.6}$$

where τ and Γ denote the intensity of the shear stresses and the shear strains, respectively, we introduce the new notation

$$b = 2\mu = \frac{2\tau}{\Gamma}, \quad \mu = \frac{\tau}{\Gamma}. \tag{1.3.7}$$

[1] The problems of the theory of elasticity of anisotropic media are not considered here. The linear theory of anisotropic media is treated by S.G. Lekhnitsky in the book Theory of elasticity of anisotropic medium (in Russian), Gostekhizdat, Moscow, 1950, whereas the book by A.E. Green and J.E. Adkins, Large elastic deformations, Claredon Press, Oxford, 1970 considers in detail the nonlinear problems of anisotropic elastic media.

Changing the notation once again we can represent the coefficient α as a sum of two terms

$$a = \lambda\vartheta + a',$$

where a' depends on temperature θ and vanishes when the temperature is equal to the temperature of the natural state. Then, due to eqs. (1.3.4) and (1.3.7)

$$\frac{1}{3}\sigma = \left(\lambda + \frac{2}{3}\mu\right)\vartheta + a' = k\vartheta + a', \tag{1.3.8}$$

where

$$k = \lambda + \frac{2}{3}\mu. \tag{1.3.9}$$

Here k and μ are the bulk modulus and shear modulus, respectively. In what follows, referring to much experimental data on the behaviour of materials under uniform compression we assume that the bulk modulus does not depend upon the strain invariants. A dependence on the change in volume was observed by Bridgman only at extremely high pressures.

At temperature θ and in the case of no external forces $(\sigma = 0)$, the strain tensor $\hat{\varepsilon}$ in the tested cube is spherical and given by

$$\hat{\varepsilon} = \alpha\theta\hat{E}, \quad I_1(\hat{\varepsilon}) = \vartheta = 3\alpha\theta, \tag{1.3.10}$$

where α denotes the coefficient of linear expansion. Substitution into eq. (1.3.8) yields

$$3k\alpha\theta + a' = 0, \quad a' = -3k\alpha\theta.$$

Now we arrive at the equality

$$a = \lambda\vartheta - 3k\alpha\theta = \lambda\vartheta - (3\lambda + 2\mu)\alpha\theta$$

and Hencky's constitutive law (1.3.2) is set in the form

$$\hat{T} = \lambda\vartheta\hat{E} + 2\mu\hat{\varepsilon} - (3\lambda + 2\mu)\alpha\theta\hat{E} \tag{1.3.11}$$

or in the equivalent form

$$\frac{1}{3}\sigma = I_1\left(\hat{T}\right) = k(\vartheta - 3\alpha\theta), \quad \operatorname{Dev}\hat{T} = 2\mu\operatorname{Dev}\hat{\varepsilon}. \tag{1.3.12}$$

Returning to expression (1.2.3) for the specific elementary work we have

$$\operatorname{Dev}\hat{T}\cdot\delta\operatorname{Dev}\hat{\varepsilon} = 2\mu\operatorname{Dev}\hat{\varepsilon}\cdot\delta\operatorname{Dev}\hat{\varepsilon} = \mu\delta\left(\operatorname{Dev}\hat{\varepsilon}\right)^2$$

and, by virtue of eqs. (A.11.9) and (A.3.6),

$$I_1\left[(\operatorname{Dev}\hat{\varepsilon})^2\right] = -2I_2(\operatorname{Dev}\hat{\varepsilon}) = \frac{\Gamma^2}{2}, \quad \delta I_1\left[(\operatorname{Dev}\hat{\varepsilon})^2\right] = \Gamma\delta\Gamma.$$

The formula for the specific elementary work of external forces is presented in the following expressive form

$$\delta' A_{(e)} = k(\vartheta - 3\alpha\theta)\,\delta\vartheta + \mu\Gamma\delta\Gamma = \frac{\sigma}{3}\delta\vartheta + \tau\delta\Gamma. \tag{1.3.13}$$

The first and second terms represent the specific elementary work of the change in volume and shape, respectively.

Let us recall that due to eqs. (A.11.6), (A.10.4) and (A.10.5)

$$\begin{aligned}\Gamma^2 &= -4I_2(\operatorname{Dev}\hat{\varepsilon}) = 4\left[\frac{1}{3}I_1^2(\hat{\varepsilon}) - I_2(\hat{\varepsilon})\right] \qquad (1.3.14)\\ &= 4\left[\frac{1}{3}\vartheta^2 - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11}) + \frac{1}{4}\left(\gamma_{12}^2 + \gamma_{23}^2 + \gamma_{31}^2\right)\right].\end{aligned}$$

3.2 Strain energy

3.2.1 Internal energy of a linearly deformed body

Here and throughout the chapters concerned with the linear theory of elasticity, the linear strain tensor is called, for brevity, the strain tensor. For the independent parameters of the state of the homogeneous isotropic Hencky's medium, we take the dilatation ϑ (the first invariant of the strain tensor), the intensity of shear strain Γ, and the temperature ϑ. A thermodynamic quantity (potential) referred to as the specific internal energy E is assumed to be a function of these parameters

$$E = E(\vartheta, \Gamma, \theta). \tag{2.1.1}$$

In accordance with the first law of thermodynamics the increment (variation) δE is equal to the sum of the specific elementary work of the external forces $\delta' A_{(e)}$ and the specific heat supply $\delta' Q$. The latter is given by the relationship

$$\delta' Q = c\delta\theta + \chi\delta\vartheta. \tag{2.1.2}$$

Here c denotes the specific heat at constant volume, i.e. at $\delta\vartheta = 0$, and $\chi\delta\vartheta$ is the heat required for a change in the volume. In what follows Θ stands for the absolute temperature, that is

$$\Theta = \Theta_0 + \theta, \quad \delta\Theta = \delta\theta, \tag{2.1.3}$$

with Θ_0 designating the absolute temperature of the body in the natural state.

Now, by virtue of eqs. (2.1.2) and (1.3.13) we have

$$\delta E = \delta' A_{(e)} + \delta' Q = [k(\vartheta - 3\alpha\theta) + \chi]\,\delta\vartheta + c\delta\theta + \mu\Gamma\delta\Gamma. \tag{2.1.4}$$

The integrability conditions for this expression, that is the conditions for the existence of E as a function of the above listed parameters, are written in the form

$$\left.\begin{aligned} &(\vartheta - 3\alpha\theta)\frac{\partial k}{\partial \Gamma} + \frac{\partial \chi}{\partial \Gamma} = \Gamma\frac{\partial \mu}{\partial \vartheta}, \qquad \frac{\partial c}{\partial \Gamma} = \Gamma\frac{\partial \mu}{\partial \theta}, \\ &\frac{\partial \chi}{\partial \theta} - 3k\alpha + (\vartheta - 3\alpha\theta)\frac{\partial k}{\partial \theta} = \frac{\partial c}{\partial \vartheta}. \end{aligned}\right\} \tag{2.1.5}$$

The second law of thermodynamics postulates the existence of a further function of the parameters of the system state which is the entropy S. In a reversible process, a variation of this quantity is determined by the equality

$$\delta S = \frac{\delta' Q}{\Theta} = \frac{c}{\Theta}\delta\theta + \frac{\chi}{\Theta}\delta\vartheta = \frac{\partial S}{\partial \Theta}\delta\theta + \frac{\partial S}{\partial \vartheta}\delta\vartheta + \frac{\partial S}{\partial \Gamma}\delta\Gamma \tag{2.1.6}$$

and the conditions of integrability of this expression are set as follows

$$\frac{\partial c}{\partial \Gamma} = 0, \quad \frac{\partial \chi}{\partial \Gamma} = 0, \quad \frac{\partial c}{\partial \vartheta} = \frac{\partial \chi}{\partial \theta} - \frac{\chi}{\Theta}. \tag{2.1.7}$$

Since k does not depend on the strain invariants it follows from the first two equalities in (2.1.5) and (2.1.7) that

$$\frac{\partial \mu}{\partial \theta} = 0, \quad \frac{\partial \mu}{\partial \vartheta} = 0, \quad \mu = \mu(\Gamma). \tag{2.1.8}$$

The remaining equalities yield

$$\chi = \left[3k\alpha - (\vartheta - 3\alpha\theta)\frac{\partial k}{\partial \theta}\right]\Theta. \tag{2.1.9}$$

Here and in what follows it is supposed that $\alpha\theta$ is a value of the order of ϑ. According to the standard assumptions in the linear theory and because of a weak dependence of k on temperature, one can replace eq. (2.1.9) by the following relationship

$$\chi = 3k\alpha\Theta. \tag{2.1.10}$$

By virtue of eqs. (2.1.5) and (2.1.7) we have

$$\frac{\partial c}{\partial \vartheta} = 0, \quad \frac{\partial c}{\partial \Gamma} = 0, \quad c = c(\theta) \tag{2.1.11}$$

implying that for the adopted approximation the specific heat at constant value depends only on the temperature.

Expressions $\delta' Q$ and δE are now written down in the form

$$\delta' Q = 3k\alpha\Theta\delta\vartheta + c(\theta)\,\delta\theta, \tag{2.1.12}$$

$$\delta E = k(\vartheta + 3\alpha\Theta_0)\,\delta\vartheta + \mu\Gamma\delta\Gamma + c(\theta)\,\delta\theta, \tag{2.1.13}$$

and under the assumptions made k is independent of the temperature.

3.2.2 Isothermal process of deformation

Provided that the temperature is kept constant during deformation, then $\theta = 0, \Theta = \Theta_0$ and due to eq. (1.3.13)

$$\delta' A_{(e)} = k\vartheta\delta\vartheta + \mu\Gamma\delta\Gamma = \frac{1}{3}\sigma\delta\vartheta + \tau\delta\Gamma. \tag{2.2.1}$$

Referring to eqs. (2.1.2), (2.1.4(and (2.1.6) we have

$$\delta E = \delta' A_{(e)} + \Theta\delta S = \delta' A_{(e)} + \delta(\Theta S) - S\delta\Theta. \tag{2.2.2}$$

The thermodynamic function (potential)

$$F = E - \Theta S \tag{2.2.3}$$

is termed the free energy of the system. Its variation, due to eq. (2.2.2), is equal to

$$\delta F = \delta' A_{(e)} - S\delta\Theta, \tag{2.2.4}$$

and thus under the isothermal process the specific elementary work of the external forces is equal to the variation of the free energy

$$\delta F = \delta' A_{(e)} = k\vartheta\delta\vartheta + \mu\Gamma\delta\Gamma. \tag{2.2.5}$$

3.2.3 Adiabatic process

Under this process $\delta' Q = 0$ and by eq. (2.1.12)

$$c(\theta)\,\delta\theta = -3k\alpha\Theta\delta\vartheta. \tag{2.3.1}$$

According to the earlier assumptions we can write

$$\begin{aligned}\frac{c(\theta)}{\Theta} &= \frac{1}{\Theta_0\left(1 + \dfrac{\theta}{\Theta_0}\right)}\left[c(0) + \theta c'(0) + \ldots\right] \\ &= \frac{c(0)}{\Theta_0} + \frac{\theta}{\Theta_0}\left[c'(0) - \frac{c(0)}{\Theta_0} + \ldots\right] \approx \frac{c_0}{\Theta_0},\end{aligned}$$

where $c_0 = c(0)$ denotes the specific heat at the temperature of the initial state ($\theta = 0, \Theta = \Theta_0$). Taking into account that $\vartheta = 0$ at $\theta = 0$ we arrive at the relationship

$$3k\alpha\delta\vartheta \approx -\frac{c_0}{\Theta_0}\delta\theta, \quad \theta = -\frac{3k\alpha\Theta_0}{c_0}\vartheta, \tag{2.3.2}$$

determining the temperature change under the adiabatic deformation process.

Due to eqs. (2.1.4) and (2.3.2) we have

$$\delta E = \delta' A_{(e)} = k(\vartheta - 3\alpha\theta)\delta\vartheta + \mu\Gamma\delta\Gamma = k\vartheta\left(1 + \frac{9k\alpha^2\Theta_0}{c_0}\right)\delta\vartheta + \mu\Gamma\delta\Gamma. \tag{2.3.3}$$

Using the notation

$$k' = k\left(1 + \frac{9k\alpha^2\Theta_0}{c_0}\right) \tag{2.3.4}$$

equation (2.3.3) is set in the form

$$\delta E = \delta' A_{(e)} = k'\vartheta\delta\vartheta + \mu\Gamma\delta\Gamma. \tag{2.3.5}$$

The coefficients k and k' are referred to as the adiabatic and isothermal bulk modulus, respectively. The bulk modulus k has the same value under adiabatic and isothermal processes.

Under a free thermal expansion when $\delta\vartheta = 3\alpha\delta\theta$, the total amount of the heat received by a volume unit is $\delta' Q = c_p\delta\theta$, where c_p denotes the specific heat at constant pressure. This parameter is measured in stress-free tests. For this reason, due to eqs. (2.1.12) and (2.1.10)

$$\delta' Q = c_p\delta\theta = c\delta\theta + 3k\alpha\Theta \cdot 3\alpha\delta\theta \approx c\delta\theta\left(1 + \frac{9k\alpha^2\Theta_0}{c_0}\right),$$

so that by eq. (2.3.4)

$$\frac{c_p}{c} = \frac{k'}{k}. \tag{2.3.6}$$

The values of this ratio for some metals at temperature 20° C are shown in Table 3.1. From this table it appears that there is no difference between the adiabatic and isothermal processes to an accuracy sufficient for technical calculations.

material	c_p/c	material	c_p/c	material	c_p/c
aluminium	1.043	manganese	1.044	cobalt	1.020
molybdenum	1.007	lead	1.067	nickel	1.021
tungsten	1.006	iron	1.016	platinum	1.020
silver	1.004	copper	1.028	gold	1.038

Table 3.1

3.2.4 Specific strain energy. Hencky's media

Based on equalities (2.2.5) and (2.3.5), we introduce into consideration a function of the strain invariants $A(\vartheta, \Gamma)$ whose variation is given by the equality

$$\delta A = k\vartheta\delta\vartheta + \mu\Gamma\delta\Gamma = k\vartheta\delta\vartheta + \tau\delta\Gamma. \tag{2.4.1}$$

For the isothermal process, A is identified with the free energy F whilst for the adiabatic process it is identified with the internal energy E and in the latter case k should be replaced by the adiabatic bulk modulus k'. In both processes one can introduce a function of the state, referred to in what follows as the specific strain energy,

$$A = \frac{1}{2}k\vartheta^2 + \int_{\Gamma_0}^{\Gamma} \tau\delta\Gamma \tag{2.4.2}$$

with practically coincident value of k for isothermal or adiabatic processes of deformation. In these processes the specific strain energy is equal to the specific work of the external forces in a continuous sequence of equilibrium states from the natural state to the actual state.

Refining expression (2.4.2) requires a knowledge of the experimental dependence

$$\tau = \mu(\Gamma)\Gamma. \tag{2.4.3}$$

Particular cases of isotropic Hencky's media are:

i) a linear elastic Hookean medium

$$\mu(\Gamma) = \text{const}; \tag{2.4.4}$$

ii) a medium with the effects of yielding when

$$\tau = \mu(\Gamma)\Gamma = \text{const} = \tau_s. \tag{2.4.5}$$

Here τ_s denotes the yield stress for the material.

The general case determines the medium with a hardening effect. Under a permanent increase in the loading, the same material can pass through three stages. Figure 3.1 schematically illustrates the behaviour of a plain steel in the plane (Γ, τ). Portion OA corresponds to a linear elastic behaviour, AB presents a portion in which the steel yields with increasing strain at unchanged $\tau = \tau_s$. The portion of hardening BC begins at $\Gamma = \Gamma_*$ and a further growth of Γ requires a growth in τ. Rigid-plastic materials actually possess no linear portion, that is the material does not deform unless $\tau = \tau_s$ and then a plastic flow begins which can be followed by a hardening. For nonlinear elastic materials, for instance copper, portions OA and AB are each absent.

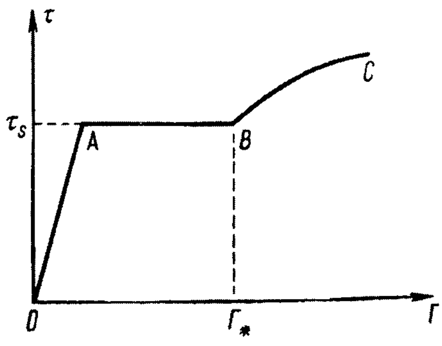

FIGURE 3.1.

Under a uniaxial state of stress, which is approximately realised in tests by means of axial forces, there is a single non-vanishing component σ_x of the stress tensor. In this case, due to eq. (2.2.11) of Chapter 1, we obtain

$$\tau^2 = \frac{1}{3}\sigma_x^2, \qquad \tau = \frac{\sigma_x}{\sqrt{3}},$$

so that referring to σ_s as that value of σ_x which corresponds to the yield stress we have

$$\tau_s = \frac{\sigma_s}{\sqrt{3}}. \tag{2.4.6}$$

In the case of pure shear τ is equal to the shear stress that does not vanish. Determining τ_s from pure shear tests realised by torsion of a thin-walled tube one can predict that in tension tests on a rod made of the same material the yield stress will occur at $\sigma_s = \sqrt{3}\tau_s$. This conclusion has been confirmed by experiments on soft materials (tests by Rosch and Eichinger etc.).

An extended form of condition (2.4.5), referred to as the Mises yield condition, is written in the form

$$\begin{aligned} \tau^2 &= -I_2\left(\operatorname{Dev}\hat{T}\right) = \qquad (2.4.7)\\ &= \frac{1}{6}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2\right]+\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2=\tau_s^2. \end{aligned}$$

Estimates for the external forces enabling the formulation of sufficient criterion for the presence of plastic zones in a loaded body and the necessary condition for their absence are given in Subsection 1.4.9.

The above is applicable to a hypothetical material, which is a physical model possessing the property of storing energy at the expense of the external forces during loading and returning the energy without loss as the initial (natural) state is restored. One of the assumptions for constructing this model was the reversibility of the process. The behaviour of many real materials is irreversible and the stored energy is partially dissipated during unloading. This fact renders the suggested model acceptable only for processes with monotonically increasing intensity of the shear stresses τ. Energy dissipation under unloading a linear-elastic (Hookean) body is negligible; this being so, the irreversibility of the process "loading-unloading" is neglected.

3.3 Generalised Hooke's law

3.3.1 *Elasticity moduli*

The constitutive law of the linear-elastic body under an isothermal deformation process ($\theta = 0$), due to eq. (1.3.11), is written down in the form

$$\hat{T} = \lambda\vartheta\hat{E} + 2\mu\hat{\varepsilon}. \tag{3.1.1}$$

Here λ and μ are the constant moduli of elasticity called the Lamé moduli. The form of this law is the same in the adiabatic process, but, by eqs. (1.3.9) and (2.3.4), λ needs to be replaced by the adiabatic modulus

$$\lambda' = \lambda + \frac{9k\alpha^2\Theta_0}{c} \tag{3.1.2}$$

which slightly differs from λ. By using eq. (3.1.1) one can easily express the strain tensor $\hat{\varepsilon}$ in terms the stress tensor $\hat{T}$. We have

$$I_1\left(\hat{T}\right) = \sigma = (3\lambda + 2\mu)\,\vartheta = 3k\vartheta, \quad \vartheta = \frac{\sigma}{3\lambda + 2\mu}, \tag{3.1.3}$$

so that

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\lambda}{3\lambda + 2\mu}\sigma\hat{E}\right). \tag{3.1.4}$$

Equalities (3.1.1) and (3.1.4) express the generalised Hooke law. The behaviour of a material is prescribed by means of two constants and this is a consequence of assumptions on the medium isotropy and smallness of the components of tensor $\nabla\mathbf{u}$ enabling one to keep only a linear term in the general quadratic dependence between the aligned tensors $\hat{T}$ and $\hat{\varepsilon}$.

Equations (3.1.1) and (3.1.4) are written in terms of the components of tensors $\hat{T}$ and $\hat{\varepsilon}$ in the following way

$$\sigma_x = \lambda\vartheta + 2\mu\varepsilon_x, \quad \tau_{xy} = \mu\gamma_{xy} \text{ etc.}, \tag{3.1.5}$$

$$\varepsilon_x = \frac{1}{2\mu}\left(\sigma_x - \frac{\lambda}{3\lambda + 2\mu}\sigma\right), \quad \gamma_{xy} = \frac{1}{\mu}\tau_{xy} \text{ etc.} \tag{3.1.6}$$

Lamé's moduli are used in theoretical papers whereas in the technical literature they are replaced by other moduli of elasticity, most commonly by Young's modulus E and Poisson's ratio ν. In order to introduce these parameters we separate the term with σ_x in eq. (3.1.6) for ε_x

$$\begin{aligned}\varepsilon_x &= \frac{1}{2\mu}\left[\sigma_x\left(1-\frac{\lambda}{3\lambda+2\mu}\right)-\frac{\lambda}{3\lambda+2\mu}\left(\sigma_y+\sigma_z\right)\right]\\ &= \frac{\lambda+\mu}{\mu\left(3\lambda+2\mu\right)}\left[\sigma_x-\frac{\lambda}{2\left(\lambda+\mu\right)}\left(\sigma_y+\sigma_z\right)\right].\end{aligned}$$

Using the notation

$$\frac{\mu\left(3\lambda+2\mu\right)}{\lambda+\mu}=E, \qquad \frac{\lambda}{2\left(\lambda+\mu\right)}=\nu \tag{3.1.7}$$

the generalised Hooke law (3.1.6) reduces to the form

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{E}\left[\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right], & \gamma_{xy} &= \frac{1}{\mu}\tau_{xy},\\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y-\nu\left(\sigma_z+\sigma_x\right)\right], & \gamma_{yz} &= \frac{1}{\mu}\tau_{yz},\\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z-\nu\left(\sigma_x+\sigma_y\right)\right], & \gamma_{zx} &= \frac{1}{\mu}\tau_{zx}.\end{aligned}\right\} \tag{3.1.8}$$

In the uniaxial state of stress in which there is a single non-vanishing component σ_x, we have

$$\varepsilon_x=\frac{\sigma_x}{E}, \quad \varepsilon_y=\varepsilon_z=-\nu\frac{\sigma_x}{E}=-\nu\varepsilon_x, \quad \gamma_{xy}=\gamma_{yz}=\gamma_{zx}=0, \tag{3.1.9}$$

and one can easily recognize the elementary law of deformation of a rod extended by an axial force: the rod extension in the axial direction is proportional to the stress with the proportionality factor E^{-1} and this strain is accompanied by a proportional transverse contraction of the rod sizes, determined by Poisson's ratio. The general case of the three-axial stretching can be interpreted as a result of superimposing three consequent uniaxial states of stress. This reasoning assumes, of course, the linearity of the deformation law.

The second group of formulae, eq. (3.1.8), expresses a proportionality of the shear strain to the shear stress at pure shear. Provided that only τ_{xy} is not equal to zero, then the only non-vanishing component is the corresponding shear strain γ_{xy}. Nonlinearity of the strain introduces an essential correction into this simple representation, see Subsection 2.6.3.

Using eq. (3.1.7), the expressions for Lamè's moduli, in terms of E and ν, are put in the form

$$\mu=\frac{E}{2\left(1+\nu\right)}, \quad \lambda=2\mu\frac{\nu}{1-2\nu}. \tag{3.1.10}$$

The shear modulus μ is often denoted as G and instead of Poisson's ratio one introduces an inverse value denoted by m

$$\mu = G, \quad m = \frac{1}{\nu}. \tag{3.1.11}$$

The first formula in (3.1.10), expressing the shear modulus in terms of E and ν, can be obtained by means of a geometric construction in which the elongation of the diagonals of a square is considered, the square's sides being subjected to shear stresses which change the right angles between the sides.

Clearly, the generalised Hooke law can be written down in terms of any pair of the introduced moduli

$$k, \quad \lambda, \quad \mu = G, \quad E, \quad \nu = \frac{1}{m}.$$

The pair μ, ν is often used. Then, relationships (3.1.1) and (3.1.4) take the form which is predominantly used in the present book

$$\hat{T} = 2\mu \left(\frac{\nu}{1-2\nu} \vartheta \hat{E} + \hat{\varepsilon} \right), \tag{3.1.12}$$

$$\hat{\varepsilon} = \frac{1}{2\mu} \left(\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} \right). \tag{3.1.13}$$

The summary table below provides the reader with formulae expressing the elasticity moduli in terms of the basic pair of moduli.

moduli	basic pair				
	λ, μ	k, μ	μ, ν	E, ν	E, μ
λ	λ	$k - \frac{2}{3}\mu$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{(E-2\mu)\mu}{3\mu-E}$
$\mu = G$	μ	μ	μ	$\frac{E}{2(1+\nu)}$	μ
k	$\lambda + \frac{2}{3}\mu$	k	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$	$\frac{E\mu}{3(3\mu-E)}$
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{9k\mu}{3k+\mu}$	$2\mu(1+\nu)$	E	E
ν	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3k-2\mu}{6k+2\mu}$	ν	ν	$\frac{1}{2}\frac{E}{\mu} - 1$

Table 3.2

3.3.2 *Specific strain energy for a linear-elastic body*

By eqs. (2.4.2) and (2.4.4), the expression for the specific strain energy in the isothermal process is set as follows

$$A = \frac{1}{2}\left(k\vartheta^2 + \mu\Gamma^2\right). \tag{3.2.1}$$

This formula can also be used for the adiabatic process if k is replaced by k'. Using transformation formulae (A.10.10) and (A.11.6) and entering moduli μ and ν, we obtain

$$A = \frac{1}{2}\left[\lambda I_1^2(\hat{\varepsilon}) + 2\mu I_1\left(\hat{\varepsilon}^2\right)\right]. \tag{3.2.2}$$

Taking into account that

$$I_1\left(\hat{\varepsilon}^2\right) = \varepsilon_{ts}\varepsilon_{st} = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \frac{1}{2}\left(\gamma_{12}^2 + \gamma_{23}^2 + \gamma_{31}^2\right),$$

we arrive at the following expression for the specific strain energy $A(\varepsilon)$ in terms of the components of tensor $\hat{\varepsilon}$

$$A(\varepsilon) = \frac{1}{2}\left[\lambda\vartheta^2 + 2\mu\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2\right) + \mu\left(\gamma_{12}^2 + \gamma_{23}^2 + \gamma_{31}^2\right)\right]. \tag{3.2.3}$$

Let us recall that in the processes under consideration a variation of the specific strain energy (equal to variation of the free energy in the first case and variation of the internal energy in the second case) can be written, due to eq. (1.2.1), in the form

$$\begin{aligned}\delta A &= \delta' A_{(e)} = \hat{T}\cdot\cdot\,\delta\hat{\varepsilon} \\ &= t_{11}\delta\varepsilon_{11} + t_{22}\delta\varepsilon_{22} + t_{33}\delta\varepsilon_{33} + t_{12}\delta\gamma_{12} + t_{23}\delta\gamma_{23} + t_{31}\delta\gamma_{31}.\end{aligned} \tag{3.2.4}$$

This representation yields the formulae

$$t_{st} = \frac{1}{2}\left(\frac{\partial A}{\partial\varepsilon_{st}} + \frac{\partial A}{\partial\varepsilon_{ts}}\right) = \begin{cases} \dfrac{\partial A}{\partial\varepsilon_{ss}}, & t = s, \\ \dfrac{\partial A}{\partial\gamma_{st}}, & t \neq s, \end{cases} \tag{3.2.5}$$

which hold not only for a isotropic linear-elastic body but also for any medium for which one can introduce the concept of strain energy as a function of the strain components determined by the external work.

In a linear-elastic (Hookean) body, A is a homogenous quadratic form of the strain components and, by Euler's theorem,

$$2A = \frac{\partial A}{\partial\varepsilon_{st}}\varepsilon_{st} = t_{st}\varepsilon_{st} = \hat{T}\cdot\cdot\,\hat{\varepsilon}.$$

Thus we are led to a bilinear representation of the specific strain energy which is denoted as

$$A(\varepsilon,\sigma) = \frac{1}{2}\hat{T}\cdot\cdot\,\hat{\varepsilon} = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right). \tag{3.2.6}$$

From this formula, by using Hooke's law in the form of eq. (3.1.13) we obtain the following expression for the specific strain energy in terms of the stress tensor denoted as

$$A(\sigma) = \frac{1}{4\mu}\hat{T} \cdot\cdot \left[\hat{T} - \frac{\nu}{1+\nu}\hat{E} I_1\left(\hat{T}\right)\right] = \frac{1}{4\mu}\left[I_1\left(\hat{T}^2\right) - \frac{\nu}{1+\nu} I_1^2\left(\hat{T}\right)\right] \tag{3.2.7}$$

or in the extended form

$$A(\sigma) = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right] + \frac{1}{2\mu}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right). \tag{3.2.8}$$

Aligned tensors appear in the bilinear equation (3.2.6) for the specific strain. For this reason, along with eq. (3.2.4) the variation can be set in the form

$$\begin{aligned} \delta A &= \hat{T} \cdot\cdot \delta\hat{\varepsilon} = \hat{\varepsilon} \cdot\cdot \delta\hat{T} \\ &= \varepsilon_x\delta\sigma_x + \varepsilon_y\delta\sigma_y + \varepsilon_z\delta\sigma_z + \gamma_{xy}\delta\tau_{xy} + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx}\,. \end{aligned} \tag{3.2.9}$$

This leads to relationships which are inverse of eq. (3.2.5)

$$\left.\begin{array}{lll} \dfrac{\partial A}{\partial \sigma_x} = \varepsilon_x, & \dfrac{\partial A}{\partial \sigma_y} = \varepsilon_y, & \dfrac{\partial A}{\partial \sigma_z} = \varepsilon_z, \\ \dfrac{\partial A}{\partial \tau_{xy}} = \gamma_{xy}, & \dfrac{\partial A}{\partial \tau_{yz}} = \gamma_{yz}, & \dfrac{\partial A}{\partial \tau_{zx}} = \gamma_{zx}, \end{array}\right\} \tag{3.2.10}$$

which are valid, similar to eq. (3.2.6), only for the Hookean body.

3.3.3 Clapeyron's formula. Limits for the elasticity moduli.

The strain energy of an elastic body is determined by the integral of the specific strain energy over the volume

$$a = \iiint\limits_V A d\tau. \tag{3.3.1}$$

This value is equal to half the work of the external forces in a sequence of equilibrium states of the linear-elastic body beginning from the natural state. The proof is based on the equality

$$\iiint\limits_V \mathbf{u} \cdot \left(\operatorname{div}\hat{T} + \rho\mathbf{K}\right) d\tau + \iint\limits_O \left(\mathbf{F} - \mathbf{n}\cdot\hat{T}\right)\cdot \mathbf{u} do = 0. \tag{3.3.2}$$

Indeed, due to eqs. (B.3.10) and (3.2.6),

$$\iiint\limits_V \mathbf{u} \cdot \operatorname{div} \hat{T} d\tau = \iiint\limits_V \operatorname{div} \left(\hat{T} \cdot \mathbf{u} \right) d\tau - \iiint\limits_V \hat{T} \cdot \cdot \hat{\varepsilon} d\tau$$
$$= \iint\limits_O \mathbf{n} \cdot \hat{T} \cdot \mathbf{u} do - \iiint\limits_V 2A d\tau$$

and inserting this result into eq. (3.3.2) leads to the required relation

$$a = \frac{1}{2} \left(\iiint\limits_V \rho \mathbf{K} \cdot \mathbf{u} d\tau + \iint\limits_O \mathbf{F} \cdot \mathbf{u} do \right). \tag{3.3.3}$$

This is Clapeyron's formula. It states that the work of the external forces is stored in the linear-elastic body in the form of the strain energy which is returned in the form of work under a slow unloading (or kinetic energy in the case of an abrupt unloading). From this energy perspective, it follows that $a > 0$. This statement is equipollent to a local, i.e. in any part of volume V, property

$$A > 0, \tag{3.3.4}$$

due to the arbitrariness of volume V.

Statement (3.3.4) is a property assigned to the elastic body, that is, there is no regions with $A < 0$ in it. In a linear-elastic body this must be ensured by the requirement imposed on the elasticity moduli

$$k > 0, \quad \mu > 0. \tag{3.3.5}$$

This is immediately evident from eq. (3.2.1): for zero shear ($\Gamma = 0$) inequality (3.3.4) requires positiveness of the bulk modulus ($k > 0$), whereas for unchanged volume ($\vartheta = 0$) it requires positiveness of the shear modulus. Inequalities (3.3.5) correspond to habitual static concepts regarding the behaviour of a solid, namely, in the pure shear stress state (Subsection 1.2.4) the shear strain and shear stress have coincident signs ($\mu > 0$) and the volume of a cube decreases under a uniform pressure ($k > 0$).

From the expression for k

$$k = \frac{2}{3} \mu \frac{1 + \nu}{1 - 2\nu}$$

it follows that the first inequality holds true for the following values of Poisson's ratio

$$-1 < \nu < \frac{1}{2}. \tag{3.3.6}$$

Elongation of a rod of material with negative ν greater than -1 would be accompanied by an increase in its transverse dimension. Existence of such materials is not impossible from an energy perspective.

Let us notice that inequalities (3.3.5) can also be written in the form

$$3\lambda + 2\mu > 0, \qquad \mu > 0. \tag{3.3.7}$$

Remark. The squares of velocities of propagation of the shear waves and compression waves in the elastic medium are respectively given by

$$\frac{\mu}{\rho}, \quad \frac{1}{\rho}(\lambda + 2\mu) = \frac{2\mu}{\rho}\frac{1-\nu}{1-2\nu}.$$

Hence $\mu > 0$ and $\nu < \frac{1}{2}$, i.e. propagation of compression waves is possible in a solid with any $\nu < 0$. The restriction $\nu > -1$ is a result of the independent requirement (3.3.4). In a hypothetical material with $\nu > -1$ the uniform compression of the cube would be accompanied by an increase in its volume.

3.3.4 Taking account of thermal terms. Free energy

Let us disregard the assumption that the process is isothermal or adiabatic. Then the variation of the specific elementary work of the external forces can not be equated to the variation of the strain energy, since the very concept of strain energy is no longer applicable. The role of strain energy is played by one of the thermodynamic potentials: either the free energy or the Gibbs potential.

Let us make use of eqs. (2.1.13) and (1.3.13) and set the equation for the specific internal energy δE in the following form

$$\begin{aligned}\delta E &= k\vartheta\delta\vartheta + \mu\Gamma\delta\Gamma + 3k\alpha\Theta_0\delta\vartheta + c\delta\vartheta \\ &= \delta' A_{(e)} + 3k\alpha(\Theta_0 + \theta)\delta\vartheta + c\delta\vartheta\end{aligned}$$

or recalling eqs. (1.2.1) and (2.1.13) we obtain

$$\delta E = \delta' A_{(e)} + 3k\alpha\Theta\delta\vartheta + c\delta\vartheta = t_{st}\delta\varepsilon_{st} + 3k\alpha\Theta\delta\vartheta + c\delta\vartheta. \tag{3.4.1}$$

Considering the internal energy and the entropy as functions of the strain components and temperature yields

$$\delta E = \delta' A_{(e)} + \delta' Q = t_{st}\delta\varepsilon_{st} + \Theta\delta S = t_{st}\delta\varepsilon_{st} + \Theta\left(\frac{\partial S}{\partial\varepsilon_{st}}\delta\varepsilon_{st} + \frac{\partial S}{\partial\Theta}\delta\Theta\right)$$

or

$$\delta E = \left(t_{st} + \Theta\frac{\partial S}{\partial\varepsilon_{st}}\right)\delta\varepsilon_{st} + \Theta\frac{\partial S}{\partial\Theta}\delta\Theta. \tag{3.4.2}$$

Comparison with eq. (3.4.1) leads to the formulae

$$\frac{\partial S}{\partial \varepsilon_{st}} = \begin{cases} 0, & s \neq t, \\ 3k\alpha, & s = t, \end{cases} \qquad \frac{\partial S}{\partial \Theta} = \frac{c}{\Theta}, \tag{3.4.3}$$

which yield the following expression for the entropy

$$S = 3k\alpha\vartheta + \int_{\Theta_0}^{\Theta} \frac{c(\Theta)}{\Theta} d\Theta, \tag{3.4.4}$$

where Θ_0 denotes the absolute temperature in the natural state of the body.

Using eq. (2.1.13), in the case of the linear-elastic body we have

$$E = \frac{1}{2}\left(k\vartheta^2 + \mu\Gamma^2\right) + 3k\alpha\Theta_0\vartheta + \int_{\Theta_0}^{\Theta} c(\Theta)\, d\Theta,$$

or

$$E = A(\varepsilon) + 3k\alpha\Theta_0\vartheta + \int_{\Theta_0}^{\Theta} c(\Theta)\, d\Theta, \tag{3.4.5}$$

with $A(\varepsilon)$ being a quadratic form of the strain components. Here $A(\varepsilon)$ is formally coincident with the specific strain energy in the isothermal process.

Now, due to eqs. (3.4.5) and (3.4.4) and the definition of the free energy (2.2.3) we have

$$F = A(\varepsilon) - 3k\alpha\vartheta\theta - \int_{\Theta_0}^{\Theta} \frac{c(\xi)}{\xi} (\Theta - \xi)\, d\xi. \tag{3.4.6}$$

The derivatives of this function with respect to the strain components determine the components of the stress tensor. Indeed, by eqs. (3.2.3) and (1.3.9) we have

$$\frac{\partial F}{\partial \varepsilon_x} = \lambda\vartheta + 2\mu\varepsilon_x - (3\lambda + 2\mu)\,\alpha\theta, \qquad \frac{\partial F}{\partial \gamma_{xy}} = \mu\gamma_{xy} \tag{3.4.7}$$

etc. This coincides with relationship (1.3.11) derived above for the case of a Hookean body

$$\hat{T} = \lambda\vartheta\hat{E} + 2\mu\hat{\varepsilon} - (3\lambda + 2\mu)\,\alpha\theta\hat{E}. \tag{3.4.8}$$

Taking the derivative of F with respect to θ leads to the expression for the entropy

$$\frac{\partial F}{\partial \theta} = -\left(3k\alpha\vartheta + \int_{\Theta_0}^{\Theta} \frac{c(\xi)}{\xi} d\xi\right) = -S \tag{3.4.9}$$

which was derived earlier. Using eq. (3.4.8) we have

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) + \alpha\theta\hat{E}. \tag{3.4.10}$$

The term

$$\hat{\varepsilon}' = \alpha\theta\hat{E} \tag{3.4.11}$$

represents the strain tensor of a free elementary cube heated to temperature θ. The surrounding medium prevents any change in the size of this cube and this leads to a state of stress with tensor $\hat{T}$ which, in turn, produces the strains given by Hooke's law for the isothermal process

$$\hat{\varepsilon}'' = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right). \tag{3.4.12}$$

This strain should be superimposed on the thermal strain (3.4.11) which explains the structure of formula (3.4.10). Strictly speaking, tensors $\hat{\varepsilon}'$ and $\hat{\varepsilon}''$ can not be termed strain tensors since the compatibility conditions are fulfilled for tensor $\hat{\varepsilon}' + \hat{\varepsilon}''$ rather than for each separate term in this sum. Let us notice in passing that, due to eq. (3.4.10)

$$\vartheta = I_1(\hat{\varepsilon}) = \frac{1}{2\mu}\frac{1-2\nu}{1+\nu}\sigma + 3\alpha\theta. \tag{3.4.13}$$

3.3.5 The Gibbs thermodynamic potential

This thermodynamic function is denoted by G and is related to the free energy by the Legendre transformation

$$G = t_{sk}\varepsilon_{sk} - F = \hat{T}\cdot\cdot\hat{\varepsilon} - F.$$

The independent variables of the Gibbs potential are the components of the stress tensor $\hat{T}$ and temperature θ. By virtue of eq. (3.4.10)

$$\begin{aligned}\hat{T}\cdot\cdot\hat{\varepsilon} = I_1\left(\hat{T}\cdot\hat{\varepsilon}\right) &= \frac{1}{2\mu}\left[I_1\left(\hat{T}^2\right) - \frac{\nu}{1+\nu}\sigma^2\right] + \alpha\theta\sigma \\ &= 2A(\sigma) + \alpha\theta\sigma,\end{aligned} \tag{3.5.1}$$

where $A(\sigma)$ is a quadratic form of the components of the stress tensor (3.2.8). It remains to express the free energy F in terms of these components. We have

$$\begin{aligned}I_1(\hat{\varepsilon}) &= I_1(\hat{\varepsilon}') + I_1(\hat{\varepsilon}''),\\ I_1^2(\hat{\varepsilon}) &= I_1^2(\hat{\varepsilon}'') + \frac{1-2\nu}{\mu(1+\nu)}3\alpha\theta\sigma + 9\alpha^2\theta^2,\\ \hat{\varepsilon}^2 &= \hat{\varepsilon}''^2 + \frac{1}{\mu}\alpha\theta\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) + \alpha^2\theta^2\hat{E},\\ I_1\left(\hat{\varepsilon}^2\right) &= I_1\left(\hat{\varepsilon}''^2\right) + \frac{1}{\mu}\alpha\theta\frac{1-2\nu}{1+\nu}\sigma + 3\alpha^2\theta^2.\end{aligned}$$

Using eq. (3.2.2) and representing λ in terms of μ and ν yields

$$\begin{aligned}A(\varepsilon) &= \frac{1}{2}\left[\lambda I_1^2\left(\hat{\varepsilon}''\right) + 2\mu I_1\left(\hat{\varepsilon}''^2\right)\right] + \alpha\theta\sigma + \mu\frac{1+\nu}{1-2\nu}3\alpha^2\theta^2 \\ &= A(\sigma) + \alpha\theta\sigma + \mu\frac{1+\nu}{1-2\nu}3\alpha^2\theta^2 \end{aligned} \tag{3.5.2}$$

and further, by eqs. (3.4.13) and (3.2.2)

$$F = A(\sigma) + \alpha\theta\sigma + \mu\frac{1+\nu}{1-2\nu}3\alpha^2\theta^2 - 2\mu\frac{1+\nu}{1-2\nu}\vartheta\alpha\theta - \int_{\Theta_0}^{\Theta}\frac{c(\xi)}{\xi}(\Theta-\xi)\,d\xi,$$

or

$$F = A(\sigma) - \mu\frac{1+\nu}{1-2\nu}3\alpha^2\theta^2 - \int_{\Theta_0}^{\Theta}\frac{c(\xi)}{\xi}(\Theta-\xi)\,d\xi. \tag{3.5.3}$$

Inserting into eq. (3.5.1) leads to the following expression for the Gibbs potential

$$G = A(\sigma) + \alpha\theta\sigma + \mu\frac{1+\nu}{1-2\nu}3\alpha^2\theta^2 + \int_{\Theta_0}^{\Theta}\frac{c(\xi)}{\xi}(\Theta-\xi)\,d\xi. \tag{3.5.4}$$

Using the property of the Legendre transformation, we arrive at the relationships inverse to (3.4.7)

$$\varepsilon_x = \frac{\partial G}{\partial\sigma_x} = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y+\sigma_z\right)\right] + \alpha\theta, \qquad \gamma_{xy} = \frac{\partial G}{\partial\tau_{xy}} = \frac{1}{\mu}\tau_{xy} \tag{3.5.5}$$

etc. The entropy is defined as follows

$$S = \frac{\partial G}{\partial\theta} = \alpha\sigma + 6\mu\alpha^2\frac{1+\nu}{1-2\nu}\theta + \int_{\Theta_0}^{\Theta}\frac{c(\xi)}{\xi}d\xi. \tag{3.5.6}$$

Clearly, the same expression follows from eqs. (3.4.4) and (3.4.13).

In the problems of thermal stresses, the free energy and the Gibbs potential play the part of the strain energy expressed respectively in terms of the strains and stresses.

Assuming the specific heat at constant volume to be independent of the temperature and the temperature change $\theta = \Theta - \Theta_0$ to be small we have

$$\int_{\Theta_0}^{\Theta}\frac{c(\xi)}{\xi}d\xi = c\ln\frac{\Theta}{\Theta_0} = c\ln\left(1+\frac{\theta}{\Theta_0}\right) \approx c\frac{\theta}{\Theta_0},$$

and referring to Table 3.2 we obtain

$$S = \alpha\sigma + 9k\alpha^2\theta + c\frac{\theta}{\Theta_0}. \tag{3.5.7}$$

In the natural state $S = 0$ and it remains equal to zero in the adiabatic process. For this reason, in this process the temperature change in a solid is equal to

$$\theta = -\frac{\alpha\sigma}{9k\alpha^2 + \dfrac{c}{\Theta_0}} = -\frac{\alpha\Theta_0}{c_p}\sigma, \tag{3.5.8}$$

c_p denoting the specific heat at constant volume, see also eqs. (2.3.2), (2.3.4) and (2.3.6). One can find lower estimates for the maximum of the absolute value for this parameter in Subsection 1.4.12.

3.3.6 Equation of thermal conductivity

We introduce into consideration the vector of heat flux $\mathbf{q}$ which is proportional to the temperature gradient and directed toward the temperature decrease

$$\mathbf{q} = -K \operatorname{grad}\theta, \tag{3.6.1}$$

where K is the thermal conductivity coefficient. Vector $\mathbf{q}$ determines the amount of heat leaving an arbitrary volume V in a unit of time across the bounding surface O

$$\iint\limits_O \mathbf{n}\cdot\mathbf{q}\,do = -\iint\limits_O \mathbf{n}\cdot K \operatorname{grad}\theta\,do = -\iiint\limits_V \operatorname{div} K \operatorname{grad}\theta\,d\tau. \tag{3.6.2}$$

The amount of heat supplied to a unit of volume in a unit of time can be expressed as follows

$$\frac{\delta' Q}{dt} = \Theta\frac{dS}{dt} = \Theta\dot{S}. \tag{3.6.3}$$

Hence

$$\iiint\limits_V \left(\Theta\dot{S} - \operatorname{div} K \operatorname{grad}\theta\right) d\tau = 0$$

and due to the arbitrariness of volume V

$$\Theta\dot{S} - \operatorname{div} K \operatorname{grad}\theta = 0. \tag{3.6.4}$$

Replacing here the entropy S by expression (3.4.4), we obtain

$$(3\lambda + 2\mu)\,\alpha\Theta\dot{\vartheta} + c\,(\Theta)\,\dot{\theta} - \operatorname{div} K \operatorname{grad}\theta = 0. \tag{3.6.5}$$

As above, $\alpha\theta$ is considered to have the order of smallness of ϑ. Assuming K as being constant, we arrive at the equation of thermal conductivity

$$\nabla^2\theta - \frac{1}{a}\dot{\theta} - \frac{\alpha\Theta_0}{K}\,(3\lambda + 2\mu)\,\dot{\vartheta} = 0, \tag{3.6.6}$$

where

$$a = \frac{K}{c\,(\Theta_0)}$$

is the coefficient of thermal diffusivity. Another form of the thermal conductivity equation is obtained by replacing S in eq. (3.6.4) by means of eq. (3.5.7) and takes the form

$$\nabla^2\theta - \frac{1}{a'}\dot{\theta} - \frac{\alpha\Theta_0}{K}\dot{\sigma} = 0, \tag{3.6.7}$$

where $a' = K/c_p$ as follows from eqs. (2.3.6) and (2.3.4).

Usage of the concept of entropy in a stationary equilibrium process for deriving the equation of non-stationary temperature distribution is based on the assumption of local equilibrium processes and slowly progressing processes.

Equations (3.6.6) and (3.6.7) differ from the classical Fourier equation of thermal conductivity

$$\nabla^2\theta - \frac{1}{a}\dot{\theta} = 0 \tag{3.6.8}$$

in the terms due to the deformation of the solid. In the general non-stationary case, the problems of thermal conductivity and elasticity are coupled: the temperature distribution depends on the strain whilst the latter is dependent upon the temperature distribution. The equations of equilibrium of the solid must be replaced by the equations governing dynamics of the solid. This effect can be considerable under abrupt changes in temperature (under a "thermal shock") otherwise it is mostly negligible. In the latter case, a "quasi-static" consideration is used, that is, the thermal conductivity equation takes the form of Fourier's equation (3.6.8) and the solid is assumed to be in equilibrium (acceleration is neglected). The problem of thermal conductivity is then solved independently of the problem of the elasticity theory.

Under a stationary temperature distribution the thermal conductivity equation takes the form

$$\nabla^2\theta = 0. \tag{3.6.9}$$

4

Governing relationships in the linear theory of elasticity

4.1 Differential equations governing the linear theory of elasticity

4.1.1 Fundamental relationships

The basic equations governing elasticity theory can be classified into three groups of relationships. The first group is presented by the equations of statics in volume V

$$\operatorname{div} \hat{T} + \rho \mathbf{K} = 0, \tag{1.1.1}$$

relating six components of the symmetric stress tensor $\hat{T}$ by three equations.

The second group of equations determines the linear strain tensor $\hat{\varepsilon}$ in terms of the displacement vector $\mathbf{u}$

$$\hat{\varepsilon} = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^* \right]. \tag{1.1.2}$$

Here we have six equations determining the components of the linear strain tensor by means of the first derivatives of the displacement vector.

The constitutive law for a linear elastic body is formulated in the third group of six equations. For an isotropic solid in an isothermal or adiabatic process this law, referred to as Hooke's law, is written in the form

$$\hat{T} = 2\mu \left(\frac{\nu}{1 - 2\nu} \vartheta \hat{E} + \hat{\varepsilon} \right) \tag{1.1.3}$$

or in the form of the inverse relation

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right). \tag{1.1.4}$$

The three groups contain a total of fifteen equations, which is the same number of unknowns - twelve components of two symmetric tensors of second rank $\hat{T}$ and $\hat{\varepsilon}$ and three components of vector $\mathbf{u}$.

4.1.2 Boundary conditions

The conditions on the surface need to be added to the system of equations (1.1.1)-(1.1.3) determining the behaviour of the linear elastic body in its volume. These conditions prescribe either the surface forces or the displacement of the surface points. These distinguish the internal problem for the elastic body bounded from outside from the external problem for an unbounded medium with a cavity or cavities. For each of these problems one states three types of problems.

In the first problem a kinematic boundary condition is posed. In volume V the displacement vector is sought such that it takes a prescribed value on the surface O bounding this volume

$$\mathbf{u}|_O = \mathbf{u}_*\left(x_1, x_2, x_3\right). \tag{1.2.1}$$

Evidently, coordinates x_1, x_2, x_3 are related by the surface equation.

The second boundary value problem is the static one. Given the distribution of surface forces $\mathbf{F}$, the boundary condition implies the equilibrium equation on the surface

$$\mathbf{n}\cdot\hat{T}\Big|_O = \mathbf{F}. \tag{1.2.2}$$

The third boundary value problem is the mixed one. A kinematic condition is posed on part O_1 of the surface, whereas on the other part O_2 a static boundary condition holds

$$\left.\begin{array}{l} \mathbf{u}|_{O_1} = \mathbf{u}_*\left(x_1, x_2, x_3\right), \\ \mathbf{n}\cdot\hat{T}\Big|_{O_2} = \mathbf{F}. \end{array}\right\} \tag{1.2.3}$$

Clearly, the above boundary conditions do not exhaust the possible variety of problems of elasticity theory. For example, not all three components of vector $\mathbf{u}$ or force $\mathbf{F}$ can be given on a certain part of the boundary. So the boundary conditions for the contact surface of the body resting on a rigid smooth bed are set as follows

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times(\mathbf{F}\times\mathbf{n}) = 0, \tag{1.2.4}$$

where $\mathbf{n}$ denotes the unit vector of the outward normal to the body surface. The first and second condition respectively express the absence of the normal component of the displacement and the tangential component of the force. In addition to this, the projection of force $\mathbf{F}$ on normal $\mathbf{n}$, i.e. the distributed reaction of the bedding $\mathbf{n} \cdot \mathbf{F}$, is not known in advance. The problem becomes considerably difficult in the case of a one-sided constraint, that is when the bed does not prevent the body displacement in direction $-\mathbf{n}$. Then the inequality $\mathbf{n} \cdot \mathbf{F} \leq 0$ needs to be added to condition (1.2.4) and on that part of surface (not known in advance) where this inequality does not hold true it should be replaced by the condition $\mathbf{F} = 0$.

Two ways of solving the problems of elasticity theory are known. The first one implies determining the displacement vector $\mathbf{u}$. Using this it is not difficult to calculate the strain tensor $\hat{\varepsilon}$ in terms of $\mathbf{u}$ and thus the stress tensor in terms of $\hat{\varepsilon}$. This is the only way when the first boundary value problem is considered. However this way is not always the simplest one and in many cases the way of solving the problem in terms of stresses is favoured. Then one poses the question of seeking a statically possible stress tensor $\hat{T}$ such that the corresponding stress tensor $\hat{\varepsilon}$ satisfies the compatibility condition (2.1.5) of Chapter 2. The displacement vector $\mathbf{u}$ is then obtained by Cesaro's formula (2.2.2) of Chapter 2.

Both ways rely on the differential equations governing elasticity theory but they do not exhaust all the possible approaches to solving the problems. Other possibilities involve using the principles of minimum energy and the direct methods of solving the variational problems.

4.1.3 Differential equations governing the linear theory of elasticity in terms of displacements

Using the fundamental relationships of Subsection 4.1.1 it is easy to obtain the differential equations for vector $\mathbf{u}$. To this end, it is sufficient to substitute the expression for the stress tensor in terms of this vector, to obtain

$$\mu \operatorname{div} \left[\frac{2\nu}{1-2\nu} \vartheta \hat{E} + \nabla \mathbf{u} + (\nabla \mathbf{u})^* \right] + \rho \mathbf{K} = 0. \tag{1.3.1}$$

Inserting

$$\operatorname{div} \vartheta \hat{E} = \hat{E} \cdot \operatorname{grad} \vartheta = \operatorname{grad} \operatorname{div} \mathbf{u}, \quad \operatorname{div} \nabla \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u},$$

$$\operatorname{div} (\nabla \mathbf{u})^* = \nabla \cdot (\nabla \mathbf{u})^* = \mathbf{i}_s \frac{\partial}{\partial x_s} \cdot \mathbf{i}_k \mathbf{i}_t \frac{\partial u_k}{\partial x_t} = \mathbf{i}_t \frac{\partial^2 u_s}{\partial x_t \partial x_s} = \operatorname{grad} \operatorname{div} \mathbf{u}$$

(see also Section B.4) into eq. (1.3.1) yields the sought-for differential equation

$$\frac{1}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{u} + \nabla^2 \mathbf{u} + \frac{\rho}{\mu} \mathbf{K} = 0. \tag{1.3.2}$$

Projecting this onto the axes of the Cartesian coordinate system we arrive at three equations

$$\left.\begin{aligned} \frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial x} + \nabla^2 u + \frac{\rho}{\mu}K_x = 0, \\ \frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial y} + \nabla^2 v + \frac{\rho}{\mu}K_y = 0, \\ \frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial z} + \nabla^2 w + \frac{\rho}{\mu}K_z = 0, \end{aligned}\right\} \tag{1.3.3}$$

where

$$\vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div}\mathbf{u}. \tag{1.3.4}$$

These equations are referred to as the differential equations of elasticity theory in terms of displacements. They were first derived by Navier in 1827 in the framework of a "single constant theory" (Poisson's ratio $\nu = 1/4$) and simultaneously by Cauchy in 1827-1828.

Equations (1.3.3) result in the following differential equation for the dilatation

$$\nabla^2\vartheta + \frac{\rho}{2\mu}\frac{1-2\nu}{1-\nu}\operatorname{div}\mathbf{K} = 0. \tag{1.3.5}$$

Recalling transformation (B.4.5) one can reset eq. (1.3.2) in another form

$$\frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\vartheta - \operatorname{rot}\operatorname{rot}\mathbf{u} + \frac{\rho}{\mu}\mathbf{K} = 0. \tag{1.3.6}$$

The analysis that follows relies on the following, easily proved, relationship

$$\nabla^2\mathbf{R}\vartheta = \mathbf{R}\nabla^2\vartheta + 2\operatorname{grad}\vartheta = -\frac{\rho}{2\mu}\frac{1-2\nu}{1-\nu}\mathbf{R}\operatorname{div}\mathbf{K} + 2\operatorname{grad}\vartheta, \tag{1.3.7}$$

where $\mathbf{R} = \mathbf{i}_s x_s$ denotes the position vector. Replacing $\operatorname{grad}\vartheta$ in eq. (1.3.2) using eq. (1.3.7) we obtain the equation in terms of displacements in the form suggested by Tedone

$$\nabla^2\left(\mathbf{u} + \frac{1}{2(1-2\nu)}\mathbf{R}\vartheta\right) + \frac{\rho}{\mu}\left(\mathbf{K} + \frac{1}{4(1-\nu)}\mathbf{R}\operatorname{div}\mathbf{K}\right) = 0. \tag{1.3.8}$$

When the volume forces are absent, dilatation ϑ, see eq. (1.3.5), is a harmonic function whereas $\mathbf{u}$ is a biharmonic vector

$$\nabla^2\vartheta = 0; \quad \nabla^4\mathbf{u} = 0: \quad \nabla^4 u = 0, \quad \nabla^4 v = 0, \quad \nabla^4 w = 0. \tag{1.3.9}$$

The latter follows directly from eq. (1.3.3). However it should be mentioned that the three biharmonic functions are not independent. Indeed, using eq.

(1.3.8), vector $\mathbf{u}$ can be represented (for $\mathbf{K} = 0$) in terms of four harmonic functions which are a harmonic vector $\mathbf{a}$ and a harmonic function ϑ

$$\mathbf{u} = \mathbf{a} - \frac{1}{2(1-2\nu)} \mathbf{R}\vartheta \tag{1.3.10}$$

related by condition (1.3.4).

On the part of the boundary where the surface forces are given, boundary condition (1.2.2) is written down in terms of the displacement vector in the following form

$$\begin{aligned} \mathbf{F} = \mathbf{n} \cdot \hat{T} &= 2\mu \left(\frac{\nu}{1-2\nu} \vartheta \mathbf{n} + \mathbf{n} \cdot \hat{\varepsilon} \right) \\ &= 2\mu \left(\frac{\nu}{1-2\nu} \vartheta \mathbf{n} + \mathbf{n} \cdot \nabla \mathbf{u} + \frac{1}{2} \mathbf{n} \times \operatorname{rot} \mathbf{u} \right), \end{aligned} \tag{1.3.11}$$

where equalities (1.2.13) and (1.2.12) of Chapter 2 were used. Projecting boundary condition (1.3.11) on the axes of the Cartesian coordinate system yields three scalar conditions

$$F_x = 2\mu \left\{ \frac{\nu}{1-2\nu} \vartheta n_x + \frac{\partial u}{\partial n} + \frac{1}{2} \left[n_y \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - n_z \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \right] \right\} \tag{1.3.12}$$

etc. Here

$$\frac{\partial u}{\partial n} = n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} + n_z \frac{\partial u}{\partial z}$$

denotes the derivative of u with respect to the normal to the surface.

4.1.4 Solution in the Papkovich-Neuber form

The difficulty with finding the particular solutions of the system of equations in terms of the displacements arises because each of the sought functions u, v and w appears in all three equations (1.3.3). This difficulty was alleviated by P.F. Papkovich (1932) and H. Neuber (1934) who suggested that the displacement should be represented in terms of harmonic functions. This enables one to use a well-known "catalogue" of particular solutions of the Laplace equation and sometimes it is even possible to reduce the problem, if not completely at least partly, to one of the classical problems of the theory of harmonic functions (the theory of potential).

One can suggest a wealth of representations of the type (1.3.10) for solving the homogeneous ($\mathbf{K} = 0$) system of equations of elasticity theory in terms of harmonic functions. Their shortcoming, remedied in the Papkovich-Neuber solution, is that the harmonic functions introduced are independent.

Let $\mathbf{B}$ be a harmonic vector whose Laplace operator vanishes, i.e.

$$\nabla^2 \mathbf{B} = 0. \tag{1.4.1}$$

The projections of this vector on the axes of the Cartesian coordinate system also satisfy the Laplace equation

$$\nabla^2 B_x = 0, \quad \nabla^2 B_y = 0, \quad \nabla^2 B_z = 0. \tag{1.4.2}$$

It is, however, erroneous to generalise this reasoning to the case of the axes of a curvilinear coordinate system, since the projections of Laplace's operator on a vector of axes of varying directions are not equal to Laplace's operator of the projections on these axes.

Assuming the volume force as being potential

$$\rho \mathbf{K} = -\operatorname{grad} \Pi \tag{1.4.3}$$

we look for the solution of eq. (1.3.2) in the form

$$\mathbf{u} = 4(1-\nu)\mathbf{B} + \operatorname{grad}\chi. \tag{1.4.4}$$

Then noting that

$$\begin{gathered} \operatorname{div}\mathbf{u} = 4(1-\nu)\operatorname{div}\mathbf{B} + \nabla^2\chi, \\ \operatorname{grad}\operatorname{div}\mathbf{u} = \operatorname{grad}\left[4(1-\nu)\operatorname{div}\mathbf{B} + \nabla^2\chi\right] \end{gathered}$$

and taking into account eq. (1.4.1) we arrive at the relationship

$$\operatorname{grad}\left[4(1-\nu)\operatorname{div}\mathbf{B} + 2(1-\nu)\nabla^2\chi - \frac{1-2\nu}{\mu}\Pi\right] = 0,$$

which can be fulfilled by requiring that χ satisfies the following equation

$$\nabla^2\chi = -2\operatorname{div}\mathbf{B} + \frac{1-2\nu}{2\mu(1-\nu)}\Pi. \tag{1.4.5}$$

The general solution of this equation consists of the sum of the solution of the equation

$$\nabla^2\chi = -2\operatorname{div}\mathbf{B} \tag{1.4.6}$$

and any particular solution χ_0 of Poisson's equation

$$\nabla^2\chi_0 = \frac{1-2\nu}{2\mu(1-\nu)}\Pi. \tag{1.4.7}$$

The particular solution of eq. (1.4.6) can be taken in the form

$$\chi = -\mathbf{R}\cdot\mathbf{B} = -(xB_x + yB_y + zB_z), \tag{1.4.8}$$

which can be easily proved by direct calculation

$$\nabla^2 \mathbf{R} \cdot \mathbf{B} = \mathbf{R} \cdot \nabla^2 \mathbf{B} + \mathbf{B} \cdot \nabla^2 \mathbf{R} + 2 \operatorname{div} \mathbf{B} = 2 \operatorname{div} \mathbf{B}, \tag{1.4.9}$$

since $\nabla^2 \mathbf{R} = 0$. The general solution of this equation is obtained by adding an arbitrary harmonic vector denoted as $-B_0$ to eq. (1.4.8). Hence

$$\chi = -(\mathbf{R} \cdot \mathbf{B} + B_0)$$

and the sought-for representation of the solution of the equation of elasticity theory is written as follows

$$\mathbf{u} = 4(1-\nu)\mathbf{B} - \operatorname{grad}(\mathbf{R} \cdot \mathbf{B} + B_0) + \operatorname{grad}\chi_0 . \tag{1.4.10}$$

The latter term is omitted if the volume forces are absent and it should be replaced by a particular solution of eq. (1.3.3) in the case of nonpotential volume forces. Such a particular solution is usually obtained with ease, and there exists a general approach for constructing this solution, see Subsection 3.7 of the present chapter.

By virtue of eqs. (B.2.12) and (B.2.9) we have

$$\nabla \mathbf{R} \cdot \mathbf{B} = \mathbf{R} \cdot (\nabla \mathbf{B})^* + \mathbf{B} = \mathbf{R} \cdot \operatorname{def} \mathbf{B} + \frac{1}{2} \mathbf{R} \times \operatorname{rot} \mathbf{B} + \mathbf{B}$$

which allows us to write down the Papkovich-Neuber solution (1.4.10) in the following forms

$$\mathbf{u} = (3-4\nu)\mathbf{B} - \mathbf{R} \cdot (\nabla \mathbf{B})^* - \operatorname{grad} B_0 + \operatorname{grad}\chi_0, \tag{1.4.11}$$

$$\mathbf{u} = (3-4\nu)\mathbf{B} - \mathbf{R} \cdot \operatorname{def} \mathbf{B} - \frac{1}{2} \mathbf{R} \times \operatorname{rot} \mathbf{B} - \operatorname{grad} B_0 + \operatorname{grad}\chi_0. \tag{1.4.12}$$

The strain tensor corresponding to solution (1.4.10) is equal to

$$\hat{\varepsilon} = 4(1-\nu) \operatorname{def} \mathbf{B} - \operatorname{def} \operatorname{grad}(\mathbf{R} \cdot \mathbf{B} + B_0) + \operatorname{def} \operatorname{grad}\chi_0.$$

Noticing that the gradient of the vector which is the gradient of a scalar ψ is a symmetric tensor, we obtain

$$\operatorname{def} \operatorname{grad}\psi = \frac{1}{2}\left[\nabla\nabla\psi + (\nabla\nabla\psi)^*\right] = \nabla\nabla\psi.$$

For this reason

$$\hat{\varepsilon} = 4(1-\nu) \operatorname{def} \mathbf{B} - \nabla\nabla(\mathbf{R} \cdot \mathbf{B} + B_0) + \nabla\nabla\chi_0 \tag{1.4.13}$$

and then

$$\vartheta = I_1(\hat{\varepsilon}) = 4(1-\nu) \operatorname{div} \mathbf{B} - \nabla^2 \mathbf{R} \cdot \mathbf{B} + \nabla^2 \chi_0$$

or referring to eqs. (1.4.9) and (1.4.7)

$$\vartheta = 2(1-2\nu)\operatorname{div}\mathbf{B} + \frac{1-2\nu}{2\mu(1-\nu)}\Pi. \tag{1.4.14}$$

By virtue of eq. (1.1.3) the stress tensor takes the form

$$\hat{T} = 2\mu\left[2\nu\hat{E}\operatorname{div}\mathbf{B} + 4(1-\nu)\operatorname{def}\mathbf{B} - \nabla\nabla(\mathbf{R}\cdot\mathbf{B} + B_0)\right] + \hat{T}^0, \tag{1.4.15}$$

where $\hat{T}^0$ is determined in terms of the volume forces

$$\hat{T}^0 = \frac{\nu}{1-\nu}\Pi\hat{E} + 2\mu\nabla\nabla\chi_0. \tag{1.4.16}$$

Making use of the relationship

$$\nabla\nabla\mathbf{R}\cdot\mathbf{B} = \mathbf{i}_s\mathbf{i}_t\frac{\partial^2}{\partial x_s\partial x_t}x_kB_k = 2\operatorname{def}\mathbf{B} + x_k\nabla\nabla B_k,$$

we can present tensor $\hat{T}$ in another form

$$\hat{T} = 2\mu\left[2\nu\hat{E}\operatorname{div}\mathbf{B} + 2(1-2\nu)\operatorname{def}\mathbf{B} - x_k\nabla\nabla B_k - \nabla\nabla B_0\right] + \hat{T}^0. \tag{1.4.17}$$

Remark 1. The original system of homogeneous equations of equilibrium in terms of displacements contain three unknown functions u, v and w. Therefore it is an acceptable assumption that it is sufficient to keep only three of the four harmonic functions B_s, B_0. Dropping B_0 (which enables the symmetry with respect to the coordinates to be preserved) we arrive at the solution

$$\mathbf{u} = 4(1-\nu)\mathbf{B} - \nabla\mathbf{R}\cdot\mathbf{B}. \tag{1.4.18}$$

However it can be proved that, in the case of a simply-connected region, the general solution of the equilibrium equations in terms of displacements can be presented in this form only under the condition $\nu \neq 0.25$.

Remark 2. It follows directly from the equilibrium equations that their solutions are the gradient of a harmonic scalar $\left(\mathbf{u} = \nabla\mathbf{B}_0, \nabla^2\mathbf{B}_0 = 0\right)$ as well as the rotor of a harmonic vector $\left(\mathbf{u} = \nabla\times\mathbf{C}, \nabla^2\mathbf{C} = 0\right)$. These solutions are insipid since they describe only deformations with unchanged volume $\left(\vartheta = \nabla\cdot\mathbf{u} = \nabla^2\mathbf{B}_0 = 0, \nabla\cdot\nabla\times\mathbf{C} = 0\right)$.

Remark 3. It is easily proved that the displacement vector in the form of I.S. Arzhanykh and M.G. Slobodyansky

$$\mathbf{u} = 4(1-\nu)\mathbf{B} + \mathbf{R}\cdot\nabla\mathbf{B} - \mathbf{R}\nabla\cdot\mathbf{B}, \tag{1.4.19}$$

where $\mathbf{B}$ is a harmonic vector, is also a solution of the equilibrium equations in terms of displacements.

Representation (1.4.19) is reset as follows

$$\mathbf{u} = 4(1-\nu)\mathbf{B} + \mathbf{R}\cdot\operatorname{def}\mathbf{B} - \frac{1}{2}\mathbf{R}\times\operatorname{rot}\mathbf{B} - \mathbf{R}\nabla\cdot\mathbf{B} \tag{1.4.20}$$

and is a solution provided that the difference between this solution and the Papkovich-Neuber solution in the form of eq. (1.4.12)

$$\mathbf{A} = 2\mathbf{R}\cdot\operatorname{def}\mathbf{B} + \mathbf{B} - \mathbf{R}\nabla\cdot\mathbf{B}$$

is also a solution of the equilibrium equations in terms of displacements. This difference can then be put in the form of the rotor of a harmonic vector $\left(\mathbf{A} = \operatorname{rot}\mathbf{C}, \nabla^2\mathbf{A} = 0\right)$. Hence it is necessary to prove that

$$\nabla\cdot\mathbf{A} = 0,\ \nabla^2\mathbf{A} = 0.$$

This follows from the relationships

$$\begin{aligned}
\nabla\cdot 2\mathbf{R}\cdot\operatorname{def}\mathbf{B} &= 2\mathbf{R}\cdot\nabla\cdot\operatorname{def}\mathbf{B} + I_1(\operatorname{def}\mathbf{B})\\
&= \mathbf{R}\cdot\nabla^2\mathbf{B} + \mathbf{R}\cdot\nabla\nabla\cdot\mathbf{B} + 2\nabla\cdot\mathbf{B},\\
\nabla\cdot(\mathbf{R}\nabla\cdot\mathbf{B}) &= 3\nabla\cdot\mathbf{B} + \mathbf{R}\cdot\nabla\nabla\cdot\mathbf{B},\\
\nabla^2\mathbf{R}\cdot\operatorname{def}\mathbf{B} &= 2\left(\nabla^2\mathbf{B} + \nabla\nabla\cdot\mathbf{B}\right),\\
\nabla^2\mathbf{R}\nabla\cdot\mathbf{B} &= 2\nabla\nabla\cdot\mathbf{B}
\end{aligned}$$

and the condition $\nabla^2\mathbf{B} = 0$.

M.G. Slobodyansky has proved that eq. (1.4.19) is the general solution of the equations of elasticity theory for a simply-connected region and eq. (1.4.18) is that for an infinite region which is external for a closed surface, both solutions being valid for any ν ($\nu = 0.25$ included).

4.1.5 The solution in terms of stresses. Beltrami's dependences

Stress tensor $\hat{T}$ satisfying the equations of statics in the volume should be chosen such that the corresponding strain tensor obeys the compatibility conditions (2.1.5) of Chapter 2

$$\operatorname{inc}\hat{\varepsilon} = \frac{1}{2\mu}\operatorname{inc}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) = 0. \tag{1.5.1}$$

Using the equation of statics one can transform this relationship to a form which is easier to view and remember. The result of this transformation is Beltrami's dependences (1892).

By virtue of eq. (2.3.2) of Chapter 2

$$\operatorname{inc}\sigma\hat{E} = \hat{E}\nabla^2\sigma - \nabla\nabla\sigma. \tag{1.5.2}$$

Then referring to formulae (B.4.15) and equilibrium equations (1.5.6) of Chapter 1 we can represent the components $\left(\operatorname{inc}\hat{T}\right)_{ik}$ of tensor $\operatorname{inc}\hat{T}$ in the form

$$\left(\operatorname{inc}\hat{T}\right)_{11} = \frac{\partial^2 t_{22}}{\partial x_3^2} + \frac{\partial^2 t_{33}}{\partial x_2^2} - 2\frac{\partial^2 t_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 t_{22}}{\partial x_3^2} + \frac{\partial^2 t_{33}}{\partial x_2^2} +$$
$$\left(\frac{\partial^2 t_{13}}{\partial x_3 \partial x_1} + \frac{\partial^2 t_{33}}{\partial x_3^2} + \rho\frac{\partial K_3}{\partial x_3}\right) + \left(\frac{\partial^2 t_{12}}{\partial x_2 \partial x_1} + \frac{\partial^2 t_{22}}{\partial x_2^2} + \rho\frac{\partial K_2}{\partial x_2}\right)$$
$$= \nabla^2 \left(t_{22} + t_{33}\right) - \frac{\partial^2}{\partial x_1^2}\left(t_{22} + t_{33}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial t_{12}}{\partial x_2} + \frac{\partial t_{13}}{\partial x_3}\right) +$$
$$\rho\left(\frac{\partial K_2}{\partial x_2} + \frac{\partial K_3}{\partial x_3}\right) = \nabla^2\left(\sigma - t_{11}\right) - \frac{\partial^2 \sigma}{\partial x_1^2} + \rho \operatorname{div}\mathbf{K} - 2\rho\frac{\partial K_1}{\partial x_1},$$

$$\left(\operatorname{inc}\hat{T}\right)_{12} = \frac{\partial}{\partial x_3}\left(\frac{\partial t_{23}}{\partial x_1} + \frac{\partial t_{31}}{\partial x_2} - \frac{\partial t_{12}}{\partial x_3}\right) - \frac{\partial^2 t_{33}}{\partial x_1 \partial x_2}$$
$$= -\frac{\partial}{\partial x_1}\left(\frac{\partial t_{12}}{\partial x_1} + \frac{\partial t_{22}}{\partial x_2} + \rho K_2\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{21}}{\partial x_2} + \rho K_1\right) -$$
$$\frac{\partial^2 t_{12}}{\partial x_3^2} - \frac{\partial^2 t_{33}}{\partial x_1 \partial x_2} = -\left[\nabla^2 t_{12} + \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} + \rho\left(\frac{\partial K_2}{\partial x_1} + \frac{\partial K_1}{\partial x_2}\right)\right].$$

All of these equations can be set in the unified form

$$\left(\operatorname{inc}\hat{T}\right)_{sk} = \delta_{sk}\left(\nabla^2\sigma + \rho \operatorname{div}\mathbf{K}\right) - \frac{\partial^2\sigma}{\partial x_s \partial x_k} - \nabla^2 t_{sk} - \rho\left(\frac{\partial K_s}{\partial x_k} + \frac{\partial K_k}{\partial x_s}\right)$$

yielding the following invariant (coordinate-free) representation

$$\operatorname{inc}\hat{T} = \left(\hat{E}\nabla^2 - \nabla\nabla\right)\sigma - \nabla^2\hat{T} + \hat{E}\rho \operatorname{div}\mathbf{K} - 2\rho \operatorname{def}\mathbf{K}. \tag{1.5.3}$$

In general, for any symmetric tensor of second rank $\hat{Q}$

$$\operatorname{inc}\hat{Q} = -\nabla^2\hat{Q} + 2\operatorname{def}\operatorname{div}\hat{Q} + \left(\hat{E}\nabla^2 - \nabla\nabla\right)I_1\left(\hat{Q}\right) - \hat{E}\nabla\cdot\nabla\cdot\hat{Q}. \tag{1.5.4}$$

Substituting now eqs. (1.5.2) and (1.5.3) into eq. (1.5.1) we obtain

$$-\nabla^2\hat{T} + \hat{E}\rho \operatorname{div}\mathbf{K} - 2\rho \operatorname{def}\mathbf{K} + \frac{1}{1+\nu}\left(\hat{E}\nabla^2 - \nabla\nabla\right)\sigma = 0. \tag{1.5.5}$$

Evaluating the first invariant of the tensor on the left hand side of this equation, we arrive at the relationship

$$\nabla^2\sigma = -\frac{1+\nu}{1-\nu}\rho \operatorname{div}\mathbf{K}, \tag{1.5.6}$$

which clearly can be obtained from eq. (1.3.5) by replacing ϑ due to eq. (3.1.3) of Chapter 3 in terms of σ.

Finally we arrive at the standard form of Beltrami's dependences

$$\nabla^2 \hat{T} + \frac{1}{1+\nu} \nabla\nabla\sigma + 2\rho \operatorname{def} \mathbf{K} + \hat{E} \frac{\nu}{1-\nu} \rho \operatorname{div} \hat{K} = 0. \tag{1.5.7}$$

When the mass forces are absent these equations take the following simple form

$$\nabla^2 \hat{T} + \frac{\nabla\nabla\sigma}{1+\nu} = 0 \tag{1.5.8}$$

or in terms of the components in the Cartesian coordinate system

$$\left.\begin{aligned} \nabla^2 \sigma_x + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial x^2} &= 0, & \nabla^2 \tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial x \partial y} &= 0, \\ \nabla^2 \sigma_y + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial y^2} &= 0, & \nabla^2 \tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial y \partial z} &= 0, \\ \nabla^2 \sigma_z + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial z^2} &= 0, & \nabla^2 \tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2 \sigma}{\partial z \partial x} &= 0. \end{aligned}\right\} \tag{1.5.9}$$

4.1.6 Krutkov's transformation

Considering the case in which mass forces are absent and following eq. (1.6.6) of Chapter 1, we represent stress tensor $\hat{T}$ in terms of the tensor of stress functions

$$\hat{T} = \operatorname{inc} \hat{\Phi}. \tag{1.6.1}$$

The equations of statics are then identically satisfied and it remains only to ensure that $\hat{\Phi}$ satisfies Beltrami's dependences (1.5.8).

Referring to eq. (1.5.2) and (1.5.6) we have

$$\nabla\nabla\sigma = -\operatorname{inc} \sigma \hat{E} \tag{1.6.2}$$

which allows us to set eq. (1.5.8) in the form

$$\operatorname{inc}\left(\nabla^2 \hat{\Phi} - \frac{\sigma}{1+\nu} \hat{E}\right) = 0. \tag{1.6.3}$$

As the tensor in the parentheses is symmetric the latter equation means that this tensor is a deformation of a vector, see Subsection 2.2.1, hence

$$\nabla^2 \hat{\Phi} - \frac{\sigma}{1+\nu} \hat{E} = \operatorname{def} \mathbf{c}. \tag{1.6.4}$$

On the other hand

$$\sigma = I_1\left(\operatorname{inc} \hat{\Phi}\right) = I_1\left(\nabla^2 \hat{\Phi}\right) - \operatorname{div}\operatorname{div} \hat{\Phi} \tag{1.6.5}$$

which can be easily proved by adding the diagonal elements of tensor inc $\hat{\Phi}$, see (B.4.15). Hence, denoting for brevity

$$I_1\left(\hat{\Phi}\right) = \Phi, \quad \operatorname{div}\hat{\Phi} = \mathbf{b} \tag{1.6.6}$$

we arrive at another form of relationship (1.6.4)

$$\nabla^2\hat{\Phi} - \frac{1}{1+\nu}\hat{E}\left(\nabla^2\Phi - \operatorname{div}\mathbf{b}\right) = \operatorname{def}\mathbf{c}. \tag{1.6.7}$$

Vector $\mathbf{c}$ can be removed by equating the traces of the tensors on both sides of the latter equation. The result is

$$I_1\left(\nabla^2\hat{\Phi}\right) = \nabla^2\Phi, \; I_1\left(\operatorname{def}\mathbf{c}\right) = \operatorname{div}\mathbf{c}, \; \nabla^2\Phi = \operatorname{div}\operatorname{grad}\Phi, \; I_1\left(\hat{E}\right) = 3.$$

For this reason

$$\operatorname{div}\left(\frac{3}{1+\nu}\mathbf{b} - \frac{2-\nu}{1+\nu}\operatorname{grad}\Phi - \mathbf{c}\right) = 0 \tag{1.6.8}$$

and the vector in the parentheses is the rotor of some vector such that

$$\mathbf{c} = \frac{3}{1+\nu}\left(\mathbf{b} - \operatorname{rot}\mathbf{q}\right) - \frac{2-\nu}{1+\nu}\operatorname{grad}\Phi.$$

Assuming that rot $\mathbf{q}$ is included in vector $\mathbf{b}$ we have now

$$\mathbf{c} = \frac{3}{1+\nu}\mathbf{b} - \frac{2-\nu}{1+\nu}\operatorname{grad}\Phi, \quad \operatorname{def}\mathbf{c} = \frac{3}{1+\nu}\operatorname{def}\mathbf{b} - \frac{2-\nu}{1+\nu}\nabla\nabla\Phi,$$

and inserting this into eq. (1.6.7) leads to a differential equations containing only operations over tensor $\hat{\Phi}$

$$\nabla^2\hat{\Phi} = \frac{1}{1+\nu}\hat{E}\left(\nabla^2\Phi - \operatorname{div}\mathbf{b}\right) + \frac{3}{1+\nu}\operatorname{def}\mathbf{b} - \frac{2-\nu}{1+\nu}\nabla\nabla\Phi. \tag{1.6.9}$$

Relying on formulae (1.5.4) we can write the expression for stress tensor $\hat{T}$ in terms of the tensor of stress functions as follows

$$\hat{T} = \operatorname{inc}\hat{\Phi} = -\nabla^2\hat{\Phi} + 2\operatorname{def}\mathbf{b} + \left(\hat{E}\nabla^2 - \nabla\nabla\right)\Phi - \hat{E}\operatorname{div}\mathbf{b}$$

or, after removing $\nabla^2\hat{\Phi}$ by means of eq. (1.6.9) we have

$$\hat{T} = \frac{\nu}{1+\nu}\hat{E}\left(\nabla^2\Phi - \operatorname{div}\mathbf{b}\right) - \frac{1-2\nu}{1+\nu}\left(\operatorname{def}\mathbf{b} - \nabla\nabla\Phi\right). \tag{1.6.10}$$

From this equation we obtain

$$\sigma = I_1\left(\hat{T}\right) = \nabla^2\Phi - \operatorname{div}\mathbf{b}, \tag{1.6.11}$$

so that the strain tensor takes the form

$$2\mu\hat{\varepsilon} = \hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E} = -\frac{1-2\nu}{1+\nu}\left(\operatorname{def}\mathbf{b} - \nabla\nabla\Phi\right)$$

or

$$2\mu\hat{\varepsilon} = \frac{1-2\nu}{1+\nu}\operatorname{def}\left(\nabla\Phi - \mathbf{b}\right). \tag{1.6.12}$$

Using this equation we obtain the displacement vector $\mathbf{u}$ with accuracy up to a rigid body displacement

$$2\mu\mathbf{u} = \frac{1-2\nu}{1+\nu}\left(\nabla\Phi - \mathbf{b}\right) = \frac{1-2\nu}{1+\nu}\left[\nabla I_1\left(\hat{\Phi}\right) - \operatorname{div}\hat{\Phi}\right]. \tag{1.6.13}$$

Formulae (1.6.10) and (1.6.13) obtained by Yu.A. Krutkov in 1949 present one of the forms of the general solution of the theory of linear elasticity. The stress tensor $\hat{T}$ and the displacement vector $\mathbf{u}$ are determined by the tensor of stress functions satisfying differential equation (1.6.9) and are dependent only on the first invariant of Φ and divergence $\mathbf{b}$ of tensor $\hat{\Phi}$. Thus, there is no need to know all of the components of the latter tensor, it is sufficient only to relate $\mathbf{b}$ and Φ with the help of eq. (1.6.9).

4.1.7 The Boussinesq-Galerkin solution

The sought-for expression for $\mathbf{b}$ in terms of Φ can be obtained by equating the divergence of both sides of equation (1.6.9). We have

$$\operatorname{div}\nabla^2\hat{\Phi} = \nabla^2\mathbf{b}, \quad \operatorname{div}\hat{E}\left(\nabla^2\Phi - \mathbf{b}\right) = \operatorname{grad}\nabla^2\Phi - \operatorname{grad}\operatorname{div}\mathbf{b},$$
$$\operatorname{div}\nabla\nabla\Phi = \nabla^2\nabla^2\Phi = \operatorname{grad}\nabla^2\Phi, \quad \operatorname{div}\operatorname{def}\mathbf{b} = \frac{1}{2}\left(\nabla^2\mathbf{b} + \operatorname{grad}\operatorname{div}\mathbf{b}\right)$$

and after substitution into eq. (1.6.9) we obtain

$$\nabla^2\mathbf{b} + \frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{b} = \frac{2(1-\nu)}{1-2\nu}\operatorname{grad}\nabla^2\Phi. \tag{1.7.1}$$

One can satisfy this relationship by introducing the representation of $\mathbf{b}$ and Φ in terms of vector $\mathbf{G}$ in the following form

$$\mathbf{b} = \nabla^2\mathbf{G}, \quad \Phi = \frac{1}{1-2\nu}\operatorname{div}\mathbf{G}. \tag{1.7.2}$$

By virtue of eq. (1.7.1), vector $\mathbf{G}$ is proved to be biharmonic

$$\nabla^4\mathbf{G} = 0. \tag{1.7.3}$$

With the help of eqs. (1.6.13) and (1.6.10) the expressions for the displacement vectors and the stress tensor in terms of vector $\mathbf{G}$ can be written down as follows

$$2\mu\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{G} - 2(1-\nu)\nabla^2\mathbf{G}, \tag{1.7.4}$$

$$\hat{T} = \nabla\nabla\operatorname{div}\mathbf{G} - 2(1-\nu)\operatorname{def}\nabla^2\mathbf{G} - \nu\hat{E}\operatorname{div}\nabla^2\mathbf{G}. \tag{1.7.5}$$

This form of the solution of the equation of elasticity theory was given by B.G. Galerkin in 1930 and was earlier known to Boussinesq (1878).

Based upon eqs. (1.7.1) and (1.6.13) we can immediately obtain the solution in the Papkovich-Neuber form, too. It is sufficient to notice that equation (1.7.1) is coincident with the equation in terms of displacements (1.3.2) if one identifies $\frac{2(1-\nu)}{1-2\nu}\mu\nabla^2\Phi$ with the potential of the volume force Π. Then, due to eq. (1.4.7) $\chi_0 = \Phi$, and setting the solution of equation (1.7.1) in the form of eq. (1.4.10)

$$\mathbf{b} = -\frac{1+\nu}{1-2\nu}\left[4(1-\nu)\mathbf{B} - \operatorname{grad}(\mathbf{R}\cdot\mathbf{B} + B_0)\right] + \nabla\Phi, \tag{1.7.6}$$

we arrive, using eq. (1.6.13), to the representation of the displacement vector in the above form (without mass forces).

It is also easy to establish the relation between vectors $\mathbf{B}$ and $\mathbf{G}$. Using eqs. (1.7.6) and (1.7.2) we have

$$\nabla^2\mathbf{G} = -\frac{1+\nu}{1-2\nu}\left[4(1-\nu)\mathbf{B} - \operatorname{grad}(\mathbf{R}\cdot\mathbf{B} + B_0)\right] + \frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\Phi,$$

so that, due to eqs. (1.7.3) and (1.4.9),

$$\frac{1+\nu}{1-2\nu}\nabla^2\mathbf{R}\cdot\mathbf{B} = -\frac{1}{1-2\nu}\operatorname{div}\nabla^2\mathbf{G},$$

$$\operatorname{div}\left[4(1-\nu)\mathbf{B} + \frac{1-2\nu}{1+\nu}\nabla^2\mathbf{G}\right] = 0,$$

and one can take that

$$4(1-\nu)\mathbf{B} = -\frac{1-2\nu}{1+\nu}\nabla^2\mathbf{G}. \tag{1.7.7}$$

4.1.8 Curvilinear coordinates

In the above subsections the fundamental relationships are presented in the invariant form of dependences between the vectorial and tensorial quantities. For this reason the corresponding formulae in the curvilinear coordinates require only a careful consideration of the rules of tensor calculus (Appendices C-E).

In the linear theory there is no need to distinguish between the basis of the initial state and that of the final state. This allows the stress tensor $\hat{T}$ to be represented in terms of its contravariant components in the vector basis $\mathbf{r}_s$ (instead of eq. (3.1.1) of Chapter 1)

$$\hat{T} = t^{sk}\mathbf{r}_s\mathbf{r}_k\,. \tag{1.8.1}$$

By eq. (3.3.4) of Chapter 1, the equilibrium equations in volume are set as follows

$$\frac{\partial}{\partial q^s}\sqrt{g}t^{sq} + \left\{ \begin{matrix} q \\ mn \end{matrix} \right\} t^{mn} + \rho\sqrt{g}K^q = 0. \tag{1.8.2}$$

The linear strain tensor is represented in terms of its covariant components by means of eq. (3.6.7) of Chapter 2, i.e.

$$\hat{\varepsilon} = \varepsilon_{st}\mathbf{r}^s\mathbf{r}^t,\ \varepsilon_{st} = \frac{1}{2}\left(\frac{\partial u_s}{\partial q^t} + \frac{\partial u_t}{\partial q^s}\right) - \left\{ \begin{matrix} r \\ st \end{matrix} \right\} u_r\,, \tag{1.8.3}$$

where u_r denotes covariant components of the displacement vector. Due to eqs. (D.7.5) and (E.4.4) the dilatation can be expressed in one of the following representations

$$\vartheta = g^{sk}\varepsilon_{sk} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}g^{rq}u_q}{\partial q^r}\,. \tag{1.8.4}$$

Writing formulae relating the contravariant components of the stress tensor with the covariant components of the strain tensor it is necessary to bear in mind that the role of tensor $\hat{E}$ in the generalised Hooke's law (1.1.3) is played now by tensor $\hat{g}$. Therefore, referring to eq. (1.8.4) we have

$$\begin{aligned} t^{sk} &= 2\mu\left(\frac{\nu}{1-2\nu}g^{sk}\vartheta + g^{sm}g^{kn}\varepsilon_{mn}\right) \\ &= 2\mu\left(\frac{\nu}{1-2\nu}g^{sk}g^{mn} + g^{sm}g^{kn}\right)\varepsilon_{mn}\,. \end{aligned} \tag{1.8.5}$$

The inverse relationships are set as follows

$$\varepsilon_{sk} = \frac{1}{2\mu}\left(g_{sm}g_{kn}t^{mn} - \frac{\nu}{1+\nu}g_{sk}\sigma\right) = \frac{1}{2\mu}\left(g_{sm}g_{kn} - \frac{\nu}{1+\nu}g_{sk}g_{mn}\right)t^{mn}\,. \tag{1.8.6}$$

The bilinear representation of the specific strain energy, eq. (3.2.6) of Chapter 2, is written in the form

$$A\left(\varepsilon,\sigma\right) = \frac{1}{2}t^{sk}\varepsilon_{sk}\,, \tag{1.8.7}$$

so that, by virtue of eqs. (1.8.5) and (1.8.6), the formulae for $A(\varepsilon)$ and $A(\sigma)$ in terms of the tensors of strains and stresses are given by

$$A(\varepsilon) = \mu\left(\frac{\nu}{1-2\nu}g^{sk}g^{mn} + g^{sm}g^{kn}\right)\varepsilon_{sk}\varepsilon_{mn}\,, \tag{1.8.8}$$

$$A(\sigma) = \frac{1}{4\mu}\left(g_{sm}g_{kn} - \frac{\nu}{1+\nu}g_{sk}g_{mn}\right)t^{sk}t^{mn}\,. \tag{1.8.9}$$

The equilibrium equations in terms of displacements, eq. (1.3.2), is obtained by utilising eq. (E.4.9) in the form

$$\frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial q^t} + g^{sk}\nabla_s\nabla_k u_t + \frac{\rho}{\mu}K_t = 0. \tag{1.8.10}$$

By virtue of eqs. (E.3.5) and (E.3.4) we have

$$\nabla^2\hat{T} = \nabla\cdot\nabla\hat{T} = \mathbf{r}^m\frac{\partial}{\partial q^m}\cdot\mathbf{r}^m\mathbf{r}_s\mathbf{r}_t\nabla_q t^{st} = g^{mq}\mathbf{r}_s\mathbf{r}_t\nabla_m\nabla_q t^{st},$$

$$\nabla\nabla\sigma = \mathbf{r}^n\frac{\partial}{\partial q^n}\mathbf{r}^k\frac{\partial\sigma}{\partial q^k} = \mathbf{r}^n\mathbf{r}^k\nabla_n\frac{\partial\sigma}{\partial q^k}\,,$$

and in the case of no mass forces the Beltrami dependences (1.5.8) take the form

$$g^{mq}\nabla_m\nabla_q t^{st} + \frac{g^{sn}g^{tk}}{1+\nu}\nabla_n\frac{\partial\sigma}{\partial q^k} = 0. \tag{1.8.11}$$

The expanded expressions for the operations of double covariant differentiation in eqs. (1.8.10) and (1.8.11) are very cumbersome.

4.1.9 Orthogonal coordinates

In this subsection, subscripts denote the physical (rather than covariant) components of vectors and tensors. The expressions for the differential operations used in what follows are collected in Section C.5.

The generalised Hooke law (in terms of the physical components) is written down as

$$t_{sk} = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\delta_{sk} + \varepsilon_{sk}\right), \tag{1.9.1}$$

the expressions for ϑ and ε_{sk} being given by formulae (C.5.3), (C.5.8) and (C.5.9). In these formulae a_s needs to be replaced by u_s which is the projection of the displacement vector on the direction of the unit vector $\mathbf{e}_s$ of the basis trihedron. Using eq. (C.5.10) the equilibrium equation (1.1.1) is set as follows

$$\frac{\partial}{\partial q^s}\left(\frac{\sqrt{g}}{H_s}t_{st}\mathbf{e}_t\right) + \rho\sqrt{g}\mathbf{K} = 0\,. \tag{1.9.2}$$

In the cylindrical coordinates, see Sections C.1 and C.7, the components of the strain tensor and the dilatation are written down in the form

$$\left.\begin{aligned}&\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\varphi = \frac{\partial v}{r\partial\varphi} + \frac{u}{r}, \qquad \varepsilon_z = \frac{\partial w}{\partial z},\\ &\gamma_{r\varphi} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial\varphi} - \frac{v}{r}, \ \gamma_{\varphi z} = \frac{1}{r}\frac{\partial w}{\partial\varphi} + \frac{\partial v}{\partial z}, \ \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},\\ &\vartheta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{r\partial\varphi} + \frac{\partial w}{\partial z},\end{aligned}\right\} \tag{1.9.3}$$

where u, v and w denote the projections of the displacement vector on axes $\mathbf{e}_r, \mathbf{e}_\varphi$ and $\mathbf{k}$ of the cylindrical coordinate system. The equilibrium equations have the form

$$\left.\begin{aligned}\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} + \frac{\partial\tau_{r\varphi}}{r\partial\varphi} + \frac{\partial\tau_{zr}}{\partial z} + \rho K_r = 0,\\ \frac{\partial\tau_{r\varphi}}{\partial r} + 2\frac{\tau_{r\varphi}}{r} + \frac{\partial\sigma_\varphi}{r\partial\varphi} + \frac{\partial\tau_{z\varphi}}{\partial z} + \rho K_\varphi = 0,\\ \frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{\partial\tau_{z\varphi}}{r\partial\varphi} + \frac{\partial\sigma_z}{\partial z} + \rho K_z = 0.\end{aligned}\right\} \tag{1.9.4}$$

The expressions for the components of the strain tensor in the spherical coordinates are more bulky, see Sections C.1 and C.8. Denoting the projections of the displacement vector on axes $\mathbf{e}_R, \mathbf{e}_\vartheta$ and $\mathbf{e}_\lambda$ by u_R, u_ϑ and u_λ respectively we have

$$\left.\begin{aligned}\varepsilon_R = \frac{\partial u_R}{\partial R}, \quad \varepsilon_\vartheta = \frac{1}{R}\frac{\partial u_\vartheta}{\partial\vartheta} + \frac{u_R}{R},\\ \varepsilon_\lambda = \frac{1}{R\sin\vartheta}\frac{\partial u_\lambda}{\partial\lambda} + \frac{u_\vartheta}{R}\cot\vartheta + \frac{u_R}{R},\\ \gamma_{\vartheta\lambda} = \frac{1}{R}\left(\frac{\partial u_\lambda}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial u_\vartheta}{\partial\lambda} - u_\lambda\cot\vartheta\right),\\ \gamma_{\lambda R} = \frac{1}{R\sin\vartheta}\frac{\partial u_R}{\partial\lambda} + \frac{\partial u_\lambda}{\partial R} - \frac{u_\lambda}{R},\end{aligned}\right\} \tag{1.9.5}$$

$$\left.\begin{aligned}\gamma_{R\vartheta} = \frac{\partial u_\vartheta}{\partial R} + \frac{1}{R}\frac{\partial u_R}{\partial\vartheta} - \frac{u_\vartheta}{R},\\ \vartheta = \frac{1}{R^2\sin\vartheta}\left[\frac{\partial}{\partial R}\left(R^2 u_R\sin\vartheta\right) + \frac{\partial}{\partial\vartheta}\left(Ru_\vartheta\sin\vartheta\right) + \frac{\partial}{\partial\lambda}\left(Ru_\lambda\right)\right].\end{aligned}\right\} \tag{1.9.6}$$

The equilibrium equations are put in the following form

$$\left.\begin{aligned}
&\frac{\partial \sigma_R}{\partial R}+\frac{1}{R}\frac{\partial \tau_{R\vartheta}}{\partial \vartheta}+\frac{1}{R\sin\vartheta}\frac{\partial \tau_{R\lambda}}{\partial \lambda}+\\
&\frac{1}{R}\left(2\sigma_R-\sigma_\vartheta-\sigma_\lambda+\tau_{R\vartheta}\cot\vartheta\right)+\rho K_R=0,\\
&\frac{\partial \tau_{R\vartheta}}{\partial R}+\frac{1}{R}\frac{\partial \sigma_\vartheta}{\partial \vartheta}+\frac{1}{R\sin\vartheta}\frac{\partial \tau_{\vartheta\lambda}}{\partial \lambda}+\\
&\frac{1}{R}\left[\left(\sigma_\vartheta-\sigma_\lambda\right)\cot\vartheta+3\tau_{R\vartheta}\right]+\rho K_\vartheta=0,\\
&\frac{\partial \tau_{R\lambda}}{\partial R}+\frac{1}{R}\frac{\partial \tau_{\vartheta\lambda}}{\partial \vartheta}+\frac{1}{R\sin\vartheta}\frac{\partial \sigma_\lambda}{\partial \lambda}+\\
&\frac{1}{R}\left(3\tau_{R\lambda}+2\tau_{\vartheta\lambda}\cot\vartheta\right)+\rho K_\lambda=0.
\end{aligned}\right\} \tag{1.9.7}$$

4.1.10 *Axisymmetric problems. Love's solution*

In the problem of equilibrium of the bodies of revolution (Section C.9) under an axial symmetry of the loading (i.e. independence of the volume and surface forces of the azimuthal angle φ) the stress tensor and the displacement vector do not depend on φ and are functions of coordinates q^1 and q^2. In other words, the state of stress is the same in all meridional planes.

Let u_1, u_2 and $v = u_\varphi$ denote the projections of the displacement vector on directions $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_\varphi$ of the basis trihedron respectively. Then

$$\left.\begin{aligned}
&\varepsilon_1=\frac{1}{H_1}\frac{\partial u_1}{\partial q^1}+\frac{u_2}{H_2}\frac{\partial \ln H_1}{\partial q^2},\quad \varepsilon_2=\frac{1}{H_2}\frac{\partial u_2}{\partial q^2}+\frac{u_1}{H_1}\frac{\partial \ln H_2}{\partial q^1},\\
&\varepsilon_\varphi=\frac{u_1}{H_1}\frac{\partial \ln r}{\partial q^1}+\frac{u_2}{H_2}\frac{\partial \ln r}{\partial q^2},\\
&\gamma_{12}=\frac{1}{H_1}\frac{\partial u_2}{\partial q^1}+\frac{1}{H_2}\frac{\partial u_1}{\partial q^2}-\frac{u_1}{H_2}\frac{\partial \ln H_1}{\partial q^2}-\frac{u_2}{H_1}\frac{\partial \ln H_2}{\partial q^1},
\end{aligned}\right\} \tag{1.10.1}$$

$$\left.\begin{aligned}
&\vartheta=\frac{1}{rH_1H_2}\left[\frac{\partial}{\partial q^1}\left(u_1H_2r\right)+\frac{\partial}{\partial q^2}\left(u_2H_1r\right)\right],\\
&\gamma_{1\varphi}=\frac{1}{H_1}\left(\frac{\partial v}{\partial q^1}-v\frac{\partial \ln r}{\partial q^1}\right),\quad \gamma_{2\varphi}=\frac{1}{H_2}\left(\frac{\partial v}{\partial q^2}-v\frac{\partial \ln r}{\partial q^2}\right).
\end{aligned}\right\} \tag{1.10.2}$$

The components of the strain tensor are seen to be split into two sets: the set for extensions and the set for two components $\gamma_{1\varphi}$ and $\gamma_{2\varphi}$. The static equations are also split into two equations for stresses $\sigma_1, \sigma_2, \tau_{12}$ and σ_φ

$$\left.\begin{aligned}
&\frac{1}{H_1 H_2 r}\left(\frac{\partial H_2 r\sigma_1}{\partial q^1} + \frac{\partial H_1 r\tau_{12}}{\partial q^2}\right) + \frac{\tau_{12}}{H_2}\frac{\partial \ln H_1}{\partial q^2} - \\
&\qquad \frac{\sigma_2}{H_1}\frac{\partial \ln H_2}{\partial q^1} - \frac{\sigma_\varphi}{H_1}\frac{\partial \ln r}{\partial q^1} + \rho K_1 = 0, \\
&\frac{1}{H_1 H_2 r}\left(\frac{\partial H_1 r\sigma_2}{\partial q^2} + \frac{\partial H_2 r\tau_{12}}{\partial q^1}\right) + \frac{\tau_{12}}{H_1}\frac{\partial \ln H_2}{\partial q^1} - \\
&\qquad \frac{\sigma_1}{H_2}\frac{\partial \ln H_1}{\partial q^2} - \frac{\sigma_\varphi}{H_2}\frac{\partial \ln r}{\partial q^2} + \rho K_2 = 0,
\end{aligned}\right\} \qquad (1.10.3)$$

and the equation for stresses $\tau_{\varphi 1}$ and $\tau_{\varphi 2}$

$$\frac{1}{H_1 H_2 r}\left(\frac{\partial H_2 r\tau_{\varphi 1}}{\partial q^1} + \frac{\partial H_1 r\tau_{\varphi 2}}{\partial q^2}\right) + \frac{\tau_{\varphi 1}}{H_1}\frac{\partial \ln r}{\partial q^1} + \frac{\tau_{\varphi 2}}{H_2}\frac{\partial \ln r}{\partial q^2} + \rho K_\varphi = 0.$$

Using the generalised Hooke law one can express the normal stresses and shear stress τ_{12} in terms of the strains (1.10.1) of the first set and shear stresses $\tau_{\varphi 1}$ and $\tau_{\varphi 2}$ in terms of the strains (1.10.2) of the second set. Consequently, the axisymmetric problem is split into two uncoupled problems. The first one is the problem of deformation in the meridional plane where components v of the displacement is absent (however normal stress σ_φ is present) and the second one is the problem of torsion. The latter problem yields displacement $v\left(q^1, q^2\right)$ which is perpendicular to the meridional plane and independent of the azimuthal angle φ.

The general solution of the problem of the axisymmetric deformation can be expressed in terms of a single biharmonic function χ referred to as Love's function. The latter is a particular case of the Boussinesq-Galerkin solution, eqs. (1.7.4) and (1.7.5), in which the biharmonic vector $\mathbf{G}$ is given by a single component along the symmetry axis

$$\mathbf{G} = \mathbf{k}\chi(r, z). \qquad (1.10.4)$$

In cylindrical coordinates using the notation of Subsection 4.1.9 we have

$$2\mu u = \frac{\partial^2 \chi}{\partial r \partial z}, \quad \nu = 0, \quad 2\mu w = \frac{\partial^2 \chi}{\partial z^2} - 2(1-\nu)\nabla^2\chi, \qquad (1.10.5)$$

$$\left.\begin{aligned}
&\sigma_r = \frac{\partial}{\partial z}\left(-\nu\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2}\right), \quad \sigma_\varphi = \frac{\partial}{\partial z}\left(-\nu\nabla^2\chi + \frac{1}{r}\frac{\partial\chi}{\partial r}\right), \\
&\sigma_z = \frac{\partial}{\partial z}\left[-(2-\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial z^2}\right], \quad \tau_{rz} = \frac{\partial}{\partial r}\left[-(1-\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial z^2}\right].
\end{aligned}\right\} \qquad (1.10.6)$$

Here

$$\nabla^2\chi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\chi}{\partial r} + \frac{\partial^2\chi}{\partial z^2}, \quad \nabla^4\chi = 0. \qquad (1.10.7)$$

Relying on these formulae it is easy to generalise the approach to the general coordinates of a body of revolution, see Section C.9. Using eq. (C.9.8) we obtain

$$2\mu u_s = \frac{1}{H_s}\frac{\partial}{\partial q^s}\left(\frac{1}{H_1^2}\frac{\partial \chi}{\partial q^1}\frac{\partial z}{\partial q^1} + \frac{1}{H_2^2}\frac{\partial \chi}{\partial q^2}\frac{\partial z}{\partial q^2}\right) - 2(1-\nu)\frac{1}{H_s}\frac{\partial z}{\partial q^s}\nabla^2\chi \qquad (s=1,2), \tag{1.10.8}$$

where, using eq. (C.5.5),

$$\nabla^2\chi = \frac{1}{H_1 H_2 r}\left[\frac{\partial}{\partial q^1}\left(\frac{rH_2}{H_1}\frac{\partial \chi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{rH_1}{H_2}\frac{\partial \chi}{\partial q^2}\right)\right]. \tag{1.10.9}$$

4.1.11 Torsion of a body of revolution

It is sufficient to set in the Papkovich-Neuber solution that the harmonic vector has a single component which has direction $\mathbf{e}_\varphi$

$$\mathbf{B} = B_\varphi \mathbf{e}_\varphi. \tag{1.11.1}$$

Then

$$\mathbf{R}\cdot\mathbf{B} = (r\mathbf{e}_r + z\mathbf{k})\cdot\mathbf{B} = 0,$$

and if the volume force is absent the displacement v is proportional to B_φ whereas displacements u_1 and u_2 vanish. Hence, $v\mathbf{e}_\varphi$ is a harmonic vector and by eqs. (B.4.19) and (C.7.5)

$$\nabla^2 v\mathbf{e}_\varphi = \mathbf{e}_\varphi\nabla^2 v + v\nabla^2\mathbf{e}_\varphi + 2\nabla v\cdot\nabla\mathbf{e}_\varphi = \mathbf{e}_\varphi\left(\nabla^2 v - \frac{v}{r^2}\right) = 0,$$

such that by virtue of eqs. (C.7.4) and (C.7.5)

$$\nabla v\cdot\nabla\mathbf{e}_\varphi = -\frac{1}{r}\frac{\mathbf{e}_s}{H_s}\frac{\partial v}{\partial q^s}\cdot\mathbf{e}_\varphi\mathbf{e}_r = 0, \qquad \nabla^2\mathbf{e}_\varphi = -\frac{1}{r^2}\mathbf{e}_\varphi.$$

Therefore, v is governed by the differential equation

$$\begin{aligned}\nabla_*^2 v &= \nabla^2 v - \frac{1}{r^2}v \\ &= \frac{1}{H_1H_2r}\left[\frac{\partial}{\partial q^1}\left(\frac{rH_2}{H_1}\frac{\partial v}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{rH_1}{H_2}\frac{\partial v}{\partial q^2}\right)\right] - \frac{v}{r^2} = 0,\end{aligned} \tag{1.11.2}$$

so that $ve^{i\varphi}$, rather than v, is a harmonic function, see eq. (C.5.5). Using eq. (1.10.2) the stresses are determined from the following formulae

$$\tau_{1\varphi} = \mu\frac{r}{H_1}\frac{\partial}{\partial q^1}\frac{v}{r}, \qquad \tau_{2\varphi} = \mu\frac{r}{H_2}\frac{\partial}{\partial q^2}\frac{v}{r}. \tag{1.11.3}$$

4.1.12 Deformation of a body of revolution

Let us drop the assumption that the loading is axisymmetic. The quantities characterising deformation of a body of revolution are periodic functions of angle φ. Then the displacement can be sought in the form of a Fourier series in variable φ. In cylindrical coordinates the general term of this series is given by

$$u = u_* (r, z) \cos n\varphi, \quad v = v_* (r, z) \sin n\varphi, \quad w = w_* (r, z) \cos n\varphi. \quad (1.12.1)$$

(Clearly, one could assume w and u to be proportional to $\sin n\varphi$ and v to be proportional to $\cos n\varphi$.) Instead of eq. (1.1.0.4) we can put in the Boussinesq-Galerkin solution that

$$\mathbf{G} = \mathbf{k}\chi (r, z) \cos n\varphi.$$

Then, by virtue of eq. (1.7.4)

$$2\mu u_* = \frac{\partial^2 \chi}{\partial r \partial z}, \quad 2\mu w_* = \frac{\partial^2 \chi}{\partial z^2} - 2(1-\nu)\nabla_*^2 \chi, \quad 2\mu v_* = -\frac{n}{r}\frac{\partial \chi}{\partial z}, \quad (1.12.2)$$

where now

$$\nabla_*^2 \chi = \nabla^2 \chi - \frac{n^2}{r^2}\chi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} - \frac{n^2}{r^2}\chi, \quad (1.12.3)$$

that is function $\chi e^{in\varphi}$ is biharmonic. By eq. (1.7.5) we have

$$\left.\begin{aligned} \sigma_r^* &= \frac{\partial}{\partial z}\left(-\nu\nabla_*^2\chi + \frac{\partial^2\chi}{\partial r^2}\right), \quad \sigma_\varphi^* = \frac{\partial}{\partial z}\left(-\nu\nabla_*^2\chi + \frac{1}{r}\frac{\partial \chi}{\partial r} - \frac{n^2}{r^2}\chi\right), \\ \sigma_z^* &= \frac{\partial}{\partial z}\left[-(2-\nu)\nabla_*^2\chi + \frac{\partial^2\chi}{\partial z^2}\right], \quad \tau_{rz}^* = \frac{\partial}{\partial r}\left[-(1-\nu)\nabla_*^2\chi + \frac{\partial^2\chi}{\partial z^2}\right], \end{aligned}\right\} \quad (1.12.4)$$

these quantities being factors in front of $\cos n\varphi$ in the expressions for the corresponding components of the stress tensor ($\sigma_r = \sigma_r^* \cos n\varphi$ etc.). The remaining components are proportional to $\sin n\varphi$

$$\tau_{r\varphi}^* = -\frac{n}{r}\frac{\partial^2 \chi}{\partial r \partial z} + \frac{n}{r^2}\frac{\partial \chi}{\partial z}, \quad \tau_{z\varphi}^* = -\frac{n}{r}\left[\frac{\partial^2 \chi}{\partial z^2} - (1-\nu)\nabla_*^2 \chi\right]. \quad (1.12.5)$$

Contrary to the axisymmetric loading case, it is not possible to split the problem into a deformation in the meridional plane and the torsion deformation.

In the general coordinates of a body of revolution the formulae for the displacements have the form

$$\left.\begin{aligned} 2\mu u_s^* &= \frac{1}{H_s}\frac{\partial}{\partial q^s}\left(\frac{1}{H_1^2}\frac{\partial \chi}{\partial q^1}\frac{\partial z}{\partial q^1}+\frac{1}{H_2^2}\frac{\partial \chi}{\partial q^2}\frac{\partial z}{\partial q^2}\right)-\frac{2(1-\nu)}{H_s}\frac{\partial z}{\partial q^s}\nabla_*^2\chi, \\ 2\mu v^* &= -\frac{n}{r}\left(\frac{1}{H_1^2}\frac{\partial \chi}{\partial q^1}\frac{\partial z}{\partial q^1}+\frac{1}{H_2^2}\frac{\partial \chi}{\partial q^2}\frac{\partial z}{\partial q^2}\right) \qquad (s=1,2)\,, \end{aligned}\right\} \tag{1.12.6}$$

where, due to eq. (C.5.5)

$$\begin{aligned} \nabla_*^2\chi &= \left(\nabla^2-\frac{n^2}{r^2}\right)\chi \qquad\qquad (1.12.7) \\ &= \frac{1}{H_1H_2r}\left[\frac{\partial}{\partial q^1}\left(\frac{H_2r}{H_1}\frac{\partial\chi}{\partial q^1}\right)+\frac{\partial}{\partial q^2}\left(\frac{H_1r}{H_2}\frac{\partial\chi}{\partial q^2}\right)\right]-\frac{n^2}{r^2}\chi. \end{aligned}$$

By eq. (1.7.5) the stress tensor is represented in the form

$$\hat{T}=\nabla\nabla\frac{\partial\chi}{\partial z}\cos n\varphi-2(1-\nu)\operatorname{def}\mathbf{k}\nabla^2(\chi\cos n\varphi)-\nu\hat{E}\frac{\partial}{\partial z}\nabla^2(\chi\cos n\varphi). \tag{1.12.8}$$

Prescribing harmonic function $\chi\cos n\varphi$ in terms of two harmonic functions $\chi_0\cos n\varphi$ and $\chi_3\cos n\varphi$

$$\chi\cos n\varphi=(\chi_0+z\chi_3)\cos n\varphi \tag{1.12.9}$$

and adopting

$$\left(\frac{\partial\chi_0}{\partial z}+\chi_3\right)\cos n\varphi=-2\mu B_0,\ \frac{\partial\chi_3}{\partial z}\cos n\varphi=-2\mu B_3, \tag{1.12.10}$$

we arrive at the representation of the stress tensor $\hat{T}$ in the Papkovich-Neuber form (1.4.17) in which only two harmonic function B_3 and B_0 are kept

$$\frac{1}{2\mu}\hat{T}=2(1-2\nu)\operatorname{def}\mathbf{k}B_3+2\nu\hat{E}\frac{\partial B_3}{\partial z}-z\nabla\nabla B_3-\nabla\nabla B_0. \tag{1.12.11}$$

Taking now

$$B_0=b_0\cos n\varphi,\ B_3=b_3\cos n\varphi, \tag{1.12.12}$$

we obtain the following formulae for the factors associated with $\cos n\varphi$ in the expressions for components $\sigma_1, \sigma_2, \sigma_\varphi$ and τ_{12} of the stress tensor

$$\left.\begin{aligned}
\frac{1}{2\mu}\sigma_1^* &= 2(1-\nu)\frac{1}{H_1^2}\frac{\partial z}{\partial q^1}\frac{\partial b_3}{\partial q^1} + 2\nu\frac{1}{H_2^2}\frac{\partial z}{\partial q^2}\frac{\partial b_3}{\partial q^2} \\
&-\frac{1}{H_1^2}\left(\frac{\partial^2 b_0}{\partial q^{1^2}} + z\frac{\partial^2 b_3}{\partial q^{1^2}}\right) + \frac{1}{H_1^2}\frac{\partial \ln H_1}{\partial q^1}\left(\frac{\partial b_0}{\partial q^1} + z\frac{\partial b_3}{\partial q^1}\right) - \\
&\qquad \frac{1}{H_2^2}\frac{\partial \ln H_1}{\partial q^2}\left(\frac{\partial b_0}{\partial q^2} + z\frac{\partial b_3}{\partial q^2}\right), \\
\frac{1}{2\mu}\sigma_2^* &= 2(1-\nu)\frac{1}{H_2^2}\frac{\partial z}{\partial q^2}\frac{\partial b_3}{\partial q^2} + 2\nu\frac{1}{H_1^2}\frac{\partial z}{\partial q^1}\frac{\partial b_3}{\partial q^1} \\
&-\frac{1}{H_1^2}\left(\frac{\partial^2 b_0}{\partial q^{2^2}} + z\frac{\partial^2 b_3}{\partial q^{2^2}}\right) + \frac{1}{H_2^2}\frac{\partial \ln H_2}{\partial q^2}\left(\frac{\partial b_0}{\partial q^2} + z\frac{\partial b_3}{\partial q^2}\right) - \\
&\qquad \frac{1}{H_1^2}\frac{\partial \ln H_2}{\partial q^1}\left(\frac{\partial b_0}{\partial q^1} + z\frac{\partial b_3}{\partial q^1}\right), \\
\frac{1}{2\mu}\sigma_\varphi^* &= 2\nu\left(\frac{1}{H_1^2}\frac{\partial z}{\partial q^1}\frac{\partial b_3}{\partial q^1} + \frac{1}{H_2^2}\frac{\partial z}{\partial q^2}\frac{\partial b_3}{\partial q^2}\right) + \frac{n^2}{r^2}(b_0 + zb_3) \\
&-\frac{1}{H_1^2}\frac{\partial \ln r}{\partial q^1}\left(\frac{\partial b_0}{\partial q^1} + z\frac{\partial b_3}{\partial q^1}\right) - \frac{1}{H_2^2}\frac{\partial \ln r}{\partial q^2}\left(\frac{\partial b_0}{\partial q^2} + z\frac{\partial b_3}{\partial q^2}\right), \\
\frac{1}{2\mu}\tau_{12}^* &= \frac{1}{H_1 H_2}(1-2\nu)\left(\frac{\partial z}{\partial q^1}\frac{\partial b_3}{\partial q^1} + \frac{\partial z}{\partial q^2}\frac{\partial b_3}{\partial q^2}\right) \\
&-\left(\frac{\partial^2 b_0}{\partial q^1 \partial q^2} + z\frac{\partial^2 b_3}{\partial q^1 \partial q^2}\right) + \frac{\partial \ln H_1}{\partial q^2}\left(\frac{\partial b_0}{\partial q^1} + z\frac{\partial b_3}{\partial q^1}\right) + \\
&\qquad \frac{\partial \ln H_2}{\partial q^1}\left(\frac{\partial b_0}{\partial q^2} + z\frac{\partial b_3}{\partial q^2}\right),
\end{aligned}\right\} \tag{1.12.13}$$

The factors associated with $\sin n\varphi$ in the expressions for components $\tau_{1\varphi}$ and $\tau_{2\varphi}$ are

$$\left.\begin{aligned}
\frac{1}{2\mu}\tau_{1\varphi}^* &= \frac{n}{rH_1}\left[-(1-2\nu)\frac{\partial z}{\partial q^1}b_3 + \left(\frac{\partial b_0}{\partial q^1} + z\frac{\partial b_3}{\partial q^1}\right) - \right. \\
&\qquad \left. \frac{\partial \ln r}{\partial q^2}(b_0 + zb_3)\right], \\
\frac{1}{2\mu}\tau_{2\varphi}^* &= \frac{n}{rH_2}\left[-(1-2\nu)\frac{\partial z}{\partial q^2}b_3 + \left(\frac{\partial b_0}{\partial q^2} + z\frac{\partial b_3}{\partial q^2}\right) - \right. \\
&\qquad \left. \frac{\partial \ln r}{\partial q^1}(b_0 + zb_3)\right].
\end{aligned}\right\} \tag{1.12.14}$$

Finally, due to eq. (1.4.10) the displacement vector is as follows

$$\mathbf{u} = 4(1-\nu)\mathbf{k}B_3 - \left(\frac{\mathbf{e}_k}{H_k}\frac{\partial}{\partial q^k} + \frac{\mathbf{e}_\varphi}{r}\frac{\partial}{\partial \varphi}\right)(zB_3 + B_0), \tag{1.12.15}$$

and, by eq. (1.12.12), is given by the formulae

$$\left.\begin{aligned} u_s &= \cos n\varphi \left[(3-4\nu) \frac{1}{H_s} \frac{\partial z}{\partial q^s} b_3 - \frac{1}{H_s} \left(\frac{\partial b_0}{\partial q^s} + z \frac{\partial b_3}{\partial q^s} \right) \right], \\ v &= \frac{n}{r} (b_0 + z b_3) \sin n\varphi. \end{aligned} \right\} \tag{1.12.16}$$

4.1.13 The Papkovich-Neuber solution for a body of revolution

Let us now complete the solution from Subsection 4.1.12 by the terms determined by projections B_x and B_y of the harmonic vector. Assuming now that

$$\mathbf{B} = \mathbf{i}_1 B_x + \mathbf{i}_2 B_y = \mathbf{e}_r B_r + \mathbf{e}_\varphi B_\varphi,$$

so that

$$\mathbf{R} \cdot \mathbf{B} = (r\mathbf{e}_r + z\mathbf{k}) \cdot \mathbf{B} = r B_r,$$

we have

$$\mathbf{u} = 4(1-\nu)(\mathbf{e}_r B_r + \mathbf{e}_\varphi B_\varphi) - \operatorname{grad} r B_r. \tag{1.13.1}$$

Similar to eq. (1.12.12) we introduce into consideration functions $b_r\left(q^1, q^2\right)$ and $b_\varphi\left(q^1, q^2\right)$

$$B_r = b_r\left(q^1, q^2\right) \cos n\varphi, \quad B_\varphi = b_\varphi\left(q^1, q^2\right) \sin n\varphi. \tag{1.13.2}$$

Then the expressions for the displacements take the form

$$\left.\begin{aligned} u_s &= \cos n\varphi \left[(3-4\nu) \frac{1}{H_s} \frac{\partial r}{\partial q^s} b_r - \frac{r}{H_s} \frac{\partial b_r}{\partial q^s} \right], \\ v &= \sin n\varphi \left[4(1-\nu) b_\varphi + n b_r \right]. \end{aligned} \right\} \tag{1.13.3}$$

Using eqs. (C.7.4), (C.7.5) and (B.4.19), the system of differential equations for functions b_r and b_φ is obtained from the relationship

$$\begin{aligned} \nabla^2 \mathbf{B} = {} & \left(\mathbf{e}_r \nabla^2 B_r + 2\nabla B_r \cdot \nabla \mathbf{e}_r + B_r \nabla^2 \mathbf{e}_r \right) + \\ & \qquad \left(\mathbf{e}_\varphi \nabla^2 B_\varphi + 2\nabla B_\varphi \cdot \nabla \mathbf{e}_\varphi + B_\varphi \nabla^2 \mathbf{e}_\varphi \right) \\ = {} & \mathbf{e}_r \left(\nabla^2 B_r - \frac{B_r}{r^2} - \frac{2}{r^2} \frac{\partial B_\varphi}{\partial \varphi} \right) + \mathbf{e}_\varphi \left(\nabla^2 B_\varphi - \frac{B_\varphi}{r^2} + \frac{2}{r^2} \frac{\partial B_r}{\partial \varphi} \right). \end{aligned}$$

The result is

$$\left.\begin{aligned} \nabla^2 b_r - \frac{n^2+1}{r^2} b_r - \frac{2n}{r^2} b_\varphi = 0, \\ \nabla^2 b_\varphi - \frac{n^2+1}{r^2} b_\varphi - \frac{2n}{r^2} b_r = 0. \end{aligned} \right\} \tag{1.13.4}$$

In the axisymmetric case ($n = 0$) it is split into two independent equations expressing that $b_r e^{i\varphi}$ and $b_\varphi e^{i\varphi}$ are harmonic functions.

The components of the stress tensor are found by the following formulae: the factors associated with $\cos n\varphi$ are

$$\left.\begin{aligned}
\frac{1}{2\mu}\sigma_1^* = {} & 2(1-\nu)\frac{r}{H_1^2}\frac{\partial b_r}{\partial q^1}\frac{\partial \ln r}{\partial q^1} + 2\nu\frac{r}{H_2^2}\frac{\partial b_r}{\partial q^2}\frac{\partial \ln r}{\partial q^2} + \\
& \frac{2\nu}{r}(nb_\varphi + b_r) - r\left(\frac{1}{H_1^2}\frac{\partial^2 b_r}{\partial q^{1^2}} - \frac{1}{H_1^2}\frac{\partial \ln H_1}{\partial q^1}\frac{\partial b_r}{\partial q^1} + \right. \\
& \left. \frac{1}{H_2^2}\frac{\partial \ln H_1}{\partial q^2}\frac{\partial b_r}{\partial q^2}\right), \\
\frac{1}{2\mu}\sigma_2^* = {} & 2(1-\nu)\frac{r}{H_2^2}\frac{\partial b_r}{\partial q^2}\frac{\partial \ln r}{\partial q^2} + 2\nu\frac{r}{H_1^2}\frac{\partial b_r}{\partial q^1}\frac{\partial \ln r}{\partial q^1} + \\
& \frac{2\nu}{r}(nb_\varphi + b_r) - r\left(\frac{1}{H_2^2}\frac{\partial^2 b_r}{\partial q^{2^2}} - \frac{1}{H_1^2}\frac{\partial \ln H_2}{\partial q^2}\frac{\partial b_r}{\partial q^2} + \right. \\
& \left. \frac{1}{H_1^2}\frac{\partial \ln H_2}{\partial q^1}\frac{\partial b_r}{\partial q^1}\right), \\
\frac{1}{2\mu}\sigma_\varphi^* = {} & -(1-2\nu)\left(\frac{r}{H_1^2}\frac{\partial \ln r}{\partial q^1}\frac{\partial b_r}{\partial q^1} + \frac{r}{H_2^2}\frac{\partial \ln r}{\partial q^2}\frac{\partial b_r}{\partial q^2}\right) + \\
& \left(n^2-1\right)\frac{b_r}{r} + 2(2-\nu)\frac{1}{r}(nb_\varphi + b_r), \\
\frac{1}{2\mu}\tau_{12}^* = {} & \frac{r}{H_1 H_2}\left[(1-2\nu)\left(\frac{\partial b_r}{\partial q^1}\frac{\partial \ln r}{\partial q^2} + \frac{\partial b_r}{\partial q^2}\frac{\partial \ln r}{\partial q^1}\right) - \right. \\
& \left. \frac{\partial^2 b_r}{\partial q^1 \partial q^2} + \frac{\partial b_r}{\partial q^2}\frac{\partial \ln H_2}{\partial q^1} + \frac{\partial b_r}{\partial q^1}\frac{\partial \ln H_1}{\partial q^2}\right]
\end{aligned}\right\} \tag{1.13.5}$$

and the factors associated with $\sin n\varphi$ are given by

$$\left.\begin{aligned}
\frac{1}{2\mu}\tau_{1\varphi}^* = {} & 2(1-\nu)\frac{1}{H_1}\left(\frac{\partial b_\varphi}{\partial q^1} - \frac{\partial \ln r}{\partial q^1}b_\varphi\right) + \\
& \frac{n}{H_1}\left[\frac{\partial b_r}{\partial q^1} - 2(1-2\nu)\frac{\partial \ln r}{\partial q^1}b_r\right], \\
\frac{1}{2\mu}\tau_{2\varphi}^* = {} & 2(1-\nu)\frac{1}{H_2}\left(\frac{\partial b_\varphi}{\partial q^2} - \frac{\partial \ln r}{\partial q^2}b_\varphi\right) + \\
& \frac{n}{H_2}\left[\frac{\partial b_r}{\partial q^2} - 2(1-2\nu)\frac{\partial \ln r}{\partial q^2}b_r\right].
\end{aligned}\right\} \tag{1.13.6}$$

4.1.14 Account of thermal components

In this case the only change in the system of the fundamental equations of elasticity theory, Subsection 4.1.1, is another form of the generalised Hooke law. Using eq. (3.4.8) of Chapter 3 and Table 3.2 in Subsection 3.3.1 we have

$$\hat{T} = 2\mu \left(\frac{\nu}{1-2\nu} \vartheta \hat{E} + \hat{\varepsilon} - \frac{1+\nu}{1-2\nu} \alpha\theta \hat{E} \right), \tag{1.14.1}$$

where θ denotes the temperature measured from the temperature of the natural state. The inverse relationship can be written down as follows

$$\hat{\varepsilon} = \frac{1}{2\mu} \left(\hat{T} - \frac{\nu}{1+\nu} \sigma \hat{E} \right) + \alpha\theta\hat{E}. \tag{1.14.2}$$

In what follows, the external mass and surface forces are assumed to be absent. The assumption that the problem of thermal conductivity can be considered separately from the problem of elasticity theory, cf. Subsection 3.3.5, does not affect the generality of the statement since the problem is linear. For a solid obeying Hooke's law this means that one can superimpose the states of stress caused by volume and surface forces as well as temperature determined separately for each of the listed factors.

Repeating the derivation of the equations in terms of displacements (1.3.2) and taking account of the thermal term in eq. (1.4.1) we obtain

$$\frac{1}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{u} + \nabla^2 \mathbf{u} - 2\frac{1+\nu}{1-2\nu} \operatorname{grad} \alpha\theta = 0. \tag{1.14.3}$$

Boundary condition (1.3.11) is set in the form

$$\frac{\nu}{1-2\nu} \mathbf{n} \operatorname{div} \mathbf{u} + n \cdot \nabla \mathbf{u} + \frac{1}{2} \mathbf{n} \times \operatorname{rot} \mathbf{u} - \frac{1+\nu}{1-2\nu} \mathbf{n} \alpha\theta = 0. \tag{1.14.4}$$

A comparison shows that the influence of the thermal term in Hooke's law can be formally reduced to prescribing mass forces with a potential proportional to the temperature

$$\Pi = 2\mu \frac{1+\nu}{1-2\nu} \alpha\theta, \tag{1.14.5}$$

and surface forces which are orthogonal to surface O of the body under consideration

$$\mathbf{F} = 2\mu \frac{1+\nu}{1-2\nu} \alpha\theta \mathbf{n} = \mathbf{n}\Pi. \tag{1.14.6}$$

Comparing expressions (1.14.5) and (1.4.7) yields that equation (1.14.3) admits a particular solution

$$\mathbf{u}_* = \operatorname{grad} \chi_0, \tag{1.14.7}$$

where χ_0 obeys Poisson's equation

$$\nabla^2 \chi_0 = \frac{1+\nu}{1-\nu} \alpha\theta. \tag{1.14.8}$$

Let us proceed to construct the differential equations in terms of stresses. The equation of statics in the case of no mass forces

$$\operatorname{div} \hat{T} = 0 \tag{1.14.9}$$

needs to be completed, due to eqs. (1.5.1) and (1.1.4.2), by the condition

$$2\mu \operatorname{inc} \hat{\varepsilon} = \operatorname{inc}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) + 2\mu \operatorname{inc} \alpha\theta\hat{E} = 0. \tag{1.14.10}$$

However, by eq. (1.5.2)

$$\operatorname{inc} \alpha\theta\hat{E} = \left(\hat{E}\nabla^2 - \nabla\nabla\right)\alpha\theta,$$

and using eq. (1.5.3) one can set condition (1.14.10) in the following form

$$\nabla^2\hat{T} + \left(\nabla\nabla - \hat{E}\nabla^2\right)\left(\frac{\sigma}{1+\nu} + 2\mu\alpha\theta\right) = 0. \tag{1.14.11}$$

Estimating the first invariant we have

$$\nabla^2\sigma = -4\mu\frac{1+\nu}{1-\nu}\alpha\nabla^2\theta, \tag{1.14.12}$$

so that

$$\nabla^2\hat{T} + \frac{1}{1+\nu}\nabla\nabla\sigma = -2\mu\alpha\left(\nabla\nabla\theta + \frac{1+\nu}{1-\nu}\hat{E}\nabla^2\theta\right). \tag{1.14.13}$$

Provided that the temperature is a linear function in the coordinates, see Subsection 2.2.3,

$$\theta = \theta_0 + \mathbf{q}\cdot\mathbf{R},$$

then the term in eq. (1.14.13), described by this formula, vanishes. For this reason, under the assumption that the surface forces vanish on the whole body's surface

$$\mathbf{n}\cdot\hat{T} = 0, \tag{1.14.14}$$

we obtain that all equations (1.14.9), (1.14.14) and (1.14.13) for tensor $\hat{T}$ are homogeneous. The solution

$$\hat{T} = 0$$

satisfies all above-mentioned boundary conditions and the equation in the volume. Besides, it is the unique solution, see Subsection 4.4.1. Hence, no thermal stresses appear in a solid under a linear law of the temperature distribution. The displacement vector is calculated using formulae (2.3.5) of Chapter 2 in which one adds the terms of the type (2.2.6) of Chapter 2 describing a rigid-body rotation of the medium

$$\mathbf{u} = \alpha\theta\left(\mathbf{R} - \mathbf{R}_0\right) - \frac{1}{2}a\mathbf{q}\left|\mathbf{R} - \mathbf{R}_0\right|^2 + \mathbf{u}_0 + \boldsymbol{\omega}_0 \times \left(\mathbf{R} - \mathbf{R}_0\right), \qquad (1.14.15)$$

with $\mathbf{R}$ denoting the position vector.

The above said takes place only under the assumption that condition (1.14.14) holds on the whole surface O. If such a displacement vector $\mathbf{u}$ is prescribed on a part of the surface and differs from that given by formula (1.14.15) then there appear stresses due to the constraint forces.

It is also easy to prove that, under a linear distribution of the temperature, displacement vector (1.14.15) satisfies differential equation (1.14.13) and boundary condition (1.14.4).

4.2 Variational principles of statics for a linear elastic body

4.2.1 Stationarity of the potential energy of the system

First, only isothermal and adiabatic processes will be considered in Subsections 4.2.1-4.2.6. The elementary work of the external forces $\delta' a_{(e)}$ can be identified as a variation of the strain energy δa which is equal to the variation of the free energy in the isothermal process or the internal energy in the adiabatic process

$$\delta' a_{(e)} = \delta a = \delta \iiint\limits_V A d\tau = \iiint\limits_V \rho \mathbf{K} \cdot \delta \mathbf{u} d\tau + \iint\limits_O \mathbf{F} \cdot \delta \mathbf{u} do. \qquad (2.1.1)$$

From this equality one can obtain three different variational principles depending upon which variables are used for the specific strain energy. Prescribing the latter by a quadratic form of the strain components $A(\varepsilon)$, see eq. (3.2.3) of Chapter 3, we arrive at the principle of minimum potential energy of the system whereas proceeding from a quadratic form of the stress components $A(\sigma)$, eq. (3.2.8) of Chapter 3, we obtain the principle of minimum complementary work. In the first principle, the displacements are varied whilst in the second principle the stress components are varied. Finally, in the mixed stationarity principle the specific strain energy is given by a bilinear form $A(\varepsilon, \sigma)$, in which both stresses and strains are varied.

Formula (2.1.1) contains the statement that the work of the external mass and surface forces due to a virtual displacement $\delta\mathbf{u}$ of particles from the equilibrium position is equal to the variation of the strain energy. One takes $\delta\mathbf{u} = 0$ on O_1, i.e. on that part of the surface where the displacements are prescribed. Thus

$$\iint_O \mathbf{F} \cdot \delta \mathbf{u} do = \iint_{O_2} \mathbf{F} \cdot \delta \mathbf{u} do.$$

Two states of the solid are considered under the same forces $\rho\mathbf{K}$ in the volume and $\mathbf{F}$ at part O_2, the latter having not overlapped O_1. The first state is the equilibrium state while the second one is a state which is infinitesimally close to it and differs from the first in a field of virtual displacement. In other words, it is assumed that in volume V

$$\delta\rho\mathbf{K} = 0, \qquad \rho\mathbf{K} \cdot \delta\mathbf{u} = \delta\left(\rho\mathbf{K} \cdot \mathbf{u}\right)$$

and on surface O_2

$$\delta\mathbf{F} = 0, \qquad \mathbf{F} \cdot \delta\mathbf{u} = \delta\left(\mathbf{F} \cdot \mathbf{u}\right).$$

It is legitimate to factor out the sign of variation beyond the integral since volume V and surface O_2 are fixed. By doing so and taking into account eq. (2.1.1) we obtain

$$\delta \left[\iiint_V A d\tau - \iiint_V \rho\mathbf{K} \cdot \mathbf{u} d\tau - \iint_{O_2} \mathbf{F} \cdot \mathbf{u} do \right] = 0. \tag{2.1.2}$$

One refers to

$$\Phi = \iiint_V A d\tau - \iiint_V \rho\mathbf{K} \cdot \mathbf{u} d\tau - \iint_{O_2} \mathbf{F} \cdot \mathbf{u} do \tag{2.1.3}$$

as the potential energy of the system. It is equal to the difference between the strain energy and the work of the prescribed external forces (they are not prescribed on O_1) calculated under the assumption that these forces, during the whole process of deformation from the natural state, retain the values which they had in the considered equilibrium state.

The potential energy Φ is a functional over $\mathbf{u}$ and its numerical value changes together with changing $\mathbf{u}$. Within this set of numerical values of Φ, a particular value of Φ, corresponding to the vector $\mathbf{u}$ in the equilibrium position of the solid, possesses a remarkable property of stationarity

$$\delta\Phi = 0. \tag{2.1.4}$$

This means that calculating Φ for the displacement field in the equilibrium position and a second time for a displacement field $\mathbf{u} + \delta\mathbf{u}$ leads to the same value provided that the calculation is carried out up to the values of order of smallness of $\delta\mathbf{u}$. Under a virtual field of displacement from the equilibrium position an increment in functional Φ is a value of higher order of smallness than $\delta\mathbf{u}$.

4.2.2 *The principle of minimum potential energy of the system*

In this subsection it is proved that the stationary value of functional Φ in the equilibrium position is a minimum.

In order to explain the forthcoming reasoning let us make precise the concept of the increment in function $F(x_1, x_2, \ldots, x_n)$ of n variables $x_1, x_2, \ldots, x_n$. Let $\delta x_1, \delta x_2, \ldots, \delta x_n$ denote the increments (or variations) of the corresponding variables. Then the increment of the function ΔF is given by

$$\begin{aligned}\Delta F &= F(x_1+\delta x_1, x_2+\delta x_2, \ldots, x_n+\delta x_n) - F(x_1, x_2, \ldots, x_n)\\ &= \delta F + \delta^2 F + \ldots,\end{aligned} \tag{2.2.1}$$

where the first variation δF denotes the term which is linear in δx_s whereas $\delta^2 F$ contains the quadratic terms etc.

$$\delta F = \frac{\partial F}{\partial x_s}\delta x_s, \quad \delta^2 F = \frac{1}{2}\frac{\partial^2 F}{\partial x_s \partial x_k}\delta x_s \delta x_k.$$

If F is a quadratic form in variables x_s then

$$F = \frac{1}{2}a_{sk}x_s x_k, \qquad \frac{\partial^2 F}{\partial x_s \partial x_k} = a_{sk}, \qquad \frac{\partial^3 F}{\partial x_k \partial x_s \partial x_t} = 0,$$

and therefore

$$\delta^2 F = \frac{1}{2}a_{sk}\delta x_s \delta x_k = F(\delta x_1, \delta x_2, \ldots, \delta x_n), \quad \delta^s F = 0 \quad (s > 2)$$

which is the second variation of the quadratic form and is equal to this form of the variations of the variables. Instead of eq. (2.2.1) we have

$$\Delta F = \delta F + F(\delta x_1, \delta x_2, \ldots, \delta x_n), \tag{2.2.2}$$

and if the form of F is linear, then

$$\Delta F = \delta F. \tag{2.2.3}$$

Returning to functional Φ we have, using eq. (2.1.3), that

$$\Delta\Phi = \iiint\limits_V \Delta A d\tau - \iiint\limits_V \Delta\rho\mathbf{K}\cdot\mathbf{u}d\tau - \iint\limits_{O_2} \Delta\mathbf{F}\cdot\mathbf{u}d\tau.$$

Since $\rho\mathbf{K}$ and $\mathbf{F}$ do not depend on $\mathbf{u}$ we obtain, referring to eq. (2.2.3), that

$$\Delta\rho\mathbf{K}\cdot\mathbf{u} = \delta\rho\mathbf{K}\cdot\mathbf{u}, \quad \Delta\mathbf{F}\cdot\mathbf{u} = \delta\mathbf{F}\cdot\mathbf{u}.$$

On the other hand, by virtue of eq. (2.2.2)

$$\Delta A = A(\varepsilon + \delta\varepsilon) - A(\varepsilon) = \delta A + A(\delta\varepsilon),$$

because A is a quadratic form of the component of the strain tensor. Finally we arrive at the equality

$$\Delta\Phi = \delta\left(\iiint\limits_V A d\tau - \iiint\limits_V \rho\mathbf{K}\cdot\mathbf{u}d\tau - \iint\limits_{O_2} \mathbf{F}\cdot\mathbf{u}do\right) + \iiint\limits_V A(\delta\varepsilon)\,d\tau,$$

so that, by eqs. (2.1.2) or (2.1.4)

$$\Delta\Phi = \iiint\limits_V A(\delta\varepsilon)\,d\tau.$$

According to eq. (3.3.4) of Chapter 3 the specific strain energy is a positive definite function, hence $A > 0$ for any nontrivial $\delta\hat{\varepsilon}$. This proves that

$$\Delta\Phi > 0,$$

that is, functional Φ increases for any deviation of the solid from the equilibrium state. In other words, this functional has a minimum at the equilibrium. Thus, we have arrived at the principle of minimum potential energy of the system: The state of equilibrium of a linear solid differs from the other possible states in that functional Φ (the potential energy of the system) has a minimum value in this state. By the word "possible" we indicate that displacements $\mathbf{u} + \delta\mathbf{u}$ continuous in volume V are coincident with $\mathbf{u}$ on that part O_1 of surface O where the displacement is given.

It follows from Clapeyron's formula, eq. (3.3.3) of Chapter 3, and eq. (2.1.3) that in the equilibrium position

$$\Phi = \Phi_{\min} = \frac{1}{2}\left[-\iiint\limits_V \rho\mathbf{K}\cdot\mathbf{u}d\tau - \iint\limits_{O_2} \mathbf{F}\cdot\mathbf{u}do + \iint\limits_{O_1} \mathbf{F}\cdot\mathbf{u}do\right]. \quad (2.2.4)$$

The problem of searching for the equilibrium state of a linear solid is thus reduced to a variational problem of determining the vector $\mathbf{u}$ rendering a minimum to functional Φ and coinciding with given values on O_1. It is known that this problem of variational calculus is equipollent to a boundary-value problem. Its differential equations and the boundary conditions are obtained from the minimised functional and are respectively Euler's equations and the natural boundary conditions corresponding to this functional.

Let us now proceed to construct this variation. Due to eq. (3.2.4) of Chapter 3 we have

$$\delta A = \frac{\partial A}{\partial \varepsilon_x}\delta\varepsilon_x + \ldots + \frac{\partial A}{\partial \gamma_{zx}}\delta\gamma_{zx} = \sigma_x\delta\varepsilon_x + \ldots + \tau_{zx}\delta\gamma_{zx},$$

where $\sigma_x, \ldots, \tau_{zx}$ are the linear forms in the strain components determined by equalities (3.1.5) of Chapter 3. In other notation, cf. eq. (B.3.10),

$$\delta A = \hat{T} \cdot\cdot\, \delta\hat{\varepsilon} = \operatorname{div}\left(\hat{T} \cdot \delta\mathbf{u}\right) - \delta\mathbf{u} \cdot \operatorname{div}\hat{T}. \tag{2.2.5}$$

Tensor $\hat{T}$ is expressed here in terms of the strain tensor. Expressing the latter in terms of the displacement vector $\mathbf{u}$ and referring to the derivation of the equations of the elasticity theory in Subsection 1.3 of the present Chapter, we have by eq. (1.3.2)

$$\operatorname{div}\hat{T} = \mu\left(\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u} + \nabla^2\mathbf{u}\right) = \mathbf{L}(\mathbf{u}) \tag{2.2.6}$$

and simultaneously by eq. (1.3.12)

$$\mathbf{n}\cdot\hat{T} = 2\mu\left(\frac{1}{1-2\nu}\mathbf{n}\operatorname{div}\mathbf{u} + \mathbf{n}\cdot\nabla\mathbf{u} + \frac{1}{2}\mathbf{n}\times\operatorname{rot}\mathbf{u}\right) = \mathbf{M}(\mathbf{u}). \tag{2.2.7}$$

By virtue of eqs. (2.2.5)-(2.2.7) we have

$$\begin{aligned}
\delta\iiint\limits_V A d\tau &= \iiint\limits_V \operatorname{div}\left(\hat{T}\cdot\delta\mathbf{u}\right)\delta\tau - \iiint\limits_V \delta\mathbf{u}\cdot\operatorname{div}\hat{T}d\tau \\
&= \iint\limits_O \mathbf{n}\cdot\hat{T}\cdot\delta\mathbf{u}do - \iiint\limits_V \delta\mathbf{u}\cdot\mathbf{L}(\mathbf{u})\,d\tau \\
&= \iint\limits_{O_2} \mathbf{M}(\mathbf{u})\cdot\delta\mathbf{u}do - \iiint\limits_V \mathbf{L}(\mathbf{u})\cdot\delta\mathbf{u}d\tau.
\end{aligned} \tag{2.2.8}$$

Here it is taken into account that vector $\mathbf{u}$ is sought within a class of functions taking a prescribed value on O_1

$$\mathbf{u}|_{O_1} = \mathbf{u}_*, \tag{2.2.9}$$

so that $\delta\mathbf{u} = 0$ on O_1.

Inserting eq. (2.2.8) into eq. (2.1.2) leads to the equality

$$\delta\Phi = -\iiint\limits_V [\mathbf{L}(\mathbf{u}) + \rho\mathbf{K}]\cdot\delta\mathbf{u}d\tau + \iint\limits_{O_2} [\mathbf{M}(\mathbf{u}) - \mathbf{F}]\cdot\delta\mathbf{u}do. \tag{2.2.10}$$

As $\delta\mathbf{u}$ is arbitrary in the volume and on the part O_2 of the surface where the displacements are not given, fulfillment of the stationarity condition (2.1.14) requires that the integrands in the volume and surface integrals are zero. We are led to the differential equation of equilibrium in terms of displacements

$$\mathbf{L}(\mathbf{u}) + \rho\mathbf{K} = 0 \tag{2.2.11}$$

and the boundary condition on O_2

$$\mathbf{M}(\mathbf{u}) = \mathbf{F}. \tag{2.2.12}$$

Naturally, we obtain the boundary-value problem in terms of displacements as Φ is a functional over $\mathbf{u}$. As mentioned above the definition of the potential energy of the system and the formulation of the minimum principle are not related to the state of stress in the system. The concept of stress is not required for this energetic principle operating with a functional over the displacement vector. Similar to Hamilton's principle from general mechanics, the principle of minimum potential energy synthesizes the properties of the physical model taking into account the experimental data for the behaviour of the stressed body.

4.2.3 Ritz's method

The variational statement of the problem of equilibrium in the form of the principle of minimum potential energy suggests the possibility of applying direct methods of variational calculus for solving problems of elasticity theory.

In Ritz's method (1909) differential equation (2.2.11) and the static boundary condition (2.2.12) are not considered since it is known in advance that they are automatically satisfied if there exists a vector $\mathbf{u}$ rendering an exact minimum to functional Φ. The approach allowing an approximate determination of this vector consists of prescribing its projections by approximate representations of the form

$$\left.\begin{aligned} u &= \sum_{k=1}^{n} a_k \varphi_k(x,y,z) + u_0(x,y,z), \\ v &= \sum_{k=1}^{n} a_{k+n} \varphi_{k+n}(x,y,z) + v_0(x,y,z), \\ w &= \sum_{k=1}^{n} a_{k+2n} \varphi_{k+2n}(x,y,z) + w_0(x,y,z), \end{aligned}\right\} \tag{2.3.1}$$

Here u_0, v_0, w_0 take given values (2.2.9) on O_1 and functions φ_s $(s = 1, 2, \ldots, 3n)$ are chosen to vanish on O_1 which ensures that the boundary condition for vector $\mathbf{u}$ is satisfied for any coefficients a_s. The system of approximating ("coordinate") functions φ_s should be taken in such a general form that any system of displacements satisfying condition (2.2.9) can be approximately represented in the form of eq. (2.3.1) for sufficiently large n. Such a system represents, for example, the products of the integer powers of the variables $x^{q_1} y^{q_2} z^{q_3}$ multiplied by a function vanishing on O_1.

After substituting chosen representations for displacements u, v, w, eq. (2.3.1), into the expression for the potential energy of the system Φ the latter becomes a sum of quadratic and linear forms in coefficients a_s and a constant term, i.e.

$$\Phi = \Phi_2 (a_1, \dots, a_{3n}) - \Phi_1 (a_1, \dots, a_{3n}) + \Phi_0 \,. \tag{2.3.2}$$

The quadratic form Φ_2 is just equal to the strain energy calculated by vector $\mathbf{u} - \mathbf{u}_0$

$$\Phi_2 = \iiint\limits_V A (\varepsilon - \varepsilon_0) \, d\tau = \frac{1}{2} \sum_{s=1}^{3n} \sum_{k=1}^{3n} C_{st} a_s a_t \tag{2.3.3}$$

and is a positive definite form of $a_1, a_2, \dots, a_{3n}$ since A is a positive definite form of the strain components. Thus, the determinant of matrix C_{st} is positive

$$|C_{st}| > 0. \tag{2.3.4}$$

By the theorem of minimum potential energy of the system, the best approximation in the chosen class of functions approximating vector $\mathbf{u}$ is provided by the value of the coefficients rendering the minimum to expression (2.3.2). This leads to the system of $3n$ linear equations

$$\frac{\partial}{\partial a_t} (\Phi_2 - \Phi_1) = 0, \quad t = 1, 2, \dots, 3n \tag{2.3.5}$$

or

$$\sum_{s=1}^{3n} C_{st} a_s = B_t, \quad t = 1, 2, \dots, 3n \tag{2.3.6}$$

with the same number of unknowns. The existence and uniqueness of the solution follows from inequality (2.3.4).

In such a manner, an approximate solution of the problem is constructed. It is reasonable to assume that, when the system of approximating functions is sufficiently general, the calculated value of the potential energy of the system tends to the minimum with the growth of n. However the "convergence with respect to energy" does not imply that the sequence of approximations (2.3.1) diverges to the sought-for solution. This particular analysis is beyond the scope of the present book and is a subject of extensive literature[1]. Under a reasonable choice of the form and number of the approximating functions the calculation yields the values of vector $\mathbf{u}$ which are close to the exact solution. Less accuracy is expected for the derivatives of the displacements found by Ritz's method and, in turn, for stresses.

[1] See for example Mikhlin S.G., Direct methods in mathematical physics (in Russian). Gostekhizdat, Moscow, 1950.

4.2.4 Galerkin's method (1915)

For boundary-value problems admitting a variational statement, in particular, for the problems of elasticity theory, this approximate method of integrating differential equations presents a modification of Ritz's method which simplifies calculation. Approximation (2.3.1) is substituted into the expression for the variation of the potential energy of the system (2.2.10) rather than into the potential energy (2.1.3). This excludes the necessity to square sum (2.3.1) for calculating A.

Replacing variations $\delta u, \delta v, \delta w$ in eq. (2.2.10) by the following expressions

$$\delta u = \sum_{k=1}^{n} \varphi_k \delta a_k, \quad \delta v = \sum_{k=1}^{n} \varphi_{k+n} \delta a_{k+n}, \quad \delta w = \sum_{k=1}^{n} \varphi_{k+2n} \delta a_{k+2n}, \qquad (2.4.1)$$

in which variations of the sought coefficients δa_s are arbitrary we arrive at the equality

$$-\delta\Phi = \sum_{k=1}^{n} \delta a_k \left\{ \iiint_V \left[L_1(\mathbf{u}) + \rho K_1\right] \varphi_k d\tau - \iint_{O_2} \left[M_1(\mathbf{u}) - F_1\right] \varphi_k do \right\} +$$
$$\sum_{k=1}^{n} \delta a_{k+n} \left\{ \iiint_V \left[L_2(\mathbf{u}) + \rho K_2\right] \varphi_{k+n} d\tau - \iint_{O_2} \left[M_2(\mathbf{u}) - F_2\right] \varphi_{k+n} do \right\} +$$
$$\sum_{k=1}^{n} \delta a_{k+2n} \left\{ \iiint_V \left[L_3(\mathbf{u}) + \rho K_3\right] \varphi_{k+2n} d\tau - \iint_{O_2} \left[M_3(\mathbf{u}) - F_3\right] \varphi_{k+2n} do \right\}$$
$$= 0, \qquad (2.4.2)$$

where L_s and M_s denote respectively the projections of vectors $\mathbf{L}$ and $\mathbf{M}$ given by eqs. (2.2.6) and (2.2.7) on the coordinate axes. Clearly, u, v, w in expressions for L_s and M_s are replaced by their representations (2.3.1).

Equating now the coefficients of the arbitrary variations δa_s to zero we obtain the following system of $3n$ linear equations for unknowns a_s

$$\iiint_V L_1(\mathbf{u}) \varphi_k d\tau - \iint_{O_2} M_1(\mathbf{u}) \varphi_k do = -\iiint_V \rho K_1 \varphi_k d\tau - \iint_{O_2} F_1 \varphi_k do,$$
$$\iiint_V L_2(\mathbf{u}) \varphi_{k+n} d\tau - \iint_{O_2} M_2(\mathbf{u}) \varphi_{k+n} do =$$
$$= -\iiint_V \rho K_2 \varphi_{k+n} d\tau - \iint_{O_2} F_2 \varphi_{k+n} do,$$

$$\iiint\limits_V L_3(\mathbf{u})\,\varphi_{k+2n}d\tau - \iint\limits_{O_2} M_3(\mathbf{u})\,\varphi_{k+2n}do =$$
$$= -\iiint\limits_V \rho K_3\varphi_{k+2n}d\tau - \iint\limits_{O_2} F_3\varphi_{k+2n}do. \quad (2.4.3)$$

Of course, it is just another form of equations (2.3.6) obtained by another sequence of calculations.

By Galerkin's equations one often understands the system of equations

$$\iiint\limits_V (L_1 + \rho K_1)\,\varphi_k d\tau = 0, \quad \iiint\limits_V (L_2 + \rho K_2)\,\varphi_{k+n}d\tau = 0,$$
$$\iiint\limits_V (L_3 + \rho K_3)\,\varphi_{k+2n}d\tau = 0, \quad (2.4.4)$$

for which only differential equations of the problem are utilised. However the choice of functions φ_s approximating the solution needs to be subjected not only to the kinematic boundary conditions but also to the static boundary conditions (2.2.12). The surface integrals in the system of equations (2.4.3) drop out and this system transforms into system (2.4.4).

4.2.5 Principle of minimum complementary work

The principle of minimum potential energy of the system was obtained by comparing the displacement field of the solid at the equilibrium state and an infinitesimally close state admitted by constraints. In the principle of minimum complementary work, two statically admissible states of stress are compared: the true one given by the stress tensor $\hat{T}$ and an infinitesimally close state with the stress tensor $\hat{T} + \delta\hat{T}$. Both states are considered under the same prescribed forces which are the volume force $\rho\mathbf{K}$ and the surface force on part O_2 of the surface O bounding the body. Hence, in volume V

$$\operatorname{div}\hat{T} + \rho\mathbf{K} = 0, \quad \operatorname{div}\left(\hat{T} + \delta\hat{T}\right) + \rho\mathbf{K} = 0 \quad (2.5.1)$$

and on surface O_2

$$\mathbf{n}\cdot\hat{T} = \mathbf{F}, \quad \mathbf{n}\cdot\left(\hat{T} + \delta\hat{T}\right) = \mathbf{F}, \quad (2.5.2)$$

so that

$$\operatorname{div}\delta\hat{T} = 0, \quad \mathbf{n}\cdot\delta\hat{T}\Big|_{O_2} = 0. \quad (2.5.3)$$

Considering the specific strain energy A as a function of the stress components (i.e. in the form of eq. (3.2.8) of Chapter 3) and accounting for eq.

(3.2.9) of Chapter 3, as well as eq. (2.5.3) we have (cf. eq. (B.3.10))

$$\delta A(\sigma) = \hat{\varepsilon} \cdot \cdot \delta \hat{T} = \operatorname{div}\left(\delta \hat{T} \cdot \mathbf{u}\right) - \mathbf{u} \cdot \operatorname{div} \delta \hat{T} = \operatorname{div}\left(\delta \hat{T} \cdot \mathbf{u}\right). \tag{2.5.4}$$

Hence

$$\delta a = \iiint\limits_V \delta A d\tau = \iiint\limits_V \operatorname{div}\left(\delta \hat{T} \cdot \mathbf{u}\right) d\tau = \iint\limits_O \mathbf{n} \cdot \delta \hat{T} \cdot \mathbf{u} do = \iint\limits_{O_1} \mathbf{u} \cdot \delta \mathbf{F} do, \tag{2.5.5}$$

where $\delta \mathbf{F}$ denotes variation of the surface force on that part O_1 of the surface where the displacement vector is prescribed. On O_1 we have

$$\delta \mathbf{u} = 0, \quad \mathbf{u} \cdot \delta \mathbf{F} = \delta(\mathbf{u} \cdot \mathbf{F}),$$

and equality (2.5.5) is reset in the form

$$\delta \left(\iiint\limits_V A(\sigma) d\tau - \iint\limits_{O_1} \mathbf{u} \cdot \mathbf{F} do \right) = 0. \tag{2.5.6}$$

The expression

$$\Psi = \iiint\limits_V A(\sigma) d\tau - \iint\limits_{O_1} \mathbf{u} \cdot \mathbf{F} do = \iiint\limits_V A(\sigma) d\tau - \iint\limits_{O_1} \mathbf{n} \cdot \hat{T} \cdot \mathbf{u} do \tag{2.5.7}$$

is termed the complimentary work whereas relationship (2.5.6) expresses the property of stationarity of this functional over the stress tensor $\hat{T}$ in the equilibrium state

$$\delta \Psi = 0. \tag{2.5.8}$$

The stationary value of the complimentary work is a minimum. Indeed, by virtue of eqs. (2.2.2) and (2.2.3)

$$\begin{aligned} \Delta \Psi &= \iiint\limits_V \delta A(\sigma) d\tau + \iiint\limits_V A(\delta\sigma) d\tau - \iint\limits_{O_1} \mathbf{u} \cdot \delta \mathbf{F} do \\ &= \delta \Psi + \iiint\limits_V A(\delta\sigma) d\tau \end{aligned}$$

and by eq. (2.5.8)

$$\Delta \Psi = \iiint\limits_V A(\delta\sigma) d\tau > 0,$$

which proves the existence of the minimum in the equilibrium state.

Hence, the equilibrium state of a linear solid differs from all statically admissible ones under the given external forces in that functional Ψ over the stress tensor $\hat{T}$, referred to as the complimentary work, has a minimum in the equilibrium state.

By eq. (2.5.7) and Clapeyron's formula, eq. (3.3.3) of Chapter 3, this minimum is equal to

$$\Psi_{\min} = \frac{1}{2}\left(\rho\mathbf{K}\cdot\mathbf{u}d\tau - \iint\limits_{O_1} \mathbf{F}\cdot\mathbf{u}do + \iint\limits_{O_2} \mathbf{F}\cdot\mathbf{u}do\right). \tag{2.5.9}$$

It was stated in Subsection 4.2.2 that Euler's equations and the natural boundary conditions of the variational problem on the minimum potential energy of the system are nothing other than the equations of equilibrium in terms of the displacements and the static boundary conditions. It is natural to expect that the principle of minimum complementary work, which is the functional over the statically admissible stress tensor $\hat{T}$, yields Beltrami's dependences and the kinematic boundary conditions (as natural boundary conditions of the variational problem).

In order to prove this, we represent the stationarity condition (2.5.8)

$$\begin{aligned}
\delta\Psi &= \iiint\limits_V \delta A(\sigma)\, d\tau - \iint\limits_{O_1} \mathbf{u}\cdot\delta\mathbf{F}do \\
&= \iiint\limits_V \left(\frac{\partial A}{\partial\sigma_x}\delta\sigma_x + \ldots + \frac{\partial A}{\partial\tau_{zx}}\delta\tau_{zx}\right) d\tau - \iint\limits_{O_1} \mathbf{u}\cdot\delta\mathbf{F}do \\
&= \iiint\limits_V \left(\varepsilon_x\delta\sigma_x + \ldots + \gamma_{zx}\delta\tau_{zx}\right) d\tau - \iint\limits_{O_1} \mathbf{u}\cdot\delta\mathbf{F}do \\
&= \iiint\limits_V \hat{\varepsilon}\cdot\cdot\,\delta\hat{T}d\tau - \iint\limits_{O_1} \mathbf{u}\cdot\delta\mathbf{F}do = 0.
\end{aligned} \tag{2.5.10}$$

Here $\varepsilon_x, \ldots \gamma_{zx}$ are the linear forms in the components of the stress tensor $\hat{T}$ determined by means of eq. (3.1.8) of Chapter 3 and expressed by formulae (3.1.8) of Chapter 3, besides, $\hat{\varepsilon}$ is the tensor given by these linear forms and eq. (1.1.4). Variations of the components of tensor $\hat{T}$, i.e. $\delta\hat{T}$, in the integrand in eq. (2.5.10) are not independent and must satisfy dependences (2.5.3). We thus arrive at the constrained problem of the calculus of variations. Following the standard approach we introduce a Lagrange vector $\boldsymbol{\lambda}$ in volume V. Representing eq. (2.5.10) in the form

$$\delta\Psi = \iiint\limits_V \left(\hat{\varepsilon}\cdot\cdot\,\delta\hat{T} + \boldsymbol{\lambda}\cdot\operatorname{div}\delta\hat{T}\right) d\tau - \iint\limits_{O_1} \mathbf{u}\cdot\delta\mathbf{F}do = 0, \tag{2.5.11}$$

we can consider all six variations $\delta\sigma_x, \dots, \delta\tau_{zx}$, constrained by three conditions (2.5.3), as being independent by a proper choice of the three components of vector λ.

Applying transformation (B.3.10)

$$\boldsymbol{\lambda} \cdot \operatorname{div} \delta\hat{T} = \operatorname{div}\left(\delta\hat{T} \cdot \boldsymbol{\lambda}\right) - \operatorname{def} \boldsymbol{\lambda} \cdot\cdot \delta\hat{T},$$

we rewrite eq. (2.5.11) as follows

$$\delta\Psi = \iiint\limits_V (\hat{\varepsilon} - \operatorname{def} \boldsymbol{\lambda}) \cdot\cdot \delta\hat{T} d\tau + \iiint\limits_V \operatorname{div}\left(\delta\hat{T} \cdot \boldsymbol{\lambda}\right) d\tau - \iint\limits_{O_1} \mathbf{u} \cdot \delta\mathbf{F} do =$$

$$\iiint\limits_V (\hat{\varepsilon} - \operatorname{def} \boldsymbol{\lambda}) \cdot\cdot \delta\hat{T} d\tau + \iint\limits_{O_2} \mathbf{n} \cdot \delta\hat{T} \cdot \boldsymbol{\lambda} do + \iint\limits_{O_1} \left(\mathbf{n} \cdot \delta\hat{T} \cdot \boldsymbol{\lambda} - \mathbf{u} \cdot \delta\mathbf{F}\right) do,$$

so that, referring to eq. (2.5.3), we obtain

$$\delta\Psi = \iiint\limits_V (\hat{\varepsilon} - \operatorname{def} \boldsymbol{\lambda}) \cdot\cdot \delta\hat{T} d\tau + \iint\limits_{O_1} \delta\mathbf{F} \cdot (\boldsymbol{\lambda} - \mathbf{u}) \, do. \tag{2.5.12}$$

Expressing now the condition for which the multipliers of variations $\delta\hat{T}$ and $\delta\mathbf{F}$ in the integrand vanish, we arrive at the relationships

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) = \operatorname{def} \boldsymbol{\lambda}, \quad (\boldsymbol{\lambda} - \mathbf{u})\Big|_{O_1} = 0. \tag{2.5.13}$$

The first equality shows that the tensor denoted here as $\hat{\varepsilon}$ is the deformation of Lagrange's vector $\boldsymbol{\lambda}$. The latter must be equal to the prescribed displacement vector on O_1 and nothing prohibits us from identifying λ with the displacement vector $\mathbf{u}$ in volume V and defining tensor $\hat{\varepsilon}$ in terms of the displacement field. The principle of complementary work does not operate with the concept of the strain tensor. Hence we must additionally identify $\boldsymbol{\lambda}$ with $\mathbf{u}$ because "the principle does not know about it".

By eq. (2.1.9) of Chapter 2. any tensor which is a deformation needs to satisfy the condition

$$\operatorname{inc} \operatorname{def} \boldsymbol{\lambda} = 0, \tag{2.5.14}$$

and thus excluding $\boldsymbol{\lambda}$ from the first formula in (2.5.13) results in the relationship

$$\operatorname{inc}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) = 0\,. \tag{2.5.15}$$

The latter, together with condition (1.1.1) stating that $\hat{T}$ is a statically admissible tensor, leads to Beltrami's dependences, see Subsection 4.1.5. This is exactly what is required. As condition (2.5.14) is met, vector $\boldsymbol{\lambda} \equiv \mathbf{u}$ can be found by means of Cesaro's formula (2.2.2) of Chapter 2.

4.2.6 Mixed stationarity principle (E. Reissner, 1961)

While formulating the principle of minimum potential energy one considers a functional over vector $\mathbf{u}$ and requires that the latter coincides with a prescribed displacement on part O_1 of the surface, the stress tensor being excluded from consideration. The Euler differential equations prove to be the equilibrium equations in terms of displacements whereas the natural boundary conditions turn out to be the equilibrium conditions (in terms of displacements) on the part of surface O_2 where the external surface forces are given. In contrast to this, the principle of minimum complementary work deals with a functional over the stress tensor $\hat{T}$ along with all statically admissible states of stress with the tensors $\hat{T}$ satisfying the necessary conditions of statics of solids in the volume and on surface O_2 where the surface forces are prescribed. The constrained boundary-value problem obtained leads to Beltrami's dependences (which add the static equations to the sufficient conditions) and the boundary conditions on part O_1 of the surface on which the displacement vector is prescribed.

The mixed stationary principle introduces a functional over the displacement vector $\mathbf{u}$ and the stress tensor $\hat{T}$ which are considered as being mutually independent. This functional is written down in the form

$$J = \iiint_V \left[\hat{T} \cdot\cdot\, \hat{\varepsilon} - A(\sigma)\right] d\tau - \iiint_V \rho \mathbf{K} \cdot \mathbf{u} d\tau - \iint_{O_1} \mathbf{n} \cdot \hat{T} \cdot (\mathbf{u} - \mathbf{u}_*)\, do - \iint_{O_2} \mathbf{F} \cdot \mathbf{u} do. \tag{2.6.1}$$

Here $\hat{\varepsilon}$ is the tensor determined in terms of vector $\mathbf{u}$ by formulae (1.1.2), $\mathbf{F}$ is the surface force given on O_2 and $\mathbf{u}_*$ is the displacement vector given on O_1. By $A(\sigma)$ we denote the specific strain energy given by quadratic form (3.2.8) of Chapter 3. The derivatives of this form with respect to the components of the stress tensor are linear functions of these components determined by the left hand side of the relationships in eq. (3.1.8) of Chapter 3. They are the components of a tensor denoted as follows

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma \hat{E}\right), \tag{2.6.2}$$

so that, due to eq. (3.2.9) of Chapter 3

$$\delta A(\sigma) = \delta \hat{T} \cdot\cdot\, \hat{\varepsilon}^*. \tag{2.6.3}$$

Now we have

$$\delta J = \iiint\limits_V \left[\delta\hat{T}\cdot\cdot(\hat{\varepsilon}-\hat{\varepsilon}^*) + \hat{T}\cdot\cdot\delta\hat{\varepsilon}\right] d\tau - \iiint\limits_V \rho\mathbf{K}\cdot\delta\mathbf{u}d\tau -$$
$$\iint\limits_{O_1} \mathbf{n}\cdot\delta\hat{T}\cdot(\mathbf{u}-\mathbf{u}_*)\,do - \iint\limits_{O_1} \mathbf{n}\cdot\hat{T}\cdot\delta\mathbf{u}do - \iint\limits_{O_2} \mathbf{F}\cdot\delta\mathbf{u}do, \qquad (2.6.4)$$

since $\delta\mathbf{F} = 0$ on O_2. It remains only to apply a well-known transformation

$$\iiint\limits_V \hat{T}\cdot\cdot\hat{\varepsilon}d\tau = \iiint\limits_V \operatorname{div}\left(\hat{T}\cdot\delta\mathbf{u}\right) d\tau - \iiint\limits_V \delta\mathbf{u}\cdot\operatorname{div}\hat{T}d\tau$$
$$= \iint\limits_{O_1} \mathbf{n}\cdot\hat{T}\cdot\delta\mathbf{u}do + \iint\limits_{O_2} \mathbf{n}\cdot\hat{T}\cdot\delta\mathbf{u}do - \iiint\limits_V \delta\mathbf{u}\cdot\operatorname{div}\hat{T}d\tau,$$

in order to write down the condition of stationarity of functional J in the form

$$\delta J = \iiint\limits_V \left[\delta\hat{T}\cdot\cdot(\hat{\varepsilon}-\hat{\varepsilon}^*) - \left(\operatorname{div}\hat{T}+\rho\mathbf{K}\right)\cdot\delta\mathbf{u}\right] d\tau -$$
$$\iint\limits_{O_1} \mathbf{n}\cdot\delta\hat{T}\cdot(\mathbf{u}-\mathbf{u}_*)\,do + \iint\limits_{O_2} \left(\mathbf{n}\cdot\hat{T}-\mathbf{F}\right)\cdot\delta\mathbf{u}do = 0. \qquad (2.6.5)$$

Due to the arbitrariness of $\delta\hat{T}$ and $\delta\mathbf{u}$ in the volume, as well as $\mathbf{n}\cdot\delta\hat{T}$ on O_1 and $\delta\mathbf{u}$ on O_2, we arrive at the equations of statics in the volume

$$\operatorname{div}\hat{T} + \rho\mathbf{K} = 0, \qquad (2.6.6)$$

the generalised Hooke law

$$\hat{\varepsilon} = \hat{\varepsilon}^* = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) \qquad (2.6.7)$$

and the boundary conditions

$$\mathbf{u}\Big|_{O_1} = \mathbf{u}_*, \quad \mathbf{n}\cdot\hat{T}\Big|_{O_2} = \mathbf{F}. \qquad (2.6.8)$$

Here the Euler equations of the variational problem on the stationarity of functional J proved to be the fundamental relationships of the linear theory of elasticity listed in Subsection 4.1.1, whilst the natural boundary conditions turned out to be the kinematic and static boundary conditions.

4.2.7 Variational principles accounting for the thermal terms

The thermal conductivity equation is considered in its classical form (the Fourier form), eq. (3.6.8) of Chapter 3. The static statement of the problem of the theory of elasticity is taken, i.e. a time-variant change in the state of stress due to the non-stationarity of the temperature field is neglected. This allows the temperature to be considered as a non-varying external factor under variation of the state of stress and formally treats the temperature field as the field of the volume forces having potential (1.14.5) and surface forces (1.14.6), see Subsection 4.1.14. It is also necessary to take into account the reaction forces on O_1 caused by constraints ensuring the prescribed displacement on this part of the surface.

According to eqs. (2.1.3), (1.14.5) and (1.1.4.6) an analogue of functional Φ in the principle of minimum potential energy is the following functional

$$\Phi_* = \iiint\limits_V A(\varepsilon)\, d\tau + 2\mu \frac{1+\nu}{1-2\nu} \left(\iiint\limits_V \mathbf{u} \cdot \operatorname{grad} \alpha\theta d\tau - \iint\limits_{O_2} \alpha\theta \mathbf{n} \cdot \mathbf{u} do \right), \tag{2.7.1}$$

where $A(\varepsilon)$ is the quadratic form of the components of the strain tensor $\hat{\varepsilon}$ given by eq. (3.2.3) of Chapter 3. Applying the following easily proved transformation

$$\iiint\limits_V \mathbf{u} \cdot \operatorname{grad} \alpha\theta d\tau = \iint\limits_O \mathbf{n} \cdot \mathbf{u} \alpha\theta d\tau - \iiint\limits_V \vartheta \alpha\theta d\tau, \tag{2.7.2}$$

eq. (2.7.1) is transformed to the form

$$\Phi_* = \iiint\limits_V F d\tau + 2\mu \frac{1+\nu}{1-2\nu} \iint\limits_{O_1} \mathbf{n} \cdot \mathbf{u} \alpha\theta do. \tag{2.7.3}$$

Here, in view of eq. (3.4.6) of Chapter 3 (see also Table 3.2 in Chapter 3) the quantity

$$F = A(\varepsilon) - 2\mu \frac{1+\nu}{1-2\nu} \vartheta \alpha\theta$$

represents a free energy of the system up to a term depending only on the temperature.

We repeat the transformations of functional (2.7.3) resulting in formula (2.2.8) and arrive at the relationship

$$\begin{aligned} \delta\Phi_* &= \iiint\limits_V \hat{T} \cdot\cdot\, \delta\hat{\varepsilon} d\tau + 2\mu \frac{1+\nu}{1-2\nu} \iint\limits_{O_1} \mathbf{n} \cdot \delta\mathbf{u} \alpha\theta do \\ &= \iint\limits_O \mathbf{n} \cdot \hat{T} \cdot \delta\mathbf{u} do - \iiint\limits_V \delta\mathbf{u} \cdot \operatorname{div} \hat{T} d\tau + 2\mu \frac{1+\nu}{1-2\nu} \iint\limits_{O_1} \mathbf{n} \cdot \delta\mathbf{u} \alpha\theta do. \end{aligned} \tag{2.7.4}$$

Here $\hat{T}$ denotes the tensor defined by eqs. (3.4.7) and (3.4.8) of Chapter 3

$$\hat{T} = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\hat{E} + \hat{\varepsilon}\right) - 2\mu\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E},$$

so that

$$\operatorname{div}\hat{T} = \mathbf{L}(\mathbf{u}) - 2\mu\frac{1+\nu}{1-2\nu}\operatorname{grad}\alpha\theta, \quad \mathbf{n}\cdot\hat{T} = \mathbf{M}(\mathbf{u}) - 2\mu\frac{1+\nu}{1-2\nu}\mathbf{n}\alpha\theta,$$

where differential operators $\mathbf{L}(\mathbf{u})$ and $\mathbf{M}(\mathbf{u})$ are introduced by eqs. (2.2.6) and (2.2.7) respectively.

Insertion into eq. (2.7.4) yields

$$\delta\Phi_* = -\iiint\limits_V \left[\mathbf{L}(\mathbf{u}) - 2\mu\frac{1+\nu}{1-2\nu}\operatorname{grad}\alpha\theta\right]\cdot\delta\mathbf{u}d\tau +$$

$$\iint\limits_{O_2}\left[\mathbf{M}(\mathbf{u}) - 2\mu\frac{1+\nu}{1-2\nu}\mathbf{n}\alpha\theta\right]\cdot\delta\mathbf{u}do + \iint\limits_{O_1}\mathbf{M}(\mathbf{u})\cdot\delta\mathbf{u}do = 0, \quad (2.7.5)$$

the latter term must vanish as $\delta\mathbf{u} = 0$ on O_1.

Thus we have arrived at the differential equation of equilibrium in terms of displacements (1.14.3) and the boundary condition (1.14.4). Repeating the reasoning of Subsection 4.2.2 one can convince oneself that functional Φ_* has a minimum at the equilibrium.

An analogue of functional Ψ in the principle of minimum complementary work is the following functional

$$\Psi_* = \iiint\limits_V G d\tau - \iint\limits_{O_1}\mathbf{F}\cdot\mathbf{u}do = \iiint\limits_V G d\tau - \iint\limits_{O_1}\mathbf{n}\cdot\hat{T}\cdot\mathbf{u}do, \quad (2.7.6)$$

where G in the Gibbs potential, eq. (3.5.4) of Chapter 3, whereas $\mathbf{F}$ denotes the vector of the surface reaction forces on O_1. Then we have

$$\delta G = \hat{\varepsilon}\cdot\cdot\,\delta\hat{T}, \quad (2.7.7)$$

where $\hat{\varepsilon}$ denotes the tensor whose components, given by eq. (3.5.5) of Chapter 3, are linear forms of the components of the stress tensor and the temperature. An expression for tensor $\hat{\varepsilon}$ is also given by formulae (3.4.10) of Chapter 3.

The proof of the stationarity and minimality of the functional in the equilibrium position does not differ from that in Subsection 4.2.5 provided that the statically admissible states of stress are considered.

4.2.8 Saint-Venant's principle. Energetic consideration

"The principle of the elastic equivalence of statically equivalent systems of forces" was first formulated for the problem of the state of stress of a prismatic rod loaded at its ends in the classical memoir "On torsion of prisms" by Saint-Venant in 1855. A more general formulation of this principle referred to as Saint-Venant's principle was given by Boussinesq in 1885. Works by Mises (1945) and Sternberg (1954) are concerned with refining Boussinesq's consideration.

Systems of forces $\mathbf{F}$ and $\mathbf{F}_1$ are termed statically equivalent when their resultant vectors and resultant moments about the same reduction point are equivalent. Clearly, the system of forces $\mathbf{F} - \mathbf{F}_1$ is statically equivalent to zero, that is, the principal vector and the principal moment are equal to zero. Saint-Venant's principle states that a system of forces, which is statically equivalent to zero and distributed over a small part of the surface of a solid, creates only a local state of stress and becomes negligibly small at distances sufficiently great in comparison with the size of the solid. For example, the state of stress in a long prismatic bar loaded only at its end cross-sections (end faces) is practically independent of the distribution of the surface forces. At a certain distance from the end faces the state of stress is determined only by the principal vector and the principal moment.

Thus one can speak about the possibility of replacing the boundary conditions by other ones under the above-specified conditions of the static equivalence and "smallness" of the loading area. Deliberately or not, an idealisation of the boundary conditions is always used for solving (correctly stated) problems of mathematical physics. In the problems of elasticity theory it is all the more unavoidable since the details of the distribution of the surface forces are most often unknown and the possibility of replacement by another distribution with the same integral properties seems to be intuitively acceptable. It is, however, clear that the above formulation of Saint-Venant's principle is of a qualitative character and needs to be completed by some quantitative estimates. One of such attempts undertaken by Zanaboni (1937) and Locatelli (1940, 1941) consisted in a comparison of the strain energy in the parts of the solid loaded by a statically equivalent system of forces, which are first taken close to the loading area and then far from it.

Let us consider a body A_1 loaded on a part of the surface by a system of forces $\mathbf{P}$ which is statically equivalent to zero and denote the strain energy of the body as $a_1(\mathbf{P})$. Let us add a body A_2 which is free of load to body A_1 on a free surface S', see Fig. 4.1 and denote the strain energy of the new body $A_1 + A_2$ by $a_{1+2}(\mathbf{P})$. Let us prove that

$$a_1(\mathbf{P}) > a_{1+2}(\mathbf{P}). \tag{2.8.1}$$

Indeed, let $\mathbf{R}_{12}$ denote a system of forces which is statically equivalent to zero and acts in cross-section S^* of body $A_1 + A_2$. Surface S^* denotes the free part of surface S of body A_1 which needs to be deformed in order to unite A_1 and A_2 into a single body $A_1 + A_2$. The strain energy of part A_2 of body $A_1 + A_2$ is $a_2(\mathbf{R}_{12})$, while part A_1 of this body possesses the

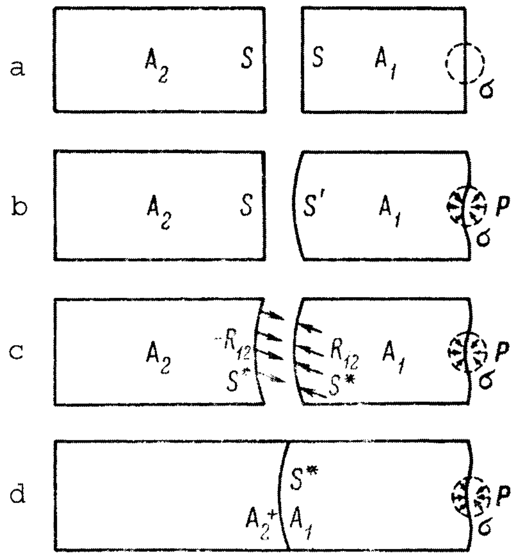

FIGURE 4.1.

following strain energy

$$a_1(\mathbf{P}) + a_1(\mathbf{R}_{12}) + a_1'(\mathbf{R}_{12}),$$

where $a_1'(\mathbf{R}_{12})$ denotes the work of the earlier applied forces $\mathbf{P}$. This work is done since the body part A_1 subjected to these forces is deformed due to forces $\mathbf{R}_{12}$ acting on S^*. Hence

$$a_{1+2}(\mathbf{P}) = a_1(\mathbf{P}) + a_2(\mathbf{R}_{12}) + a_1(\mathbf{R}_{12}) + a_1'(\mathbf{R}_{12}). \qquad (2.8.2)$$

The steps of the reasoning are illustrated in Fig. 4.1a-d.

The true state of equilibrium in which forces $\mathbf{P}$ produce a state of stress resulting in the system of forces $\mathbf{R}_{12}$ in cross-section S^* is compared with the state in which this system (at the same $\mathbf{P}$) is replaced by a proportionally changed system of forces $(1+\varepsilon)\mathbf{R}_{12}$. Notice that we deal with a linear system of static equations describing the behaviour of body A_2 loaded by a system $\mathbf{R}_{12}$ on S^* which is statically equivalent to zero. Thus the system of forces $(1+\varepsilon)\mathbf{R}_{12}$ is due to a statically admissible system of stresses and this allows the principle of minimum complementary work to be applied.

Under the above proportional change in forces $\mathbf{R}_{12}$, the strain energies $a_1(\mathbf{R}_{12})$ and $a_2(\mathbf{R}_{12})$ become equal to $(1+\varepsilon)^2 a_1(\mathbf{R}_{12})$ and $(1+\varepsilon)^2 a_2(\mathbf{R}_{12})$ respectively. Next, $a_1'(\mathbf{R}_{12})$ should be replaced by $(1+\varepsilon)a_1'(\mathbf{R}_{12})$ because $1+\varepsilon$ implies that only strains change and forces $\mathbf{P}$ remain unchanged. The expression for the varied strain energy of body $A_1 + A_2$ takes the form

$$a_{1+2}'(\mathbf{P}) = a_1(\mathbf{P}) + (1+\varepsilon)^2 a_1(\mathbf{R}_{12}) + (1+\varepsilon)^2 a_2(\mathbf{R}_{12}) + (1+\varepsilon)a_1'(\mathbf{R}_{12})$$

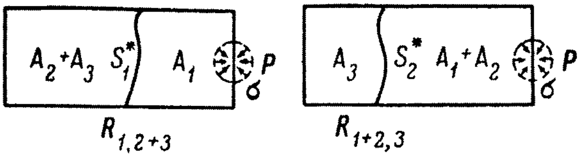

FIGURE 4.2.

and hence

$$\Delta a_{1+2}(\mathbf{P}) = a'_{1+2}(\mathbf{P}) - a_{1+2}(\mathbf{P}) \tag{2.8.3}$$
$$= \varepsilon\left[2a_1(\mathbf{R}_{12}) + 2a_2(\mathbf{R}_{12}) + a'_1(\mathbf{R}_{12})\right] + \varepsilon^2\left[a_1(\mathbf{R}_{12}) + a_2(\mathbf{R}_{12})\right]$$

Here $a_1(\mathbf{R}_{12}) > 0, a_2(\mathbf{R}_{12}) > 0$ and by the theorem on minimum complementary work, the difference (2.8.3) must remain positive regardless of the sign of ε. For this reason

$$2a_1(\mathbf{R}_{12}) + 2a_2(\mathbf{R}_{12}) + a'_1(\mathbf{R}_{12}) = 0 \tag{2.8.4}$$

and expression (2.8.2) is set in the form

$$a_{1+2}(\mathbf{P}) = a_1(\mathbf{P}) - a_1(\mathbf{R}_{12}) - a_2(\mathbf{R}_{12}) < a_1(\mathbf{P}). \tag{2.8.5}$$

An equality sign is impossible here as forces $\mathbf{R}_{12}$ appear if $\mathbf{P}$ is present. Statement (2.8.1) is thus proved. Continuing the imaginary process of adding part A_3 etc. to body $A_1 + A_2$ we have

$$a_1(\mathbf{P}) > a_{1+2}(\mathbf{P}) > a_{1+2+3}(\mathbf{P}) > \dots \tag{2.8.6}$$

The second step of the consideration implies comparing the strain energies due to systems of forces $\mathbf{R}_{1,2+3}$ and $\mathbf{R}_{1+2,3}$. As follows from this notion and Fig. 4.2, the first system of forces appears in cross-section S_1^* of body A_1 attached to unloaded body $A_2 + A_3$, whilst the second system appears in cross-section S_2^* between bodies $A_1 + A_2$ and A_3 and is more remote from the loading place $\mathbf{P}$ than the first.

For the second way of generating the body, we have by eq. (2.8.5)

$$a_{1+2+3}(\mathbf{P}) = a_{1+2}(\mathbf{P}) - a_{1+2}(\mathbf{R}_{1+2,3}) - a_3(\mathbf{R}_{1+2,3}) \tag{2.8.7}$$
$$= a_1(\mathbf{P}) - a_1(\mathbf{R}_{12}) - a_2(\mathbf{R}_{12}) - a_{1+2}(\mathbf{R}_{1+2,3}) - a_3(\mathbf{R}_{1+2,3}),$$

whereas for the first way

$$a_{1+2+3}(\mathbf{P}) = a_1(\mathbf{P}) - a_1(\mathbf{R}_{1,2+3}) - a_{2+3}(\mathbf{R}_{1,2+3}). \tag{2.8.8}$$

From these equalities we have

$$a_1(\mathbf{R}_{1,2+3}) + a_{2+3}(\mathbf{R}_{1,2+3}) = a_{1+2}(\mathbf{R}_{1+2,3}) + a_3(\mathbf{R}_{1+2,3}) + a_1(\mathbf{R}_{12}) + a_2(\mathbf{R}_{12})$$

or, as one would expect,

$$a_1(\mathbf{R}_{1,2+3}) + a_{2+3}(\mathbf{R}_{1,2+3}) > a_{1+2}(\mathbf{R}_{1+2,3}) + a_3(\mathbf{R}_{1+2,3}). \qquad (2.8.9)$$

Functional $a(\mathbf{R})$ is equal to the strain energy of the solid and is calculated in terms of the stresses caused by forces $\mathbf{R}$ appearing in cross-section S due to the body being loaded by a system of forces $\mathbf{P}$ which is statically equivalent to zero. The proved inequality (2.8.9) suggests that $a(\mathbf{R})$ decreases with the distance from the loading place. Since $a(\mathbf{R})$ is a positive definite functional, it can be taken as an integral measure of the stresses and the estimates obtained indicate that these stresses decrease with a growth in the distance from the applied loading and serve as a confirmation of Saint-Venant's principle.

Other estimates for stresses will be discussed later, in Subsections 5.2.12-5.2.14.

4.3 Reciprocity theorem. The potentials of elasticity theory

4.3.1 Formulation and proof of the reciprocity theorem (Betti, 1872)

Let us consider two equilibrium states of a linear solid, these states being called the first and the second in what follows. The displacement vectors $\mathbf{u}'$ and $\mathbf{u}''$ corresponding to these states determine the strain tensors

$$\hat{\varepsilon}' = \operatorname{def}\mathbf{u}', \quad \hat{\varepsilon}'' = \operatorname{def}\mathbf{u}'', \qquad (3.1.1)$$

which in turn determine the stress tensors

$$\hat{T}' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta'\hat{E} + \hat{\varepsilon}'\right), \quad \hat{T}'' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta''\hat{E} + \hat{\varepsilon}''\right). \qquad (3.1.2)$$

We can now find the mass and surface forces which can be applied to the solid in order to realise the above states

$$\rho\mathbf{K}' = -\operatorname{div}\hat{T}', \quad \mathbf{F}' = \mathbf{n}\cdot\hat{T}'; \quad \rho\mathbf{K}'' = -\operatorname{div}\hat{T}'', \quad \mathbf{F}'' = \mathbf{n}\cdot\hat{T}''. \qquad (3.1.3)$$

The objective is to prove that the work done by the external forces of the first state in the displacements of the second state is equal to the work of the external forces of the second state in the displacements of the first state:

$$\iiint\limits_V \rho\mathbf{K}'\cdot\mathbf{u}''d\tau + \iint\limits_O \mathbf{F}'\cdot\mathbf{u}''do = \iiint\limits_V \rho\mathbf{K}''\cdot\mathbf{u}'d\tau + \iint\limits_O \mathbf{F}''\cdot\mathbf{u}'do. \qquad (3.1.4)$$

Replacing the forces on the left hand side of this equality by expressions (3.1.3) in terms of the stress tensors and using transformations (B.3.10) and (B.5.5) we have

$$-\iiint\limits_V \mathbf{u}'' \cdot \operatorname{div} \hat{T}' d\tau + \iint\limits_O \mathbf{n} \cdot \hat{T}' \cdot \mathbf{u}'' do =$$
$$= \iiint\limits_V \left[-\mathbf{u}'' \cdot \operatorname{div} \hat{T}' + \operatorname{div} \left(\mathbf{u}'' \cdot \hat{T}' \right) \right] d\tau = \iiint\limits_V \hat{T}' \cdot \cdot \hat{\varepsilon}'' d\tau,$$

so that

$$\iiint\limits_V \rho \mathbf{K}' \cdot \mathbf{u}'' d\tau + \iint\limits_O \mathbf{F}' \cdot \mathbf{u}'' do = \iiint\limits_V \hat{T}' \cdot \cdot \hat{\varepsilon}'' d\tau \tag{3.1.5}$$

and similarly

$$\iiint\limits_V \rho \mathbf{K}'' \cdot \mathbf{u}' d\tau + \iint\limits_O \mathbf{F}'' \cdot \mathbf{u}' do = \iiint\limits_V \hat{T}'' \cdot \cdot \hat{\varepsilon}' d\tau. \tag{3.1.6}$$

It remains to prove that the right hand sides of these equations are equal. This follows from eq. (3.1.2)

$$\hat{T}' \cdot \cdot \hat{\varepsilon}'' = 2\mu \left(\frac{\nu}{1-2\nu} \vartheta' \hat{E} + \hat{\varepsilon}' \right) \cdot \cdot \hat{\varepsilon}'' = 2\mu \left(\frac{\nu}{1-2\nu} \vartheta' \vartheta'' + \hat{\varepsilon}' \cdot \cdot \hat{\varepsilon}'' \right), \tag{3.1.7}$$

as $\hat{E} \cdot \cdot \hat{\varepsilon}'' = I_1 \left(\hat{\varepsilon}'' \right) = \vartheta''$ and additionally

$$\hat{\varepsilon}' \cdot \cdot \hat{\varepsilon}'' = I_1 \left(\hat{\varepsilon}' \cdot \hat{\varepsilon}'' \right) = \varepsilon'_{sk} \varepsilon''_{ks}.$$

Thus, the quantities of the first and second states appear equally on the right hand side of eq. (3.1.6) which proves the theorem.

The formulation of the reciprocity theorem becomes more complicated for multiple-connected volumes provided that multiple-valued displacements are possible, see Subsection 5.3 of the present chapter.

4.3.2 The influence tensor. Maxwell's theorem

Let a solid be loaded by a unit concentrated force $\mathbf{e}$ at point Q, this force being balanced by the constraint forces. The constraints are assumed to be ideal, i.e. the work done by the constraint forces for any displacement of solid's points is equal to zero.

The displacement vector of point M of the solid denoted as $\mathbf{u}(M, Q)$ is represented in the form

$$\mathbf{u}(M, Q) = \hat{G}(M, Q) \cdot \mathbf{e}. \tag{3.2.1}$$

Here $\hat{G}(M,Q)$ is a second rank tensor referred to as the influence tensor. Component $G_{sk}(M,Q)$ is the projection of the displacement of point M on direction $\mathbf{i}_s$ due to unit force applied along $\mathbf{i}_k$.

Let us agree to refer to points Q and M as the points of source and observation, respectively. We consider now two states of the solid: the first implies that Q and M are respectively the points of source and observation, i.e.

$$\mathbf{u}(M,Q) = \hat{G}(M,Q) \cdot \mathbf{e}_Q,$$

whereas the second state means that M and Q are respectively the points of source and observation, that is

$$\mathbf{u}(Q,M) = \hat{G}(Q,M) \cdot \mathbf{e}_M.$$

In accordance with the reciprocity theorem, the work of force $\mathbf{e}_M$ in the displacement of point M due to force $\mathbf{e}_Q$ is equal to the work of force $\mathbf{e}_Q$ in the displacement of point Q due to force $\mathbf{e}_M$

$$\mathbf{e}_M \cdot \mathbf{u}(M,Q) = \mathbf{e}_Q \cdot \mathbf{u}(Q,M) \tag{3.2.2}$$

or alternatively

$$\mathbf{e}_M \cdot \hat{G}(M,Q) \cdot \mathbf{e}_Q = \mathbf{e}_Q \cdot \hat{G}(Q,M) \cdot \mathbf{e}_M\,. \tag{3.2.3}$$

This expresses the property of the influence tensor

$$\hat{G}(M,Q) = \hat{G}^*(Q,M) \tag{3.2.4}$$

referred to as Maxwell's theorem. As always, an asterisk denotes the operation of transposing a tensor, so that

$$G_{sk}(M,Q) = G_{ks}(Q,M)\ , \tag{3.2.5}$$

which is required.

Knowledge of the influence tensor $\hat{G}(M,Q)$ allows one to represent the displacement vector in terms of any given mass and surface forces

$$\mathbf{u}(M) = \iiint\limits_V \hat{G}(M,Q) \cdot \rho\mathbf{K}(Q)\, d\tau_Q + \iint\limits_O \hat{G}(M,Q_O) \cdot \mathbf{F}(Q_O)\, do_{Q_O}\,. \tag{3.2.6}$$

Clearly, an efficient construction of the influence tensor is of the same order of difficulty as solving the boundary value problem. The solution is easy to find for an unbounded space for which the boundary conditions are not needed, see Subsection 3.5 of the present chapter.

4.3.3 Application of the reciprocity theorem

As the first state one usually takes a particular simple state of stress. Given external forces of the second state (these must be a system of forces which are statically equivalent to zero), the reciprocity theorem enables certain averaged quantities corresponding to this state to be determined.

Let us prescribe the displacement vector of the first state by an affine transformation

$$\mathbf{u}' = \hat{\Lambda} \cdot \mathbf{R}, \tag{3.3.1}$$

where $\hat{\Lambda}$ is a constant tensor of the second rank. By eqs. (1.2.3)-(1.2.5) of Chapter 2

$$d\mathbf{u}' = (\nabla \mathbf{u}')^* \cdot d\mathbf{R} = d\mathbf{R} \cdot \nabla \mathbf{u}' = \hat{\Lambda} \cdot d\mathbf{R} = d\mathbf{R} \cdot \hat{\Lambda}^*,$$

so that, referring to eq. (1.2.13) of Chapter 2, we have

$$\hat{\varepsilon}' = \frac{1}{2}\left(\hat{\Lambda} + \hat{\Lambda}^*\right), \quad \hat{\Lambda} = \hat{\varepsilon}' + \hat{\Omega}' \tag{3.3.2}$$

and then

$$\hat{T}' = 2\mu\left(\frac{\nu}{1-2\nu}\hat{E}\vartheta' + \hat{\varepsilon}'\right), \quad \rho\mathbf{K}' = 0, \quad \mathbf{F}' = 2\mu\left(\frac{\nu}{1-2\nu}\mathbf{n}\vartheta' + \hat{\varepsilon}' \cdot \mathbf{n}\right). \tag{3.3.3}$$

By the reciprocity theorem (omitting the primes corresponding to the quantities of the second state) we obtain

$$\iiint\limits_V \rho\mathbf{K} \cdot \hat{\Lambda} \cdot \mathbf{R} d\tau + \iint\limits_O \mathbf{F} \cdot \hat{\Lambda} \cdot \mathbf{R} do = \\ = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta' \iint\limits_O \mathbf{n} \cdot \mathbf{u} do + \iint\limits_O \mathbf{n} \cdot \hat{\varepsilon}' \cdot \mathbf{u}\vartheta do\right). \tag{3.3.4}$$

Tensor $\hat{\Lambda}$ on the left hand side can be replaced by $\hat{\varepsilon}'$ since

$$\iiint\limits_V \rho\mathbf{K} \cdot \hat{\Omega} \cdot \mathbf{R} d\tau + \iint\limits_O \mathbf{F} \cdot \hat{\Omega} \cdot \mathbf{R} do = \\ = \iiint\limits_V \rho\mathbf{K} \cdot (\boldsymbol{\omega} \times \mathbf{R})\, d\tau + \iint\limits_O \mathbf{F} \cdot (\boldsymbol{\omega} \times \mathbf{R})\, do \\ = \boldsymbol{\omega} \cdot \left(\iiint\limits_V \mathbf{R} \times \rho\mathbf{K} d\tau + \iint\limits_O \mathbf{R} \times \mathbf{F} do\right) = 0,$$

where $\boldsymbol{\omega}$ denotes the vector accompanying tensor $\hat{\Lambda}$, see Section A.4, and the expression in parentheses is the principal moment of the external forces of the second state which is equal to zero. In addition to this

$$\iint\limits_O \mathbf{n} \cdot \mathbf{u} do = \iiint\limits_V \operatorname{div} \mathbf{u} d\tau = \iiint\limits_V \vartheta d\tau,$$

$$\iint\limits_O \mathbf{n} \cdot \hat{\varepsilon}' \cdot \mathbf{u} do = \iiint\limits_V \operatorname{div} \left(\hat{\varepsilon}' \cdot \mathbf{u}\right) d\tau = \iiint\limits_V \mathbf{u} \cdot \operatorname{div} \hat{\varepsilon}' d\tau + \hat{\varepsilon}' \cdot\cdot \iiint\limits_V \hat{\varepsilon} d\tau.$$

Denoting the volume-averaged value of a quantity by subscript m we have

$$\frac{1}{V} \iiint\limits_V \vartheta d\tau = \vartheta_m, \quad \hat{\varepsilon}' \cdot\cdot \frac{1}{V} \iiint\limits_V \hat{\varepsilon} d\tau = \hat{\varepsilon}' \cdot\cdot \hat{\varepsilon}_m = I_1 \left(\hat{\varepsilon}' \cdot \hat{\varepsilon}_m\right).$$

We arrive at the equality

$$\frac{\nu}{1-2\nu} \vartheta' \vartheta_m + I_1 \left(\hat{\varepsilon}' \cdot \hat{\varepsilon}_m\right) = \frac{1}{2\mu V} \left(\iiint\limits_V \rho \mathbf{K} \cdot \hat{\varepsilon}' \cdot \mathbf{R} d\tau + \iint\limits_O \mathbf{F} \cdot \hat{\varepsilon}' \cdot \mathbf{R} do \right) \tag{3.3.5}$$

which is transformed by means of the identity

$$\hat{Q} \cdot\cdot \mathbf{ab} = I_1 \left(\hat{Q} \cdot \mathbf{ab}\right) = I_1 \left(\mathbf{a} \cdot \hat{Q}^* \mathbf{b}\right) = \mathbf{a} \cdot \hat{Q}^* \cdot \mathbf{b}$$

to the following form

$$\frac{\nu}{1-2\nu} \vartheta' \vartheta_m + \hat{\varepsilon}' \cdot\cdot \hat{\varepsilon}_m = \frac{1}{2\mu V} \hat{\varepsilon}' \cdot\cdot \left(\iiint\limits_V \rho \mathbf{KR} d\tau + \iint\limits_O \mathbf{FR} do \right). \tag{3.3.6}$$

Assuming that $\hat{\varepsilon}' = \hat{E}$ we have $\vartheta' = 3$, $\hat{\varepsilon}' \cdot\cdot \hat{\varepsilon}_m = \vartheta_m$, $\hat{\varepsilon}' \cdot\cdot \rho \mathbf{KR} = \rho \mathbf{K} \cdot \mathbf{R}$ and so on. We are then lead to the following expression for the averaged dilatation

$$\vartheta_m = \frac{1-2\nu}{2\mu V (1+\nu)} \left(\iiint\limits_V \rho \mathbf{K} \cdot \mathbf{R} d\tau + \iint\limits_O \mathbf{F} \cdot \mathbf{R} do \right). \tag{3.3.7}$$

Inserting this into eq. (3.3.6) results into the relationship

$$\hat{\varepsilon}' \cdot\cdot \left[\hat{\varepsilon}_m - \frac{1}{2\mu V} \left(\iiint\limits_V \rho \mathbf{KR} d\tau + \iint\limits_O \mathbf{FR} do \right) \right] + \frac{\nu \vartheta'}{1+\nu} \frac{1}{2\mu V} \left(\iiint\limits_V \rho \mathbf{K} \cdot \mathbf{R} d\tau + \iint\limits_O \mathbf{F} \cdot \mathbf{R} do \right) = 0. \tag{3.3.8}$$

For example, for $\hat{\varepsilon}' = \mathbf{i}_1\mathbf{i}_1, \vartheta' = 1$ we obtain the following expression for the average extension

$$(\varepsilon_{11})_m = \frac{1}{2\mu V}\left[\iiint\limits_V \rho\left(xK_x - \frac{\nu}{1+\nu}\mathbf{R}\cdot\mathbf{K}\right)d\tau + \iint\limits_O \left(xF_x - \frac{\nu}{1+\nu}\mathbf{R}\cdot\mathbf{F}\right)do\right], \qquad (3.3.9)$$

whereas assuming $\hat{\varepsilon}' = \mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1, \vartheta' = 0$ we obtain the average value of the shear

$$(\gamma_{12})_m = \frac{1}{2\mu V}\left(\iiint\limits_V \rho\left(xK_y + yK_x\right)d\tau + \iint\limits_O \left(xF_y + yF_x\right)do\right). \qquad (3.3.10)$$

Now it is easy to find averaged values of the stresses

$$(\sigma_x)_m = \frac{1}{V}\left(\iiint\limits_V \rho xK_x d\tau + \iint\limits_O xF_x do\right),$$

$$(\tau_{xy})_m = \frac{1}{V}\left(\iiint\limits_V \rho yK_x d\tau + \iint\limits_O yF_x do\right)$$

etc. These expressions were obtained in Subsection 1.4.3 by using solely the equations of statics and are valid for any continuum rather than for a Hookean body. Using the derived equations and Hooke's law one can obtain the above formulae for averaged values of the components of the strain tensor in an elastic solid.

Taking the displacement vector $\mathbf{u}'$ in the form of the quadratic form of the coordinates and using the reciprocity theorem one can obtain the formulae of Subsection 1.4.4 for the first order moments of the stresses.

Prescribing components of $\hat{\varepsilon}'$ in the form of quadratic forms of the coordinates means 36 coefficients related by six conditions of the strain compatibility, see eq. (2.1.5) of Chapter 2. Using the reciprocity theorem in the form of eq. (3.1.5)

$$\iiint\limits_V t_{sk}\varepsilon'_{sk} d\tau = \iiint\limits_V \mathbf{K}\cdot\mathbf{u}' d\tau + \iint\limits_O \mathbf{F}\cdot\mathbf{u}' do$$

leads to thirty equations corresponding to the number of independent coefficients of these forms. The number of unknown second order moments of stresses

$$\iiint_V t_{ik} x_r x_s d\tau$$

is equal to thirty six. The same result was obtained in Subsection 4.4.1.

4.3.4 The reciprocity theorem taking account of thermal terms

The derivation of formulae (3.1.5) and (3.1.6) is not affected by the presence of thermal terms in the expression for the stress tensor (1.14.1). However the thermal terms should be taken into account while calculating the right hand sides of these formulae

$$\left.\begin{aligned} \hat{T}' \cdot\cdot \hat{\varepsilon}'' &= 2\mu \left(\frac{\nu}{1-2\nu} \vartheta' \vartheta'' + \hat{\varepsilon}' \cdot \hat{\varepsilon}'' - 2\mu\alpha \frac{1+\nu}{1-2\nu} \theta' \vartheta'' \right), \\ \hat{T}'' \cdot\cdot \hat{\varepsilon}' &= 2\mu \left(\frac{\nu}{1-2\nu} \vartheta'' \vartheta' + \hat{\varepsilon}'' \cdot \hat{\varepsilon}' - 2\mu\alpha \frac{1+\nu}{1-2\nu} \theta'' \vartheta' \right). \end{aligned}\right\} \tag{3.4.1}$$

Instead of eq. (3.1.4) we obtain

$$\iiint_V \rho \mathbf{K}' \cdot \mathbf{u}'' d\tau + \iint_O \mathbf{F}' \cdot \mathbf{u}'' do + 2\mu\alpha \frac{1+\nu}{1-2\nu} \iiint_V \theta' \vartheta'' d\tau =$$

$$= \iiint_V \rho \mathbf{K}'' \cdot \mathbf{u}' d\tau + \iint_O \mathbf{F}'' \cdot \mathbf{u}' do + 2\mu\alpha \frac{1+\nu}{1-2\nu} \iiint_V \theta'' \vartheta' d\tau. \tag{3.4.2}$$

It is evident that the same result can be obtained by a formal replacement of the temperature field by the volume and surface forces (1.14.5) and (1.14.6)

$$2\mu\alpha \frac{1+\nu}{1-2\nu} \left(- \iiint_V \mathbf{u}'' \cdot \operatorname{grad} \theta' d\tau + \iint_O \mathbf{n} \cdot \mathbf{u}'' \theta' do \right) =$$

$$= 2\mu\alpha \frac{1+\nu}{1-2\nu} \iiint_V \theta' \operatorname{div} \mathbf{u}'' d\tau,$$

which is required.

By way of an example let us consider the state of stress due to a unit force applied at point Q as the first state for which it is assumed that $\theta' = 0$. Let the forces vanish in the second state, i.e. $\rho\mathbf{K}' = 0, \mathbf{F}'' = 0$ and the temperature be θ. Applying formulae (3.4.2) and taking into account eq. (3.2.1) leads to the equality

$$\mathbf{e}_Q \cdot \mathbf{u}(Q) = 2\mu\alpha \frac{1+\nu}{1-2\nu} \iiint_V \theta(M) \operatorname{div}_M \hat{G}(M, Q) \cdot \mathbf{e}_Q d\tau_M,$$

and, since $\mathbf{e}_Q$ is an arbitrarily directed vector, we have

$$\mathbf{u}(Q) = 2\mu\alpha\frac{1+\nu}{1-2\nu}\iiint\limits_V \theta(M)\,\mathrm{div}_M\,\hat{G}(M,Q)\,d\tau_M. \qquad (3.4.3)$$

Thus, a known influence tensor allows determination of the displacement field in terms of the temperature field.

4.3.5 The influence tensor of an unbounded medium

Let us perform an imaginary cut of a finite volume V_i bounded by surface O from an unbounded elastic medium and let V_e denote the remaining volume with cavity V_i.

A unit concentrated force $\mathbf{e}$ is applied at point Q of the elastic solid and yields a state of stress described by tensor $\hat{T}$. The equations of statics for volume V_i are written down in the form

$$\iint\limits_O \mathbf{n}\cdot\hat{T}do + \mathbf{e} = 0, \quad \iint\limits_O \mathbf{R}\times\left(\mathbf{n}\cdot\hat{T}\right)do = 0, \quad Q \subset V_i, \qquad (3.5.1)$$

$$\iint\limits_O \mathbf{n}\cdot\hat{T}do = 0, \quad \iint\limits_O \mathbf{R}\times\left(\mathbf{n}\cdot\hat{T}\right)do = 0, \quad Q \subset V_e. \qquad (3.5.2)$$

Here $\mathbf{R}$ designates the position vector with the origin at point Q. If $\mathbf{r}_M$ and $\mathbf{r}_Q$ denotes the position vectors of the points of observation M and origin Q, then

$$\mathbf{R} = \mathbf{r}_M - \mathbf{r}_Q. \qquad (3.5.3)$$

A sphere of radius R with the centre at point Q is taken as surface O. This assumption does not limit our consideration since the value of the integrals in eq. (3.5.1) is the same on any surface enclosing sphere O. Equation (3.5.1) takes now the form

$$R^2\iint\limits_{O^*} \mathbf{n}\cdot\hat{T}do^* + \mathbf{e} = 0, \qquad (3.5.4)$$

where O^* and do^* denote a sphere of a unit radius and an element of its surface, respectively. It follows from the latter equation that the principal vector of the stresses on any surface enclosing point Q is equal to $\mathbf{e}$, i.e. it has a value which is independent of R. This is possible only if the components of tensor $\hat{T}$ decrease as R^{-2}. This implies that the magnitude of the displacement vector must decrease as R^{-1}.

This reasoning suggests a particular character of the solution. The harmonic vector $\mathbf{B}$ in the Papkovich-Neuber solution (1.4.10) should be taken in the form

$$\mathbf{B} = \frac{A}{R}\mathbf{e},$$

since R^{-1} is the only harmonic function having such a decreasing character at infinity, vector $\mathbf{e}$ must appear in the solution and A must denote a constant determined from condition (3.5.1). Introducing harmonic vector B_0 is redundant and the displacement vector, by eq. (1.4.10), is represented as follows

$$\mathbf{u} = A\left[(3-4\nu)\frac{\mathbf{e}}{R} + \frac{\mathbf{e}\cdot\mathbf{R}}{R^3}\mathbf{R}\right], \tag{3.5.5}$$

since

$$\nabla\frac{1}{R}\mathbf{e}\cdot\mathbf{R} = \frac{\mathbf{e}}{R} - \frac{1}{R^3}\mathbf{R}\mathbf{e}\cdot\mathbf{R}.$$

The stress tensor is calculated by eq. (1.4.15). The result is

$$\begin{gathered}\operatorname{div}\mathbf{B} = -\frac{A}{R^3}\mathbf{e}\cdot\mathbf{R}, \quad \nabla\mathbf{B} = -A\frac{\mathbf{R}\mathbf{e}}{R^3}, \quad \operatorname{def}\mathbf{B} = -\frac{1}{2}\frac{A}{R^3}\left(\mathbf{R}\mathbf{e}+\mathbf{e}\mathbf{R}\right), \\ \nabla\nabla\mathbf{R}\cdot\mathbf{B} = -A\left[\frac{1}{R^3}\left(\mathbf{R}\mathbf{e}+\mathbf{e}\mathbf{R}\right) + \hat{E}\frac{\mathbf{e}\cdot\mathbf{R}}{R^3} - \frac{3}{R^5}\mathbf{e}\cdot\mathbf{R}\mathbf{R}\mathbf{R}\right],\end{gathered}$$

and substitution yields

$$\hat{T} = \frac{2\mu A}{R^3}\left[(1-2\nu)\left(\hat{E}\mathbf{e}\cdot\mathbf{R} - \mathbf{e}\mathbf{R} - \mathbf{R}\mathbf{e}\right) - \frac{3\mathbf{e}\cdot\mathbf{R}}{R^2}\mathbf{R}\mathbf{R}\right]. \tag{3.5.6}$$

The equality

$$-\mathbf{e} = \frac{2\mu A}{R^3}\iint\limits_O\left[(1-2\nu)\left(\mathbf{n}\mathbf{e}\cdot\mathbf{R} - \mathbf{n}\cdot\mathbf{e}\mathbf{R} - \mathbf{n}\cdot\mathbf{R}\mathbf{e}\right) - \frac{3\mathbf{e}\cdot\mathbf{R}}{R^2}\mathbf{n}\cdot\mathbf{R}\mathbf{R}\right]do$$

serves for determining A. On the surface of sphere O

$$\mathbf{n} = R^{-1}\mathbf{R}, \quad \mathbf{n}\mathbf{e}\cdot\mathbf{R} - \mathbf{n}\cdot\mathbf{e}\mathbf{R} = 0, \quad \mathbf{n}\cdot\mathbf{R} = R, \quad \frac{\mathbf{e}\cdot\mathbf{R}}{R^2}\mathbf{n}\cdot\mathbf{R}\mathbf{R} = \mathbf{e}\cdot\mathbf{R}\mathbf{n},$$

so that

$$\mathbf{e} = \frac{2\mu A}{R^3}\left[(1-2\nu)R\mathbf{e}\iint\limits_O do + 3\iint\limits_O \mathbf{n}\mathbf{e}\cdot\mathbf{R}do\right].$$

Since

$$\iint\limits_O do = 4\pi R^2, \iint\limits_O \mathbf{ne}\cdot\mathbf{R}do = \iiint\limits_V \nabla\mathbf{e}\cdot\mathbf{R}d\tau = \mathbf{e}\iiint\limits_V d\tau = \frac{4}{3}\pi R^3\mathbf{e},$$

we find

$$A = \frac{1}{16\pi\mu(1-\nu)}. \tag{3.5.7}$$

Inserting the displacement vector in the form (3.2.1) we arrive at the equality

$$\mathbf{u}(M,Q) = \hat{U}(M,Q)\cdot\mathbf{e}, \tag{3.5.8}$$

where the influence tensor $\hat{U}(M,Q)$ for an unbounded elastic medium, referred to as the Kelvin-Somigliana tensor, is given, due to eq. (3.5.5), by the formula

$$\hat{U} = \frac{1}{16\pi\mu(1-\nu)R}\left[(3-4\nu)\hat{E} + \frac{\mathbf{RR}}{R^2}\right] = \frac{1}{4\pi\mu}\left(\frac{\hat{E}}{R} - \frac{\nabla\nabla R}{4(1-\nu)}\right). \tag{3.5.9}$$

The stress tensor at point M on an elementary surface with normal $\mathbf{n}_M$ is equal to

$$\mathbf{n}_M\cdot\hat{T} = \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)(\mathbf{n}_M\mathbf{e}\cdot\mathbf{R} - \mathbf{n}_M\cdot\mathbf{eR} - \mathbf{n}_M\cdot\mathbf{Re}) - 3\frac{\mathbf{e}\cdot\mathbf{R}}{R^2}\mathbf{n}_M\cdot\mathbf{RR}\right] \tag{3.5.10}$$

and can be represented in the form

$$\mathbf{n}_M\cdot\hat{T} = \hat{\Phi}(M,Q)\cdot\mathbf{e}, \tag{3.5.11}$$

where the tensor of "force" influence is determined by the following formula

$$\hat{\Phi}(M,Q) = \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)\left(\mathbf{n}_M\mathbf{R} - \mathbf{R}\mathbf{n}_M - \mathbf{n}_M\cdot\mathbf{R}\hat{E}\right) - 3\frac{\mathbf{n}_M\cdot\mathbf{R}}{R^2}\mathbf{RR}\right] = \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)(\mathbf{n}_M\mathbf{R} - \mathbf{R}\mathbf{n}_M) - 2(1-\nu)\hat{E}\mathbf{n}_M\cdot\mathbf{R} - \mathbf{R}^3\mathbf{n}_M\cdot\mathbf{R}\nabla\nabla\frac{1}{R}\right]. \tag{3.5.12}$$

The equations of statics (3.5.1) and (3.5.2) are now set in the form

$$\iint\limits_O \hat{\Phi}(M,Q)\,do_M = \begin{cases} -\hat{E}, & Q\subset V_i, \\ 0, & Q\subset V_e, \end{cases} \tag{3.5.13}$$

and for any point Q we have

$$\iint\limits_{O} \mathbf{R} \times \hat{\Phi}(M,Q)\, do_M = 0. \tag{3.5.14}$$

An extended form of eq. (3.5.13) is as follows

$$\begin{aligned}
&\iint\limits_{O} \hat{\Phi}(M,Q)\, do_M = \\
&= \frac{1}{8\pi(1-\nu)} \iint\limits_{O} \left[(1-2\nu)\frac{1}{R^3}(\mathbf{n}_M\mathbf{R} - \mathbf{R}\mathbf{n}_M) - \mathbf{n}_M \cdot \mathbf{R}\nabla\nabla\frac{1}{R} \right] do_M - \\
&\qquad \frac{1}{4\pi}\hat{E} \iint\limits_{O} \frac{\mathbf{n}_M \cdot \mathbf{R}}{R^3} do_M = \begin{cases} -\hat{E}, & Q \subset V_i, \\ 0, & Q \subset V_e. \end{cases}
\end{aligned} \tag{3.5.15}$$

According to the Gauss theorem on the double layer potential of the unit density distributed over a closed surface

$$\iint\limits_{O} \frac{\cos\varphi}{R^2} do = \iint\limits_{O} \frac{\mathbf{n}_M \cdot \mathbf{R}}{R^3} do_M = \begin{cases} 4\pi, & Q \subset V_i, \\ 2\pi, & Q \subset O, \\ 0, & Q \subset V_e, \end{cases} \tag{3.5.16}$$

and thus

$$\iint\limits_{O} \left[(1-2\nu)\frac{1}{R^3}(\mathbf{n}_M\mathbf{R} - \mathbf{R}\mathbf{n}_M) - \mathbf{n}_M \cdot \mathbf{R}\nabla\nabla\frac{1}{R} \right] do_M = 0 \tag{3.5.17}$$

for any position of point Q, also on O. This allows eq. (3.5.13) to be set in the form

$$\iint\limits_{O} \hat{\Phi}(M,Q)\, do_M = -\hat{E}\delta(Q), \tag{3.5.18}$$

where $\delta(Q)$ is the function of the position of point Q

$$\delta(Q) = \begin{cases} 1, & Q \subset V_i, \\ \frac{1}{2}, & Q \subset O, \\ 0, & Q \subset V_e. \end{cases} \tag{3.5.19}$$

Equality (3.5.18) is repeatedly used in what follows and is referred to as the generalised Gauss theorem.

Let us also notice that relationships (3.5.14) and (3.5.17) can be directly proved with relative ease.

4.3.6 The potentials of the elasticity theory

In order to generalise the classic definition of the theory of Newtonian potential we introduce into consideration two potentials of elasticity theory.

The first potential is analogous to the single layer potential and is determined by the vector

$$\mathbf{A}(Q) = \iint_O \mathbf{a}(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0}, \tag{3.6.1}$$

where $\hat{U}(M_0, Q)$ is the Kelvin-Somigliana tensor (3.5.9). It is assumed that O belongs to a class of Lyapunov surfaces and $\mathbf{a}(M_0)$ is the layer density prescribed on O. Vector $\mathbf{A}(Q)$ is a function which is continuous in the whole space (the normal derivative $n \cdot \nabla \mathbf{A}$ has a discontinuity under passage through the layer) and satisfies the homogeneous equations of the elasticity theory in terms of displacements in V_i and V_e

$$\frac{1}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{A} + \nabla^2 \mathbf{A} = 0. \tag{3.6.2}$$

The limiting values of potential $\mathbf{A}(Q)$ from inside and outside denoted respectively as

$$\mathbf{A}_i(Q_0) = \lim_{V_i \supset Q \to Q_0} \mathbf{A}(Q), \quad \mathbf{A}_e(Q_0) = \lim_{V_e \supset Q \to Q_0} \mathbf{A}(Q) \tag{3.6.3}$$

are equal to the "direct value" determined by the improper convergent integral

$$\mathbf{A}(Q_0) = \iint_O \mathbf{a}(M_0) \cdot \hat{U}(M_0, Q_0)\, do_{M_0}. \tag{3.6.4}$$

The notion $V_i \supset Q \to Q_0$ ($V_e \supset Q \to Q_0$) means that point Q approaches point Q_0 on the layer remaining in V_i (V_e) along the normal to Q. In summary

$$\mathbf{A}_i(Q_0) = \mathbf{A}(Q_0) = \mathbf{A}_e(Q_0). \tag{3.6.5}$$

The second potential of elasticity theory possesses the properties of the double layer potential. It is determined by the vector

$$\mathbf{B}(Q) = \iint_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0}, \tag{3.6.6}$$

where $\hat{\Phi}(M_0, Q)$ is the tensor of force influence (3.5.12). This vector also satisfies the homogeneous equations in terms of displacements in V_i and V_e.

Similar to the earlier introduced vector $\mathbf{a}(M_0)$ the density vector $\mathbf{b}(M_0)$ is assumed to satisfy Hölder's condition

$$|\mathbf{b}(M_0') - \mathbf{b}(M_0'')| < A\left|\mathbf{r}_{M_0'} - \mathbf{r}_{M_0''}\right|^{\gamma} \tag{3.6.7}$$

with a positive constant γ. The integral

$$\mathbf{B}(Q_0) = \iint\limits_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0}, \tag{3.6.8}$$

referred to as the direct value of potential $\mathbf{B}(Q)$, is understood as a principal value of the integral. Let us remind ourselves that the principal value of the integral is defined as the limit of the integral over surface $O - O(Q_0, \varepsilon)$ at $\varepsilon \to 0$, where $O(Q_0, \varepsilon)$ denotes the vicinity of point Q_0 on O of diameter 2ε. The limiting values $\mathbf{B}_i(Q_0)$ and $\mathbf{B}_e(Q_0)$ of potential $\mathbf{B}(Q)$

$$\mathbf{B}_i(Q_0) = \lim_{V_i \supset Q \to Q_0} \mathbf{B}(Q), \quad \mathbf{B}_e(Q_0) = \lim_{V_e \supset Q \to Q_0} \mathbf{B}(Q), \tag{3.6.9}$$

are not equal to each other and do not coincide with its direct value, that is, potential $\mathbf{B}(Q)$ experiences a jump while passing through the layer. The Newtonian double layer potential

$$W(Q) = \iint\limits_O \rho(M_0) \frac{\mathbf{n}_{M_0} \cdot \mathbf{R}}{R^3} do_{M_0}$$

(the density is denoted as $\rho(M_0)$) possesses the same property. However it is proved that the limiting values are equal to the direct value if Q_0 is the point on O where the density vanishes, i.e. for $\rho(Q_0) = 0$

$$W(Q_0) = \iint\limits_O \rho(M_0) \frac{\mathbf{n}_{M_0} \cdot \mathbf{R}}{R^3} do_{M_0} = W_i(Q_0) = W_e(Q_0).$$

The second potential of the elasticity theory has this particular property of the Newtonian potential, that is, if $\mathbf{b}(Q_0) = 0$, then

$$\mathbf{B}(Q_0) = \mathbf{B}_i(Q_0) = \mathbf{B}_e(Q_0). \tag{3.6.10}$$

Referring to the generalised Gauss theorem (3.5.18) we write down the following equality

$$\begin{aligned}
\mathbf{B}(Q) = &\iint\limits_O [\mathbf{b}(M_0) - \mathbf{b}(Q_0)] \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} + \\
&\mathbf{b}(Q_0) \cdot \iint\limits_O \hat{\Phi}(M_0, Q)\, do_{M_0} = \\
= &\iint\limits_O [\mathbf{b}(M_0) - \mathbf{b}(Q_0)] \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} - \begin{cases} \mathbf{b}(Q_0), & Q \subset V_i, \\ 0, & Q \subset V_e. \end{cases}
\end{aligned} \tag{3.6.11}$$

The integral

$$\iint\limits_{O} [\mathbf{b}(M_0) - \mathbf{b}(Q_0)] \cdot \hat{\Phi}(M_0, Q)\, do_{M_0}$$

represents a potential of the type of $\mathbf{B}(Q)$ but with the density

$$\mathbf{b}_*(M_0) = \mathbf{b}(M_0) - \mathbf{b}(Q_0)$$

which vanishes at point Q_0. For this reason, using eqs. (3.6.10), (3.6.11), (3.6.8) and (3.5.18) we obtain

$$\begin{aligned}
\mathbf{B}_i(Q_0) &= \iint\limits_{O} [\mathbf{b}(M_0) - \mathbf{b}(Q_0)] \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} - \mathbf{b}(Q_0) \\
&= \mathbf{B}(Q_0) + \frac{1}{2}\mathbf{b}(Q_0) - \mathbf{b}(Q_0) = \mathbf{B}(Q_0) - \frac{1}{2}\mathbf{b}(Q_0), \\
\mathbf{B}_e(Q_0) &= \iint\limits_{O} [\mathbf{b}(M_0) - \mathbf{b}(Q_0)] \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = \mathbf{B}(Q_0) + \frac{1}{2}\mathbf{b}(Q_0).
\end{aligned}$$

Therefore, we arrived at the Plemelj formulae for the second potential of the elasticity theory

$$\mathbf{B}_i(Q_0) = \mathbf{B}(Q_0) - \frac{1}{2}\mathbf{b}(Q_0), \tag{3.6.12}$$

$$\mathbf{B}_e(Q_0) = \mathbf{B}(Q_0) + \frac{1}{2}\mathbf{b}(Q_0). \tag{3.6.13}$$

4.3.7 Determining the displacement field for given external forces and displacement vector of the surface

Let a unit concentrated force $\mathbf{e}$ be applied at point Q of an unbounded elastic media, then by eq. (3.5.11) the surface forces

$$\left(\mathbf{n} \cdot \hat{T}\right)_0 = \hat{\Phi}(M_0, Q) \cdot \mathbf{e} \tag{3.7.1}$$

appear on surface O bounding volume V_i which is an imaginary cut of the medium. The displacement vector in this volume is equal to

$$\mathbf{u}(M, Q) = \hat{U}(M, Q) \cdot \mathbf{e}. \tag{3.7.2}$$

This particular state of volume V_i is taken as being the first state in the reciprocity theorem. The state of the same body subjected to external volume ($\rho\mathbf{K}$) and surface ($\mathbf{F}$) forces is understood as the second state. The displacement vector in this state is denoted by $\mathbf{u}(M)$.

The work done by the forces of the first state in the displacement of the second state is

$$a' = \begin{cases} \left[\mathbf{u}(Q) + \iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} \right] \cdot \mathbf{e}, & Q \subset V_i, \\ \iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} \cdot \mathbf{e}, & Q \subset V_e. \end{cases} \tag{3.7.3}$$

Referring to Plemelj's formulae (3.6.12) and (3.6.13) we have in both cases

$$\lim_{V_i \supset Q \to Q_0} a' = \lim_{V_e \supset Q \to Q_0} a' = \left[\frac{1}{2}\mathbf{u}(Q_0) + \iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} \right] \cdot \mathbf{e},$$

so that recalling definition (3.5.19) of function $\delta(Q)$ we have

$$a' = \left[\delta(Q)\, \mathbf{u}(Q) + \iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} \right] \cdot \mathbf{e} \tag{3.7.4}$$

provided that, when $Q \subset O$, the integral on the right hand side is understood as a principal value

$$\iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0}. \tag{3.7.5}$$

The work of the forces of the second state done in the displacement of the first state is equal to

$$a'' = \left[\iiint\limits_{V_i} \rho \mathbf{K}(M) \cdot \hat{U}(M, Q)\, d\tau_M + \iint\limits_O \mathbf{F}(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0} \right] \cdot \mathbf{e}, \tag{3.7.6}$$

and, by virtue of eq. (3.6.5) this formula retains its validity for both $Q \subset V_i, Q \subset V_e$ and $Q \subset O$. In the latter case the improper integral converges since the singularity of tensor $\hat{U}(M_0, Q)$ as a function of point M_0 is weak at point Q_0 (as R^{-1}).

Applying the reciprocity theorem leads to the following relationship

$$\delta(Q)\, \mathbf{u}(Q) = \iint\limits_O \mathbf{F}(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0} - \iint\limits_O \mathbf{u}(M_0) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} + \iiint\limits_{V_i} \rho \mathbf{K}(M) \cdot \hat{U}(M, Q)\, d\tau_M, \tag{3.7.7}$$

since the arbitrarily prescribed vector $\mathbf{e}$ can be omitted. The last term in this formula

$$\mathbf{u}_* (Q) = \iiint\limits_{V_i} \rho \mathbf{K} (M) \cdot \hat{U} (M, Q) \, d\tau_M \tag{3.7.8}$$

presents the particular solution of the equilibrium equations in terms of displacements corresponding to the volume forces. This proves that this solution can be determined by a quadrature for any prescribed volume force, see Subsection 4.1.4.

Relationship (3.7.7) determines the displacement vector for given external force $\mathbf{F}$ on surface O and the displacement vector $\mathbf{u}$. For this reason, it is not the solution of the boundary value problem.

Let us prove that when the second case is a natural state, i.e. for $\rho\mathbf{K} = 0$ and $\mathbf{F} = 0$, relationship (3.7.7) is satisfied by a rigid body displacement

$$\mathbf{u} (M) = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_M = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q + \boldsymbol{\omega} \times \mathbf{R}. \tag{3.7.9}$$

Indeed, noticing that

$$(\boldsymbol{\omega} \times \mathbf{R}) \cdot \hat{\Phi} (M_0, Q_0) = \boldsymbol{\omega} \cdot \left[\mathbf{R} \times \hat{\Phi} (M_0, Q)\right]$$

we have

$$-\iint\limits_{O} \mathbf{u} (M_0) \cdot \hat{\Phi} (M_0, Q) \, do_{M_0} =$$

$$= - (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q) \cdot \iint\limits_{O} \hat{\Phi} (M_0, Q) \, do_{M_0} - \boldsymbol{\omega} \cdot \iint\limits_{O} \mathbf{R} \times \hat{\Phi} (M_0, Q) \, do_{M_0}$$

and referring to eqs. (3.5.14) and (3.5.18) we obtain

$$-\iint\limits_{O} \mathbf{u} (M_0) \cdot \hat{\Phi} (M_0, Q) \, do_{M_0} = (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q) \, \delta (Q), \tag{3.7.10}$$

which is required.

4.3.8 On behaviour of the potential of the elasticity theory at infinity

When point $Q \subset V_e$ is at a considerable distance from surface O, then

$$\mathbf{R} = \mathbf{r}_M - \mathbf{r}_Q \approx -\mathbf{r}_Q, \; R \approx r_Q \tag{3.8.1}$$

and the kernel (3.5.9) of the first potential is set in the form

$$\hat{U} (M_0, Q) \underset{V_e \supset Q \to Q_\infty}{\approx} \frac{1}{16\pi\mu (1 - \nu) r_Q} \left[(3 - 4\nu) \hat{E} + \mathbf{e}_Q \mathbf{e}_Q\right], \quad \mathbf{e}_Q = \frac{1}{r_Q} \mathbf{r}_Q.$$

Due to symmetry of tensor $\hat{U}$, the factors $\hat{U}$ and $\mathbf{a}$ are interchangeable in formula (3.6.1), thus

$$\underset{V_e \supset Q \to Q_\infty}{\mathbf{A}(Q)} = \frac{1}{16\pi\mu(1-\nu) r_Q}\left[(3-4\nu)\hat{E} + \mathbf{e}_Q\mathbf{e}_Q\right] \cdot \iint_O \mathbf{a}(M_0)\, do_{M_0}. \tag{3.8.2}$$

By eq. (3.5.8) the latter formula presents the displacement vector at point Q due to the force

$$\iint_O \mathbf{a}(M_0)\, do_{M_0}$$

applied at the origin of the coordinate system. This force can be treated as the principal vector of the system of forces on the surface of a small volume V_i provided that the latter tends to zero. The displacement due to such a system of forces and thus the first potential decreases at infinity as r_Q^{-1}.

Under the same replacement (3.8.1) the kernel (3.5.12) of the second potential is given by

$$\underset{V_e \supset Q \to Q_\infty}{\hat{\Phi}(M_0, Q)} = \frac{1}{8\pi(1-\nu) r_Q^2}\left[(1-2\nu)\left(\mathbf{e}_Q\mathbf{n}_M - \mathbf{n}_M\mathbf{e}_Q + \mathbf{n}_M \cdot \mathbf{e}_Q\hat{E}\right) + 3\mathbf{n}_M \cdot \mathbf{e}_Q\mathbf{e}_Q\mathbf{e}_Q\right]$$

and the potential takes the form

$$\mathbf{B}(Q) = \frac{1}{8\pi(1-\nu)}\left[(1-2\nu)\iint_O \left(\mathbf{n}_M\mathbf{b} - \mathbf{b}\cdot\mathbf{n}_M\hat{E} + \mathbf{b}\mathbf{n}_M\right) do_{M_0} - 3\mathbf{e}_Q \cdot \iint_O \mathbf{b}\mathbf{n}_M do_{M_0} \cdot \mathbf{e}_Q\hat{E}\right] \cdot \frac{\mathbf{e}_Q}{r_Q^2}. \tag{3.8.3}$$

This represents the displacement of point Q_∞ caused by a system of forces distributed over the surface of a small volume V_i when the latter tends to zero. The principal vector of this system of forces is equal to zero, otherwise the displacement decreases as r_Q^{-1} rather than r_Q^{-2} when Q is at a distance well away from V_i. The second potential, similar to the double layer potential, behaves as r_Q^{-2} at great distances, see also Subsection 5.1.3.

4.4 Theorems on uniqueness and existence of solutions

4.4.1 Kirchhoff's theorem

The fundamental system of equations and boundary conditions of elasticity theory is given in Subsection 4.1.1. The following assumptions are introduced: (i) the initial state is the natural one, (ii) constants μ and ν in the generalised Hooke law satisfy inequalities (3.3.5) and (3.3.6) of Chapter 3 ensuring positiveness of the specific strain energy, the latter vanishing only in the natural state, and (iii) while formulating boundary conditions one adopts the standard assumption of the linear theory of elasticity, that is, that the surface O bounding the elastic solid in the equilibrium state coincides with that in the natural state.

The Kirchhoff theorem states that, under the listed conditions, the solution of the boundary value problem is unique. Indeed, assuming existence of two different solutions $\mathbf{u}', \hat{T}'$ and $\mathbf{u}'', \hat{T}''$ for the same volume forces in V and surface forces on O_2 as well as the same displacement vector on O_1 we would obtain that the differences

$$\mathbf{u} = \mathbf{u}'' - \mathbf{u}', \quad \hat{T} = \hat{T}' - \hat{T}'' \tag{4.1.1}$$

are the solution of the homogeneous boundary value problem

$$\begin{gathered}\operatorname{div}\hat{T} = 0, \hat{T} = \hat{T}^*, \\ \hat{T} = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\hat{E} + \hat{\varepsilon}\right), \\ \hat{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^*\right],\end{gathered} \tag{4.1.2}$$

$$\mathbf{u}|_{O_1} = 0, \quad \mathbf{n}\cdot\hat{T}\Big|_{O_2} = 0. \tag{4.1.3}$$

It follows from these equations, cf. (B.3.10), that

$$\begin{aligned}\iiint\limits_V \mathbf{u}\cdot\operatorname{div}\hat{T}d\tau &= \iiint\limits_V \operatorname{div}\left(\hat{T}\cdot\mathbf{u}\right)d\tau - \iiint\limits_V \hat{T}\cdot\cdot\hat{\varepsilon}d\tau \\ &= \iint\limits_O \mathbf{n}\cdot\hat{T}\cdot\mathbf{u}do - \iiint\limits_V \hat{T}\cdot\cdot\hat{\varepsilon}d\tau = 0.\end{aligned}$$

By virtue of eq. (4.1.3) the surface integral vanishes, thus

$$\iiint\limits_V \hat{T}\cdot\cdot\hat{\varepsilon}d\tau = 2\iiint\limits_V Ad\tau = 0, \tag{4.1.4}$$

and since the strain energy is positive definite we have

$$A \equiv 0, \quad \hat{\varepsilon} \equiv 0, \quad \hat{T} \equiv 0. \tag{4.1.5}$$

By eq. (4.1.1) we obtain

$$\mathbf{u}' = \mathbf{u}'', \quad \hat{T}' = \hat{T}'', \tag{4.1.6}$$

which contradicts the assumption regarding the existence of two different solutions.

In the case of the mixed boundary value problem (1.2.3) as well as for the first boundary value problem (1.2.1) it follows from eq. (4.1.5) that $\mathbf{u} \equiv 0$ while in the second boundary value problem the displacement vector is determined up to a rigid body displacement

$$\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0). \tag{4.1.7}$$

Remark 1. The Kirchhoff theorem reflects the properties of the equations of the linear theory of elasticity. In particular, the linear theory is insufficient for predicting the coexistence of the different equilibrium states under the same loading conditions, for example in the problem of the bending of axially loaded rods. It is essential for the proof that the change in the body form is negligible. If this assumption is not made, then for any of the possible equilibrium states one should state the following kinematic boundary conditions

$$\mathbf{u}'|_{O_1'} = \mathbf{u}_*, \quad \mathbf{u}''|_{O_1''} = \mathbf{u}_*.$$

Here $O_1' = O_1''$ since prescribing displacement $\mathbf{u}_*$ determines the same shape of this part of the surface in the deformed state. However, the part of the surface on which the surface forces are prescribed does not retain its shape, i.e. $O_2' \neq O_2''$, so that $\mathbf{N}' \neq \mathbf{N}''$ and the static boundary conditions should be set as follows

$$\mathbf{N}' \cdot \hat{T}'\Big|_{O_2'} = \mathbf{F}, \quad \mathbf{N}'' \cdot \hat{T}''\Big|_{O_2''} = \mathbf{F}.$$

For this reason, the boundary condition (4.1.3) for the difference in the solutions does not hold.

Remark 2. It follows from the proof of the theorem that there appears no state of stress in the solid if the external forces are absent. This does not contradict the possibility of the existence of stresses in an unloaded simply-connected elastic volume from which a wedge-shape is removed and then the cut surfaces are joined together. The field of displacement which is continuous together with its derivatives and enables the solid to return to the natural state from this state is not feasible in such a body. Under these conditions the above proof of Kirchhoff's theorem fails for no other reason than it is not possible to transform the volume integral into a surface one since this transformation assumes continuity of $\hat{T}, \mathbf{u}$ and their first derivatives.

Remark 3. In a multiply-connected volume the continuity of the tensors of strain and stress is also guaranteed in the case when the displacement caused by Volterra's distortion is not single-valued, see Subsection 2.2.4. In the above statement the Kirchhoff theorem is also not valid. It is completed by the requirement that solutions $\mathbf{u}'$ and $\mathbf{u}''$ have coincident cyclic constant vectors $\mathbf{b}$ and $\mathbf{c}$, i.e. the solutions must have the same distortion. Then vector $\mathbf{u} = \mathbf{u}' - \mathbf{u}''$ is a continuous and single-valued function and the above proof remains valid, see Section 5 of this chapter.

Remark 4. Kirchhoff's theorem states that the solution is unique if it exists. A proof of the existence of the solution of the first and second boundary value problems is considered in Subsections 4.2-4.8 of this chapter.

Remark 5. Kirchhoff's theorem does not exclude the existence of discontinuous solutions of the homogeneous boundary value problems for which the displacements (or the surface forces in the second boundary value problem) vanish on the body surface. A solution which is continuous or even analytical in body's volume can be constructed for values of Poisson's ratio ν beyond the admissible interval of its values, i.e. for $\nu > 1/2$ and $\nu < -1$.

Homogeneous boundary value problems for a hollow sphere bounded by concentric spheres $R = R_0$ and $R = R_1$ can serve as an example. The solution can be constructed by means of the biharmonic functions of Love (see Subsection 4.1.10) of the type

$$\chi = \sum_s \left(A_s R^{s+2} + \frac{B_s}{R^{s-1}} + C_s R^s + \frac{D_s}{R^{s+1}} \right) P_s(\mu) \quad (\mu = \cos\vartheta), \tag{4.1.8}$$

where $P_s(\mu)$ is the solution of Legendre's equation

$$\frac{d}{d\mu}\left(1-\mu^2\right)\frac{dP_s}{d\mu} + s(s+1)P_s = 0.$$

Subscript s is determined by the condition of the existence of a nontrivial solution of the homogeneous boundary value problem, i.e. the determinant Δ of the system of linear homogeneous equations for unknown coefficients A_s, B_s, C_s, D_s obtained from the boundary conditions. It depends on $s, \nu, R_0/R_1$ and the roots of the equation

$$\Delta\left(s, \nu; \frac{R_0}{R_1}\right) = 0 \tag{4.1.9}$$

(transcendental with respect to s) are complex-valued in the admissible interval of values of ν.

Prescribing integer values of s, i.e. $s = n \geq 2$, one obtains the solutions continuous in the whole volume. For these values of s and for all R_0/R_1

eq. (4.1.9) yields the values of parameter ν eq. (4.1.9) lying outside the admissible interval.

4.4.2 Integral equations of the first boundary value problem

The solution is set in the form of the second potential of the elasticity theory (3.6.6) with the unknown density vector $\mathbf{b}(M_0)$

$$\mathbf{v}(Q) = \mathbf{u}(Q) - \mathbf{u}_*(Q) = \iint_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = \mathbf{B}(Q). \quad (4.2.1)$$

By $\mathbf{u}_*(Q)$ we denote a particular solution corresponding to the mass forces. When this solution is given by eq. (3.7.8) then vector $\mathbf{v}(Q)$, due to eq. (4.2.1), is the solution of the homogeneous equations of the elasticity theory in terms of displacements both at $Q \subset V_i$ and $Q \subset V_e$. The value $\mathbf{v}(Q_0)$ on surface O is given.

Referring to eqs. (3.6.12) and (3.6.13) we write the following equalities

$$\underset{V_i \supset Q \to Q_0}{\mathbf{v}}(Q) = \mathbf{v}(Q_0) = \mathbf{B}_i(Q_0) = \mathbf{B}_0(Q_0) - \frac{1}{2}\mathbf{b}(Q_0), \quad (4.2.2)$$

$$\underset{V_e \supset Q \to Q_0}{\mathbf{v}}(Q) = \mathbf{v}(Q_0) = \mathbf{B}_e(Q_0) = \mathbf{B}_0(Q_0) + \frac{1}{2}\mathbf{b}(Q_0), \quad (4.2.3)$$

where $\mathbf{B}_0(Q_0)$ denotes the direct value of potential $\mathbf{B}(Q)$. We thus arrive at the following integral equations of the first internal ($\mathrm{I}^{(i)}$) and first external ($\mathrm{I}^{(e)}$) boundary value problems

$$\mathrm{I}^{(i)} \quad \frac{1}{2}\mathbf{b}(Q_0) - \iint_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = -\mathbf{v}(Q_0), \quad (4.2.4)$$

$$\mathrm{I}^{(e)} \quad \frac{1}{2}\mathbf{b}(Q_0) + \iint_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = \mathbf{v}(Q_0). \quad (4.2.5)$$

Remark 1. Let us consider the case $\mathrm{I}^{(i)}$ and let the displacement vector $\mathbf{v}(Q_0)$ be a rigid body displacement

$$\mathbf{v}(Q_0) = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0}. \quad (4.2.6)$$

The solution of integral equation (4.2.4)

$$\mathbf{b}(M_0) = -(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{M_0}) \quad (4.2.7)$$

follows immediately from eq. (3.7.10) if $\delta(Q) = 1/2$. Indeed, putting eq. (4.2.7) as follows

$$\mathbf{b}(M_0) = -(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0}) - \boldsymbol{\omega} \times \mathbf{R}$$

yields

$$\iint_O \mathbf{b}(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = \frac{1}{2}(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0}), \tag{4.2.8}$$

and the above said follows when expressions (4.2.7) and (4.2.8) are substituted into eq. (4.2.4). Using eq. (4.2.1) and taking $\delta(Q) = 1$ we obtain

$$\mathbf{v}(Q) = -\iint_O (\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{M_0}) \cdot \hat{\Phi}(M_0, Q)\, do_{M_0} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q. \tag{4.2.9}$$

Thus, when a rigid body displacement of surface O is prescribed, the whole volume V_i moves as a rigid body and the state of stress is absent. By Kirchhoff's theorem, this solution is unique.

Remark 2. It follows from the above analysis that vector

$$\mathbf{b}^0(M_0) = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{M_0}, \tag{4.2.10}$$

with $\mathbf{v}_0, \boldsymbol{\omega}$ being arbitrary constant vectors, is a solution of the homogeneous integral equation

$$\mathrm{I}_0^{(e)} \quad \frac{1}{2}\mathbf{b}^0(Q_0) - \iint_O \mathbf{b}^0(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = 0. \tag{4.2.11}$$

Hence, the density vector $\mathbf{b}(M_0)$ in problem $\mathrm{I}^{(e)}$ can be determined only up to term (4.2.10).

Remark 3. In accordance with eq. (3.8.3) the displacement vector (4.2.1) decreases not slower than R^{-1} in the external problem. This solution can be obtained if the principal vector of the forces applied on O is equal to zero under the prescribed surface displacement $\mathbf{v}(Q_0)$. For this reason, the solution of the first external problem in terms of the second potential (3.6.6) does not exist for arbitrary $\mathbf{v}(Q_0)$.

A similar phenomenon is known from electrostatics. The problem of determining the field of the electric potential $\mathbf{v}(Q)$ vanishing at infinity in terms of the prescribed distribution $\mathbf{v}(Q_0)$ on the conducting surface O reduces to the external problem of Dirichlet. The solution of the latter problem can be represented by a double layer potential only under the condition that the total charge on O is equal to zero. For this reason the way that the problem is solved is to superimpose the double layer potential by a solution of the Robin problem. In this problem the potential on O is constant whilst its value in V_e is presented by a single layer potential. Clearly, the solution of the first external boundary value problem of the theory of elasticity reduces to an analogous "elastostatic Robin's problem".

4.4.3 Integral equations of the second boundary value problem

The solution of the homogeneous equations of the elasticity theory in terms of displacements is sought in the form of the first potential (3.6.1)

$$\mathbf{v}(Q) = \mathbf{u}_Q - \mathbf{u}_*(Q) = \iint\limits_O \mathbf{a}(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0} = \mathbf{A}(Q) \qquad (4.3.1)$$

with an unknown density vector $\mathbf{a}(M_0)$. As shown in Subsection 4.3.5 the stress tensor is as follows

$$\hat{T}(Q) = \frac{1}{8\pi(1-\nu)} \iint\limits_O \frac{do_{M_0}}{R^3} \left[(1-2\nu)\left(\mathbf{Ra} + \mathbf{aR} - \hat{E}\mathbf{R}\cdot\mathbf{a}\right) + 3\frac{\mathbf{RR}}{r^2}\mathbf{R}\cdot\mathbf{a}\right]. \qquad (4.3.2)$$

Here

$$\mathbf{R} = \mathbf{r}_{M_0} - \mathbf{r}_Q, \qquad (4.3.3)$$

and the difference in the sign from eq. (3.5.12) is attributable to the fact that the differentiation in eq. (4.3.2) is carried out with respect to the coordinates of point Q. The validity of differentiation in eq. (4.3.1) is beyond question since point Q is not located on O, that is $R \neq 0$.

The stress vector on the elementary surface with normal $\mathbf{n}_Q$ is determined by the equality

$$\mathbf{n}_Q \cdot \hat{T}(Q) = \iint\limits_O \hat{\Psi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0}, \qquad (4.3.4)$$

where eq. (4.3.2) suggests that the following nonsymmetric tensor of the second rank should be introduced

$$\hat{\Psi}(M_0, Q) = \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)\left(\mathbf{n}_Q\cdot\mathbf{R}\hat{E} + \mathbf{Rn}_Q - \mathbf{n}_Q\mathbf{R}\right) + 3\frac{\mathbf{n}_Q\cdot\mathbf{R}}{R^2}\mathbf{RR}\right] = \frac{1}{8\pi(1-\nu)}\left[2(1-\nu)\frac{\mathbf{n}_Q\cdot\mathbf{R}}{R^3}\hat{E} + \frac{1-2\nu}{R^3}\left(\mathbf{Rn}_Q - \mathbf{n}_Q\mathbf{R}\right) + \mathbf{n}_Q\cdot\mathbf{R}\nabla\nabla\frac{1}{R}\right]. \qquad (4.3.5)$$

This tensor differs essentially from $\hat{\Phi}(M_0, Q)$ in that its definition contains the normal vector at point Q rather than at M_0. Due to eq. (3.5.12) the sum

$$\hat{\Psi}(M_0, Q) + \hat{\Phi}(M_0, Q) = \frac{1}{8\pi(1-\nu)} \left\{ \frac{2(1-\nu)}{R^3} (\mathbf{n}_Q - \mathbf{n}_{M_0}) \cdot \mathbf{R}\hat{E} + \right.$$

$$\left. \frac{1-2\nu}{R^3} [\mathbf{R}(\mathbf{n}_Q - \mathbf{n}_{M_0}) - (\mathbf{n}_Q - \mathbf{n}_{M_0})\mathbf{R}] + (\mathbf{n}_Q - \mathbf{n}_{M_0}) \cdot \mathbf{R}\nabla\nabla\frac{1}{R} \right\} \tag{4.3.6}$$

implies a kernel of the potential with a weak singularity (of the kind R^{-1})

$$\mathbf{C}(Q) = \iint_O \left[\hat{\Psi}(M_0, Q) + \hat{\Phi}(M_0, Q)\right] \cdot \mathbf{a}(M_0)\, do_{M_0}$$

whose limiting inward and outward values are coincident with its direct value (similar to the case of the first potential)

$$\lim_{V_i \supset Q \to Q_0} \mathbf{C}(Q) = \mathbf{C}_i(Q) = \lim_{V_e \supset Q \to Q_0} \mathbf{C}(Q) = \mathbf{C}_e(Q) = \mathbf{C}^0(Q_0).$$

For this reason,

$$\lim_{V_i \supset Q \to Q_0} \iint_O \hat{\Psi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0} =$$

$$= \iint_O \left[\hat{\Psi}(M_0, Q_0) + \hat{\Phi}(M_0, Q_0)\right] \cdot \mathbf{a}(M_0)\, do_{M_0} -$$

$$\lim_{V_i \supset Q \to Q_0} \iint_O \hat{\Phi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0}.$$

However, similar to eq. (4.2.2)

$$\lim_{V_i \supset Q \to Q_0} \iint_O \hat{\Phi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0} = \iint_O \hat{\Phi}(M_0, Q_0) \cdot \mathbf{a}(M_0)\, do_{M_0} -$$

$$\frac{1}{2}\mathbf{a}(Q_0),$$

so that

$$\lim_{V_i \supset Q \to Q_0} \iint_O \hat{\Psi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0} = \iint_O \hat{\Psi}(M_0, Q_0) \cdot \mathbf{a}(M_0)\, do_{M_0} +$$

$$\frac{1}{2}\mathbf{a}(Q_0) \tag{4.3.7}$$

and by analogy

$$\lim_{V_e \supset Q \to Q_0} \iint\limits_O \hat{\Psi}(M_0, Q) \cdot \mathbf{a}(M_0)\, do_{M_0} = \iint\limits_O \hat{\Psi}(M_0, Q_0) \cdot \mathbf{a}(M_0)\, do_{M_0} - \frac{1}{2}\mathbf{a}(Q_0). \quad (4.3.8)$$

Comparing eqs. (4.3.5) and (3.5.12) and accounting for $-\mathbf{r_Q} - \mathbf{r}_M = -\mathbf{R}$, we have

$$\hat{\Psi}(M_0, Q_0) = \hat{\Phi}(Q_0, M_0). \quad (4.3.9)$$

We introduce the distributed surface forces prescribed on O

$$\begin{aligned} \lim_{V_i \supset Q \to Q_0} \mathbf{n}_Q \cdot \hat{T} &= \left(\mathbf{n}_Q \cdot \hat{T}\right)_i = \mathbf{F}(Q_0), \\ \lim_{V_e \supset Q \to Q_0} \mathbf{n}_Q \cdot \hat{T} &= \left(\mathbf{n}_Q \cdot \hat{T}\right)_e = \mathbf{F}(Q_0), \end{aligned} \quad (4.3.10)$$

where in both cases $\mathbf{n}_Q$ denotes the unit vector of the normal external to volume V_i. Referring to eqs. (4.3.4), (4.3.7), (4.3.8) and (4.3.9) we arrive at the integral equations of the first internal ($\mathrm{II}^{(i)}$) and second external ($\mathrm{II}^{(e)}$) boundary value problems

$$\mathrm{II}^{(i)} \quad \frac{1}{2}\mathbf{a}(Q_0) + \iint\limits_O \hat{\Phi}(M_0, Q_0) \cdot \mathbf{a}(M_0)\, do_{M_0} = \mathbf{F}(Q_0), \quad (4.3.11)$$

$$\mathrm{II}^{(e)} \quad \frac{1}{2}\mathbf{a}(Q_0) - \iint\limits_O \hat{\Phi}(M_0, Q_0) \cdot \mathbf{a}(M_0)\, do_{M_0} = -\mathbf{F}(Q_0). \quad (4.3.12)$$

Let us notice that the surface force in the second line of eq. (4.3.10) has the opposite sign to that in the standard definition of the scalar product of normal vector external to V_i and the stress tensor.

Remark. Making use of eq. (3.5.9) we can rewrite the displacement vector (4.3.1) in the following form

$$\mathbf{v}(Q) = \frac{1}{4\pi\mu}\left[\iint\limits_O \frac{\mathbf{a}(M_0)}{R} do_{M_0} + \frac{1}{4(1-\nu)} \operatorname{grad}_Q \iint\limits_O \mathbf{R} \cdot \frac{\mathbf{a}(M_0)}{R} do_{M_0}\right]. \quad (4.3.13)$$

Replacing $\mathbf{R}$ by $\mathbf{r}_{M_0} - \mathbf{r_Q}$ we obtain

$$\mathbf{v}(Q) = \frac{1}{4\pi\mu}\left[\iint\limits_O \frac{\mathbf{a}(M_0)}{R} do_{M_0} - \frac{1}{4(1-\nu)} \operatorname{grad}_Q \mathbf{r}_Q \cdot \iint\limits_O \frac{\mathbf{a}(M_0)}{R} do_{M_0} + \frac{1}{4(1-\nu)} \operatorname{grad}_Q \iint\limits_O \frac{\mathbf{r}_{M_0} \cdot \mathbf{a}(M_0)}{R} do_{M_0}\right]. \quad (4.3.14)$$

This is the solution in the Papkovich-Neuber form, eq. (1.4.10), provided that the following potentials

$$
\mathbf{B} = \frac{1}{16\pi\mu(1-\nu)} \iint\limits_{O} \frac{\mathbf{a}(M_0)}{R} do_{M_0},
$$

$$
B_0 = -\frac{1}{16\pi\mu(1-\nu)} \iint\limits_{O} \frac{\mathbf{r}(M_0)\cdot\mathbf{a}(M_0)}{R} do_{M_0}. \tag{4.3.15}
$$

are understood as the harmonic vector and scalar, respectively.

4.4.4 *Comparison of the integral equations of the first and second boundary value problems*

Let us rewrite the integral equations of Subsections 4.4.2 and 4.4.3 in the following sequence

$$
\left.
\begin{array}{ll}
\mathrm{I}^{(i)} & \dfrac{1}{2}\mathbf{b}(Q_0) - \displaystyle\iint\limits_{O} \mathbf{b}(M_0)\cdot\hat{\Phi}(M_0,Q_0)\, do_{M_0} = -\mathbf{v}(Q_0), \\
\mathrm{II}^{(e)} & \dfrac{1}{2}\mathbf{a}(Q_0) - \displaystyle\iint\limits_{O} \hat{\Phi}(Q_0,M_0)\cdot\mathbf{a}(M_0)\, do_{M_0} = -\mathbf{F}(Q_0),
\end{array}
\right\} \tag{4.4.1}
$$

$$
\left.
\begin{array}{ll}
\mathrm{I}^{(e)} & \dfrac{1}{2}\mathbf{b}(Q_0) + \displaystyle\iint\limits_{O} \mathbf{b}(M_0)\cdot\hat{\Phi}(M_0,Q_0)\, do_{M_0} = \mathbf{v}(Q_0), \\
\mathrm{II}^{(i)} & \dfrac{1}{2}\mathbf{a}(Q_0) + \displaystyle\iint\limits_{O} \hat{\Phi}(Q_0,M_0)\cdot\mathbf{a}(M_0)\, do_{M_0} = \mathbf{F}(Q_0).
\end{array}
\right\} \tag{4.4.2}
$$

It has been mentioned above that the surface integrals are understood as the principal values, that is, the equations are singular. The basic theorems and Fredholm's alternative can be proved to be applicable for the values of constants μ and ν for which the specific strain energy is positive, see eqs. (3.3.5) and (3.3.6) of Chapter 3.

Integral equations comprising system (4.4.1) are conjugate, the same is valid for system (4.4.2). The corresponding systems of the homogeneous equations can be written in the form

$$
\left.
\begin{array}{ll}
\mathrm{I}_0^{(i)}, \mathrm{I}_0^{(e)} & \dfrac{1}{2}\mathbf{b}^0(Q_0) - \lambda \displaystyle\iint\limits_{O} \mathbf{b}^0(M_0)\cdot\hat{\Phi}(M_0,Q_0)\, do_{M_0} = 0, \\
\mathrm{II}_0^{(e)}, \mathrm{II}_0^{(i)} & \dfrac{1}{2}\mathbf{a}^0(Q_0) - \lambda \displaystyle\iint\limits_{O} \hat{\Phi}(Q_0,M_0)\cdot\mathbf{a}^0(M_0)\, do_{M_0} = 0,
\end{array}
\right\} \tag{4.4.3}
$$

where $\lambda = 1$ for problems $\mathrm{I}_0^{(i)}, \mathrm{II}_0^{(e)}$, and $\lambda = -1$ for problems $\mathrm{I}_0^{(e)}, \mathrm{II}_0^{(i)}$. The eigenvalues of the conjugate integral equations are known to coincide, i.e. these equations have simultaneously (for the same λ) either a trivial solution or nontrivial proper solutions. According to Fredholm's alternative it is known that in the first case the corresponding inhomogeneous equation has only one solution, whereas in the second case this equation possesses no solution for an arbitrary right hand side and may have non-unique solutions under certain conditions imposed on the right hand side.

In what follows it is proved that $\lambda = 1$ is not the eigenvalue of the system of conjugate equations (4.4.3). Thus, the first internal $(\mathrm{I}^{(i)})$ and second external $(\mathrm{II}^{(e)})$ problems have a unique solution for an arbitrarily prescribed right hand side.

On the contrary, for $\lambda = -1$ the homogeneous equation $\mathrm{I}_0^{(e)}$ has a nontrivial family of solutions (4.2.10) depending on two arbitrary constant vectors (six constant scalars). Hence, the homogeneous equation $(\mathrm{II}_0^{(i)})$ also has a nontrivial family of solutions depending on six constants. Therefore, problems $\mathrm{II}^{(i)}$ and $\mathrm{I}^{(e)}$ have in general no solutions. It is easy to understand inasmuch as the free term $\mathbf{F}(Q)$ in problem $\mathrm{II}^{(i)}$ determining the distribution of the surface forces must satisfy the equations of statics and then the displacement vector is determined up to a rigid body displacement. The very statement of problem $\mathrm{I}^{(e)}$ imposes an essential restriction on prescribing vector $\mathbf{v}(Q_0)$, see Remark 3 in Subsection 4.4.2.

The problems (4.4.1) and (4.4.2) are considered in detail below.

4.4.5 Theorem on the existence of solutions to the second external and first internal problems

Let the homogeneous integral equation

$$\frac{1}{2}\mathbf{a}(Q_0) - \iint\limits_O \hat{\Phi}(Q_0, M_0) \cdot \mathbf{a}(M_0)\, do_{M_0} = 0 \tag{4.5.1}$$

have a nontrivial solution $\mathbf{a}^0(Q_0)$. Due to eq. (4.3.1) the displacement vector $\mathbf{v}(Q)$ determined by the first potential

$$\mathbf{v}(Q) = \mathbf{A}\left(Q\,; \mathbf{a}^0\right) = \iint\limits_O \mathbf{a}^0(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0} \tag{4.5.2}$$

satisfies the homogeneous equations of the elasticity theory in terms of displacements. At infinity it decreases not slower than R^{-1} whereas the surface forces calculated by eqs. (4.4.1) and (4.5.1) vanishes on O and have the order of R^{-2} on the surface of the sphere of a sufficiently large radius R. The double strain energy given by Clapeyron's formula (3.3.3) of Chapter 3 is as follows

$$\iiint\limits_{V_e} \hat{T} \cdot\cdot\, \hat{\varepsilon}\, d\tau = \iint\limits_{\Omega} \mathbf{v} \cdot \mathbf{F} do + \iint\limits_O \mathbf{v} \cdot \mathbf{F} do = R^2 \iint\limits_{\Omega^*} \mathbf{v} \cdot \mathbf{F} do^* + \iint\limits_O \mathbf{v} \cdot \mathbf{F} do,$$

where do^* denotes a surface element of sphere $\Omega^*$of a unit radius. The integrand decreases not slower than R^{-3} so that

$$\iiint_{V_e} \hat{T} \cdot\cdot\, \hat{\varepsilon} d\tau = \iint_O \mathbf{v} \cdot \mathbf{F} do = 0,$$

and $\hat{\varepsilon} = 0$ because the specific strain energy is positive definite. This is why $\mathbf{v}(Q)$ may be only a rigid body displacement. However, it vanishes at infinity, thus

$$\mathbf{v}(Q) = \mathbf{A}\left(Q, \mathbf{a}^0\right) = 0, \qquad Q \subset V_e. \tag{4.5.3}$$

It remains only to prove that this equality contradicts the assumption $\mathbf{a}^0 \neq 0$. To this end, we note that it follows from the continuity of the first potential and eq. (4.5.3) that

$$\mathbf{A}_e(Q_0) = \mathbf{A}_i(Q_0) = \mathbf{v}(Q_0) = 0. \tag{4.5.4}$$

Turning once again to Clapeyron's formula we obtain

$$\iiint_{V_e} \hat{T} \cdot\cdot\, \hat{\varepsilon} d\tau = \iint_O \mathbf{v}(Q_0) \cdot \mathbf{F} do_{M_0} = 0,$$

so that $\hat{\varepsilon} = 0$ and, by eq. (4.5.4), we have

$$\mathbf{v}(Q) = 0, \qquad Q \subset V_i. \tag{4.5.5}$$

The surface forces corresponding to zero displacement vector vanish, and by eqs. (4.3.10)-(4.3.12) we obtain

$$\left(\mathbf{n}_Q \cdot \hat{T}\right)_i - \left(\mathbf{n}_Q \cdot \hat{T}\right)_e = \mathbf{a}^0(Q_0) = 0, \tag{4.5.6}$$

which is the required result. Integral equation $\mathrm{II}_0^{(e)}$, and thus the conjugate equation $\mathrm{I}_0^{(i)}$ admits only a trivial solution and $\lambda = 1$ is not the eigenvalue of these equations. This proves the existence and uniqueness of the solution of problems $\mathrm{I}^{(i)}$ and $\mathrm{II}^{(e)}$ for an arbitrary displacement vector $\mathbf{v}(Q_0)$ on O in the first problem and an arbitrary surface force $\mathbf{F}(Q_0)$ in the second problem.

4.4.6 The second internal boundary value problem ($II^{(i)}$)

The homogeneous integral equation corresponding to this problem

$$\mathrm{II}_0^{(i)} \quad \frac{1}{2}\mathbf{a}^0(Q_0) + \iint_O \hat{\Phi}(Q_0, M_0) \cdot \mathbf{a}^0(M_0)\, do_{M_0} = 0 \tag{4.6.1}$$

is conjugate to eq. (4.2.11):

$$\mathrm{I}_0^{(e)} \quad \frac{1}{2}\mathbf{b}^0(Q_0) + \iint\limits_O \mathbf{b}^0(M_0)\cdot\hat{\Phi}(M_0,Q_0)\,do_{M_0} = 0. \tag{4.6.2}$$

The latter possesses a nontrivial solution (4.2.10), thus, the first one has a nontrivial solution, too. Proceeding now to the inhomogeneous integral equation (4.4.2) of problem $\mathrm{II}^{(i)}$ we have

$$\iint\limits_O \mathbf{b}^0(Q_0)\cdot\mathbf{F}(Q_0)\,do_{Q_0} = \frac{1}{2}\iint\limits_O \mathbf{b}^0(Q_0)\cdot\mathbf{a}(Q_0)\,do_{Q_0} + \iint\limits_O do_{M_0}\left[\iint\limits_O \mathbf{b}^0(Q_0)\cdot\hat{\Phi}(Q_0,M_0)\,do_{Q_0}\right]\cdot\mathbf{a}(M_0).$$

The internal integral is equal to $-\frac{1}{2}\mathbf{b}^0(M_0)$ so that

$$\iint\limits_O \mathbf{b}^0(Q_0)\cdot\mathbf{F}(Q_0)\,do_{Q_0} = \frac{1}{2}\left[\iint\limits_O \mathbf{b}^0(Q_0)\cdot\mathbf{a}(Q_0)\,do_{Q_0} - \iint\limits_O \mathbf{b}^0(M_0)\cdot\mathbf{a}(M_0)\,do_{M_0}\right] = 0.$$

This proves one of the Fredholm theorems which states that problem $\mathrm{II}^{(i)}$ may have a solution if the prescribed distribution of the surface force $\mathbf{F}(Q_0)$ is orthogonal to the family of eigensolutions of the conjugate integral equation (4.6.2)

$$\iint\limits_O \mathbf{b}^0(Q_0)\cdot\mathbf{F}(Q_0)\,do_{Q_0} = 0. \tag{4.6.3}$$

Inserting the expression for $\mathbf{b}^0(Q_0)$ we have

$$\mathbf{u}_0\cdot\iint\limits_O \mathbf{F}(Q_0)\,do_{Q_0} + \boldsymbol{\omega}\cdot\iint\limits_O \mathbf{r}_{Q_0}\times\mathbf{F}(Q_0)\,do_{Q_0} = 0,$$

and due to the arbitrariness of vectors $\mathbf{u}_0$ and $\boldsymbol{\omega}$ we arrive at the expected static conditions requiring that the principal vector and the principal moment of the surface forces in problem $\mathrm{II}^{(i)}$ vanish, i.e.

$$\mathbf{V} = \iint\limits_O \mathbf{F}(Q_0)\,do_{Q_0} = 0, \quad \mathbf{m}^0 = \iint\limits_O \mathbf{r}(Q_0)\times\mathbf{F}(Q_0)\,do_{Q_0} = 0. \tag{4.6.4}$$

When these conditions are fulfilled then the displacement vector $\mathbf{v}(Q)$ is determined up to a rigid body displacement, the latter being the eigensolution of the conjugate equation (4.6.2) in accordance with Fredholm's theorem.

4.4.7 Elastostatic Robin's problem

Let the nontrivial eigensolution $\mathbf{a}^0(Q_0)$ of problem $\mathrm{II}_0^{(i)}$ be taken as the density of the first potential for problem $\mathrm{II}^{(e)}$. By virtue of eqs. (4.4.1), (4.6.1) and (4.3.10), the distribution of the surface forces corresponding to this density is given by

$$
\begin{aligned}
-\mathbf{F}(Q_0) &= -\left(\mathbf{n}_Q \cdot \hat{T}\right)_e \\
&= \frac{1}{2}\mathbf{a}^0(Q_0) - \iint\limits_O \hat{\Phi}(Q_0, M_0) \cdot \mathbf{a}^0(M_0)\, do_{M_0} = \mathbf{a}^0(Q_0).
\end{aligned} \tag{4.7.1}
$$

This provides us with a mechanical interpretation of the eigensolution of the second internal problem.

Let the first potential of the elasticity theory obtained by means of density $\mathbf{a}^0(M_0)$ be denoted as

$$
\mathbf{w}(Q) = \iint\limits_O \mathbf{a}^0(M_0) \cdot \hat{U}(M_0, Q)\, do_{M_0}. \tag{4.7.2}
$$

This function is continuous in the whole space and determines the displacement vector

$$
\mathbf{w}(Q) = \begin{cases} \mathbf{w}_i(Q), & Q \subset V_i, \\ \mathbf{w}(Q_0), & Q \subset O, \\ \mathbf{w}_e(Q), & Q \subset V_e. \end{cases} \tag{4.7.3}
$$

The surface force on O, which is obtained by eq. (4.3.11) and corresponds to the displacement vector $\mathbf{w}_i(Q)$ of problem $\mathrm{II}^{(i)}$, is zero, i.e.

$$
\frac{1}{2}\mathbf{a}^0(Q_0) + \iint\limits_O \hat{\Phi}(Q_0, M_0) \cdot \mathbf{a}^0(M_0)\, do_{M_0} = \left[\mathbf{n}_Q \cdot \hat{T}(\mathbf{w}_i)\right]_0 = 0, \tag{4.7.4}
$$

and follows from definition (4.6.1) of density $\mathbf{a}^0(Q_0)$. In the case of no surface force the displacement $\mathbf{w}_i(Q)$ of the second internal problem may only be a rigid body displacement

$$
\mathbf{w}_i(Q) = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q \tag{4.7.4}
$$

and due to continuity of the single layer potential (4.7.2), we have

$$
\mathbf{w}_i(Q_0) = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0} = \mathbf{w}_e(Q_0). \tag{4.7.6}
$$

Let us imagine a rigid body filled in cavity V_i of an unbounded medium. Let the displacement of this rigid body be given by vector (4.7.5). It results in a displacement field $\mathbf{w}_e(Q)$ in V_e which is described by the first potential

(4.7.2). The eigensolution $-\mathbf{a}^0(Q_0)$ of problem $\mathrm{II}_0^{(i)}$ (with a minus sign) determines the reaction force of the medium to displacement of the rigid body since $\mathbf{n}_Q$ in eq. (4.7.1) denotes a unit vector of the normal directed into V_e.

This problem of the state of stress in the solid appearing due to a displacement of a fill-in rigid body is analogous to Robin's problem in electrostatics. A constant potential on a conducting surface corresponds to a rigid body displacement of volume V_i and a zero field of voltage corresponds to a zero state of stress in volume V_i. Robin's problem reduces to seeking the charge distribution on conductor O from a homogeneous integral equation for density of the single layer potential while the corresponding elastostatic Robin's problem reduces to searching for the eigenvector $\mathbf{a}^0(Q_0)$ of problem $\mathrm{II}_0^{(i)}$. The existence of the solution of elastostatic Robin's problem is ensured by the existence of the nontrivial eigensolution of integral equation $\mathrm{II}_0^{(i)}$.

The principal vector and the principal moment of the system of forces which need to be applied to the rigid body filled into the medium for providing displacement (4.7.5) are determined from the static equations

$$\mathbf{V} = \iint\limits_O \mathbf{a}^0(Q_0)\, do_{Q_0}, \qquad \mathbf{m}^C = \iint\limits_O \mathbf{r}(Q_0) \times \mathbf{a}^0(Q_0)\, do_{Q_0}. \tag{4.7.7}$$

Let

$$\overset{k}{\mathbf{a}}, \quad \overset{k+3}{\mathbf{a}}, \quad (k = 1, 2, 3) \tag{4.7.8}$$

denote distributions of the surface forces on O caused by a unit force $\overset{k}{\mathbf{V}} = \mathbf{i}_k$ and respectively a unit moment $\overset{k+3}{\mathbf{m}} = \mathbf{i}_k$, both being directed along axis Cx_k and applied to the rigid body. Then by eq. (4.7.7)

$$\left.\begin{aligned} &\iint\limits_O \overset{k}{\mathbf{a}}\, do_{Q_0} \cdot \mathbf{i}_r = \delta_{kr}, && \iint\limits_O \mathbf{r}(Q_0) \times \overset{k}{\mathbf{a}}\, do_{Q_0} \cdot \mathbf{i}_r = 0, \\ &\iint\limits_O \overset{k+3}{\mathbf{a}}\, do_{Q_0} \cdot \mathbf{i}_r = 0, && \iint\limits_O \mathbf{r}(Q_0) \times \overset{k+3}{\mathbf{a}}\, do_{Q_0} \cdot \mathbf{i}_r = \delta_{kr}, \end{aligned}\right\} \tag{4.7.9}$$

since axis Cx_k is the action line for the resultant force $\overset{k}{\mathbf{a}}$ and the force distribution $\overset{k+3}{\mathbf{a}}$ is statically equivalent to zero.

Let

$$\overset{k}{\mathbf{u}} = \mathbf{i}_k, \qquad \overset{k+3}{\mathbf{u}} = \mathbf{i}_k \times \mathbf{r}(Q_0) \tag{4.7.10}$$

denote the system of eigensolutions of integral equation $\mathrm{I}_0^{(e)}$. It is evident that any displacement of the rigid body soldered into V_i is a linear combination of these elementary displacements. Formulae (4.7.9) are rewritten in the form

$$\iint\limits_{O} \overset{k}{\mathbf{a}} \cdot \overset{r}{\mathbf{u}}\, do_{Q_0} = \delta_{kr} \qquad (k, r = 1, 2, \dots, 6). \tag{4.7.11}$$

This determines the distribution of the surface forces which are the eigensolutions of integral equations $\mathrm{II}_0^{(i)}$ and orthonormalised with system (4.7.10) of the eigensolutions of problem $\mathrm{I}_0^{(e)}$, see eq. (4.2.11).

4.4.8 The first external boundary value problem ($I^{(e)}$)

According to Fredholm's theorem (Subsection 4.4.6) the integral equation (4.4.2) of this problem has a solution only under the condition of orthogonality of the free term to any eigensolution $\mathbf{a}^0(Q_0)$ of problem $\mathrm{II}_0^{(i)}$:

$$\iint\limits_{O} \mathbf{v}(Q_0) \cdot \mathbf{a}^0(Q_0)\, do_{Q_0} = 0. \tag{4.8.1}$$

As mentioned in Remark 3 of Subsection 4.4.2 this condition stems from representing $\mathbf{v}(Q_0)$ in the form of the second potential of the elasticity theory rather than the essence of the problem. By eq. (3.8.3), this vector decreases not slower than R^{-2} with the distance from O, whereas the required decrease is not slower than R^{-1}.

Instead of the prescribed distribution $\mathbf{v}(Q_0)$ on O we introduce an auxiliary vector

$$\mathbf{v}^*(Q) = \mathbf{v}(Q_0) - \sum_{r=1}^{6} D_r \overset{r}{\mathbf{u}}, \tag{4.8.2}$$

with $\overset{r}{\mathbf{u}}$ denoting the elementary system of eigensolutions (4.7.10) of integral equation $\mathrm{I}_0^{(e)}$.

Condition (4.8.1) will hold for any eigenvector $\mathbf{a}^0(Q_0)$ if it holds for any of vectors (4.7.8). Referring to eq. (4.7.11) we have

$$\iint\limits_{O} \mathbf{v}^*(Q_0) \cdot \overset{k}{\mathbf{a}}(Q_0)\, do_{Q_0} = \iint\limits_{O} \mathbf{v}(Q_0) \cdot \overset{k}{\mathbf{a}}(Q_0)\, do_{Q_0} - D_k = 0. \tag{4.8.3}$$

Coefficients D_r are determined now. Assuming

$$\left.\begin{aligned} \mathbf{u}_0 &= \sum_{k=1}^{3} \mathbf{i}_k \iint\limits_{O} \mathbf{v}(Q_0) \cdot \overset{k}{\mathbf{a}}(Q_0)\, do_{Q_0}, \\ \boldsymbol{\omega} &= \sum_{k=1}^{3} \mathbf{i}_k \iint\limits_{O} \mathbf{v}(Q_0) \cdot \overset{k+3}{\mathbf{a}}(Q_0)\, do_{Q_0} \end{aligned}\right\} \tag{4.8.4}$$

we put eq. (4.8.2) in the form

$$\mathbf{v}^*(Q) = \mathbf{v}(Q_0) - (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_Q). \tag{4.8.5}$$

Looking for solution $\mathbf{v}^*(Q_0)$ in the form (4.2.1) of the second potential of the elasticity theory we arrive, instead of eq. (4.2.5), at the following integral equation

$$\frac{1}{2}\mathbf{b}^0(Q_0) + \iint\limits_O \mathbf{b}^0(M_0) \cdot \hat{\Phi}(M_0, Q_0)\, do_{M_0} = \mathbf{v}^*(Q_0), \tag{4.8.6}$$

which possesses a solution since the condition of orthogonality of the free term and the eigenvector of problem $\mathrm{II}_0^{(i)}$ is fulfilled.

It remains only to construct the first potential $\mathbf{w}_e(Q_0)$ in V_e. This potential solves the elastostatic Robin's problem corresponding to prescribing the displacement vector

$$\mathbf{w}_e(Q_0) = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0} \tag{4.8.7}$$

on O. The solution of the first external boundary value problem is presented in the form

$$\mathbf{v}(Q) = \mathbf{v}^*(Q) + \mathbf{w}_e(Q). \tag{4.8.8}$$

Indeed, this solution satisfies the homogeneous equations of elasticity theory in V_e (as each potential satisfy these) and, besides, by eqs. (4.8.5) and (4.8.7), on O

$$\mathbf{v}(Q_0) = \mathbf{v}(Q_0) - (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0}) + (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}_{Q_0}) = \mathbf{v}(Q_0),$$

which completes the proof. The uniqueness of the solution is ensured by Kirchhoff's theorem.

4.5 State of stress in a double-connected volume

4.5.1 Overview of the content

It is assumed in what follows that the components of the strain tensor $\hat{\varepsilon}$ are single-valued continuous functions of coordinates having continuous partial derivatives of the first and second order and satisfying the compatibility condition (2.1.5) of Chapter 2. Let us agree to refer to such a deformation as a regular one.

Under a regular deformation of an elastic solid in a single-connected volume, the displacement vector $\mathbf{u}$ and the linear rotation vector $\boldsymbol{\omega}$ obtained by means of the strain tensor are also single-valued and continuous. According to Kirchhoff's theorem (Subsection 4.4.1) this volume is in the natural state if it is not subjected to external forces. This is not to say that this is the case for a double-connected volume (torus, hollow cylinder) where a state of stress may appear under a regular deformation and absent external forces. A double-connected volume which was originally in the natural state is stressed by means of Volterra's distortion, see Subsection 2.2.4. A thin layer of material is removed by two congruent cuts and the ends of the single-connected volume obtained are rigidly connected and the congruent surfaces are called "the barrier". The characteristics of the distortion are two cyclic constant vectors **c** and **b** referred to in what follows as the translational and rotational vectors of distortion. They determine the translation and rotation which is needed for one of the ends after cutting in order to make it coincident with the other congruous end.

Prescribing external forces acting on the elastic solid in a single-connected volume determines the state of stress and the single-valued displacement vector whereas in a double-connected volume determining state of stress is only possible when it is *a priori* known that the vectors of distortion vanish.

Jumps in the rotation vector $\boldsymbol{\omega}$ and displacement vector $\mathbf{u}$ on the barrier are determined by Weingarten's formulae in terms of the distortion vectors **c** and **b** whose components are termed by Volterra as the barrier constants. For a double-connected volume the formulation of Kirchhoff's theorem must be completed by the requirement of prescribing six barrier constants. Provided that the elastic medium fills in a double-connected volume and the deformation is regular then the state of stress is determined not only by the external forces but also by the six barrier constants. This will be proved in Subsection 4.5.2 by constructing the state of stress in an unloaded body in terms of prescribed vectors **c** and **b**. A modified formulation of the reciprocity theorem in a double-connected volume is given in Subsection 4.5.3, whilst Subsections 4.5.4 and 4.5.5 are concerned with the expressions for the strain energy due to distortion. The boundary value problem of the theory of distortion is obtained in Subsection 4.5.6. Examples related to the distortion in a hollow cylinder are studied in Subsection 5.7.3.

4.5.2 Determination of the state of stress in terms of the barrier constants

Let us consider a double-connected volume V_i and a volume V_i^* bounded by surface S. A barrier σ is a part of S and transforms V_i into a single-connected volume. The surface of volume V_i is denoted by O. The volumes outside of O and S are designated respectively as V_e and $V_e^* = V_e + V_i - V_i^*$, see Fig. 4.3.

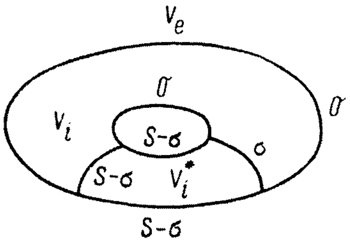

FIGURE 4.3.

Let the barrier constants $\mathbf{c}$ and $\mathbf{b}$ be prescribed. Referring to eqs. (3.7.9) and (3.7.10) we have

$$\delta(Q)(\mathbf{c}+\mathbf{b}\times\mathbf{r}_Q) = -\iint_S (\mathbf{c}+\mathbf{b}\times\mathbf{r}_{M_S})\cdot\hat{\Phi}(M_S,Q)\,do_{M_S}, \tag{5.2.1}$$

where $\delta(Q)=1$ for $Q\subset V_i^*$ and $\delta(Q)=0$ for $Q\subset V_e^*$. The left hand side of this equality can be represented as a jump in the displacement vector

$$\mathbf{v}(Q) = \begin{cases} \mathbf{c}+\mathbf{b}\times\mathbf{r}_Q, & Q\subset V_i^*, \\ 0, & Q\subset V_e^* \end{cases} \tag{5.2.2}$$

when passing over barrier. The strain tensor $\hat{\varepsilon}(\mathbf{v}_Q)$ obtained from this vector vanishes identically

$$\hat{\varepsilon}(\mathbf{v}(Q)) = 0 \quad (Q\subset V_i^*,\ Q\subset S,\ Q\subset V_e^*). \tag{5.2.3}$$

Hence,

$$\mathbf{v}(Q) = -\iint_S (\mathbf{c}+\mathbf{b}\times\mathbf{r}_{M_S})\cdot\hat{\Phi}(M_S,Q)\,do_{M_S}. \tag{5.2.4}$$

The integral on the right hand side is the sum of two integrals

$$\mathbf{u}(Q) = -\iint_\sigma (\mathbf{c}+\mathbf{b}\times\mathbf{r}_{M_\sigma})\cdot\hat{\Phi}(M_S,Q)\,do_{M_\sigma}, \tag{5.2.5}$$

$$\mathbf{u}'(Q) = -\iint_{S-\sigma} (\mathbf{c}+\mathbf{b}\times\mathbf{r}_{M_\sigma})\cdot\hat{\Phi}(M_S,Q)\,do_{M_S}. \tag{5.2.6}$$

Vectors $\mathbf{u}'(Q)$ and $\mathbf{u}(Q)$ remain continuous when passing over σ and $S-\sigma$, respectively. The same is valid for strain tensors $\hat{\varepsilon}(\mathbf{u}'(Q))$ and $\hat{\varepsilon}(\mathbf{u}(Q))$ obtained in terms of $\mathbf{u}'$ and $\mathbf{u}$, respectively. By eq. (5.2.3)

$$\hat{\varepsilon}(\mathbf{v}(Q)) = \hat{\varepsilon}(\mathbf{u}(Q)) + \hat{\varepsilon}(\mathbf{u}'(Q)) = 0.$$

Tensor $\hat{\varepsilon}(\mathbf{u}'(Q))$ remains continuous when passing over σ, hence, tensor $\hat{\varepsilon}(\mathbf{u}(Q))$ is also continuous on σ. Due to the continuity of vector $\mathbf{u}'(Q)$ on σ the vector

$$\mathbf{u}(Q) = \mathbf{v}(Q) - \mathbf{u}'(Q)$$

is continuous everywhere except for barrier σ where it exhibits the same discontinuity as $\mathbf{v}(Q)$. Hence, by eq. (5.2.4)

$$\mathbf{u}^{+}(Q_\sigma) - \mathbf{u}^{-}(Q_\sigma) = \mathbf{c} + \mathbf{b} \times \mathbf{r}_{Q_\sigma}. \tag{5.2.7}$$

Here, according to eq. (2.4.6) of Chapter 2, superscripts + and − indicate values of $\mathbf{u}$ "under" and "over" the barrier, see Fig. 2.2.

By eq. (5.2.5) the displacement vector $\mathbf{u}(Q)$ of particles of the solid is set in the form

$$\mathbf{u}(Q) = -\mathbf{c} \cdot \iint_\sigma \hat{\Phi}(M_\sigma, Q)\, do_{M_\sigma} - \mathbf{b} \cdot \iint_\sigma \mathbf{r}_{M_\sigma} \times \hat{\Phi}(M_\sigma, Q)\, do_{M_\sigma} \tag{5.2.8}$$

and satisfies the equations of the elasticity theory in terms of displacements. The corresponding tensors of strain $\hat{\varepsilon}(\mathbf{u}(Q))$ and stress $\hat{T}(\mathbf{u}(Q))$ are continuous everywhere. Vector $\mathbf{u}(Q)$ is continuous everywhere, except on barrier σ, and experiences the required discontinuity (5.2.7).

Tensor $\hat{T}(\mathbf{u}(Q))$ determines the surface forces $\mathbf{n} \cdot \hat{T}(\mathbf{u}(Q))$ on surface O of the double-connected volume. This system of forces is statically equivalent to zero since $\mathbf{u}(Q)$ results in an equilibrium state of stress.

Let us now determine the state of stress $\hat{T}^*$ in volume V_i caused by the surface forces $-\mathbf{n} \cdot \hat{T}(\mathbf{u}(Q))$ in the case of no distortion. By the theorem of Subsection 4.4.6, such a state of stress exists and is uniquely determined since the sought displacement vector $\mathbf{u}^*$ is continuous and unique and the system of the surface forces $-\mathbf{n} \cdot \hat{T}(\mathbf{u}(Q))$ is statically equivalent to zero. Superimposing the states of stress $\hat{T}(\mathbf{u}(Q))$ and $\hat{T}^*$ presents a state of stress in the double-connected volume determined by the distortion only since the external forces are absent in this state.

4.5.3 The reciprocity theorem

Let us apply Clapeyron's formula (3.3.3) of Chapter 3 to the single-connected elastic solid which is obtained from a double-connected body by means of a barrier

$$2a = \iiint_V \rho \mathbf{K} \cdot \mathbf{u}\, d\tau + \iint_O \mathbf{F} \cdot \mathbf{u}\, do + \iint_{\sigma^+} \mathbf{n}^+ \cdot \hat{T}^+ \cdot \mathbf{u}^+ do + \iint_{\sigma^-} \mathbf{n}^- \cdot \hat{T}^- \cdot \mathbf{u}^- do, \tag{5.3.1}$$

where $\mathbf{n}^+ = -\mathbf{n}^-$ denotes the unit vector of the normal to the barrier directed into the body cut by the barrier. In addition to this, $\hat{T}^+ = \hat{T}^- = \hat{T}$ on the barrier. Referring to eq. (5.2.7) and assuming that there are no external forces we obtain

$$2a = \iint_\sigma \mathbf{n}^+ \cdot \hat{T} \cdot (\mathbf{c} + \mathbf{b} \times \mathbf{r})\, do = \mathbf{Q} \cdot \mathbf{c} + \mathbf{m}^0 \cdot \mathbf{b}. \tag{5.3.2}$$

Here $\mathbf{Q}$ and $\mathbf{m}^0$ denote the principal vector and the principal moment about the origin of the chosen coordinate system of the stresses due to distortion in the selected volume

$$\mathbf{Q} = \iint_\sigma \mathbf{n}^+ \cdot \hat{T} do, \qquad \mathbf{m}^0 = \iint_\sigma \mathbf{r} \times \left(\mathbf{n}^+ \cdot \hat{T}\right) do. \tag{5.3.3}$$

Considering two states of the elastic double-connected volume, namely, the first caused by the mass and surface forces $(\rho\mathbf{K}', \mathbf{F}')$ under distortion $\mathbf{c}', \mathbf{b}'$ and the second with $\rho\mathbf{K}'', \mathbf{F}'', \mathbf{c}'', \mathbf{b}''$, one obtains by virtue of the reciprocity theorem that

$$\iiint_V \rho\mathbf{K}' \cdot \mathbf{u}'' d\tau + \iint_O \mathbf{F}' \cdot \mathbf{u}'' do + \mathbf{Q}' \cdot \mathbf{c}'' + \mathbf{m}^{O'} \cdot \mathbf{b}'' =$$
$$= \iiint_V \rho\mathbf{K}'' \cdot \mathbf{u}' d\tau + \iint_O \mathbf{F}'' \cdot \mathbf{u}' do + \mathbf{Q}'' \cdot \mathbf{c}' + \mathbf{m}^{O''} \cdot \mathbf{b}'. \tag{5.3.4}$$

In particular, when the external forces are absent in the first state and the distortions are absent in the second state we arrive at the following relationship (Colonnetti, 1912)

$$\iiint_V \rho\mathbf{K}'' \cdot \mathbf{u}' d\tau + \iint_O \mathbf{F}'' \cdot \mathbf{u}' do + \mathbf{Q}'' \cdot \mathbf{c}' + \mathbf{m}^{O''} \cdot \mathbf{b}' = 0. \tag{5.3.5}$$

When the external forces are absent in both states, then

$$\mathbf{Q}' \cdot \mathbf{c}'' + \mathbf{m}^{O'} \cdot \mathbf{b}'' = \mathbf{Q}'' \cdot \mathbf{c}' + \mathbf{m}^{O''} \cdot \mathbf{b}'. \tag{5.3.6}$$

In particular, considering the translational distortion $\mathbf{c}$ as the first state and the rotational distortion $\mathbf{b}$ as the second state we have $\mathbf{c}' = \mathbf{c}, \mathbf{b}' = 0$, $\mathbf{c}'' = 0, \mathbf{b}'' = \mathbf{b}$ and, by eq. (5.3.6)

$$\mathbf{b} \cdot \mathbf{m}_*^O = \mathbf{c} \cdot \mathbf{Q}_*, \tag{5.3.7}$$

where $\mathbf{Q}_*$ and $\mathbf{m}_*^O$ are the principal vector and the principal vector of the stresses due to the rotational ($\mathbf{b}$) and translational ($\mathbf{c}$) distortions respectively.

4.5.4 The strain energy of distortion

In a linear elastic solid the principal vector $\mathbf{Q}$ and the principal moment $\mathbf{m}^0$ of the stresses on the barrier due to the distortion are linear vectors determining the distortion of vectors $\mathbf{c}$ and $\mathbf{b}$

$$\mathbf{Q} = \hat{C} \cdot \mathbf{c} + \hat{M} \cdot \mathbf{b}, \qquad \mathbf{m}^0 = \hat{N} \cdot \mathbf{c} + \hat{B} \cdot \mathbf{b}, \tag{5.4.1}$$

where $\hat{C}, \hat{M}, \hat{N}, \hat{B}$ are tensors of the second rank. Terms $\hat{M} \cdot \mathbf{b}$ and $\hat{N} \cdot \mathbf{c}$ present the vectors denoted in eqs. (5.3.7) as $\mathbf{Q}_*$ and $\mathbf{m}_*^O$ respectively. Hence

$$\mathbf{c} \cdot \hat{M} \cdot \mathbf{b} = \mathbf{b} \cdot \hat{N} \cdot \mathbf{c}, \tag{5.4.2}$$

that is, tensor $\hat{N}$ is a transpose of $\hat{M}$

$$\hat{N} = \hat{M}^*. \tag{5.4.3}$$

The strain energy of distortion (5.3.2) is now set in the form

$$a = \frac{1}{2} \left(\mathbf{c} \cdot \hat{C} \cdot \mathbf{c} + 2\mathbf{c} \cdot \hat{M} \cdot \mathbf{b} + \mathbf{b} \cdot \hat{B} \cdot \mathbf{b} \right). \tag{5.4.4}$$

In general, this expression contains 21 constants since tensors $\hat{C}$ and $\hat{B}$ in eq. (5.4.1) are symmetric which follows from the reciprocity theorem. Indeed, considering two states, for instance, translational distortion on the same barrier, we have

$$\begin{aligned} \mathbf{c}' &= c_1 \mathbf{i}_1, \quad \mathbf{b}' = 0, \quad \mathbf{Q}' = \hat{C} \cdot \mathbf{c}' = c_1 \hat{C} \cdot \mathbf{i}_1, \\ \mathbf{c}'' &= c_2 \mathbf{i}_2, \quad \mathbf{b}'' = 0, \quad \mathbf{Q}'' = \hat{C} \cdot \mathbf{c}'' = c_2 \hat{C} \cdot \mathbf{i}_2 \end{aligned}$$

and by eq. (5.3.6)

$$\mathbf{i}_1 \cdot \hat{C} \cdot \mathbf{i}_2 = \mathbf{i}_2 \cdot \hat{C} \cdot \mathbf{i}_1,$$

which is what we set out to prove. Notice, that the value of the potential energy of the distortion depends, generally speaking, on choice of the barrier, i.e. on the location of the distortion.

4.5.5 The case of a body of revolution

Let axis Ox_3 be the axis of revolution and the barrier be a plane region σ of intersection of the body by the meridional plane. Let σ_0 denote the barrier formed by plane Ox_3x_1. We introduce into consideration the trihedron of unit vectors of the cylindrical coordinate system $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{k}$, see Section C.7. Because of the symmetry, the state of stress on barrier σ_0 due to distortion $\mathbf{c}^0, \mathbf{b}^0$ coincides with that due to $\mathbf{c}, \mathbf{b}$ provided that the position of vectors

$\mathbf{c}, \mathbf{b}$ with respect to axes $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{k}$ is coincident with that of vectors $\mathbf{c}^0, \mathbf{b}^0$ with respect to axes $\mathbf{e}_r^0, \mathbf{e}_\varphi^0, \mathbf{k}$. Introducing the rotation tensors, see Section A.8

$$\hat{A} = \mathbf{e}_r^0 \mathbf{e}_r + \mathbf{e}_\varphi^0 \mathbf{e}_\varphi + \mathbf{kk}, \qquad \hat{A}^* = \mathbf{e}_r \mathbf{e}_r^0 + \mathbf{e}_\varphi \mathbf{e}_\varphi^0 + \mathbf{kk} \tag{5.5.1}$$

we have

$$\mathbf{c} = \hat{A}^* \cdot \mathbf{c}^0 = \mathbf{c}^0 \cdot \hat{A}, \qquad \mathbf{b} = \hat{A}^* \cdot \mathbf{b}^0 = \mathbf{b}^0 \cdot \hat{A}. \tag{5.5.2}$$

Let us insert the values of $\mathbf{c}^0, \mathbf{b}^0$ into the expression for the strain energy of distortion which is evidently the same under these two distortions. By eq. (5.4.4) we have

$$\begin{aligned} 2a &= \mathbf{c}^0 \cdot \hat{C}^0 \cdot \mathbf{c}^0 + 2\mathbf{c}^0 \cdot \hat{M}^0 \cdot \mathbf{b}^0 + \mathbf{b}^0 \cdot \hat{B}^0 \cdot \mathbf{b}^0 \\ &= \mathbf{c} \cdot \hat{C} \cdot \mathbf{c} + 2\mathbf{c} \cdot \hat{M} \cdot \mathbf{b} + \mathbf{b} \cdot \hat{B} \cdot \mathbf{b} \qquad\qquad (5.5.3) \\ &= \mathbf{c}^0 \cdot \hat{A} \cdot \hat{C} \cdot \hat{A}^* \cdot \mathbf{c}^0 + 2\mathbf{c}^0 \cdot \hat{A} \cdot \hat{M} \cdot \hat{A}^* \cdot \mathbf{b}^0 + \mathbf{b}^0 \cdot \hat{A} \cdot \hat{B} \cdot \hat{A}^* \cdot \mathbf{b}^0, \end{aligned}$$

so that

$$\hat{C}^0 = \hat{A} \cdot \hat{C} \cdot \hat{A}^*, \quad \hat{M}^0 = \hat{A} \cdot \hat{M} \cdot \hat{A}^*, \quad \hat{B}^0 = \hat{A} \cdot \hat{B} \cdot \hat{A}^*. \tag{5.5.4}$$

The constant tensors $\hat{C}, \hat{M}, \hat{B}$ are independent of angle φ (i.e. of the position of the barrier) which follows from the second expression for the strain energy (5.5.3).

By virtue of eqs. (5.5.1) and (C.7.3) we have

$$\frac{d\hat{A}}{d\varphi} = \mathbf{e}_r^0 \mathbf{e}_\varphi - \mathbf{e}_\varphi^0 \mathbf{e}_r, \quad \frac{d\hat{A}^*}{d\varphi} = \mathbf{e}_\varphi \mathbf{e}_r^0 - \mathbf{e}_r \mathbf{e}_\varphi^0$$

which allows the derivative of tensor $\hat{A} \cdot \hat{P} \cdot \hat{A}^*$ with respect to φ ($\hat{P}$ is a constant tensor of the second rank, for example $\hat{C}, \hat{M}$ or $\hat{B}$) to be determined

$$\begin{aligned} \frac{d}{d\varphi} \hat{A} \cdot \hat{P} \cdot \hat{A}^* &= \left(\mathbf{e}_r^0 \mathbf{e}_\varphi - \mathbf{e}_\varphi^0 \mathbf{e}_r\right) \cdot \hat{P} \cdot \hat{A}^* + \hat{A} \cdot \hat{P} \cdot \left(\mathbf{e}_\varphi \mathbf{e}_r^0 - \mathbf{e}_r \mathbf{e}_\varphi^0\right) \\ &= \left(\mathbf{e}_r^0 \mathbf{e}_r^0 - \mathbf{e}_\varphi^0 \mathbf{e}_\varphi^0\right)(P_{21} + P_{12}) + \left(\mathbf{e}_\varphi^0 \mathbf{e}_r^0 + \mathbf{e}_r^0 \mathbf{e}_\varphi^0\right)(P_{22} - P_{11}) + \\ &\qquad \mathbf{e}_r^0 \mathbf{k} P_{23} + \mathbf{k} \mathbf{e}_r^0 P_{32} - \mathbf{e}_\varphi^0 \mathbf{k} P_{13} - \mathbf{k} \mathbf{e}_\varphi^0 P_{31}. \end{aligned}$$

The conditions when this tensor is zero are set as follows

$$P_{11} = P_{22}, \quad P_{21} = -P_{12}, \quad P_{23} = P_{32} = P_{13} = P_{31} = 0,$$

so that

$$\hat{P} = P_{11} \left(\mathbf{e}_r \mathbf{e}_r + \mathbf{e}_\varphi \mathbf{e}_\varphi\right) + P_{33} \mathbf{kk} + P_{12} \left(\mathbf{e}_r \mathbf{e}_\varphi - \mathbf{e}_\varphi \mathbf{e}_r\right),$$

and if tensor $\hat{P}$ is symmetric, i.e. $\hat{P} = \hat{P}^*$, then $P_{12} = 0$. Being applied to $\hat{C}, \hat{B}$ and $\hat{M}$ the latter formula yields

$$\left.\begin{aligned} \hat{C} &= C_{11}\left(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\varphi\mathbf{e}_\varphi\right) + C_{33}\mathbf{kk}, \\ \hat{B} &= B_{11}\left(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\varphi\mathbf{e}_\varphi\right) + B_{33}\mathbf{kk}, \\ \hat{M} &= M_{11}\left(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\varphi\mathbf{e}_\varphi\right) + M_{33}\mathbf{kk} + M_{12}\left(\mathbf{e}_r\mathbf{e}_\varphi - \mathbf{e}_\varphi\mathbf{e}_r\right) \end{aligned}\right\} \tag{5.5.5}$$

and the strain energy of the distortion (5.4.4) is as follows

$$\begin{aligned} 2a = C_{11}\left(c_1^2 + c_2^2\right) + C_{33}c_3^2 + 2M_{11}\left(b_1c_1 + b_2c_2\right) + \\ 2M_{12}\left(c_1b_2 - c_2b_1\right) + 2M_{33}c_3b_3 + B_{11}\left(b_1^2 + b_2^2\right) + B_{33}b_3^2. \end{aligned} \tag{5.5.6}$$

A further simplification is possible if one takes into account the indifference of the strain energy of the distortion with respect to rotational b_3 and translational c_2 distortions.

Let the only non-vanishing components be b_3 and c_3. Then by eq. (5.5.6)

$$2a = C_{33}c_3^2 + 2M_{33}c_3b_3 + B_{33}b_3^2$$

and this expression must not change under changes in the sign of the relative rotation b_3 of the connected ends about the symmetry axis. For this reason $M_{33} = 0$. The same reasoning can be applied to the case of $c_2 \neq 0, b_2 \neq 0$ and yields $M_{11} = 0$, since changing the sign of the translational distortion c_2 (perpendicular to the barrier) has also no influence on the value of the strain energy of the distortion. Thus

$$M_{33} = 0, \qquad M_{11} = 0. \tag{5.5.7}$$

Now let only $c_2 \neq 0$. Because of the symmetry the shear stresses $t_{12} = \tau_{r\varphi}$ and $t_{23} = \tau_{\varphi z}$ are absent in this state of stress, that is due to eq. (5.3.3)

$$\begin{aligned} \mathbf{Q} = \mathbf{e}_\varphi \iint\limits_\sigma t_{22}do = Q_2\mathbf{e}_\varphi, \quad \mathbf{m}^O &= \iint\limits_\sigma \left(r\mathbf{e}_r + x_3\mathbf{k}\right) \times \mathbf{e}_\varphi t_{22}do \\ &= \mathbf{k}\iint\limits_\sigma rt_{22}do - \mathbf{e}_r \iint\limits_\sigma x_3t_{22}do = \mathbf{k}m_3^0 - \mathbf{e}_r m_1^0. \end{aligned} \tag{5.5.8}$$

Using eqs. (5.4.1), (5.5.5) and (5.5.6) we have

$$\begin{aligned} \mathbf{Q} = C_{11}c_2\mathbf{e}_\varphi = Q_2\mathbf{e}_\varphi, \quad \mathbf{m}^O = M_{12}\left(\mathbf{e}_\varphi\mathbf{e}_r - \mathbf{e}_r\mathbf{e}_\varphi\right)\cdot\mathbf{c} = -M_{12}c_2\mathbf{e}_r, \\ a = \frac{1}{2}C_{11}c_2^2, \end{aligned} \tag{5.5.9}$$

which means that $Q_2 \neq 0$ and moment $\mathbf{m}^O$ has the direction of $\mathbf{e}_r$, i.e. $m_2^0 = m_3^0 = 0$.

Let the centre of moments O be placed at point O_* of axis Ox_3, then by eq. (5.5.8)

$$x_3 = x_3^* + h, \qquad m_1^0 = m_1^{O_*} + hQ_2,$$

and h can be taken such that $m_1^{O_*} = 0$, which yields

$$h = \frac{m_1^0}{Q_2}.$$

Volterra referred to this point O_* on axis x_3 as central. Provided that the body has a symmetry plane which is perpendicular to the axis of revolution, the point of intersection of this axis with the symmetry plane is the central one. Choosing this as being the centre of moments we have by means of eq. (5.5.8) that

$$\mathbf{m}^{O_*} = -M_{12} c_2 \mathbf{e}_r = 0, \qquad M_{12} = 0, \tag{5.5.10}$$

and, by eq. (5.5.7), tensor $\hat{M}$ vanishes. Expression (5.5.6) for the strain energy of distortion is reduced to the form

$$a = \frac{1}{2}\left[C_{11}\left(c_1^2 + c_2^2\right) + C_{33}c_3^2 + B_{11}\left(b_1^2 + b_2^2\right) + B_{33}b_3^2\right]. \tag{5.5.11}$$

Only four constants appear here. By virtue of eq. (5.4.1) the principal vector and the principal moment of the stresses in the meridional section about the central point are given by

$$\left.\begin{aligned} \mathbf{Q} &= \hat{C}\cdot\mathbf{c} = C_{11}\left(c_1\mathbf{e}_r + c_2\mathbf{e}_\varphi\right) + C_{33}c_3\mathbf{k}, \\ \mathbf{m}^{O_*} &= \hat{B}\cdot\mathbf{b} = B_{11}\left(b_1\mathbf{e}_r + b_2\mathbf{e}_\varphi\right) + B_{33}b_3\mathbf{k}. \end{aligned}\right\} \tag{5.5.12}$$

Thus, in any elastic double-connected solid possessing rotational symmetry, any elementary distortion results in a corresponding force provided that the central point is taken as the centre of moments. The stresses caused by a translational distortion are statically equivalent to a resultant force through the central point while those due to a rotational distortion are statically equivalent to a moment.

4.5.6 Boundary value problem for a double-connected body of revolution

The following vector

$$v_* = \frac{1}{2\pi}\left(\mathbf{c} + \mathbf{b}\times\mathbf{R}\right)\varphi \quad \left(\varphi = \arctan\frac{x_2}{x_1},\ \mathbf{R} = \mathbf{e}_r r + \mathbf{k}x_3 = \mathbf{i}_s x_s\right) \tag{5.6.1}$$

is multiple-valued (which is required in the case of a body of revolution). However the strains calculated by this vector are single-valued and continuous in the region except for axis x_3. Indeed, the gradient of this vector and its transpose are equal to

$$\left.\begin{aligned} \nabla v_* &= \frac{1}{2\pi}\left[-\hat{E}\times\mathbf{b}\varphi+\frac{1}{r}\mathbf{e}_\varphi(\mathbf{c}+\mathbf{b}\times\mathbf{R})\right], \\ (\nabla v_*)^* &= \frac{1}{2\pi}\left[\mathbf{b}\times\hat{E}\varphi+\frac{1}{r}(\mathbf{c}+\mathbf{b}\times\mathbf{R})\mathbf{e}_\varphi\right] \end{aligned}\right\} \tag{5.6.2}$$

and furthermore

$$\operatorname{def}\mathbf{v}_* = \frac{1}{4\pi r}(\mathbf{e}_\varphi\mathbf{c}+\mathbf{c}\mathbf{e}_\varphi+\mathbf{e}_\varphi\mathbf{b}\times\mathbf{R}+\mathbf{b}\times\mathbf{R}\mathbf{e}_\varphi), \tag{5.6.3}$$

so that

$$\mathbf{b}\times\hat{E}-\hat{E}\times\mathbf{b} = e_{kst}(\mathbf{i}_t\mathbf{i}_s+\mathbf{i}_s\mathbf{i}_t)b_k = 0. \tag{5.6.4}$$

Vector $\mathbf{v}_*$ does not satisfy the homogeneous equations of equilibrium in terms of displacements. Hence we introduce a correcting vector $\mathbf{v}$ which is single-valued and continuous in the region with excluded axis Ox_3 and require that

$$\mathbf{u} = \mathbf{v}_* + \mathbf{v} \tag{5.6.5}$$

is the particular solution of these equations. Direct calculation shows that this solution is as follows

$$\mathbf{u} = \frac{1}{2\pi}\left\{(\mathbf{c}+\mathbf{b}\times\mathbf{R})\varphi+\left[\mathbf{k}\times\mathbf{c}+(\mathbf{k}\times\mathbf{b})\times\mathbf{R}+\frac{1-2\nu}{2(1-\nu)}b_3 r\mathbf{e}_r\right]\ln r\right\}. \tag{5.6.6}$$

The stress tensor corresponding to this vector is equal to

$$\hat{T}(\mathbf{u}) = \frac{\mu}{2\pi}\left\{b_3\left[\frac{2\nu}{1-2\nu}+\frac{\nu}{1-\nu}(1+2\ln r)\right]\hat{E}+\right. \tag{5.6.7}$$
$$\frac{1}{r}\left[\mathbf{e}_\varphi\mathbf{c}+\mathbf{c}\mathbf{e}_\varphi+\mathbf{e}_\varphi\mathbf{b}\times\mathbf{R}+\mathbf{b}\times\mathbf{R}\mathbf{e}_\varphi+\mathbf{e}_r\mathbf{k}\times\mathbf{c}+\mathbf{k}\times\mathbf{c}\mathbf{e}_r+\right.$$
$$\left.\left.\mathbf{e}_r(\mathbf{k}\times\mathbf{b})\times\mathbf{R}+(\mathbf{k}\times\mathbf{b})\times\mathbf{R}\mathbf{e}_r\right]+\frac{1-2\nu}{1-\nu}b_3\left[\mathbf{e}_r\mathbf{e}_r(1+\ln r)+\mathbf{e}_\varphi\mathbf{e}_\varphi\ln r\right]\right\}.$$

The boundary value problem of Volterra's theory of distortion reduces to obtaining the displacement vector $\mathbf{U}$ from the homogeneous equations of equilibrium and the following boundary condition

$$\mathbf{n}\cdot\hat{T}(\mathbf{U}) = -\mathbf{n}\cdot\hat{T}(\mathbf{u}) \tag{5.6.8}$$

on surface O of the double-connected volume.

The formulae obtained are cumbersome because the general case of distortion is considered. In the case of rotational distortion about the axis of symmetry in which only $b_3 \neq 0$ we have

$$\mathbf{u} = \frac{1}{2\pi} b_3 \left(r\varphi \mathbf{e}_\varphi + \frac{1-2\nu}{2(1-\nu)} r \ln r \mathbf{e}_r \right), \tag{5.6.9}$$

and only normal stresses appear

$$\left.\begin{aligned} \sigma_r &= \frac{\mu}{2\pi} b_3 \left(\frac{1}{1-2\nu} + \frac{1}{1-\nu} \ln r \right), \\ \sigma_\varphi &= \frac{\mu}{2\pi} b_3 \left(\frac{2-3\nu}{(1-\nu)(1-2\nu)} + \frac{1}{1-\nu} \ln r \right), \\ \sigma_z &= \frac{\mu}{2\pi} b_3 \left(\frac{3\nu - 4\nu^2}{(1-\nu)(1-2\nu)} + \frac{2\nu}{1-\nu} \ln r \right). \end{aligned}\right\} \tag{5.6.10}$$

Simple formulae are also obtained for the translational distortion c_2

$$\left.\begin{aligned} &\mathbf{u} = \frac{c_2}{2\pi} \left[(\mathbf{e}_r \sin\varphi + \mathbf{e}_\varphi \cos\varphi)\, \varphi - (\mathbf{e}_r \cos\varphi - \mathbf{e}_\varphi \sin\varphi) \ln r \right], \\ &\sigma_r = -\frac{\mu}{\pi} c_2 \frac{\cos\varphi}{r}, \quad \sigma_\varphi = \frac{\mu}{\pi} c_2 \frac{\cos\varphi}{r}, \quad \tau_{r\varphi} = \frac{\mu}{\pi} c_2 \frac{\sin\varphi}{r}. \end{aligned}\right\} \tag{5.6.11}$$

The solution of these problems for a hollow cylinder is obtained in Subsection 5.7.3.

Part III

Special problems of the linear theory of elasticity

5

Three-dimensional problems in the theory of elasticity

5.1 Unbounded elastic medium

5.1.1 Singularities due to concentrated forces

The displacement of the "point of observation" M in an unbounded elastic medium subjected to a concentrated force $\mathbf{P}$ applied at the "point of source" Q is determined by means of the Kelvin-Somigliana formula, eq. (3.5.9) of Chapter 4,

$$\mathbf{u}\,(M,Q) = \hat{U}\,(M,Q)\cdot\mathbf{P}. \qquad (1.1.1)$$

Here

$$\hat{U} = \frac{1}{16\pi\mu\,(1-\nu)\,R}\left[(3-4\nu)\,\hat{E} + \frac{\mathbf{RR}}{R^2}\right], \quad \mathbf{R} = \mathbf{r}_M - \mathbf{r}_Q, \;\; R = \left|\mathbf{r}_M - \mathbf{r}_Q\right|. \qquad (1.1.2)$$

Placing point Q' close to Q such that

$$\mathbf{r}'_Q = \mathbf{r}_Q + \boldsymbol{\rho} \qquad (1.1.3)$$

and carrying out the calculation up to terms of the first order in ρ we have

$$\left.\begin{array}{c} \mathbf{R}' = \mathbf{r}_M - \mathbf{r}_{Q'} = \mathbf{R} - \boldsymbol{\rho}, \\ R' = \left|\mathbf{R} - \boldsymbol{\rho}\right| = \left(R^2 - 2\mathbf{R}\cdot\boldsymbol{\rho} + \rho^2\right)^{1/2} \approx R\left(1 - \dfrac{\mathbf{R}\cdot\boldsymbol{\rho}}{R^2}\right). \end{array}\right\} \qquad (1.1.4)$$

Under this displacement of the source point the Kelvin-Somigliana tensor and the displacement vector are presented in the form

$$\hat{U}(M, Q') = \hat{U}(M, Q) + \frac{1}{16\pi\mu(1-\nu)R^3}\left[(3-4\nu)\hat{E}\mathbf{R}\cdot\boldsymbol{\rho} - \mathbf{R}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{R} + 3\frac{\mathbf{R}\mathbf{R}}{R^2}\mathbf{R}\cdot\boldsymbol{\rho}\right], \quad (1.1.5)$$

$$\mathbf{u}(M, Q') = \mathbf{u}(M, Q) + \frac{1}{16\pi\mu(1-\nu)R^3}\times \left[(3-4\nu)\mathbf{R}\cdot\boldsymbol{\rho}\mathbf{P} - \mathbf{R}\boldsymbol{\rho}\cdot\mathbf{P} - \boldsymbol{\rho}\mathbf{R}\cdot\mathbf{P} + 3\mathbf{R}\frac{\mathbf{R}\cdot\boldsymbol{\rho}\mathbf{R}\cdot\mathbf{P}}{R^2}\right]. \quad (1.1.6)$$

In these expressions dyadic $\boldsymbol{\rho}\mathbf{R}$ is presented as a sum of the symmetric and skew-symmetric parts, the symmetric part, in turn, being a sum of the deviator and the spherical tensor

$$\boldsymbol{\rho}\mathbf{R} = \hat{p} + \hat{\Omega}, \quad \hat{p} = \frac{1}{2}(\boldsymbol{\rho}\mathbf{P} + \mathbf{P}\boldsymbol{\rho}), \quad \hat{\Omega} = \frac{1}{2}(\boldsymbol{\rho}\mathbf{P} - \mathbf{P}\boldsymbol{\rho}), \quad (1.1.7)$$

$$\hat{p} = \operatorname{Dev}\hat{p} + \frac{1}{3}\hat{E}I_1(\hat{p}) = \operatorname{Dev}\hat{p} + \frac{1}{3}\boldsymbol{\rho}\cdot\mathbf{P}\hat{E}. \quad (1.1.8)$$

Taking into account the following relationships

$$\mathbf{R}\cdot\hat{\Omega}\cdot\mathbf{R} = 0, \quad \mathbf{R}\cdot\boldsymbol{\rho}\mathbf{P}\cdot\mathbf{R} = \mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R} + \frac{1}{3}R^2\boldsymbol{\rho}\cdot\mathbf{P},$$

$$\mathbf{R}\cdot\hat{\Omega} = \frac{1}{2}(\mathbf{R}\cdot\boldsymbol{\rho}\mathbf{P} - \mathbf{R}\cdot\mathbf{P}\boldsymbol{\rho}) = \frac{1}{2}(\boldsymbol{\rho}\times\mathbf{P})\times\mathbf{R} = \frac{1}{2}\mathbf{m}^Q(\mathbf{P})\times\mathbf{R},$$

where $\mathbf{m}^Q(\mathbf{P})$ denotes the moment of force $\mathbf{P}$ about point Q one can set formula (1.1.6) in the form

$$\mathbf{u}(M, Q') = \mathbf{u}(M, Q) + \frac{1}{8\pi\mu R^3}\mathbf{m}^Q(\mathbf{P})\times\mathbf{R} + \frac{1-2\nu}{24\pi\mu(1-\nu)}\boldsymbol{\rho}\cdot\mathbf{P}\frac{\mathbf{R}}{R^3} + \frac{1-2\nu}{8\pi\mu(1-\nu)R^3}\left[\mathbf{R}\cdot\operatorname{Dev}\hat{p} + \frac{3}{2(1-2\nu)R^2}\mathbf{R}\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}\right]. \quad (1.1.9)$$

Let us assume that $\rho \to 0$ and $\mathbf{P} \to \infty$ so that the components of dyadic $\boldsymbol{\rho}\mathbf{P}$ are finite and are referred to $\hat{p}, \mathbf{m}^Q(\mathbf{P}), \boldsymbol{\rho}\cdot\mathbf{P}$ at point Q as a result of the limit process as the force tensor, concentrated moment and the intensity of the centre of expansion, respectively. Introduction of these "force singularities" allows the following interpretation of each term in eq. (1.1.9):

a) the displacement caused by force $\mathbf{P}$ at point Q

$$\mathbf{u}_1(M, Q) = \hat{U}(M, Q)\cdot\mathbf{P}; \quad (1.1.10)$$

b) the displacement due to the concentrated moment at point Q

$$\mathbf{u}_2\left(M,Q\right)=\frac{1}{8\pi\mu R^3}\mathbf{m}^Q\left(\mathbf{P}\right)\times\mathbf{R};\tag{1.1.11}$$

c) the displacement due to the centre of expansion at point Q

$$\mathbf{u}_3\left(M,Q\right)=\frac{1-2\nu}{24\pi\mu\left(1-\nu\right)R^3}\boldsymbol{\rho}\cdot\mathbf{P}\frac{\mathbf{R}}{R^3}=-\frac{1-2\nu}{24\pi\mu\left(1-\nu\right)}\boldsymbol{\rho}\cdot\mathbf{P}\nabla\frac{1}{R};\tag{1.1.12}$$

d) the displacement caused by the force tensor

$$\mathbf{u}_4\left(M,Q\right)=\frac{1-2\nu}{8\pi\mu\left(1-\nu\right)R^3}\left[\mathbf{R}\cdot\operatorname{Dev}\hat{p}+\frac{3}{2\left(1-2\nu\right)R^2}\mathbf{RR}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}\right].\tag{1.1.13}$$

With the growth of distance R from the source point the displacement caused by a concentrated force decreases as R^{-1}, while that due other concentrated singularities decreases as R^{-2}.

5.1.2 The system of forces distributed in a small volume. Lauricella's formula

Let a system of forces $\mathbf{P}_1,\mathbf{P}_2,\dots,\mathbf{P}_n$ be applied in the vicinity of point Q at points $Q_1,Q_2,\dots,Q_n$ with the position vectors $\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\dots,\boldsymbol{\rho}_n$, each having the origin at point Q. The displacement of point M is thus a geometric sum of displacements (1.1.9) corresponding to each individual force. We introduce into consideration:

a) the principal vector $\mathbf{P}$ of the system of forces

$$\mathbf{P}=\sum_{i=1}^{n}\mathbf{P}_i,\tag{1.2.1}$$

b) the principal moment about point Q

$$\mathbf{m}^Q=\sum_{i=1}^{n}\mathbf{m}^Q\left(\mathbf{P}_i\right),\tag{1.2.2}$$

c) the tensor of the system of forces

$$\hat{p}=\sum_{i=1}^{n}\hat{p}_i=\frac{1}{2}\sum_{i=1}^{n}\left(\boldsymbol{\rho}_i\mathbf{P}_i+\mathbf{P}_i\boldsymbol{\rho}_i\right)\tag{1.2.3}$$

and d) its first invariant

$$I_1\left(\hat{p}\right)=\sum_{i=1}^{n}I_1\left(\hat{p}_i\right)=\sum_{i=1}^{n}\boldsymbol{\rho}_i\cdot\mathbf{P}_i.\tag{1.2.4}$$

If the forces are distributed along a line, on a surface or in a volume, the above sums are replaced by the corresponding integrals.

The displacement vector at point M is presented in the form

$$\mathbf{u}(M,Q) = \hat{U}(M,Q)\cdot\mathbf{P} + \frac{1}{8\pi\mu R^3}\mathbf{m}^Q\times\mathbf{R} + \frac{1-2\nu}{24\pi\mu(1-\nu)R^3}I_1(\hat{p})\mathbf{R} + \frac{1-2\nu}{8\pi\mu(1-\nu)R^3}\left[\mathbf{R}\cdot\operatorname{Dev}\hat{p} + \frac{3}{2(1-2\nu)R^2}\mathbf{RR}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}\right]. \quad (1.2.5)$$

Let us consider the case of a force dipole, which is a system of two equal but oppositely directed forces with a common action line. Let the direction of this line be given by unit vector $\mathbf{e}$. then

$$\mathbf{P}_1 = -\mathbf{e}P, \quad \mathbf{P}_2 = \mathbf{e}P, \quad \boldsymbol{\rho}_1 = 0, \quad \boldsymbol{\rho}_2 = \mathbf{e}\rho,$$

where the product $\rho P = \sigma$ and the tensor $\sigma\mathbf{ee}$ are named the intensity and the moment of the dipole, respectively. In formulae (1.2.1)-(1.2.4)

$$\mathbf{P} = 0, \quad \mathbf{m}^Q = 0, \quad \hat{p} = \sigma\mathbf{ee}, \quad I_1(\hat{p}) = \sigma, \quad \operatorname{Dev}\hat{p} = \left(\mathbf{ee} - \frac{1}{3}\hat{E}\right)\sigma \quad (1.2.6)$$

and, by virtue of eq. (1.2.5), the displacement at point M caused by the dipole at point Q is as follows

$$\mathbf{u}(M,Q) = \frac{\sigma}{8\pi\mu(1-\nu)R^3}\left\{\frac{1}{2}\left[3\frac{(\mathbf{R}\cdot\mathbf{e})^2}{R^2} - 1\right]\mathbf{R} + (1-2\nu)\,\mathbf{eR}\cdot\mathbf{e}\right\}. \quad (1.2.7)$$

The force tensor defined by three dipoles of equal intensity σ in three mutually orthogonal directions is an spherical one:

$$\hat{p} = \sigma(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3) = \sigma\hat{E}, \quad I_1(p) = 3\sigma = q, \quad \operatorname{Dev}\hat{p} = 0.$$

Such a singularity is referred to as the centre of expansion and its intensity is denoted by q. Due to eq. (1.2.5) the corresponding displacement is equal to

$$\mathbf{u}(M,Q) = \frac{(1-2\nu)q}{24\pi\mu(1-\nu)R^3}\mathbf{R} = -\frac{1-2\nu}{24\pi\mu(1-\nu)}q\nabla\frac{1}{R}. \quad (1.2.8)$$

The state of stress caused by the centre of expansion is thus given by

$$\left.\begin{aligned} \hat{\varepsilon} &= -\frac{(1-2\nu)q}{24\pi\mu(1-\nu)}\nabla\nabla\frac{1}{R}, \quad \vartheta = -\frac{(1-2\nu)q}{24\pi\mu(1-\nu)}\nabla^2\frac{1}{R} = 0, \\ \hat{T} &= -\frac{1-2\nu}{12\pi(1-\nu)}q\nabla\nabla\frac{1}{R} = \frac{1-2\nu}{12\pi(1-\nu)}\frac{q}{R^3}\left(\hat{E} - 3\frac{\mathbf{RR}}{R^2}\right). \end{aligned}\right\} \quad (1.2.9)$$

The components of the stress tensor in the spherical coordinate system, see eq. (1.9.4) of Chapter 4, can be written in the form

$$\sigma_R = -\frac{1-2\nu}{6\pi(1-\nu)}\frac{q}{R^3}, \quad \sigma_\vartheta = \sigma_\lambda = \frac{1-2\nu}{12\pi(1-\nu)}\frac{q}{R^3}, \quad \tau_{R\vartheta} = \tau_{\vartheta\lambda} = \tau_{\lambda R} = 0. \tag{1.2.10}$$

Such a state of stress occurs in a solid with a cavity of radius R_0 whose surface is loaded by a normal stress of intensity

$$p = \frac{1-2\nu}{6\pi(1-\nu)}\frac{q}{R_0^3}.$$

For this radial-symmetric state of stress the displacements and the stresses are as follows

$$u_R = \frac{pR_0^3}{4\mu R^2}, \quad u_\vartheta = u_\lambda = 0, \quad \sigma_R = -p\frac{R_0^3}{R^3}, \quad \sigma_\vartheta = \sigma_\lambda = p\frac{R_0^3}{2R^3}. \tag{1.2.11}$$

If three dipoles correspond to the mutually orthogonal directions and the sum of their intensities vanish, then the displacement due to this system is determined only by the fourth term in eq. (1.2.5) since in this case the force tensor is a deviator.

The displacement caused by a pair is not equal to the displacement due to its moment since the second term in eq. (1.2.5) presents the displacement due to the pairs with the vanishing force tensor. This singularity is referred to as the centre of rotation and it can be imagined to be a set of four forces which are equal in magnitude, lie in the same plane and forming the pairs of the same direction of rotation, i.e.

$$\begin{array}{llll} \mathbf{P}_1 = -\mathbf{e}_2 P, & \mathbf{P}_2 = \mathbf{e}_2 P, & \boldsymbol{\rho}_1 = 0, & \boldsymbol{\rho}_2 = h\mathbf{e}_1, \\ \mathbf{P}_3 = \mathbf{e}_1 P, & \mathbf{P}_4 = -\mathbf{e}_1 P, & \boldsymbol{\rho}_3 = 0, & \boldsymbol{\rho}_4 = h\mathbf{e}_2. \end{array}$$

For such a system of forces

$$\mathbf{P} = 0, \ \mathbf{m}^Q = \boldsymbol{\rho}_2 \times \mathbf{P}_2 + \boldsymbol{\rho}_4 \times \mathbf{P}_4 = 2hP\mathbf{e}_1 \times \mathbf{e}_2 = 2hP\mathbf{e}_3 = m\mathbf{e}_3, \ \hat{p} = 0,$$

where m denotes the algebraic sum of moments of the pairs. By eq. (1.2.5) the displacement given by the centre of rotation is as follows

$$\mathbf{u}(M,Q) = \frac{m}{8\pi\mu R^3}\mathbf{e}_3 \times \mathbf{R} = \frac{R_0^3}{R^3}\boldsymbol{\theta} \times \mathbf{R} = -R_0^3\boldsymbol{\theta} \times \nabla\frac{1}{R}; \ \boldsymbol{\theta} = \frac{m}{8\pi\mu R_0^3}\mathbf{e}_3. \tag{1.2.12}$$

Such a distribution of the displacement takes place in an elastic medium provided that a rigid sphere of radius R_0 placed in the medium experiences a rotation described by vector $\boldsymbol{\theta}$. The simple elastostatic Robin's problem

(Subsection 4.4.7) yields this solution. The stress vector on the elementary surface with the normal vector $\mathbf{n}$ is equal to

$$\mathbf{n} \cdot \hat{T} = 3\mu \frac{R_0^3}{R^3} (\boldsymbol{\theta} \times \mathbf{n} - 2\boldsymbol{\theta} \times \mathbf{e}_R \mathbf{n} \cdot \mathbf{e}_R + \mathbf{n} \times \mathbf{e}_R \boldsymbol{\theta} \cdot \mathbf{e}_R) \tag{1.2.13}$$

and on the surface of sphere $R = R_0$ with the external normal vector $\mathbf{n} = \mathbf{e}_R$

$$\left(\mathbf{n} \cdot \hat{T}\right)_{R=R_0} = 3\mu \mathbf{n} \times \boldsymbol{\theta}. \tag{1.2.14}$$

The principal vector of this system of forces vanishes and the principal moment is given by

$$\begin{aligned}\mathbf{m}^Q &= 3\mu \iint\limits_O \mathbf{R} \times (\mathbf{n} \times \boldsymbol{\theta})\, do = 3\mu \iint\limits_O (\mathbf{n}\mathbf{R} \cdot \boldsymbol{\theta} - \boldsymbol{\theta} R)\, do \\ &= 3\mu \left(\iiint\limits_V \nabla \mathbf{R} \cdot \boldsymbol{\theta} d\tau - 4\pi R_0^3 \boldsymbol{\theta} \right) = -8\pi\mu R_0^3 \boldsymbol{\theta} = -m\mathbf{e}_3 \end{aligned} \tag{1.2.15}$$

as $\nabla \mathbf{R} \cdot \boldsymbol{\theta} = 0$. A moment with the opposite sign needs to be applied to the sphere and transmitted through the surface of the cavity. Therefore, we arrive at the anticipated result (1.2.12).

These examples demonstrate the possibility of constructing force systems (a force, a centre of rotation, a centre of expansion, a force dipole) corresponding to each of the introduced singularities alone. This proves that each of the four sets of the terms in eq. (1.2.5) represents a particular solution of the equations of the elasticity theory which is continuous together with its derivatives in any region with the excluded singularity point.

The concept of singularities determined by the force tensor was used by Lauricella (1895) for representing the components of the strain tensor of a solid in terms of the external forces. The derivation of Lauricella's formula is based upon applying the Betty reciprocity theorem to the following two states. The first state is created by the surface forces $\mathbf{F}$ in the case of absent volume forces, $\mathbf{u}$ and $\hat{T}$ denoting the displacement vector and the stress tensor respectively. The second state $\mathbf{u}^*, \hat{T}^*$ is prescribed by (i) the force tensor at point Q which determines the displacement vector $\mathbf{u}_1^*$ and the stress tensor $\hat{T}_1^*$, and (ii) a superposition of the state of stress $\mathbf{u}_2^*, \hat{T}_2^*$ which ensures vanishing stresses on surface O of the body. The displacement vector and the stress tensor are respectively equal to

$$\mathbf{u}^* = \mathbf{u}_1^* + \mathbf{u}_2^*, \quad \hat{T}^* = \hat{T}_1^* + \hat{T}_2^*, \tag{1.2.16}$$

so that

$$\mathbf{n} \cdot \hat{T}^* \Big|_O = 0, \tag{1.2.17}$$

which defines functions $\mathbf{u}_2^*, \hat{T}_2^*$ which are bounded and continuous in the body volume.

The reciprocity theorem is applied to the volume bounded externally by the body surface O and internally by sphere Σ with a centre at Q. The unit vector of the external normal to sphere (and internal to the considered volume) is denoted by $\mathbf{n} = R^{-1}\mathbf{R}$. Then referring to eq. (1.2.17) we have

$$\iint_O \mathbf{F} \cdot \mathbf{u}^* do - \iint_\Sigma \mathbf{n} \cdot \hat{T} \cdot \mathbf{u}_1^* do - \iint_\Sigma \mathbf{n} \cdot \hat{T} \cdot \mathbf{u}_2^* do = \\ = -\iint_\Sigma \mathbf{n} \cdot \hat{T}_1^* \cdot \mathbf{u} do - \iint_\Sigma \mathbf{n} \cdot \hat{T}_2^* \cdot \mathbf{u} do,$$

and applying the reciprocity theorem results in the following relationship

$$\iint_O \mathbf{F} \cdot \mathbf{u}^* do = \iint_\Sigma \mathbf{n} \cdot \left(\hat{T} \cdot \mathbf{u}_1^* - \hat{T}_1^* \cdot \mathbf{u}\right) do. \tag{1.2.18}$$

Equation (1.2.5) suggests that the displacements and the stresses caused by the singularity of force tensor's type tend to infinity as R^{-2} and R^{-3} at point Q respectively. As it will be clear from the forthcoming analysis it is sufficient to accept that

$$\hat{T} = \hat{T}_Q, \quad \mathbf{u} = \mathbf{u}_Q + \mathbf{R} \cdot (\nabla \mathbf{u})_Q \tag{1.2.19}$$

in the volume v of the sphere since the terms of the higher order in R vanish when $R \to 0$.

Turning to eq. (1.2.5) and transforming the volume integrals into surface integrals we have

$$\iint_\Sigma \mathbf{n} \cdot \hat{T} \cdot \mathbf{u}_1^* do = \frac{1}{8\pi\mu(1-\nu)R^3} \left[(1-2\nu)\sigma \iiint_v \nabla \cdot \hat{T} \cdot \mathbf{R} d\tau + \right. \\ \left. (1-2\nu) \iiint_v \nabla \cdot \hat{T} \cdot \mathbf{R} \cdot \operatorname{Dev} \hat{p} d\tau + \frac{3}{2R^2} \iiint_v \nabla \cdot \hat{T} \cdot \mathbf{RR} \cdot \operatorname{Dev} \hat{p} \cdot \mathbf{R} d\tau \right],$$

where $3\sigma = I_1(\hat{p})$. Referring to eq. (B.3.10) and taking into account that $\nabla \cdot \hat{T} = 0$ we obtain

$$\nabla \cdot \hat{T} \cdot \mathbf{R} = \hat{T} \cdot\cdot \hat{E}, \quad \nabla \cdot \hat{T} \cdot \mathbf{R} \cdot \operatorname{Dev} \hat{p} = \hat{T} \cdot\cdot \operatorname{Dev} \hat{p}, \\ \nabla \cdot \hat{T} \cdot \mathbf{RR} \cdot \operatorname{Dev} \hat{p} \cdot \mathbf{R} = \hat{T} \cdot\cdot \left(\hat{E}\mathbf{R} \cdot \operatorname{Dev} \hat{p} \cdot \mathbf{R} + 2\mathbf{R} \cdot \operatorname{Dev} \hat{p}\mathbf{R}\right)$$

and furthermore

$$\iiint\limits_v \nabla\cdot\hat{T}\cdot\mathbf{R}d\tau = \frac{4}{3}\pi R^3 I_1\left(\hat{T}\right),$$

$$\iiint\limits_v \nabla\cdot\hat{T}\cdot\mathbf{R}\cdot\operatorname{Dev}\hat{p}d\tau = \frac{4}{3}\pi R^3\hat{T}\cdot\cdot\operatorname{Dev}\hat{p}.$$

It is easy to see that

$$\frac{3}{R^5}\iiint\limits_v \mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}d\tau = \frac{4\pi}{5}I_1\left(\operatorname{Dev}\hat{p}\right) = 0,$$

$$\frac{3}{R^5}\iiint\limits_v \mathbf{R}\cdot\operatorname{Dev}\hat{p}\mathbf{R}d\tau = \frac{4\pi}{5}\operatorname{Dev}\hat{p}$$

and thus

$$\frac{3}{2R^5}\iiint\limits_v \nabla\cdot\hat{T}\cdot\mathbf{RR}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}d\tau = \frac{4\pi}{5}\hat{T}\cdot\cdot\operatorname{Dev}\hat{p}.$$

Replacing tensor $\hat{T}$ and its first invariant by the following expressions

$$\hat{T} = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\hat{E}+\hat{\varepsilon}\right),\quad I_1\left(\hat{T}\right) = 2\mu\frac{1+\nu}{1-2\nu}\vartheta,$$

we find

$$\iint\limits_\Sigma \mathbf{n}\cdot\hat{T}\cdot\mathbf{u}_1^* do = \frac{1}{15\left(1-\nu\right)}\left[5\sigma\left(1+\nu\right)\vartheta + \left(8-10\nu\right)\hat{\varepsilon}\cdot\cdot\operatorname{Dev}\hat{p}\right].$$

Proceeding to the second term in formula (1.2.18) we notice that the stress tensor calculated in terms of $\mathbf{u}_1^*$ is equal to

$$\hat{T}_1^* = \frac{1}{4\pi\left(1-\nu\right)R^3}\left\{\left(1-2\nu\right)\sigma\left(\hat{E}-3\frac{\mathbf{RR}}{R^2}\right)+\left(1-2\nu\right)\operatorname{Dev}\hat{p}+\right.$$
$$\left.\frac{3}{2R^2}\left[\left(1-2\nu\right)\hat{E}-5\frac{\mathbf{RR}}{R^2}\right]\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}+\frac{3\nu}{R^2}\left(\mathbf{RR}\cdot\operatorname{Dev}\hat{p}+\mathbf{R}\cdot\operatorname{Dev}\hat{p}\mathbf{R}\right)\right\}$$

and therefore

$$\mathbf{n}\cdot\mathbf{T}_1^* = \frac{1}{4\pi\left(1-\nu\right)R^3}\times$$
$$\left[-2\left(1-2\nu\right)\sigma\mathbf{n}\cdot\operatorname{Dev}\hat{p}+\left(1+\nu\right)\mathbf{n}\cdot\operatorname{Dev}\hat{p}-\frac{6}{R^2}\mathbf{nR}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}\right].$$

Now we have

$$\iint\limits_{\Sigma} \mathbf{n} \cdot \hat{T}_1^* \cdot \mathbf{u} do = \frac{1}{15(1-\nu)} \left[-10(1-2\nu)\sigma\vartheta + (-7+5\nu)\hat{\varepsilon}\cdot\cdot\operatorname{Dev}\hat{p}\right],$$

and the sought-for expression (1.2.18) yields the form

$$\iint\limits_{O} \mathbf{F} \cdot \mathbf{u}^* do = \sigma\vartheta + \hat{\varepsilon}\cdot\cdot\operatorname{Dev}\hat{p} = \hat{\varepsilon}\cdot\cdot\hat{p}. \tag{1.2.20}$$

In particular, for a centre of expansion $\operatorname{Dev}\hat{p} = 0$, that is, putting $\sigma = 1$ we obtain

$$\vartheta = \iint\limits_{O} \mathbf{F} \cdot \mathbf{u}^* do. \tag{1.2.21}$$

For a force dipole $\mathbf{ee}$ one has $\sigma = 1/3$ and by eqs. (1.2.20) and (1.2.6) the extension along the dipole axis is given by

$$\mathbf{e} \cdot \hat{\varepsilon} \cdot \mathbf{e} = \iint\limits_{O} \mathbf{F} \cdot \mathbf{u}^* do. \tag{1.2.22}$$

Finally, considering the singularity given by forces $-h\mathbf{e}_k, -h\mathbf{e}_s$ at point Q and $h\mathbf{e}_k, h\mathbf{e}_s$ at points $h^{-1}\mathbf{e}_s, h^{-1}\mathbf{e}_k$ $(s \neq k)$, respectively, we have

$$\hat{p} = (\mathbf{e}_k\mathbf{e}_s + \mathbf{e}_s\mathbf{e}_k), \quad I_1(\hat{p}) = 0, \quad \hat{p} = \operatorname{Dev}\hat{p}$$

and, by virtue of eq. (1.2.20), we arrive at the expressions for the shears

$$2\mathbf{e}_k \cdot \hat{\varepsilon} \cdot \mathbf{e}_s = \gamma_{ks} = \iint\limits_{O} \mathbf{F} \cdot \mathbf{u}^* do. \tag{1.2.23}$$

The integrals in these formulae can be estimated only when we have the displacement vector $\mathbf{u}_2^*$, the sum $\mathbf{u}_1^* + \mathbf{u}_2^*$ being an analogue of Green's function corresponding to this particular singularity.

5.1.3 Interpretation of the second potential of elasticity theory

The displacement vector in the first external boundary value problem $\left(\mathrm{I}^{(e)}\right)$ of the theory of elasticity was presented in the form of the second potential of elasticity theory which is an analogue of the double layer potential. In order to reconcile the previous denotation with the present one it is necessary to interchange letters M and Q in formula (4.2.1) of Chapter 4. Then recalling expression (3.5.12) of Chapter 4 we have

$$\mathbf{v}(M) = \frac{1}{8\pi(1-\nu)} \iint\limits_{O} \left[(1-2\nu)(-\mathbf{Rb}\cdot\mathbf{n} + \mathbf{nb}\cdot\mathbf{R} + \mathbf{bn}\cdot\mathbf{R}) + \right.$$

$$\left. 3\frac{\mathbf{R}}{R^2}\mathbf{R}\cdot\mathbf{bn}\cdot\mathbf{R}\right] \frac{do}{R^3}. \tag{1.3.1}$$

According to eq. (1.1.7) the force tensor is represented by the dyadic $\mathbf{nb}$

$$\mathbf{nb} = \hat{p} + \hat{\Omega}, \quad \mathbf{bn} = \hat{p} - \hat{\Omega},$$

and the integrand is written down as follows

$$\frac{1}{R^3}\left[(1-2\nu)\left(-\mathbf{R}I_1(\hat{p}) + 2\hat{p}\cdot\mathbf{R}\right) + 3\frac{\mathbf{R}}{R^2}\mathbf{R}\cdot\hat{p}\cdot\mathbf{R}\right] =$$
$$= -\frac{2}{3}(1+\nu)I_1(\hat{p})\frac{1}{R} - \left[2(1-2\nu)\operatorname{Dev}\hat{p}\cdot\nabla\frac{1}{R} + 3\frac{\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}}{R^2}\nabla\frac{1}{R}\right].$$

Therefore

$$\mathbf{v}(M) = -\frac{1+\nu}{12\pi(1-\nu)}\iint\limits_O I_1(\hat{p})\nabla\frac{1}{R}do - \tag{1.3.2}$$
$$\frac{1-2\nu}{4\pi(1-\nu)}\iint\limits_O\left[\operatorname{Dev}\hat{p} + \frac{3}{2(1-2\nu)R^2}\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}\hat{E}\right]\cdot\nabla\frac{1}{R}do.$$

A comparison with eq. (1.2.5) shows that the double layer in the elasticity theory is formed by the distribution of the centres of expansion and the force dipoles over the surface O, the force and moment singularities being absent. This incompleteness of the force system explains why problem $\mathrm{I}^{(e)}$ can not be solved with the help of the second potential only.

5.1.4 Boussinesq's potentials

Distribution of the singularities along the lines, over the surfaces and in the volumes) yields particular solutions of the equations of the elasticity theory for the unbounded medium with the removed geometric body under consideration. The solution of the boundary value problem for a bounded body is obtained by means of combining the constructed solutions.

In what follows we consider two examples of constructing the particular solutions determined by the distributions of the centers of expansions and the centres of rotation along a half-line. Its direction is given by the unit vector $\mathbf{e}$ and the position of the point on it is described by abscissa λ measured from the origin O of the half-line. The position vector $\mathbf{R}'$ of point M of the medium having the origin at the current point of the half-line is set in the form

$$\mathbf{R}' = \mathbf{R} - \mathbf{e}\lambda, \tag{1.4.1}$$

where $\mathbf{R}$ is measured from the origin O. Referring to eqs. (1.2.8) and (1.2.12) we arrive at the particular solutions

$$\mathbf{u}(M) = -A\int_0^\infty d\lambda\nabla_M\cdot\frac{1}{R'}, \quad \mathbf{u}(M) = -\mathbf{C}\times\int_0^\infty d\lambda\nabla_M\frac{1}{R'}. \tag{1.4.2}$$

The first solution corresponds to the distribution of the centres of expansion while the second is due to the centres of rotation. The constants, i.e. scalar A and vector $\mathbf{C}$ characterises the intensity of these singularities. We have

$$\int_0^\infty d\lambda \nabla_M \frac{1}{R'} = \nabla_M \int_0^\infty \frac{d\lambda}{R'} = \nabla_M \int_0^\infty \left(R^2 - 2\lambda \mathbf{e}\cdot\mathbf{R} + \lambda^2\right)^{-1/2} d\lambda$$
$$= \nabla_M \ln\left(R' + \lambda - \mathbf{R}\cdot\mathbf{e}\right) \Big|_0^\infty ,$$

Here the calculation of the gradient should be performed first and then limits are substituted. The term corresponding to the upper limit $\lambda = \infty$ vanishes and the result is

$$\int_0^\infty d\lambda \nabla_M \frac{1}{R'} = -\nabla_M \ln\left(R - \mathbf{R}\cdot\mathbf{e}\right) = -\frac{\mathbf{R} - R\mathbf{e}}{R\left(R - \mathbf{R}\cdot\mathbf{e}\right)}.$$

Thus, introducing into consideration the first Boussinesq potential

$$\Phi_1 = \ln\left(R - \mathbf{R}\cdot\mathbf{e}\right), \tag{1.4.3}$$

we arrive at the following representations of the displacement vector

$$\mathbf{u} = A\nabla\Phi_1 = A\frac{\mathbf{R} - R\mathbf{e}}{R\left(R - \mathbf{R}\cdot\mathbf{e}\right)}, \quad \mathbf{u} = \mathbf{C}\times\nabla\Phi_1 \tag{1.4.4}$$

in any region with the excluded half-line. In this region Φ_1 is a harmonic function and it can be directly proved that it satisfies Laplace's equation however there is no need to do this. In the case of no mass forces the displacement vector in elasticity theory can be represented in the form of the gradient of a scalar. This scalar is a harmonic function and can be identified, for instance, with the harmonic scalar B_0 of the Papkovich-Neuber solution.

Boussinesq's potential (1.4.3) increases as $\ln R$ with the growth of R, that is the corresponding displacement vector decreases as R^{-1}. The stress tensor calculated in terms of Φ_1 is given by

$$\hat{T} = 2A\mu\nabla\nabla\Phi_1 = -\frac{2\mu A}{R^2\left(R - \mathbf{R}\cdot\mathbf{e}\right)}\left\{\frac{1}{R}\mathbf{RR} - \hat{E}R + \right.$$
$$\left. \frac{1}{R - \mathbf{R}\cdot\mathbf{e}}\left[\mathbf{RR} - R\left(\mathbf{Re} + \mathbf{eR}\right) + R^2\mathbf{ee}\right]\right\}, \tag{1.4.5}$$

and provided that axis Oz serves as the line of the expansion centres the components of the stress tensor are as follows

$$\left.\begin{aligned}
&\frac{1}{2\mu}\sigma_x = \frac{A}{R(R+z)}\left[1 - \frac{x^2(2R+z)}{R^2(R+z)}\right],\\
&\frac{1}{2\mu}\sigma_y = \frac{A}{R(R+z)}\left[1 - \frac{y^2(2R+z)}{R^2(R+z)}\right],\\
&\frac{1}{2\mu}\tau_{xy} = -\frac{Axy(2R+z)}{R^3(R+z)^2},\quad \frac{1}{2\mu}\tau_{xz} = -A\frac{x}{R^3},\\
&\frac{1}{2\mu}\tau_{yz} = -A\frac{y}{R^3},\quad \frac{1}{2\mu}\sigma_z = -A\frac{z}{R^3}.
\end{aligned}\right\} \tag{1.4.6}$$

The simplicity of the expression for the stress components on the surfaces perpendicular to axis z shows that potential Φ_1 is an appropriate means for solving the problem of the state of stress in the elastic half-space $z > 0$.

Using the spherical coordinate system and directing $\mathbf{e}$ to the "south pole" of the sphere yields

$$\left.\begin{aligned}
&\frac{1}{2\mu}\sigma_R = -\frac{A}{R^2},\quad \frac{1}{2\mu}\tau_{R\vartheta} = \frac{A\sin\vartheta}{R^2(1+\cos\vartheta)},\quad \tau_{R\lambda} = 0,\\
&\frac{1}{2\mu}\sigma_\vartheta = \frac{A\cos\vartheta}{R^2(1+\cos\vartheta)},\quad \tau_{\vartheta\lambda} = 0,\quad \frac{1}{2\mu}\sigma_\lambda = \frac{A}{R^2(1+\cos\vartheta)}.
\end{aligned}\right\} \tag{1.4.7}$$

Another Boussinesq's potential is used for solving the boundary value problems

$$\Phi_2 = \int^z \ln(R+z)\,dz = z\ln(R+z) - R,\quad \mathbf{u} = \nabla\Phi_2. \tag{1.4.8}$$

Of course, it is a harmonic function in the region with the removed negative axis z.

5.1.5 *Thermoelastic displacements*

Referring to formulae (3.4.3) and (3.5.9) of Chapter 4 we have

$$\mathbf{u}(Q) = 2\mu\alpha\frac{1+\nu}{1-2\nu}\iiint\limits_{V_i} \theta(M)\,\mathrm{div}_M\,\hat{U}(M,Q)\,d\tau_M.$$

Here

$$\mathrm{div}_M\,\hat{U}(M,Q) = \frac{1}{4\pi\mu}\left(\nabla_M\frac{1}{R} - \frac{1}{4(1-\nu)}\nabla_M^2\frac{\mathbf{R}}{R}\right) = \frac{1-2\nu}{8\pi\mu(1-\nu)}\nabla_M\frac{1}{R} \tag{1.5.1}$$

and the expression for the displacement vector can be presented in the form

$$\mathbf{u}(Q) = \alpha \frac{1+\nu}{4\pi(1-\nu)} \iiint\limits_{V_i} \theta(M) \nabla_M \left(\frac{1}{R}\right) d\tau_M = -\nabla_Q \chi, \tag{1.5.2}$$

where a new potential is introduced into consideration

$$\chi = \alpha \frac{1+\nu}{4\pi(1-\nu)} \iiint\limits_{V_i} \frac{\theta(M)}{R} d\tau_M. \tag{1.5.3}$$

Here θ denotes the variation in the temperature around the constant value in the natural state and V_i is the volume in which the temperature distribution is given, i.e. $\theta = 0$ outside of this volume. According to eq. (1.1.12) the same field of the displacement vector in the unbounded elastic medium is caused by the distribution of the centres of expansions having intensity proportional to θ in volume V_i. Function χ is a Newtonian potential of the attracting masses with density proportional to the temperature. The first derivatives of this potential (the components of the force of attraction or the components of the displacement vector in the present case) are continuous in the whole space (under the assumption of continuous density). A jump in the second derivatives under the passage through surface O from the outside (i.e. from volume V_e) into volume V_i is given by the well-known equation

$$(\nabla\nabla\chi)_e - (\nabla\nabla\chi)_i = \alpha \frac{1+\nu}{1-\nu} \theta_0 \mathbf{nn}, \tag{1.5.4}$$

where $\mathbf{n}$ is the unit vector of the external normal to O and θ_0 denotes the value of θ on O. Potential χ satisfies Laplace's equation outside volume V_i and Poisson's equation in volume V_i, that is

$$\nabla^2 \chi = \begin{cases} 0, & Q \subset V_e, \\ -\dfrac{1+\nu}{1-\nu} \alpha\theta, & Q \subset V_i. \end{cases} \tag{1.5.5}$$

If $Q \subset V_i$, the stress tensor determined by potential χ, eq. (1.1.4.1) of Chapter 4, is equal to

$$\begin{aligned} \hat{T} &= -2\mu \left(\frac{\nu}{1-2\nu} \hat{E} \nabla^2 \chi + \nabla\nabla\chi \right) - 2\mu \frac{1+\nu}{1-2\nu} \alpha\theta \hat{E} \\ &= -2\mu \left(\frac{1+\nu}{1-\nu} \alpha\theta \hat{E} + \nabla\nabla\chi \right), \end{aligned} \tag{1.5.6}$$

whilst the temperature term is omitted if $Q \subset V_e$. By virtue of eq. (1.5.4) we obtain

$$\left(\hat{T}^{(e)} - \hat{T}^{(i)} \right)_O = 2\mu \frac{1+\nu}{1-\nu} \alpha\theta_O \left(\hat{E} - \mathbf{nn} \right), \tag{1.5.7}$$

which means that the stress tensor is continuous on surface O

$$\left[\mathbf{n}\cdot\left(\hat{T}^{(e)}-\hat{T}^{(i)}\right)\right]_O=0, \tag{1.5.8}$$

whereas on the surfaces perpendicular to the boundary (i.e. on the surfaces with the normal $\mathbf{n}^*$ where $\mathbf{n}^*\cdot\mathbf{n}=0$) this vector experiences a jump in the normal component at the points of the boundary

$$\left[\mathbf{n}^*\cdot\left(\hat{T}^{(e)}-\hat{T}^{(i)}\right)\cdot\mathbf{n}^*\right]_O=2\mu\frac{1+\nu}{1-\nu}\alpha\theta_O. \tag{1.5.9}$$

The tangential components are continuous.

Let V_i be the volume of the sphere of radius a heated to a constant temperature θ^0. The theory of the Newtonian potential suggests that

$$\chi=\begin{cases}\dfrac{1+\nu}{2(1-\nu)}\alpha\theta^0\left(a^2-\dfrac{1}{3}R^2\right), & Q\subset V_i,\\[2mm] \dfrac{1+\nu}{3(1-\nu)}\alpha\theta^0\dfrac{a^3}{R}, & Q\subset V_e,\end{cases} \tag{1.5.10}$$

and by eq. (1.5.6)

$$\hat{T}^{(i)}=-\frac{4}{3}\mu\frac{1+\nu}{1-\nu}\alpha\theta^0\hat{E},\quad \hat{T}^{(e)}=\frac{2}{3}\mu\frac{1+\nu}{1-\nu}\alpha\theta^0\left(\hat{E}-3\mathbf{e}_R\mathbf{e}_R\right)\frac{a^3}{R^3}. \tag{1.5.11}$$

In accordance with the above-said

$$\begin{aligned}\sigma_R^{(i)}\Big|_{R=a}&=\sigma_R^{(e)}\Big|_{R=a}&&=-\frac{4}{3}\mu\frac{1+\nu}{1-\nu}\alpha\theta^0,\\ \sigma_\vartheta^{(i)}\Big|_{R=a}&=\sigma_\lambda^{(i)}\Big|_{R=a}&&=-\frac{4}{3}\mu\frac{1+\nu}{1-\nu}\alpha\theta^0,\\ \sigma_\vartheta^{(e)}\Big|_{R=a}&=\sigma_\lambda^{(e)}\Big|_{R=a}&&=\frac{2}{3}\mu\frac{1+\nu}{1-\nu}\alpha\theta^0,\end{aligned}$$

so that σ_R is continuous in the whole space while σ_ϑ and σ_λ experience a discontinuity determined by formula (1.5.9).

5.1.6 *The state of stress due to an inclusion*

Increasing the temperature of the elementary volume is not the only means of obtaining the so-called free deformation, i.e. the deformation followed by no stresses. One can imagine other physical processes accompanied by the deformation. Eshelby assigned the crystal twinning, martensite transformation and phase transition to another elementary cell. However a state of stress appears in the elastic solid due to deformation $\hat{\varepsilon}^0$ in volume V_i which would be a free deformation in this volume if this volume was free.

As a result the whole solid is deformed and the strain tensor is related to the stress tensor by the relationship

$$\hat{\varepsilon} = \frac{1}{2\mu}\left(\hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}\right) + \hat{\varepsilon}^0 \tag{1.6.1}$$

since the stresses are caused by "strain" $\hat{\varepsilon} - \hat{\varepsilon}^0$. Relationship (1.6.1) is a natural generalisation of Hooke's law, eq. (3.4.10) of Chapter 3, and includes the temperature term $\hat{E}\alpha\vartheta$, which can be substantiated by the reasoning of Hooke's law, see the end of Subsection 3.3.4. It follows from eq. (1.6.1) that

$$\sigma = I_1\left(\hat{T}\right) = 2\mu\frac{1+\nu}{1-2\nu}\left(\vartheta - \vartheta^0\right), \quad \vartheta^0 = I_1\left(\hat{\varepsilon}^0\right) \tag{1.6.2}$$

and then

$$\begin{aligned}\hat{T} &= 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\hat{E} + \hat{\varepsilon}\right) - 2\mu\left(\frac{\nu}{1-2\nu}\vartheta^0\hat{E} + \hat{\varepsilon}^0\right)\\ &= 2\mu\left(\frac{\nu}{1-2\nu}\vartheta\hat{E} + \hat{\varepsilon}\right) - \hat{T}^0.\end{aligned} \tag{1.6.3}$$

Here $\hat{T}^0$ denotes a "stress tensor" which is formally related to tensor $\hat{\varepsilon}^0$ by Hooke's law

$$\hat{T}^0 = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta^0\hat{E} + \hat{\varepsilon}^0\right). \tag{1.6.4}$$

Introducing this "stress tensor" only reduces the formulae since the free deformation is not accompanied by the stresses.

Let us consider two states of the elastic solid. In the first state a unit concentrated force $\mathbf{e}'_Q$ is applied at point Q whereas in the second state the stresses are due to a free deformation in the case of no external forces.

Referring to eq. (3.1.5) of Chapter 4 and considering volume $V = V_i + V_e$ we obtain

$$\begin{aligned}\mathbf{e}'_Q \cdot \mathbf{u}''(Q) &= \iiint\limits_V \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau - \iint\limits_S \mathbf{n}\cdot\hat{\mathbf{T}}'\cdot\mathbf{u}'' do = \iiint\limits_V \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau\\ &= \iiint\limits_{V_e} \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau + \iiint\limits_{V_i} \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau\end{aligned} \tag{1.6.5}$$

as the integral over the surface S of volume V tends to zero in the case of an unbounded expansion of this volume.

The external forces are absent in the second state hence

$$\iiint\limits_V \hat{T}'' \cdot\cdot \hat{\varepsilon}' d\tau = \iiint\limits_{V_e} \hat{T}'' \cdot\cdot \hat{\varepsilon}' d\tau + \iiint\limits_{V_i} \hat{T}'' \cdot\cdot \hat{\varepsilon}' d\tau = 0. \tag{1.6.6}$$

Stress tensor $\hat{T}'$ in the whole volume V and tensor $\hat{T}''$ in volume V_e are determined with the help of Hooke's law

$$\hat{T}' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta'\hat{E} + \hat{\varepsilon}'\right), \quad \hat{T}'' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta''\hat{E} + \hat{\varepsilon}''\right),$$

whilst by eq. (1.6.3) in volume V_i we have

$$\hat{T}'' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta''\hat{E} + \hat{\varepsilon}''\right) - \hat{T}^0.$$

Hence

$$\hat{T}' \cdot\cdot \hat{\varepsilon}'' = 2\mu\left(\tfrac{\nu}{1-2\nu}\vartheta'\vartheta'' + \hat{\varepsilon}' \cdot\cdot \hat{\varepsilon}''\right) \quad \text{in } V,$$

$$\hat{T}'' \cdot\cdot \hat{\varepsilon}' = 2\mu\left(\frac{\nu}{1-2\nu}\vartheta''\vartheta' + \hat{\varepsilon}'' \cdot\cdot \hat{\varepsilon}'\right) + \left\{\begin{array}{cl} -\hat{T}^0 \cdot\cdot \hat{\varepsilon}' & \text{in } V_i, \\ 0 & \text{in } V_e \end{array}\right.$$

or

$$\hat{T}'' \cdot\cdot \hat{\varepsilon}' = \left\{\begin{array}{cl} \hat{T}' \cdot\cdot \hat{\varepsilon}'' & \text{in } V_e, \\ \hat{T}' \cdot\cdot \hat{\varepsilon}'' - \hat{T}^0 \cdot\cdot \hat{\varepsilon}' & \text{in } V_i. \end{array}\right.$$

Returning to eqs. (1.6.6) and (1.6.5) we have

$$\iiint\limits_{V_e} \hat{T}'' \cdot\cdot \hat{\varepsilon}' d\tau + \iiint\limits_{V_i} \hat{T}'' \cdot\cdot \hat{\varepsilon}' d\tau = \iiint\limits_{V_e} \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau + \iiint\limits_{V_i} \hat{T}' \cdot\cdot \hat{\varepsilon}'' d\tau -$$
$$\iiint\limits_{V_i} \hat{T}^0 \cdot\cdot \hat{\varepsilon}' d\tau = \mathbf{e}'_Q \cdot \mathbf{u}''(Q) - \iiint\limits_{V_i} \hat{T}^0 \cdot\cdot \hat{\varepsilon}' d\tau = 0,$$

so that

$$\mathbf{e}'_Q \cdot \mathbf{u}''(Q) = \iiint\limits_{V_i} \hat{T}^0 \cdot\cdot \hat{\varepsilon}' d\tau.$$

Expressing $\hat{\varepsilon}'$ in terms of the Kelvin-Somigliana tensor, eq. (3.5.9) of Chapter 4, we obtain

$$\mathbf{e}'_Q \cdot \mathbf{u}''(Q) = \frac{1}{2}\iiint\limits_{V_i} \hat{T}^0 \cdot\cdot \left[\nabla\hat{U}' \cdot \mathbf{e}'_Q + \left(\nabla\hat{U}'\right)^* \cdot \mathbf{e}'_Q\right] d\tau$$
$$= \iiint\limits_{V_i} \left(\hat{T}^0 \cdot\cdot \nabla\hat{U}'\right) d\tau \cdot \mathbf{e}'_Q$$

and dropping the arbitrary prescribed vector $\mathbf{e}'_Q$ and primes (the latter are unnecessary now) we arrive at the following equality

$$\mathbf{u}(Q) = \iiint\limits_{V_i} \hat{T}^0 \cdot\cdot \nabla_M \hat{U}(M,Q)\, d\tau_M. \tag{1.6.7}$$

The integration is carried out over the volume of the inclusion subjected to the free deformation. Moreover we take this deformation to be homogeneous, i.e. $\hat{\varepsilon}^0$ and thus $\hat{T}^0$ are constant. Then recalling eq. (3.5.9) of Chapter 4 for the Kelvin-Somigliana tensor we obtain the Eshelby formula

$$\mathbf{u}(Q) = -\frac{1}{4\pi\mu}\hat{T}^0 \cdot\cdot \left[\nabla_Q \hat{E} \iiint\limits_{V_i} \frac{d\tau}{R} - \frac{1}{4(1-\nu)}\nabla_Q\nabla_Q\nabla_Q \iiint\limits_{V_i} R d\tau\right]. \tag{1.6.8}$$

Introducing the potentials

$$\varphi = \iiint\limits_{V_i} \frac{d\tau}{R}, \quad \psi = \iiint\limits_{V_i} R d\tau \tag{1.6.9}$$

we can rewrite Eshelby's formula as follows

$$\mathbf{u}(Q) = -\frac{1}{4\pi\mu}\hat{T}^0 \cdot\cdot \left[\nabla_Q \hat{E}\varphi - \frac{1}{4(1-\nu)}\nabla_Q\nabla_Q\nabla_Q\psi\right]. \tag{1.6.10}$$

Function φ is a Newtonian potential of the attracting masses of the unit density

$$\nabla^2\psi = 2\varphi \tag{1.6.11}$$

and by virtue of eqs. (1.5.3) and (1.5.4)

$$\nabla^2\varphi = \begin{cases} 0, & Q \subset V_e, \\ -4\pi, & Q \subset V_i, \end{cases} \quad \nabla^4\psi = \begin{cases} 0, & Q \subset V_e, \\ -8\pi, & Q \subset V_i, \end{cases} \tag{1.6.12}$$

The relationships for the discontinuities on surface O of volume V_i are analogous to those in eq. (1.5.4) and can be presented in the form

$$\left.\begin{aligned} (\nabla\nabla\varphi)_e - (\nabla\nabla\varphi)_i &= 4\pi\mathbf{nn}, \\ (\nabla\nabla\nabla\nabla\psi)_e - (\nabla\nabla\nabla\nabla\psi)_i &= 8\pi\mathbf{nnnn} \end{aligned}\right\} \tag{1.6.13}$$

since the components of tensor $\nabla\nabla\psi$ are Newtonian potentials with the density equal to the corresponding components of tensor $-(4\pi)^{-1}\nabla\nabla 2\varphi$.

The components of the strain tensor found by means of eq. (1.6.8) are equal to

$$\varepsilon_{sk} = -\frac{1}{4\pi\mu}\left[\frac{1}{2}\left(t^0_{qs}\frac{\partial^2\varphi}{\partial x_k \partial x_q} + t^0_{qk}\frac{\partial^2\varphi}{\partial x_s \partial x_q}\right) - \frac{1}{4(1-\nu)}t^0_{qr}\frac{\partial^4\psi}{\partial x_k \partial x_s \partial x_q \partial x_r}\right] \tag{1.6.14}$$

In particular, when tensor $\hat{T}^0$ is spherical, then

$$t_{qs}^0 = \frac{1}{3}\sigma^0\delta_{qs} = \frac{1}{3}2\mu\frac{1+\nu}{1-2\nu}\vartheta^0\delta_{qs} \tag{1.6.15}$$

and according to eq. (1.6.11)

$$\varepsilon_{sk} = -\frac{1}{4\pi}\frac{1+\nu}{3(1-\nu)}\vartheta^0\frac{\partial^2\varphi}{\partial x_s \partial x_k},$$

$$\vartheta = -\frac{1}{4\pi}\frac{1+\nu}{3(1-v)}\vartheta^0\nabla^2\varphi = \left\{\begin{array}{ll} 0, & Q \subset V_e, \\ \dfrac{1+\nu}{3(1-\nu)}\vartheta^0, & Q \subset V_i. \end{array}\right. \tag{1.16.16}$$

For example, for $\nu = 0.25$ and under the temperature process $\vartheta^0 = 3\alpha\theta$ the dilatation constrained by the surrounding medium comprises only 5/9 of the free dilatation and is absent in the surrounding medium.

In the general case, in the medium surrounding the inclusion, the stress components are calculated in terms of the strain components (1.6.14) by means of Hooke's law in its standard form (eq. (1.1.3) of Chapter 3) and in the inclusion they are given by eq. (1.6.3). Calculation requires both potentials φ and ψ. Indeed, by eqs. (1.6.11) and (1.6.14) we have

$$\begin{aligned} \vartheta &= -\frac{1-2\nu}{8\pi\mu(1-\nu)}t_{qs}^0\frac{\partial^2\varphi}{\partial x_q \partial x_s} = -\frac{1-2\nu}{8\pi\mu(1-\nu)}\hat{T}^0\cdot\cdot\nabla\nabla\varphi \\ &= -\frac{1-2\nu}{8\pi\mu(1-\nu)}\operatorname{Dev}\hat{T}^0\cdot\cdot\nabla\nabla\varphi + \left\{\begin{array}{ll} \dfrac{1-2\nu}{2\mu(1-\nu)}\dfrac{1}{3}\sigma^0, & Q \subset V_i, \\ 0, & Q \subset V_e. \end{array}\right. \end{aligned} \tag{1.6.17}$$

From this equation and eqs. (1.6.13), (1.6.15) we obtain the jump in ϑ on the surface of the inclusion

$$\vartheta_e - \vartheta_i = -\frac{1+\nu}{1-\nu}\frac{1}{3}\vartheta^0 - \frac{1-2\nu}{1-\nu}\operatorname{Dev}\hat{\varepsilon}^0\cdot\cdot\mathbf{nn} \tag{1.6.18}$$

as $\operatorname{Dev}\hat{T}^0 = 2\mu\operatorname{Dev}\hat{\varepsilon}^0$. If $\operatorname{Dev}\hat{\varepsilon}^0 = 0$ we return to eq. (1.6.16).

5.2 Elastic half-space

5.2.1 The problems of Boussinesq and Cerruti

The objective is to search for the state of stress in an elastic half-space $z > 0$ (i.e. in an elastic solid bounded by plane $z = 0$) under a prescribed distribution of the surface forces in this plane

$$z = 0: \quad \mathbf{F} = \mathbf{n}\cdot\hat{T} = -\mathbf{i}_3\cdot\hat{T} \tag{2.1.1}$$

or

$$z=0: \quad \left.\begin{aligned} F_x &= q_1(x,y) = -\tau_{xz}, \\ F_y &= q_2(x,y) = -\tau_{yz}, \\ F_z &= p(x,y) = -\sigma_z. \end{aligned}\right\} \tag{2.1.2}$$

It is assumed that the mass forces are absent and the principal vector of the surface forces is bounded

$$\mathbf{V} = \iint_\Omega \mathbf{F} do \quad (do = dxdy), \tag{2.1.3}$$

where Ω denotes the region of loading on plane $z = 0$. At the specified conditions it is required that the sought solution, for the displacement vector and the stresses, decreases respectively not slower than R^{-1} and R^{-2} as $R \to \infty$.

There exist several strategies for solving this classical problem, as considered by Boussinesq and Cerruti. In the particular Boussinesq problem, in which $q_1 = q_2 = 0$ and the loading is due to a concentrated force $\mathbf{Q}$ normal to the half-space boundary, the solution is easily obtained by superposition of the state of stress (1.4.6) caused by a special line of the compression centres on the state of stress in an unbounded elastic medium caused by a concentrated force (the Kelvin-Somigliana solution, Subsection 4.3.5). The generalisation to the general case of the normal loading $p(x,y)$ is evident. Another approach applies the Papkovich-Neuber solution, Subsection 4.1.4, and can be generalised to the general problem of Boussinesq and Cerruti, that is, the case of loading (2.1.2).

5.2.2 *The particular Boussinesq problem*

In an unbounded elastic medium a force applied at the origin of the coordinate system and having the direction of axis Oz produces the following state of stress, cf. eq. (3.5.6) of Chapter 4

$$\hat{T} = \frac{2\mu C}{R^3}\left[(1-2\nu)\left(\hat{E}z - \mathbf{i}_3\mathbf{R} - \mathbf{R}\mathbf{i}_3\right) - \frac{3z}{R^2}\mathbf{R}\mathbf{R}\right] \quad (\mathbf{R} = \mathbf{i}_1 x + \mathbf{i}_2 y + \mathbf{i}_3 z), \tag{2.2.1}$$

where C is a proportionality factor to be determined in what follows. The stress vector on plane $z = 0$ is equal to

$$\mathbf{n}\cdot\hat{T} = -\mathbf{i}_3\cdot\hat{T} = \frac{2\mu C}{R_0^3}(1-2\nu)\mathbf{R}_0 \quad (\mathbf{R}_0 = \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2). \tag{2.2.2}$$

In accordance with eq. (1.4.6) this distribution coincides qualitatively with that given by Boussinesq's potential (1.4.3)

$$\mathbf{n}\cdot\hat{T} = 2\mu A\frac{\mathbf{R}_0}{R_0^3} \tag{2.2.3}$$

and the requirement of zero stresses on this plane can be satisfied if the constants A and C are related by the equality

$$A = -C(1-2\nu).$$

Using eqs. (3.5.5) and (3.5.6) of Chapter 4, as well as eqs. (1.4.4) and (1.4.5) we obtain

$$\mathbf{u} = C\left\{\frac{1}{R}\left[(3-4\nu)\,\mathbf{i}_3 + \frac{z}{R^3}\mathbf{R}\right] - (1-2\nu)\,\nabla\ln(R+z)\right\}, \qquad (2.2.4)$$

$$\hat{T} = 2\mu C\left\{\frac{1}{R^3}\left[(1-2\nu)\left(\hat{E}z - \mathbf{i}_3\mathbf{R} - \mathbf{R}\mathbf{i}_3\right) - \frac{3\mathbf{R}\mathbf{R}}{R^2}z\right] + \frac{1-2\nu}{R^2(R+z)}\left[\frac{\mathbf{R}\mathbf{R}}{R} - \hat{E}R + \frac{1}{R+z}(\mathbf{R}+\mathbf{i}_3R)(\mathbf{R}+\mathbf{i}_3R)\right]\right\}. \qquad (2.2.5)$$

The constant C is determined from the equilibrium equation for a half-sphere of radius R with the centre at the point where force $\mathbf{i}_3Q$ is applied

$$\mathbf{i}_3Q + R^2\iint_{O_*}\mathbf{e}_R\cdot\hat{T}\,do_* = 0. \qquad (2.2.6)$$

Here $do_* = \sin\vartheta d\vartheta d\lambda$ denotes the area element of the surface of the unit half-sphere O_* and

$$\mathbf{e}_R = \mathbf{R}R^{-1} = \mathbf{i}_3\cos\vartheta + \sin\vartheta\,(\mathbf{i}_1\cos\lambda + \mathbf{i}_2\sin\lambda).$$

Then we have

$$\begin{aligned} R^2\mathbf{e}_R\cdot\hat{T} &= 2\mu C\left[-(1-2\nu)\,\mathbf{i}_3 - 3\mathbf{e}_R\cos\vartheta + \frac{1-2\nu}{1+\cos\vartheta}(\mathbf{e}_R+\mathbf{i}_3)\right] \\ &= -6\mu C\mathbf{i}_3\cos^2\vartheta + \ldots, \end{aligned}$$

where the dots denote the terms which do not contribute to the integral in eq. (2.2.6). The result is

$$Q - 6\mu C\int_0^{2\pi}d\lambda\int_0^{\pi/2}\cos^2\vartheta\sin\vartheta d\vartheta = 0, \quad C = \frac{Q}{4\pi\mu},$$

and this accomplishes the solution of the particular Boussinesq problem. The expressions for the displacements are reduced to the form

$$\left.\begin{aligned} u &= \frac{Q}{4\pi\mu}\frac{x}{R}\left(\frac{z}{R^2} - \frac{1-2\nu}{R+z}\right), \\ v &= \frac{Q}{4\pi\mu}\frac{y}{R}\left(\frac{z}{R^2} - \frac{1-2\nu}{R+z}\right), \\ w &= \frac{Q}{4\pi\mu R}\left[\frac{z^2}{R^2} + 2(1-\nu)\right]. \end{aligned}\right\} \qquad (2.2.7)$$

The stresses on the planes parallel to the boundary of the half-space have a simple form and turn out to be independent of Poisson's ratio

$$\tau_{xz} = -\frac{3Q}{2\pi}\frac{xz^2}{R^5}, \quad \tau_{yz} = -\frac{3Q}{2\pi}\frac{yz^2}{R^5}, \quad \sigma_z = -\frac{3Q}{2\pi}\frac{z^3}{R^5}. \tag{2.2.8}$$

5.2.3 *The distributed normal load*

The solution of the above problem is easily generalised to an arbitrary number of forces $Q_i\mathbf{i}_3$ which are normal to the boundary $z = 0$ and applied at points with the coordinates $(x_i, y_i, 0)$

$$\left.\begin{array}{l} \mathbf{u} = \dfrac{1-\nu}{\pi\mu}\mathbf{i}_3 \displaystyle\sum_{i=1}^{n} \frac{Q_i}{R_i} - \frac{1}{4\pi\mu}\operatorname{grad}\sum_{i=1} Q_i\left[\frac{z}{R_i} + (1-2\nu)\ln(R_i + z)\right], \\ \mathbf{R}_i = \mathbf{i}_1(x - x_i) + \mathbf{i}_2(y - y_i) + \mathbf{i}_3 z. \end{array}\right\} \tag{2.3.1}$$

The case of the distributed load $p(x, y)$ is obtained by replacing Q_i by $p(x', y')\,do'$ and integrating over the loading region Ω. The potentials

$$\omega(x, y, z) = \iint\limits_{\Omega} \frac{p(x', y')}{R'}do', \tag{2.3.2}$$

$$\omega_1(x, y, z) = \iint\limits_{\Omega} p(x', y')\ln(R' + z)\,do', \tag{2.3.3}$$

are introduced into consideration, where

$$\mathbf{R}' = \mathbf{i}_1(x - x_i) + \mathbf{i}_2(y - y_i) + \mathbf{i}_3 z, \; R' = \left[(x - x')^2 + (y - y\prime)^2 + z^2\right]^{1/2}.$$

In terms of these potentials the displacement vector is represented as follows

$$\mathbf{u} = \frac{1-\nu}{\pi\mu}\mathbf{i}_3\omega - \frac{1}{4\pi\mu}\operatorname{grad}\left[z\omega + (1-2\nu)\,\omega_1\right], \tag{2.3.4}$$

and the stresses on the surfaces perpendicular to axis z are given by

$$\tau_{xz} = -\frac{z}{2\pi}\frac{\partial^2\omega}{\partial x\partial z}, \quad \tau_{yz} = -\frac{z}{2\pi}\frac{\partial^2\omega}{\partial y\partial z}, \quad \sigma_z = \frac{1}{2\pi}\left(\frac{\partial\omega}{\partial z} - z\frac{\partial^2\omega}{\partial z^2}\right). \tag{2.3.5}$$

Function $\omega(x, y, z)$ is the simple layer potential distributed over the loading area with density $p(x, y)$. This function is continuous everywhere, including region Ω, and for large distances from Ω it decreases as PR^{-1}, where P denotes the principal vector of the surface forces

$$P = \iint\limits_{\Omega} p(x, y)\,do$$

and $R = \left(x^2 + y^2 + z^2\right)^{1/2} \to \infty$.

It is known that the derivative of the simple layer potential with respect to the normal to the surface of the layer experiences a jump under a passage of the point through this surface. In particular, for the layer distributed over region Ω on plane $z = 0$ we have

$$\left.\frac{\partial \omega}{\partial z}\right|_{z\to\pm 0} = \begin{cases} \mp 2\pi p\,(x,y), & (x,y) \subset \Omega, \\ 0, & (x,y) \not\subset \Omega. \end{cases} \tag{2.3.6}$$

It follows once again from the latter result that the obtained solution satisfies the boundary condition (2.1.2) for $q_1 = q_2 = 0$.

Function $\omega_1\,(x,y,z)$ is harmonic in the half-space $z > 0$ and increases as

$$P \ln (R + z) \tag{2.3.7}$$

along with increasing R. However the displacement vector depends only on the first derivatives of ω_1 with respect to the coordinates and decreases as R^{-1} as $R \to \infty$. Let us notice the equality

$$\frac{\partial \omega_1}{\partial z} = \omega \tag{2.3.8}$$

determining, together with the condition at infinity (2.3.7), function ω_1 in terms of ω up to an inessential additive constant.

5.2.4 Use of the Papkovich-Neuber functions to solve the Boussinesq-Cerruti problem

The expressions for the components of the stress tensor in terms of these functions are set, by means of eq. (1.4.17) of Chapter 4, in the form

$$\begin{aligned}
\frac{\tau_{xz}}{2\mu} &= (1-2\nu)\left(\frac{\partial B_1}{\partial z} + \frac{\partial B_3}{\partial x}\right) - \left(x\frac{\partial^2 B_1}{\partial z \partial x} + y\frac{\partial^2 B_2}{\partial z \partial x} + z\frac{\partial^2 B_3}{\partial z \partial x} + \frac{\partial^2 B_0}{\partial z \partial x}\right), \\
\frac{\tau_{yz}}{2\mu} &= (1-2\nu)\left(\frac{\partial B_2}{\partial z} + \frac{\partial B_3}{\partial y}\right) - \left(x\frac{\partial^2 B_1}{\partial y \partial z} + y\frac{\partial^2 B_2}{\partial y \partial z} + z\frac{\partial^2 B_3}{\partial y \partial z} + \frac{\partial^2 B_0}{\partial y \partial z}\right), \\
\frac{\sigma_z}{2\mu} &= 2\,(1-2\nu)\,\frac{\partial B_3}{\partial z} + 2\nu\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) - \\
&\qquad \left(x\frac{\partial^2 B_1}{\partial z^2} + y\frac{\partial^2 B_2}{\partial z^2} + z\frac{\partial^2 B_3}{\partial z^2} + \frac{\partial^2 B_0}{\partial z^2}\right).
\end{aligned} \tag{2.4.1}$$

If only the tangential loading is present it is sufficient to take

$$B_1 = 0, \quad B_2 = 0, \quad \frac{\partial B_0}{\partial z} = (1-2\nu)\,B_3. \tag{2.4.2}$$

Then

$$\frac{1}{2\mu}\tau_{xz} = -z\frac{\partial^2 B_3}{\partial x \partial z}, \quad \frac{1}{2\mu}\tau_{yz} = -z\frac{\partial^2 B_3}{\partial y \partial z}, \quad \frac{1}{2\mu}\sigma_z = \frac{\partial B_3}{\partial z} - z\frac{\partial^2 B_3}{\partial z^2}. \quad (2.4.3)$$

The boundary conditions for the tangential stresses are satisfied automatically

$$(\tau_{xz})_{z=0} = (\tau_{yz})_{z=0} = 0 \quad (2.4.4)$$

and the remaining boundary condition

$$\frac{1}{2\mu}(\sigma_z)_{z=0} = -\frac{1}{2\mu}p(x,y) = \left(\frac{\partial B_3}{\partial z}\right)_{z=0} \quad (2.4.5)$$

suggests that, according to eq. (2.3.6), B_3 is a simple layer potential with the density $(4\pi\mu)^{-1} p(x,y)$. Function B_0 is determined by the second equality in eq. (2.4.2) and the condition of vanishing derivatives at infinity. Evidently, B_3 and B_0 differ from ω and ω_1 only in constant factors

$$B_3 = \frac{1}{4\pi\mu}\omega, \quad B_0 = \frac{1}{4\pi\mu}(1-2\nu)\,\omega_1. \quad (2.4.6)$$

Proceeding to the general boundary value problem (2.1.2) we put expressions for τ_{xz} and τ_{yz} in another form

$$\left.\begin{aligned} \frac{1}{2\mu}\tau_{xz} &= 2(1-\nu)\frac{\partial B_1}{\partial z} + (1-2\nu)\frac{\partial B_3}{\partial x} - \\ &\qquad \frac{\partial}{\partial x}\left(x\frac{\partial B_1}{\partial z} + y\frac{\partial B_2}{\partial z} + z\frac{\partial B_3}{\partial z} + \frac{\partial B_0}{\partial z}\right), \\ \frac{1}{2\mu}\tau_{yz} &= 2(1-\nu)\frac{\partial B_2}{\partial z} + (1-2\nu)\frac{\partial B_3}{\partial y} - \\ &\qquad \frac{\partial}{\partial y}\left(x\frac{\partial B_1}{\partial z} + y\frac{\partial B_2}{\partial z} + z\frac{\partial B_3}{\partial z} + \frac{\partial B_0}{\partial z}\right). \end{aligned}\right\} \quad (2.4.7)$$

The scalar B_0 can be taken as follows

$$B_0 = -(xB_1 + yB_2 + zB_3) - 2(1-\nu)\int_z^\infty B_3\,dz \quad (2.4.8)$$

provided that the right hand side of this relationship satisfies Laplace's equation

$$\nabla^2(xB_1 + yB_2 + zB_3) = 2\,\mathrm{div}\,\mathbf{B} = 0, \quad \frac{\partial B_3}{\partial z} = -\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y}\right). \quad (2.4.9)$$

By virtue of eq. (2.4.7) we then arrive at the simple boundary conditions

$$4\mu(1-\nu)\frac{\partial B_1}{\partial z} = -q_1(x,y), \quad 4\mu(1-\nu)\frac{\partial B_2}{\partial z} = -q_2(x,y) \quad (2.4.10)$$

which enable B_1 and B_2 to be defined as the simple layer potentials

$$B_1 = \frac{1}{8\pi\mu(1-\nu)}\varphi_1(x,y,z), \quad B_2 = \frac{1}{8\pi\mu(1-\nu)}\varphi_2(x,y,z), \tag{2.4.11}$$

where

$$\varphi_i(x,y,z) = \iint\limits_{\Omega} \frac{q_i(x',y')}{R'} do' \quad (i=1,2). \tag{2.4.12}$$

Due to eq. (2.4.9) we also have

$$\left.\begin{aligned} B_3 &= -\frac{1}{8\pi\mu(1-\nu)}\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial x}\right), \\ \psi_i(x,y,z) &= \iint\limits_{\Omega} q_i(x',y')\ln(R'+z)\, do'. \end{aligned}\right\} \tag{2.4.13}$$

By eqs. (2.4.1) and (2.4.9) the normal stress σ_z takes the form

$$\sigma_z = 4(1-\nu)\mu\frac{\partial B_3}{\partial z}. \tag{2.4.14}$$

It remains to satisfy the third boundary condition in eq. (2.1.2). This leads to the considered problem of the state of stress in the half-space in which the shear stresses are absent on the boundary $z=0$ and the normal stresses are equal to

$$\sigma_z\big|_{z=0} = -p(x,y) - 4(1-\nu)\mu\frac{\partial B_3}{\partial z}. \tag{2.4.15}$$

According to eqs. (2.4.2) and (2.4.6) the harmonic Papkovich functions B_i^* solving this problem are given by the following equalities

$$\left.\begin{aligned} &B_1^* = 0, \quad B_2^* = 0, \quad B_3^* = \frac{1}{4\pi\mu}\omega - 2(1-\nu)B_3, \\ &\frac{\partial B_0^*}{\partial z} = (1-2\nu)B_3^*. \end{aligned}\right\} \tag{2.4.16}$$

The original boundary value problem is solved by superimposing these solutions, to yield the following expressions for the stresses

$$\left.\begin{aligned} \tau_{xz} &= \frac{1}{2\pi}\frac{\partial\varphi_1}{\partial z} - \frac{1}{2(1+\nu)}z\frac{\partial\sigma}{\partial x}, \\ \tau_{yz} &= \frac{1}{2\pi}\frac{\partial\varphi_2}{\partial z} - \frac{1}{2(1+\nu)}z\frac{\partial\sigma}{\partial y}, \\ \sigma_z &= \frac{1}{2\pi}\frac{\partial\omega}{\partial z} - \frac{1}{2(1+\nu)}z\frac{\partial\sigma}{\partial z}, \end{aligned}\right\} \tag{2.4.17}$$

where

$$\sigma = \frac{1+\nu}{\pi}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} + \frac{\partial \omega}{\partial z}\right). \tag{2.4.18}$$

It will be proved below that σ is a sum of three normal stresses. The displacement vector is determined by superimposing the above solutions

$$\mathbf{u} = 4(1-\nu)(\mathbf{B}+\mathbf{B}^*) - \operatorname{grad}\left[\mathbf{R}\cdot(\mathbf{B}+\mathbf{B}^*) + (B_0 + B_0^*)\right].$$

The result is presented in the form

$$\left.\begin{aligned}
2\pi\mu u &= \varphi_1 - \frac{1}{2}z\frac{\partial}{\partial x}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) + \nu\frac{\partial}{\partial x}\left(\frac{\partial \chi_1}{\partial x} + \frac{\partial \chi_2}{\partial y}\right) - \\
&\qquad \frac{1}{2}z\frac{\partial \omega}{\partial x} - \frac{1}{2}(1-2\nu)\frac{\partial \omega_1}{\partial x}, \\
2\pi\mu v &= \varphi_2 - \frac{1}{2}z\frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) + \nu\frac{\partial}{\partial y}\left(\frac{\partial \chi_1}{\partial x} + \frac{\partial \chi_2}{\partial y}\right) - \\
&\qquad \frac{1}{2}z\frac{\partial \omega}{\partial y} - \frac{1}{2}(1-2\nu)\frac{\partial \omega_1}{\partial y}, \\
2\pi\mu w &= \frac{1}{2}(1-2\nu)\varphi_2\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) - \frac{1}{2}z\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}\right) + \\
&\qquad (1-\nu)\omega - \frac{1}{2}z\frac{\partial \omega}{\partial z},
\end{aligned}\right\} \tag{2.4.19}$$

where the following potentials

$$\chi_i(x,y,z) = \iint_\Omega q_i(x',y')\left[z\ln(R'+z) - R'\right]do' \quad (i=1,2) \tag{2.4.20}$$

are introduced. The dilatation ϑ obtained by means of the above equations is given by

$$\vartheta = \frac{1-2\nu}{2\pi\mu}\left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} + \frac{\partial \omega}{\partial z}\right), \tag{2.4.21}$$

that also leads to eq. (2.4.18).

Formulae (2.4.19) present the solution of the Boussinesq-Cerruti problem.

5.2.5 *The influence tensor in elastic half-space*

We seek the state of stress in the elastic half-space $z > 0$ caused by a concentrated force applied at point Q.

Into consideration are introduced the point $Q_*(0,0,-h)$ and the force $\mathbf{P}_*$

$$\mathbf{P}_* = P_1\mathbf{i}_1 + P_2\mathbf{i}_2 - P_3\mathbf{i}_3 = \mathbf{P} - 2\mathbf{i}_3 P_3 \tag{2.5.1}$$

which are the mirror mapping of point Q and force $\mathbf{P}$ in plane $z = 0$. The position vectors of the point of observation $M(x, y, z)$ having origins at points Q and Q_* are denoted respectively as

$$\mathbf{R} = \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 (z - h), \quad \mathbf{R}_* = \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 (z + h). \tag{2.5.2}$$

The sought state of stress is presented as a sum of three states: two states $\hat{T}^0$ and $\hat{T}^0_*$ in the unbounded elastic space caused by the concentrated forces $\mathbf{P}$ and $\mathbf{P}_*$ at points Q and Q_*, respectively, and state $\hat{T}'$, the latter having no singularities in the half-space $z > 0$ and being chosen in such a way that the boundary $z = 0$ is free of loading due to the state $\hat{T}^0 + \hat{T}^0_*$

$$\mathbf{i}_3 \cdot \hat{T} = \mathbf{i}_3 \cdot \left(\hat{T}^0 + \hat{T}^0_* + \hat{T}' \right)\Big|_{z=0} = 0. \tag{2.5.3}$$

By virtue of eqs. (3.5.6) and (3.5.7) of Chapter 4 the tensors $\hat{T}^0$ and $\hat{T}^0_*$ are given by

$$\left.\begin{aligned} \hat{T}^0 &= \frac{1}{8\pi(1-\nu)R^3}\left[(1-2\nu)\left(\hat{E}\mathbf{P}\cdot\mathbf{R} - \mathbf{P}\mathbf{R} - \mathbf{R}\mathbf{P}\right) - \frac{3\mathbf{R}\mathbf{R}}{R^2}\mathbf{P}\cdot\mathbf{R}\right], \\ \hat{T}^0_* &= \frac{1}{8\pi(1-\nu)R_*^3}\left[(1-2\nu)\left(\hat{E}\mathbf{P}_*\cdot\mathbf{R}_* - \mathbf{P}_*\mathbf{R}_* - \mathbf{R}_*\mathbf{P}_*\right) - \right. \\ &\qquad\qquad \left. \frac{3\mathbf{R}_*\mathbf{R}_*}{R_*^2}\mathbf{P}_*\cdot\mathbf{R}_*\right]. \end{aligned}\right\} \tag{2.5.4}$$

On the plane $z = 0$

$$R = R_* = R_0 = \sqrt{x^2 + y^2 + h^2},$$

so that

$$\begin{aligned} -\left(\mathbf{i}_3\cdot\hat{T}'\right)\Big|_{z=0} = \frac{1}{4\pi(1-\nu)R_0^3}\mathbf{i}_3\left[(1-2\nu)(P_1x + P_2y + P_3h) - \right. \\ \left. \frac{3h^2}{R_0^2}(P_1x + P_2y - P_3h)\right] \end{aligned} \tag{2.5.5}$$

and the boundary condition (2.5.3) leads to the problem of the state of stress in the half-space under the normal loading of the bounding plane. Clearly, this is to be expected because of the symmetry. The boundary conditions are written down in the form

$$\left.\begin{aligned} z = 0: \quad \sigma'_z = -\frac{P_3 h}{4\pi(1-\nu)}\left(\frac{1-2\nu}{R_0^3} + \frac{3h^2}{R_0^5}\right) - \\ \frac{P_1x + P_2y}{4\pi(1-\nu)}\left(\frac{1-2\nu}{R_0^3} - \frac{3h^2}{R_0^5}\right), \quad \tau'_{xz} = 0, \quad \tau'_{yz} = 0. \end{aligned}\right\} \tag{2.5.6}$$

Let us next consider each of the groups of terms appearing in the boundary condition (2.5.6). The pairs of harmonic functions solving these problems are denoted by ω', ω_1' and ω'', ω_1'', see Subsection 5.2.3.

By eq. (2.3.5) we have

$$\begin{aligned}\left.\frac{\partial \omega'}{\partial z}\right|_{z\to 0} &= -\frac{P_3 h}{2(1-\nu)}\left(\frac{1-2\nu}{R_0^3}+\frac{3h^2}{R_0^5}\right)\\ &= -\frac{P_3 h}{R_0^3}-\frac{P_3 h}{2(1-\nu)}\left(-\frac{1}{R_0^3}+\frac{3h^2}{R_0^5}\right)\end{aligned} \tag{2.5.7}$$

and

$$\frac{\partial}{\partial z}\frac{1}{R_*} = -\frac{z+h}{R_*^3}, \quad \frac{\partial^2}{\partial z^2}\frac{1}{R_*} = -\frac{1}{R_*^3}+\frac{3(z+h)^2}{R_*^5}.$$

This allows one to rewrite eq. (2.5.7) as follows

$$\left.\frac{\partial \omega'}{\partial z}\right|_{z=0} = \left.\left[P_3\frac{\partial}{\partial z}\frac{1}{R_*}-\frac{P_3 h}{2(1-\nu)}\frac{\partial^2}{\partial z^2}\frac{1}{R_*}\right]\right|_{z=0}. \tag{2.5.8}$$

Both $\partial\omega'/\partial z$ and the function in the brackets are harmonic in the region $z>0$ and are coincident on the boundary of this region. Hence equality (2.5.8) is fulfilled in the whole half-space $z>0$. For this reason, referring to eq. (2.3.8) we obtain

$$\omega' = \frac{P_3}{R_*}-\frac{P_3 h}{2(1-\nu)}\frac{\partial}{\partial z}\frac{1}{R_*}, \quad \omega_1' = P_3\ln(R_*+z+h)-\frac{P_3 h}{2(1-\nu)}\frac{1}{R_*}. \tag{2.5.9}$$

An analogous calculation is performed for the second pair of terms in the boundary condition (2.5.6), to give

$$\frac{x}{R_*^3} = -\frac{\partial}{\partial x}\frac{1}{R_*} = -\frac{\partial^2}{\partial x\partial z}\ln(R_*+z+h), \quad \frac{3x(z+h)}{R_*^5} = \frac{\partial^2}{\partial x\partial z}\frac{1}{R_*}.$$

Referring to eq. (1.4.8) we have

$$\left.\begin{aligned}\omega'' &= \frac{1}{2(1-\nu)}\left(P_1\frac{\partial}{\partial x}+P_2\frac{\partial}{\partial y}\right)\left[(1-2\nu)\ln(R_*+z+h)+\frac{h}{R_*}\right],\\ \omega_1'' &= \frac{1}{2(1-\nu)}\left(P_1\frac{\partial}{\partial x}+P_2\frac{\partial}{\partial y}\right)\times\\ &\quad \{(1-2\nu)(z+h)[\ln(R_*+z+h)-R_*]+h\ln(R_*+z+h)\}.\end{aligned}\right\} \tag{2.5.10}$$

The solution to the problem is thus given by the potentials

$$\left.\begin{aligned}
\omega = \omega' + \omega'' = \frac{1}{2(1-\nu)} \frac{P_3}{R_*} + \\
\frac{1-2\nu}{2(1-\nu)} \mathbf{P} \cdot \nabla \ln (R_* + z + h) + \frac{h}{2(1-\nu)} \mathbf{P}_* \cdot \nabla \frac{1}{R_*}, \\
\omega_1 = \omega_1' + \omega_1'' = \frac{1}{2(1-\nu)} P_3 \ln (R_* + z + h) + \\
\frac{1-2\nu}{2(1-\nu)} \mathbf{P} \cdot \nabla [(z+h) \ln (R_* + z + h) - R_*] + \\
\frac{h}{2(1-\nu)} \mathbf{P}_* \cdot \nabla \ln (R_* + z + h).
\end{aligned}\right\} \qquad (2.5.11)$$

The displacement vector is calculated by means of formulae (3.5.8) and (3.5.9) of Chapter 4 and eq. (2.3.4). This solves the problem of constructing the influence tensor for the elastic half-space.

5.2.6 *Thermal stresses in the elastic half-space*

In what follows relationship (3.4.3) is applied under the assumption that $\hat{G}(M, Q)$ denotes the influence tensor of the elastic half-space. It is sufficient to know the divergence of this tensor which is equal to the sum of divergences of displacements $\mathbf{u}^0, \mathbf{u}_*^0, \mathbf{u}'$ corresponding to the stress tensors $\hat{T}^0, \hat{T}_*^0, \hat{T}'$ introduced in Subsection 5.2.5. The expression for $\operatorname{div} \mathbf{u}^0$ is given by eq. (1.5.1) and $\operatorname{div} \mathbf{u}_*^0$ is obtained from the latter by replacing $\mathbf{R}$ by $\mathbf{R}_*$. The result is

$$\begin{aligned}
\operatorname{div} \mathbf{u}^0 (M, Q) &= \frac{1-2\nu}{8\pi\mu(1-\nu)} \nabla \frac{1}{R} \cdot \mathbf{P}, \qquad \operatorname{div} \mathbf{u}_*^0 (M, Q) = \\
&= \frac{1-2\nu}{8\pi\mu(1-\nu)} \nabla \frac{1}{R_*} \cdot \mathbf{P}_* = \frac{1-2\nu}{8\pi\mu(1-\nu)} \left(\nabla \frac{1}{R_*} \cdot \mathbf{P} - 2P_3 \frac{\partial}{\partial z} \frac{1}{R_*} \right).
\end{aligned}$$

The divergence of displacement vector $\mathbf{u}'$ is found by means of formulae (2.5.11) and (2.3.4)

$$\begin{aligned}
\operatorname{div} \mathbf{u}' &= \frac{1-2\nu}{2\pi\mu} \frac{\partial \omega}{\partial z} \\
&= \frac{1-2\nu}{4\pi\mu(1-\nu)} \left[P_3 \frac{\partial}{\partial z} \frac{1}{R_*} + (1-2\nu) \mathbf{P} \cdot \nabla \frac{1}{R_*} + h \mathbf{P}_* \cdot \nabla \frac{\partial}{\partial z} \frac{1}{R_*} \right].
\end{aligned}$$

Adding these expressions and substituting $\mathbf{P}_*$ from eq. (2.5.1) we obtain

$$\operatorname{div} \mathbf{u} = \frac{1-2\nu}{4\pi\mu(1-\nu)} \left[\frac{1}{2} \nabla \left(\frac{1}{R} + \frac{3-4\nu}{R_*} \right) + h \nabla \frac{\partial}{\partial z} \frac{1}{R_*} - 2h \frac{\partial^2}{\partial z^2} \frac{1}{R_*} \mathbf{i}_3 \right] \cdot \mathbf{P}. \tag{2.6.1}$$

$$\left.\begin{aligned} R &= \left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{1/2}, \\ R_* &= \left[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2\right]^{1/2}. \end{aligned}\right\} \tag{2.6.2}$$

By eq. (3.4.3) of Chapter 4 we have

$$\mathbf{u}(\xi,\eta,\zeta) = \mathbf{u}(Q) = \frac{1+\nu}{2\pi(1-\nu)}\alpha \iiint\limits_V \theta(M)\left[\frac{1}{2}\operatorname{grad}_M\left(\frac{1}{R}+\frac{3-4\nu}{R_*}\right)\right.$$
$$\left. +\zeta\frac{\partial}{\partial z}\nabla_M\frac{1}{R_*} - 2\mathbf{i}_3\zeta\frac{\partial^2}{\partial z^2}\frac{1}{R_*}\right] d\tau_M. \tag{2.6.3}$$

Here V denotes a heated volume which completely lies in the half-space $z>0$ and $\theta(M)$ denotes a temperature distribution in this volume. The gradients are calculated at point $M(x,y,z)$ which is the source point now. Taking into account the relationships

$$\left.\begin{aligned} &\operatorname{grad}_M\frac{1}{R} = -\operatorname{grad}_Q\frac{1}{R}, \quad \frac{\partial}{\partial z}\frac{1}{R_*} = \frac{\partial}{\partial\zeta}\frac{1}{R_*}, \\ &\operatorname{grad}_M\frac{1}{R_*} = -\operatorname{grad}_Q\frac{1}{R_*} + 2\mathbf{i}_3\frac{\partial}{\partial\zeta}\frac{1}{R_*} \end{aligned}\right\} \tag{2.6.4}$$

we can put eq. (2.6.3) in the form

$$\mathbf{u}(Q) = \nabla_Q\chi_1 + (3-4\nu)\left(\nabla_Q - 2\mathbf{i}_3\frac{\partial}{\partial\zeta}\right)\chi_2 + 2\zeta\frac{\partial}{\partial\zeta}\nabla_Q\chi_2, \tag{2.6.5}$$

where the following potentials are introduced

$$\chi_1 = -\frac{1+\nu}{4\pi(1-\nu)}\alpha\iiint\limits_V \frac{\theta(M)}{R}d\tau_M, \quad \chi_2 = -\frac{1+\nu}{4\pi(1-\nu)}\alpha\iiint\limits_V \frac{\theta(M)}{R_*}d\tau_M. \tag{2.6.6}$$

The term determined by potential χ_1 described the field of displacement which was considered in Subsection 5.1.5 for the unbounded elastic medium. Function χ_2 is harmonic in the half-space $z>0$ and the stresses calculated in terms of χ_2

$$\left.\begin{aligned} \tau_{xz} &= 2\mu\left(2z\frac{\partial^3\chi_2}{\partial x\partial z^2} + \frac{\partial^2\chi_2}{\partial x\partial z}\right), \\ \tau_{yz} &= 2\mu\left(2z\frac{\partial^3\chi_2}{\partial y\partial z^2} + \frac{\partial^2\chi_2}{\partial y\partial z}\right), \\ \sigma_z &= 2\mu\left(2z\frac{\partial^3\chi_2}{\partial z^3} - \frac{\partial^2\chi_2}{\partial z^2}\right) \end{aligned}\right\} \tag{2.6.7}$$

annihilate the stresses due to potential χ_1 on plane $z=0$.

5.2.7 The case of the steady-state temperature

In a steady-state regime, temperature $\theta(x,y,z)$ is a harmonic function in the half-space $z>0$. Its value on the boundary $z=0$ is assumed to be given by

$$\theta(x,y,0)=\begin{cases}\theta_0(x,y,0), & (x,y)\subset\Omega,\\ 0, & (x,y)\not\subset\Omega.\end{cases} \tag{2.7.1}$$

Introducing into consideration the simple layer potential with the density $-\theta_0/2\pi$

$$\Phi(x,y,z)=-\frac{1}{2\pi}\iint\limits_{\Omega}\frac{\theta_0(x',y',0)}{R'}do',\ R'=\left[(x-x')^2+(y-y')^2+z^2\right]^{1/2} \tag{2.7.2}$$

one can write the solution of the problem of thermal conductivity for the half-space in the form

$$\theta(x,y,z)=\frac{\partial\Phi}{\partial z}. \tag{2.7.3}$$

Indeed, the function determined by this equation is harmonic and satisfies boundary condition (2.7.1) which follows from eq. (2.3.6).

Proceeding to solve the problem of the elasticity theory we use two harmonic functions (B_3 and B_0) in the Papkovich-Neuber solution

$$\mathbf{u}=4(1-\nu)\,\mathbf{i}_3B_3-\nabla(zB_3+B_0)+\nabla\psi. \tag{2.7.4}$$

The latter term takes into account the temperature field and function ψ is a particular solution of eq. (1.1.4.8) of Chapter 4

$$\nabla^2\psi=\frac{1+\nu}{1-\nu}\alpha\theta. \tag{2.7.5}$$

The stress components on the surfaces perpendicular to axis z are calculated using eq. (2.4.1) and taking account of $\nabla\psi$ in eq. (2.7.4)

$$\tau_{xz}=\mu\frac{\partial M}{\partial x},\ \tau_{yz}=\mu\frac{\partial M}{\partial y},\ \sigma_z=\mu\left(4\frac{\partial B_3}{\partial z}-2\frac{1+\nu}{1-\nu}\alpha\theta+\frac{\partial M}{\partial z}\right), \tag{2.7.6}$$

where for brevity

$$M=4(1-\nu)B_2-2\frac{\partial}{\partial z}(zB_3+B_0)+2\frac{\partial\psi}{\partial z}. \tag{2.7.7}$$

It is possible to choose B_3 and B_0 in such a way that M vanishes. To this end we take

$$zB_3=\psi,\ \ \frac{\partial B_0}{\partial z}=2(1-\nu)B_3. \tag{2.7.8}$$

It follows from the first equality and eqs. (2.7.5) and (2.7.3) that

$$\nabla^2 z B_3 = 2\frac{\partial B_3}{\partial z} = \frac{1+\nu}{1-\nu}\alpha\theta, \quad B_3 = \frac{1+\nu}{2(1-\nu)}\alpha\Phi(x,y,z), \tag{2.7.9}$$

which is possible as Φ is a harmonic function. Returning to formulae (2.7.6) we arrive at a result which was difficult to anticipate: under a steady-state thermal regime the thermal stresses are absent on the planes parallel to the boundary

$$\tau_{xz} = 0, \quad \tau_{yz} = 0, \quad \sigma_z = 0 \quad (z \geq 0). \tag{2.7.10}$$

This property retains in the problem of thermal stresses in the elastic layer under a stead-state temperature.

By virtue of eqs. (2.7.8) and (2.7.2) we find

$$B_0 = -\frac{1}{2\pi}(1+\nu)\alpha \iint\limits_{\Omega} \theta_0(x',y',0)\ln(R'+z)\,do', \tag{2.7.11}$$

and, referring to eq. (2.7.4), we can write down the expressions for the projections of the displacement vector

$$u = -\frac{\partial B_0}{\partial x}, \quad v = -\frac{\partial B_0}{\partial y}, \quad w = \frac{\partial B_0}{\partial z}. \tag{2.7.12}$$

The temperature is also determined in terms of this harmonic function B_0

$$(1+\nu)\alpha\theta(x,y,z) = \frac{\partial^2 B_0}{\partial z^2}. \tag{2.7.13}$$

The non-trivial components of the stress tensor are obtained by eq. (1.14.1) of Chapter 4 and are given by

$$\sigma_x = 2\mu\frac{\partial^2 B_0}{\partial y^2}, \quad \sigma_y = 2\mu\frac{\partial^2 B_0}{\partial x^2}, \quad \tau_{xy} = -2\mu\frac{\partial^2 B_0}{\partial x \partial y}. \tag{2.7.14}$$

5.2.8 Calculation of the simple layer potential for the plane region

As shown above the solution of the problem of the state of stress in the elastic half-space essentially depends on knowledge of the potential of the layer distributed over the plane region. First of all, the simple layer potential is required since the more complex potentials are determined in terms of this potential by integration over x.

Let $M^1(x,y,0)$ denote the projection of the observation point $M(x,y,z)$ on the plane $z=0$. Taking M^1 as being the origin of the polar coordinate system (ρ,λ) we have

$$x'-x = \rho\cos\lambda, \; y'-y = \rho\sin\lambda, \; do' = \rho d\rho d\lambda, \; R'^2 = \rho^2 + z^2. \tag{2.8.1}$$

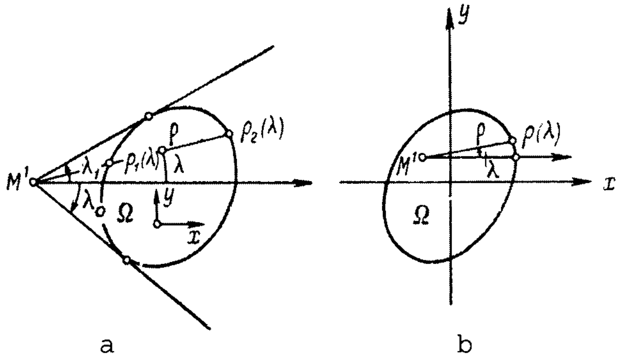

FIGURE 5.1.

The expression for the simple layer potential (2.3.2) is now set in the form

$$\omega(x,y,z) = \iint_\Omega \frac{p(x',y')}{R'} do' \iint_\Omega p(x+\rho\cos\lambda, y+\rho\sin\lambda) \frac{\rho d\rho d\lambda}{\sqrt{\rho^2+z^2}}$$

$$= \int_{\lambda_0}^{\lambda_1} d\lambda \int_{\rho_1(\lambda)}^{\rho_2(\lambda)} p(x+\rho\cos\lambda, y+\rho\sin\lambda) \frac{\rho d\rho}{\sqrt{\rho^2+z^2}}. \tag{2.8.2}$$

Figure 5.1a explains the notation for $M^1 \not\subset \Omega$. If $M^1 \subset \Omega$ then it is necessary to take $\rho_1(\lambda) = 0, \rho_2(\lambda) = \rho(\lambda), \lambda_0 = 0, \lambda_1 = 2\pi$, see Fig. 5.1b.

The calculation is simplified if the value of the potentials is sought at points on the plane $z = 0$. Then

$$\omega(x,y,0) = \int_{\lambda_0}^{\lambda_1} \Theta(x,y,\lambda)\, d\lambda, \tag{2.8.3}$$

where

$$\Theta(x,y,\lambda) = \int_{\rho_1(\lambda)}^{\rho_2(\lambda)} p(x+\rho\cos\lambda, y+\rho\sin\lambda)\, d\rho. \tag{2.8.4}$$

When the density is constant ($p = \text{const}$) then

$$\omega(x,y,z) = \int_{\lambda_0}^{\lambda_1} \left(\sqrt{\rho_2^2(\lambda)+z^2} - \sqrt{\rho_1^2(\lambda)+z^2}\right) d\lambda \tag{2.8.5}$$

and, in particular, on plane $z = 0$

$$\omega(x, y, 0) = p \int_{\lambda_0}^{\lambda_1} [\rho_2(\lambda) - \rho_1(\lambda)]\, d\lambda. \tag{2.8.6}$$

For example, calculating the potential of a circular region Ω of radius a yields

$$\omega(r, 0) = \begin{cases} 4paE\left(\dfrac{r}{a}\right), & M \subset \Omega, \\ 4pr\left[E\left(\dfrac{a}{r}\right) - \left(1 - \dfrac{a^2}{r^2}\right) K\left(\dfrac{a}{r}\right)\right], & M \not\subset \Omega, \end{cases} \tag{2.8.7}$$

where r is the distance between the observation point $M(x, y, 0)$ and the centre of the disc, while $K(k)$ and $E(k)$ denote the complete elliptic integrals of the first and second kind respectively.

5.2.9 Dirichlet's problem for the half-space

Function $W_*(x, y, z)$ is harmonic in the half-space $z > 0$, represents the double layer potential of density $\mu(x, y)$ distributed over region Ω in the plane $z = 0$ and is given by

$$W_*(x, y, z) = z \iint_\Omega \frac{\mu(\xi, \eta)}{R'^3} d\xi d\eta = -\frac{\partial}{\partial z} \iint_\Omega \frac{\mu(\xi, \eta)}{R'} d\xi d\eta = -\frac{\partial \omega}{\partial z}, \tag{2.9.1}$$

where $R' = \left[(x - \xi)^2 + (y - \eta)^2 + z^2\right]^{1/2}$ and function

$$\omega(x, y, z) = \iint_\Omega \frac{\mu(\xi, \eta)}{R'} d\xi d\eta \tag{2.9.2}$$

is the simple layer potential of the same density $\mu(x, y)$. By eq. (2.3.6)

$$W_*(x, y, z)|_{z \to \pm 0} = -\left.\frac{\partial \omega}{\partial z}\right|_{z \to \pm 0} = \begin{cases} \pm 2\pi\mu(x, y), & (x, y) \subset \Omega, \\ 0, & (x, y) \not\subset \Omega. \end{cases}$$

For this reason the harmonic function

$$W(x, y, z) = \frac{z}{2\pi} \iint_\Omega \frac{\mu(\xi, \eta)}{R'^3} d\xi d\eta \tag{2.9.3}$$

yields the solution of Dirichlet's problem in the half-space $z > 0$

$$z = 0: \quad W(x, y, z) = \begin{cases} \mu(x, y), & (x, y) \subset \Omega, \\ 0, & (x, y) \not\subset \Omega. \end{cases} \tag{2.9.4}$$

Let the observation point $M(x,y,z)$ lie within the cylinder having the base Ω and the generating line parallel to axis z. First, we consider the case of the density which is constant on Ω, then by eqs. (2.8.5) and (2.8.6)

$$\omega(x,y,z) = \mu \int_0^{2\pi} \left(\sqrt{\rho^2(\lambda) + z^2} - z\right) d\lambda,$$

$$W(x,y,z) = \mu - \frac{\mu}{2\pi} \int_0^{2\pi} \frac{z d\lambda}{\sqrt{\rho^2(\lambda) + z^2}}$$

and furthermore

$$\frac{\partial W}{\partial z} = -\frac{\mu}{2\pi} \int_0^{2\pi} \frac{\rho^2(\lambda) d\lambda}{[\rho^2(\lambda) + z^2]^{3/2}}, \quad \left.\frac{\partial W}{\partial z}\right|_{z=0} = -\frac{\mu}{2\pi} \int_0^{2\pi} \frac{d\lambda}{\rho(\lambda)}. \tag{2.9.5}$$

Let us notice that the right hand side is not constant as $\rho(\lambda)$ depends on the choice of the origin $(x,y,0)$ of the polar coordinate system ρ, λ. For the arbitrary density one can put eq. (2.8.2) in the form

$$\omega(x,y,z) = \int_0^{2\pi} d\lambda \int_0^{\rho(\lambda)} [\mu(x + \rho\cos\lambda, y + \rho\sin\lambda) - \mu(x,y)] \frac{\rho d\rho}{\sqrt{\rho^2 + z^2}} +$$
$$\mu(x,y) \int_0^{2\pi} \left(\sqrt{\rho^2(\lambda) + z^2} - z\right) d\lambda,$$

to obtain

$$W(x,y,z) = \frac{z}{2\pi} \int_0^{2\pi} d\lambda \int_0^{\rho(\lambda)} [\mu(x + \rho\cos\lambda, y + \rho\sin\lambda) - \mu(x,y)] \times$$
$$\frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} + \mu(x,y) - \frac{\mu(x,y)}{2\pi} \int_0^{2\pi} \frac{z d\lambda}{\sqrt{\rho^2(\lambda) + z^2}} \tag{2.9.6}$$

and

$$\frac{\partial W}{\partial z} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \int_0^{\rho(\lambda)} [\mu(x + \rho\cos\lambda, y + \rho\sin\lambda) - \mu(x,y)] \frac{\rho^3 - 2\rho z^2}{(\rho^2 + z^2)^{5/2}} d\rho$$
$$- \frac{\mu(x,y)}{2\pi} \int_0^{2\pi} \frac{\rho^2(\lambda) d\lambda}{[\rho^2(\lambda) + z^2]^{3/2}}.$$

Assuming

$$\mu(x+\rho\cos\lambda, y+\rho\sin\lambda) - \mu(x,y) =$$
$$= \rho\left(\frac{\partial\mu}{\partial x}\cos\lambda + \frac{\partial\mu}{\partial y}\sin\lambda\right) + \rho^2 g(\rho,\lambda;x,y), \quad (2.9.7)$$

where $g(\rho,\lambda;x,y)$ is finite at $\rho = 0$ and taking into account that

$$\int_0^{\rho(\lambda)} \frac{\rho^4 - 2\rho^2 z^2}{(\rho^2+z^2)^{5/2}} d\rho = -\frac{\rho(\lambda)}{\sqrt{\rho^2(\lambda)+z^2}} -$$
$$\frac{\rho^3(\lambda)}{[\rho^2(\lambda)+z^2]^{3/2}} + \ln\left[\rho(\lambda) + \sqrt{\rho^2(\lambda) z + z^2}\right] - \ln z,$$

we easily obtain

$$\left.\frac{\partial W}{\partial z}\right|_{z\to 0} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\mu}{\partial x}\cos\lambda + \frac{\partial\mu}{\partial y}\sin\lambda\right)\ln\rho(\lambda)\,d\lambda -$$
$$\frac{1}{2\pi}\mu(x,y)\int_0^{2\pi}\frac{d\lambda}{\rho(\lambda)} + \frac{1}{2\pi}\int_0^{2\pi} d\lambda \int_0^{\rho(\lambda)} g(\rho,\lambda;x,y)\,d\rho. \quad (2.9.8)$$

This proves that if the density can be represented in the form of eq. (2.9.7) the normal derivative of the double layer potential is bounded when the observation point passes to point $(x,y) \subset \Omega$ remaining within the above cylinder.

Referring to eqs. (2.9.1) and (2.8.2) we also have

$$\left.\frac{\partial W}{\partial z}\right|_{z\to 0} = \frac{1}{2\pi}\int_{\lambda_0}^{\lambda_1} d\lambda \int_{\rho_1(\lambda)}^{\rho_2(\lambda)} \mu(x+\rho\cos\lambda, y+\rho\sin\lambda)\frac{d\rho}{\rho^2}, \ (x,y) \not\subset \Omega \quad (2.9.9)$$

and the normal derivative is also bounded. It vanishes at infinity since $\lambda_1 - \lambda_0 \to 0$ for $\sqrt{x^2+y^2} \to \infty$.

5.2.10 *The first boundary value problem for the half-space*

The displacements are assumed to be given on the plane $z = 0$

$$z = 0: \quad u = u_0(x,y), \quad v = v_0(x,y), \quad w = w_0(x,y). \quad (2.10.1)$$

The solution of the equations of elasticity is taken in Tedone's form, eq. (1.3.10) of Chapter 4

$$u = a_1 - \frac{x\vartheta}{2(1-2\nu)}, \ v = a_2 - \frac{y\vartheta}{2(1-2\nu)}, \ w = a_3 - \frac{z\vartheta}{2(1-2\nu)}, \quad (2.10.2)$$

where a_1, a_2, a_3 and ϑ are harmonic functions. Referring to eq. (2.9.3) we obtain from the third equation in (2.10.2)

$$a_3|_{z=0} = w_0(x,y), \quad a_3 = \frac{z}{2\pi}\iint \frac{w_0(\xi,\eta)}{R'^3} d\xi d\eta. \tag{2.10.3}$$

This determines function $a_3(x,y,z)$ and its derivatives in the region $z > 0$. Now we have

$$\frac{\partial w}{\partial z} = \vartheta - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial a_3}{\partial z} - \frac{1}{2(1-2\nu)}\left(\vartheta + z\frac{\partial \vartheta}{\partial z}\right), \tag{2.10.4}$$

so that

$$\vartheta|_{z=0} = \frac{2(1-2\nu)}{3-4\nu}\left(\left.\frac{\partial a_3}{\partial z}\right|_{z=0} + \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right). \tag{2.10.5}$$

Using eq. (2.9.3) we find

$$\vartheta(x,y,z) = \frac{z}{2\pi}\iint \frac{\vartheta(\xi,\eta,0)}{R'^3} d\xi d\eta. \tag{2.10.6}$$

Therefore

$$\left.\begin{aligned} a_1 &= \frac{z}{2\pi}\iint \left[u_0(\xi,\eta) + \frac{\xi\vartheta(\xi,\eta,0)}{2(1-2\nu)}\right]\frac{d\xi d\eta}{R'^3}, \\ a_2 &= \frac{z}{2\pi}\iint \left[v_0(\xi,\eta) + \frac{\eta\vartheta(\xi,\eta,0)}{2(1-2\nu)}\right]\frac{d\xi d\eta}{R'^3}, \end{aligned}\right\} \tag{2.10.7}$$

and it remains to substitute the obtained expressions for the harmonic functions a_1, a_2, a_3, ϑ into eq. (2.10.2).

5.2.11 Mixed problems for the half-space

It is assumed that the displacements u, v and the normal stress σ_z are prescribed on the plane $z = 0$

$$z = 0: \quad u = u_0(x,y), \quad v = v_0(x,y), \quad \sigma_z = \sigma_z^0(x,y). \tag{2.11.1}$$

We have

$$\left.\begin{aligned} \frac{\sigma_z}{2\mu} &= \frac{\nu}{1-2\nu}\vartheta + \frac{\partial w}{\partial z} = \frac{1-\nu}{1-2\nu}\vartheta - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \\ \vartheta &= \frac{1-2\nu}{1-\nu}\left(\frac{\sigma_z}{2\mu} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \end{aligned}\right\} \tag{2.11.2}$$

and the harmonic function given on the boundary

$$z = 0: \quad \vartheta(x,y,0) = \frac{1-2\nu}{1-\nu}\left(\frac{\sigma_z^0}{2\mu} + \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) \tag{2.11.3}$$

is determined by the solution (2.9.3) of Dirichlet's problem

$$\vartheta(x,y,z) = \frac{z}{2\pi} \iint \frac{\vartheta(\xi,\eta,0)}{R'^3} d\xi d\eta. \tag{2.11.4}$$

By virtue of eqs. (2.10.4) and (2.11.3) we have

$$z=0: \quad \left.\frac{\partial a_3}{\partial z}\right|_{z=0} = \frac{3-4\nu}{2(1-\nu)}\frac{\sigma_z^0}{2\mu} + \frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = \rho(x,y)$$

and the harmonic function a_3 is presented by the simple layer potential

$$a_3(x,y,z) = -\frac{1}{2\pi} \iint \frac{\rho(\xi,\eta)}{R'} d\xi d\eta, \tag{2.11.5}$$

whereas a_1 and a_2 are given by eq. (2.10.7), where $\vartheta(\xi,\eta,0)$ needs to be replaced according to eq. (2.11.3).

Let us proceed to another mixed problem, namely the shear stresses and displacement w are prescribed on the plane $z=0$

$$z=0: \quad \tau_{xz} = \tau_{xz}^0(x,y), \quad \tau_{yz} = \tau_{yz}^0(x,y), \quad w = w_0(x,y). \tag{2.11.6}$$

Using eq. (2.10.2) and solving Dirichlet's problem for the harmonic function a_3 we have

$$a_3(x,y) = \frac{z}{2\pi} \iint \frac{w_0(\xi,\eta)}{R'^3} d\xi d\eta. \tag{2.11.7}$$

Referring to the equation in displacements, eq. (1.3.3) of Chapter 4, we obtain

$$\begin{aligned}\frac{\partial\vartheta}{\partial z} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial z^2} \\ &= \frac{1}{\mu}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) - 2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \nabla^2 w,\end{aligned}$$

so that

$$\begin{aligned} z=0: \quad \left.\frac{\partial\vartheta}{\partial z}\right|_{z=0} = \frac{1-2\nu}{2(1-\nu)}\Bigg[\frac{1}{\mu}\left(\frac{\partial \tau_{xz}^0}{\partial x} + \frac{\partial \tau_{yz}^0}{\partial y}\right) - \\ 2\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right)\Bigg] = \rho(x,y) \end{aligned} \tag{2.11.8}$$

and the harmonic function ϑ is determined as the simple layer potential

$$\vartheta(x,y,z) = -\frac{1}{2\pi} \iint \frac{\rho(\xi,\eta)}{R'} d\xi d\eta. \tag{2.11.9}$$

Equations (2.11.7) and (2.10.2) yield displacement $w(x, y, z)$. Harmonic functions a_1 and a_2 are determined as the simple layer potentials by means of the conditions

$$\left.\frac{\partial a_1}{\partial z}\right|_{z=0} = \frac{1}{\mu}\tau^0_{xz} - \frac{\partial w_0}{\partial x} + \frac{x\rho(x,y)}{2(1-\nu)},$$
$$\left.\frac{\partial a_2}{\partial z}\right|_{z=0} = \frac{1}{\mu}\tau^0_{yz} - \frac{\partial w_0}{\partial y} + \frac{y\rho(x,y)}{2(1-\nu)}.$$

Finally we obtain

$$\left.\begin{aligned} a_1 &= -\frac{1}{2\pi}\iint\left[\frac{1}{\mu}\tau^0_{xz}(\xi,\eta) - \frac{\partial w_0}{\partial \xi} + \frac{\xi\rho(\xi,\eta)}{2(1-2\nu)}\right]\frac{d\xi d\eta}{R'}, \\ a_2 &= -\frac{1}{2\pi}\iint\left[\frac{1}{\mu}\tau^0_{yz}(\xi,\eta) - \frac{\partial w_0}{\partial \eta} + \frac{\eta\rho(\xi,\eta)}{2(1-2\nu)}\right]\frac{d\xi d\eta}{R'}. \end{aligned}\right\} \tag{2.11.10}$$

The solutions obtained assume that loadings $\tau^0_{xz}(x,y)$ and $\tau^0_{yz}(x,y)$ are differentiable whilst displacement $w_0(x,y)$ is twice differentiable.

5.2.12 On Saint-Venant's principle. Mises's formulation

Subsections 1.1 and 1.2 of this chapter deal with the loading by forces distributed in a small volume and the state of stress in the unbounded elastic medium at large distances from the loading. The action of these forces was shown to be replaceable by the integral characteristics of these forces, i.e. the principal force, the principal moment and the force tensor, provided that the consideration is restricted to values of the first order in linear dimensions of this volume. At large distances from the loading the stresses due to the principal moment and the force tensor are of the same order. Now it will be shown that the same phenomenon is observed in the elastic half-space $z > 0$ when the forces are localised on a small area o of the boundary $z = 0$.

Let the origin of the coordinate system be chosen at point O of the surface o. The position radii of the (arbitrarily taken) source point Q and the observation point M are denoted by $\boldsymbol{\rho} = \mathbf{i}_1\xi + \mathbf{i}_2\eta$ and $\mathbf{R} = \overrightarrow{OM}$, respectively, then $\mathbf{R}' = \overrightarrow{QM} = \mathbf{R} - \boldsymbol{\rho}$ where $\rho \le \varepsilon$, ε being the radius of the circle having the centre at point O and covering surface o. Similar to eq. (1.1.4) we obtain up to values of the order of ε/R that

$$\frac{1}{R'} = \frac{1}{R} + \frac{1}{R^3}\boldsymbol{\rho}\cdot\mathbf{R}. \tag{2.12.1}$$

Let us view the potentials $\varphi_1, \varphi_2, \omega$ introduced in Subsection 5.2.4 as being the components of the following vector

$$\boldsymbol{\Phi} = \iint\limits_o \mathbf{f}(\xi,\eta)\frac{do}{R'} = \frac{1}{R}\iint\limits_o \mathbf{f}(\xi,\eta)\,do + \frac{\mathbf{R}}{R^3}\cdot\iint\limits_o \boldsymbol{\rho}\mathbf{f}(\xi,\eta)\,do, \tag{2.12.2}$$

where the density vector $\mathbf{f}(\xi,\eta)$ presents the surface force whose components are denoted by q_1, q_2, p. The integrals in eq. (2.12.2) imply that the principal vector $\mathbf{F}$ and the force tensor of the system of forces

$$\mathbf{F} = \iint\limits_{O} \mathbf{f}(\xi,\eta)\, do, \tag{2.12.3}$$

$$\iint\limits_{O} \boldsymbol{\rho}\mathbf{f}(\xi,\eta)\, do = \frac{1}{2}\iint\limits_{O} (\boldsymbol{\rho}\mathbf{f} + \mathbf{f}\boldsymbol{\rho})\, do + \frac{1}{2}\iint\limits_{O} (\boldsymbol{\rho}\mathbf{f} - \mathbf{f}\boldsymbol{\rho})\, do = \hat{p} + \hat{\Omega},$$

where the force tensor is split into the symmetric ($\hat{p}$) and skew-symmetric ($\hat{\Omega}$) parts. In addition to this

$$\mathbf{R}\cdot\hat{\Omega} = \frac{1}{2}\mathbf{R}\cdot\iint\limits_{O} (\boldsymbol{\rho}\mathbf{f} - \mathbf{f}\boldsymbol{\rho})\, do = \frac{1}{2}\iint\limits_{O} (\boldsymbol{\rho}\times\mathbf{f})\, do \times \mathbf{R} = \frac{1}{2}\mathbf{m}^0 \times \mathbf{R}, \tag{2.12.4}$$

where

$$\mathbf{m}^0 = \iint\limits_{O} \boldsymbol{\rho}\times\mathbf{f}\, do \tag{2.12.5}$$

is the principal moment of the system of forces $\mathbf{f}$ about point O. Hence, within the approximation (2.12.1)

$$\Phi = \frac{1}{R}\mathbf{F} + \frac{1}{2R^3}\mathbf{m}^0\times\mathbf{R} + \frac{1}{R^3}\mathbf{R}\cdot\hat{p}. \tag{2.12.6}$$

For the sake of brevity we restrict our consideration to the stress vector on the surfaces perpendicular to axis z. Due to eqs. (2.4.17) and (2.4.18) we have

$$\mathbf{k}\cdot\hat{T} = \frac{1}{2\pi}\left(\mathbf{k}\cdot\nabla\Phi - z\nabla\nabla\cdot\Phi\right). \tag{2.12.7}$$

The calculation yields

$$\nabla\Phi = -\frac{1}{R^3}\mathbf{R}\mathbf{F} - \frac{1}{2R^3}\mathbf{m}^0\times\hat{E} - \frac{3}{2R^5}\mathbf{R}\mathbf{m}^0\times\mathbf{R} - \frac{3}{R^5}\mathbf{R}\mathbf{R}\cdot\hat{p} + \frac{1}{R^3}\hat{p},$$

$\hat{E}$ denoting the unit tensor. Hence,

$$\begin{aligned}\nabla\cdot\Phi &= -\frac{1}{R^3}\mathbf{R}\cdot\mathbf{F} - \frac{3}{R^5}\mathbf{R}\cdot\hat{p}\cdot\mathbf{R} + \frac{1}{R^3}I_1(\hat{p})\\ &= -\frac{1}{R^3}\mathbf{R}\cdot\mathbf{F} - \frac{3}{R^5}\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R}.\end{aligned}$$

Then we have

$$\nabla\nabla\cdot\Phi = -\frac{1}{R^3}\mathbf{F} + \frac{3}{R^5}\mathbf{R}\mathbf{R}\cdot\mathbf{F} + \frac{15}{R^7}\mathbf{R}\mathbf{R}\cdot\operatorname{Dev}\hat{p}\cdot\mathbf{R} - \frac{6}{R^5}\mathbf{R}\cdot\operatorname{Dev}\hat{p},$$

$$k\cdot\nabla\Phi = -\frac{z}{R^3}\mathbf{F} - \frac{1}{2R^3}\mathbf{k}\times\mathbf{m}^0 - \frac{3z}{2R^5}\mathbf{m}^0\times\mathbf{R} - \frac{3z}{2R^5}\mathbf{R}\cdot\hat{p} + \frac{1}{R^3}\mathbf{k}\cdot\hat{p},$$

and the latter expression is simplified since

$$\mathbf{k}\cdot\hat{p}-\frac{1}{2}\mathbf{k}\times\mathbf{m}^0=\frac{1}{2}\mathbf{k}\cdot\iint_Q \mathbf{f}\boldsymbol{\rho}do-\frac{1}{2}\mathbf{k}\times\iint_Q \boldsymbol{\rho}\times\mathbf{f}do=0$$

because $\mathbf{k}\cdot\boldsymbol{\rho}=0$. Insertion into eq. (2.12.7) yields

$$\mathbf{k}\cdot\hat{T}=\frac{z}{2\pi R^5}\left(-3\mathbf{RR}\cdot\mathbf{F}-\frac{3}{2}\mathbf{m}^0\times\mathbf{R}-\mathbf{R}I_1(\hat{p})+\right.$$
$$\left.3\,\mathrm{Dev}\,\hat{p}\cdot\mathbf{R}-\frac{15}{R^2}\mathbf{RR}\cdot\mathrm{Dev}\,\hat{p}\cdot\mathbf{R}\right).\qquad(2.12.8)$$

Naturally, this expression vanishes at $z=0$ as the observation point must lie outside the loading area. Similar to the case of the unbounded space, the stress vector is expressed in terms of the principal vector, the principal moment, the first invariant and the deviator of the force tensor.

The term determined by the principal vector has the order of $F/R^2=\sigma$ whereas all remaining terms are of the order of $\sigma/R\varepsilon$, irrespective of whether the system is statically equivalent or not, that is whether $\mathbf{m}^0=0$ or not for $\mathbf{F}=0$. This forces one to accept a more careful formulation of Saint-Venant's principle (Subsection 4.2.8) suggested by Mises in 1945: in solids the values of stresses caused by forces distributed over small parts of its boundary decrease with increasing distance from these parts if the loading of each part is statically equipollent to zero.

5.2.13 Superstatic system of forces

A statically equivalent system of forces is called superstatic if its force tensor vanishes:

$$\mathbf{F}=0,\quad \mathbf{m}^O=0,\quad \hat{p}=0.\qquad(2.13.1)$$

If the boundary of the half-space is loaded by a superstatic system of forces on a small surface o all terms in eq. (2.12.8) are equal to zero. At a sufficient distance from the loading area the stresses have at least the order $(\varepsilon/R)^2\sigma$ which were not taken into account in approximation (2.12.1).

An example of the superstatic system of forces is a system of forces statically equivalent to zero and normal to the boundary of the half-space (similar to the particular Boussinesq problem). Then $f_1=f_2=0$ and

$$m_1^O=\iint_O \rho_2 f_2 do=0,\quad m_2^O=-\iint_O \rho_1 f_3 do=0,\quad m_3^O=0,$$

whereas the remaining components $\iint_O f_k\rho_s do$ of the force tensor vanish as $\rho_3=0$.

A more general example is the system of forces remaining statically equivalent to zero under any rotation of the forces comprising the system. Let the rotated vector of force $\mathbf{f}$ be denoted by $\grave{\mathbf{f}}$, then referring to eqs. (A.8.1) and (A.8.2) we have $\grave{\mathbf{f}} = \mathbf{f} \cdot \hat{A}$ where $\hat{A}$ denotes the rotation tensor. Let us take for simplicity that the forces are rotated through 90° about axis $\mathbf{i}_s$, then

$$\mathbf{i}'_k = \mathbf{i}_s \times \mathbf{i}_k, \quad \hat{A} = \mathbf{i}_k \mathbf{i}'_k = -\mathbf{i}_k \mathbf{i}_k \times \mathbf{i}_s = -\hat{E} \times \mathbf{i}_s,$$
$$\grave{\mathbf{f}} = -\mathbf{f} \cdot \hat{E} \times \mathbf{i}_s = \mathbf{i}_s \times \mathbf{f}$$

and due to the above condition

$$\begin{aligned} \mathbf{m}'^O &= \iint_O \boldsymbol{\rho} \times \grave{\mathbf{f}} do = \iint_O \boldsymbol{\rho} \times (\mathbf{i}_s \times \mathbf{f})\, do \\ &= \mathbf{i}_s \iint_O \boldsymbol{\rho} \cdot \mathbf{f} do - \iint_O \mathbf{f} \boldsymbol{\rho} do \cdot \mathbf{i}_s = 0. \end{aligned}$$

By eq. (2.12.3) the force tensor is symmetric for $\mathbf{m}^O = 0$ hence

$$\iint_O \mathbf{f} \boldsymbol{\rho} do = \hat{p}, \quad \mathbf{i}_s I_1(\hat{p}) - \hat{p} \cdot \mathbf{i}_s = 0, \tag{2.13.2}$$

and it follows from this relationship that

$$I_1(\hat{p}) = p_{11} + p_{22} + p_{33} = \mathbf{i}_s \cdot \hat{p} \cdot \mathbf{i}_s = p_{ss} \quad (s = 1, 2, 3),$$

so that

$$p_{11} = p_{22} = p_{33} = 0. \tag{2.13.3}$$

According to eq. (2.13.2) we have for $s \neq k$ that

$$\mathbf{i}_k \cdot \mathbf{i}_s I_1(\hat{p}) = 0 = \mathbf{i}_k \cdot \hat{p} \cdot \mathbf{i}_s = p_{ks}, \tag{2.13.4}$$

which proves the statement. For instance, the force dipole, Fig. 5.2a is not a superstatic system of forces whilst the pairs shown in Fig. 5.2b are an example of a superstatic system of forces.

5.2.14 Sternberg's theorem (1954)

The estimates of the rate of decreasing stresses in the elastic half-space obtained in Subsection 5.2.12 are valid for the case of an elastic body of finite dimensions bounded by a surface with the continuous curvature.

It is assumed that the body's surface is loaded on several regions having linear dimensions of the order of $\varepsilon \ll 1$ where the unit implies a characteristic length of the body. The forces distributed over a region are bounded,

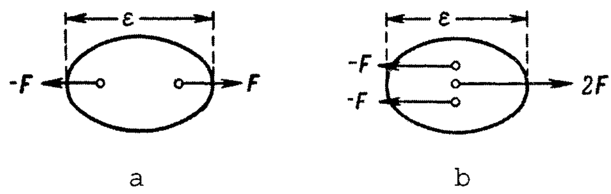

FIGURE 5.2.

thus the orders of values of the principal vector, the principal moment and the force tensor are $\varepsilon^2, \varepsilon^3$ and ε^3 respectively. Let $\sigma(x, \varepsilon)$ denote a stress component caused by the loading on one of the regions at the observation point at the distance $x \geq 1$. The order of this value is ε^m, i.e.

$$\sigma(x, \varepsilon) = O(\varepsilon^m).$$

Sternberg has proved the following statements: (i) $m \geq 2$ if the principal vector of the system of forces on the considered region does not vanish; (ii) $m \geq 3$ if it vanishes as well as in the case of a system which is statically equivalent to zero (that is, if the principal moment vanishes, too) and (iii) $m \geq 4$ if the system of forces is superstatic.

The proof follows immediately from Lauricella's formulae (1.2.20)-(1.2.23). Let O_k be one of the loading regions, M_k^0 be a fixed point and M_k be any point in this region, so that $\overrightarrow{M_k^0 M_k} = \rho_k$ and $\rho_k \leq \varepsilon$. Presenting vector $\mathbf{u}^*(M_k)$ in the form

$$\mathbf{u}^*(M_k) = \mathbf{u}_k^{*^0} + \boldsymbol{\rho}_k \cdot \nabla \mathbf{u}_k^{*^0} + \ldots$$
$$\left(\mathbf{u}_k^{*^0} = \mathbf{u}^*(M_k^0), \ \nabla \mathbf{u}_k^{*^0} = [\nabla \mathbf{u}_k^*(M)]|_{\rho_k=0}\right), \quad (2.14.1)$$

we have

$$\begin{aligned} \mathbf{u}^*(M_k) \cdot \mathbf{F}^{(k)} &= \mathbf{u}_k^{*^0} \cdot \mathbf{F}^{(k)} + \boldsymbol{\rho}_k \cdot \nabla \mathbf{u}_k^{*^0} \cdot \mathbf{F}^{(k)} + \ldots \\ &= \mathbf{u}_k^{*^0} \cdot \mathbf{F}^{(k)} + \nabla \mathbf{u}_k^{*^0} \cdot\cdot \mathbf{F}^{(k)} \boldsymbol{\rho}_k + \ldots, \end{aligned}$$

since the second term is the first invariant of the product of tensor $\nabla \mathbf{u}_k^{*^0}$ and tensor $\mathbf{F}^{(k)} \boldsymbol{\rho}_k$. By virtue of eq. (1.2.20)

$$\hat{\varepsilon}^{(k)} \cdot\cdot \hat{p} = \mathbf{u}_k^{*^0} \cdot \iint_{O_k} \mathbf{F}^{(k)} do + \nabla \mathbf{u}_k^{*^0} \cdot\cdot \iint_{O_k} \mathbf{F}^{(k)} \boldsymbol{\rho}_k do + \ldots, \quad (2.14.2)$$

and it remains to notice that the first and the second terms have the order of the loading area (ε^2) and ε^3, respectively. This provides one with the above estimate of the strains and the values of the stresses at the points lying at distances $x \gg \varepsilon$. Let us recall that tensor $\hat{p}$ and vector $\nabla \mathbf{u}_k^{*^0}(M)$ are auxiliary means for deriving Lauricella's formula and they are not related to the distribution of forces over O_k and the strain $\hat{\varepsilon}^{(k)}$ determined by this distribution.

5.3 Equilibrium of the elastic sphere

5.3.1 Statement of the problem

The solution to the first and the second boundary value problems for the sphere is sought in the form suggested by E. Trefftz

$$\mathbf{u} = \mathbf{U} + \left(R^2 - R_0^2\right) \nabla \Psi. \tag{3.1.1}$$

Here R_0 is the radius of the sphere, $\mathbf{R} = \mathbf{i}_k x_k$ is the position vector, $\mathbf{U} = \mathbf{i}_s U_s$ is a harmonic vector and Ψ is a harmonic scalar

$$\nabla^2 U_s = 0, \quad \nabla^2 \Psi = 0. \tag{3.1.2}$$

Using the Papkovich-Neuber form does not lead rapidly to the result, especially for the first boundary value problem.

The equalities relating the harmonic functions U_s and Ψ follow from the equations of the theory of elasticity in displacements. When the volume forces are absent the latter are set in the form

$$(1 - 2\nu) \nabla^2 \mathbf{u} + \nabla \nabla \cdot \mathbf{u} = 0. \tag{3.1.3}$$

By virtue of eqs. (3.1.1) and (3.1.2) we have

$$\nabla^2 \mathbf{u} = \nabla^2 R^2 \nabla \Psi = 6 \nabla \Psi + 4 \mathbf{R} \cdot \nabla \nabla \Psi = \nabla \left(2\Psi + 4\mathbf{R} \cdot \nabla \Psi\right), \tag{3.1.4}$$
$$\nabla \nabla \cdot \mathbf{u} = \nabla \left(\nabla \cdot \mathbf{U} + 2\mathbf{R} \cdot \nabla \Psi\right), \tag{3.1.5}$$

and substitution into eq. (3.1.3) leads to the required relationship

$$(1 - 2\nu) \Psi + (3 - 4\nu) \mathbf{R} \cdot \nabla \Psi + \frac{1}{2} \nabla \cdot \mathbf{U} = 0. \tag{3.1.6}$$

The strain tensor and the dilatation are as follows

$$\left.\begin{aligned} &\hat{\varepsilon} = \operatorname{def} \mathbf{U} + \mathbf{R} \nabla \Psi + (\nabla \Psi) \mathbf{R} + \left(R^2 - R_0^2\right) \nabla \nabla \Psi, \\ &\vartheta = I_1(\hat{\varepsilon}) = \nabla \cdot \mathbf{U} + 2\mathbf{R} \cdot \nabla \Psi. \end{aligned}\right\} \tag{3.1.7}$$

Now it is easy to construct the expression for the stress vector on the surface of the sphere $R = \text{const}$

$$\mathbf{P}_R = \frac{\mathbf{R}}{R} \cdot \hat{T} = \frac{2\mu}{R} \left[\frac{\nu}{1 - 2\nu} \mathbf{R} \nabla \cdot \mathbf{U} + \frac{1}{1 - 2\nu} \mathbf{R}\mathbf{R} \cdot \nabla \Psi + \right.$$
$$\left. \mathbf{R} \cdot \operatorname{def} \mathbf{U} + \mathbf{R}^2 \nabla \Psi + \left(R^2 - R_0^2\right) \mathbf{R} \cdot \nabla \nabla \Psi \right] \tag{3.1.8}$$

or

$$\frac{1}{2\mu} R\mathbf{P}_R = \Pi + \left(R^2 - R_0^2\right) \mathbf{R} \cdot \nabla\nabla\Psi, \tag{3.1.9}$$

where we introduced the vector

$$\Pi = \frac{1}{1-2\nu} \mathbf{R}\nabla \cdot \mathbf{U} + \frac{1}{1-2\nu} \mathbf{R}\mathbf{R} \cdot \nabla\Psi + \mathbf{R} \cdot \operatorname{def} \mathbf{U} + \mathbf{R}^2 \nabla\Psi, \tag{3.1.10}$$

that is harmonic. Indeed, using eqs. (3.1.4) and (3.1.5) we have

$$\begin{aligned} &\nabla^2 \mathbf{R}\nabla \cdot \mathbf{U} = 2\nabla\nabla \cdot \mathbf{U}; \quad \nabla^2 \mathbf{R}\mathbf{R} \cdot \nabla\Psi = 2\nabla\left(\mathbf{R} \cdot \nabla\Psi\right); \\ &\nabla^2 \mathbf{R} \cdot \operatorname{def} \mathbf{U} = \nabla\nabla \cdot \mathbf{U}; \quad \nabla^2 R^2 \nabla\Psi = 2\nabla\Psi + 4\nabla\mathbf{R} \cdot \nabla\Psi, \end{aligned}$$

and insertion into eq. (3.1.10) yields

$$\nabla^2 \Pi = \frac{2}{1-2\nu} \nabla \left[\frac{1}{2} \nabla \cdot \mathbf{U} + (1-2\nu)\Psi + (3-4\nu)\mathbf{R} \cdot \nabla\Psi\right].$$

The value in the brackets is equal to zero, thus

$$\nabla^2 \Pi = 0. \tag{3.1.11}$$

5.3.2 The first boundary value problem

According to Section D.4 the displacement vector $\mathbf{u}$ prescribed on the sphere surface O is presented by the series in terms of the Laplace spherical vectors

$$\begin{aligned} \mathbf{u}|_{R=R_0} &= \sum_{n=0}^{\infty} \mathbf{Y}_n(\mu, \lambda) \qquad (3.2.1) \\ &= \sum_{n=0}^{\infty} \left[\mathbf{a}_{n0} P_n(\mu) + \sum_{m=1}^{n} \left(\mathbf{a}_{nm} \cos m\lambda + \mathbf{b}_{nm} \sin m\lambda\right) P_n^m(\mu)\right]. \end{aligned}$$

By eq. (3.1.1) the harmonic vector $\mathbf{U}$ and the displacement vector $\mathbf{u}$ on surface O are equal to

$$\mathbf{U}|_{R=R_0} = \mathbf{u}|_{R=R_0} = \sum_{n=0}^{\infty} \mathbf{Y}_n(\mu, \lambda). \tag{3.2.2}$$

Referring to eq. (F.4.2) we have

$$\mathbf{U} = \sum_{n=0}^{\infty} \left(\frac{R}{R_0}\right)^n \mathbf{Y}_n(\mu, \lambda) = \sum_{n=0}^{\infty} \mathbf{U}_n \qquad (R < R_0), \tag{3.2.3}$$

$$\mathbf{U} = \sum_{n=0}^{\infty} \left(\frac{R_0}{R}\right)^{n+1} \mathbf{Y}_n(\mu, \lambda) = \sum_{n=0}^{\infty} \mathbf{U}_{-(n+1)} \quad (R > R_0). \tag{3.2.4}$$

Here $\mathbf{U}_n$ and $\mathbf{U}_{-(n+1)}$ are the homogeneous harmonic vectors of the power of n and $-(n+1)$ respectively. The harmonic scalar Ψ is also sought as a series in terms the homogeneous harmonic polynomials of power n for the internal problem and $-(n+1)$ for the external problem

$$\Psi = \sum_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} \left(\frac{R}{R_0}\right)^n \mathbf{Z}_n(\mu, \lambda) \qquad (R < R_0), \tag{3.2.5}$$

$$\Psi = \sum_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} \left(\frac{R_0}{R}\right)^{n+1} \mathbf{Z}_{-(n+1)}(\mu, \lambda) \quad (R > R_0). \tag{3.2.6}$$

Here and in what follows while considering the boundary value problems for the sphere we will apply Euler's theorem on homogeneous functions, eq. (F.2.2)

$$\mathbf{R} \cdot \nabla \mathbf{U}_n = n\mathbf{U}_n, \qquad \mathbf{R} \cdot \nabla \Psi_n = n\Psi_n. \tag{3.2.7}$$

Using this theorem and eq. (3.1.6) we obtain

$$(1 - 2\nu)\Psi_{n-1} + (3 - 4\nu)(n-1)\Psi_{n-1} + \frac{1}{2}\nabla \cdot \mathbf{U}_n = 0,$$

so that

$$\Psi_{n-1} = -\frac{1}{2}\frac{\nabla \cdot \mathbf{U}_n}{3n - 2 - 2\nu(2n-1)} \tag{3.2.8}$$

and furthermore

$$\nabla \Psi_{n-1} = -\frac{1}{2}\frac{\operatorname{grad}\operatorname{div}\mathbf{U}_n}{3n - 2 - 2\nu(2n-1)}, \tag{3.2.9}$$

where $\nabla \cdot \mathbf{U}_0 = \nabla \cdot \mathbf{a}_{00} = 0$. By eq. (3.1.1) we have

$$\mathbf{u} = \sum_{n=0}^{\infty} \left[\mathbf{U}_n + \left(R^2 - R_0^2\right)\nabla \Psi_{n-1}\right],$$

where $\Psi_{-1} = 0$. Hence, the solution of the internal problem is presented by the series

$$\mathbf{u} = \sum_{n=0}^{\infty} \left[\mathbf{U}_n + \frac{1}{2}\left(R_0^2 - R^2\right)\frac{\operatorname{grad}\operatorname{div}\mathbf{U}_n}{3n - 2 - 2\nu(2n-1)}\right] \quad (R < R_0). \tag{3.2.10}$$

The solution of the external problem is obtained when we replace n by $-(n+1)$

$$\mathbf{u} = \sum_{n=0}^{\infty} \left[\mathbf{U}_{-(n+1)} + \frac{1}{2}\left(R^2 - R_0^2\right)\frac{\operatorname{grad}\operatorname{div}\mathbf{U}_{-(n+1)}}{3n + 5 - 2\nu(2n+3)}\right] \quad (R > R_0). \tag{3.2.11}$$

Here, by eqs. (3.2.3) and (3.2.4)

$$\mathbf{U}_n = \left(\frac{R}{R_0}\right)^n \mathbf{Y}_n(\mu, \lambda), \quad \mathbf{U}_{-(n+1)} = \left(\frac{R_0}{R}\right)^{n+1} \mathbf{Y}_n(\mu, \lambda) \tag{3.2.12}$$

and vectors $\mathbf{Y}_n(\mu, \lambda)$ are the Laplace spherical vectors given by expansion (3.2.1).

5.3.3 The elastostatic Robin's problem for the sphere

In accordance with Subsection 4.4.7 the problem is concerned with the state of stress in the elastic medium provided that a rigid sphere placed in the medium is subjected to the following small displacement

$$\mathbf{u}^* = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{R}, \tag{3.3.1}$$

where $\mathbf{u}_0$ and $\boldsymbol{\omega}$ are constant-valued vectors. The points of cavity O which forms a sphere of radius $R = R_0$ bounding the medium have the displacement

$$\mathbf{u}|_{R=R_0} = \mathbf{u}^*|_{R=R_0} = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{R}_0, \tag{3.3.2}$$

where $\mathbf{R}_0$ denotes the position vector of the point on O. The right hand side of eq. (3.3.2) is already presented as a sum of Laplace's spherical vectors of the zeroth and first order

$$\mathbf{Y}_0 = \mathbf{u}_0, \quad \mathbf{Y}_1 = \boldsymbol{\omega} \times \mathbf{R}_0.$$

Hence, referring to eq. (3.2.4) we have

$$\mathbf{U}_{-1} = \frac{R_0}{R}\mathbf{u}_0, \quad \mathbf{U}_{-2} = \frac{R_0^3}{R^3}\boldsymbol{\omega} \times \mathbf{R} = -R_0^3 \boldsymbol{\omega} \times \nabla \frac{1}{R}. \tag{3.3.3}$$

The general expression (3.2.11) can be set as follows

$$\mathbf{u} = \mathbf{u}_{-1} + \mathbf{u}_{-2}, \tag{3.3.4}$$

with

$$\mathbf{u}_{-1} = \mathbf{U}_{-1} + \frac{1}{2}\left(R^2 - R_0^2\right) \frac{\operatorname{grad}\operatorname{div}\mathbf{U}_{-1}}{5 - 6\nu},$$
$$\mathbf{u}_{-2} = \mathbf{U}_{-2} + \frac{1}{2}\left(R^2 - R_0^2\right) \frac{\operatorname{grad}\operatorname{div}\mathbf{U}_{-2}}{8 - 10\nu}.$$

Now we have

$$\operatorname{div}\mathbf{U}_{-1} = -\frac{R_0}{R^3}\mathbf{R} \cdot \mathbf{u}_0,$$
$$\operatorname{grad}\operatorname{div}\mathbf{U}_{-1} = \frac{3R_0}{R^5}\mathbf{R}\mathbf{R} \cdot \mathbf{u}_0 - \frac{R_0}{R^3}\mathbf{u}_0; \quad \operatorname{div}\mathbf{U}_{-2} = 0$$

and the solution of the problem is given by the following formulae

$$\mathbf{u}_{-1} = \frac{R_0}{R}\mathbf{u}_0 + \frac{R^2 - R_0^2}{2(5-6\nu)}\frac{R_0}{R^3}\left(3\frac{\mathbf{R}}{R^2}\mathbf{R}\cdot\mathbf{u}_0 - \mathbf{u}_0\right), \tag{3.3.5}$$

$$\mathbf{u}_{-2} = \frac{R_0^3}{R^3}\boldsymbol{\omega}\times\mathbf{R}. \tag{3.3.6}$$

The latter formula determines the displacement due to the centre of rotation, see eq. (1.2.12). This displacement results is the stress vector of the surface of the sphere and the principal moment of the forces needed to be applied to the sphere for the required rotation $\boldsymbol{\omega}$, the principal vector of these forces being equal to zero.

The calculation of the displacements caused by the translational displacement $\mathbf{u}_0$ of the sphere is more cumbersome. The result is

$$\hat{\varepsilon} = \operatorname{def}\mathbf{u}_{-1} = \frac{3R_0}{2(5-6\nu)R^3}\left\{-\left[(1-2\nu)+\frac{R_0^2}{R^2}\right](\mathbf{R}\mathbf{u}_0+\mathbf{u}_0\mathbf{R}) + \left[\hat{E}\left(1-\frac{R_0^2}{R^2}\right) - 3\frac{\mathbf{R}\mathbf{R}}{R^2} + 5\frac{\mathbf{R}\mathbf{R}}{R^2}\frac{R_0^2}{R^2}\right]\mathbf{R}\cdot\mathbf{u}_0\right\},$$

$$\operatorname{div}\mathbf{u}_{-1} = -\frac{1-2\nu}{5-6\nu}\frac{3R_0}{R^3}\mathbf{R}\cdot\mathbf{u}_0$$

and the stress tensor on the surface O is as follows

$$\left.\frac{1}{2\mu}\hat{T}\right|_0 = -\frac{3}{5-6\nu}\frac{1}{R_0^2}\left[(1-\nu)(\mathbf{R}_0\mathbf{u}_0+\mathbf{u}_0\mathbf{R}_0) + \nu\hat{E}\mathbf{R}_0\cdot\mathbf{u}_0 - \frac{\mathbf{R}_0\mathbf{R}_0}{R_0^2}\mathbf{R}_0\cdot\mathbf{u}_0\right]. \tag{3.3.7}$$

The traction vector on this surface is given by the formula

$$\frac{1}{2\mu}\hat{T}\cdot\mathbf{n} = -\frac{1}{2\mu}\hat{T}\cdot\frac{\mathbf{R}_0}{R_0} = \frac{3(1-\nu)}{(5-6\nu)R_0}\mathbf{u}_0. \tag{3.3.8}$$

Thus, the principal vector of these forces is equal to

$$\mathbf{V} = \frac{24\pi\mu(1-\nu)}{5-6\nu}R_0\mathbf{u}_0, \tag{3.3.9}$$

with the principal moment being equal to zero.

5.3.4 Thermal stresses in the sphere

A rigid case is placed around an elastic sphere and a steady-state temperature distribution θ on the surface of the elastic sphere is assumed to be prescribed. In other words the solution must satisfy the following condition

$$\theta|_{R=R_0} = \sum_{n=0}^{\infty} Z_n(\mu,\lambda), \quad \mathbf{u}|_{R=R_0} = 0. \tag{3.4.1}$$

Here $Z_n(\mu,\lambda)$ denote Laplace's spherical functions and the temperature θ which is a harmonic functions for $R < R_0$ is expanded into a series in terms of these functions

$$\theta = \sum_{n=0}^{\infty} \left(\frac{R}{R_0}\right)^n Z_n(\mu,\lambda) = \sum_{n=0}^{\infty} \theta_n. \tag{3.4.2}$$

According to eqs. (1.14.7) and (1.14.8) of Chapter 4, the particular solution u_* of the equilibrium equations in terms of displacements corresponding to the thermal term is given by

$$u_* = \nabla\chi, \quad \nabla^2\chi = \alpha\frac{1+\nu}{1-\nu}\theta = \alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\theta_n. \tag{3.4.3}$$

It is sufficient to find any particular solution of this equations which is sought in the form

$$\chi = \sum_{n=0}^{\infty}\chi_n, \quad \nabla^2\chi_n = \alpha\frac{1+\nu}{1-\nu}\theta_n.$$

The right hand side is a homogeneous harmonic polynomial. Assuming

$$\chi_n = A_n R^2\theta_n, \quad \nabla^2\chi_n = A_n\left(6\theta_n + 4\mathbf{R}\cdot\nabla\chi_n\right) = (6+4n)A_n\chi_n,$$

we have

$$\chi_n = \frac{1}{2}\alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}R^2\frac{\theta_n}{2n+3},$$

so that

$$\mathbf{u}_* = \nabla\chi = \frac{1}{2}\alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\frac{1}{2n+3}\left(2\mathbf{R}\cdot\theta_n + R^2\nabla\theta_n\right).$$

The solution is sought as the following sum

$$\mathbf{u} = \mathbf{v} + \mathbf{u}_* = \mathbf{v} + \alpha\frac{1+\nu}{1-\nu}\left(\mathbf{R}\sum_{n=0}^{\infty}\frac{\theta_n}{2n+3} + \frac{1}{2}R^2\sum_{n=0}^{\infty}\frac{\nabla\theta_n}{2n+3}\right), \tag{3.4.4}$$

where vector $\mathbf{v}$ is a solution of the homogeneous equilibrium equations in terms of displacements and, due to eq. (3.4.1), satisfies the boundary condition

$$\mathbf{v}|_{R=R_0} = -\alpha\frac{1+\nu}{1-\nu}\left[\mathbf{R}_0\sum_{n=0}^{\infty}\frac{Z_n(\mu,\lambda)}{2n+3} + \frac{1}{2}R_0^2\sum_{n=0}^{\infty}\frac{1}{2n+3}\left(\nabla\theta_n\right)|_{R=R_0}\right]. \tag{3.4.5}$$

Vector $\nabla\theta_n$ is the gradient of a harmonic vector and thus is a harmonic vector. For this reason, its value on the surface of the sphere is a Laplace spherical vector of the order $n-1$. Splitting the sought vector $\mathbf{v}$ into two terms

$$\mathbf{v} = -\alpha\frac{1+\nu}{1-\nu}\left(\mathbf{v}^{(1)} + \mathbf{v}^{(2)}\right),$$

determined by the boundary conditions

$$\mathbf{v}^{(1)}\Big|_{R=R_0} = R_0\sum_{n=0}^{\infty}\mathbf{e}_R\frac{Z_n(\mu,\lambda)}{2n+3}, \quad \mathbf{v}^{(2)} = \frac{1}{2}R_0^2\sum_{n=0}^{\infty}\frac{\nabla\theta_n}{2n+3}, \tag{3.4.6}$$

referring to eq. (3.2.10) and taking into account that $\operatorname{div}\nabla\theta_n = \nabla^2\theta_n = 0$, one can set the expression for $\mathbf{v}^{(2)}$ in the form

$$\mathbf{v}^{(2)} = \frac{1}{2}R_0^2\sum_{n=0}^{\infty}\frac{\nabla\theta_n}{2n+3}. \tag{3.4.7}$$

The problem is thus reduced to determining vector $\mathbf{v}^{(1)}$. Restricting our consideration to the case of the symmetric distribution of the temperature over the surface of the sphere we have $Z_n = a_{n0}P_n(\mu)$. The vector

$$\mathbf{e}_R P_n(\mu) = \left[(\mathbf{i}_1\cos\lambda + \mathbf{i}_2\sin\lambda)\sin\theta + \mathbf{i}_3\cos\theta\right]P_n(\mu)$$

is now needed to be replaced by the expansion in terms of Laplace's spherical functions. To this aim, use is made of the recurrent formulae

$$(2n+1)\mu P_n(\mu) = (n+1)P_{n+1}(\mu) + nP_{n-1}(\mu),$$
$$(2n+1)\mu P_n(\mu) = P'_{n+1}(\mu) - P'_{n-1}(\mu).$$

Then we have

$$\mathbf{e}_R P_n(\mu) = \frac{1}{2n+1}\left[(\mathbf{i}_1\cos\lambda + \mathbf{i}_2\sin\lambda)P'_{n+1}(\mu) + (n+1)\mathbf{i}_3P_{n+1}(\mu)\right] + \frac{1}{2n+1}\left[-(\mathbf{i}_1\cos\lambda + \mathbf{i}_2\sin\lambda)P'_{n-1}(\mu) + n\mathbf{i}_3P_{n-1}(\mu)\right].$$

The expressions in the brackets are the Laplace spherical vectors, see eq. (F.2.10). Introducing the notion

$$\left.\begin{aligned}\mathbf{Y}^{*}_{n+1}(\mu,\lambda) &= \frac{1}{2n+1}\left[\mathbf{e}_r P^1_{n+1}(\mu) + \mathbf{k}(n+1)P_{n+1}(\mu)\right],\\ \mathbf{Y}^{**}_{n-1} &= \frac{1}{2n+1}\left[-\mathbf{e}_r P^1_{n-1}(\mu) + \mathbf{k}nP_{n-1}(\mu)\right]\end{aligned}\right\} \tag{3.4.8}$$

we can write the boundary condition (3.4.6) for vector $\mathbf{v}^{(1)}$ in the form

$$\mathbf{v}^{(1)}\Big|_{R=R_0} = R_0\left(\sum_{n=0}^{\infty}\frac{a_{n0}}{2n+3}\mathbf{Y}^{*}_{n+1} + \sum_{n=0}^{\infty}\frac{a_{n0}}{2n+3}\mathbf{Y}^{**}_{n-1}\right). \tag{3.4.9}$$

Here $\mathbf{e}_r$ and $\mathbf{k}$ denote the unit vectors of the cylindrical coordinate system $(\mathbf{e}_r = \mathbf{i}_1 \cos\lambda + \mathbf{i}_2 \sin\lambda, \mathbf{k} = \mathbf{i}_3)$. The component $Z_n = a_{n0}P_n(\mu)$ in the boundary condition (3.4.6) is described by two harmonic vectors

$$\mathbf{U}^*_{n+1} = \frac{a_{n0}R_0}{2n+3}\left(\frac{R}{R_0}\right)^{n+1}\mathbf{Y}^*_{n+1}, \quad \mathbf{U}^{**}_{n-1} = \frac{a_{n0}R_0}{2n+3}\left(\frac{R}{R_0}\right)^{n-1}\mathbf{Y}^{**}_{n-1}, \tag{3.4.10}$$

where $\mathbf{U}^{**}_{-1} = 0$. By eqs. (3.2.10) and (3.4.4)-(3.4.7), the solution of the problem is given as follows

$$\mathbf{u} = \alpha\frac{1+\nu}{1-\nu}\left\{\mathbf{R}\sum_{n=0}^{\infty}\frac{\theta_n}{2n+3} + \frac{1}{2}\left(R^2 - R_0^2\right)\sum_{n=0}^{\infty}\frac{\nabla\theta_n}{2n+3} - \sum_{n=0}^{\infty}\mathbf{U}^*_{n+1} - \right.$$
$$\sum_{n=1}^{\infty}\mathbf{U}^{**}_{n-1} + \frac{1}{2}\left(R^2 - R_0^2\right)\left[\sum_{n=0}^{\infty}\frac{\nabla\nabla\cdot\mathbf{U}^*_{n+1}}{3n+1-2\nu(2n+1)} + \right.$$
$$\left.\left.\sum_{n=1}^{\infty}\frac{\nabla\nabla\cdot\mathbf{U}^{**}_{n-1}}{3n-5-2\nu(2n-3)}\right]\right\}, \tag{3.4.11}$$

where

$$\theta_n = a_{n0}\left(\frac{R}{R_0}\right)^n P_n(\mu). \tag{3.4.12}$$

For example, let the surface temperature be the following linear function

$$\theta|_{R=R_0} = \theta^0 + \frac{\theta^1 - \theta^0}{2R_0}z = \theta^0 + \frac{\theta^1 - \theta^0}{2}P_1(\mu), a_{00} = \theta_0, a_{10} = \frac{1}{2}\left(\theta^1 - \theta^0\right)$$

then the temperature in the solid obeys the linear law as well

$$\theta = \theta^0 + \frac{\theta^1 - \theta^0}{2}\frac{R}{R_0}P_1(\mu) = \theta^0 + \frac{\theta^1 - \theta^0}{2R_0}z \quad (R < R_0).$$

Calculation using the above formulae yields

$$\mathbf{u} = \alpha\frac{1+\nu}{4-6\nu}\frac{\theta^1 - \theta^0}{2R_0}\left(R^2 - R_0^2\right)\mathbf{k},$$

which can be easily proved by means of eq. (1.14.3) of Chapter 4. The stresses are found with the help of eq. (1.14.1) of Chapter 4.

5.3.5 The second boundary value problem for the sphere

The stress vector on the surface of any sphere which is concentric with sphere O of radius R_0 is denoted as $\mathbf{P}_R$ in Subsection 5.3.1. Vector $\mathbf{P}_R$

is prescribed on sphere O and can be presented in the form of a series in terms of Laplace's spherical vectors

$$R\mathbf{P}_R|_{R=R_0} = \sum_{n=0}^{\infty} \mathbf{Y}_n(\mu,\lambda). \tag{3.5.1}$$

By virtue of eqs. (3.1.9) and (3.1.11), on surface O this series coincides with the vector

$$\Pi = \frac{1}{2G}\sum_{n=0}^{\infty}\left(\frac{R}{R_0}\right)^n \mathbf{Y}_n(\mu,\lambda) = \sum_{n=0}^{\infty}\Pi_n, \tag{3.5.2}$$

which is harmonic inside the sphere and the vector

$$\Pi = \frac{1}{2G}\sum_{n=0}^{\infty}\left(\frac{R_0}{R}\right)^{n+1} \mathbf{Y}_n(\mu,\lambda) = \sum_{n=0}^{\infty}\Pi_{-(n+1)}, \tag{3.5.3}$$

which is harmonic outside of it. The shear modulus is denoted here as G.

The solution of the external problem is known to be obtained by replacing n by $-(n+1)$ in the solution for the internal problem. For this reason, the latter is considered first. Due to eq. (3.1.9) determining $\mathbf{P}_R$ requires the harmonic scalar Ψ to be found. It can be obtained by excluding the harmonic vector $\mathbf{U}$ from equalities (3.1.6) and (3.1.10). Taking into account that $\nabla^2\mathbf{U} = 0$ and $\nabla^2\Psi = 0$ we obtain from eq. (3.1.10) that

$$\nabla\cdot\Pi = \frac{1}{1-2\nu}\Big[(1+\nu)\nabla\cdot\mathbf{U} + \\ \frac{1}{2}\mathbf{R}\cdot\nabla\nabla\cdot\mathbf{U} + (5-4\nu)\mathbf{R}\cdot\nabla\Psi + \mathbf{R}\cdot\nabla\mathbf{R}\cdot\nabla\Psi\Big].$$

Replacing here $\nabla\cdot\mathbf{U}$ by means of eq. (3.1.6) we have

$$\nabla\cdot\Pi = -2\left[(1+\nu)\Psi + (1+2\nu)\mathbf{R}\cdot\nabla\Psi + \mathbf{R}\cdot\nabla\mathbf{R}\cdot\nabla\Psi\right].$$

Representing the harmonic scalar Ψ by a sum of the harmonic polynomials

$$\Psi = \sum_{k=0}^{\infty}\Psi_k$$

we obtain

$$\mathbf{R}\cdot\nabla\Psi = \sum_{k=1}^{\infty}k\Psi_k, \quad \mathbf{R}\cdot\nabla\mathbf{R}\cdot\nabla\Psi = \sum_{k=1}^{\infty}k^2\Psi_k,$$

$$\nabla\cdot\Pi = \sum_{n=0}^{\infty}\nabla\cdot\Pi_n = -2\sum_{k=0}^{\infty}\left[k^2 + (1+2\nu)k + (1+\nu)\right]\Psi_k.$$

The left hand side contains a sum of the harmonic polynomials of power $n-1$, that is, taking $k=n-1$ we have

$$\left.\begin{aligned}\Psi_{n-1} &= -\frac{1}{2}\frac{\nabla\cdot\Pi_n}{n^2-(1-2\nu)\,n+(1-\nu)},\\ \Psi &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\nabla\cdot\Pi_n}{n^2-(1-2\nu)\,n+1-\nu}\end{aligned}\right\}\tag{3.5.4}$$

and taking into account that $\nabla\Psi_{n-1}$ is a harmonic vector of power $n-2$ we obtain by means of eq. (3.1.9) that

$$\mathbf{P}_R=\frac{2G}{R}\sum_{n=0}^{\infty}\left[\Pi_n+\frac{1}{2}\left(R^2-R_0^2\right)\frac{(n-2)\,\nabla\nabla\cdot\Pi_n}{n^2-(1-2\nu)\,n+1-\nu}\right].\tag{3.5.5}$$

Using eqs. (3.1.6) and (3.1.1) we find

$$\left.\begin{aligned}\nabla\cdot\mathbf{U} &= \sum_{n=0}^{\infty}\frac{(3-4\nu)\,n-2\,(1-\nu)}{n^2-(1-2\nu)\,n+1-\nu}\nabla\cdot\Pi_n,\\ \vartheta=\nabla\cdot\mathbf{u} &= (1-2\nu)\sum_{n=0}^{\infty}\frac{(2n-1)\,\nabla\cdot\Pi_n}{n^2-(1-2\nu)\,n+1-\nu}.\end{aligned}\right\}\tag{3.5.6}$$

The latter formula yields the sum of the normal stresses

$$\sigma=\sigma_R+\sigma_\vartheta+\sigma_\lambda=2G\,(1+\nu)\sum_{n=0}^{\infty}\frac{(2n-1)\,\nabla\cdot\Pi_n}{n^2-(1-2\nu)\,n+1-\nu}.\tag{3.5.7}$$

The system of the external forces acting on the sphere needs to be statically equivalent to zero, i.e. the principal vector $\mathbf{V}$ and the principal moment $\mathbf{m}^0$ must vanish

$$\mathbf{V}=\iint\limits_{O}\mathbf{P}_R|_{R=R_0}\,do=0,\quad \mathbf{m}^0=\iint\limits_{O}\mathbf{R}_0\times\mathbf{P}_R|_{R=R_0}\,do=0.\tag{3.5.8}$$

It is known that the integral of the product of two surface Laplace's vectors of different orders over the surface of the sphere is equal to zero. Thus, when estimating the integrals in eq. (3.5.8) it is necessary to keep only the terms $\mathbf{Y}_0$ and $\mathbf{Y}_1$ in the first and the second integrals, respectively. The result is

$$\begin{aligned}\mathbf{V}=4\pi R_0\mathbf{Y}_0=0,\quad \mathbf{m} &= \iint\limits_{O}\frac{\mathbf{R}_0}{R_0}\times\mathbf{Y}_1do=\iint\limits_{O}\mathbf{n}\times\mathbf{Y}_1do\\ &= 2G\iiint\limits_{V}\nabla\times\Pi_1d\tau=\frac{8\pi}{3}GR_0^3\nabla\times\Pi_1=0,\end{aligned}\tag{3.5.9}$$

as $\nabla \times \Pi_1$ is a constant-valued vector. Hence, the constant term $\mathbf{Y}_0$ must vanish in the expansion of the vector of the external forces whilst the term $R_0^{-1}\mathbf{Y}_1$ must be subjected to the condition

$$2G\Pi_1 = \frac{R}{R_0}\mathbf{Y}_1 = \nabla H, \quad R_0^{-1}\mathbf{Y}_1 = R^{-1}\nabla H, \tag{3.5.10}$$

where H is a homogeneous quadratic form in coordinates x, y, z. The summation in formulae (3.5.5)-(3.5.7) should begin with $n = 1$ whereas that in the second group in eq. (3.5.5) from $n = 3$. Replacing n by $-(n+1)$ in expansions (3.5.5) and (3.5.7) we arrive at the solution for the external problem

$$\mathbf{P}_R = \frac{2G}{R}\sum_{n=0}^{\infty}\left[\Pi_{-(n+1)} + \frac{1}{2}\left(R^2 - R_0^2\right)\frac{(n+3)\,\nabla\nabla\cdot\Pi_{-(n+1)}}{n^2 + (3-2\nu)\,n + 3\,(1-\nu)}\right], \tag{3.5.11}$$

$$\sigma = \sigma_R + \sigma_\vartheta + \sigma_\lambda = -2G\,(1+\nu)\sum_{n=0}^{\infty}\frac{(2n+3)\,\nabla\cdot\Pi_{-(n+1)}}{n^2 + (3-2\nu)\,n + 3\,(1-\nu)}. \tag{3.5.12}$$

5.3.6 Calculation of the displacement vector

Formulae (3.5.5) and (3.5.7) determine the sum of the normal stresses and the stress vector on the surface of any sphere $R = R_0$ which is concentric to O. It is more difficult to find the displacement vector. Using the formulae

$$\nabla \times \mathbf{a}\varphi = \varphi\nabla \times \mathbf{a} + \nabla\varphi \times \mathbf{a}, \quad \nabla \times \mathbf{R}\cdot \operatorname{def}\mathbf{U} = \frac{1}{2}\mathbf{R}\cdot\nabla\nabla \times \mathbf{U}$$

we obtain by means of eqs. (3.1.10) and (3.1.6)

$$\begin{aligned}\operatorname{rot}\Pi = {} & \frac{\nu}{1-2\nu}\mathbf{R} \times \nabla\nabla\cdot\mathbf{U} + \frac{1}{2}\mathbf{R}\cdot\nabla\nabla\times\mathbf{U} - \\ & \frac{1}{1-2\nu}\mathbf{R}\times\nabla\mathbf{R}\cdot\nabla\Psi + 2\mathbf{R}\times\nabla\Psi \\ = {} & 2\,(1+\nu)\,\mathbf{R}\times\nabla\Psi - (1-4\nu)\,\mathbf{R}\times\nabla\left(\mathbf{R}\cdot\nabla\Psi\right) + \frac{1}{2}\mathbf{R}\cdot\nabla\nabla\times\mathbf{U}.\end{aligned} \tag{3.6.1}$$

Taking into account that

$$\mathbf{R}\cdot\nabla\nabla\times\mathbf{U}_n = (n-1)\,\nabla\times\mathbf{U}_n$$

and recalling eq. (3.5.4) we find that for $n \neq 1$

$$\nabla\times\mathbf{U}_n = \frac{1}{n-1}\left[2\nabla\times\Pi_n - \frac{n\,(1-4\nu) - 3 + 2\nu}{n^2 - (1-2\nu)\,n + 1 - \nu}\mathbf{R}\times\nabla\nabla\cdot\Pi_n\right]. \tag{3.6.2}$$

Assuming

$$\operatorname{def}\mathbf{U} = \nabla\mathbf{U} + \hat{\Omega},$$
$$\mathbf{R}\cdot\operatorname{def}\mathbf{U} = \mathbf{R}\cdot\nabla\mathbf{U} - \hat{\Omega}\cdot\mathbf{R} = \mathbf{R}\cdot\nabla\mathbf{U} + \frac{1}{2}\mathbf{R}\times(\nabla\times\mathbf{U})$$

in eq. (3.1.10) and utilising eq. (3.1.6) we obtain

$$\Pi = -2\nu\mathbf{R}\Psi + (1-4\nu)\,\mathbf{R}\mathbf{R}\cdot\nabla\Psi + R^2\nabla\Psi + \mathbf{R}\cdot\nabla\mathbf{U} + \frac{1}{2}\mathbf{R}\times(\nabla\times\mathbf{U}),$$

so that

$$\Pi_n = [-2\nu + (n-1)(1-4\nu)]\,\mathbf{R}\Psi_{n-1} + R^2\nabla\Psi_{n-1} + n\mathbf{U}_n + \frac{1}{2}\mathbf{R}\times(\nabla\times\mathbf{U}).$$

Eliminating $\nabla\times\mathbf{U}$ and Ψ_{n-1} by means of eqs. (3.6.2) and (3.5.4), respectively, we arrive at the equality

$$\mathbf{U}_n = \frac{1}{n}\Pi_n - \frac{1}{n(n-1)}\mathbf{R}\times(\nabla\Pi\times_n) + \frac{n(1-4\nu)-2(1-\nu)}{n\left[n^2-(1-2\nu)n+1-\nu\right]}\mathbf{R}\nabla\cdot\Pi_n + \frac{1-\nu+2n\nu}{n(n-1)\left[n^2-(1-2\nu)n+1-\nu\right]}R^2\nabla\nabla\cdot\Pi_n \quad (n\neq 1), \tag{3.6.3}$$

and by eqs. (3.1.1) and (3.5.4) the displacement vector takes the form

$$\mathbf{u} = \sum_{n=2}^{\infty}\left[\mathbf{U}_n + \frac{1}{2}\left(R_0^2 - R^2\right)\frac{\nabla\nabla\cdot\Pi_n}{n^2-(1-2\nu)n+1-\nu}\right] + \mathbf{u}_1. \tag{3.6.4}$$

It is clear that the term corresponding to $n = 0$ describes the rigid body displacement of the sphere. The term $\mathbf{u}_1$ represents a vector which depends linearly on the coordinates and can be set as follows

$$\mathbf{u}_1 = \hat{A}\cdot\mathbf{R} = \frac{1}{2}\left(\hat{A}+\hat{A}^*\right)\cdot\mathbf{R} + \frac{1}{2}\left(\hat{A}-\hat{A}^*\right)\cdot\mathbf{R}. \tag{3.6.5}$$

Here $\hat{A}$ is a constant tensor of the second rank which can be taken as being symmetric $\left(\hat{A} = \hat{A}^*\right)$ since the skew-symmetric part would add the term

$$\frac{1}{2}\left(\hat{A}-\hat{A}^*\right)\cdot\mathbf{R} = \hat{\Omega}\cdot\mathbf{R} = \omega\times\mathbf{R}$$

describing a rigid body rotation. The stress tensor $\hat{T}^{(1)}$ corresponding to $\mathbf{u}_1$ is equal to

$$\hat{T}^{(1)} = 2G\left[\frac{\nu}{1-2\nu}\hat{E}I_1\left(\hat{A}\right) + \hat{A}\right], \tag{3.6.6}$$

so that

$$\mathbf{R} \cdot \hat{T}^{(1)} = R\mathbf{P}_R = 2G\Pi_1 = 2G\left[\frac{\nu}{1-2\nu}\mathbf{R}I_1\left(\hat{A}\right) + \mathbf{u}_1\right]. \tag{3.6.7}$$

The first invariant of tensor $\hat{A}$ is determined by eqs. (3.6.6) and (3.5.7)

$$I_1\left(\hat{A}\right) = \frac{1}{2G}\frac{1-2\nu}{1+\nu}I_1\left(\hat{T}^{(1)}\right) = \frac{1-2\nu}{1+\nu}\nabla \cdot \Pi_1,$$

so that

$$\mathbf{u}_1 = \Pi_1 - \frac{\nu}{1+\nu}\mathbf{R}\nabla \cdot \Pi_1. \tag{3.6.8}$$

In the case of the external problem, replacing n by $-(n+1)$ in eq. (3.6.3) we have

$$\mathbf{U}_{-(n+1)} = -\left\{\frac{1}{n+1}\Pi_{-(n+1)} + \frac{1}{(n+1)(n+2)}\mathbf{R} \times \operatorname{rot}\Pi_{-(n+1)} - \right.$$
$$\frac{1}{(n+1)\left[n^2 + (3-2\nu)n + 3(1-\nu)\right]}\left[\{n(1-4\nu) + 3(1-2\nu)\}\mathbf{R}\nabla \cdot \Pi_{-(n+1)}\right.$$
$$\left.\left. - \frac{(2n+3)\nu - 1}{n+2}R^2\nabla\nabla \cdot \Pi_{-(n+1)}\right]\right\} \tag{3.6.9}$$

for all $n = 0, 1, 2, \ldots$ and the displacement vector is as follows

$$\mathbf{u} = \sum_{n=0}^{\infty}\left[\mathbf{U}_{-(n+1)} - \frac{1}{2}\left(R^2 - R_0^2\right)\frac{\nabla\nabla \cdot \Pi_{-(n+1)}}{n^2 + (3-2\nu)n + 3(1-\nu)}\right]. \tag{3.6.10}$$

5.3.7 The state of stress at the centre of the sphere

The stress vector on an arbitrary oriented surface at the centre of the sphere ($R = 0$) is given by eq. (3.5.5)

$$\begin{aligned}(\mathbf{P}_R)_{R=0} &= 2G\left(\frac{1}{R}\Pi_1 + \frac{R_0^2}{2(7+5\nu)}\frac{\nabla\nabla \cdot \Pi_3}{R}\right)_{R\to 0} \\ &= \frac{1}{R_0}\left[\mathbf{Y}_1 + \frac{1}{2(7+5\nu)}\left(\frac{\nabla \cdot \nabla R^3\mathbf{Y}_3}{R}\right)_{R=0}\right],\end{aligned} \tag{3.7.1}$$

that is, in order to determine the stresses at the centre of the sphere it is sufficient to know only the first and third terms of expansion (3.5.1) of the load in a series in terms of Laplace's spherical vectors.

5.3.8 Thermal stresses

Here we assume surface O of the sphere is free in the steady-state thermal regime. Similar to Subsection 5.3.4 the displacement vector is presented in the form

$$\mathbf{u} = \mathbf{v} + \nabla\chi, \tag{3.8.1}$$

where χ implies a particular solution of Poisson's equation (3.4.4) and $\mathbf{v}$ denotes the vector determined from the homogeneous equilibrium equations in terms of displacements. By virtue of eq. (1.14.1) of Chapter 4 the stress tensor is equal to

$$\hat{T} = \hat{T}(\mathbf{u}) - 2G\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E} = \hat{T}(\mathbf{v}) + \hat{T}(\nabla\chi) - 2G\frac{1+\nu}{1-2\nu}\alpha\theta\hat{E},$$

where the operation $\hat{T}$ over vector $\mathbf{a}$ is defined by the equality

$$\hat{T}(\mathbf{a}) = 2G\left(\frac{\nu}{1-2\nu}\hat{E}\operatorname{div}\mathbf{a} + \operatorname{def}\mathbf{a}\right),$$

so that

$$\hat{T}(\nabla\chi) = 2G\left(\frac{\nu}{1-2\nu}\hat{E}\nabla^2\chi + \nabla\nabla\chi\right).$$

Referring now to eq. (1.1.4.8) of Chapter 4 and eq. (3.1.8) we have

$$\hat{T} = \hat{T}(\mathbf{v}) + 2G\left(\nabla\nabla\chi - \nabla^2\chi\hat{E}\right) = \hat{T}(\mathbf{v}) + 2G\left(\nabla\nabla\chi - \frac{1+\nu}{1-\nu}\alpha\theta\hat{E}\right),$$

$$R\mathbf{P}_R = \mathbf{R}\cdot\hat{T} = R\mathbf{P}_R(v) + 2G\left(\mathbf{R}\cdot\nabla\nabla\chi - \frac{1+\nu}{1-\nu}\alpha\theta\mathbf{R}\right).$$

As shown in Subsection 5.3.4

$$\nabla\chi = \sum_{n=0}^{\infty}\nabla\chi_n = \alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\frac{1}{2n+3}\left(\mathbf{R}\theta_n + \frac{1}{2}R^2\nabla\theta_n\right),$$

where θ_n denotes the homogeneous harmonic polynomials of the $n-th$ power, in terms of which the harmonic function θ is expanded, and χ_n are the homogeneous polynomials of the $(n+2)-th$ power, so that

$$\mathbf{R}\cdot\nabla\nabla\chi_n = (n+1)\nabla\chi_n = \alpha\frac{1+\nu}{1-\nu}\frac{n+1}{2n+3}\left(\mathbf{R}\theta_n + \frac{1}{2}R^2\cdot\nabla\theta_n\right)$$

and furthermore

$$\frac{1}{2G}R\mathbf{P}_R = \frac{1}{2G}R\mathbf{P}_R(\mathbf{v}) + \alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\left[\frac{1}{2}\frac{n+1}{2n+3}R^2\nabla\theta_n - \frac{n+2}{2n+3}\mathbf{R}\theta_n\right]. \tag{3.8.2}$$

Vector $\mathbf{v}$ is determined by the boundary condition

$$\frac{1}{2G}R_0\,\mathbf{P}_R(\mathbf{v})|_{R=R_0} = \\ = -\alpha\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\left[\frac{1}{2}\frac{n+1}{2n+3}R_0^2(\nabla\theta_n)_{R=R_0} - \frac{n+2}{2n+3}\mathbf{R}_0(\theta_n)_{R=R_0}\right] \quad (3.8.3)$$

and the harmonic vector $\Pi(\mathbf{v})$ is as follows

$$\Pi(\mathbf{v}) = -\alpha\frac{1+\nu}{1-\nu}\left\{\sum_{n=0}^{\infty}\frac{1}{2}\frac{n+1}{2n+3}R_0^2\nabla\theta_n - \right. \\ \left. R_0\sum_{n=0}^{\infty}\frac{(n+2)\,a_{n0}}{2n+3}\left[\left(\frac{R}{R_0}\right)^{n+1}\mathbf{Y}^*_{n+1} + \left(\frac{R}{R_0}\right)^{n-1}\mathbf{Y}^{**}_{n-1}\right]\right\}, \quad (3.8.4)$$

where the representation (3.4.8) of vector $(e_R\theta_n)_{R=R_0}$ in terms of Laplace's spherical vectors was used, see Subsection 5.3.4. By eq. (3.5.5) we have

$$\frac{1}{2G}R\mathbf{P}_R = \frac{1}{2}\alpha\frac{1+\nu}{1-\nu}\left(R^2 - R_0^2\right)\sum_{n=0}^{\infty}\frac{n+1}{2n+3}\nabla\theta_n - \\ \alpha\frac{1+\nu}{1-\nu}R_0\sum_{n=0}^{\infty}\frac{n+2}{2n+3}\left[\frac{\mathbf{R}}{R_0}\theta_n - \left(\Pi^*_{n+1} + \Pi^{**}_{n-1}\right)\right] + \\ \frac{1}{2}\alpha\frac{1+\nu}{1-\nu}R_0\left(R_0^2 - R^2\right)\sum_{n=0}^{\infty}\frac{n+2}{2n+3}\left[\frac{(n-1)\,\nabla\nabla\cdot\Pi^*_{n+1}}{n^2+n(1+2\nu)+1+\nu} + \right. \\ \left. \frac{(n-3)\,\nabla\nabla\cdot\Pi^{**}_{n-1}}{n^2-(3-2\nu)\,n+3(1-\nu)}\right], \quad (3.8.5)$$

where the harmonic vectors are denoted as follows

$$\Pi^*_{n+1} = a_{n0}\left(\frac{R}{R_0}\right)^{n+1}\mathbf{Y}^*_{n+1}, \quad \Pi^{**}_{n-1} = a_{n0}\left(\frac{R}{R_0}\right)^{n-1}\mathbf{Y}^{**}_{n-1}. \quad (3.8.6)$$

It is easy to prove that under the linear distribution of temperature

$$\theta = a_{00} + \frac{a_{10}}{R_0}z, \quad a_{k0} = 0, \quad k = 2, 3, \dots,$$

vector $\mathbf{P}_R$ vanishes, cf. Subsection 4.1.14. For this reason, the summation must begin with $n = 2$.

The stress at the centre ($R = 0$) is determined by prescribing only the second term in expansion (3.4.12) of the temperature in a series in terms of the harmonic polynomials. The remaining terms, among them the terms with a_{40}, vanish at $R = 0$. Calculation using formulae (3.8.5), (3.8.6), (3.4.8) and (F.2.17) yields

$$(\mathbf{P}_R)_{R=R_0} = \frac{1}{5}\alpha G\frac{1+\nu}{1-\nu}a_{20}\frac{14-5\nu}{r+5\nu}\left(2\mathbf{k}\cos\vartheta - \mathbf{e}_r\right). \quad (3.8.7)$$

5.3.9 *The state of stress in the vicinity of a spherical cavity*

Far away from the cavity the state of stress is assumed to be uniform and is given by a constant tensor $\hat{T}^\infty$. When the cavity is present, the stress tensor $\hat{T}$ is given by the following sum

$$\hat{T} = \hat{T}^\infty + \hat{T}^*, \tag{3.9.1}$$

where $\hat{T}^*$ denotes a tensor which vanishes at infinity (for $R \to \infty$) and describes the perturbation of the state of stress due to the cavity. The cavity surface $R = R_0$ is assumed to be free of load, thus

$$\left(\mathbf{e}_r \cdot \hat{T}\right)_{R=R_0} = 0, \quad \left(\mathbf{e}_r \cdot \hat{T}^*\right)_{R=R_0} = -\left(\mathbf{e}_r \cdot \hat{T}^\infty\right)_{R=R_0}$$

or

$$\begin{aligned} R\mathbf{P}^*_R|_{R=R_0} &= -R_0\mathbf{e}_r \cdot \hat{T}^\infty = -R_0\mathbf{e}_r \cdot \mathbf{i}_s\mathbf{i}_k t^\infty_{sk} \\ &= -R_0\mathbf{i}_k \left[\sin\vartheta \left(t^\infty_{1k}\cos\lambda + t^\infty_{2k}\sin\lambda\right) + t^\infty_{3k}\cos\vartheta\right] = -\mathbf{Y}_1(\mu,\lambda). \end{aligned} \tag{3.9.2}$$

The harmonic vector Π^* is determined by the equality

$$\Pi^* = \Pi^*_{-2} = -\frac{1}{2G}\left(\frac{R_0}{R_0}\right)^2 \mathbf{Y}^*_1(\mu,\lambda) = -\frac{1}{2G}\left(\frac{R_0}{R_0}\right)^3 \mathbf{R}\cdot\hat{T}^\infty. \tag{3.9.3}$$

An expression for the displacement vector is obtained from eq. (3.6.10) in which only one term $n = 1$ is present

$$\begin{aligned} \mathbf{u}^* = \frac{1}{4G}R_0^3 \Bigg[&\frac{\mathbf{R}\cdot\hat{T}^\infty}{R^3} + \frac{1}{3}\mathbf{R}\times\left(\nabla\times\frac{\mathbf{R}\cdot\hat{T}^\infty}{R^3}\right) - \frac{4-10\nu}{7-5\nu}\mathbf{R}\nabla\cdot\frac{\mathbf{R}\cdot\hat{T}^\infty}{R^3} \\ &+ \frac{5\nu-1}{3(7-5\nu)}R^2\nabla\nabla\cdot\frac{\mathbf{R}\cdot\hat{T}^\infty}{R^3} + \frac{R^2-R_0^2}{7-5\nu}\nabla\nabla\cdot\frac{\mathbf{R}\cdot\hat{T}^\infty}{R^3}\Bigg]. \end{aligned} \tag{3.9.4}$$

Let us consider a particular case $\hat{T}^\infty = \mathbf{i}_3\mathbf{i}_3\sigma_z^\infty$ corresponding to the case of a stretched rod with a spherical cavity whose diameter is very small in comparison with the rod diameter. A rather straightforward, although cumbersome, calculation by means of eq. (3.9.4) leads to the relative simple formulae

$$\begin{aligned} \sigma_\vartheta &= \frac{\sigma_z^\infty}{2(7-5\nu)}\left(27 - 15\nu - 30\cos^2\vartheta\right), \\ \sigma_\lambda &= \frac{\sigma_z^\infty}{2(7-5\nu)}\left(15\nu - 3 - 30\nu\cos^2\vartheta\right). \end{aligned}$$

In particular for $\vartheta = \pi/2$

$$\begin{aligned} \sigma_\vartheta = \sigma_z &= \frac{27-15\nu}{2(7-5\nu)}\sigma_z^\infty \approx 2,07\sigma_z^\infty, \\ \sigma_\lambda &= \frac{15\nu-3}{2(7-5\nu)}\sigma_z^\infty \approx 0,19\sigma_z^\infty \quad \left(\nu = \frac{1}{3}\right) \end{aligned}$$

and for $\vartheta = 0$

$$\sigma_\vartheta = \sigma_\lambda = -\frac{3+15\nu}{2(7-5\nu)}\sigma_z^\infty \approx -0,75\sigma_z^\infty.$$

The maximum tensile stress is at the equator of the cavity and is 2.07 times greater than the nominal stress σ_z^∞, the compressed stresses $0,75\sigma_z^\infty$ appear at the poles of the cavity. The stress concentration is of the local character, for example, in the equatorial plane $\vartheta = \pi/2$

$$\sigma_z = \sigma_\vartheta = \sigma_z^\infty \left[1 + \frac{4-5\nu}{2(7-5\nu)}\left(\frac{R_0}{R}\right)^3 + \frac{9}{2(7-5\nu)}\left(\frac{R_0}{R}\right)^5\right].$$

For $R = R_0$ we obtain the above value of $2.07\sigma_z^\infty$, however the stress reduces to $1.03\sigma_z^\infty$ at $R = 2R_0$.

5.3.10 *The state of stress in the vicinity of a small spherical cavity in a twisted cylindrical rod*

The problems with non-uniform states of stress at infinity are considered by analogy. For instance, in a cylindrical rod subjected to torsion

$$\tau_{xz}^\infty = -\frac{M_z}{I_p}y, \quad \tau_{yz}^\infty = \frac{M_z}{I_p}y$$

or

$$\hat{T}^\infty = \frac{M_z}{I_p}\left[-(\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_1\mathbf{i}_3)\,y + (\mathbf{i}_2\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_2)\,x\right], \tag{3.10.1}$$

where M_z denotes the torque and I_p is the polar moment of inertia of the rod.

Assuming as above

$$\hat{T} = \hat{T}^\infty + \hat{T}^*$$

we have

$$\begin{aligned}(R\mathbf{P}_R^*)_{R=R_0} &= -R_0\mathbf{e}_R \cdot \hat{T}^\infty = -\frac{M_z R_0^2}{I_p}\sin\vartheta\cos\vartheta\,(-\mathbf{i}_1\sin\lambda + \mathbf{i}_2\cos\lambda)\\ &= -\frac{M_z R_0^2}{3I_p}P_2^1(\mu)\,(-\mathbf{i}_1\sin\lambda + \mathbf{i}_2\cos\lambda) = -\mathbf{Y}_2(\mu,\lambda).\end{aligned}$$

For this reason

$$\Pi^* = \Pi_{-3}^* = -\frac{1}{2G}\left(\frac{R_0}{R}\right)^3\mathbf{Y}_2(\mu,\lambda) = -\frac{M_z}{2GI_p}\left(\frac{R_0}{R}\right)^5 z\,(-\mathbf{i}_1 y + \mathbf{i}_2 x)$$

or

$$\Pi^* = -\frac{1}{2}\left(\frac{R_0}{R}\right)^5 \mathbf{u}^\infty, \quad \mathbf{u}^\infty = \frac{M_z}{GI_p}\left(-\mathbf{i}_1 y + \mathbf{i}_2 x\right) z, \tag{3.10.2}$$

where $\mathbf{u}^\infty$ denotes the displacement vector at an infinite distance from the cavity. Only the $n = 2$ term is kept in expansion (3.6.10). Taking into account that

$$\operatorname{div} \frac{\mathbf{u}^\infty}{R^5} = 0$$

we obtain

$$\mathbf{u}^* = \frac{R_0^5}{6}\left(\frac{\mathbf{u}^\infty}{R^5} + \frac{1}{4}\mathbf{R} \times \operatorname{rot} \frac{\mathbf{u}^\infty}{R^5}\right) = \frac{1}{4}\left(\frac{R_0}{R}\right)^5 \mathbf{u}^\infty. \tag{3.10.3}$$

In spherical coordinates we have

$$\mathbf{u} = \mathbf{u}^* + \mathbf{u}^\infty = -\frac{M_z}{GI_p} R^2 \left[1 + \frac{1}{4}\left(\frac{R_0}{R}\right)^5\right] \mathbf{e}_\lambda \sin\vartheta \cos\vartheta = u_\lambda \mathbf{e}_\lambda \tag{3.10.4}$$

and the non-vanishing stresses are

$$\left.\begin{aligned} \tau_{\vartheta\lambda} &= G\left(\frac{1}{R}\frac{\partial u_\lambda}{\partial \vartheta} - \frac{u_\lambda}{R}\cot\vartheta\right) = \frac{M_z}{I_p} R\left(1 + \frac{1}{4}\frac{R_0^5}{R^5}\right)\sin^2\vartheta, \\ \tau_{R\lambda} &= G\left(\frac{\partial u_\lambda}{\partial R} - \frac{u_\lambda}{R}\right) = -\frac{M_z}{I_p} R\left(1 - \frac{R_0^5}{R^5}\right)\sin\vartheta\cos\vartheta. \end{aligned}\right\} \tag{3.10.5}$$

On the surface of the cavity

$$\tau_{\vartheta\lambda} = \frac{5}{4}\frac{M_z}{I_p} R_0 \sin^2\vartheta, \quad \tau_{R\lambda} = 0. \tag{3.10.6}$$

The distortion of the state of stress has a sharply defined local character. The maximum shear stress exceeds the nominal stress $M_z R_0 / I_p$ by 25%.

The simplicity of the obtained solution is explained by the fact that the problem of torsion of the body of revolution reduces to the only displacement u_λ, the product $u_\lambda e^{i\lambda}$ being a harmonic function, see Subsection 4.1.11.

5.3.11 Action of the mass forces

Given the mass forces with potential Φ, the particular solution of the equilibrium equations in terms of displacements is determined by eqs. (1.4.7) and (1.4.10) of Chapter 4

$$\mathbf{u} = \operatorname{grad} \chi, \quad \nabla^2 \chi = \frac{1 - 2\nu}{2G(1-\nu)}\Phi. \tag{3.11.1}$$

In what follows we consider some special cases of Φ.

1. $\Phi = \Phi(r), r = \sqrt{x^2 + y^2}$. In this case χ is sought as a function of r only

$$\left.\begin{aligned} &\nabla^2 \chi = \frac{1}{r}\frac{d}{dr} r \frac{d\chi}{dr} = \frac{1-2\nu}{2G(1-\nu)} \Phi(r), \\ &\chi = \frac{1-2\nu}{2G(1-\nu)} \left(\int^r \frac{dr}{r} \int^r r\Phi(r)\, dr + C_1 \ln r + C_2 \right), \\ &\mathbf{u} = \nabla\chi = \frac{1-2\nu}{2G(1-\nu)} \mathbf{e}_r \left(\frac{1}{r} \int^r r\Phi(r)\, dr + \frac{C_1}{r} \right) \end{aligned}\right\} \tag{3.11.2}$$

and by eq. (1.1.3) of Chapter 4

$$\hat{T} = \frac{1}{1-\nu} \left\{ \hat{E}\nu\Phi(r) + \right. \tag{3.11.3}$$

$$\left. (1-2\nu) \left[\mathbf{e}_r\mathbf{e}_r \Phi(r) - \frac{1}{r^2} (\mathbf{e}_r\mathbf{e}_r - \mathbf{e}_\varphi\mathbf{e}_\varphi) \left(\int^r r\Phi(r)\, dr + C \right) \right] \right\},$$

where $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{k}$ denote the unit vectors of the cylindrical coordinate system, see Section C.7.

For example, when a solid rotates about the motionless axis Oz with a constant angular velocity ω, the potential of the centrifugal force is given by eq. (1.2.6) of Chapter 1

$$\Phi = -\frac{1}{2}\frac{\gamma}{g}\omega^2 r^2 = -\frac{1}{2}\gamma \left| \boldsymbol{\omega} \times R \right|^2, \tag{3.11.4}$$

and the particular solution for the displacement vector, which is bounded at $r = 0$, is as follows

$$\mathbf{u} = -\frac{1-2\nu}{2G(1-\nu)} \frac{\gamma\omega^2}{8g} r^3 \mathbf{e}_r, \tag{3.11.5}$$

and the non-trivial components of the stress tensor are

$$\sigma_r = -\frac{\gamma\omega^2}{8g}\frac{3-2\nu}{1-\nu} r^2, \quad \sigma_\varphi = -\frac{\gamma\omega^2}{8g}\frac{1+2\nu}{1-\nu} r^2, \quad \sigma_z = -\frac{\gamma\omega^2}{2g}\frac{\nu}{1-\nu} r^2. \tag{3.11.6}$$

2. $\Phi = \Phi(R), r = \sqrt{x^2 + y^2 + z^2}$. Assuming χ to be a function of R and using eq. (3.11.1) we have

$$\frac{1}{R^2}\frac{d}{dR} R^2 \frac{d\chi}{dR} = \frac{1-2\nu}{2G(1-\nu)} \Phi(R),$$

$$\chi = \frac{1-2\nu}{2G(1-\nu)} \left(\int^R \frac{dR}{R^2} \int^R R^2 \Phi(R)\, dR - \frac{C_1}{R} + C_2 \right),$$

so that

$$\mathbf{u} = \frac{1-2\nu}{2G(1-\nu)}\mathbf{e}_R \left(\frac{1}{R^2} \int^R R^2 \Phi(R)\, dR + C_1 \right), \tag{3.11.7}$$

$$\hat{T} = \frac{1}{1-\nu} \left\{ \hat{E}\nu\Phi(R) + (1-2\nu) \left[\mathbf{e}_R\mathbf{e}_R\Phi(R) - \right.\right.$$
$$\left.\left. (2\mathbf{e}_R\mathbf{e}_R - \mathbf{e}_\vartheta\mathbf{e}_\vartheta - \mathbf{e}_\lambda\mathbf{e}_\lambda) \left(\frac{1}{R^3} \int^R R^2 \Phi(R)\, dR + \frac{C}{R} \right) \right] \right\}. \tag{3.11.8}$$

3. Potential Φ is a harmonic function which is presented by a series in terms of the homogeneous harmonic polynomials

$$\Phi = \sum_n \Phi_n(x, y, z). \tag{3.11.9}$$

Similar to Subsection 5.3.8 we find

$$\chi = \frac{1-2\nu}{4G(1-\nu)} R^2 \sum_n \frac{\Phi_n(x,y,z)}{2n+3}. \tag{3.11.10}$$

5.3.12 An attracting sphere

As follows from the theory of the Newtonian potential, the attracting force acting on a unit mass particle of the sphere is directed to the centre of the sphere and is proportional to the radius of the particle

$$\rho\mathbf{K} = -\frac{\gamma}{R_0}\mathbf{R} = -\operatorname{grad}\Phi, \quad \Phi = \frac{\gamma R^2}{2R_0}. \tag{3.12.1}$$

Here R_0 is the radius of the sphere and γ denotes the value of the volume force on the surface of the sphere, i.e. for the Earth it is the weight of a unit volume. The particular solution (3.11.7) is as follows

$$\mathbf{u}^* = \frac{1-2\nu}{20G(1-\nu)} \frac{\gamma R^2}{R_0}\mathbf{R}, \tag{3.12.2}$$

and by eqs. (3.11.8) and (3.1.8) we find

$$\left. \begin{aligned} \mathbf{R}\cdot\hat{T}^* &= R\mathbf{P}_R^* = \frac{3-\nu}{10(1-\nu)}\frac{\gamma R^2}{R_0}\mathbf{R}, \\ \Pi^* &= \frac{3-\nu}{20G(1-\nu)}\gamma R_0 \mathbf{R} = \Pi_1^* = \frac{R}{2GR_0}\mathbf{Y}_1. \end{aligned} \right\} \tag{3.12.3}$$

The surface of the sphere is not loaded and, by virtue of eq. (3.6.8), one needs to superimpose the following solution

$$\mathbf{u}_{**} = -\Pi_1^* + \frac{\nu}{1+\nu}\mathbf{R}\nabla\cdot\Pi_1^* = -\frac{3-\nu}{20G(1-\nu)}\frac{1-2\nu}{1+\nu}\gamma R_0 \mathbf{R}$$

on the solution (3.12.2). The result is

$$\mathbf{u} = \mathbf{u}_* + \mathbf{u}_{**} = \frac{1-2\nu}{20G\,(1-\nu)}\gamma R_0 \mathbf{R}\left(\frac{R^2}{R_0^2} - \frac{3-\nu}{1+\nu}\right). \tag{3.12.4}$$

The stress vector in the centre of the sphere is obtained by means of eq. (3.7.1)

$$(\mathbf{P}_R)_{R=0} = -\frac{3-\nu}{10\,(1-\nu)}\gamma R_0 \mathbf{e}_R. \tag{3.12.5}$$

For the Earth $\gamma = 5.53 \cdot 10^3\ kg/m^3$ and $R_0 = 6.37 \cdot 10^6\ m$ and the stress calculated with the help of the latter formula turns out to be unrealistically large. This indicates that the methods of the linear theory of elasticity are not applicable to the considered problem.

5.3.13 A rotating sphere

The stress tensor $\hat{T}$ and the displacement vector $\mathbf{u}$ are put in the form

$$\hat{T} = \hat{T}^0 + \hat{T}^*, \quad \mathbf{u} = \mathbf{u}^0 + \mathbf{u}^*, \tag{3.13.1}$$

where $\hat{T}^0$ and $\mathbf{u}^0$ are the particular solutions (3.11.6) and (3.11.5) corresponding to the centrifugal forces. Writing $\hat{T}^0$ in the form

$$\hat{T}^0 = -\frac{\gamma\omega^2}{8g\,(1-\nu)}\left[\mathbf{e}_R\mathbf{e}_R\,(3-2\nu) + \mathbf{e}_\lambda\mathbf{e}_\lambda\,(1+2\nu) + 4\mathbf{k}\mathbf{k}\nu\right] R^2\left(1-\mu^2\right) \tag{3.13.2}$$

and referring to eqs. (F.2.12) and (F.2.16) we have

$$\begin{aligned}\mathbf{R}\cdot\hat{T}^0 = -\frac{\gamma\omega^2}{20\,(1-\nu)\,g}R^3\Big\{&\left[2\,(3-2\nu)\,\mathbf{e}_r P_1^1(\mu) + (3+2\nu)\,\mathbf{k}P_1(\mu)\right] - \\ &\left[\frac{1}{3}\,(3-2\nu)\,\mathbf{e}_r P_3^1(\mu) + (3+2\nu)\,\mathbf{k}P_3(\mu)\right]\Big\},\end{aligned} \tag{3.13.3}$$

where the unit vector $\mathbf{e}_r$ is introduced, cf. Subsection 5.3.4, so that

$$\mathbf{e}_R = \mathbf{e}_r \sin\vartheta + \mathbf{k}\cos\vartheta.$$

The surface $R = R_0$ of the sphere is free of loads, i.e.

$$\left(\mathbf{R}\cdot\hat{T}\right)_{R=R_0} = \left(\mathbf{R}\cdot\hat{T}^0 + \mathbf{R}\cdot\hat{T}^*\right)_{R=R_0} = 0$$

or

$$\left(R\mathbf{P}_R^*\right)_{R=R_0} = -\left(\mathbf{R}\cdot\mathbf{P}_R^0\right)_{R=R_0} = \left(\Pi_1 + \Pi_3\right)_{R=R_0}.$$

Here Π_1 and Π_3 are harmonic vectors for $R < R_0$

$$\Pi_1 = \frac{R}{R_0} \mathbf{Y}_1(\mu, \lambda), \quad \Pi_3 = \left(\frac{R}{R_0}\right)^3 \mathbf{Y}_3(\mu, \lambda) \tag{3.13.4}$$

and $\mathbf{Y}_1$ and $\mathbf{Y}_3$ are Laplace's spherical vectors given by eq. (3.13.3)

$$\left.\begin{aligned} \mathbf{Y}_1 &= A\left[2(3-2\nu)\,\mathbf{e}_r P_1^1(\mu) + (3+2\nu)\,\mathbf{k} P_1(\mu)\right], \\ \mathbf{Y}_3 &= -A\left[\frac{1}{3}(3-2\nu)\,\mathbf{e}_r P_3^1(\mu) + (3+2\nu)\,\mathbf{k} P_3(\mu)\right], \end{aligned}\right\} \tag{3.13.5}$$

with

$$A = \frac{\gamma\omega^2 R_0^3}{40G(1-\nu)\,g}.$$

The displacement vector is determined by means of formulae (3.6.3), (3.6.4), (3.6.8) and (3.13.1), (3.11.5). More general formulae for the case of the ellipsoid of rotation are derived in Subsection 5.4.5. At the pole and the equator of the sphere the displacement vector is equal to

$$\left.\begin{aligned} \mathbf{u}|_{\substack{R=R_0\\ \vartheta=0}} &= \frac{\gamma\omega^2}{2gG} R_0^3 \left[\frac{2(1-2\nu)}{15(1+\nu)} - \frac{2(2+\nu)}{3(7+5\nu)}\right]\mathbf{k}, \\ \mathbf{u}|_{\substack{R=R_0\\ \vartheta=\pi/2}} &= \frac{\gamma\omega^2}{2gG} R_0^3 \left[\frac{2(1-2\nu)}{15(1+\nu)} + \frac{2+\nu}{3(7+5\nu)}\right]\mathbf{e}_r, \end{aligned}\right\} \tag{3.13.6}$$

where $\mathbf{k}$ and $\mathbf{e}_r$ are the unit vectors of the cylindrical coordinate system. Applying these equations to the Earth and taking the contraction at the poles $\varepsilon_R = 1/300$ and $\nu = 1/3$ we obtain $G \approx 2.6 \cdot 10^{10} N/m^2$ which approximately corresponds to the shear modulus of glass.

The stress vector on the surface of the sphere $R < R_0$ determined by eq. (3.5.5) is equal to

$$\begin{aligned} \mathbf{P}_R &= \frac{\gamma\omega^2}{20g(1-\nu)}\left(R_0^2 - R^2\right)\Bigg[2(3-2\nu)\,\mathbf{e}_r P_1^1(\mu) + (3+2\nu)\,\mathbf{k} P_1(\mu) + \\ &\quad \frac{\nabla\nabla\cdot R^3\mathbf{Y}_3}{(14+10\nu)\,RA}\Bigg] = \frac{\gamma\omega^2}{20g(1-\nu)}\left\{\left[2(3-2\nu) + \frac{21-2\nu}{14+10\nu}\right]\mathbf{e}_r \sin\vartheta + \right. \\ &\quad \left.\left(3+2\nu - \frac{21-2\nu}{7+5\nu}\right)\mathbf{k}\cos\vartheta\right\}\left(R_0^2 - R^2\right). \end{aligned} \tag{3.13.7}$$

Besides,

$$\mathbf{P}_R = \mathbf{e}_R \cdot \hat{T} = \mathbf{e}_r \cdot \hat{T}\sin\vartheta + \mathbf{k}\cdot\hat{T}\cos\vartheta,$$

and thus at the equator and the pole of the sphere

$$\left.\begin{aligned} \left(\mathbf{e}_r\cdot\hat{T}\right)_{\vartheta=\pi/2} &= \frac{\gamma\omega^2}{20g(1-\nu)}\left(R_0^2 - R^2\right)\left[2(3-2\nu) + \frac{21-2\nu}{14+10\nu}\right]\mathbf{e}_r, \\ \left(\mathbf{k}\cdot\hat{T}\right)_{\vartheta=0} &= \frac{\gamma\omega^2}{20g(1-\nu)}\left(R_0^2 - R^2\right)\left(3+2\nu - \frac{21-2\nu}{7+5\nu}\right)\mathbf{k}. \end{aligned}\right\} \tag{3.13.8}$$

Vectors $\left(\mathbf{e}_r \cdot \hat{T}\right)_{\vartheta=\pi/2}$ and $\left(\mathbf{k} \cdot \hat{T}\right)_{\vartheta=0}$ have the directions of $\mathbf{e}_r$ and $\mathbf{k}$, respectively, that is, at the equator and the pole the shear stresses are absent and the above formulae yield normal stresses σ_r and σ_z, respectively.

The sum of the normal stresses in the centre of the sphere is found with the help of eq. (3.5.7)

$$\sigma = (\sigma_R + \sigma_\vartheta + \sigma_\lambda)_{R=0} = 2G\,(\nabla \cdot \mathrm{II}_1)_{R=0} = \frac{\gamma\omega^2 R_0^2}{20g\,(1-\nu)}\,(15 - 6\nu)\,. \tag{3.13.9}$$

From this equation and eq. (3.13.8) we have

$$\left.\begin{aligned} (\sigma_r)_{R=0} = (\sigma_\lambda)_{R=0} &= \frac{\gamma\omega^2 R_0^2}{20g\,(1-\nu)}\left[2\,(3-2\nu) + \frac{21-2\nu}{14+10\nu}\right], \\ (\sigma_z)_{R=0} &= \frac{\gamma\omega^2 R_0^2}{20g\,(1-\nu)}\left(3 + 2\nu - \frac{21-2\nu}{7+5\nu}\right). \end{aligned}\right\} \tag{3.13.10}$$

The coincidence of normal stresses σ_r and σ_λ at the centre of the sphere also follows from the symmetry of the problem.

5.3.14 Action of concentrated forces

We consider the state of stress in the sphere loaded by a system of concentrated forces applied at points $R_0\mathbf{e}_R^{(i)}$ of the surface. This system of forces is assumed to be in equilibrium, i.e.

$$\sum_{r=1}^{N} \mathbf{Q}_i = 0, \quad \sum_{r=1}^{N} \mathbf{e}_R^{(i)} \times \mathbf{Q}_i = 0. \tag{3.14.1}$$

The plane passing through the position radius of the point of application of force $\mathbf{Q}_i$ and the line of action of this force intersects the sphere along the meridional plane π_i^0. The meridional plane π_i passes through the point of application of force $\mathbf{Q}_i$ and the observation point $R\mathbf{e}_R$. The angle in plane π_i between vectors $\mathbf{e}_R^{(i)}$ and $\mathbf{e}_R$ is denoted by θ_i

$$\gamma_i = \cos\theta_i = \mathbf{e}_R^{(i)} \cdot \mathbf{e}_R = \cos\vartheta\cos\vartheta_i + \sin\vartheta\sin\vartheta_i\cos(\lambda - \lambda_i)\,, \tag{3.14.2}$$

where $(R_i, \vartheta_i, \lambda_i)$ and (R, ϑ, λ) are the spherical coordinates of force $\mathbf{Q}_i$ and the observation point, respectively. It is evident that

$$\nabla\gamma_i = \nabla\mathbf{e}_R^{(i)} \cdot \mathbf{e}_R = \nabla\mathbf{e}_R^{(i)} \cdot \frac{\mathbf{R}}{R} = \mathbf{e}_R^{(i)} \cdot \frac{\hat{E}}{R} - \mathbf{e}_R^{(i)} \cdot \frac{\mathbf{R}\mathbf{R}}{R^3} = \frac{1}{R}\left(\mathbf{e}_R^{(i)} - \gamma_i\mathbf{e}_R\right). \tag{3.14.3}$$

The vector of the concentrated force $\mathbf{Q}_i$ is expanded in terms of Laplace's spherical surface vectors by means of a limiting passage $\varepsilon_i \to 0$ from the following distributed force

$$\mathbf{Q}_i(\theta_i) = \begin{cases} 0, & \varepsilon_i < \theta_i < \pi, \\ \dfrac{\mathbf{Q}_i}{\pi R_0^2 \varepsilon_i^2}, & 0 < \theta_i < \varepsilon_i. \end{cases}$$

Referring to eq. (F.4.8) we have

$$\mathbf{Q}_i(\theta_i) = \lim_{\varepsilon_i \to 0} \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\gamma_i) \int_{\cos\varepsilon_i}^{1} \frac{\mathbf{Q}_i}{\pi R_0^2 \varepsilon_i^2} P_n(\gamma_i') \, d\gamma_i'.$$

Furthermore,

$$(2n+1) P_n = P_{n+1}' - P_{n-1}',$$

$$(2n+1) \int_{\cos\varepsilon_i}^{1} P_n(\gamma_i') \, d\gamma_i' = P_{n-1}(\cos\varepsilon_i) - P_{n+1}(\cos\varepsilon_i),$$

since $P_n(1) = 1$. Next

$$\lim_{\varepsilon_i \to 0} \frac{1}{\varepsilon_i^2} \left[P_{n-1}(\cos\varepsilon_i) - P_{n+1}(\cos\varepsilon_i)\right] =$$

$$= \lim_{\varepsilon_i \to 0} \frac{1}{2\varepsilon_i} \frac{d\cos\varepsilon_i}{d\varepsilon_i} \left[P_{n-1}'(1) - P_{n+1}'(1)\right] = \frac{2n+1}{2},$$

and thus the concentrated force is represented by the following (divergent) series in terms of Laplace's polynomials $P_n(\gamma_i)$

$$\mathbf{Q}_i = \frac{1}{4\pi R_0^2} \sum_{n=0}^{\infty} (2n+1) P_n(\gamma_i).$$

Referring to eqs. (3.5.1) and (3.5.2) we obtain

$$\Pi = \sum_{n=0}^{\infty} \Pi_n = \frac{1}{8G\pi R_0} \sum_{n=0}^{\infty} (2n+1) \left(\frac{R}{R_0}\right)^n \sum_{i=1}^{N} \mathbf{Q}_i P_n(\gamma_i). \qquad (3.14.4)$$

Let us notice that the series

$$\sum_{n=0}^{\infty} (2n+1) \left(\frac{R}{R_0}\right)^n P_n(\gamma_i)$$

converges to yield

$$\frac{R_0 \left(R_0^2 - R^2\right)}{\left(R_0^2 + R^2 - 2RR_0\gamma_i\right)^{3/2}},$$

so that

$$\Pi = \frac{R_0^2 - R^2}{8\pi G} \sum_{i=1}^{N} \mathbf{Q}_i \left(R_0^2 + R^2 - 2RR_0\gamma_i\right)^{-3/2}. \tag{3.14.5}$$

Returning to eq. (3.14.4) we obtain

$$\Pi_n = \frac{2n+1}{8\pi G R_0^{n+1}} R^n \sum_{i=1}^{N} \mathbf{Q}_i P_n(\gamma_i)$$

and then referring to eq. (3.14.3) we obtain

$$\begin{aligned} \nabla \cdot R^n P_n(\gamma_i)\, \mathbf{Q}_i &= R^{n-1} \left[nP_n \mathbf{e}_R + P_n' \left(\mathbf{e}_R^{(i)} - \gamma_i \mathbf{e}_R \right) \right] \cdot \mathbf{Q}_i \\ &= R^{n-1} \left(P_n' \mathbf{e}_R^{(i)} - P_{n-1}' \mathbf{e}_R \right) \cdot \mathbf{Q}_i, \end{aligned} \tag{3.14.6}$$

since

$$nP_n - \gamma_i P_n' = -P_{n-1}'. \tag{3.14.7}$$

In accordance with eq. (3.5.7) the sum of the normal stresses is presented by the series

$$\sigma = \frac{1+\nu}{4\pi R_0^2} \sum_{n=1}^{\infty} \frac{4n^2 - 1}{n^2 - (1-2\nu)n + (1-\nu)} \left(\frac{R}{R_0}\right)^{n-1} \times \sum_{i=1}^{N} \mathbf{Q}_i \cdot \left(P_n' \mathbf{e}_R^{(i)} - P_{n-1}' \mathbf{e}_R \right). \tag{3.14.8}$$

Now we have

$$\nabla\nabla \cdot R^n P_n(\gamma_i)\, \mathbf{Q}_i = R^{n-2} \Big\{ (n-1)\, \mathbf{e}_R \left(P_n' \mathbf{e}_R^{(i)} - P_{n-1}' \mathbf{e}_R \right) + P_n'' \left(\mathbf{e}_R^{(i)} - \gamma \mathbf{e}_R \right) \mathbf{e}_R^{(i)} - P_{n-1}'' \left(\mathbf{e}_R^{(i)} - \mathbf{e}_R \gamma_i \right) \mathbf{e}_R - P_{n-1}' \left(\hat{E} - \mathbf{e}_R \mathbf{e}_R \right) \Big\} \cdot \mathbf{Q}_i.$$

Applying eq. (3.14.7) once again we can set this result in the form

$$\nabla\nabla \cdot R^n P_n(\gamma_i)\, \mathbf{Q}_i = R^{n-2} \Big[-\hat{E} P_{n-1}' + \mathbf{e}_R \mathbf{e}_R P_{n-2}'' + \mathbf{e}_R^{(i)} \mathbf{e}_R^{(i)} P_n'' - P_{n-1}'' \left(\mathbf{e}_R^{(i)} \mathbf{e}_R + \mathbf{e}_R \mathbf{e}_R^{(i)} \right) \Big] \cdot \mathbf{Q}_i \tag{3.14.9}$$

and, by virtue of eq. (3.5.5), the distribution of stresses on the surfaces $R = \text{const}$ is set as follows

$$\mathbf{P}_R = \frac{1}{4\pi R_0^2} \sum_{n=1}^{\infty} (2n+1) \left(\frac{R}{R_0} \right)^{n-1} \sum_{i=1}^{N} \mathbf{Q}_i P_n(\gamma_i) +$$
$$\frac{1}{4\pi R_0^2} \sum_{n=3}^{\infty} \frac{(2n+1)(n-2)}{2[n^2 - (1-2\nu)n + 1 - \nu]} \left[\left(\frac{R}{R_0} \right)^{n-3} - \left(\frac{R}{R_0} \right)^{n-1} \right] \times$$
$$\sum_{i=1}^{N} \Big[-\hat{E} P'_{n-1}(\gamma_i) + \mathbf{e}_R \mathbf{e}_R P''_{n-2}(\gamma_i) + \mathbf{e}_R^{(i)} \mathbf{e}_R^{(i)} P''_n(\gamma_i) -$$
$$P''_{n-1}(\gamma_i) \left(\mathbf{e}_R^{(i)} \mathbf{e}_R + \mathbf{e}_R \mathbf{e}_R^{(i)} \right) \Big] \cdot \mathbf{Q}_i. \qquad (3.14.10)$$

Using eqs. (F.2.12) and (3.7.1) we find that at the centre of the sphere

$$\mathbf{P}_R|_{R=0} = \frac{21}{8\pi R_0^2 (7+5\nu)} \sum_{i=1}^{N} \Big[\left(1 + \frac{10\nu}{7} \right) \mathbf{Q}_i \gamma_i +$$
$$5\gamma_i \mathbf{Q}_i \cdot \mathbf{e}_R^{(i)} \mathbf{e}_R^{(i)} - \mathbf{Q}_i \cdot \mathbf{e}_R^{(i)} \mathbf{e}_R - \mathbf{Q}_i \cdot \mathbf{e}_R \mathbf{e}_R^{(i)} \Big]. \qquad (3.14.11)$$

For example, in the case of a sphere compressed by two concentrated forces at the poles we have

$$\begin{aligned} &\mathbf{Q}_1 = -\mathbf{k}Q, \quad \mathbf{e}_R^{(1)} = \mathbf{k}, \quad \gamma_1 = \mathbf{e}_R \cdot \mathbf{k} = \cos\vartheta, \\ &\mathbf{Q}_2 = \mathbf{k}Q, \quad \mathbf{e}_R^{(2)} = -\mathbf{k}, \quad \gamma_2 = -\mathbf{e}_R \cdot \mathbf{k} = -\cos\vartheta \end{aligned}$$

and by eq. (3.14.11)

$$(\mathbf{P}_R)_{R=0} = \frac{21Q}{4\pi R_0^2 (7+5\nu)} \left[\mathbf{e}_R - \left(5 + \frac{10\nu}{7} \right) \mathbf{k} \cos\vartheta \right],$$

so that the normal stresses on the elementary surfaces which are parallel and perpendicular to the forces are respectively given by ($\nu = 1/3$)

$$\sigma_x = \frac{21Q}{4\pi R_0^2 (7+5\nu)} \approx \frac{Q}{\pi R_0^2} 0,605, \quad \sigma_z = -\frac{42+15\nu}{2\pi R_0^2 (7+5\nu)} Q \approx -2,71 \frac{Q}{\pi R_0^2}.$$

5.3.15 The distributed load case

In this case the sums in eqs. (3.14.10) and (3.14.8) should be replaced by integrals. Provided that the loading is prescribed by vector $\mathbf{P}_R^0(\mu', \lambda')$ we have

$$\mathbf{P}_R = \frac{1}{4\pi}\sum_{n=1}^{\infty}(2n+1)\left(\frac{R}{R_0}\right)^{n-1}\int_0^{2\pi} d\lambda' \int_{-1}^{1} d\mu' \mathbf{P}_R^0(\mu',\lambda') P_n(\gamma) +$$

$$\frac{1}{4}\sum_{n=3}^{\infty}\frac{(2n+1)(n-2)}{2\left[n^2-(1-2\nu)n+1-\nu\right]}\left[\left(\frac{R}{R_0}\right)^{n-3}-\left(\frac{R}{R_0}\right)^{n-1}\right]\times$$

$$\int_0^{2\pi} d\lambda' \int_{-1}^{1} d\mu' \mathbf{P}_R^0(\mu',\lambda')\cdot\left[-\hat{E}P'_{n-1}(\gamma)+\mathbf{e}_R\mathbf{e}_R P''_{n-2}(\gamma)+\right.$$

$$\left.\mathbf{e}'_R\mathbf{e}'_R P''_n(\gamma)+\left(\mathbf{e}'_R\mathbf{e}_R+\mathbf{e}_R\mathbf{e}'_R\right)P''_{n-1}(\gamma)\right], \quad (3.15.1)$$

where $\gamma = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\lambda-\lambda')$. The sum of the normal stresses is the following sum

$$\sigma = \frac{1+\nu}{4\pi}\sum_{n=1}^{\infty}\frac{4n^2-1}{n^2-(1-2\nu)n+(1-\nu)}\left(\frac{R}{R_0}\right)^{n-1}\times$$

$$\int_0^{2\pi} d\lambda' \int_{-1}^{1} d\mu' \mathbf{P}_R^0(\mu',\lambda')\cdot\left[\mathbf{e}'_R P'_n(\gamma)-\mathbf{e}_R P'_{n-1}(\gamma)\right]. \quad (3.15.2)$$

Referring to eq. (3.14.5) we notice that the first group of terms in eq. (3.5.1) can be expressed as Poisson's integral

$$\frac{1}{4\pi R}R_0^2\left(R_0^2-R^2\right)\int_0^{2\pi} d\lambda' \int_{-1}^{1} d\mu' \frac{\mathbf{P}_R^0(\mu',\lambda')}{\left(R^2+R_0^2-2RR_0\gamma\right)^{3/2}}.$$

5.4 Bodies of revolution

5.4.1 Integral equation of equilibrium

We use the notation of Section C.9 and consider a body of revolution with the unloaded lateral surface $q^2 = q_0^2$. The surfaces $q^1 = \pm q_0^1$ which are orthogonal to it are termed the end faces. It is assumed that the surface $q^2 = q_*^2$ degenerates to the axis of rotation Oz on which

$$r\left(q^1, q_*^2\right) = 0 \quad (4.1.1)$$

and the surface $q^1 = 0$ on plane $z = 0$ is a domain bounded by the circle of radius b

$$z\left(0, q^2\right) = 0, \quad r\left(0, q^2\right) \le r\left(0, q_0^2\right) = b. \quad (4.1.2)$$

The stress vector on the end faces is equal to

$$\mathbf{e}_1 \cdot \hat{T} = \sigma_1 \mathbf{e}_1 + \tau_{12}\mathbf{e}_2 + \tau_{1\varphi}\mathbf{e}_{1\varphi} \tag{4.1.3}$$

and the distribution of the stresses on it is statically equivalent to the principal vector

$$\mathbf{V} = \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} \left(\sigma_1 \mathbf{e}_1 + \tau_{12}\mathbf{e}_2 + \tau_{1\varphi}\mathbf{e}_{1\varphi}\right) H_2 r dq^2 \tag{4.1.4}$$

and the principal moment

$$\mathbf{m}^O = \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} \left(r\mathbf{e}_r + \mathbf{k}z\right) \times \left(\sigma_1 \mathbf{e}_1 + \tau_{12}\mathbf{e}_2 + \tau_{1\varphi}\mathbf{e}_{1\varphi}\right) H_2 r dq^2. \tag{4.1.5}$$

Here $H_2 r d\varphi dq^2$ denotes the area element of the surface $q^1 = \text{const}$.

Let us put

$$\mathbf{V} = \mathbf{i}_1 V_x + \mathbf{i}_2 V_y + \mathbf{k} V_z, \quad \mathbf{m}^O = m_x \mathbf{i}_1 + m_y \mathbf{i}_2 + m_z \mathbf{k}, \tag{4.1.6}$$

where, following the beam theory terminology, V_x, V_y are the transverse forces and V_z is the force of tension, whereas m_x, m_y are the bending moments and m_z is the torque. Using eq. (C.9.8) we have

$$\left.\begin{aligned}
V_x &= \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} \left[\left(-\sigma_1 \frac{\partial z}{\partial q^2} + \tau_{12}\frac{\partial r}{\partial q^2}\right)\cos\varphi - \tau_{1\varphi} H_2 \sin\varphi\right] r dq^2, \\
V_y &= \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} \left[\left(-\sigma_1 \frac{\partial z}{\partial q^2} + \tau_{12}\frac{\partial r}{\partial q^2}\right)\sin\varphi + \tau_{1\varphi} H_2 \cos\varphi\right] r dq^2, \\
V_z &= \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} \left(\sigma_1 \frac{\partial r}{\partial q^2} + \tau_{12}\frac{\partial z}{\partial q^2}\right) r dq^2.
\end{aligned}\right\} \tag{4.1.7}$$

Next

$$\mathbf{m}^O = \int_0^{2\pi} d\varphi \int_{q_*^2}^{q_0^2} r dq^2 \left\{\left[-\sigma_1\left(r\frac{\partial r}{\partial q^2} + z\frac{\partial z}{\partial q^2}\right) + \tau_{12}\left(\frac{\partial r}{\partial q^2} z - \frac{\partial z}{\partial q^2} r\right)\right]\mathbf{e}_{1\varphi} + H_2 \tau_{1\varphi}\left(r\mathbf{k} - z\mathbf{e}_r\right)\right\},$$

so that

$$\left.\begin{aligned}
m_x &= -\int\limits_0^{2\pi} d\varphi \int\limits_{q_*^2}^{q_0^2} r dq^2 \left\{ \left[-\sigma_1 \left(r\frac{\partial r}{\partial q^2} + z\frac{\partial z}{\partial q^2} \right) + \right.\right. \\
&\quad \left.\left. \tau_{12} \left(\frac{\partial r}{\partial q^2} z - \frac{\partial z}{\partial q^2} r \right) \right] \sin\varphi + H_2 \tau_{1\varphi} z \cos\varphi \right\}, \\
m_y &= \int\limits_0^{2\pi} d\varphi \int\limits_{q_*^2}^{q_0^2} r dq^2 \left\{ \left[-\sigma_1 \left(r\frac{\partial r}{\partial q^2} + z\frac{\partial z}{\partial q^2} \right) + \right.\right. \\
&\quad \left.\left. \tau_{12} \left(\frac{\partial r}{\partial q^2} z - \frac{\partial z}{\partial q^2} r \right) \right] \cos\varphi - H_2 \tau_{1\varphi} z \sin\varphi \right\}, \\
m_z &= \int\limits_0^{2\pi} d\varphi \int\limits_{q_*^2}^{q_0^2} r^2 H_2 \tau_{1\varphi} dq^2.
\end{aligned}\right\} \qquad (4.1.8)$$

The boundary conditions on the side surface are set in the form

$$q^2 = q_0^2; \quad \tau_{12} = 0, \quad \sigma_2 = 0, \quad \tau_{2\varphi} = 0. \qquad (4.1.9)$$

As the side surface is free, the principal vector $\mathbf{V}$ and the principal moment $\mathbf{m}^O$ of the stresses on the face surface do not depend on q^1. This follows from the static reasoning: the forces applied to the body bounded by two arbitrary surfaces $q^1 = \text{const}$ and $q^2 = q_0^2$ are in equilibrium, so that their principal vector and principal moment are the same for any surface q^1. The only requirement imposed on the stresses on these surfaces is that of the static equivalence to the prescribed $\mathbf{V}$ and $\mathbf{m}^O$.

The posed problem is split into four essentially different problems: (i) tension by the axial force V_z, (ii) torsion by torque m_z, (iii) bending by moment m_x (or m_y) and (iv) bending by force V_x (or V_y). As formulae (4.1.7) and (4.1.8) suggest, the state of stress in problems (i) and (ii) can be taken as being axisymmetric, where in the problem of tension the non-vanishing stresses are $\sigma_1, \sigma_2, \sigma_\varphi, \tau_{12}$ and non-vanishing displacements are u_1, u_2 while in the problem of torsion only stresses $\tau_{1\varphi}, \tau_{2\varphi}$ and displacement u_φ do not vanish. The problems of bending are more complicated because all components of the stress tensor and the displacement vector do not vanish. According to eqs. (4.7.1) and (4.8.1) in the problem of bending by force V_x (V_y) and moment m_x (m_y), the stresses and the displacements

$$\sigma_1, \sigma_2, \sigma_\varphi, \tau_{12}, u_1, u_2 \text{ are proportional to } \cos\varphi\,(\sin\varphi),$$
$$\tau_{1\varphi}, \tau_{2\varphi}, u_\varphi \text{ are proportional to } \sin\varphi\,(\cos\varphi).$$

It is convenient to introduce into consideration the "nominal" stresses. They are determined by means of the elementary theory for the stresses in a circular rod of radius $b = r\left(0, q_0^2\right)$

$$q_x = \frac{V_x}{\pi b^2}, \quad q_z = \frac{V_z}{\pi b^2}; \quad p_y = \frac{m_y b}{I_y} = \frac{4m_y}{\pi b^3}, \quad p_z = \frac{m_z b}{I_p} = \frac{2m_z}{\pi b^3}.$$

After integrating over φ, formulae (4.1.7) and (4.1.8) can be written down in the form

$$q_x = \frac{1}{b^2} \int_{q_*^2}^{q_0^2} \left(-\sigma_1^* \frac{\partial z}{\partial q^2} + \tau_{12}^* \frac{\partial r}{\partial q^2} - \tau_{1\varphi}^* H_2 \right) r dq^2, \tag{4.1.10}$$

$$q_z = \frac{2}{b^2} \int_{q_*^2}^{q_0^2} \left(\sigma_1 \frac{\partial r}{\partial q^2} + \tau_{12} \frac{\partial z}{\partial q^2} \right) r dq^2, \tag{4.1.11}$$

$$p_x = -\frac{4}{b^3} \int_{q_*^2}^{q_0^2} \left[-\sigma_1^* \left(r \frac{\partial r}{\partial q^2} + z \frac{\partial z}{\partial q^2} \right) + \right.$$
$$\left. \tau_{12}^* \left(\frac{\partial r}{\partial q^2} z - \frac{\partial z}{\partial q^2} r \right) + z H_2 \tau_{1\varphi}^* \right] r dq^2, \tag{4.1.12}$$

$$p_z = \frac{4}{b^3} \int_{q_*^2}^{q_0^2} r^2 H_2 \tau_{1\varphi} dq^2. \tag{4.1.13}$$

Here, analogous to Subsections 4.1.11-4.1.13, an asterisk denotes the factors in front of $\cos\varphi$ and $\sin\varphi$ in the expressions for the stresses in asymmetric problems of bending.

The integrals in eqs. (4.1.10) and (4.1.12) show that if $R = \sqrt{r^2 + z^2} \to \infty$ then in the problems of tension and force bending, the stresses decrease not slower than R^{-2}, whereas in the problem of torsion and moment bending they decrease not slower than R^{-3}.

Since $q_x, \ldots, p_z$ are constants which are independent of q^1 one can take $q^1 = 0, z = 0$, cf. eq. (4.1.2), and thus $\partial z / \partial q^2 = 0$. This results in the following equalities

$$q_x = \frac{1}{b^2} \int_{q_*^2}^{q_0^2} \left[\left(\tau_{12}^* \frac{\partial r}{\partial q^2} - \tau_{1\varphi}^* H_2 \right) r \right]_{q^1=0} dq^2, \tag{4.1.14}$$

$$q_z = \frac{2}{b^2} \int_{q_*^2}^{q_0^2} \left(\sigma_1 \frac{\partial r^2}{\partial q^2} \right)_{q^1=0} dq^2, \tag{4.1.15}$$

$$p_x = -\frac{4}{b^3} \int\limits_{q_*^2}^{q_0^2} \left[r \left(-\sigma_1^* r \frac{\partial r}{\partial q^2} + \tau_{12}^* r H_2 \right) \right]_{q^1=0} dq^2, \tag{4.1.16}$$

$$p_z = \frac{4}{b^3} \int\limits_{q_*^2}^{q_0^2} \left(r^2 H_2 \tau_{1\varphi} \right)_{q^1=0} dq^2. \tag{4.1.17}$$

5.4.2 Tension of the hyperboloid of revolution of one nappe

In order to determine the stress concentration in the neck of the surface of the cylindric rod, Neuber considered a number of problems of equilibrium of bodies, bounded by a surface of a hyperboloid of revolution of one nappe, Fig. 5.3. The end face surfaces are loaded and the side surfaces of the hyperboloid are free. The present subsection is concerned with the problem of tension, whilst the problems of torsion and bending are considered later.

We introduce the curvilinear orthogonal coordinates of Section C.10

$$q^1 = s, \quad q^2 = \mu, \quad q^3 = \varphi; \quad r = a\sqrt{1+s^2}\sqrt{1-\mu^2}, \quad z = as\mu,$$

where, due to eq. (C.10.7), $-\infty \le s \le \infty, 0 \le \mu \le 1$ and on hyperboloid's axis $q^2 = q_*^2 = \mu = 1$ and on the unloaded surface of the hyperboloid $q^2 = q_0^2 = \mu_0$. The boundary conditions (4.1.9) are written in the form

$$\mu = \mu_0 : \quad \sigma_2 = 0, \quad \tau_{12} = 0, \tag{4.2.1}$$

whereas the stresses on the part of the surface of any ellipsoid $s = s_0$ bounded by the surface $\mu = \mu_0$ satisfy condition (4.1.11) or (4.1.15)

$$\begin{aligned} q_z &= \frac{2\sqrt{1+s^2}}{1-\mu_0^2} \int\limits_1^{\mu_0} \left(-\sigma_1 \mu \sqrt{1+s^2} + \tau_{12} s \sqrt{1-\mu^2} \right) d\mu \\ &= \frac{2}{1-\mu_0^2} \int\limits_{\mu_0}^{1} (\sigma_1)_{s=0} \, \mu d\mu, \end{aligned} \tag{4.2.2}$$

as $b^2 = a^2 \left(1 - \mu_0^2\right)$. With growth of s the stresses decrease not slower than s^{-2}.

Using formulae (1.12.13) of Chapter 4 we have the following expressions for the stresses in terms of the axially symmetric harmonic functions b_0

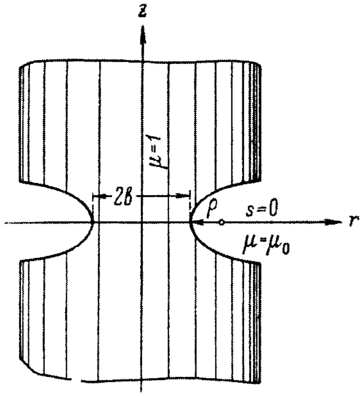

FIGURE 5.3.

and b_3

$$
\left.
\begin{aligned}
\frac{a\sigma_1}{2G} &= \frac{1-\mu^2}{s^2+\mu^2}\left[2(1-\nu)\frac{1+s^2}{1-\mu^2}\mu\frac{\partial b_3}{\partial s}+\right.\\
&\quad 2\nu s\frac{\partial b_3}{\partial \mu}-\frac{1+s^2}{1-\mu^2}\left(\frac{\partial^2 b_0}{\partial s^2}+\mu s\frac{\partial^2 b_3}{\partial s^2}\right)+\\
&\quad \left.\frac{s}{s^2+\mu^2}\left(\frac{\partial b_0}{\partial s}+\mu s\frac{\partial b_3}{\partial s}\right)-\frac{\mu}{s^2+\mu^2}\left(\frac{\partial b_0}{\partial \mu}+\mu s\frac{\partial b_3}{\partial \mu}\right)\right],\\
\frac{a\sigma_2}{2G} &= \frac{1-\mu^2}{s^2+\mu^2}\left[2(1-\nu)s\frac{\partial b_3}{\partial \mu}+2\nu\frac{1+s^2}{1-\mu^2}\mu\frac{\partial b_3}{\partial s}-\right.\\
&\quad \left(\frac{\partial^2 b_0}{\partial \mu^2}+\mu s\frac{\partial^2 b_3}{\partial \mu^2}\right)+\frac{\mu(1+s^2)}{(s^2+\mu^2)(1-\mu^2)}\left(\frac{\partial b_0}{\partial \mu}+\mu s\frac{\partial b_3}{\partial \mu}\right)-\\
&\quad \left.\frac{s(1+s^2)}{(s^2+\mu^2)(1-\mu^2)}\left(\frac{\partial b_0}{\partial s}+\mu s\frac{\partial b_3}{\partial s}\right)\right],\\
\frac{a\sigma_\varphi}{2G} &= \frac{1-\mu^2}{s^2+\mu^2}\left[2\nu\left(\frac{1+s^2}{1-\mu^2}\mu\frac{\partial b_3}{\partial s}+s\frac{\partial b_3}{\partial \mu}\right)-\right.\\
&\quad \left.\frac{s}{1-\mu^2}\left(\frac{\partial b_0}{\partial s}+\mu s\frac{\partial b_3}{\partial s}\right)+\frac{\mu}{1+s^2}\left(\frac{\partial b_0}{\partial \mu}+\mu s\frac{\partial b_3}{\partial \mu}\right)\right],\\
\frac{a\tau_{12}}{2G} &= \frac{\sqrt{(1-\mu^2)(1+s^2)}}{s^2+\mu^2}\left[(1-2\nu)\left(\mu\frac{\partial b_3}{\partial \mu}+s\frac{\partial b_0}{\partial \mu}\right)-\right.\\
&\quad \left(\frac{\partial^2 b_0}{\partial \mu \partial s}+\mu s\frac{\partial^2 b_3}{\partial \mu \partial s}\right)+\frac{\mu}{s^2+\mu^2}\left(\frac{\partial b_0}{\partial s}+\mu s\frac{\partial b_3}{\partial s}\right)+\\
&\quad \left.\frac{s}{s^2+\mu^2}\left(\frac{\partial b_0}{\partial \mu}+\mu s\frac{\partial b_3}{\partial \mu}\right)\right].
\end{aligned}
\right\}
\quad (4.2.3)
$$

It follows from these expressions that with growth of s function b_0 must increase not faster than $\ln s$ and b_3 must decrease not slower than s^{-1}.

The axially symmetric harmonic function increasing as $\ln s$ is

$$\ln\frac{r}{a}=\frac{1}{2}\ln\left(1+s^2\right)+\frac{1}{2}\ln\left(1-\mu^2\right). \tag{4.2.4}$$

It is not bounded on the hyperboloid's axis for $\mu=1$ but this singularity can be removed by adding $Q_0(\mu)$ to expression (4.2.4). For this reason, one of the solutions in b_0 is

$$\varphi(s,\mu)=\ln\frac{r}{a}+Q_0(\mu)=\frac{1}{2}\ln\left(1+s^2\right)+\ln(1+\mu). \tag{4.2.5}$$

Due to eqs. (F.3.7) and (F.1.8), the axisymmetric solutions of Laplace's equation decreasing as s^{-1} and s^{-2} are given by

$$q_0(s)P_0(\mu)=\arctan s,\quad q_1(s)P_1(\mu)=(s\arctan s-1)\mu. \tag{4.2.6}$$

The first function is taken as b_3 and the second one is included into b_0. In other words, we assume

$$\left.\begin{aligned}b_3&=\frac{a}{2G}C\arctan s,\\ b_0&=\frac{a}{2G}\left\{A\left[\ln\sqrt{1+s^2}+\ln(1+\mu)\right]+B(s\arctan s-1)\mu\right\}.\end{aligned}\right\} \tag{4.2.7}$$

Using formulae (4.2.3) we can set the boundary conditions (4.2.1) (which must be satisfied for any s) in the form

$$\left.\begin{aligned}-\left[\frac{A}{1+\mu_0}-C(1-2\nu)\right]+\frac{1}{\mu_0^2+s^2}\left(A-B-C\mu_0^2\right)&=0,\\ \left[\frac{A}{1+\mu_0}-C(1-2\nu)\right]+\frac{1}{\mu_0^2+s^2}\left(A-B-C\mu_0^2\right)&=0.\end{aligned}\right\} \tag{4.2.8}$$

It is no surprise that for the correct solution the second condition in eq. (4.2.1), i.e. $\tau_{12}=0$, is automatically fulfilled if the first one ($\sigma_2=0$) is met and vice versa. For such a solution, the stresses distributed over the surfaces of two arbitrary taken ellipsoids bounded by the piece of the surfaces $s_1\leq s\leq s_2$ of the hyperboloid $\mu=\mu_0$ are balanced. On this piece $\sigma_2=0$ and thus, the stresses τ_{12} are statically equivalent to zero on it. Because of the arbitrariness of s_1 and s_2 we have $\tau_{12}=0$.

Thus we obtain

$$\left.\begin{aligned}C(1-2\nu)&=\frac{A}{1+\mu_0},\qquad B=A-C\mu_0^2,\\ \sigma_1\big|_{s=0}&=\frac{1}{\mu}\left[B-2(1-\nu)C+\frac{1}{\mu^2}(B-A)\right]\end{aligned}\right\} \tag{4.2.9}$$

and from eq. (4.2.2)

$$C = -\frac{1}{2} q_z \frac{1+\mu_0}{1+2\nu\mu_0+\mu_0^2}. \tag{4.2.10}$$

This completely defines the state of stress. The coefficients of concentration of stresses σ_1 and σ_φ are defined by the ratio of these stresses at the deepest point of the neck to the nominal stress

$$\left.\begin{aligned} k_1 &= \frac{1}{q_z}(\sigma_1)_{\substack{s=0\\ \mu=\mu_0}} = \frac{1}{2\mu_0}\frac{1+\mu_0}{1+2\nu\mu_0+\mu_0^2}\left[2+\mu_0^2-(1-2\nu)\mu_0\right], \\ k_\varphi &= \frac{1}{q_z}(\sigma_\varphi)_{\substack{s=0\\ \mu=\mu_0}} = \frac{1-\mu_0^2}{1+2\nu\mu_0+\mu_0^2}\left(\frac{\nu}{\mu_0}+\frac{1}{2}\right). \end{aligned}\right\} \tag{4.2.11}$$

For example, for $\mu_0 = 0.2$ and $\nu = 0.3$ we have $k_1 = 5.08$ and $k_\varphi = 1.65$. In the latter formulae under μ_0 one understands the curvature of the meridian at the critical point

$$\frac{a}{\rho} = \frac{1-\mu_0^2}{\mu_0^2}.$$

The stresses decrease rapidly with increasing distance from the critical point. This allows one to estimate the concentration coefficient at the point of maximum curvature of the external neck for any form of the meridian with the help of expression (4.2.11) for the hyperboloid.

5.4.3 Torsion of the hyperboloid

This is the simplest problem among the considered problems for hyperboloids since according to Subsection 4.1.11 the solution reduces to searching for a single harmonic function $ve^{i\varphi}$ ensuring that v/r decreases not slower than s^{-3} as s increases. The solution is given by the function

$$\begin{aligned} v &= aCq_1^1(s)P_1^1(\mu) = aC\left(\frac{s}{1+s^2} - \arctan s\right)\sqrt{1+s^2}\sqrt{1-\mu^2} \\ &= Cr\left(\frac{s}{1+s^2} - \arctan s\right), \end{aligned} \tag{4.3.1}$$

and by eq. (1.11.3) of Chapter 4

$$\frac{1}{G}\tau_{1\varphi} = \frac{2C}{1+s^2}\sqrt{\frac{1-\mu^2}{s^2+\mu^2}}, \quad \tau_{2\varphi} = 0. \tag{4.3.2}$$

The boundary condition on the side surface, eq. (4.1.9), is fulfilled and the constant C, determined by eq. (4.1.7), is given by

$$C = \frac{3}{8}\frac{p_z}{G}\frac{\sqrt{1-\mu_0^2}\,(1+\mu_0)}{(2+\mu_0)(1-\mu_0)}. \tag{4.3.3}$$

The coefficient of concentration of stress $\tau_{1\varphi}$ is as follows

$$\left(\frac{\tau_{1\varphi}}{p_z}\right)_{\substack{s=0\\ \mu=\mu_0}} = \frac{3}{4}\frac{(1+\mu_0)^2}{\mu_0(2+\mu_0)}. \tag{4.3.4}$$

5.4.4 *Bending of the hyperboloid*

The problems of bending of the body of revolution is not axially symmetric and the harmonic functions B_0 and B_3 should be taken to be proportional to $\cos\varphi$. In the case of bending by force V_x and moment m_y the solution also contains the harmonic function B_1, that is

$$B_r = B_1\cos\varphi, \quad B_\varphi = -B_1\sin\varphi,$$

thus, taking

$$B_r = b_r\left(q^1, q^2\right)\cos\varphi, \quad B_\varphi = b_\varphi\left(q^1, q^2\right)\sin\varphi$$

we have in formulae (1.13.5) and (1.13.6) of Chapter 4

$$n = 1, \quad b_r = -b_\varphi = B_1, \tag{4.4.1}$$

with b_r and b_φ being axially symmetric harmonic functions (see also eq. (1.13.4) of Chapter 4) as well as $b_0 e^{i\varphi}$ and $b_3 e^{i\varphi}$.

We consider formulae (1.12.13), (1.12.14) and (1.13.5), (1.13.6) of Chapter 4 and take into account that r, z, H_2 increase in proportion to s and H_1 remains bounded with the growth of s whilst the stresses in the problem of bending by a force (a moment) decreases as s^{-2} $\left(s^{-3}\right)$. This imposes certain restrictions onto the order of growth of functions b_0, b_3 and $b_r = -b_\varphi$.

1. *Bending by force.* In this case the harmonic function $b_0 e^{i\omega t}$ is needed to be completed by a term which remains bounded at $s \to \infty$. By eqs. (F.3.12), (F.3.16) and (F.3.17), the terms

$$\tilde{p}_0^1(s)\tilde{P}_0^1(\mu) = \frac{s\mu}{\sqrt{(1+s^2)(1-\mu^2)}}, \quad \tilde{p}_0^1(s)Q_0^1(\mu) = \frac{s}{\sqrt{(1+s^2)(1-\mu^2)}} \tag{4.4.2}$$

possess the property that their difference is bounded on the hyperboloid axis, i.e. for $\mu = 1$

$$\varphi_1(s,\mu) = \frac{s}{\sqrt{1+s^2}}\sqrt{\frac{1-\mu}{1+\mu}}. \tag{4.4.3}$$

An additional term of the type (4.3.1) is added to b_0. Next, function b_3 must decrease as s^{-1}. Such a solution bounded on the hyperboloid axis is given by

$$\varphi_2(s,\mu) = -q_0^1(s)\left[Q_0^1(\mu) - \tilde{P}_0^1(\mu)\right] = \frac{1}{\sqrt{1+s^2}}\sqrt{\frac{1-\mu}{1+\mu}}. \tag{4.4.4}$$

Finally, b_r is an axially symmetric harmonic function decreasing as s^{-1}. This function is $q_0(s) = \arctan s$. Thus, the solution to the problem is constructed by means of the following functions

$$\left.\begin{aligned} b_0 &= A\frac{s}{\sqrt{1+s^2}}\sqrt{\frac{1-\mu}{1+\mu}} + B\left(\frac{s}{1+s^2} - \arctan s\right) r, \\ b_3 &= C\frac{1}{\sqrt{1+s^2}}\sqrt{\frac{1-\mu}{1+\mu}}, \quad b_r = -b_\varphi = D\arctan s. \end{aligned}\right\} \tag{4.4.5}$$

Three constants are sufficient to fulfill the three boundary conditions in eq. (4.1.9), whereas the fourth equation is given by condition (4.1.14).

2. *Bending by moment m_y.* The set of functions for the solution of the problem is given by the functions

$$\left.\begin{aligned} b_0 &= A\frac{1}{\sqrt{1+s^2}}\sqrt{\frac{1-\mu}{1+\mu}} + Bq_2^1(s)\,P_2^1(\mu), \\ b_3 &= C\left(\frac{s}{1+s^2} - \arctan s\right) r, \\ b_r &= -b_\varphi = DP_1(\mu)\,q_1(s) = D\mu\,(s\arctan s - 1), \end{aligned}\right\} \tag{4.4.6}$$

where due to eqs. (F.3.7), (F.3.11) and (F.2.16)

$$q_2^1(s) = 3\sqrt{1+s^2}\left[s\arctan s - 1 - \frac{1}{3(1+s^2)}\right], \quad P_2^1(\mu) = 3\mu\sqrt{1-\mu^2}.$$

Further calculations are omitted as they would require a great deal of space. By analogy one considers the problems of the state of stress of an elastic half-space having a cavity bounded by a surface of contracted ellipsoid of revolution when the state of stress is prescribed at infinity. A method of solving the more general problem in which the surface of the cavity is a triaxial ellipsoid is demonstrated in Section 5.5.5.

5.4.5 Rotating ellipsoid of revolution

It is assumed that the ellipsoid rotates about the axis of symmetry Oz. A particular solution corresponding to the mass centrifugal force is assumed to be taken in accordance with formulae (3.11.5) and (3.11.6). The solution of this axially symmetric problem is constructed with the help of biharmonic function of Love χ, see Subsection 4.1.10. We use the cylindric coordinates r, z since applying the degenerated elliptic coordinates would complicate the solution.

The state of stress $\hat{T}^0$ given by formulae (3.11.6) is superimposed by an axially symmetric state of stress $\hat{T}^*$ such that

$$\mathbf{n}\cdot\hat{T} = \mathbf{n}\cdot\hat{T}^0 + \mathbf{n}\cdot\hat{T}^* = 0$$

or in an expanded form

$$\sigma_r^* n_r + \tau_{rz}^* n_z = A r^2 n_r, \quad \tau_{rz}^* n_r + \sigma_r^* n_z = A_1 r^2 n_z, \tag{4.5.1}$$

$$A = \frac{\omega^2 \gamma (3 - 2\nu)}{8g(1-\nu)}, \quad A_1 = \frac{\omega^2 \gamma \nu}{2g(1-\nu)}. \tag{4.5.2}$$

Here n_r and n_z denote the projections of the unit vector $\mathbf{n}$ in the directions $\mathbf{e}_r$ and $\mathbf{k}$ respectively, i.e.

$$\mathbf{n} = \frac{1}{H_s}\left(\frac{\partial r}{\partial s}\mathbf{e}_r + \frac{\partial z}{\partial s}\mathbf{k}\right), \quad \frac{n_z}{n_r} = \frac{\partial z/\partial s}{\partial r/\partial s} = \frac{z}{r}\frac{s^2 \pm 1}{s^2} = \frac{z}{r}\frac{a^2}{c^2} = \frac{z}{r\alpha},$$

where $\alpha = c^2/a^2$ is the square of the ratio of the semi-axes of the ellipsoid which can be both oblong and oblate, see Section C.10. By eq. (1.10.6) of Chapter 4 we arrive at the boundary conditions

$$\left.\begin{aligned} \alpha r\frac{\partial}{\partial z}\left(-\nu\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2}\right) + z\frac{\partial}{\partial r}\left[-(1-\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial z^2}\right] &= A\alpha r^3, \\ \alpha r\frac{\partial}{\partial r}\left(-(1-\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2}\right) + z\frac{\partial}{\partial z}\left[-(2-\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial z^2}\right] &= A_1 r^2 z \end{aligned}\right\} \tag{4.5.3}$$

which need to be satisfied on the surface of the ellipsoid

$$z^2 = c^2 - \alpha r^2. \tag{4.5.4}$$

It is easy to understand that the biharmonic function χ must be an odd function of z. Then the right hand sides in eq. (4.5.3) contain (after cancelling out r and z) only terms which are even with respect to z and are eliminated by means of eq. (4.5.4).

Axisymmetric functions $R^n P_n(\mu)$ which are harmonic in the ellipsoid are represented by the following polynomials of r and z

$$RP_1(\mu) = z, \quad R^2 P_2(\mu) = z^2 - \frac{1}{2}r^2, \quad R^3 P_3(\mu) = z^3 - \frac{3}{2}zr^3,$$
$$R^4 P_4(\mu) = z^4 - 3z^2 r^2 + \frac{3}{8}r^4, \quad R^5 P_5(\mu) = z^5 - 5z^3 r^2 + \frac{15}{8}r^4 z,$$

the inessential numerical factors being dropped. In what follows these polynomials are denoted as $\varphi_1, \varphi_2, \ldots, \varphi_5$. The products of a harmonic function and z or $z^2 + r^2 = R^2$ are biharmonic functions. For this reason, along with φ_3 and φ_5 function χ contains also

$$\left(r^2 + z^2\right)\varphi_1, \quad z\varphi_2, \quad \left(r^2 + z^2\right)\varphi_3, \quad z\varphi_4, \ldots$$

It turns out to be sufficient to adopt

$$\chi = C_3 c^2 \varphi_3 + C_5 \varphi_5 + D_1 c^2 z r^2 + D_2\left(z^3 r^2 - \frac{3}{4}r^4 z\right), \tag{4.5.5}$$

where the third and fourth biharmonic terms are linear combinations of the above functions.

After having determined the constants we arrive at the following expressions for the components of tensor $\hat{T}^*$

$$\left.\begin{aligned}
\frac{1}{Q}\sigma_r^* &= \left\{(3-2\nu)\left[4\alpha(1+\nu)+2+\nu\right]+4\nu\left(\frac{11}{4}+\frac{5}{2}\nu\right)\right\}\left(c^2-z^2\right)+\\
&\quad \left\{(3-2\nu)\left[\alpha\nu+\frac{1}{4}(3+\nu)\right]+4\nu\left[\frac{1}{2}(1-\alpha)\nu-\frac{3}{4}\alpha\right]\right\}r^2+\\
&\quad a^2\left\{2(3-2\nu)-4\nu\left[\frac{\nu}{2}+2\alpha(1+\nu)\right]\right\},\\
\frac{1}{Q}\sigma_\varphi^* &= \left\{(3-2\nu)\left[4\alpha(1+\nu)+2+\nu\right]+4\nu\left(\frac{11}{4}+\frac{5}{2}\nu\right)\right\}\left(c^2-z^2\right)+\\
&\quad \left\{(3-2\nu)\left[\left(3\alpha+\frac{3}{4}\right)\nu+\frac{1}{4}\right]+4\nu\left(\frac{\alpha\nu}{2}+2\nu-\frac{\alpha}{4}\right)\right\}r^2+\\
&\quad a^2\left\{2(3-2\nu)-4\nu\left[\frac{\nu}{2}+2\alpha(1+\nu)\right]\right\},\\
\frac{1}{Q}\sigma_z^* &= \left\{-2(3-2\nu)+4\nu\left[\frac{\nu}{2}+2\alpha(1+\nu)\right]\right\}\left(c^2-z^2\right)+\\
&\quad \left\{(3-2\nu)4\alpha+4\nu\left[(2+\nu)\alpha+\frac{11+\nu}{4}\right]\right\}r^2.\\
\frac{1}{Q}\tau_{rz}^* &= \left\{-2(3-2\nu)+4\nu\left[\frac{\nu}{2}+2\alpha(1+\nu)\right]\right\}rz,
\end{aligned}\right\} \tag{4.5.6}$$

where

$$Q=\frac{\omega^2\gamma}{8g(1-\nu)\Delta},\quad \Delta=4\alpha^2(1+\nu)+2\alpha(1+\nu)+\frac{11+\nu}{4}. \tag{4.5.7}$$

The projections of the displacement vector $\mathbf{u}$ in the directions $\mathbf{e}_r$ and $\mathbf{k}$ are given by the formulae

$$\begin{aligned}
\frac{2G}{Q}u_r &= a^2 r\frac{1-\nu}{1+\nu}\left[2(3-2\nu)-4\nu\left(\frac{\nu}{2}+2\alpha+2\alpha\nu\right)\right]+\\
&\quad r\left(c^2-z^2\right)\left\{(3-2\nu)\left[2-\nu+4\alpha(1-\nu)\right]+4\nu\left(\frac{11}{4}-2\alpha\nu-3\nu\right)\right\}+\\
&\quad r^3\left\{(3-2\nu)\left[\frac{1}{4}(1-\nu)-\alpha\nu\right]-4\nu\left(\frac{3}{4}\nu+\frac{\alpha}{4}+\frac{1}{2}\nu\alpha\right)\right\}+\frac{2G}{Q}u_r^0,
\end{aligned} \tag{4.5.8}$$

$$\frac{2G}{Q}w = a^2 z\frac{\nu}{1+\nu}\left[-4\left(3-2\nu\right)+4\nu\left(\nu+4\alpha+4\alpha\nu\right)\right]+$$
$$z\left(c^2-\frac{z^2}{3}\right)\left\{-2\left(3-2\nu\right)\left[1+\left(4\alpha+1\right)\nu+4\nu\left(2\alpha-5\nu\right)\right]\right\}+$$
$$zr^2\left\{-\left(3-2\nu\right)\left[\nu-4\alpha\left(1-\nu\right)\right]+4\nu\left(\frac{11-10\nu}{4}+2\alpha\right)\right\}. \quad (4.5.9)$$

In particular, at the poles and the equator of the ellipsoid of revolution

$$\left(2Gw\right)_{\substack{z=c\\r=0}} = \frac{\omega^2\gamma a^2 c}{8g\left(1-\nu\right)\Delta}\left\{\frac{\nu}{1+\nu}\left[-4\left(3-2\nu\right)+4\nu\left(\nu+4\alpha+4\alpha\nu\right)\right]+\right.$$
$$\left.\frac{2}{3}\alpha\left[-2\left(3-2\nu\right)\left(1+4\alpha\nu+\nu\right)+4\nu\left(2\alpha-5\nu\right)\right]\right\}, \quad (4.5.10)$$

$$\left(2Gu_r\right)_{\substack{z=0\\r=a}} = \frac{\omega^2\gamma a^3}{8g\left(1-\nu\right)\Delta}\left\{\frac{1-\nu}{1+\nu}\left[2\left(3-2\nu\right)-4\nu\left(\frac{\nu}{2}+2\alpha+2\alpha\nu\right)\right]+\right.$$
$$\left[\left(3-2\nu\right)\left(\frac{1}{4}-\frac{\nu}{4}-\alpha\nu\right)-4\nu\left(\frac{3}{4}\nu+\frac{\alpha}{4}+\frac{1}{2}\nu\alpha\right)\right]-$$
$$\left.\left(1-2\nu\right)\left[4\alpha^2\left(1+\nu\right)+2\alpha\left(1+\nu\right)+\frac{11+\nu}{4}\right]\right\}. \quad (4.5.11)$$

When $\alpha = 1$ we return to formulae (3.13.10) for the sphere. The case of the rotating thin oblate ellipsoid considered by C. Chree (1895) can be obtained from the above equations.

5.5 Ellipsoid

5.5.1 *Elastostatic Robin's problem for the three-axial ellipsoid*

The statement of the problem is given in Subsection 4.4.7 while Subsection 5.3.3 suggests the solution in the simplest case of the displacement of a rigid sphere in the unbounded elastic medium. This problem is considered here under the assumption that the rigid body is the three-axial ellipsoid

$$\frac{x_1^2}{a^2\rho_0^2}+\frac{x_2^2}{a^2\left(\rho_0^2-e^2\right)}+\frac{x_3^2}{a^2\left(\rho_0^2-1\right)}-1=0 \quad (5.1.1)$$

with the semi-axes $a\rho_0, a\sqrt{\rho_0^2-e^2}, a\sqrt{\rho_0^2-1}$. For the solution we use the potentials of the simple layer on the ellipsoid listed in (F.8.3) which, for brevity, are denoted here as $\psi_i\left(x,y,z\right)$. On the surface of the ellipsoid and within it these potentials are as follows

$$\psi_0 = 1; \quad \psi_s = x_s \ \left(s=1,2,3\right), \ \psi_4 = x_2x_3, \ \psi_5 = x_3x_1, \ \psi_6 = x_1x_2, \quad (5.1.2)$$

whereas outside the ellipsoid they are

$$\psi_0 = \frac{\omega_0(\rho)}{\omega_0(\rho_0)}, \quad \psi_s = \frac{\omega_0(\rho)}{\omega_0(\rho_0)} x_s \quad (s = 1, 2, 3),$$

$$\psi_4 = \frac{\omega_4(\rho)}{\omega_4(\rho_0)} x_2 x_3, \quad \psi_5 = \frac{\omega_5(\rho)}{\omega_5(\rho_0)} x_3 x_1, \quad \psi_6 = \frac{\omega_6(\rho)}{\omega_6(\rho_0)} x_1 x_2. \tag{5.1.3}$$

Functions $\omega_s(\rho)$ are given by the elliptic integrals, eqs. (F.7.5)-(F.7.9), in which $\omega_1^{(k)}$ are replaced by ω_s for $s = 1, 2, 3$ and $\omega_2^{(k)}$ are replaced by ω_s for $s = 4, 5, 6$.

In what follows we consider first the case of the translatory displacement

$$\rho = \rho_0 : \quad \mathbf{u} = \mathbf{u}^0 \tag{5.1.5}$$

of the ellipsoid and then the case of the displacement due to the rotation

$$\rho = \rho_0 : \quad \mathbf{u} = \boldsymbol{\theta} \times \mathbf{R}_0, \tag{5.1.6}$$

where $\mathbf{R}_0$ denotes the position vector of the point on the surface (5.1.1).

5.5.2 *Translatory displacement*

In order to construct the harmonic vector $\mathbf{B}$ and the harmonic scalar B_0 in the Papkovich-Neuber solution (1.4.10) of Chapter 4 written here as follows

$$u_s = (3 - 4\nu) B_s - \sum_{k=1}^{3} x_k \frac{\partial B_k}{\partial x_s} - \frac{\partial B_0}{\partial x_s} \quad (s = 1, 2, 3) \tag{5.2.1}$$

we use potentials $\psi_0, \psi_1, \psi_2, \psi_3$. We take

$$B_s = C_s \psi_0, \quad B_0 = M_1 \psi_1 + M_2 \psi_2 + M_3 \psi_3 \quad (s = 1, 2, 3) \tag{5.2.2}$$

and the six constants C_s, M_s are sufficient to satisfy the boundary conditions (5.1.5). Indeed, an extended form of equalities (5.2.1) is

$$\begin{aligned} u_s = (3 - 4\nu) C_s \frac{\omega_0(\rho)}{\omega_0(\rho_0)} - M_s \frac{\omega_s(\rho)}{\omega_s(\rho_0)} + \\ \frac{1}{\Delta(\rho)} \frac{\partial \rho}{\partial x_s} \left\{ x_1 \left[\frac{C_1}{\omega_0(\rho_0)} + \frac{M_1}{\omega_1(\rho_0)\rho^2} \right] + x_2 \left[\frac{C_2}{\omega_0(\rho_0)} + \frac{M_2}{\omega_2(\rho_0)(\rho^2 - e^2)} \right] \right. \\ \left. + x_3 \left[\frac{C_3}{\omega_0(\rho_0)} + \frac{M_3}{\omega_3(\rho_0)(\rho^2 - 1)} \right] \right\}, \end{aligned} \tag{5.2.3}$$

and it remains only to take

$$\left. \begin{aligned} & u_s^0 = (3 - 4\nu) C_s - M_s \quad (s = 1, 2, 3), \\ & \frac{C_1}{\omega_0(\rho_0)} + \frac{M_1}{\omega_1(\rho_0)\rho_0^2} = 0, \quad \frac{C_2}{\omega_0(\rho_0)} + \frac{M_2}{\omega_2(\rho_0)(\rho_0^2 - e^2)} = 0, \\ & \frac{C_3}{\omega_0(\rho_0)} + \frac{M_3}{\omega_3(\rho_0)(\rho_0^2 - 1)} = 0. \end{aligned} \right\} \tag{5.2.4}$$

This allows us to find the constants C_s, M_s. Using relationships (C.11.26) we arrive at the following expressions for the displacement

$$\left.\begin{aligned} u_1 &= \frac{\sigma_1(\rho)}{\sigma_1(\rho_0)}u_1^0 + \frac{\rho^2-\rho_0^2}{\rho^2}x_1\Phi, \\ u_2 &= \frac{\sigma_2(\rho)}{\sigma_2(\rho_0)}u_2^0 + \frac{\rho^2-\rho_0^2}{\rho^2-e^2}x_2\Phi, \\ u_3 &= \frac{\sigma_3(\rho)}{\sigma_3(\rho_0)}u_3^0 + \frac{\rho^2-\rho_0^2}{\rho^2-1}x_3\Phi, \end{aligned}\right\} \tag{5.2.5}$$

where we denote

$$\left.\begin{aligned} \sigma_1(\rho) &= (3-4\nu)\,\omega_0(\rho) + \rho_0^2\omega_1(\rho), \\ \sigma_2(\rho) &= (3-4\nu)\,\omega_0(\rho) + \left(\rho_0^2-e^2\right)\omega_2(\rho), \\ \sigma_3(\rho) &= (3-4\nu)\,\omega_0(\rho) + \left(\rho_0^2-1\right)\omega_3(\rho), \end{aligned}\right\} \tag{5.2.6}$$

$$\Phi = \frac{1}{\rho\Delta(\rho)D^2}\left[\frac{u_1^0}{\sigma_1(\rho_0)}\frac{x_1}{\rho^2} + \frac{u_2^0}{\sigma_2(\rho_0)}\frac{x_2}{\rho^2-e^2} + \frac{u_3^0}{\sigma_3(\rho_0)}\frac{x_3}{\rho^2-1}\right] \tag{5.2.7}$$

and

$$\left.\begin{aligned} D^2 &= \frac{x_1^2}{\rho^4} + \frac{x_2^2}{\left(\rho^2-e^2\right)^2} + \frac{x_3^2}{\left(\rho^2-1\right)^2}, \\ \rho\Delta(\rho)\,D &= \Delta(\rho)\,H_\rho = a\sqrt{\left(\rho^2-\mu^2\right)\left(\rho^2-\nu^2\right)}. \end{aligned}\right\} \tag{5.2.8}$$

5.5.3 Distribution of stresses over the surface of the ellipsoid

Given the displacement vector one can find the stress tensor at any point of the medium. Its expression is very cumbersome, that is why we restrict our consideration to determining the stress vector t_n on the surface of the ellipsoid (at $\rho = \rho_0$). Taking into account that

$$\sigma_s'(\rho_0) = -\frac{4(1-\nu)}{\Delta(\rho_0)} \quad (s = 1,2,3), \tag{5.3.1}$$

we have for $\rho = \rho_0$

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x_s} &= \left[-\frac{4(1-\nu)}{\Delta(\rho_0)}\frac{u_1^0}{\sigma_1(\rho_0)} + \frac{2\rho_0}{\rho_0^2}x_1\Phi^0\right]\frac{\partial\rho}{\partial x_s} = \Psi_1\frac{\partial\rho}{\partial x_s}, \\ \frac{\partial u_2}{\partial x_s} &= \left[-\frac{4(1-\nu)}{\Delta(\rho_0)}\frac{u_2^0}{\sigma_2(\rho_0)} + \frac{2\rho_0}{\rho_0^2-e^2}x_2\Phi^0\right]\frac{\partial\rho}{\partial x_s} = \Psi_2\frac{\partial\rho}{\partial x_s}, \\ \frac{\partial u_3}{\partial x_s} &= \left[-\frac{4(1-\nu)}{\Delta(\rho_0)}\frac{u_3^0}{\sigma_3(\rho_0)} + \frac{2\rho_0}{\rho_0^2-1}x_3\Phi^0\right]\frac{\partial\rho}{\partial x_s} = \Psi_3\frac{\partial\rho}{\partial x_s}. \end{aligned}\right\} \tag{5.3.2}$$

Noticing that on the surface of the ellipsoid $n_s = \frac{\partial \rho}{\partial x_s}\rho_0 D_0$ where n_s denotes the cosine of the angle between the unit vector of the normal to the surface of the ellipsoid $\rho = \rho_0$ and axis x_s we have

$$\frac{t_{n1}}{2G} = \frac{1}{2G}\left(\sigma_1 n_1 + \tau_{12} n_2 + \tau_{13} n_3\right) = \frac{\nu}{1-2\nu}\rho_0 D_0 \frac{\partial \rho}{\partial x_1}\sum_{s=1}^{3}\frac{\partial \rho}{\partial x_s} +$$
$$\Psi_1 \frac{\partial \rho}{\partial x_1} n_1 + \frac{1}{2}\left(\Psi_1 \frac{\partial \rho}{\partial x_2} + \Psi_2 \frac{\partial \rho}{\partial x_1}\right) n_2 + \frac{1}{2}\left(\Psi_1 \frac{\partial \rho}{\partial x_3} + \Psi_3 \frac{\partial \rho}{\partial x_2}\right) n_3$$
$$= \frac{1}{2(1-2\nu)}\frac{\partial \rho}{\partial x_1}\rho_0 D_0 \sum_{s=1}^{3}\Psi_s \frac{\partial \rho}{\partial x_s} + \frac{1}{2\rho_0 D_0}\Psi_1. \tag{5.3.3}$$

Referring once again to formulae (C.11.26) we obtain

$$\sum_{s=1}^{3}\Psi_s \frac{\partial \rho}{\partial x_s} = -\frac{4(1-\nu)}{\rho_0 \Delta(\rho_0) D_0^2}\left[\frac{u_1}{\sigma_1(\rho_0)}\frac{x_1}{\rho_0^2} + \frac{u_2^0}{\sigma_2(\rho_0)}\frac{x_2}{\rho_0^2 - e^2} + \right.$$
$$\left.\frac{u_3^0}{\sigma_3(\rho_0)}\frac{x_3}{\rho_0^2 - 1}\right] + \frac{2\Phi^0}{D_0^2}\left(\frac{x_1^2}{\rho_0^4} + \frac{x_2^2}{\left(\rho_0^2 - e^2\right)^2} + \frac{x_3^2}{\left(\rho_0^2 - 1\right)^2}\right),$$

so that by eqs. (5.2.7) and (5.2.8)

$$\sum_{s=1}^{3}\Psi_s \frac{\partial \rho}{\partial x_s} = -2(1-2\nu)\Phi^0. \tag{5.3.4}$$

Insertion into eq. (5.3.3) yields

$$\left.\begin{aligned}
\frac{1}{2G}t_{n1} &= \frac{1}{\rho_0 D_0}\left(\frac{1}{2}\Psi_1 - \frac{\rho_0 x_1}{\rho_0^2}\Phi^0\right) = -\frac{2(1-\nu)}{\rho_0 D_0 \Delta(\rho_0)}\frac{u_1^0}{\sigma_1(\rho_0)},\\
\frac{1}{2G}t_{n2} &= \frac{1}{\rho_0 D_0}\left(\frac{1}{2}\Psi_2 - \frac{\rho_0 x_2}{\rho_0^2 - e^2}\Phi^0\right) = -\frac{2(1-\nu)}{\rho_0 D_0 \Delta(\rho_0)}\frac{u_2^0}{\sigma_2(\rho_0)},\\
\frac{1}{2G}t_{n3} &= \frac{1}{\rho_0 D_0}\left(\frac{1}{2}\Psi_3 - \frac{\rho_0 x_3}{\rho_0^2 - 1}\Phi^0\right) = -\frac{2(1-\nu)}{\rho_0 D_0 \Delta(\rho_0)}\frac{u_3^0}{\sigma_3(\rho_0)}
\end{aligned}\right\} \tag{5.3.5}$$

or, due to eqs. (C.11.21) and (C.11.22),

$$t_{ns} = -\frac{4G(1-\nu)}{a\sqrt{\left(\rho_0^2 - \mu^2\right)\left(\rho_0^2 - \nu^2\right)}}\frac{u_s^0}{\sigma_s(\rho_0)} \quad (s = 1, 2, 3). \tag{5.3.6}$$

This unexpectedly simple result derived by direct calculation can be obtained immediately if we use the Papkovich-Neuber vector, eq. (4.3.15) of Chapter 4, in which the density $\mathbf{a}(M_0)$ is the sought stress vector $-\mathbf{t}_n$ on the surface of the cavity in the elastic medium which follows from eq.

(4.7.1) of Chapter 4. In our case, the projections of vector $\mathbf{B}$ at $\rho = \rho_0$, by eqs. (5.2.2) and (5.2.5), are given by

$$B_s = \frac{u_s^0}{\sigma_s(\rho_0)} \omega_0(\rho) \tag{5.3.7}$$

and they have the same constant values on the ellipsoid surface and in the ellipsoid. Referring to eq. (4.3.15) of Chapter 4 we obtain

$$-\frac{1}{16\pi G(1-\nu)} \iint\limits_O \frac{t_{ns}}{R} do = \begin{cases} B_s^{(e)} = \dfrac{u_s^0}{\sigma_s(\rho_0)} \omega_0(\rho), & \rho > \rho_0, \\ B_s^{(i)} = \dfrac{u_s^0}{\sigma_s(\rho_0)} \omega_0(\rho_0), & \rho < \rho_0. \end{cases} \tag{5.3.8}$$

By a theorem on the jump of the normal derivative of the simple layer we have

$$\left(\frac{\partial B_s^{(i)}}{\partial n} - \frac{\partial B_s^{(e)}}{\partial n} \right)_{\rho=\rho_0} = -4\pi \frac{t_{ns}}{16\pi G(1-\nu)}$$

or

$$t_{ns} = 4G(1-\nu) \left(\frac{\partial B_s^{(e)}}{\partial n} \right)_{\rho=\rho_0} = 4G(1-\nu) \frac{u_s^0}{\sigma_s(\rho_0)} \left(\frac{\partial \omega_0(\rho)}{H_\rho \partial \rho} \right)_{\rho=\rho_0}$$
$$= -\frac{4G(1-\nu)}{\sigma_s(\rho_0)} \frac{u_s^0}{H_\rho^0 \Delta(\rho_0)},$$

which is required.

Equation

$$Q_s + \iint\limits_O t_{ns} do = 0$$

determines the projections of force $\mathbf{Q}$ which should be applied to the ellipsoid to ensure displacement $\mathbf{u}^0$. By virtue of eqs. (C.11.22) and (C.11.23) we have

$$do = H_\mu H_\nu d\mu d\nu, \qquad \frac{do}{\Delta(\rho_0) H_\rho^0} = \frac{a(\mu^2 - \nu^2)}{\Delta_1(\mu)\Delta(\nu)} d\mu d\nu,$$

so that by eq. (5.3.6)

$$Q_s = 32G(1-\nu) \frac{a u_s^0}{\sigma_s(\rho_0)} \int_e^1 \frac{d\mu}{\Delta_1(\mu)} \int_0^e \frac{\mu^2 - \nu^2}{\Delta(\nu)} d\nu,$$

where the integration is carried out within the octant of the ellipsoid. The estimation yields

$$\int_e^1 \frac{\mu^2 d\mu}{\Delta_1(\mu)} \int_0^e \frac{d\nu}{\Delta(\nu)} = E(e')\,K(e),$$

$$-\int_0^e \frac{\nu^2}{\Delta(\nu)} d\nu \int_e^1 \frac{d\mu}{\Delta_1(\mu)} = K(e')\left[\int_0^e \sqrt{\frac{1-\nu^2}{e^2-\nu^2}} d\nu - K(e)\right]$$
$$= K(e')\,E(e) - K(e')\,K(e) \quad \left(e' = \sqrt{1-e^2}\right),$$

so that due to the Legendre relationship

$$\int_e^1 \frac{d\mu}{\Delta_1(\mu)} \int_0^e \frac{\mu^2-\nu^2}{\Delta(\nu)} d\nu = E(e')\,K(e) + K(e')\,E(e) - K(e')\,K(e) = \frac{\pi}{2}.$$

Finally we have

$$Q_s = 16\pi G(1-\nu)\frac{a u_s^0}{\sigma_s(\rho_0)} \quad (s = 1,2,3). \tag{5.3.9}$$

5.5.4 Rotational displacement

Let us prescribe the Papkovich-Neuber harmonic functions by the equalities

$$\left.\begin{aligned} B_1 &= D_1\theta_2 x_3\omega_3(\rho) - D_1'\theta_3 x_2\omega_2(\rho), \\ B_2 &= D_2\theta_3 x_1\omega_1(\rho) - D_2'\theta_1 x_3\omega_3(\rho), \\ B_3 &= D_3\theta_1 x_2\omega_2(\rho) - D_3'\theta_3 x_1\omega_1(\rho), \end{aligned}\right\} \tag{5.4.1}$$

$$B_0 = N_1\theta_1 x_2 x_3\omega_4(\rho) + N_2\theta_2 x_3 x_1\omega_5(\rho) + N_3\theta_3 x_1 x_2\omega_6(\rho), \tag{5.4.2}$$

so that

$$\begin{aligned} x_1B_1 + x_2B_2 + x_3B_3 + B_0 = {} & \theta_1 x_2 x_3\left[D_3\omega_2(\rho) - D_2'\omega_3(\rho) + N_1\omega_4(\rho)\right] + \\ & \theta_2 x_3 x_1\left[D_1\omega_3(\rho) - D_3'\omega_1(\rho) + N_2\omega_5(\rho)\right] + \\ & \theta_3 x_1 x_2\left[D_2\omega_1(\rho) - D_1'\omega_2(\rho) + N_3\omega_6(\rho)\right]. \end{aligned} \tag{5.4.3}$$

Component u_1 of the displacement vector is presented by the formula

$$
\begin{aligned}
u_1 = {} & \theta_2 x_3 \left[(3-4\nu) D_1 \omega_3(\rho) + D_3' \omega_1(\rho) - N_2 \omega_5(\rho)\right] - \\
& \theta_3 x_2 \left[(3-4\nu) D_1' \omega_2(\rho) + D_2 \omega_1(\rho) + N_3 \omega_6(\rho)\right] + \\
& \frac{1}{\Delta(\rho)} \frac{\partial \rho}{\partial x_1} \left\{ \theta_1 x_2 x_3 \left[\frac{D_3}{\rho^2 - e^2} - \frac{D_2'}{\rho^2 - 1} + \frac{N_1}{(\rho^2-1)(\rho^2-e^2)} \right] + \right. \\
& \theta_2 x_3 x_1 \left[\frac{D_1'}{\rho^2 - 1} - \frac{D_3'}{\rho^2} + \frac{N_2}{\rho^2(\rho^2-1)} \right] + \\
& \left. \theta_3 x_1 x_2 \left[\frac{D_2'}{\rho^2} - \frac{D_1}{\rho^2 - e^2} + \frac{N_2}{(\rho^2-e^2)\rho^2} \right] \right\}. \qquad (5.4.4)
\end{aligned}
$$

The expressions for u_2 and u_3 are written down by analogy.

The boundary conditions (5.1.6)

$$
\rho = \rho_0: \quad u_1 = \theta_2 x_3 - \theta_3 x_2, \; u_2 = \theta_3 x_1 - \theta_1 x_3, \; u_3 = \theta_1 x_2 - \theta_2 x_1
$$

lead to a system of nine equations for the same number of unknown constants D_s, D_s', N_s. It is split into three independent systems corresponding to the rotations about each of the ellipsoid axes. For instance, the system generated by rotation θ_1 about axis x_1 is as follows

$$
\left.
\begin{aligned}
& (3-4\nu) D_3 \omega_2(\rho_0) + D_2' \omega_3(\rho_0) - N_1 \omega_4(\rho_0) = 1, \\
& (3-4\nu) D_2' \omega_3(\rho_0) + D_3 \omega_2(\rho_0) - N_1 \omega_4(\rho_0) = 1, \\
& \frac{N_1}{(\rho^2-1)(\rho^2-e^2)} + \frac{D_3}{\rho^2-e^2} - \frac{D_2'}{\rho^2-1} = 0.
\end{aligned}
\right\} \qquad (5.4.5)
$$

Omitting the calculation which utilises relationships (F.7.7)-(F.7.9) we present the final expressions for the displacement in the elastic medium

$$
\left.
\begin{aligned}
u_1 &= \theta_2 x_3 \frac{\delta_2(\rho)}{\delta_2(\rho_0)} - \theta_3 x_2 \frac{\delta_3^*(\rho)}{\delta_3(\rho_0)} + \frac{\rho_0^2 - \rho^2}{\rho^2} x_1 \Omega, \\
u_2 &= \theta_3 x_1 \frac{\delta_3(\rho)}{\delta_3(\rho_0)} - \theta_1 x_3 \frac{\delta_1^*(\rho)}{\delta_1(\rho_0)} + \frac{\rho_0^2 - \rho^2}{\rho^2 - e^2} x_2 \Omega, \\
u_3 &= \theta_1 x_2 \frac{\delta_1(\rho)}{\delta_1(\rho_0)} - \theta_2 x_1 \frac{\delta_2^*(\rho)}{\delta_2(\rho_0)} + \frac{\rho_0^2 - e^2}{\rho^2 - 1} x_3 \Omega.
\end{aligned}
\right\} \qquad (5.4.6)
$$

Here

$$
\left.
\begin{aligned}
\delta_1(\rho) &= 2(1-2\nu)\,\omega_2(\rho)\,\omega_3(\rho_0) + \\
& \qquad \omega_4(\rho_0)\left[\left(\rho_0^2 - e^2\right)\omega_2(\rho) + \left(\rho_0^2 - 1\right)\omega_3(\rho)\right], \\
\delta_2(\rho) &= 2(1-2\nu)\,\omega_3(\rho)\,\omega_1(\rho_0) + \omega_5(\rho_0)\left[\left(\rho_0^2 - 1\right)\omega_3(\rho) + \rho_0^2 \omega_1(\rho)\right], \\
\delta_3(\rho) &= 2(1-2\nu)\,\omega_1(\rho)\,\omega_2(\rho_0) + \omega_6(\rho_0)\left[\rho_0^2 \omega_1(\rho) + \left(\rho_0^2 - e^2\right)\omega_2(\rho)\right],
\end{aligned}
\right\} \qquad (5.4.7)
$$

and functions $\delta_s^*(\rho)$ differ from $\delta_s(\rho)$ in replacing the indices in the first term, for example

$$\delta_1^*(\rho) = 2(1-2\nu)\,\omega_2(\rho_0)\,\omega_3(\rho) + \omega_4(\rho_0)\left[\left(\rho_0^2 - e^2\right)\omega_2(\rho) + \left(\rho_0^2 - 1\right)\omega_3(\rho)\right]. \quad (5.4.8)$$

Clearly, $\delta_s^*(\rho_0) = \delta_s(\rho_0)$. In solution (5.4.6) function Ω is determined as follows

$$\Omega = \frac{1}{\rho\Delta(\rho)D^2}\left[\frac{\theta_1}{\delta_1(\rho_0)}\frac{\omega_2(\rho_0)-\omega_3(\rho_0)}{(\rho_0^2-e^2)(\rho_0^2-1)}x_2x_3 + \frac{\theta_2}{\delta_2(\rho_0)}\frac{\omega_3(\rho_0)-\omega_1(\rho_0)}{(\rho_0^2-1)\rho^2}x_3x_1 + \frac{\theta_3}{\delta_3(\rho_0)}\frac{\omega_1(\rho_0)-\omega_2(\rho_0)}{\rho^2(\rho^2-e^2)}x_1x_2\right]. \quad (5.4.9)$$

5.5.5 *Distribution of stresses over the surface of the ellipsoid*

The calculation is carried out under the assumption that only $\theta_1 \neq 0$ and the symmetry reasoning suggests the form of the formulae in the general case.

By eqs. (5.4.1) and (5.4.5) we have

$$B_1 = 0, \quad B_2 = -\theta_1 D_2' x_3 \omega_3(\rho), \quad B_3 = \theta_1 D_3 x_2 \omega_2(\rho), \quad (5.5.1)$$

where

$$\left.\begin{aligned} D_2' &= \frac{1}{2(1-\nu)\delta_1(\rho_0)}\left[\left(\rho_0^2-1\right)\omega_4(\rho_0) + (1-2\nu)\,\omega_2(\rho_0)\right], \\ D_3 &= \frac{1}{2(1-\nu)\delta_1(\rho_0)}\left[\left(\rho_0^2-e^2\right)\omega_4(\rho_0) + (1-2\nu)\,\omega_3(\rho_0)\right]. \end{aligned}\right\} \quad (5.5.2)$$

Similar to eq. (5.3.8) we have

$$-\frac{1}{16\pi G(1-\nu)}\iint\limits_O \frac{t_{n2}}{R}\,do = \begin{cases} -\theta_1 D_2' x_3\omega_3(\rho) = B_2^{(e)}, & \rho > \rho_0, \\ -\theta_1 D_2' x_3\omega_3(\rho_0) = B_2^{(i)}, & \rho < \rho_0, \end{cases}$$

as well as

$$-\frac{1}{16\pi G(1-\nu)}\iint\limits_O \frac{t_{n3}}{R}\,do = \begin{cases} \theta_1 D_3 x_2\omega_2(\rho) = B_3^{(e)}, & \rho > \rho_0, \\ \theta_1 D_3 x_2\omega_2(\rho_0) = B_3^{(i)}, & \rho < \rho_0, \end{cases}$$

and the application of the theorem on the normal derivative of the simple layer potential results in the following expressions for the components of the stress vector on the surface of the ellipsoid

$$\left.\begin{aligned} t_{n2} &= \frac{2G\theta_1 x_3}{\delta_1(\rho_0)H_\rho^0\Delta(\rho_0)}\left[\omega_4(\rho_0) + \frac{1-2\nu}{\rho_0^2-1}\omega_2(\rho_0)\right], \\ t_{n3} &= \frac{2G\theta_1 x_2}{\delta_1(\rho_0)H_\rho^0\Delta(\rho_0)}\left[\omega_4(\rho_0) + \frac{1-2\nu}{\rho_0^2-e^2}\omega_3(\rho_0)\right]. \end{aligned}\right\} \quad (5.5.3)$$

The moments m_s^O about axes x_s which should be applied to the ellipsoid in order to provide the rotation $\boldsymbol{\theta}$ are given by the equalities

$$\left.\begin{aligned}
m_1^O &= -\iint\limits_O (x_2 t_{n3} - x_3 t_{n2})\, do = \frac{2G\theta_1}{\delta_1(\rho_0)} \{(1-2\nu)[\omega_2(\rho_0) J_3 + \\
&\qquad \omega_3(\rho_0) J_2] + \omega_4(\rho_0)\left[\left(\rho_0^2 - e^2\right) J_2 + \left(\rho_0^2 - 1\right) J_3\right]\}, \\
m_2^O &= -\iint\limits_O (x_3 t_{n1} - x_1 t_{n3})\, do = \frac{2G\theta_2}{\delta_2(\rho_0)} \{(1-2\nu)[\omega_3(\rho_0) J_1 + \\
&\qquad \omega_1(\rho_0) J_3] + \omega_5(\rho_0)\left[\left(\rho_0^2 - 1\right) J_3 + \rho_0^2 J_1\right]\}, \\
m_3^O &= -\iint\limits_O (x_1 t_{n2} - x_2 t_{n1})\, do = \frac{2G\theta_3}{\delta_3(\rho_0)} \{(1-2\nu)[\omega_1(\rho_0) J_2 + \\
&\qquad \omega_2(\rho_0) J_1] + \omega_6(\rho_0)\left[\rho_0^2 J_1 + \left(\rho_0^2 - e^2\right) J_2\right]\},
\end{aligned}\right\} \tag{5.5.4}$$

where

$$J_1 = \frac{1}{\rho_0^2 \Delta(\rho_0)} \iint\limits_O x_1^2 \frac{do}{H_\rho^0}, \quad J_2 = \frac{1}{(\rho_0^2 - e^2)\Delta(\rho_0)} \iint\limits_O x_2^2 \frac{do}{H_\rho^0},$$
$$J_3 = \frac{1}{(\rho_0^2 - 1)\Delta(\rho_0)} \iint\limits_O x_3^2 \frac{do}{H_\rho^0}.$$

Taking into account the equalities

$$n_1 = \frac{x_1}{\rho_0 H_\rho^0}, \quad n_2 = \frac{x_2 \rho_0}{(\rho_0^2 - e^2) H_\rho^0}, \quad n_3 = \frac{x_2 \rho_0}{(\rho_0^2 - 1) H_\rho^0}, \tag{5.5.5}$$

we obtain

$$\frac{x_1^2}{H_\rho^0} = \rho_0 n_1 x_1, \quad \frac{x_2^2}{H_\rho^0} = \frac{\rho_0^2 - e^2}{\rho_0} n_2 x_2, \quad \frac{x_2^2}{H_\rho^0} = \frac{\rho_0^2 - 1}{\rho_0} n_3 x_3. \tag{5.5.6}$$

At the same time

$$\begin{aligned}
\iint\limits_O n_1 x_1 do = \iint\limits_O n_2 x_2 do = \iint\limits_O n_3 x_3 do = \iiint\limits_V d\tau \\
= \frac{4}{3}\pi abc = \frac{4}{3}\pi a^3 \rho_0 \sqrt{(\rho_0^2 - 1)(\rho_0^2 - e^2)},
\end{aligned}$$

so that

$$J_1 = J_2 = J_3 = \frac{4}{3}\pi a^3,$$

and expressions (5.5.4) are reduced to the form

$$m_s^0 = \frac{8\pi a^3}{3} G\theta_s \gamma_s(\rho_0) \qquad (s = 1, 2, 3), \tag{5.5.7}$$

where

$$\left.\begin{aligned}
\gamma_1(\rho_0) &= \frac{1}{\delta_1(\rho_0)}\{(1-2\nu)[\omega_2(\rho_0)+\omega_3(\rho_0)]+\\
&\qquad\qquad (2\rho_0^2-1-e^2)\,\omega_4(\rho_0)\},\\
\gamma_2(\rho_0) &= \frac{1}{\delta_2(\rho_0)}\{(1-2\nu)[\omega_3(\rho_0)+\omega_1(\rho_0)]+(2\rho_0^2-1)\,\omega_5(\rho_0)\},\\
\gamma_3(\rho_0) &= \frac{1}{\delta_3(\rho_0)}\{(1-2\nu)[\omega_1(\rho_0)+\omega_2(\rho_0)]+(2\rho_0^2-e^2)\,\omega_6(\rho_0)\},
\end{aligned}\right\} \tag{5.5.8}$$

5.5.6 *An ellipsoidal cavity in the unbounded elastic medium*

The state of stress at an infinite distance from the cavity is given by the tensor

$$\hat{T}^{\infty} = \mathbf{i}_1\mathbf{i}_1 q_1 + \mathbf{i}_2\mathbf{i}_2 q_2 + \mathbf{i}_3\mathbf{i}_3 q_3, \tag{5.6.1}$$

whose principal axes are parallel to the axes of the ellipsoidal surface of the cavity

$$D_1 = \frac{x^2}{\rho_0^2} + \frac{y^2}{\rho_0^2 - e^2} + \frac{z^2}{\rho_0^2 - 1} - a^2 = 0. \tag{5.6.2}$$

The stress tensor $\hat{T}$ is presented by the sum of tensor $\hat{T}^{\infty}$ and the correcting tensor $\hat{T}^*$

$$\hat{T} = \hat{T}^{\infty} + \hat{T}^*, \tag{5.6.3}$$

The latter is determined by the boundary condition

$$\rho = \rho_0: \quad \mathbf{n}\cdot\hat{T}^* = -\mathbf{n}\cdot\hat{T}^{\infty} = -(n_1\mathbf{i}_1 q_1 + n_2\mathbf{i}_2 q_2 + n_3\mathbf{i}_3 q_3) \tag{5.6.4}$$

which expresses that the cavity surface is not loaded. The projections of the external normal to this surface on the coordinate axes are denoted by n_s.

The displacement vector corresponding to the correcting tensor $\hat{T}^*$ is presented in terms of the Papkovich-Neuber harmonic functions B_s, B_0 by formula (1.4.10) of Chapter 4

$$u_s = 4(1-\nu)B_s - \frac{\partial}{\partial x_s}(B_1x_1 + B_2x_2 + B_3x_3 + B_0).$$

Referring also to formula (1.4.17) of Chapter 4 one can reduce the boundary conditions (5.6.4) to the following form

$$-\frac{1}{2G}n_s q_s = n_s \operatorname{div}\mathbf{B} + (1-2\nu)\left(\frac{\partial B_s}{\partial n} - n_s\operatorname{div}\mathbf{B} + \sum_{k=1}^{3}\frac{\partial B_k}{\partial x_s}n_k\right) - \\ \sum_{k=1}^{3} x_k \frac{\partial}{\partial n}\frac{\partial B_k}{\partial x_s} - \frac{\partial}{\partial n}\frac{\partial B_0}{\partial x_s} \quad (s = 1, 2, 3), \tag{5.6.5}$$

where $x_1 = x, x_2 = y, x_3 = z$. It is natural to prescribe the harmonic functions B_s by the potentials (F.8.9) which are proportional to the coordinates on the surface $\rho = \rho_0$

$$B_1 = A_1 x \omega_1(\rho), \quad B_2 = A_2 y \omega_2(\rho), \quad B_3 = A_3 z \omega_3(\rho), \tag{5.6.6}$$

functions $\omega_{3s}(\rho)$ being determined by the elliptic integrals. Let the harmonic scalar B_0 be taken in the form

$$B_0 = C_1 \left(\rho_0^2 - \sigma_1\right)^2 F_4 \omega_4(\rho) + C_2 \left(\rho_0^2 - \sigma_2\right)^2 F_5 \omega_5(\rho) + A\omega + A_0 a^2 \omega_0(\rho). \tag{5.6.7}$$

Here $F_4\omega_4(\rho)$ and $F_5\omega_5(\rho)$ are the potentials denoted in Section F.8 as $F_2^{(4)}\omega_2^{(4)}$ and $F_2^{(5)}\omega_2^{(5)}$, respectively

$$\left.\begin{aligned} F_4 &= \frac{x^2}{\sigma_1} + \frac{y^2}{\sigma_1 - e^2} + \frac{z^2}{\sigma_1 - 1} - a^2, \quad F_5 = \frac{x^2}{\sigma_2} + \frac{y^2}{\sigma_2 - e^2} + \frac{z^2}{\sigma_2 - 1} - a^2, \\ \omega_4(\rho) &= \int_\rho^\infty \frac{d\lambda}{\left(\lambda^2 - \sigma_1\right)^2 \Delta(\lambda)}, \qquad \omega_5(\rho) = \int_\rho^\infty \frac{d\lambda}{\left(\lambda^2 - \sigma_2\right)^2 \Delta(\lambda)}, \end{aligned}\right\} \tag{5.6.8}$$

and ω is the Newtonian potential (F.8.20):

$$\begin{aligned} \omega &= \int_\rho^\infty \frac{d\lambda}{\Delta(\lambda)} \left(a^2 - \frac{x^2}{\lambda^2} - \frac{y^2}{\lambda^2 - e^2} - \frac{z^2}{\lambda^2 - 1}\right) \\ &= a^2\omega_0 - x^2\omega_1 - y^2\omega_2 - z^2\omega_3. \end{aligned} \tag{5.6.9}$$

Equalities (C.11.19), (C.11.21) and (C.11.26) and the expressions for the projections of the normal vector

$$\left.\begin{aligned} n_1 &= \frac{x}{\rho H_\rho} = \frac{x}{\rho^2\sqrt{D_2}}, \quad n_2 = \frac{y\rho}{(\rho^2 - e^2) H_\rho} = \frac{y}{(\rho^2 - e^2)\sqrt{D_2}}, \\ n_3 &= \frac{z\rho}{(\rho^2 - 1) H_\rho} = \frac{z}{(\rho^2 - 1)\sqrt{D_2}} \end{aligned}\right\} \tag{5.6.10}$$

are essentially used for transforming the boundary conditions (5.6.5). Here D_2 is the form denoted in Section C.11 by D_ρ^2

$$D_2 = \frac{x^2}{\rho^4} + \frac{y^2}{\left(\rho^2 - e^2\right)^2} + \frac{z^2}{\left(\rho^2 - 1\right)^2}. \tag{5.6.11}$$

We also introduce into consideration the following form

$$D_3 = \frac{x^2}{\rho^6} + \frac{y^2}{\left(\rho^2 - e^2\right)^3} + \frac{z^2}{\left(\rho^2 - 1\right)^3} = -\frac{1}{2\rho}\frac{\partial D_2}{\partial \rho}, \tag{5.6.12}$$

which can be easily proved by means of the above equalities. For brevity we also introduce the following denotations

$$\Phi_1(x,y,z) = A_1\frac{x^2}{\rho^2} + A_2\frac{y^2}{\rho^2-e^2} + A_3\frac{z^2}{\rho^2-1} + A_0a^2, \tag{5.6.13}$$

$$\Phi_2(x,y,z) = A_1\frac{x^2}{\rho^4} + A_2\frac{y^2}{(\rho^2-e^2)^2} + A_3\frac{z^2}{(\rho^2-1)^2}, \tag{5.6.14}$$

$$G_i = \frac{x^2}{\rho^2\sigma_i} + \frac{y^2}{(\rho^2-e^2)(\sigma_i-e^2)} + \frac{z^2}{(\rho^2-1)(\sigma_i-1)} \quad (i=1,2). \tag{5.6.15}$$

The expressions for the first derivatives of the functions containing ρ and the Cartesian coordinates are rather simple. For instance, using eqs. (F.7.5) and (C.11.26) we have

$$\begin{aligned}\frac{\partial}{\partial x}x\omega_1(\rho) &= \omega_1(\rho) + x\frac{\partial\omega_1}{\partial\rho}\frac{\partial\rho}{\partial x}\\ &= \omega_1(\rho) - \frac{x}{\rho^2\Delta(\rho)}\frac{\partial\rho}{\partial x} = \omega_1(\rho) - \frac{x^2}{\rho^5 D_2\Delta(\rho)}.\end{aligned} \tag{5.6.16}$$

The derivative with respect to the normal is easy to find

$$\frac{\partial}{\partial n}x\omega_1(\rho) = n_1\omega_1(\rho) + \frac{x}{H_\rho}\frac{\partial\omega_1}{\partial\rho} = n_1\left(\omega_1 - \frac{1}{\rho\Delta(\rho)}\right). \tag{5.6.17}$$

By means of formulae (5.6.10) one can carry out the transformation of the type

$$xn_2 = \frac{xy}{(\rho^2-e^2)\sqrt{D_2}} = n_1 y\frac{\rho^2}{\rho^2-e^2}$$

etc. This enables one to factor out n_1 (as well as n_2 and n_3) on the right hand side of the first (second and third) boundary condition (5.6.5). The expressions for the second derivatives are more cumbersome. This calculation is alleviated by the fact that the second derivatives in the boundary conditions (5.6.5) are the derivatives with respect to the normal of the first derivatives of the Papkovich-Neuber function taken by means of eq. (5.6.17). The calculations relating to potential (5.6.9) are especially simple. Referring to eq. (5.6.2), which is valid due to eq. (C.11.9) for any ρ, we have

$$\frac{\partial\omega}{\partial x_s} = -2x_s\omega_s, \quad \frac{\partial}{\partial n}\frac{\partial\omega}{\partial x_s} = -2n_s\left(\omega_s - \frac{1}{\rho\Delta(\rho)}\right). \tag{5.6.18}$$

It is worth noting that one of the terms in B_0 is redundant since these four functions are related by a linear relationship (F.8.16).

5.5.7 *The boundary conditions*

Each of three boundary conditions (5.6.5) contains two sets of terms. The first set (I_s) has quantities which are constant on the cavity surface $\rho = \rho_0$ whereas the second set contains the terms depending not only on ρ_0 but also on the Cartesian coordinates x, y, z of the point of this surface. As a result we arrive at the equality

$$
\begin{aligned}
\frac{q_s}{4G} = &-\left\{(1-2\nu) A_s\omega_s + \nu\left(A_1\omega_1 + A_2\omega_2 + A_3\omega_3 + \frac{A_s}{\rho\Delta}\right) + \right. \\
&A\left(\omega_s - \frac{1}{\rho\Delta}\right) - \left[C_1\frac{\left(\rho^2-\sigma_1\right)^2}{\sigma_1-\alpha_s}\omega_4 + C_2\frac{\left(\rho^2-\sigma_2\right)^2}{\sigma_2-\alpha_s}\omega_5\right] + \\
&\left.\left(\frac{C_1}{\sigma_1-\alpha_s} + \frac{C_2}{\sigma_2-\alpha_s}\right)\frac{\rho^2-\alpha_s}{\rho\Delta}\right\} + \\
&\frac{1}{\rho^3\Delta^3 D_2}\left\{\left[\chi_s(\rho) - \frac{D_3}{D_2}\rho^2\Delta^2\right]\left(\Phi_1 + C_1F_4 + C_2F_5\right) + \right. \\
&\left. 2\rho^2\Delta^2\left(\frac{C_1}{\rho^2-\sigma_1}F_4 + \frac{C_2}{\rho^2-\sigma_2}F_5\right) + \rho^2\Delta^2\left(\Phi_2 - C_1G_1 - C_2G_2\right)\right\}
\end{aligned}
\tag{5.7.1}
$$

where $\rho = \rho_0, \alpha_1 = 0, \alpha_2 = e^2, \alpha_3 = 1, s = 1,2,3$ and

$$
\left.
\begin{aligned}
&2\chi_s = 3\rho^4 - 2\left(1+e^2\right)\rho^2 + e^2 + \frac{\rho^2\Delta^2}{\rho^2-\alpha_s}, \\
&\Delta^2(\rho) = \left(\rho^2-1\right)\left(\rho^2-e^2\right), \\
&\chi_1 = 2\rho^4 - \frac{3}{2}\left(1+e^2\right)\rho^2 + e^2, \\
&\chi_2 = 2\rho^4 - \left(\frac{3}{2}+e^2\right)\rho^2 + \frac{1}{2}e^2, \\
&\chi_3 = 2\rho^4 - \left(1+\frac{3}{2}e^2\right)\rho^2 + \frac{1}{2}e^2.
\end{aligned}
\right\}
\tag{5.7.2}
$$

It is necessary to require that the quadratic polynomial

$$\Phi_1 + C_1F_4 + C_2F_5$$

has a factor D_2. Taking into account eq. (5.6.2) this condition is set in the form

$$\Phi_1 + C_1F_4 + C_2F_5 = \varkappa D_2 + \lambda D_1 \tag{5.7.3}$$

and leads to the four equations

$$\frac{A_s}{\rho^2-\alpha_s} + \frac{C_1}{\sigma_1-\alpha_s} + \frac{C_2}{\sigma_2-\alpha_s} = \frac{\varkappa}{\left(\rho^2-\alpha_s\right)^2} + \frac{\lambda}{\rho^2-\alpha_s} \tag{5.7.4}$$

$$(s = 1,2,3),$$

$$C_1 + C_2 = \lambda + A_0. \tag{5.7.5}$$

The second set of the terms in eq. (5.7.1) is now expressed in the form

$$\mathrm{II}_s = \frac{1}{\rho^3\Delta^3}\varkappa\chi_s(\rho) + \left[-\varkappa D_3 + 2\left(\frac{C_1}{\rho^2-\sigma_1}F_4 + \frac{C_2}{\rho^2-\sigma_2}F_5\right) + \Phi_2 - C_1G_1 - C_2G_2\right]\frac{1}{D_2\rho\Delta}, \quad (5.7.6)$$

that is, the term in the square brackets is equated to λD_2. Indeed, taking

$$-\varkappa D_3 + 2\left(\frac{C_1}{\rho^2-\sigma_1}F_4 + \frac{C_2}{\rho^2-\sigma_2}F_5\right) + \Phi_2 - C_1G_1 - C_2G_2 - \lambda D_2 = 2\mu D_1 = 0$$

we arrive at the system of four equations

$$\frac{C_1}{\rho^2-\sigma_1} + \frac{C_2}{\rho^2-\sigma_2} = \mu, \quad (5.7.7)$$

$$-\frac{\varkappa}{(\rho^2-\alpha_s)^3} + \frac{A_s-\lambda}{(\rho^2-\alpha_s)^2} + 2\left[\frac{C_1}{(\rho^2-\sigma_1)(\sigma_1-\alpha_s)} + \frac{C_2}{(\rho^2-\sigma_2)(\sigma_2-\alpha_s)}\right] - \frac{1}{\rho^2-\alpha_s}\left(\frac{C_1}{\sigma_1-\alpha_s} - \frac{C_2}{\sigma_2-\alpha_s}\right) = \frac{2\mu}{\rho^2-\alpha_s} \quad (s=1,2,3), \quad (5.7.8)$$

the system of equations (5.7.8) being identically satisfied by virtue of eqs. (5.7.4) and (5.7.7). Indeed, due to eq. (5.7.4) it can be written in the form

$$\frac{C_1}{\sigma_1-\alpha_s}\left(\frac{1}{\rho^2-\sigma_1} - \frac{1}{\rho^2-\alpha_s}\right) + \frac{C_2}{\sigma_2-\alpha_s}\left(\frac{1}{\rho^2-\sigma_2} - \frac{1}{\rho^2-\alpha_s}\right) = \frac{\mu}{\rho^2-\alpha_s}$$

i.e. it reduces to eq. (5.7.7).

The boundary conditions (5.7.1) yield now the system of equations whose right hand side is constant for $\rho = \rho_0$

$$-\frac{q_s}{4G} = (1-2\nu)A_s\omega_s + \nu\left(A_1\omega_1 + A_2\omega_2 + A_3\omega_3 + \frac{A_s}{\rho\Delta}\right) + A\left(\omega_s - \frac{1}{\rho\Delta}\right) - \frac{\varkappa\chi_s}{\rho^3\Delta^3} - \frac{\lambda}{\rho\Delta} - \left[C_1\frac{(\rho^2-\sigma_1)^2}{\sigma_1-\alpha_s}\omega_4 + C_2\frac{(\rho^2-\sigma_2)^2}{\sigma_2-\alpha_s}\omega_5\right] + \left(\frac{C_1}{\sigma_1-\alpha_s} + \frac{C_2}{\sigma_2-\alpha_s}\right)\frac{\rho^2-\alpha_s}{\rho\Delta}, \quad (5.7.9)$$

which is required.

5.5.8 Expressing the constants in terms of three parameters

Determining the constants C_1 and C_2 by means of eqs. (5.7.5) and (5.7.7) we have

$$\left.\begin{aligned} C_1 &= \frac{1}{\sigma_2 - \sigma_1}\left[(\lambda + A_0)\left(\rho^2 - \sigma_1\right) - \mu\left(\rho^2 - \sigma_1\right)\left(\rho^2 - \sigma_2\right)\right], \\ C_2 &= \frac{1}{\sigma_2 - \sigma_1}\left[-(\lambda + A_0)\left(\rho^2 - \sigma_2\right) + \mu\left(\rho^2 - \sigma_1\right)\left(\rho^2 - \sigma_2\right)\right], \end{aligned}\right\} \tag{5.8.1}$$

where $\rho = \rho_0$ here and in what follows. Due to eq. (5.7.4)

$$\frac{A_s - \lambda}{\rho^2 - \alpha_s} - \frac{\varkappa}{\left(\rho^2 - \alpha_s\right)^2} = \\ = \frac{1}{(\sigma_1 - \alpha_s)(\sigma_2 - \alpha_s)}\left[-(\lambda + A_0)\left(\rho^2 - \alpha_s\right) + \mu\left(\rho^2 - \sigma_1\right)\left(\rho^2 - \sigma_2\right)\right].$$

Using definition (F.6.8) of parameters σ_1 and σ_2 we obtain

$$\sum_{s=1}^{3} \frac{1}{\sigma_k - \alpha_s} = 0, \ \sigma_1\sigma_2 = \frac{1}{3}e^2, \ \sigma_1 + \sigma_2 = \frac{2}{3}\left(1 + e^2\right) \quad (k = 1, 2). \tag{5.8.2}$$

It follows that the constants A_s must be related by the equality

$$\sum_{s=1}^{3}\left[\frac{A_s - \lambda}{\rho^2 - \alpha_s} - \frac{\varkappa}{\left(\rho^2 - \alpha_s\right)^2}\right] = 0,$$

which can only be satisfied if

$$\lambda + A_0 = 0. \tag{5.8.3}$$

By virtue of eq. (5.8.1)

$$C_1 = -C_2 = C = \frac{\mu}{\sigma_1 - \sigma_2}\left(\rho^2 - \sigma_1\right)\left(\rho^2 - \sigma_2\right) \tag{5.8.4}$$

and parameter μ can be replaced by C. Using eq. (5.7.4) we have

$$A_s = \lambda + \frac{\varkappa}{\rho^2 - \alpha_s} + C(\sigma_1 - \sigma_2)\frac{\rho^2 - \alpha_s}{(\sigma_1 - \alpha_s)(\sigma_2 - \alpha_s)} \quad (s = 1, 2, 3) \tag{5.8.5}$$

or more fully

$$\left.\begin{aligned} A_1 &= \lambda + \frac{\varkappa}{\rho_0^2} + \frac{3C}{e^2}(\sigma_1 - \sigma_2)\rho_0^2, \\ A_2 &= \lambda + \frac{\varkappa}{\rho_0^2 - e^2} - \frac{3C}{e^2(1 - e^2)}(\sigma_1 - \sigma_2)\left(\rho_0^2 - e^2\right), \\ A_2 &= \lambda + \frac{\varkappa}{\rho_0^2 - 1} + \frac{3C}{1 - e^2}(\sigma_1 - \sigma_2)\left(\rho_0^2 - 1\right). \end{aligned}\right\} \tag{5.8.6}$$

The constants A_s are expressed in terms of three parameters. However the boundary conditions contain the fourth parameter A which is redundant. Indeed, turning to the original representation of the displacements in terms of the harmonic functions B_s, B_0

$$u_s = 4(1-\nu) A_s x_s \omega_s(\rho) - \frac{\partial \Psi}{\partial x_s}, \quad \Psi = \sum_{k=1}^{3} A_k x_k^2 \omega_k + B_0,$$

and referring to eqs. (5.6.7), (5.8.3), (5.8.6) and (5.6.9) we obtain the following result

$$\begin{aligned}\Psi = (A-\lambda)\omega + \sum_{k=1}^{3} \frac{\varkappa}{\rho_0^2 - \alpha_k} x_k^2 \omega_k + \\ C\left[\left(\rho_0^2 - \sigma_1\right)^2 F_4 \omega_4 - \left(\rho_0^2 - \sigma_2\right)^2 F_5 \omega_5\right] + \\ C(\sigma_1 - \sigma_2) \sum_{k=1}^{3} \frac{\rho^2 - \alpha_k}{(\sigma_1 - \alpha_k)(\sigma_2 - \alpha_k)} \omega_k x_k^2. \quad (5.8.7)\end{aligned}$$

Due to eq. (5.6.18) the expressions for the displacement contain two constants λ and A only in the following combination

$$2\left[(1-2\nu)\lambda + A\right] x_s \omega_s$$

which indicates the redundancy of A.

The sum of three elliptic integrals can be written down in the following form

$$\begin{aligned}\omega_1 + \omega_2 + \omega_3 &= \int_{\rho}^{\infty} \frac{d\lambda}{\Delta(\lambda)} \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2 - e^2} + \frac{1}{\lambda^2 - 1} \right) \\ &= \int_{\rho}^{\infty} \frac{d\lambda}{\Delta(\lambda)} \left(\frac{1}{\lambda^2} + \frac{2\lambda^2 - (1+e^2)}{\Delta^2(\lambda)} \right) = \left[\rho \Delta(\rho)\right]^{-1}, \quad (5.8.8)\end{aligned}$$

which is easy to prove by direct differentiation.

The boundary conditions (5.7.9) are now presented by three equations containing the same number of unknowns $\lambda, \varkappa, C$. In the case $\rho = \rho_0$ we have

$$
\begin{aligned}
-\frac{q_s}{4G} = {} & \lambda(1-2\nu)\left(\omega_s - \frac{1}{\rho\Delta}\right) + \varkappa\left[(1-2\nu)\frac{\omega_s}{\rho_0^2-\alpha_s} + \right. \\
& \left.\nu\left(\frac{\omega_1}{\rho^2} + \frac{\omega_2}{\rho^2-e^2} + \frac{\omega_3}{\rho^2-1}\right)\right] + \frac{1}{\rho\Delta}\left(\frac{\nu}{\rho^2-\alpha_s} - \frac{\chi_s(\rho)}{\rho^2\Delta^2}\right) + \\
& C(\sigma_1-\sigma_2)\left\{(1-2\nu)\frac{\rho^2-\alpha_s}{(\sigma_1-\alpha_s)(\sigma_2-\alpha_s)}\omega_s + 3\nu\left(\frac{\rho^2}{e^2}\omega_1 - \right.\right. \\
& \left.\frac{\rho^2-e^2}{e^2(1-e^2)}\omega_2 + \frac{\rho^2-1}{1-e^2}\omega_3\right) - \frac{1-\nu}{(\sigma_1-\alpha_s)(\sigma_2-\alpha_s)}\frac{\rho^2-\alpha_s}{\rho\Delta} + \\
& \left.\frac{1}{\sigma_2-\sigma_1}\int_\rho^\infty\left[\frac{(\rho^2-\sigma_1)^2}{\sigma_1-\alpha_s}\frac{1}{(\lambda^2-\sigma_1)^2} - \frac{(\rho^2-\sigma_2)^2}{\sigma_2-\alpha_s}\frac{1}{(\lambda^2-\sigma_2)^2}\right]\frac{d\lambda}{\Delta(\lambda)}\right\}
\end{aligned}
\tag{5.8.9}
$$

The displacements are given by the following relationships (where, of course, a distinction should be made between ρ and ρ_0)

$$
\begin{aligned}
u_s = {} & 2(1-2\nu)\left[\lambda + \frac{\varkappa}{\rho_0^2-\alpha_s} + C\frac{(\sigma_1-\sigma_2)(\rho_0^2-\alpha_s)}{(\sigma_1-\alpha_s)(\sigma_2-\alpha_s)}\right]x_s\omega_s(\rho) - \\
& 2C\left[\frac{(\rho_0^2-\sigma_1)^2}{\sigma_1-\alpha_s}\omega_4(\rho) - \frac{(\rho_0^2-\sigma_2)^2}{\sigma_2-\alpha_s}\omega_5(\rho)\right]x_s + \\
& \frac{1}{\Delta(\rho)}\frac{\partial\rho}{\partial x_s}\left\{\varkappa\left[\frac{x_1^2}{\rho_0^2\rho^2} + \frac{x_2^2}{(\rho_0^2-e^2)(\rho^2-e^2)} + \frac{x_3^2}{(\rho_0^2-1)(\rho^2-1)}\right] + \right. \\
& 3C(\sigma_1-\sigma_2)\left[\frac{\rho_0^2}{e^2}\frac{x_1^2}{\rho^2} - \frac{\rho_0^2-e^2}{e^2(1-e^2)}\frac{x_2^2}{\rho^2-e^2} + \frac{\rho_0^2-1}{1-e^2}\frac{x_3^2}{\rho^2-1}\right] + \\
& \left.C\left[\frac{(\rho_0^2-\sigma_1)^2}{(\rho^2-\sigma_1)^2}F_4 - \frac{(\rho_0^2-\sigma_2)^2}{(\rho^2-\sigma_2)}F_5\right]\right\}.
\end{aligned}
\tag{5.8.10}
$$

Here

$$
\alpha_1 = 0, \quad \alpha_2 = e^2, \quad \alpha_3 = 1, \qquad \frac{\partial\rho}{\partial x_s} = \frac{x_s}{\rho(\rho-\alpha_s)}\frac{1}{D^2},
$$

$$
(\sigma_1-\alpha_1)(\sigma_2-\alpha_1) = \frac{e^2}{3}, \quad (\sigma_1-\alpha_2)(\sigma_2-\alpha_2) = -\frac{1}{3}e^2(1-e^2),
$$

$$
(\sigma_1-\alpha_3)(\sigma_2-\alpha_3) = \frac{1}{3}(1-e^2).
$$

5.5.9 *A spheroidal cavity in the elastic medium*

Let the surface of the cavity be an ellipsoid of revolution and the field of the stress tensor $\hat{T}^\infty$ be symmetric about the axis of revolution of this ellipsoid.

If $e = 0$ then the surface of the cavity is an oblate ellipsoid of revolution about axis z. Assuming

$$\rho^2 = 1 + s^2, \quad d\rho = \frac{sds}{\sqrt{1+s^2}}$$

we have

$$\Delta(\rho) = s\sqrt{1+s^2}, \quad \rho\Delta(\rho) = s\left(1+s^2\right)$$

and referring to eqs. (F.3.7), (F.3.13) and (F.7.5) we obtain

$$\left.\begin{aligned}
\omega_0 &= \int_s^\infty \frac{d\lambda}{1+\lambda^2} = \arctan s = q_0(s), \\
\omega_1 = \omega_2 &= \int_s^\infty \frac{d\lambda}{\left(1+\lambda^2\right)^2} = \frac{1}{2}\left(\arctan s - \frac{s}{1+s^2}\right) = \frac{1}{2}\frac{q_1^1(s)}{\sqrt{1+s^2}}, \\
\omega_3 &= \int_s^\infty \frac{d\lambda}{\lambda^2\left(1+\lambda^2\right)} = \frac{1}{2}\left(\frac{1}{s} - \arctan s\right) = -\frac{1}{s}q_1(s).
\end{aligned}\right\} \tag{5.9.1}$$

Under the assumption $q_1 = q_2$ the problem becomes axially symmetric and the second boundary condition (5.8.9) is coincident with the first. Hence it is sufficient to keep only two constants λ and $\varkappa$. Then we arrive at the boundary conditions: at $s = s_0$

$$\begin{aligned}
\frac{q_1}{4G} &= \lambda(1-2\nu)(\omega_1+\omega_3) - \varkappa\left[\frac{\omega_1(s_0)}{1+s_0^2} + \nu\frac{\omega_3(s_0)}{s_0^2} + \frac{2\nu s_0^2 - 4s_0^2 - 1}{2s_0^3\left(1+s_0^2\right)^2}\right], \\
\frac{q_3}{4G} &= 2\lambda(1-2\nu)\,\omega_1(s_0) + 2\varkappa\left[\omega_1(s_0)\left(\frac{1-\nu}{s_0^2} - \frac{\nu}{s_0^2+1}\right) + \frac{1}{2s_0\left(1+s_0^2\right)^2}\right]
\end{aligned} \tag{5.9.2}$$

The displacements are determined by formulae (5.8.10)

$$\begin{aligned}
u_r &= 2(1-2\nu)\left(\lambda + \frac{\varkappa}{1+s_0^2}\right) r\omega_1(s) + \\
&\qquad \frac{\varkappa}{s\left(1+s^2\right)^2 D_2} r\left[\frac{r^2}{\left(s_0^2+1\right)\left(s^2+1\right)} + \frac{z^2}{s_0^2 s^2}\right], \\
w &= 2(1-2\nu)\left(\lambda + \frac{\varkappa}{s_0^2}\right) z\omega_3(s) + \\
&\qquad \frac{\varkappa}{s^3\left(1+s^2\right) D_2} z\left[\frac{r^2}{\left(s_0^2+1\right)\left(s^2+1\right)} + \frac{z^2}{s_0^2 s^2}\right], \\
D_2 &= \frac{r^2}{\left(s^2+1\right)^2} + \frac{z^2}{s^4}.
\end{aligned} \tag{5.9.3}$$

In the case of an oblong ellipsoid of revolution $\rho = s, e = 1, \Delta(\rho) = s^2 - 1$ and by eqs. (F.3.3) and (F.3.11) we have

$$\left.\begin{aligned}
\omega_0(s) &= \int_s^\infty \frac{d\lambda}{\lambda^2 - 1} = \frac{1}{2}\ln\frac{s+1}{s-1} = Q_0(s), \\
\omega_1(s) &= \int_s^\infty \frac{d\lambda}{\lambda^2(\lambda^2 - 1)} = \frac{1}{s} - \frac{1}{2}\ln\frac{s+1}{s-1} = \frac{1}{s}Q_1(s), \\
\omega_2(s) = \omega_3(s) &= \int_s^\infty \frac{d\lambda}{(\lambda^2 - 1)^2} = \frac{1}{2}\left(\frac{s}{s^2-1} - \frac{1}{2}\ln\frac{s+1}{s-1}\right) \\
&= -\frac{1}{2}\frac{Q_1^1(s)}{\sqrt{s^2-1}}.
\end{aligned}\right\} \tag{5.9.4}$$

For $q_2 = q_3$ we also arrive at two equations for the constants λ and $\varkappa$.

5.5.10 *A circular slot in elastic medium*

For small s_0 we represent the coefficients of equations in (5.9.2) by the power series

$$\omega_1(s_0) = \frac{\pi}{4} - s_0 + \ldots, \quad \omega_3(s_0) = \frac{1}{s_0} - \frac{\pi}{2} + s_0 + \ldots$$

and arrive at the system of equations

$$\begin{aligned}
\frac{q_1}{4G} &= (1-2\nu)\lambda\left(-\frac{\pi}{4} + \frac{1}{s_0} + \ldots\right) - \varkappa\left[\frac{\nu}{s_0^2}\left(\frac{1}{s_0} - \frac{\pi}{2}\right) - \frac{1}{2s_0^3} + \ldots\right], \\
\frac{q_3}{8G} &= (1-2\nu)\lambda\left(\frac{\pi}{4} - s_0 + \ldots\right) + \varkappa\left[\frac{1}{s_0^2}(1-\nu)\left(\frac{\pi}{4} - s_0\right) + \frac{1}{2s_0} + \ldots\right],
\end{aligned}$$

which can be satisfied by expanding the unknowns λ and $\varkappa$ in the series

$$\lambda = \lambda_0 + \lambda_1 s_0 + \ldots, \quad \varkappa = \varkappa_2 s_0^2 + \ldots \tag{5.10.1}$$

where

$$\lambda_0 = -\frac{1}{2\pi G}q_3, \quad \varkappa_2 = \frac{1}{\pi G}q_3 \quad \text{etc.} \tag{5.10.2}$$

The case of a circular slot in elastic medium is described by $s_0 \to 0, \lambda = \lambda_0, \varkappa = \varkappa_2 s_0^2$. The solution does not depend on q_1 since the presence of the slot does not affect the state of stress due to the loading which is parallel to the plane of the slot (the correcting tensor is zero).

For the obtained values of the constants λ and $\varkappa$ the correcting vector of displacement (5.9.3) is given by the projections u_r and w on the directions

$\mathbf{e}_r$ and $\mathbf{k}$ of the cylindrical coordinate system

$$\left.\begin{aligned} Gu_r &= -\frac{q_3}{\pi}\left[(1-2\nu)\,r\omega_1(s) - \frac{z^2 r}{(1+s^2)^2 s^3 D_2}\right],\\ Gw &= \frac{q_3}{\pi}\left[(1-2\nu)\,z\omega_3(s) + \frac{z^3}{(1+s^2)\,s^5 D_2}\right],\\ D_2 &= \frac{r^2}{(1+s^2)^2} + \frac{z^2}{s^4}, \end{aligned}\right\} \tag{5.10.3}$$

the cylindrical coordinates being expressed in terms of the spheroidal coordinates μ and s (the coordinates of the compressed spheroid) by formulae (C.10.1)

$$r = a\sqrt{1+s^2}\sqrt{1-\mu^2}, \quad z = as\mu. \tag{5.10.4}$$

It allows relationships (5.10.3) to be set in another form

$$\left.\begin{aligned} \frac{G\pi}{q_3}u_r &= -(1-2\nu)\,r\omega_1(s) + a\sqrt{\frac{1-\mu^2}{1+s^2}}\frac{\mu^2 s}{s^2+\mu^2},\\ \frac{G\pi}{q_3}w &= (1-2\nu)\,z\omega_3(s) + a\frac{\mu^3}{s^2+\mu^2}. \end{aligned}\right\} \tag{5.10.5}$$

It is known that $\mu = 0, s \neq 0$ describes the part of plane $z = 0$ outside the circle $r = a, \mu = 0$, whilst $\mu \neq 0, s = 0$ describes that inside the circle. On the circle $s = 0, \mu = 0$. Thus displacement w in the slot plane $z = 0$ is a continuous function of r, namely

$$w = \frac{q_3}{\pi G}\begin{cases} a\mu = \sqrt{a^2 - r^2}, & r < a \quad (s = 0),\\ 0, & r > a \quad (\mu = 0). \end{cases} \tag{5.10.6}$$

To determine the stresses one uses the formulae for differentiation

$$\left.\begin{aligned} \frac{\partial s}{\partial z} &= \frac{\mu(1+s^2)}{a(s^2+\mu^2)}, & \frac{\partial \mu}{\partial z} &= \frac{s(1-\mu^2)}{a(s^2+\mu^2)},\\ \frac{\partial s}{\partial r} &= \frac{sr}{a^2(s^2+\mu^2)}, & \frac{\partial \mu}{\partial r} &= -\frac{\mu r}{a^2(s^2+\mu^2)}. \end{aligned}\right\} \tag{5.10.7}$$

The shear stress τ_{rz} is equal to zero in the whole plane of the slot

$$\frac{1}{G}\tau_{rz}\Big|_{z=0} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)_{z=0} = \begin{cases} 0, & s = 0, \quad \mu \neq 0,\\ 0, & s \neq 0, \quad \mu = 0. \end{cases} \tag{5.10.8}$$

Further we find

$$\frac{G\pi}{q_3}\operatorname{div}\mathbf{u} = (1-2\nu)\left[\omega_3 - \frac{1}{s} - 2\omega_1 + \frac{s}{s^2+\mu^2}\left(1+\frac{1-\mu^2}{1+s^2}\right)\right] + \dots,$$

$$\frac{G\pi}{q_3}\frac{\partial w}{\partial z} = (1-2\nu)\left(\omega_3 - \frac{1}{s} + \frac{s}{s^2+\mu^2}\right) + \dots,$$

$$\frac{G\pi}{q_3}\frac{\partial u}{\partial r} = (1-2\nu)\left[\omega_1 - \frac{s\left(1-\mu^2\right)}{\left(1+s^2\right)\left(s^2+\mu^2\right)}\right] + \dots,$$

$$\frac{G\pi}{q_3}\frac{u}{r} = -(1-2\nu)\,\omega_1 + \dots,$$

where the terms vanishing in plane $z=0$, i.e. for $s=0$ or $\mu=0$, are not written down. They remain continuous when the focal circle $s=0, \mu=0$ is approached while the point remains in plane $z=0$.

In plane $z=0$ we have

$$\sigma_z = \begin{cases} -q_3, & s=0, \\ \dfrac{2q_3}{\pi}\left[(1-\nu)\,\omega_3 - 2\nu\omega_1 + \dfrac{\nu}{s\left(1+s^2\right)}\right], & \mu=0, \end{cases} \tag{5.10.9}$$

and when approaching the focal circle $s=0$ from the side of $s>0$, i.e. $r>a$, the normal stress experiences a jump

$$\sigma_z\big|_{\substack{\mu=0\\ s\to 0}} - \sigma_z\big|_{\substack{\mu=0\\ s=0}} = \frac{2q_3}{\pi}\frac{1}{s} = \frac{2q_3}{\pi}\frac{a}{\sqrt{a^2-r^2}}. \tag{5.10.10}$$

The state of stress considered here occurs in the elastic half space covered by a rigid smooth plate with a circular opening $r \le a$. The pressure q_3 is distributed within the circle $r \le a$ and the plate admits no normal displacement w. However it does not hinder displacement u_r in its plane.

It is also easy to obtain the distribution of stresses σ_r and σ_φ in plane $z=0$.

5.5.11 An elliptic slot in an elastic medium

The problem of the state of stress in an elastic medium with an elliptic slot is considered by analogy. The slot lies in plane $z=0$ and is bounded by the focal ellipse E_0, see eq. (C.1.16). The solution of the system of equations (5.8.9) is sought in the form of series in terms of parameter $\sqrt{\rho_0^2-1}=\varepsilon$

$$\lambda = \lambda_0 + \dots, \quad \varkappa = \varkappa_2\varepsilon^2 + \dots, \quad C = C_0 + \dots,$$

where it is sufficient to take only the first terms of these series. Taking into account that, due to eqs. (5.8.8) and (5.7.2),

$$\omega_3 = \frac{1}{\rho\Delta} - (\omega_1+\omega_2), \quad \chi_1(1) = \chi_2(1) = \frac{1}{2}\left(1-e^2\right),$$

$$\chi_3(1) = 1-e^2, \quad \Delta = \sqrt{1-e^2}\varepsilon,$$

and remaining within the above approximation we can put the first two equations in (5.8.9) in the form

$$
\begin{aligned}
0 &= \lambda_0 (1-2\nu) + \frac{1}{2}\varkappa_2 (1-2\nu) + \frac{3C_0}{e^2}(1-\nu)(\sigma_1-\sigma_2), \\
0 &= \lambda_0 (1-2\nu) + \frac{1}{2}\varkappa_2 (1-2\nu) - \frac{3C_0}{e^2}(1-\nu)(\sigma_1-\sigma_2).
\end{aligned}
$$

It follows that

$$
C_0 = 0, \quad \lambda_0 = -\frac{1}{2}\varkappa_2,
$$

and the third equation in (5.8.9) yields

$$
\varkappa_2 = \frac{q_3}{2G(\omega_1+\omega_2)}, \quad \lambda_0 = -\frac{q_3}{4G(\omega_1+\omega_2)}. \tag{5.11.1}
$$

Here

$$
\left.
\begin{aligned}
\omega_1 &= \int_1^\infty \frac{d\lambda}{\lambda^2 \Delta(\lambda)} = \frac{1}{e^2}\left[K(e) - E(e)\right], \\
\omega_2 &= \int_1^\infty \frac{d\lambda}{\left(\lambda^2 - e^2\right)\Delta(\lambda)} = \frac{E(e) - \left(1-e^2\right)K(e)}{e^2\left(1-e^2\right)},
\end{aligned}
\right\} \tag{5.11.2}
$$

where K and E are the complete elliptic integrals of the first and second kind respectively, so that

$$
\omega_1 + \omega_2 = \frac{E}{1-e^2} \tag{5.11.3}
$$

and $e = 0$ yields formulae (5.10.2) for the case of the circular slot.

The solution of the problem which is the expression for the displacement vector (5.8.10) is set in a form analogous to eq. (5.10.3)

$$
\left.
\begin{aligned}
Gu &= \frac{q_3\left(1-e^2\right)}{2E(e)}\left[-(1-2\nu)\, x\omega_1(\rho) + \frac{xz^2}{\rho^3\Delta(\rho)\left(\rho^2-1\right)D_2}\right], \\
Gv &= \frac{q_3\left(1-e^2\right)}{2E(e)}\left[-(1-2\nu)\, y\omega_2(\rho) + \frac{yz^2}{\rho^3\Delta(\rho)\, D_2}\right], \\
Gw &= \frac{q_3\left(1-e^2\right)}{2E(e)}\left[(1-2\nu)\, z\omega_3(\rho) + \frac{z^3}{\rho\left(\rho^2-1\right)^2\Delta(\rho)\, D_2}\right].
\end{aligned}
\right\} \tag{5.11.4}
$$

This solution is also written down as follows

$$\left.\begin{aligned} u &= -2A\left[(1-2\nu)\,x\omega_1(\rho) + z^2\frac{\partial \omega_3}{\partial x}\right],\\ v &= -2A\left[(1-2\nu)\,y\omega_2(\rho) + z^2\frac{\partial \omega_3}{\partial y}\right],\\ w &= 2A\left[(1-2\nu)\,z\omega_3(\rho) - z^2\frac{\partial \omega_3}{\partial z}\right], \end{aligned}\right\} \tag{5.11.5}$$

$$A = \frac{q_3\left(1-e^2\right)}{4GE(e)}, \tag{5.11.6}$$

and it can be expressed in terms of the Papkovich-Neuber functions

$$B_1 = -Ax\omega_1(\rho),\ B_2 = -Ay\omega_2(\rho),\ B_3 = Az\omega_3(\rho),\ B_0 = Aa^2\omega_0(\rho). \tag{5.11.7}$$

Let us prove that this solution satisfies all of the conditions of the problem.

Indeed, referring to formulae (C.11.26) and (5.8.8) and accounting for the equality

$$\omega_3(\rho) = \frac{1}{1-e^2}\left[-E(\varphi, e) + \frac{\rho^2-e^2}{\rho\Delta(\rho)}\right],\quad \varphi = \arcsin\frac{1}{\rho},$$

we have

$$\begin{aligned} \vartheta &= 2(1-2\nu)\nabla\cdot\mathbf{B} = -2A\left\{\omega_1+\omega_2-\omega_3-\right.\\ &\left.\frac{1}{\rho\Delta(\rho)D_2}\left[\frac{x^2}{\rho^4}+\frac{y^2}{(\rho^2-e^2)^2}-\frac{z^2}{(\rho^2-1)^2}\right]\right\}(1-2\nu)\\ &= -\frac{4A(1-2\nu)}{1-e^2}\left[E(\varphi,e)-\frac{\rho^2-e^2}{\rho\Delta(\rho)}+\frac{z^2\left(1-e^2\right)}{\rho\Delta(\rho)\left(\rho^2-1\right)^2D_2}\right] \end{aligned}$$

as well as

$$\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= -2A\left\{(1-2\nu)\left[\omega_1+\omega_2-\frac{1}{\rho\Delta(\rho)D_2}\left(\frac{x^2}{\rho^4}+\frac{y^2}{(\rho^2-e^2)^2}\right)\right]\right.\\ &\qquad\left.+z^3\left(\frac{\partial^2\omega_3}{\partial x^2}+\frac{\partial^2\omega_3}{\partial y^2}\right)\right\}\\ &= -2A\left[(1-2\nu)\left(-\omega_3+\frac{z^2}{(\rho^2-1)\rho\Delta(\rho)D_2}\right)+z^2\left(\frac{\partial^2\omega_3}{\partial x^2}+\frac{\partial^2\omega_3}{\partial y^2}\right)\right]. \end{aligned}$$

The values of the parameters in these formulae within ellipse $E_0\,(\rho = 1)$ and outside it $(\mu = 1)$ are given by

$$\frac{z^2}{\rho^2-1} = \frac{a^2}{1-e^2}\left(1-\mu^2\right)\left(1-\nu^2\right) = \begin{cases} 0 & \text{outside } E_0, \\ \dfrac{a^2}{1-e^2}\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{a^2\left(1-e^2\right)}\right) & \text{inside } E_0, \end{cases}$$

$$z\omega_3 = \frac{z}{1-e^2}\left[-E(\varphi,e)+\frac{\rho^2-e^2}{\rho\Delta(\rho)}\right] = \begin{cases} 0 & \text{outside } E_0, \\ \dfrac{a}{\sqrt{1-e^2}}\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{a^2\left(1-e^2\right)}\right)^{1/2} & \text{inside } E_0, \end{cases}$$

$$-z^2\frac{\partial\omega_3}{\partial z} = \frac{z^3}{\rho\Delta(\rho)\left(\rho^2-1\right)^2 D_2} = \begin{cases} 0 & \text{outside } E_0, \\ \dfrac{a}{\sqrt{1-e^2}}\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{a^2\left(1-e^2\right)}\right)^{1/2} & \text{inside } E_0, \end{cases}$$

$$\vartheta = \begin{cases} -\dfrac{4A(1-2\nu)}{1-e^2}\left[E(\varphi,e)-\dfrac{\rho^2-e^2}{\rho\Delta(\rho)}\right] & \text{outside } E_0, \\ -\dfrac{4A(1-2\nu)}{1-e^2}E(e) & \text{inside } E_0. \end{cases}$$

In the whole plane $z = 0$

$$z^2\left(\frac{\partial^2\omega_3}{\partial x^2}+\frac{\partial^2\omega_3}{\partial y^2}\right) = 0,$$

that is, in this plane

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = \frac{\partial w}{\partial z} = \frac{1}{2}\vartheta \quad (\text{at } z = 0).$$

For this reason, the normal stress $\sigma_z|_{z=0}$ calculated by taking into account the stress at infinity

$$\sigma_z|_{z=0} = q_3 + 2G\left(\frac{\vartheta\nu}{1-2\nu}+\frac{\partial w}{\partial z}\right) = q_3 + \frac{G}{1-2\nu}\vartheta$$

is equal to

$$\sigma_z\big|_{z=0} = \begin{cases} \dfrac{q_3}{E(e)}\left[E(e) - E(\varphi, e) + \dfrac{1}{\rho}\sqrt{\dfrac{\rho^2 - e^2}{\rho^2 - 1}}\right] & \text{outside } E_0, \\ 0 & \text{inside } E_0. \end{cases} \tag{5.11.8}$$

On ellipse E_0 this stress experiences a jump.

Let us also prove that the shear stresses τ_{xz} and τ_{yz} are absent in the whole plane $z = 0$

$$\begin{aligned} \frac{1}{G}\tau_{xz}\bigg|_{z=0} &= 2A\left[(1 - 2\nu)\left(z\frac{\partial\omega_3}{\partial x} - x\frac{\partial\omega_1}{\partial z}\right) - 2z\frac{\partial\omega_3}{\partial x} - 2z^2\frac{\partial^2\omega_3}{\partial x\partial z}\right]\bigg|_{z=0} \\ &= -4A\frac{\partial}{\partial x}\left[\left(z\omega_3 + z^2\frac{\partial\omega_3}{\partial z}\right)\bigg|_{z=0}\right] = 0, \end{aligned}$$

since, by virtue of the above, the expression in the brackets vanishes.

Let us notice the expression for w

$$w = \begin{cases} 0 & \text{outside } E_0, \\ \dfrac{q_3\sqrt{1 - e^2}}{GE(e)}a(1 - \nu)\left[1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{a^2(1 - e^2)}\right]^{1/2} & \text{inside } E_0 \end{cases}$$

which is easily obtained from the above equations. The displacement remains continuous on E_0.

5.6 Contact problems

5.6.1 *The problem of the rigid die. Boundary condition*

Contact problems in the theory of elasticity are concerned with the state of stress which appears in elastic bodies pressed against each other. In particular, one of the bodies can be rigid (a rigid die) whereas the elastic body can be presented by an elastic half-space. Under some additional assumptions the solution of this simple problem turns out to be sufficient for constructing solutions to a more general Hertz's problem of the contact of two elastic bodies.

Let plane Oxy bound the half-space and axis Oz be directed into the half-space. The die base pressed against the half-space can be either flat or have the form of a convex surface S, see Fig. 5.4. The die is related to the system of axes $O\xi\eta$ whose origin lies on surface S and axis $O\xi$ is directed into the die along the normal to this surface. In the initial state, in which the die is not loaded, the origins of the systems of axes $O\xi\eta\zeta$ and $Oxyz$ are coincident. Axes ξ and x as well as η and y coincide whereas axes ζ

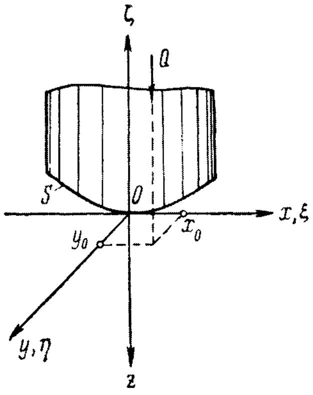

FIGURE 5.4.

and z have opposite directions. For this reason, the first system of axes is left-handed and the second one is right-handed.

In the coordinate system $O\xi\eta\zeta$ the equation for surface S of the die base has the form

$$\zeta = \varphi(\xi, \eta), \tag{6.1.1}$$

and for the adopted system of axes

$$\varphi(0,0) = 0, \quad \left(\frac{\partial \varphi}{\partial \xi}\right)_{\xi=\eta=0} = 0, \quad \left(\frac{\partial \varphi}{\partial \eta}\right)_{\xi=\eta=0} = 0. \tag{6.1.2}$$

In the case of a plane die the equation for the bounded plane is simply

$$\zeta = 0. \tag{6.1.3}$$

Under the loading the die moves and becomes immersed in the elastic medium. The quantities characterising the displacement of the die are assumed to have the same order of smallness as the displacement of the points of the medium.

Let region Ω consist of points in plane Oxy which, after deformation, lie on the displaced surface S of the die base. As always, the boundary conditions are related to the undeformed surface of the elastic body, i.e. to plane $z = 0$. The die base is assumed to be absolutely smooth, thus it is taken that the shear stresses τ_{zx} and τ_{yz} are absent in the whole plane $z = 0$

$$z = 0: \quad \tau_{zx} = 0, \quad \tau_{yz} = 0. \tag{6.1.4}$$

The normal stresses are absent in plane $z = 0$ outside the region Ω of contact of the die and the medium. At points of region Ω, the elastic medium is subjected to the compression load $p(x,y)$, so that

$$\sigma_z = \begin{cases} 0, & z \not\subset \Omega, \\ -p(x,y), & z \subset \Omega. \end{cases} \tag{6.1.5}$$

Evidently, function $p(x,y)$ is not given in advance and is the main variable of the problem. Under conditions (6.1.4) and (6.1.5) the equilibrium of the die is possible only if the resultant force Q is parallel to axis Oz. Denoting the coordinates of the point of intersection of the line of action of this force with plane Oxy by x_0 and y_0, we can write down the following equilibrium equations for the die

$$Q = \iint\limits_{\Omega} p(x,y)\,do,\ x_0 Q = \iint\limits_{\Omega} xp(x,y)\,do,\ y_0 Q = \iint\limits_{\Omega} yp(x,y)\,do$$
$$(do = dxdy)\,. \tag{6.1.6}$$

These are the integral equations which the unknown pressure $p(x,y)$ must satisfy.

Let us proceed to the boundary condition for displacement w of the points of region Ω. It must be expressed in terms of the quantities determining the displacement of the die. Under force Q the latter will move in translation and rotate. The translation displacement δ will be parallel to axis z and the rotation will occur about a certain axis in plane Oxy, the projections of the vector of small rotation being denoted as β_x and β_y. Three parameters δ, β_x, β_y are needed to obtain the expressions for the displacement of the point of surface S of the die base and the expressions for the coordinate of points x_s, y_s, z_s in the coordinate system $Oxyz$ are required. The table of the direction cosines of the angles between this system and $O\xi\eta\zeta$ is as follows

	ξ	η	ζ
x	1	0	$-\beta_y$
y	0	1	β_x
z	$-\beta_y$	β_x	-1

The unusual signs in this table is due to the fact that one system is right-handed while the second is left-handed. In the system $Oxyz$ the coordinates of the origin of the system $O\xi\eta\zeta$ are $0, 0, \delta$, the formulae for the transformation of the point with coordinates $\xi, \eta, \zeta = \varphi(\xi,\eta)$ on S are given by

$$\left.\begin{aligned} x_s &= 0 + \xi - \beta_y\zeta = \xi - \beta_y\varphi(\xi,\eta)\,, \\ y_s &= 0 + \eta + \beta_x\zeta = \eta + \beta_x\varphi(\xi,\eta)\,, \\ z_s &= \delta - \zeta - \beta_y\xi + \beta_x\eta = \delta - \varphi(\xi,\eta) - \beta_y\xi + \beta_x\eta. \end{aligned}\right\} \tag{6.1.7}$$

It follows from equalities (6.1.2) that quantity $\varphi(\xi,\eta)$ is of the second order of smallness with respect to the values characterising the extent of the contact surface and this allows us to neglect the products $\beta_x\varphi$ and $\beta_y\varphi$ in eq. (6.1.7). Therefore

$$x_s = \xi, \quad y_s = \eta, \quad z_s = \delta - \beta_y x_s + \beta_x y_s - \varphi(x_s, y_s). \tag{6.1.8}$$

Let $(x, y, 0)$ be the point of region Ω which, after deformation, becomes point (x_s, y_s, z_s) of S, i.e.

$$x_S = x + u, \quad y_S = y + v, \quad z_S = w, \tag{6.1.9}$$

where u, v, w denote the projections of the displacement of point $(x, y, 0)$ on Ω. By eqs. (6.1.8) and (6.1.9) we have

$$\left.\begin{array}{l} u = \xi - x, \quad v = \eta - y, \\ w = \delta - \beta_y (x + u) + \beta_x (y + v) - \varphi(x + u, y + v). \end{array}\right\} \tag{6.1.10}$$

In the latter equation we neglect the products $\beta_x v, \beta_y v$ and also assume that

$$\varphi(x + u, y + v) \approx \varphi(x, y).$$

Then we arrive at the sought boundary condition

$$z = 0, \quad (x, y) \subset \Omega: \quad w = \delta - \beta_y x + \beta_y y - \varphi(x, y). \tag{6.1.11}$$

In the case of a plane die this condition simplifies and takes the form

$$z = 0, \quad (x, y) \subset \Omega: \quad w = \delta - \beta_y x + \beta_y y. \tag{6.1.12}$$

The problem for the die is thus reduced to the mixed boundary value problem of elasticity theory: firstly, shear stresses τ_{zx} and τ_{yz} vanish in the whole plane $z = 0$, secondly, outside region Ω of this plane the normal stresses vanish, and thirdly, the normal displacement w of the point of region Ω is given. The values of δ, β_x, β_y are not know *a priori* and the equilibrium equations (6.1.6) are used to determine them.

The above-said can also be elucidated in the following way: the points of region Ω in plane $z = 0$ gain the normal displacement w according to the prescribed law (6.1.11) or (6.1.12). To this aim, the normal displacement $p(x, y)$ is distributed over area Ω, the law of distribution being unknown in advance. The die is placed into the "hollow" and is pressed by a vertical force Q in order to ensure equilibrium.

If the distortion due to the rotation is neglected then, for the plane die, region Ω is determined by the form of its cross-section which is normal to axis ζ. Normal stress σ_z is discontinuous on the contour of this region. If the die is not plane and its surface has no sharp corners (i.e. $\partial\varphi/\partial\xi, \partial\varphi/\partial\eta$ are continuous), the contour C of region Ω is given by the condition

$$p(x_C, y_C) = 0. \tag{6.1.13}$$

By virtue of eq. (6.1.5) the normal stress is continuous in the whole plane $z = 0$. This condition is due to the fact that the medium is fitted smoothly to the die base when no sharp corners on the die surface are present. Under conditions (6.1.4)-(6.1.6), (6.1.11) the problem for the die with a non-plane base admits a family of solutions depending on a single parameter which is determined by the requirement of a smooth fit of the medium to the die surface (6.1.13).

Quantity $w(x_C, y_C)$ is the displacement of the point of the medium along contour C of region Ω in plane $z = 0$. According to eq. (6.1.11) the sinking of the die into the medium is given by

$$\delta_1 = \delta - w(x_C, y_C) = \beta_y x_C - \beta_y y_C + \varphi(x_C, y_C), \tag{6.1.14}$$

expressing that the displacement δ of the die is a sum of the sinking into the medium and the displacement of the medium at the points of curve C.

The die must be pressed against the whole surface of the contact so that the sought pressure distribution satisfies the condition

$$p(x, y) \geq 0, \ (x, y) \subset \Omega, \tag{6.1.15}$$

the equality holding only on contour C of region Ω. This condition imposes a restriction on the location of the line of action of force Q acting on the die.

5.6.2 A method of solving the problem for a rigid die

Subsection 5.2.3 deals with the Boussinesq problem of the state of stress of the elastic half-space whose boundary $z = 0$ is free of the tangential stresses τ_{zx}, τ_{zy} whereas the normal stress is prescribed on it. The solution reduced to searching for the harmonic function ω which was determined by the simple layer potential distributed over region Ω with density equal to the normal pressure $p(x, y)$

$$\omega = \iint\limits_{\Omega} \frac{p(x', y')}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} do' \quad (do' = dx'dy'). \tag{6.2.1}$$

The shear and normal stresses on the surfaces $z = \text{const}$ are given by eq. (2.3.5)

$$\tau_{zx} = -\frac{1}{2\pi} z \frac{\partial^2 \omega}{\partial x \partial z}, \quad \tau_{yz} = -\frac{1}{2\pi} z \frac{\partial^2 \omega}{\partial y \partial z}, \quad \sigma_z = \frac{1}{2\pi}\left(\frac{\partial \omega}{\partial z} - z\frac{\partial^2 \omega}{\partial z^2}\right), \tag{6.2.2}$$

and the displacements are due to eq. (2.3.4)

$$u = -\frac{1}{4\pi G}\left[z\frac{\partial \omega}{\partial x} + (1-2\nu)\frac{\partial \tilde{\omega}}{\partial x}\right], \quad v = -\frac{1}{4\pi G}\left[z\frac{\partial \omega}{\partial y} + (1-2\nu)\frac{\partial \tilde{\omega}}{\partial y}\right], \tag{6.2.3}$$

$$w = \frac{1-\nu}{2\pi G}\omega - \frac{1}{4\pi G}z\frac{\partial \omega}{\partial z}. \tag{6.2.4}$$

It is known that the normal derivative of the simple layer potential distributed over a plane region is given by eq. (2.3.6)

$$z = 0: \quad \left.\frac{\partial \omega}{\partial z}\right|_{z\to+0} = \begin{cases} -2\pi p(x,y), & (x,y) \subset \Omega, \\ 0, & (x,y) \not\subset \Omega. \end{cases} \tag{6.2.5}$$

Referring to eq. (6.2.2) we can conclude that the solution found by means of potential ω satisfies conditions (6.1.4) and (6.1.5) of the problem of a rigid die, density $p(x,y)$ being subjected to condition (6.1.11). By means of eq. (6.2.4) this condition reduces to the integral equation of the first kind for the sought distribution of normal pressure

$$w(x,y,0) = \delta - \beta_y x + \beta_x y - \varphi(x,y) = \frac{1-\nu}{2\pi G}\iint\limits_{\Omega} \frac{p(x',y')\,do'}{\sqrt{(x-x')^2 + (y-y')^2}} \tag{6.2.6}$$

or by eq. (6.1.12) in the case of a plane die

$$w(x,y,0) = \delta - \beta_y x + \beta_x y = \frac{1-\nu}{2\pi G}\iint\limits_{\Omega} \frac{p(x',y')\,do'}{\sqrt{(x-x')^2 + (y-y')^2}}. \tag{6.2.7}$$

The closed form solution can be obtained under the assumption that the contact region Ω is an elliptic area bounded by ellipse E_0

$$E_0: \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} - 1 = 0. \tag{6.2.8}$$

For the plane die, the semi-axes $a, a\sqrt{1-e^2}$ are given by the form of the contact surface. In the case of a non-plane die the equation for surface S is presented by the expansion in a power series which starts, by eq. (6.1.2), from the terms of the second degree in ξ and η

$$\varphi(\xi,\eta) = \frac{1}{2}\left[\left(\frac{\partial^2\varphi}{\partial^2\xi}\right)_0 \xi^2 + 2\left(\frac{\partial^2\varphi}{\partial\xi\partial\eta}\right)_0 \xi\eta + \left(\frac{\partial^2\varphi}{\partial\eta^2}\right)_0 \eta^2\right] + \ldots$$

By means of an appropriate choice of directions of axes ξ and η the term with $\xi\eta$ can be made equal to zero. Then

$$\zeta = \frac{\xi^2}{2R_1} + \frac{\eta^2}{2R_2} + \ldots \tag{6.2.9}$$

Here R_1^{-1} and R_2^{-1} are the curvatures of the principal normal sections of surface S at the point of its tangency to the plane bounding the half-space. It is assumed that they are positive and R_1 denotes the larger of the two curvature radii.

Considering only the local effects we restrict ourselves by the second order terms (6.2.9). This means that surface S is approximated by an elliptic paraboloid in the neigbourhood where S is tangent to plane $z = 0$. The boundary condition (6.2.6) is now set in the form

$$\delta - \frac{1}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right) = \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} \frac{p(x', y')\, do'}{\sqrt{(x-x')^2 + (y-y')^2}}, \tag{6.2.10}$$

that is, in the case of a non-plane die the analysis is limited to the case of the translatory displacement $\left(\beta_y = \beta_x = 0\right)$. As mentioned above, the integration region Ω is assumed to lie in ellipse E_0 whose parameters a and e are not known in advance. Eventually they are determined in terms of the pressing force Q and curvatures R_1^{-1}, R_2^{-1} of the contact surface. Pressure $p(x, y)$ is assumed to vanish on E_0, i.e. condition (6.1.13) is satisfied.

It is necessary to distinguish the border of the contact area (ellipse E_0) from the contour of the cross-section of the die by plane $\zeta = \text{const}$.

The elliptic plate having a "top" $(z > 0)$ and a "bottom" $(z < 0)$ and bounded by a focal ellipse E_0 is one of the coordinate planes $\rho = 1$ of the family of ellipsoids $\rho = \text{const}$ in the system of elliptic coordinates ρ, μ, ν, see Section C.11 and in particular eq. (C.11.16). Hence it is reasonable to introduce into consideration a simple layer potential $\omega(x, y, z; \rho_0)$ on the surface of ellipsoid Ω_* $(\rho = \rho_0 = 1)$, this potential being a continuous harmonic function determined by $\omega(x, y, z; \rho_0)$ on Ω_*. According to eq. (6.2.6) for the plane die one can take

$$\text{on } \Omega_*: \quad \omega(x, y, z; \rho_0) = \frac{2\pi G}{1-\nu}\left(\delta - \beta_y x + \beta_x y\right) \tag{6.2.11}$$

and, due to eq. (6.2.6) for the non-plane die,

$$\text{on } \Omega_*: \quad \omega(x, y, z; \rho_0) = \frac{2\pi G}{1-\nu}\left(\delta - \frac{x^2}{2R_1} - \frac{y^2}{2R_2}\right). \tag{6.2.12}$$

Having constructed solutions for the internal and external Dirichlet's problems ($\omega_i(x, y, z; \rho_0)$ and $\omega_e(x, y, z; \rho_0)$ respectively) we arrive at the following function

$$\omega(x, y, z; \rho_0) = \begin{cases} \omega_i(x, y, z; \rho_0), & \rho < \rho_0, \\ \omega_e(x, y, z; \rho_0), & \rho > \rho_0, \end{cases} \tag{6.2.13}$$

which is continuous in the whole space and vanishes at infinity. It is a simple layer potential with density $p(x, y, z)$ defined as follows

$$\left(\frac{\partial \omega_e}{\partial n} - \frac{\partial \omega_i}{\partial n}\right)_{(x,y,z)\subset\Omega_*} = -4\pi p. \tag{6.2.14}$$

As plane $z = 0$ is a plane of symmetry of ellipsoid $\rho = \rho_0$ and the values of ω_i and ω_e do not depend on z (they are equal to $\omega(x, y, 0; \rho_0)$) the density p is even with respect to z.

In order to determine the simple layer potential $\omega(x, y, z)$ on the elliptic plate it remains to carry out the limiting process

$$\omega(x, y, z) = \lim_{\rho_0 \to 1} \omega_i(x, y, z; \rho_0) = \lim_{\rho_0 \to 1} \omega_e(x, y, z; \rho_0). \tag{6.2.15}$$

Potential $\omega(x, y, z)$ satisfies conditions (6.1.11) or (6.1.12) on the surface of plate Ω. While calculating the density it is necessary to take into account that two elements of ellipsoid Ω_* contribute to each area element, the elements of Ω_* being symmetric about plane $z = 0$ and having the same density. Hence, the density of the layer on Ω obtained by considering formula (6.2.14) at its limit must be doubled:

$$p(x, y) = -\frac{1}{2\pi} \lim_{\rho_0 \to 1} \left(\frac{\partial \omega_e(x, y, z; \rho_0)}{\partial n} - \frac{\partial \omega_i(x, y, z; \rho_0)}{\partial n} \right). \tag{6.2.16}$$

This determines the pressure on the surface of contact of the die and the elastic medium.

The suggested strategy allows us to avoid direct consideration of integral equations of the first kind (6.2.6) and (6.2.7). Besides, estimating integral (6.2.1) is not needed because function ω is constructed by means of eq. (6.2.15) which requires only the solution of the external Dirichlet's problem for the ellipsoid, see Section F.8.

The forces and moments applied to a non-plane die

Assuming the solution of the problem of the plane die to be known one can obtain the expressions for the forces and moments which must be applied to the die with the base $\zeta = \varphi(\xi, \eta)$ in order to ensure the translation displacement δ and rotations β_x, β_y. The cross-section of the plane die must have the size and the form of the contact surface (region Ω in plane $z = 0$ which is unknown in advance) of the non-plane die.

Let $q_0(x, y)$ denote the distribution of pressure over the base of the plane die when the latter is subjected to translatory displacement $\delta^0 = 1$ and the rotation is absent $\left(\beta_x^0 = 0, \beta_y^0 = 0\right)$. The principal vector and the principal moment of this distribution are denoted as

$$\left.\begin{aligned} Q_0 = \iint_\Omega q_0(x, y)\, do, \quad \iint_\Omega y q_0(x, y)\, do = y_0 Q_0, \\ -\iint_\Omega x q_0(x, y)\, do = -x_0 Q_0. \end{aligned}\right\} \tag{6.2.17}$$

By analogy we introduce the pressure distributions $q_1(x, y)$ and $q_2(x, y)$ ensuring the rotations $\beta_x^{(1)} = 1, \beta_y^{(1)} = 0$ and $\beta_x^{(2)} = 0, \beta_y^{(2)} = 1$, respectively, and absence of translation displacement of the die $\left(\delta^{(1)} = 0, \delta^{(2)} = 0\right)$. The principal vectors and the principal moments of these distributions are written down in the form $(s = 1, 2)$

$$\left.\begin{aligned} Q_s = \iint\limits_{\Omega} q_s(x, y)\, do, \quad \iint\limits_{\Omega} y q_s(x, y)\, do = y_s Q_s, \\ -\iint\limits_{\Omega} x q_s(x, y)\, do = -x_s Q_s. \end{aligned}\right\} \tag{6.2.18}$$

From these definitions and the reciprocity theorem the symmetry of the matrix

$$\begin{Vmatrix} Q_0 & y_0 Q_0 & -x_0 Q_0 \\ Q_1 & y_1 Q_1 & -x_1 Q_1 \\ Q_2 & y_2 Q_2 & -x_2 Q_2 \end{Vmatrix}. \tag{6.2.19}$$

follows. Referring to eq. (6.2.7) we have

$$\left.\begin{aligned} 1 &= \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} \frac{q_0(x', y')\, do'}{\sqrt{(x-x')^2 + (y-y')^2}}, \\ y &= \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} \frac{q_1(x', y')\, do'}{\sqrt{(x-x')^2 + (y-y')^2}}, \\ -x &= \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} \frac{q_2(x', y')\, do'}{\sqrt{(x-x')^2 + (y-y')^2}}. \end{aligned}\right\} \tag{6.2.20}$$

Considering the non-plane die and introducing the principal vector P and the principal moments m_1, m_2 of the forces applied to the die we have

$$P = \iint\limits_{\Omega} p(x, y)\, do, \quad m_1 = \iint\limits_{\Omega} y p(x, y)\, do, \quad m_2 = -\iint\limits_{\Omega} x p(x, y)\, do, \tag{6.2.21}$$

where $p(x, y)$ designates the distribution of pressure over the surface of contact of the die and the elastic half-space. By virtue of eq. (6.2.6)

$$\delta - \beta_y x' + \beta_x y' - \varphi(x', y') = \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} \frac{p(x, y)\, do}{\sqrt{(x'-x)^2 + (y'-y)^2}}, \tag{6.2.22}$$

where δ, β_x, β_y denote the translation displacement and the angles of rotation of the die subjected to this force and these moments.

Turning to eq. (6.2.21) and the first relationship in (6.2.20) we obtain

$$P = \iint\limits_{\Omega} p(x,y)\,do = \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} p(x,y)\,do \iint\limits_{\Omega} \frac{q_0(x',y')\,do'}{\sqrt{(x-x')^2+(y-y')^2}}$$
$$= \frac{1-\nu}{2\pi G} \iint\limits_{\Omega} q_0(x',y')\,do' \iint\limits_{\Omega} \frac{p(x,y)\,do}{\sqrt{(x'-x)^2+(y'-y)^2}}$$

or, by eqs. (6.2.22) and (6.2.17),

$$P + \iint\limits_{\Omega} q_0(x',y')\,\varphi(x',y')\,do' = Q_0\left(\delta + \beta_x y_0 - \beta_y x_0\right). \tag{6.2.23}$$

By analogy, using the second and the third equalities in eq. (6.2.20) as well as eqs. (6.2.18), (6.2.21), (6.2.22) we arrive at the relationships

$$\left.\begin{aligned} m_1 + \iint\limits_{\Omega} q_1(x',y')\,\varphi(x',y')\,do' &= Q_1\left(\delta + \beta_x y_1 - \beta_y x_1\right), \\ m_2 + \iint\limits_{\Omega} q_2(x',y')\,\varphi(x',y')\,do' &= Q_2\left(\delta + \beta_x y_2 - \beta_y x_2\right). \end{aligned}\right\} \tag{6.2.24}$$

Equations (6.2.23) and (6.2.24) do not yet solve the posed problem since we do not know the integration region Ω which is the cross-section of the introduced plane die. Evidently, the prescribed $(\delta, \beta_x, \beta_y, \varphi(x,y))$ and the sought (P, m_1, m_2) parameters must be independent of the parameters describing the form and size of Ω. This reasoning provides one with a means for determining Ω.

The suggested approach to searching for forces and moments does not require the distribution of pressure $p(x,y)$ over the base of the non-plane die. Regretfully, it is efficiently applicable only to the case of the die of an elliptic (in particular circular) die since the required closed form solutions of the integral equations of the second kind (6.2.20) are known only for the plane elliptic (circular) die.

5.6.3 A plane die with an elliptic base

Functions $\omega_i(x,y,z;\rho_0)$ and $\omega_e(x,y,z;\rho_0)$ are determined by formulae (F.8.9)

$$\omega_i = \frac{2\pi G}{1-\nu}\left(\delta - \beta_y x + \beta_x y\right),$$
$$\omega_e = \frac{2\pi G}{1-\nu}\left[\delta\frac{\omega_0(\rho)}{\omega_0(\rho_0)} - \beta_y x\frac{\omega_1(\rho)}{\omega_1(\rho_0)} + \beta_x y\frac{\omega_2(\rho)}{\omega_2(\rho_0)}\right],$$

so that by eq. (6.2.15)

$$\omega = \frac{2\pi G}{1-\nu}\left[\delta\frac{\omega_0(\rho)}{\omega_0(1)} - \beta_y x\frac{\omega_1(\rho)}{\omega_1(1)} + \beta_x y\frac{\omega_2(\rho)}{\omega_2(1)}\right]. \tag{6.3.1}$$

For determination of the distribution of pressure over the die area we have eq. (F.8.15) where

$$F(x,y,0) = \frac{2\pi G}{1-\nu}\left(\delta - \beta_y x + \beta_x y\right),$$

and value $R(1)$ for each term in this expression is given by formulae (F.6.2) and (F.6.4). Then we have

$$p(x,y) = \frac{G}{(1-\nu)\,a\sqrt{1-e^2}}\left[\frac{\delta}{\omega_0(1)} - \beta_y\frac{x}{\omega_1(1)} + \beta_x\frac{y}{(1-e^2)\,\omega_2(1)}\right]\times \left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right)^{-1/2}. \tag{6.3.2}$$

Constants δ, β_x, β_y are obtained from the equilibrium equations (6.1.6)

$$\left.\begin{aligned} Q &= \frac{G}{(1-\nu)\,a\sqrt{1-e^2}}\frac{\delta}{\omega_0(1)}\iint\limits_{\Omega}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2}do \\ &= \frac{2\pi aG}{1-\nu}\frac{\delta}{\omega_0(1)}, \\ x_0 Q &= -\frac{G}{(1-\nu)\,a\sqrt{1-e^2}}\frac{\beta_y}{\omega_1(1)}\iint\limits_{\Omega}x^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2}do \\ &= -\frac{2\pi a^3 G}{3(1-\nu)}\frac{\beta_y}{\omega_1(1)}, \\ y_0 Q &= \frac{G}{(1-\nu)\,a\,(1-e^2)^{3/2}}\frac{\beta_x}{\omega_2(1)}\iint\limits_{\Omega}y^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2}do \\ &= \frac{2\pi a^3 G}{3(1-\nu)}\frac{\beta_x}{\omega_2(1)}. \end{aligned}\right\} \tag{6.3.3}$$

The values of $\omega_s(1)$ are determined in terms of the complete elliptic integrals $K(e)$ and $E(e)$ with modulus e

$$K(e) = \int\limits_0^{\pi/2}\frac{d\psi}{1-e^2\sin^2\psi}, \quad E(e) = \int\limits_0^{\pi/2}\sqrt{1-e^2\sin^2\psi}\,d\psi$$

and are given by

$$\left.\begin{aligned}
\omega_0(1) &= \int_1^\infty \frac{d\lambda}{\Delta(\lambda)} = \int_1^\infty \frac{d\lambda}{\sqrt{(\lambda^2-e^2)(\lambda^2-1)}} = K(e),\\
\omega_1(1) &= \int_1^\infty \frac{d\lambda}{\lambda^2\Delta(\lambda)} = \frac{1}{e^2}\left[K(e)-E(e)\right] = D(e),\\
\omega_2(1) &= \int_1^\infty \frac{d\lambda}{(\lambda^2-e^2)\Delta(\lambda)} = \frac{1}{e^2}\left[\frac{E(e)}{1-e^2}-K(e)\right] = \frac{B(e)}{1-e^2},
\end{aligned}\right\} \tag{6.3.4}$$

where the denotations $D(e)$ and $B(e)$ for the above combinations of the complete elliptic integrals $K(e)$ and $E(e)$ are suggested by Janke and Emde[1].

The result is

$$\left.\begin{aligned}
\delta = \frac{Q(1-\nu)}{2\pi aG}K(e), \quad \beta_x = 3\frac{Q(1-\nu)}{2\pi a^3 G}y_0\frac{B(e)}{1-e^2},\\
\beta_y = -3\frac{Q(1-\nu)}{2\pi a^3 G}x_0 D(e).
\end{aligned}\right\} \tag{6.3.5}$$

The pressure distribution is now presented in the form

$$p(x,y) = \frac{1}{2}p_m\left[1+\frac{3xx_0}{a^2}+\frac{3yy_0}{a^2(1-e^2)}\right]\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right)^{-1/2}, \tag{6.3.6}$$

where p_m denotes the mean pressure

$$p_m = \frac{Q}{\pi a^2\sqrt{1-e^2}}. \tag{6.3.7}$$

Pressure p is equal to the half of the mean pressure at the centre of the die and increases without bound when approaching the contour of the loading region with a sharp corner of the die. The die is pressed against the elastic half-space over the whole contact surface if the line of action of force Q passes through the elliptic cylinder with semi-axes $\frac{1}{3}a$, $\frac{1}{3}a\sqrt{1-e^2}$.

Expression (6.3.1) for potential ω is written down in the form

$$\begin{aligned}
\omega &= \frac{Q}{a}\left[\omega_0(\rho)+\frac{3xx_0}{a^2}\omega_1(\rho)+\frac{3yy_0}{a^2}\omega_2(\rho)\right]\\
&= \frac{Q}{a}\int_\rho^\infty \frac{d\lambda}{\Delta(\lambda)}\left(1+\frac{3xx_0}{a^2\lambda^2}+\frac{3yy_0}{a^2(\lambda^2-e^2)}\right).
\end{aligned} \tag{6.3.8}$$

[1] Janke E., Emde F., Lösch F. Taffeln höherer Funktionen. Teubner, Stuttgart, 1960.

5.6.4 Displacements and stresses

In order to find the components u, v of the displacement vector we need the derivatives

$$\frac{\partial \tilde{\omega}}{\partial x} = -\int_z^\infty \frac{\partial \omega}{\partial x} dz, \quad \frac{\partial \tilde{\omega}}{\partial y} = -\int_z^\infty \frac{\partial \omega}{\partial y} dz. \tag{6.4.1}$$

Here, due to eqs. (6.3.8) and (C.1.26)

$$\left. \begin{aligned} \frac{\partial \omega}{\partial x} &= \frac{3Qx_0}{a^3}\omega_1(\rho) - \frac{Q}{a\Delta(\rho)}\left[1 + \frac{3xx_0}{a^2\rho^2} + \frac{3yy_0}{a^2(\rho^2 - e^2)}\right]\frac{x}{\rho^3 D_\rho^2}, \\ \frac{\partial \omega}{\partial y} &= \frac{3Qy_0}{a^3}\omega_2(\rho) - \frac{Q}{a\Delta(\rho)}\left[1 + \frac{3xx_0}{a^2\rho^2} + \frac{3yy_0}{a^2(\rho^2 - e^2)}\right]\frac{y}{\rho^2(\rho^2 - e^2) D_\rho^2} \end{aligned} \right\} \tag{6.4.2}$$

We arrive at an estimation of integrals of the type

$$\int_z^\infty dz \int_\rho^\infty \chi(\lambda)\, d\lambda, \quad \int_z^\infty f(x, y, \rho) \frac{dz}{D_\rho^2 \Delta(\rho)}.$$

By eqs. (C.11.9) and (C.11.26)

$$\left. \begin{aligned} z(\lambda) = a\sqrt{\lambda^2 - 1}\left[1 - \frac{x^2}{a^2\lambda^2} - \frac{y^2}{a^2(\lambda^2 - e^2)}\right]^{1/2} \\ = a\sqrt{\lambda^2 - 1}\gamma(x, y, \lambda) \\ \frac{dz}{D_\rho^2 \Delta(\rho)} = \frac{\rho(\rho^2 - 1)}{z\Delta(\rho)} d\rho = \rho\sqrt{\frac{\rho^2 - 1}{\rho^2 - e^2}}\frac{d\rho}{z}, \end{aligned} \right\} \tag{6.4.3}$$

the second equality being obtained by differentiating eq. (C.11.9) with respect to ρ for constant x and y. Thus,

$$-\int_z^\infty dz \int_\rho^\infty \chi(\lambda)\, d\lambda = \int_\rho^\infty [z - z(\lambda)]\, \chi(\lambda)\, d\lambda,$$

$$\int_z^\infty f(x, y, \rho) \frac{dz}{D^2 \Delta(\rho)} = \int_\rho^\infty \frac{f(x, y, \lambda)}{z(\lambda)} \sqrt{\frac{\lambda^2 - 1}{\lambda^2 - e^2}} \lambda d\lambda.$$

Applying these equalities we obtain by means of eq. (6.4.2)

$$\left.\begin{aligned}
\frac{\partial \tilde{\omega}}{\partial x} &= \frac{3Qx_0}{a^3}\int_\rho^\infty [z - z(\lambda)] \frac{d\lambda}{\lambda^2 \Delta(\lambda)} + \\
&\frac{Qx}{a^2}\int_\rho^\infty \left[1 + \frac{3xx_0}{a^2\lambda^2} + \frac{3yy_0}{a^2(\lambda^2 - e^2)}\right] \frac{d\lambda}{\lambda^2\sqrt{\lambda^2 - e^2}\gamma(x,y,\lambda)}, \\
\frac{\partial \tilde{\omega}}{\partial y} &= \frac{3Qy_0}{a^3}\int_\rho^\infty [z - z(\lambda)] \frac{d\lambda}{(\lambda^2 - e^2)\Delta(\lambda)} + \\
&\frac{Qy}{a^2}\int_\rho^\infty \left[1 + \frac{3xx_0}{a^2\lambda^2} + \frac{3yy_0}{a^2(\lambda^2 - e^2)}\right] \frac{d\lambda}{(\lambda^2 - e^2)^{3/2}\gamma(x,y,\lambda)}.
\end{aligned}\right\} \tag{6.4.4}$$

In the case of a centrally loaded die ($x_0 = y_0 = 0$) the expressions for the displacement constructed with the help of eqs. (6.2.3) and (6.2.4) take the form

$$\begin{aligned}
u &= \frac{Qx}{4\pi Ga^2}\left[\frac{z\Delta(\rho)}{a\rho(\rho^2 - \mu^2)(\rho^2 - \nu^2)} - \frac{m-2}{m}\int_\rho^\infty \frac{d\lambda}{\lambda^2(\lambda^2 - e^2)\gamma(x,y,\lambda)}\right], \\
v &= \frac{Qy}{4\pi Ga^2}\left[\frac{z\rho\Delta(\rho)}{a(\rho^2 - e^2)(\rho^2 - \mu^2)(\rho^2 - \nu^2)} - \right. \\
&\qquad\qquad \left. \frac{m-2}{m}\int_\rho^\infty \frac{d\lambda}{(\lambda^2 - e^2)^{3/2}\gamma(x,y,\lambda)}\right], \\
w &= \frac{Qz}{4\pi Ga^3}\frac{z\rho\Delta(\rho)}{(\rho^2 - 1)(\rho^2 - \mu^2)(\rho^2 - \nu^2)} + \frac{Q(m-1)}{4\pi Gma}\int_\rho^\infty \frac{d\lambda}{\Delta(\lambda)},
\end{aligned} \tag{6.4.5}$$

where $m = 1/\nu$. Here we used relationships (C.11.21) and (C.11.12):

$$D_\rho^2 = \frac{H_\rho^2}{\rho^2} = a^2\frac{(\rho^2 - \nu^2)(\rho^2 - \mu^2)}{\rho^2(\rho^2 - e^2)(\rho^2 - 1)}.$$

The integrals appearing in the expressions for u and v are estimated easily.

The dilatation is calculated by the following formula

$$\vartheta = \frac{m-2}{2\pi Gm}\frac{\partial \omega}{\partial z} = -\frac{Q(m-2)}{4\pi Gma^3}\frac{z\rho\Delta(\rho)}{(\rho^2 - 1)(\rho^2 - \mu^2)(\rho^2 - \nu^2)}. \tag{6.4.6}$$

At points of axis z we have

$$\nu = 0, \quad \mu = e, \quad z = a\sqrt{\rho^2 - 1}, \tag{6.4.7}$$

and calculating stresses at points of this axis leads to the formulae

$$\left.\begin{aligned}\sigma_x &= \frac{1}{2}p_m\sqrt{\frac{1-e^2}{\rho^2-e^2}}\left[\frac{m-2}{me^2}\left(\rho-\sqrt{\rho^2-e^2}\right)-\frac{1}{\rho^3}\right],\\ \sigma_y &= \frac{1}{2}p_m\frac{\sqrt{1-e^2}}{\rho}\left[\frac{m-2}{me^2}\left(\rho-\sqrt{\rho^2-e^2}\right)-\frac{1-e^2}{\left(\rho^2-e^2\right)^{3/2}}\right],\\ \sigma_z &= -\frac{1}{2}p_m\frac{1}{\rho}\sqrt{\frac{1-e^2}{\rho^2-e^2}}\left(\frac{\rho^2-1}{\rho^2}+\frac{\rho^2-1}{\rho^2-e^2}+1\right).\end{aligned}\right\} \tag{6.4.8}$$

The shear stresses are absent on axis z.

5.6.5 *A non-plane die*

By eq. (6.2.12) the boundary condition for potential ω is presented in the form

$$z=0, \quad (x,y)\subset\Omega: \quad \omega(x,y,0)=\frac{2\pi G}{1-\nu}\left(\delta-\frac{x^2}{2R_1}-\frac{y^2}{2R_2}\right), \tag{6.5.1}$$

where, due to eq. (6.1.13), the density of this potential must vanish on ellipse E_0 bounding the region Ω. As shown in Section F.8 this condition is fulfilled by the potential

$$\omega=\frac{1}{2}C\int_\rho^\infty\frac{d\lambda}{\Delta(\lambda)}\left[1-\frac{x^2}{a^2\lambda^2}-\frac{y^2}{a^2\left(\lambda^2-e^2\right)}-\frac{z^2}{a^2\left(\lambda^2-1\right)}\right]$$

with the density

$$p(x,y)=\frac{C}{2\pi a\sqrt{1-e^2}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1-e^2\right)}},$$

see eq. (F.8.18). Determining C using equilibrium equation (6.1.6) we obtain

$$Q=\iint_\Omega p(x,y)\,do=\frac{1}{3}aC, \quad C=\frac{3Q}{a},$$

so that

$$p(x,y)=\frac{3}{2}p_m\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1-e^2\right)}}, \quad p_m=\frac{Q}{\pi a^2\sqrt{1-e^2}}. \tag{6.5.2}$$

The maximum pressure at the centre of the area is equal to 1.5 times the mean pressure p_m.

Potential ω is presented in the form

$$\omega = \frac{3Q}{2a} \int_{\rho}^{\infty} \frac{d\lambda}{\Delta(\lambda)} \left(1 - \frac{x^2}{a^2\lambda^2} - \frac{y^2}{a^2(\lambda^2 - e^2)} - \frac{z^2}{a^2(\lambda^2 - 1)} \right), \qquad (6.5.3)$$

the constants a and e must be determined from condition (6.5.1) taking the following form

$$\frac{2\pi G}{1-\nu} \left(\delta - \frac{x^2}{2R_1} - \frac{y^2}{2R_2} \right) = \frac{3Q}{2a} \int_{\rho}^{\infty} \frac{d\lambda}{\Delta(\lambda)} \left(1 - \frac{x^2}{a^2\lambda^2} - \frac{y^2}{a^2(\lambda^2 - e^2)} \right)$$
$$= \frac{3Q}{2a} \left[K(e) - \frac{x^2}{a^2} D(e) - \frac{y^2}{a^2} \frac{B(e)}{1-e^2} \right], \qquad (6.5.4)$$

where formulae (6.3.4) are used. Finally we arrive at the equalities

$$\delta = \frac{3Q(1-\nu)}{4\pi a G} K(e), \quad \frac{1}{R_1} = \frac{3Q(1-\nu)}{2\pi a^3 G} D(e), \quad \frac{1}{R_2} = \frac{3Q(1-\nu)}{2\pi a^3 G} \frac{B(e)}{1-e^2}. \qquad (6.5.5)$$

These determine the translation displacement δ of the die, the major semi-axis and the eccentricity of the contact area. The latter can be found from the relationship

$$\frac{R_2}{R_1} = \frac{(1-e^2) D(e)}{B(e)} = \frac{(1-e^2)[K(e) - E(e)]}{E(e) - (1-e^2) K(e)}, \qquad (6.5.6)$$

and then a and δ can be found

$$a = \left[\frac{Q(1-\nu)}{G} R_1 \right]^{1/3} \alpha_a, \quad \delta = \left[\frac{Q(1-\nu)}{G\sqrt{R_1}} \right]^{2/3} \alpha_\delta, \qquad (6.5.7)$$

where

$$\alpha_a = \left[\frac{3}{2\pi D(e)} \right]^{1/3}, \quad \alpha_\delta = \left[\frac{9}{32\pi^2 D(e)} \right]^{1/3} K(e). \qquad (6.5.8)$$

Table 5.1 collects the values of $R_2/R_1, \alpha_a, \alpha_\delta$ for some values of e^2. Taking R_2/R_1 and using this table one can find e^2 and then α_a and α_δ.

e^2	0	0,05	0,10	0,15	0,20	0,25	0,30
R_2/R_1	1	0,963	0,925	0,885	0,846	0,806	0,765
α_a	0,722	0,726	0,731	0,736	0,741	0,747	0,753
α_δ	0,520	0,523	0,526	0,530	0,534	0,538	0,543
e^2	0,35	0,40	0,45	0,50	0,55	0,60	0,65
R_2/R_1	0,724	0,682	0,637	0,594	0,549	0,502	0,454
α_a	0,760	0,767	0,775	0,783	0,793	0,803	0,815
α_δ	0,547	0,553	0,559	0,565	0,571	0,580	0,589
e^2	0,70	0,75	0,80	0,85	0,90	0,95	1
R_2/R_1	0,405	0,353	0,297	0,238	0,174	0,101	0
α_a	0,829	0,844	0,863	0,888	0,921	0,975	–
α_δ	0,597	0,609	0,623	0,642	0,668	0,713	–

Table 5.1

Figure 5.5 displays the dependence of R_2/R_1 versus e^2. Displacement δ of the die turns out to be proportional to $Q^{1/3}$. This result is unusual for the linear theory of elasticity and is explained by the fact that both the force and the contact area increase with the growth of the force.

Determination of forces and moments acting on non-plane die with an elliptic base

The approach of Subsection 5.6.2 is applied here.

By eqs. (6.3.2) and (6.2.20) we have

$$\left.\begin{aligned} q_0(x,y) &= \frac{G}{(1-\nu)\,a\sqrt{1-e^2}\omega_0(1)}\left[1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right]^{-1/2}, \\ q_1(x,y) &= \frac{G}{(1-\nu)\,a(1-e^2)^{2/3}\omega_2(1)}y\left[1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right]^{-1/2}, \\ q_2(x,y) &= \frac{-G}{(1-\nu)\,a\sqrt{1-e^2}\omega_1(1)}x\left[1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right]^{-1/2} \end{aligned}\right\} \tag{6.5.9}$$

Only the diagonal elements of matrix (6.2.19) differ from zero and, by virtue of eqs. (6.3.3), (6.2.17) and (6.2.18), they are given by

$$Q_0 = \frac{2\pi a G}{(1-\nu)\,\omega_0(1)}, \quad y_1 Q_1 = \frac{2\pi a^3 G}{3(1-\nu)\,\omega_2(1)}, \quad x_2 Q_2 = -\frac{2\pi a^3 G}{3(1-\nu)\,\omega_1(1)}. \tag{6.5.10}$$

In accordance with eq. (6.2.9)

$$\varphi(x,y) = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}. \tag{6.5.11}$$

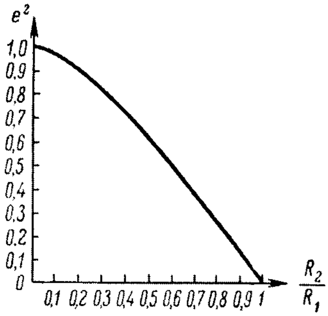

FIGURE 5.5.

Turning now to eqs. (6.3.3), we obtain by means of eq. (6.2.23)

$$P + \frac{\pi G}{3(1-\nu)\,\omega_0(1)} \frac{a^3}{R_1}\left(1 + \frac{1-e^2}{R_2} R_1\right) = \frac{2\pi a G}{(1-\nu)\,\omega_0(1)}\delta, \tag{6.5.12}$$

where $\omega_0(1) = K(e)$ and values P, R_1, R_2, δ are independent of a and e^2. Taking into account the relationships

$$2\frac{dK}{de^2} = \frac{B(e)}{1-e^2}, \quad B(e) + D(e) = K(e)$$

and taking derivatives of eq. (6.5.12) with respect to a and e^2 we find

$$\frac{a^2}{R_1}\left(1 + \frac{1-e^2}{R_2} R_1\right) = 2\delta, \quad \frac{D(e)}{R_2} = \frac{B(e)}{(1-e^2)\,R_1}. \tag{6.5.13}$$

Here the second equality is coincident with (6.5.6) while the first one yields

$$\frac{a^2}{R_1} = 2\delta\frac{D(e)}{K(e)}. \tag{6.5.14}$$

It is easy to obtain relationships (6.5.5) from this equation and eq. (6.5.12).

The integrands in formulae (6.2.24) are odd with respect to y and x respectively. Thus the expressions for the moments take the form

$$m_1 = \frac{2\pi a^3 G}{3(1-\nu)\,\omega^2(1)}\beta_x, \quad m_2 = \frac{2\pi a^3 G}{3(1-\nu)\,\omega_1(1)}\beta_y, \tag{6.5.15}$$

which is formally coincident with eq. (6.5.3). However it is necessary to bear in mind that a^2 is not prescribed now.

5.6.6 *Displacements and stresses*

Now we have, see also eq. (5.6.18)

$$\frac{\partial\omega}{\partial x} = -\frac{3Qx}{a^3}\omega_1(\rho), \quad \frac{\partial\omega}{\partial y} = -\frac{3Qy}{a^3}\omega_2(\rho), \quad \frac{\partial\omega}{\partial z} = -\frac{3Qz}{a^3}\omega_3(\rho). \tag{6.6.1}$$

Using eq. (6.4.4) we find

$$\left.\begin{aligned}\frac{\partial\tilde{\omega}}{\partial x} &= -\frac{3Qx}{a^3}\int\limits_{\rho}^{\infty}\frac{d\lambda}{\lambda^2\Delta(\lambda)}\left[z - z(l)\right],\\ \frac{\partial\tilde{\omega}}{\partial y} &= -\frac{3Qy}{a^3}\int\limits_{\rho}^{\infty}\frac{d\lambda}{\left(\lambda^2 - e^2\right)\Delta(\lambda)}\left[z - z(l)\right],\end{aligned}\right\} \tag{6.6.2}$$

where $\omega(x, y, z) = \dfrac{\partial\tilde{\omega}}{\partial z}$ and the expressions for the displacements, due to eqs. (6.2.3) and (6.2.4), are set in the form

$$\left.\begin{aligned}u &= \frac{3Qx}{4\pi Ga^3}\left[2(1-\nu)z\omega_1(\rho) - (1-2\nu)\int\limits_{\rho}^{\infty}\frac{z(\lambda)\,d\lambda}{\lambda^2\Delta(\lambda)}\right],\\ v &= \frac{3Qy}{4\pi Ga^3}\left[2(1-\nu)z\omega_2(\rho) - (1-2\nu)\int\limits_{\rho}^{\infty}\frac{z(\lambda)\,d\lambda}{\left(\lambda^2-e^2\right)\Delta(\lambda)}\right],\\ w &= \frac{3Q(1-\nu)}{4\pi Ga}\omega + \frac{3Qz^2}{4\pi Ga^3}\omega_3(\rho),\end{aligned}\right\} \tag{6.6.3}$$

where $z(\lambda)$ is given by eq. (6.4.3) and the integrals

$$\int\limits_{\rho}^{\infty}\frac{z(\lambda)\,d\lambda}{\lambda^2\Delta(\lambda)} = a\int\limits_{\rho}^{\infty}\frac{\gamma(x,y,\lambda)\,d\lambda}{\lambda^2\sqrt{\lambda^2-e^2}}, \quad \int\limits_{\rho}^{\infty}\frac{z(\lambda)\,d\lambda}{\left(\lambda^2-e^2\right)\Delta(\lambda)} = a\int\limits_{\rho}^{\infty}\frac{\gamma(x,y,\lambda)\,d\lambda}{\left(\lambda^2-e^2\right)^{3/2}}$$

are expressed in terms of the elementary functions.

When the displacements are obtained we find the stresses. Let us show only the results for the centre of the contact area and the contour.

At the centre

$$\sigma_x = -\frac{3}{2}p_m\frac{2\nu a + b}{a+b}, \quad \sigma_y = -\frac{3}{2}p_m\frac{2\nu b + a}{a+b}, \quad \sigma_z = -\frac{3}{2}p_m, \tag{6.6.4}$$

where $b = a\sqrt{1-e^2}$. On the contour

$$\left.\begin{aligned}\sigma_x = -\sigma_y &= -\frac{1-2\nu}{e^2}\frac{3}{2}p_m\sqrt{1-e^2}\times\\ &\qquad\left[1 - \frac{x}{2ae}\ln\frac{a+ex}{a-ex} - \frac{y}{ae}\arctan\frac{ey}{a(1-e^2)}\right],\\ \tau_{xy} &= -\frac{1-2\nu}{e^2}\frac{3}{2}p_m\sqrt{1-e^2}\frac{xy}{a^2}\left[\frac{x}{2ae}\ln\frac{a+ex}{a-ex} - \frac{y}{ae}\arctan\frac{ey}{a(1-e^2)}\right],\\ \sigma_z &= \tau_{xz} = \tau_{yz} = 0.\end{aligned}\right\} \tag{6.6.5}$$

In particular, at the ends of the major and minor axes of the ellipse, we have respectively

$$\left.\begin{aligned} \sigma_x = -\sigma_y = -(1-2\nu)\frac{\sqrt{1-e^2}}{e^2}\frac{3}{2}p_m\left(1-\frac{1}{2e}\ln\frac{1+e}{1-e}\right) \\ (x=a,\ y=0), \\ \sigma_x = -\sigma_y = -(1-2\nu)\frac{\sqrt{1-e^2}}{e^2}\frac{3}{2}p_m\left(1-\frac{\sqrt{1-e^2}}{e}\arctan\frac{e}{\sqrt{1-e^2}}\right) \\ (x=0,\ y=b). \end{aligned}\right\} \tag{6.6.6}$$

5.6.7 Contact of two surfaces

Two bodies bounded by convex surfaces S_1, S_2 and contacting at point O are considered. Taking this point as the origin of the coordinate system we direct axes z_1, z_2 which are perpendicular to the common tangent plane Π to surfaces S_1 and S_2 at point O into each of the bodies. Axes x_1, y_1 (x_2, y_2) of system Ox_1y_1z (Ox_2y_2z) of the first (second) body are directed in plane Π along the principal normal sections of surface S_1 (S_2). In the neigbourhood of point O the equations for surfaces S_1, S_2 in terms of these axes are presented in the form

$$z_1 = \frac{x_1^2}{2R_1'} + \frac{y_1^2}{2R_2'} + \ldots, \qquad z_2 = \frac{x_2^2}{2R_1''} + \frac{y_2^2}{2R_2''} + \ldots, \tag{6.7.1}$$

where $1/R_1', 1/R_2'$ denote the principal curvatures of surface S_1 which are positive if the corresponding centre of curvature locates within the body, i.e. on the positive axis z. By analogy, $1/R_1'', 1/R_2''$ are introduced for surface S_2. In what follows we limit the study to considering local effects in the contact area which allows us to retain in eq. (6.7.1) only the shown terms.

The distance between two points M_1 and M_2 of surfaces S_1 and S_2 lying on the same perpendicular to plane Π is equal to

$$z = z_1 + z_2 = \frac{x_1^2}{2R_1'} + \frac{y_1^2}{2R_2'} + \frac{x_2^2}{2R_1''} + \frac{y_2^2}{2R_2''}, \tag{6.7.2}$$

and clearly $z > 0$. Figure 5.6 shows two possible arrangements of surfaces S_1 and S_2, namely the cases of their external and internal contacts.

The further consideration is aimed at representing z in the form

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}. \tag{6.7.3}$$

To this end, a new system of axes Oxy is introduced. Denoting the angles between axes x_1 and x_2 with axis x by ω_1 and ω_2, respectively, and using

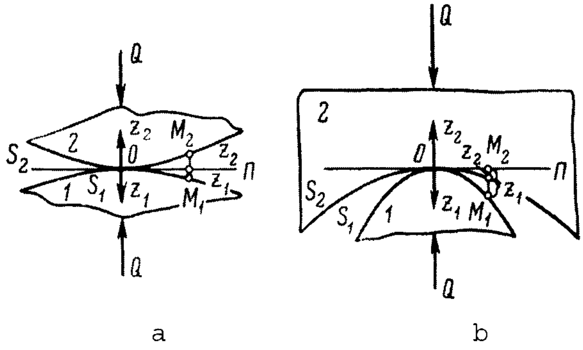

FIGURE 5.6.

the coordinate transformation we obtain

$$x_1 = x\cos\omega_1 + y\sin\omega_1, \quad y_1 = -x\sin\omega_1 + y\cos\omega_1,$$
$$x_2 = x\cos\omega_2 + y\sin\omega_2, \quad y_2 = -x\sin\omega_2 + y\cos\omega_2$$

and the expression for z takes the form

$$z = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + \frac{1}{2}xy\left(g_1\sin 2\omega_1 + g_2\sin 2\omega_2\right), \tag{6.7.4}$$

where

$$\left.\begin{aligned} \frac{1}{R_1} &= \frac{\cos^2\omega_1}{R_1'} + \frac{\sin^2\omega_1}{R_2'} + \frac{\cos^2\omega_2}{R_1''} + \frac{\sin^2\omega_2}{R_2''}, \\ \frac{1}{R_2} &= \frac{\sin^2\omega_1}{R_1'} + \frac{\cos^2\omega_1}{R_2'} + \frac{\sin^2\omega_2}{R_1''} + \frac{\cos^2\omega_2}{R_2''}, \\ g_1 &= \frac{1}{R_1'} - \frac{1}{R_2'}, \quad g_2 = \frac{1}{R_1''} - \frac{1}{R_2''}. \end{aligned}\right\} \tag{6.7.5}$$

Let α denote the angle between axes x_1 and x_2, then

$$\alpha = \omega_2 - \omega_1 \tag{6.7.6}$$

and introduce the mean curvatures of surfaces S_1 and S_2 at point O

$$2H_1 = \frac{1}{R_1'} + \frac{1}{R_2'}, \quad 2H_2 = \frac{1}{R_1''} + \frac{1}{R_2''}, \tag{6.7.7}$$

so that

$$\frac{1}{R_1} + \frac{1}{R_2} = 2\left(H_1 + H_2\right). \tag{6.7.8}$$

Let us take the value

$$\omega = \frac{1}{2}\left(\omega_1 + \omega_2\right) = \omega_1 + \frac{\alpha}{2} = \omega_2 - \frac{\alpha}{2} \tag{6.7.9}$$

in such a way that the term with xy in eq. (6.7.4) vanishes

$$g_1 \sin 2\omega_1 + g_2 \sin 2\omega_2 = (g_1 + g_2) \sin 2\omega \cos\alpha - (g_1 - g_2) \cos 2\omega \sin\alpha = 0. \tag{6.7.10}$$

By virtue of eqs. (6.7.5) and (6.7.9) we have

$$\left.\begin{aligned} \frac{1}{R_1} &= H_1 + H_2 + \frac{1}{2}\left[(g_1 + g_2)\cos 2\omega \cos\alpha + (g_1 - g_2)\sin 2\omega \sin\alpha\right], \\ \frac{1}{R_2} &= H_1 + H_2 - \frac{1}{2}\left[(g_1 + g_2)\cos 2\omega \cos\alpha + (g_1 - g_2)\sin 2\omega \sin\alpha\right]. \end{aligned}\right\} \tag{6.7.11}$$

From eq. (6.7.10) and the first equation in (6.7.11) we obtain

$$\left.\begin{aligned} \cos 2\omega &= \frac{2}{\Delta^2}\left(\frac{1}{R_1} - H_1 - H_2\right)(g_1 + g_2)\cos\alpha, \\ \sin 2\omega &= \frac{2}{\Delta^2}\left(\frac{1}{R_1} - H_1 - H_2\right)(g_1 - g_2)\sin\alpha, \end{aligned}\right\} \tag{6.7.12}$$

where

$$\Delta = \left(g_1^2 + g_2^2 + 2g_1g_2\cos 2\alpha\right)^{1/2}. \tag{6.7.13}$$

We determine now $1/R_1$ from the condition $\sin^2 2\omega + \cos^2 2\omega = 1$ and then $1/R_2$ is determined from eq. (6.7.8)

$$\frac{1}{R_1} = H_1 + H_2 - \frac{1}{2}\Delta, \quad \frac{1}{R_2} = H_1 + H_2 + \frac{1}{2}\Delta, \tag{6.7.14}$$

where R_1 denotes the larger of the two values R_1 and R_2. Using eq. (6.7.12) yields

$$\cos 2\omega = -\frac{1}{\Delta}(g_1 + g_2)\cos\alpha, \quad \sin 2\omega = -\frac{1}{2}(g_1 - g_2)\sin\alpha. \tag{6.7.15}$$

Thus the system of axes Oxy is determined which enables us to represent quadratic form (6.7.4) as a sum of two squares (6.7.3). Also the factors of this form ($1/2R_1$ and $1/2R_2$) are determined, both being positive as $z > 0$ for any values of variables x and y.

In the particular case of the surfaces of revolution with the parallel axes under the external (Fig. 5.7a) and internal (Fig. 5.7b) contact, we have $\alpha = 0$ and by eqs. (6.7.13) and (6.7.5)

$$\Delta = \left|\frac{1}{R_1'} - \frac{1}{R_2'} + \frac{1}{R_1''} - \frac{1}{R_2''}\right|.$$

For

$$\Delta = \frac{1}{R_2'} - \frac{1}{R_1'} + \frac{1}{R_2''} - \frac{1}{R_1''}$$

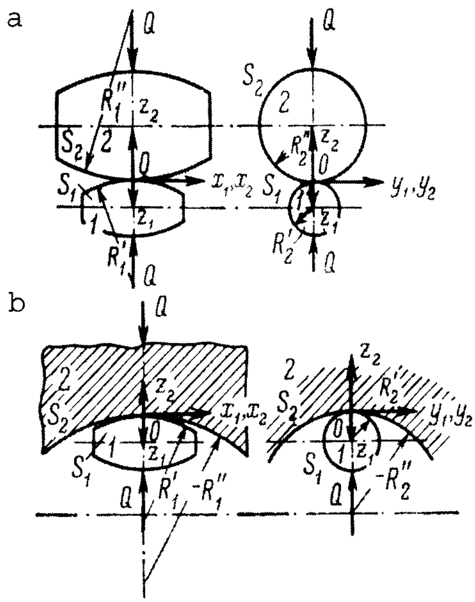

FIGURE 5.7.

we have

$$\omega = 0, \quad \frac{1}{R_1} = \frac{1}{R'_1} + \frac{1}{R''_1}, \quad \frac{1}{R_2} = \frac{1}{R'_2} + \frac{1}{R''_2}. \tag{6.7.16}$$

If

$$\Delta = \frac{1}{R'_1} - \frac{1}{R'_2} + \frac{1}{R''_1} - \frac{1}{R''_2},$$

then

$$\omega = \frac{\pi}{2}, \quad \frac{1}{R_1} = \frac{1}{R'_2} + \frac{1}{R''_2}, \quad \frac{1}{R_2} = \frac{1}{R'_1} + \frac{1}{R''_1} \tag{6.7.17}$$

and in both cases $R_1 > R_2$.

The case of the surfaces of revolution with the perpendicular axes is shown in Fig. 5.8a and Fig. 5.8b for the external and internal contact, respectively. Now $\alpha = \pi/2$ and

$$\Delta = |g_1 - g_2| = \left| \frac{1}{R'_1} - \frac{1}{R'_2} - \frac{1}{R''_1} + \frac{1}{R''_2} \right|$$

and for

$$\frac{1}{R'_1} - \frac{1}{R'_2} - \frac{1}{R''_1} + \frac{1}{R''_2} \gtrless 0$$

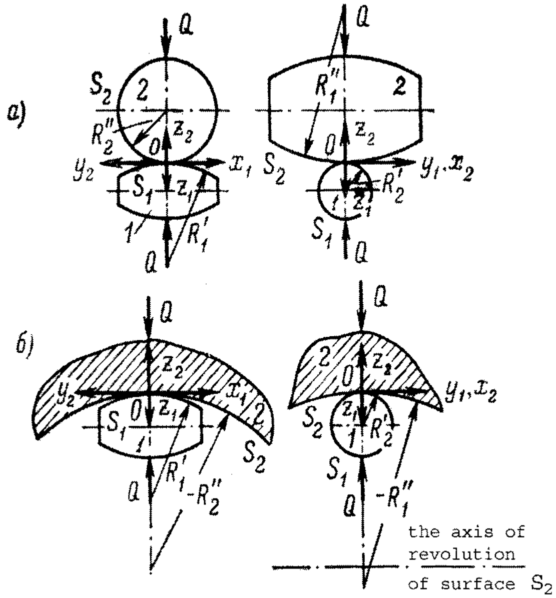

FIGURE 5.8.

we have respectively

$$\omega = \frac{3\pi}{4}, \quad \frac{1}{R_1} = \frac{1}{R_2'} + \frac{1}{R_1''}, \quad \frac{1}{R_2} = \frac{1}{R_1'} + \frac{1}{R_2''}, \tag{6.7.18}$$

$$\omega = \frac{\pi}{4}, \quad \frac{1}{R_1} = \frac{1}{R_1'} + \frac{1}{R_2''}, \quad \frac{1}{R_2} = \frac{1}{R_2'} + \frac{1}{R_1''}. \tag{6.7.19}$$

Also the case of contact of the surfaces of revolution at a point on the axis of revolution z is of interest. In this case

$$\frac{1}{R_1'} = \frac{1}{R_2'} = \frac{1}{R'}, \quad \frac{1}{R_1''} = \frac{1}{R_2''} = \frac{1}{R''}.$$

Angle α is arbitrary and $g_1 = g_2 = 0, \Delta = 0$. For the external contact we have

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R'} + \frac{1}{R''}, \tag{6.7.20}$$

and this formula remains valid for the internal contact, however the larger of the absolute values R', R'' is negative.

For example, for two spheres of radii R' and R'' making contacting outside we have by eq. (6.7.20)

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R'} + \frac{1}{R''},$$

whereas for a sphere of radius R' in a spherical cavity of radius R''

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R'} - \frac{1}{R''}.$$

When two cylinders with the perpendicular axes are contacting, then

$$\frac{1}{R_1} = \frac{1}{a}, \quad \frac{1}{R_2} = \frac{1}{b} \quad (a > b).$$

In the case of a sphere S_1 of radius R in a cylindrical groove S_2 of radius $r > R$

$$\frac{1}{R_1} = \frac{1}{R} - \frac{1}{r}, \quad \frac{1}{R_2} = \frac{1}{R},$$

axis x being directed perpendicular to the generator of the cylinder.

5.6.8 *Hertz's problem in the compression of elastic bodies*

Two bodies are pressed against each other by forces Q whose line of action is perpendicular to the joint tangent plane Π of bodies' surfaces S_1 and S_2 and intersects it at point O. Under the action of forces Q the bodies are deformed in the region adjacent to the place of contact and are brought closer together. Let $-\delta_1$ and $-\delta_2$ denote respectively the projections of the translation displacement of the first and second bodies on axes z_1 and z_2, each axis being directed into the corresponding body. One can also determine δ_1 and δ_2 as the displacements of the points of the first and second bodies provided that these points are well away from the contact place. The value

$$\delta = \delta_1 + \delta_2 \tag{6.8.1}$$

is referred to as the approach.

Let us consider two points M_1 and M_2 of the first and second bodies, respectively, which lie on the joint perpendicular to plane Π in the region adjacent to the contact place. In the system of axes $Oxyz_1$ and $Oxyz_2$ introduced in Subsection 5.6.7 the coordinates of these points before the deformations are (z_1, x, y) and (z_2, x, y). When the bodies are deformed, the projections of displacements of points M_1 and M_2 on axes z_1 and z_2 are w_1 and w_2, respectively. Simultaneously points M_1 and M_2 move together with the bodies and take positions M_1' and M_2', respectively. Thus, after the deformation the coordinates w_1' and w_2' of points will be as follows

$$z_1' = z_1 + w_1 - \delta_1, \quad z_2' = z_2 + w_2 - \delta_2 \tag{6.8.2}$$

and distance $M_1'M_2'$ we be equal to

$$z' = z_1' + z_2' = z_1 + z_2 + w_1 + w_2 - (\delta_1 + \delta_2) \tag{6.8.3}$$

or by eqs. (6.8.1) and (6.7.3)

$$z' = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + w_1 + w_2 - \delta. \tag{6.8.4}$$

For those points M_1 and M_2 of the first and second bodies which will be in contact after the deformation, this distance will be zero whereas this distance is positive for the points in the neigbourhood of the contact place. The surface of the contact is can therefore be determined as a locus where

$$z' = 0, \quad w_1 + w_2 = \delta - \frac{x^2}{R_1} - \frac{y^2}{R_2}, \tag{6.8.5}$$

whilst for the points outside the contact surface

$$z' > 0, \quad w_1 + w_2 > \delta - \frac{x^2}{R_1} - \frac{y^2}{R_2}. \tag{6.8.6}$$

There is a normal stress $p(x, y)$ on the contact surface while the shear stresses are assumed to be absent. Furthermore, it is assumed that when considering the local effects in the vicinity of the contact the contacting bodies can be replaced by two elastic half-spaces pressed against each other on the plane surface Ω lying in plane Π which is the tangent plane of surfaces S_1 and S_2 at point O and separates the half-spaces. On this plane $z_1 = z_2 = 0$. Similar to Subsection 5.6.5 the contact surface is the area within the ellipse

$$E_0: \quad 1 - \frac{x^2}{a^2} - \frac{y^2}{a^2(1-e^2)} = 0, \tag{6.8.7}$$

whose axes x and y are defined in Subsection 5.6.7. Pressure $p(x, y)$ is assumed to be absent on E_0.

The state of stress in each of the half-spaces is determined by means of function $\omega_i(x, y, z_i)$ $(i = 1, 2)$ which is a simple layer potential distributed over area Ω with intensity $p(x, y)$. Due to eq. (6.2.1) we have

$$\omega_i(x, y, z_i) = \iint\limits_{\Omega} \frac{p(x', y')\, do'}{\left[(x - x')^2 + (y - y')^2 + z_i^2\right]^{1/2}}, \tag{6.8.8}$$

and the consideration can be limited to the single potential

$$\omega(x, y, z) = \iint\limits_{\Omega} \frac{p(x', y')\, do'}{\left[(x - x')^2 + (y - y')^2 + z_i^2\right]^{1/2}}, \tag{6.8.9}$$

assuming z to be positive in each of the half-spaces. When calculating the displacements by formulae (6.2.3) and (6.2.4) it is necessary to take the corresponding values G_1, ν_1 or G_2, ν_2.

In accordance with eq. (6.2.4) on the contact area

$$w_1 = \frac{1-\nu_1}{2\pi G_1}\omega(x,y,0), \quad w_2 = \frac{1-\nu_2}{2\pi G_2}\omega(x,y,0), \tag{6.8.10}$$

and thus by eq. (6.8.5) on Ω

$$\omega = \frac{2\pi}{\vartheta_1+\vartheta_2}\left(\delta - \frac{x^2}{2R_1} - \frac{y^2}{2R_2}\right), \tag{6.8.11}$$

where

$$\vartheta_i = \frac{1-\nu_i}{G_i} \quad (i=1,2). \tag{6.8.12}$$

Potential ω is determined by condition (6.8.11) and the requirement of vanishing density $p(x,y)$ on contour E_0 of area Ω. This problem differs from the problem of Subsection 5.6.5 on the action of a non-plane die on the elastic half-space only in replacing

$$\frac{1-\nu}{G} \quad \text{by} \quad \vartheta_1+\vartheta_2 = \frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}. \tag{6.8.13}$$

For this reason, the results of the problem of die are immediately applicable to the problem of the elastic bodies pressed against each other by forces Q. A way of solving the problem is as follows:

1) The values $1/R_1$ and $1/R_2$ and the directions of axes x, y (i.e. angle ω) are determined by means of eqs. (6.7.14) and (6.7.15) in terms of the given curvatures $(1/R_1', 1/R_2')$, $(1/R_1'', 1/R_2'')$ of surfaces S_1, S_2 of the bodies contacting at the point of tangency O and angle α.

2) The eccentricity of the contact area ε is determined due to eq. (6.5.6).

3) The major semi-axis of the ellipse a and the approach δ are found by the formulae

$$a = \left[QR_1(\vartheta_1+\vartheta_2)\right]^{1/3}\alpha_a, \quad \delta = \left[\frac{Q}{\sqrt{R_1}}(\vartheta_1+\vartheta_2)\right]^{2/3}\alpha_\delta, \tag{6.8.14}$$

where the functions of the eccentricity α_a, α_δ are given by eq. (6.5.8).

4) The displacements are determined by formulae (6.6.3) by replacing ν, G with ν_i, G_i $(i=1,2)$. Clearly, when calculating the stresses by means of the formulae of Subsection 5.6.6 it is necessary to replace ν by ν_i.

5.7 Equilibrium of an elastic circular cylinder

5.7.1 Differential equation of equilibrium of a circular cylinder

In what follows we restrict ourselves to considering the cases of axially symmetric and bending deformation of a cylinder. In the first case the axial w, radial u and circumferential v displacements are functions of the cylindrical coordinates r and z. For the bending deformation, components w and u are taken to be proportional to $\cos\varphi$ whereas v is proportional to $\sin\varphi$, with φ denoting the azimuthal angle. The general case of proportionality to $\cos n\varphi$ and $\sin n\varphi$ is not considered here. Instead of r, z we introduce the following non-dimensional coordinates

$$x = \frac{r}{a}, \quad \zeta = \frac{z}{a}, \tag{7.1.1}$$

where a denotes the external radius of the cylinder. For a hollow cylinder of length $2l$ and internal radius b we have

$$\frac{b}{a} = x_1 \leq x < 1, \quad -L \leq \zeta \leq L = \frac{l}{a}. \tag{7.1.2}$$

As already mentioned in Subsection 4.1.10 the axially symmetric case is split into the problem of the meridional deformation and the problem of torsion. The solution of the first can be expressed in terms of three Papkovich-Neuber's functions. Keeping the notation of Subsections 4.1.12 and 4.1.13 we denote these functions by b_0 and b_3 (they are harmonic), The third function is denoted by b_r and the product $b_r e^{i\varphi}$ is a harmonic function

$$\nabla^2 b_0 = 0, \quad \nabla^2 b_3 = 0, \quad \nabla^2 b_r - \frac{b_r}{x^2} = 0, \tag{7.1.3}$$

where eq. (7.1.1) is used

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial \zeta^2}. \tag{7.1.4}$$

By eqs. (1.12.16) and (1.13.3) of Chapter 4 these displacements are expressed in terms of these functions by the formulae

$$\left.\begin{aligned} u &= a\left[-\frac{\partial b_0}{\partial x} + \zeta\frac{\partial b_3}{\partial x} + (3-4\nu)\, b_r - x\frac{\partial b_r}{\partial x}\right], \\ w &= a\left[(3-4\nu)\, b_3 - \frac{\partial b_0}{\partial \zeta} + \zeta\frac{\partial b_3}{\partial \zeta} - x\frac{\partial b_r}{\partial \zeta}\right], \end{aligned}\right\} \tag{7.1.5}$$

$(n = 0)$ and the non-trivial stresses given by eqs. (1.12.13) and (1.13.5) of Chapter 4 are equal to

$$\left.\begin{aligned} \frac{1}{2G}\sigma_r &= 2\nu\frac{\partial b_3}{\partial \zeta} - \left(\frac{\partial^2 b_0}{\partial x^2} + \zeta\frac{\partial^2 b_3}{\partial x^2}\right) + 2(1-\nu)\frac{\partial b_r}{\partial x} + 2\nu\frac{b_r}{x} - x\frac{\partial^2 b_r}{\partial x^2}, \\ \frac{1}{2G}\sigma_z &= 2(1-\nu)\frac{\partial b_3}{\partial \zeta} - \left(\frac{\partial^2 b_0}{\partial \zeta^2} + \zeta\frac{\partial^2 b_3}{\partial \zeta^2}\right) + 2\nu\left(\frac{\partial b_r}{\partial x} + \frac{b_r}{x}\right) - x\frac{\partial^2 b_r}{\partial \zeta^2}, \\ \frac{1}{2G}\sigma_\varphi &= 2\nu\frac{\partial b_3}{\partial \zeta} - \frac{1}{x}\left(\frac{\partial b_0}{\partial x} + \zeta\frac{\partial b_3}{\partial x}\right) - (1-2\nu)\frac{\partial b_r}{\partial x} + (3-2\nu)\frac{b_r}{x}, \\ \frac{1}{2G}\tau_{rz} &= (1-2\nu)\frac{\partial b_3}{\partial x} - \left(\frac{\partial^2 b_0}{\partial x\partial \zeta} + \zeta\frac{\partial^2 b_3}{\partial x\partial \zeta}\right) + (1-2\nu)\frac{\partial b_r}{\partial \zeta} - x\frac{\partial^2 b_r}{\partial \zeta\partial x}. \end{aligned}\right\} \tag{7.1.6}$$

Only displacement v does not vanish in the problem of torsion. Using eq. (1.11.3) of Chapter 4, the non-trivial stresses $\tau_{r\varphi}$ and $\tau_{z\varphi}$ are given by the formulae

$$\frac{a}{G}\tau_{r\varphi} = x\frac{\partial}{\partial x}\frac{v}{x}, \quad \frac{a}{G}\tau_{z\varphi} = \frac{\partial v}{\partial \zeta}, \tag{7.1.7}$$

where $ve^{i\varphi}$ is a harmonic function, i.e.

$$\nabla^2 v - \frac{v}{x^2} = 0. \tag{7.1.8}$$

In the case of bending deformation one introduces four functions b_0, b_3, b_r and b_φ. Functions $b_0 e^{i\varphi}$ and $b_3 e^{i\varphi}$ are harmonic. Due to eq. (1.13.4) of Chapter 4 the differential equations for functions b_r and b_φ are ($n = 1$)

$$\nabla^2 b_r - \frac{2b_r}{x^2} - \frac{2b_\varphi}{x^2} = 0, \quad \nabla^2 b_\varphi - \frac{2b_\varphi}{x^2} - \frac{2b_r}{x^2} = 0$$

and their half-sum and half-difference are as follows

$$p = \frac{1}{2}\left(b_r + b_\varphi\right), \quad q = \frac{1}{2}\left(b_r - b_\varphi\right), \quad b_r = p + q, \quad b_\varphi = p - q. \tag{7.1.9}$$

These functions satisfy the differential equations

$$\nabla^2 p - \frac{4}{x^2} p = 0, \quad \nabla^2 q = 0, \quad \nabla^2 b_0 - \frac{b_0}{x^2} = 0, \quad \nabla^2 b_3 - \frac{b_3}{x^2} = 0. \tag{7.1.10}$$

By virtue of eqs. (1.12.16) and (1.13.3) of Chapter 4 the expressions for the displacements reduce to the form

$$\left.\begin{aligned}
u &= a\cos\varphi\left[-\left(\frac{\partial b_0}{\partial x} + \zeta\frac{\partial b_3}{\partial x}\right) + (3 - 4\nu)(p + q) - x\frac{\partial (p+q)}{\partial x}\right], \\
w &= a\cos\varphi\left[(3 - 4\nu) b_3 - \left(\frac{\partial b_0}{\partial \zeta} + \zeta\frac{\partial b_3}{\partial \zeta}\right) - x\frac{\partial (p+q)}{\partial x}\right], \\
v &= a\sin\varphi\left[\frac{1}{x}\left(b_0 + \zeta b_3\right) + (5 - 4\nu) p - (3 - 4\nu) q\right].
\end{aligned}\right\} \tag{7.1.11}$$

The stresses are given by eqs. (1.12.13), (1.12.14) and (1.13.5) and (1.13.16) of Chapter 4

$$
\left.
\begin{aligned}
\frac{\sigma_r}{2G} &= \cos\varphi\left[2\nu\frac{\partial b_3}{\partial\zeta} - \left(\frac{\partial^2 b_0}{\partial x^2} + \zeta\frac{\partial^2 b_3}{\partial x^2}\right) + 2(1-\nu)\frac{\partial(p+q)}{\partial x} - \right.\\
&\qquad\left. x\frac{\partial^2(p+q)}{\partial x^2} + 4\nu\frac{p}{x}\right],\\
\frac{\sigma_z}{2G} &= \cos\varphi\left[2\nu 2(1-\nu)\frac{\partial b_3}{\partial\zeta} - \left(\frac{\partial^2 b_0}{\partial\zeta^2} + \zeta\frac{\partial^2 b_3}{\partial\zeta^2}\right) + \right.\\
&\qquad\left. 2\nu\frac{\partial(p+q)}{\partial x} - x\frac{\partial^2(p+q)}{\partial\zeta^2} + 4\nu\frac{p}{x}\right],\\
\frac{\sigma_\varphi}{2G} &= \cos\varphi\left[2\nu\frac{\partial b_3}{\partial\zeta} - \frac{1}{x}\left(\frac{\partial b_0}{\partial x} + \zeta\frac{\partial b_3}{\partial x}\right) + \frac{1}{x^2}(b_0+\zeta b_3) - \right.\\
&\qquad\left. (1-2\nu)\frac{\partial(p+q)}{\partial x} + 4(2-\nu)\frac{p}{x}\right],\\
\frac{\tau_{rz}}{2G} &= \cos\varphi\left[(1-2\nu)\frac{\partial b_3}{\partial x} - \left(\frac{\partial^2 b_0}{\partial x\partial\zeta} + \zeta\frac{\partial^2 b_3}{\partial x\partial\zeta}\right) + \right.\\
&\qquad\left. (1-2\nu)\frac{\partial(p+q)}{\partial\zeta} - x\frac{\partial^2(p+q)}{\partial\zeta\partial x}\right],\\
\frac{\tau_{r\varphi}}{2G} &= \sin\varphi\left[\frac{1}{x}\left(\frac{\partial b_0}{\partial x} + \zeta\frac{\partial b_3}{\partial x}\right) - \frac{1}{x^2}(b_0+\zeta b_3) + \right.\\
&\qquad\left. (3-2\nu)\frac{\partial p}{\partial x} - 4(1-\nu)\frac{p}{x} - (1-2\nu)\frac{\partial q}{\partial x}\right],\\
\frac{\tau_{z\varphi}}{2G} &= \sin\varphi\left[-(1-2\nu)\frac{b_3}{x} + \frac{1}{x}\left(\frac{\partial b_0}{\partial\zeta} + \zeta\frac{\partial b_3}{\partial\zeta}\right) + \right.\\
&\qquad\left. (3-2\nu)\frac{\partial p}{\partial\zeta} - (1-2\nu)\frac{\partial q}{\partial\zeta}\right].
\end{aligned}
\right\}
\tag{7.1.12}
$$

In the axisymmetric case, the equations of statics in the cylindrical coordinates are given by eq. (1.10.3) of Chapter 4, namely, for meridional deformation

$$
\left.
\begin{aligned}
&\frac{\partial\sigma_r}{\partial x} + \frac{\sigma_r - \sigma_\varphi}{x} + \frac{\partial\tau_{rz}}{\partial\zeta} + \rho a K_r = 0,\\
&\frac{\partial\sigma_z}{\partial\zeta} + \frac{\partial\tau_{rz}}{\partial x} + \frac{\tau_{rz}}{x} + \rho a K_z = 0
\end{aligned}
\right\}
\tag{7.1.13}
$$

and for torsion

$$
\frac{\partial\tau_{r\varphi}}{\partial x} + \frac{\partial\tau_{z\varphi}}{\partial\zeta} + \frac{2\tau_{r\varphi}}{x} + \rho a K_\varphi = 0. \tag{7.1.14}
$$

Evidently, solutions (7.1.6) and thus (7.1.7) of Chapter 5 are presented in the form ($n = 1$)

$$\left.\begin{aligned}\frac{\partial \sigma_r}{\partial x} + \frac{\sigma_r - \sigma_\varphi}{x} + \frac{\partial \tau_{rz}}{\partial \zeta} + \frac{\tau'_{r\varphi}}{x} + \rho a K_r = 0,\\ \frac{\partial \sigma_z}{\partial \zeta} + \frac{\partial \tau_{rz}}{\partial x} + \frac{\tau_{rz}}{x} + \frac{\tau'_{z\varphi}}{x} + \rho a K_z = 0,\\ \frac{\partial \tau_{r\varphi}}{\partial x} + \frac{2\tau_{r\varphi}}{x} + \frac{\partial \tau_{z\varphi}}{\partial \zeta} - \frac{1}{x}\sigma'_\varphi + \rho a K_\varphi = 0,\end{aligned}\right\} \tag{7.1.15}$$

where a prime indicates that $\sin\varphi$ in the expressions for $\tau_{r\varphi}$ and $\tau_{z\varphi}$ should be replaced by $\cos\varphi$ whilst $\cos\varphi$ in the expression for σ_φ should be replaced by $\sin\varphi$.

In the axially symmetric case, the stresses distributed over the cross-section of the cylinder result in the axial force and the torque

$$Z = 2\pi a^2 \int_{x_1}^{1} \sigma_z x\,dx, \quad m_z = 2\pi a^3 \int_{x_1}^{1} \tau_{z\varphi} x^2\,dx. \tag{7.1.16}$$

With the help of the equilibrium equations (7.1.13) and (7.1.14) these values are easily expressed in terms of the stresses on the internal and external surfaces of the hollow cylinder

$$\left.\begin{aligned} Z &= Z_0 + 2\pi a^2 \int_0^{\zeta} \left[x_1 \left(\tau_{rz}\right)_{x=x_1} - \left(\tau_{rz}\right)_{x=1}\right] d\zeta,\\ m_z &= m_z^O + 2\pi a^3 \int_0^{\zeta} \left[x_1^2 \left(\tau_{r\varphi}\right)_{x=x_1} - \left(\tau_{r\varphi}\right)_{x=1}\right] d\zeta,\end{aligned}\right\} \tag{7.1.17}$$

where Z_0 and m_z^O denote the axial force and the torque in the section $\zeta = 0$.

Under bending deformation the stresses $\sigma_z, \tau_{rz}, \tau_{z\varphi}$ in the cross-section are statically equivalent to the transverse force X and the bending moment m_y about axis y in plane $z = 0$

$$X = \pi a^2 \int_{x_1}^{1} \left(\tau^*_{rz} - \tau^*_{z\varphi}\right) x\,dx, \quad m_y = a\zeta X - \pi a^3 \int_{x_1}^{1} x^2 \sigma^*_z\,dx, \tag{7.1.18}$$

where asterisks denote the factors in front of $\cos\varphi, \sin\varphi$ in the corresponding expressions for the stresses. Expressing X, m_y by means of the equilibrium equations (7.1.15) in terms of the stresses on surfaces $x = x_1, x = 1$

we arrive at the formulae

$$\left.\begin{aligned} X &= X_0 + \pi a^2 \int\limits_0^{\zeta} \left[x_1 \left(\sigma_r^* - \tau_{r\varphi}^*\right)_{x=x_1} - \left(\sigma_r^* - \tau_{r\varphi}^*\right)_{x=1}\right] d\zeta, \\ m_y &= m_y^O + \pi a^3 \int\limits_0^{\zeta} x_1 \left\{\left[\zeta \left(\sigma_r^* - \tau_{r\varphi}^*\right) - x_1 \tau_{rz}^*\right]_{x=x_1} - \right. \\ &\qquad\qquad \left. \left[\zeta \left(\sigma_r^* - \tau_{r\varphi}^*\right) - \tau_{rz}^*\right]_{x=1}\right\} d\zeta. \end{aligned}\right\} \quad (7.1.19)$$

These expressions are easily obtained by considering the equilibrium of a finite part of the cylinder between the cross-sections $\zeta = 0$ and $\zeta = \zeta$.

5.7.2 *Lamé's problem for a hollow cylinder*

We consider the axially symmetric problem of the state of stress in a hollow cylinder loaded by a normal pressure uniformly distributed over the lateral surface

$$x = 1: \quad \sigma_r = -p_0, \quad \tau_{rz} = 0; \quad x = x_1: \quad \sigma_r = -p_1, \quad \tau_{rz} = 0. \quad (7.2.1)$$

In the Papkovich-Neuber solution it is sufficient to keep a single function b_r and assume it to be independent of ζ. By eq. (7.1.3) we have

$$b_r'' + \frac{1}{x} b_r' - \frac{1}{x^2} b_r = 0,$$

so that

$$b_r = C_1 x + \frac{C_2}{x}. \quad (7.2.2)$$

For this reason

$$\frac{1}{2G}\sigma_r = 2C_1 - 4(1-\nu)\frac{C_2}{x^2}, \quad \frac{\sigma_\varphi}{2G} = 2C_1 + 4(1-\nu)\frac{C_2}{x^2}, \quad \frac{\sigma_z}{2G} = 4\nu C_1. \quad (7.2.3)$$

Determining C_1 and C_2 with the help of the boundary conditions (7.2.1) we obtain

$$\left.\begin{aligned} \sigma_r &= \frac{1}{\left(1 - x_1^2\right) x^2} \left[\left(x^2 - 1\right) p_1 x_1^2 - \left(x^2 - x_1^2\right) p_0\right], \\ \sigma_\varphi &= \frac{1}{\left(1 - x_1^2\right) x^2} \left[\left(x^2 + 1\right) p_1 x_1^2 - \left(x^2 + x_1^2\right) p_0\right], \\ \sigma_z &= 2\nu \frac{p_1 x_1^2 - p_0}{1 - x_1^2}, \quad \tau_{rz} = 0. \end{aligned}\right\} \quad (7.2.4)$$

Using eq. (7.1.15) we find the displacements

$$2Gu = \frac{a}{1-x_1^2}\left[(1-2\nu)\left(p_1x_1^2 - p_0\right)x + (p_1 - p_0)\frac{x_1^2}{x}\right], \quad w = 0. \quad (7.2.5)$$

The state of stress obtained is realised in an elastic cylinder subjected to a uniform pressure from both inside and outside and placed between two motionless rigid and smooth plates. The latter do not admit axial displacement of the particles on the faces of the cylinder (i.e. $w = 0$) however they do not prevent the radial displacement, i.e. $\tau_{rz} = 0$. The reaction forces of these walls result in the normal stress σ_z which is uniformly distributed over the face surfaces.

The case of the cylinder whose ends can freely move in the axial direction $(\sigma_z = 0, w \neq 0)$ is obtained by superimposing the uniform axial compression

$$\sigma_z^0 = -2\nu\frac{p_1x_1^2 - p_0}{1-x_1^2}$$

which is opposite in sign to σ_z on the obtained state of stress. The resulting displacement

$$u^0 = -\frac{\sigma_z^0 \nu a}{E}x, \quad w^0 = \frac{\sigma_z^0 a}{E}\zeta \quad (E = 2G(1+\nu)),$$

does not cause any additional stresses $\sigma_r, \sigma_\varphi, \tau_{rz}$. The solution of Lamé's problem for a cylinder with freely moving ends takes the final form

$$\left.\begin{aligned} 2Gu &= \frac{a}{1-x_1^2}\left[\frac{1-\nu}{1+\nu}\left(p_1x_1^2 - p_0\right)x + (p_1 - p_0)\frac{x_1^2}{x}\right], \\ 2Gw &= -\frac{2a\nu}{1+\nu}\frac{p_1x_1^2 - p_0}{1-x_1^2}\zeta \end{aligned}\right\} \quad (7.2.6)$$

with $\sigma_z = 0$ and the other stresses are given by formulae (7.2.4).

5.7.3 *Distortion in the hollow cylinder*

The statement of the problem is given in Subsection 4.5.6. Let us consider the simplest cases of the rotational distortion b_3 about axis Oz and a translational distortion c_2 in the direction $\mathbf{e}_\varphi$.

1. *Rotational distortion* b_3. The problem is to determine the state of stress $\hat{T}'$ which being superimposed on the state of stress (5.6.10) of Chapter 4 makes the surfaces $x = 1$ and $x = x_1$ of the hollow cylinder free of stress σ_r

$$\left.\begin{aligned} x = 1: \quad & \frac{1}{2G}\sigma_r' = -\frac{b_3}{4\pi}\frac{1}{1-2\nu}; \\ x = x_1: \quad & \frac{1}{2G}\sigma_r' = -\frac{b_3}{4\pi}\left(\frac{1}{1-2\nu} + \frac{\ln x_1}{1-\nu}\right). \end{aligned}\right\} \quad (7.3.1)$$

This is Lamé's problem in which

$$p_0 = \frac{Gb_3}{2\pi}\frac{1}{1-2\nu}, \quad p_1 = \frac{Gb_3}{2\pi}\left(\frac{1}{1-2\nu} + \frac{\ln x_1}{1-\nu}\right).$$

By eqs. (7.2.4) and (5.6.10) the stresses are as follows

$$\left.\begin{aligned}
\Sigma_r &= \sigma_r + \sigma_r' = \frac{b_3 G}{2\pi(1-\nu)}\left(\ln x - \frac{1-x^2}{1-x_1^2}\frac{x_1^2}{x^2}\ln x_1\right),\\
\Sigma_\varphi &= \sigma_\varphi + \sigma_\varphi' = \frac{b_3 G}{2\pi(1-\nu)}\left(1 + \ln x + \frac{1+x^2}{1-x_1^2}\frac{x_1^2}{x^2}\ln x_1\right),\\
\Sigma_z &= \sigma_z + \sigma_z' = \frac{b_3 G\nu}{2\pi(1-\nu)}\left(1 + 2\ln x + \frac{2x_1^2}{1-x_1^2}\ln x_1\right).
\end{aligned}\right\} \tag{7.3.2}$$

This state of stress occurs in a cylinder which is subjected to distortion b_3 and placed between two rigid and smooth plates. The latter do not admit the axial displacement w and produce stress Σ_z at the ends of the cylinder. Searching the state of stress for free ends requires superimposing an additional state which eliminates the existing stress Σ_z at the ends of the cylinder and causes no stresses in the cylindrical surfaces $x = 1$ and $x = x_1$. A closed form solution is apparently not feasible.

2. *Translational distortion* c_2. The state of stress (5.6.11) of Chapter 4 needs to be superimposed by the state of stress of bending determined by the boundary conditions

$$\left.\begin{aligned}
x = 1:& \quad \frac{\sigma_r'}{2G} = \frac{c_2}{2\pi}\cos\varphi, \quad \frac{\tau_{r\varphi}'}{2G} = -\frac{c_2}{2\pi}\sin\varphi,\\
x = x_1:& \quad \frac{\sigma_r'}{2G} = \frac{c_2}{2\pi x_1}\cos\varphi, \quad \frac{\tau_{r\varphi}'}{2G} = -\frac{c_2}{2\pi x_1}\sin\varphi.
\end{aligned}\right\} \tag{7.3.3}$$

Here we use functions p and q which do not depend upon ζ and satisfy the differential equations (7.1.10)

$$p'' + \frac{1}{x}p' - \frac{4}{x^2}p = 0, \quad q'' + \frac{1}{x}q' = 0.$$

Their particular solutions are as follows

$$p_1 = x^2, \quad p_2 = \frac{1}{x^2}, \quad q_1 = \ln x, \quad q_2 = 1,$$

and the stresses, due to eq. (7.1.12), are equal to

$$\begin{aligned}
\sigma_r' &= \left[A_1 x + \frac{A_2}{x^3} + A_3(3-2\nu)\frac{1}{x}\right]\cos\varphi,\\
\sigma_\varphi' &= \left[3A_1 x - \frac{A_2}{x^3} - A_3(1-2\nu)\frac{1}{x}\right]\cos\varphi,\\
\tau_{r\varphi}' &= \left[A_1 x + \frac{A_2}{x^3} - A_3(1-2\nu)\frac{1}{x}\right]\sin\varphi.
\end{aligned}$$

The integration constants are determined by the boundary conditions (7.3.3)

$$\text{for } x=1,\ x=x_1:\quad A_1x^2+\frac{A_2}{x^2}+A_3(3-2\nu)=\frac{c_2G}{\pi},$$
$$A_1x^2+\frac{A_2}{x^2}-A_3(1-2\nu)=-\frac{c_2G}{\pi}.$$

They yield

$$A_3=\frac{c_2G}{2\pi(1-\nu)},$$

and the stresses in the cylinder subjected to the distortion are written down as follows

$$\Sigma_r=\sigma_r+\sigma_r'=\left(A_1x+\frac{A_2}{x^3}+\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\right)\cos\varphi,$$
$$\Sigma_\varphi=\sigma_\varphi+\sigma_\varphi'=\left(3A_1x-\frac{A_2}{x^3}+\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\right)\cos\varphi,$$
$$T_{r\varphi}=\tau_{r\varphi}+\tau_{r\varphi}'=\left(A_1x+\frac{A_2}{x^3}+\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\right)\sin\varphi.$$

The factors in front of $\cos\varphi$ and $\sin\varphi$ in the expressions for Σ_r and $T_{r\varphi}$ are the same and it remains only to fulfill the remaining boundary conditions

$$A_1+A_2=-\frac{c_2G}{2\pi(1-\nu)},\quad A_1x_1^2+\frac{A_2}{x_1^2}=-\frac{c_2G}{2\pi(1-\nu)}.$$

The following stresses are obtained (see also eq. (7.1.4) of Chapter 7)

$$\left.\begin{aligned}
\Sigma_r&=\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\left(1-\frac{x^2+\dfrac{x_1^2}{x^2}}{1+x_1^2}\right)\cos\varphi,\\
\Sigma_\varphi&=\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\left(1-\frac{3x^2-\dfrac{x_1^2}{x^2}}{1+x_1^2}\right)\cos\varphi,\\
T_{r\varphi}&=\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\left(1-\frac{x^2+\dfrac{x_1^2}{x^2}}{1+x_1^2}\right)\sin\varphi,\\
\Sigma_z&=\frac{c_2G}{2\pi(1-\nu)}\frac{1}{x}\left(1-\frac{2x^2}{1+x_1^2}\right)\cos\varphi.
\end{aligned}\right\}\tag{7.3.4}$$

This state of stress occurs in an elastic cylinder which is subjected to translational distortion in the cross-section $\varphi=0$ and placed between two rigid and smooth plates preventing axial displacement. Here, as in the previous case, the closed form solution to the problem of annihilating stresses on the end faces of the cylinder is very difficult.

5.7.4 Polynomial solutions to the problem of equilibrium of the cylinder

Subsection 5.7.1 presents formulae for stresses and displacements in a cylinder subjected to axisymmetric deformation and bending deformation in terms of the harmonic functions of two sorts: axially symmetric functions (depending on x, ζ) and functions of x, ζ multiplied by $\varepsilon^{i\varphi}$. In the present subsection we suggest constructing these solutions in the form of homogeneous polynomials of x, ζ for the solid cylinder with terms containing singularities on axis $z = 0$ (for $x = 0$) for the case of the hollow cylinder.

1. Axially symmetric harmonic functions. In the case of a solid cylinder the harmonic polynomials in cylindrical coordinates take the form

$$\varphi_n (x, \zeta) = R^n P_n (\mu), \quad R^2 = x^2 + \zeta^2, \quad \mu = \cos\vartheta = \frac{\zeta}{R}, \tag{7.4.1}$$

where $P_n (\mu)$ are Legendre's polynomials (F.2.11). In particular, we have

$$\begin{gathered} \varphi_0 = 1, \quad \varphi_1 = \zeta, \quad \varphi_2 = \frac{1}{2} \left(2\zeta^2 - x^2\right), \quad \varphi_3 = \frac{1}{2} \left(2\zeta^3 - 3\zeta x^2\right), \\ \varphi_4 = \frac{1}{8} \left(8\zeta^4 - 24\zeta^2 x^2 + 3x^4\right), \quad \varphi_5 = \frac{1}{8}\zeta \left(8\zeta^4 - 40\zeta^2 x^2 + 15x^4\right) \text{ etc.} \end{gathered} \tag{7.4.2}$$

In the case of a hollow cylinder we add the solutions of the type

$$\psi_n (x, \zeta) = \varphi_n (x, \zeta) \ln x + \chi_n (x, \zeta), \tag{7.4.3}$$

where χ_n is determined from the condition

$$\nabla^2 \psi_n = 0, \quad \nabla^2 \chi_n = -\nabla^2 \varphi_n \ln x = -2\nabla\varphi_n \cdot \nabla \ln x.$$

Here we used the rule of taking the Laplace operator of the product and accounted for $\nabla^2 \varphi_n = 0, \nabla^2 \ln x = 0$. Returning to the spherical coordinates we have

$$\nabla\varphi_n = R^{n-1} \left(nP_n \mathbf{e}_R - \sin\vartheta P_n' \mathbf{e}_\vartheta\right), \quad \nabla \ln x = \frac{1}{R} \left(\mathbf{e}_R + \mathbf{e}_\vartheta \cot\vartheta\right).$$

Using the known recurrent relationship for Legendre's polynomials we obtain

$$\nabla^2 \chi_n = -2R^{n-2} \left(nP_n - \mu P_n'\right) = 2R^{n-2} P_{n-1}' (\mu).$$

Thus assuming

$$\chi_n = R^n S_n (\mu) \tag{7.4.4}$$

we arrive at the non-homogeneous Legendre's equation

$$\left[\left(1 - \mu^2\right) S_n' (\mu)\right]' + n (n+1) S_n (\mu) = 2P_{n-1}' (\mu).$$

The right hand side is a polynomials of μ^{n-2} which can be presented in terms of Legendre's polynomials as follows

$$\left.\begin{aligned}P'_{n-1}(\mu) &= (2n-3)P_{n-2}(\mu) + (2n-7)P_{n-4}(\mu) + \ldots \\ &\ldots + 9P_4(\mu) + 5P_2(\mu) + P_0(\mu) \quad (\text{n is even}), \\ P'_{n-1}(\mu) &= (2n-3)P_{n-2}(\mu) + (2n-7)P_{n-4}(\mu) + \ldots \\ &\ldots + 11P_5(\mu) + 7P_3(\mu) + 3P_1(\mu) \quad (\text{n is odd}).\end{aligned}\right\} \tag{7.4.5}$$

Setting Legendre's equation for polynomials $P_\nu(\mu)$ in the form

$$\left[\left(1-\mu^2\right)P'_\nu(\mu)\right]' + n(n+1)P_\nu(\mu) = \left[n(n+1) - \nu(\nu+1)\right]P_\nu(\mu)$$

and looking for $S_n(\mu)$ in the form

$$S_n(\mu) = \sum_0^{n-2} a_\nu P_\nu(\mu)$$

we arrive at the relationship

$$\sum_0^{n-2} a_\nu P_\nu(\mu)\left[n(n+1) - \nu(\nu+1)\right] = 2P'_{n-1}(\mu)$$

which allows one to determine a_ν. Using eq. (7.4.5) we find

$$\left.\begin{aligned}\frac{1}{2}S_n(\mu) &= \frac{2n-3}{n(n+1)-(n-2)(n-1)}P_{n-2}(\mu) + \\ &\frac{2n-7}{n(n+1)-(n-4)(n-3)}P_{n-4}(\mu) + \ldots + \frac{9}{n(n+1)-4\cdot 5}P_4(\mu) + \\ &\frac{5}{n(n+1)-2\cdot 3}P_2(\mu) + \frac{P_0(\mu)}{n(n+1)} \quad (\text{n is even}), \\ \frac{1}{2}S_n(\mu) &= \frac{2n-3}{n(n+1)-(n-2)(n-1)}P_{n-2}(\mu) + \\ &\frac{2n-7}{n(n+1)-(n-4)(n-3)}P_{n-4}(\mu) + \ldots + \frac{11}{n(n+1)-5\cdot 6}P_5(\mu) + \\ &\frac{7}{n(n+1)-3\cdot 4}P_3(\mu) + \frac{3}{n(n+1)-1\cdot 2}P_1(\mu) \quad (\text{n is odd}).\end{aligned}\right\} \tag{7.4.6}$$

By virtue of these formulae and eqs. (7.4.3), (7.4.4) we obtain

$$\psi_0 = \ln x, \quad \psi_1 = \zeta\ln x, \quad \psi_2 = \varphi_2\ln x + \frac{1}{3}\left(x^2+\zeta^2\right), \tag{7.4.7}$$

$$\psi_3 = \varphi_3\ln x + \frac{3}{5}\zeta\left(\zeta^2+x^2\right), \; \psi_4 = \varphi_4\ln x + \frac{3}{70}\left(x^2+\zeta^2\right)\left(19\zeta^2-6x^2\right) \text{ etc.}$$

2. Polynomial solutions proportional to $\cos\varphi$. In cylindric coordinates, the expressions for the harmonic polynomials proportional to $\cos\varphi$ have the appearance

$$R^n P_n^1(\mu)\cos\varphi = R^n\sin\vartheta\cos\varphi P'_n(\mu) = xR^{n-1}P'_n(\mu).$$

Denoting

$$\varphi_n^1(x,\zeta) = R^{n-1} P_n'(\mu) \tag{7.4.8}$$

so that

$$\left.\begin{aligned} \varphi_1^1 = 1, \quad \varphi_2^1 = 3\zeta, \quad \varphi_3^1 = \frac{3}{2}\left(4\zeta^2 - x^2\right), \\ \varphi_4^1 = \frac{5}{2}\zeta\left(4\zeta^2 - 3x^2\right), \quad \varphi_5^1 = \frac{15}{8}\left(8\zeta^4 - 12\zeta^2 x^2 + x^4\right) \end{aligned}\right\} \tag{7.4.9}$$

etc. we can set the polynomial solutions for the solid cylinder in the form

$$\Phi_n^1 = x\varphi_n^1(x,\zeta). \tag{7.4.10}$$

In the case of the hollow cylinder the solutions with singularities on axis $z = 0$ are added. The solutions are thus sought in the form

$$\Psi_n^1 = \psi_n^1(x,\zeta)\cos\varphi, \quad \psi_n^1(x,\zeta) = \frac{\varphi_n^1(x,\zeta)}{x} + \rho_n^1(x,\zeta)\, x\ln x + \vartheta_n^1(x,\zeta) \tag{7.4.11}$$

by means of the following condition

$$\nabla^2\Psi_n^1 = 0, \ \left(\nabla^2 - \frac{1}{x^2}\right)\psi_n^1 = \left(\nabla^2 - \frac{1}{x^2}\right)\left(\frac{\varphi_n^1}{x} + \rho_n^1 x\ln x + \vartheta_n^1\right) = 0. \tag{7.4.12}$$

We find successively

$$\left.\begin{aligned} \psi_1^1 = \frac{1}{x}, \quad \psi_2^1 = \frac{\varphi_2^1}{x}, \quad \psi_3^1 = \frac{\varphi_3^1}{x} - 6x\ln x, \quad \psi_4^1 = \frac{\varphi_4^1}{x} - 30\zeta x\ln x, \\ \psi_5^1 = \frac{\varphi_5^1}{x} - 45\left(2\zeta^2 - \frac{x^2}{2}\right)x\ln x - \frac{105}{8}x^3 \quad \text{etc.} \end{aligned}\right\} \tag{7.4.13}$$

3. An example. Let us consider a cylinder loaded by a normal pressure which is linearly distributed over the external and internal surfaces

$$\left.\begin{aligned} x = 1: \quad \sigma_r = -q_0\zeta, \quad \tau_{rz} = 0, \\ x = x_1: \quad \sigma_r = -q_1\zeta, \quad \tau_{rz} = 0. \end{aligned}\right\} \tag{7.4.14}$$

The solution is presented in terms of the axially symmetric harmonic functions b_0 and b_3

$$b_0 = A\varphi_3(x,\zeta) + B\psi_1(x,\zeta), \quad b_3 = C\varphi_2(x,\zeta) + D\psi_0(x,\zeta). \tag{7.4.15}$$

Using eqs. (7.1.6), (7.4.2), (7.4.5) and determining the constants from the boundary conditions (7.4.14) we obtain the stresses

$$\left.\begin{aligned}\sigma_r &= -\frac{\zeta}{1-x_1^2}\left(q_0\frac{x^2-x_1^2}{x^2} - q_1 x_1^2\frac{x^2-1}{x^2}\right),\\ \sigma_\varphi &= -\frac{\zeta}{1-x_1^2}\left(q_0\frac{x^2+x_1^2}{x^2} - q_1 x_1^2\frac{x^2+1}{x^2}\right),\\ \tau_{rz} &= 0, \quad \sigma_z = 0.\end{aligned}\right\} \tag{7.4.16}$$

This solution corresponds to the case of the unloaded face ends of the cylinder. Taking the derivative with respect to ζ we arrive at solution (7.2.4) of Lamé's problem in which q_0, q_1 is replaced by p_0, p_1.

Prescribing the solution in the form (7.4.15) enables considering the case of loading the side surfaces by the constant shear stresses

$$\begin{aligned} x = 1: &\quad \sigma_r = 0, \quad \tau_{rz} = \tau_0,\\ x = x_1: &\quad \sigma_r = 0, \quad \tau_{rz} = -\tau_1.\end{aligned} \tag{7.4.17}$$

The stresses are as follows

$$\left.\begin{aligned}&\sigma_r = 0, \quad \sigma_\varphi = 0,\\ &\tau_{rz} = \frac{\tau_0+\tau_1 x}{1-x_1^2}x - \frac{\tau_0 x_1^2+\tau_1 x_1}{1-x_1^2}\frac{1}{x},\\ &\sigma_z = -\frac{2(\tau_0+\tau_1 x_1)}{1-x_1^2}\zeta.\end{aligned}\right\} \tag{7.4.18}$$

This state of stress is realised approximately in the cross-sections which are well away from the end faces of the long cylinder whose end $\zeta = 0$ is free of the normal stress and the end $\zeta = l$ is compressed by the loads with the resultant force

$$Z = -2\pi a l\left(\tau_0 + x_1\tau_1\right),$$

see also eq. (7.1.17).

5.7.5 *Torsion of a cylinder subjected to forces distributed over the end faces*

The problem is to determine displacement v from differential equation (7.1.8) with the boundary conditions on the side surfaces of the cylinder

$$x = 1: \quad \frac{\partial}{\partial x}\frac{v}{x} = 0; \qquad x = x_1: \quad \frac{\partial}{\partial x}\frac{v}{x} = 0 \tag{7.5.1}$$

and on the end faces

$$\zeta = \pm L: \quad \frac{1}{G}\tau_{z\varphi} = \frac{1}{a}\frac{\partial v}{\partial\zeta} = f(x). \tag{7.5.2}$$

The distribution of the shear stresses is assumed to be the same on both end faces. The solution is sought in the form

$$\frac{v}{a} = \alpha x\zeta + \sum_{s=1}^{\infty} \frac{\sinh \mu_s \zeta}{\mu_s \cosh \mu_s L} g_s(x), \tag{7.5.3}$$

where α and μ_s are the constants which must be determined. Using eq. (7.1.7)

$$\left.\begin{aligned} \frac{1}{G}\tau_{r\varphi} &= \sum_{s=1}^{\infty} \frac{\sinh \mu_s \zeta}{\mu_s \cosh \mu_s L} x \left(\frac{g_s(x)}{x}\right)', \\ \frac{1}{G}\tau_{z\varphi} &= \alpha x + \sum_{s=1}^{\infty} \frac{\cosh \mu_s \zeta}{\cosh \mu_s L} g_s(x), \end{aligned}\right\} \tag{7.5.4}$$

and the boundary conditions (7.5.1) and (7.5.2) leads to the requirements

$$\left(\frac{g_s(x)}{x}\right)'_{x=1} = 0, \quad \left(\frac{g_s(x)}{x}\right)'_{x=x_1} = 0, \tag{7.5.5}$$

$$\sum_{s=1}^{\infty} g(x) = f(x) - \alpha x. \tag{7.5.6}$$

Functions $g_s(x)$ are determined by means of eq. (7.1.8) from Bessel's differential equation

$$g_s''(x) + \frac{1}{x} g_s'(x) + \left(\mu_s^2 - \frac{1}{x^2}\right) g_s(x) = 0,$$

whose general solution is the cylinder function

$$g_s(x) = Z_1(\mu_s x) = c_1^s J_1(\mu_s x) + c_2^s N_1(\mu_s x),$$

where J_1 and N_1 are Bessel and Neumann functions of the first order. Using eq. (7.5.5) and the well-known rule of differentiation we have

$$\left.\begin{aligned} Z_2(\mu_s x_1) &= c_1^s J_2(\mu_s x_1) + c_2^s N_2(\mu_s x_1) = 0, \\ Z_2(\mu_s) &= c_1^s J_2(\mu_s) + c_2^s N_2(\mu_s) = 0. \end{aligned}\right\} \tag{7.5.7}$$

The values of μ_s are the roots of the determinant of this system

$$J_2(\mu_s x_1) N_2(\mu_s) - J_2(\mu_s) N_2(\mu_s x_1) = 0 \tag{7.5.8}$$

and $g_s(x)$ is given by

$$g_s(x) = C_s [J_1(\mu_s x) N_2(\mu_s) - N_1(\mu_s x) J_2(\mu_s)] = C_s Z_1(\mu_s x). \tag{7.5.9}$$

Thus the problem is reduced to determining the constants C_n by means of condition (7.5.6). The constant α is determined by the torque

$$m_z = 2\pi a^3 \int_{x_1}^{1} \tau_{z\varphi} x^2 dx = \frac{1}{2}\pi a^3 G \left(1 - x_1^4\right) \alpha, \tag{7.5.10}$$

since the remaining terms, as eqs. (7.4.4) and (7.5.7) suggest, do not affect the expression for the torque

$$\int_{x_1}^{1} Z_1 (\mu_s x) x^2 dx = \frac{1}{\mu_s} \left[x^2 Z_2 (\mu_s x)\right]\Big|_{x_1}^{1} = 0. \tag{7.5.11}$$

The orthogonality of the system of functions $\sqrt{x} Z_1 (\mu_s x)$ is easily proved. To this end, it is sufficient to take the well-known formula

$$\left(\mu_s^2 - \mu_k^2\right) \int Z_1 (\mu_s x) Z_1 (\mu_k x) x dx =$$
$$= \mu_k x Z_1 (\mu_s x) Z_0 (\mu_k x) - \mu_s x Z_1 (\mu_k x) Z_0 (\mu_s x),$$

substitute

$$Z_0 (\mu x) = \frac{2}{\mu x} Z_1 (\mu x) - Z_2 (\mu x)$$

and take into account eq. (7.5.7). For $\mu_s \neq \mu_k$ we obtain

$$\int_{x_1}^{1} Z_1 (\mu_s x) Z_1 (\mu_k x) x dx = 0.$$

Additionally

$$N_s^2 = \int_{x_1}^{1} Z_1^2 (\mu_s x) x dx = \frac{1}{2} \left[Z_1^2 (\mu_s) - x_1^2 Z_1^2 (\mu_s x_1)\right]. \tag{7.5.12}$$

Accounting for eq. (7.5.11) we obtain from boundary condition (7.5.6) that

$$C_s = \frac{1}{N_s^2} \int_{x_1}^{1} f(x) Z_1 (\mu_s x) x dx, \tag{7.5.13}$$

and the solution of the problem is presented in the form

$$\frac{v}{a} = \alpha x \zeta + \sum_{s=1}^{\infty} C_s \frac{\sinh \mu_s \zeta}{\mu_s \cosh \mu_s L} Z_1 (\mu_s x), \tag{7.5.14}$$

where $Z_1(\mu_s x)$ is determined from eq. (7.5.9). The table of the roots of the transcendental equation (7.5.8) for several values of x is in the handbook[2].

x_1	μ_1	μ_2	μ_3	μ_4	μ_5
$\frac{5}{6}$	15,807	31,466	47,157	62,857	78,560
$\frac{2}{3}$	6,474	12,665	18,916	25,182	31,456
$\frac{1}{2}$	3,407	6,428	9,523	12,640	15,767

Table 5.2

In formula (7.5.14) the axial coordinate is taken from the mid-section of the cylinder. Denoting the axial coordinate from the "upper" end face by ζ_1 we have $\zeta + \zeta_1 = L$, thus

$$\frac{\sinh \mu_s \zeta}{\cosh \mu_s L} = \tanh \mu_s L \cosh \mu_s \zeta_1 - \sinh \mu_s \zeta_1.$$

Even for a cylinder whose length is equal to its diameter ($L = 1$) and for the values of μ_s listed in Table 5.2 we have $\tanh \approx 1$ and hence

$$\frac{\sinh \mu_s \zeta}{\cosh \mu_s L} \approx e^{-\mu_s \zeta_1},$$

that is, the terms of series (7.5.14) decrease exponentially from the end faces. The obtained solution describes the effect of any distribution of stress $\tau_{z\varphi}$ which is statically equivalent to torque m_z and shows that the influence of the law of distribution of the stress decreases exponentially as the distance from the end faces increases. Saint-Venant's principle is validated here with higher accuracy than can be expected from the general estimates of Subsection 5.2.14.

For a solid cylinder

$$\frac{v}{a} = \alpha x \zeta + \sum_{s=1}^{\infty} C_s \frac{\sinh \mu_s \zeta}{\mu_s \cosh \mu_s L} J_1(\mu_s x), \tag{7.5.15}$$

where μ_s denotes the roots of the equation $J_2(\mu) = 0$ which are as follows

$$\mu_1 = 5.136, \quad \mu_2 = 8.417, \quad \mu_3 = 11.620, \quad \mu_4 = 14.796 \quad \text{etc.}$$

and the constants C_s are given by the formula

$$C_s = \frac{1}{N_s^2} \int_0^1 f(x) J_1(\mu_s x) x dx, \quad N_s^2 = \frac{1}{2} J_1^2(\mu_s). \tag{7.5.16}$$

[2] Janke E., Emde F., Lösch F. Taffeln höherer Funktionen. Teubner, Stuttgart, 1960.

5.7.6 Solutions in terms of Bessel functions

As shown in Subsection 5.7.1 the Papkovich-Neuber functions solving the problems of equilibrium of the elastic cylinder under radial symmetric deformation are presented in terms of the harmonic functions

$$b_0(x,\zeta), \quad b_3(x,\zeta), \quad b_r(x,\zeta)\, e^{i\varphi}, \quad b_\varphi(x,\zeta)\, e^{i\varphi}. \tag{7.6.1}$$

In the case of the meridional deformation only two of the three functions b_0, b_3, b_r are needed. Function b_φ which is proportional to displacement v serves to solve the problem of torsion.

The harmonic functions of the same type

$$q(x,\zeta), \quad b_0(x,\zeta)\, e^{i\varphi}, \quad b_3(x,\zeta)\, e^{i\varphi}, \quad p(x,\zeta)\, e^{2i\varphi} \tag{7.6.2}$$

are applied to solve the problems of bending. However using all four functions is not necessary.

In this subsection the harmonic functions

$$f_n(x,\zeta)\, e^{in\varphi}$$

are presented with the help of the products

$$g_n(x)\, e^{\mu\zeta + in\varphi} \quad (n = 0, 1, 2) \tag{7.6.3}$$

ensuring separation of the variables in Laplace's equation. The result is the following differential equation for $g_n(x)$

$$g_n''(x) + \frac{1}{x} g_n'(x) + \left(\mu^2 - \frac{n^2}{x^2}\right) g_n(x) = 0. \tag{7.6.4}$$

The necessity of satisfying the boundary conditions on the side faces $(x = 1, x = x_1)$ and on the end face $(\zeta = -L, \zeta = L)$ independently of each other does not admit a closed form solution in the form of a series with the factors obtained after a finite number of operations. Except for the case of the axisymmetric torsion, the problem reduces to infinite systems of linear equations for these factors. Under an appropriate choice of the basic functions these systems turn out to be quite regular (or regular) and admit the application of approximate methods.

Neglecting some terms allows one to avoid this complicated way. When the length of the cylinder is sufficiently large $(2L \gg 1)$, one can use the set of solutions (7.6.3) for a pure imaginary μ, which exactly satisfy the boundary conditions and approximately satisfy the conditions of the end faces. The system of forces distributed over the end faces is replaced by a statically equivalent system with a known solution for the free side surfaces. Usually this aim is achieved by superimposing the solution of St.-Venant's problem (Chapter 6). In the latter problem the boundary conditions on the end faces are fulfilled integrally, that is, the principal vector and the principal moment of the forces distributed over the end faces have the prescribed values whereas the side surface remains unloaded.

This way of solving the problem for a long cylinder is substantiated by the St.-Venant principle, Subsection 4.2.8, which states that the obtained state of stress may differ from the sought one only in local disturbances decreasing with growth of the distance from the end faces. In particular, this is confirmed by the torsion problem example in Subsection 5.7.5. Term $\alpha x \zeta$ in the displacement (7.5.14) presents the solution of the St.-Venant problem and the series determines the local disturbance of the state of stress in the vicinity of the end faces, see also Subsections 5.7.8 and 5.7.9. It can be added that the practical value of "St.-Venant's solutions" is that the details of the law of the stress distribution can often be ignored.

The second extreme case is concerned with a short cylinder $(2L \ll 1)$, i.e. a circular (solid or annular) plate. What was said above about the long cylinder can be repeated in "an inverse order". By using the set of solutions (7.6.3) for real values of μ one constructs the solution which strictly satisfies the conditions of loading the end faces of the cylinder, the boundary conditions on the side faces being satisfied "on average". The appearing problems are essentially related to the theory of bending of plates which is beyond of the scope of this book.

The case of a cylinder with length comparable with the diameter $(L \approx 1)$ is the most difficult. It is apparently not possible to suggest a general approach to the problem different from reduction to the infinite systems of linear equations.

In what follows we consider only the long cylinder loaded on the side surfaces. The end faces are assumed to be free of loads.

When $\mu = i\beta$, β being real, the solution of the differential equation

$$g_n''(x) + \frac{1}{x} g_n'(x) - \left(\beta^2 + \frac{n^2}{x^2}\right) g_n(x) = 0 \tag{7.6.5}$$

is set in the form

$$g_n(x) = C_1^{(n)} I_n(\beta x) + C_2^{(n)} K_n(\beta x) \quad (n = 0, 1, 2). \tag{7.6.6}$$

Here $I_n(\beta x) = i^{-n} J_n(i\beta x)$ denotes the Bessel function of argument $i\beta x$ and $K_n(\beta x)$ is the McDonald function. The latter has a singularity on the cylinder axis $(x = 0)$ and thus is excluded when considering the case of a solid cylinder.

Considering the case of axisymmetric deformation we take

$$b_r = g_1(x) \cos \beta\zeta, \quad b_0 = g_0(x) \cos \beta\zeta. \tag{7.6.7}$$

Removing the second derivatives from the expressions for stresses σ_r, τ_{rz} with the help of eq. (7.6.5) and using eq. (7.1.6) we obtain

$$\left.\begin{aligned} \frac{1}{2G}\sigma_r &= \left\{\frac{1}{x}g_0'(x) - \beta^2 g_0(x) + (3-2\nu)\,g_1'(x) - \right. \\ &\qquad \left.\left[\beta^2 + (1-2\nu)\frac{1}{x^2}\right] x g_1(x)\right\}\cos\beta\zeta, \\ \frac{1}{2G}\tau_{rz} &= \left[g_0'(x) + x g_1'(x) - (1-2\nu)\,g_1(x)\right]\beta\sin\beta\zeta. \end{aligned}\right\} \tag{7.6.8}$$

When the side surfaces are loaded as follows

$$\left.\begin{aligned} x = 1: &\quad \sigma_r = -p\cos\beta\zeta, \quad \tau_{rz} = q\sin\beta\zeta, \\ x = x_1: &\quad \sigma_r = -p'\cos\beta\zeta, \quad \tau_{rz} = q'\sin\beta\zeta \end{aligned}\right\} \tag{7.6.9}$$

we assume

$$\left.\begin{aligned} g_0(x) &= D_1 I_0(\beta x) + D_2 K_0(\beta x), \\ g_1(x) &= C_1 I_1(\beta x) + C_2 K_1(\beta x). \end{aligned}\right\} \tag{7.6.10}$$

Using the formulae for differentiation (a prime denotes the derivative)

$$\begin{aligned} I_0'(\beta x) &= \beta I_1(\beta x), & I_1'(\beta x) &= \beta I_0(\beta x) - \frac{1}{x}I_1(\beta x), \\ K_0'(\beta x) &= -\beta K_1(\beta x), & K_1'(\beta x) &= -\beta K_0(\beta x) - \frac{1}{x}K_1(\beta x) \end{aligned}$$

we obtain four equations for the four constants D_i, C_i. These equations are not written down here because they are very cumbersome. We only notice that the determinant of the system in the particular case of the solid cylinder ($x_1 = 0$) is equal to

$$\Delta(\beta) = \beta\psi(\beta), \quad \psi(\beta) = \beta^2\left[I_0^2(\beta) - I_1^2(\beta)\right] - 2(1-\nu)\,I_1^2(\beta). \tag{7.6.11}$$

When coefficients D_i, C_i are determined one constructs expressions for the radial and axial displacements and all components of the stress tensor by means of eqs. (7.1.5) and (7.1.6). This solution is generalised to the case of an arbitrary loading on the lateral surface of the cylinder which is symmetric about the mid-section $\zeta = 0$ of the cylinder. Then σ_r and τ_{rz} are respectively even and odd functions of ζ and the values on the boundary are presented by the trigonometric series

$$\left.\begin{aligned} x = 1: &\quad \sigma_r = -p_0 - \sum_{k=1}^{\infty} p_k \cos\frac{k\pi\zeta}{L}, \quad \tau_{rz} = \sum_{k=1}^{\infty} q_k \sin\frac{k\pi\zeta}{L}, \\ x = x_1: &\quad \sigma_r = -p_0' - \sum_{k=1}^{\infty} p_k' \cos\frac{k\pi\zeta}{L}, \quad \tau_{rz} = \sum_{k=1}^{\infty} q_k' \sin\frac{k\pi\zeta}{L}. \end{aligned}\right\} \tag{7.6.12}$$

The solution for the constant terms $-p_0$ and $-p_0'$ are given by the formulae of Lamé's problem of Subsection 5.7.2 and each term of the series describes the solution for which $\beta = k\pi/l$.

Denoting the axial force in the cross-section of the cylinder by Z we have by eq. (7.1.17)

$$\frac{dZ}{d\zeta} = 2\pi a^2 \frac{\partial}{\partial \zeta} \int\limits_{x_1}^{1} \sigma_z x dx = 2\pi a^2 \left[x_1 \left(\tau_{rz}\right)_{x=x_1} - \left(\tau_{rz}\right)_{x=1} \right]$$

and, since σ_r and τ_{rz} are proportional to $\cos\beta\zeta$ and $\sin\beta\zeta$ respectively we obtain

$$Z(\zeta) = 2\pi a^2 L \sum_{k=1}^{\infty} \frac{1}{k\pi} \left(q_k - x_1 q_k'\right) \cos\frac{k\pi\zeta}{L} + 2\pi a^2 \nu \frac{p_0' x_1^2 - p_0}{1 - x_1^2},$$

where the constant term is due to the solution of Lamé's problem. The axial forces on the end faces are equal to

$$Z^* = 2\pi a^2 L \sum_{k=1}^{\infty} \frac{(-1)^k}{k\pi} \left(q_k - q_k' x_1\right) + 2\pi a^2 \nu \frac{p_0' x_1^2 - p_0}{1 - x_1^2}, \tag{7.6.13}$$

and the system of forces on the end faces can be statically equivalent to zero when the end faces are loaded by uniformly distributed normal stresses of the following intensity

$$\sigma_z^0 = -\frac{Z^*}{\pi a^2 \left(1 - x_1^2\right)}. \tag{7.6.14}$$

The constructed solution determines the state of stress in a cylinder of length $2aL$ with accuracy up to the local disturbance near the ends. Strictly speaking, the obtained solution is that of the problem of an infinitely long cylinder whose side surface is subjected to the load given by the periodic function (7.6.12). One can also represent the load not by a series but by a Fourier integral by prescribing the load outside the interval $-L \le \zeta \le L$ in an arbitrary way. For example by assuming that the load vanishes for $|\zeta| > L$.

The exact solution requires all forces to be removed from the end surfaces even though these forces are statically equivalent to zero. The difficulty of this problem was mentioned above. In what follows we consider an approach of a partial fulfillment of this requirement with the help of the "homogeneous solutions".

The case of loading which is skew-symmetric about the mid-section of the cylinder is considered by analogy. It is necessary to replace $\cos\beta\zeta$,$\sin\beta\zeta$ in formulae (7.6.7) and (7.6.8) by $\sin\beta\zeta, -\cos\beta\zeta$. The general case of the loading can be studied by means of superposition of the symmetric and skew-symmetric loadings.

5.7.7 Filon's problem

In this problem the solid cylinder is loaded by tangential forces of the constant intensity q which are uniformly distributed over two parts of the side surface

$$\zeta_0 < \zeta < \zeta_0 + b, \quad -\zeta_0 > \zeta > -(\zeta_0 + b).$$

This scheme can describe tension of a cylindric specimen subjected to a tensile force

$$P = 2\pi a^2 qb.$$

The uniform distribution of the shear stress over the loading area is assumed.

The boundary conditions (7.6.12) are now set in the following form

$$x = 1: \quad \sigma_r = 0, \quad \tau_{rz} = \tau(\zeta), \tag{7.7.1}$$

where $\tau(\zeta)$ is an odd function prescribed for $0 \le gz \le L$ as follows

$$\tau(\zeta) = \begin{cases} 0, & 0 < \zeta < \zeta_0, \\ q, & \zeta_0 < \zeta < \zeta_0 + b, \\ 0, & \zeta_0 + b < \zeta < L. \end{cases} \tag{7.7.2}$$

The coefficients of the series expansion of this function in terms of sine functions are equal to

$$q_k = \frac{2}{L}\int_0^L \tau(\zeta)\sin s_k\zeta d\zeta = \frac{2q}{k\pi}\left[\cos s_k\zeta_0 - \cos s_k(\zeta_0 + b)\right], \quad s_k = \frac{k\pi}{L}.$$

Using eqs. (7.6.6) and (7.6.7) we present functions b_r and b_0 by the trigonometric series

$$b_r = \sum_{k=1}^{\infty} C_k I_1(s_k x)\cos s_k\zeta, \quad b_0 = \sum_{k=1}^{\infty} D_k I_0(s_k x)\cos s_k\zeta \tag{7.7.3}$$

and, with the help of the boundary conditions (7.7.1), we arrive at the equations

$$\left.\begin{array}{r} C_k\left\{(3-2\nu)s_k I_0(s_k) - \left[4(1-\nu) + s_k^2\right]I_1(s_k)\right\} + \\ D_k s_k\left[I_1(s_k) - s_k I_0(s_k)\right] = 0, \\ C_k\left[s_k I_0(s_k) - 2(1-\nu)I_1(s_k)\right] + D_k s_k I_1(s_k) = \dfrac{q_k}{2Gs_k}, \\ k = 1, 2, \ldots \end{array}\right\} \tag{7.7.4}$$

determining the unknown coefficients C_k, D_k. When b_r and b_0 are obtained we utilise eqs. (7.1.5) and (7.1.6) for constructing expressions for the stresses and displacements. By virtue of eq. (7.6.14)

$$\sigma_z^0 = 2L\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\pi}q_k = \frac{4qL}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}\left[\cos\frac{k\pi\zeta_0}{L} - \cos\frac{k\pi(\zeta_0+b)}{L}\right]$$
$$= \frac{qb}{L}(2\zeta_0+b), \tag{7.7.5}$$

where the summation is performed by means of the following series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos k\alpha}{k^2} = \frac{\pi^2}{12} - \frac{\alpha^2}{4}.$$

Adding this stress which is uniformly distributed over the section of the cylinder to stress σ_z calculated by means of the obtained solution in the form of trigonometric series, we arrive at the following stress distribution

$$\sigma_z = \sigma_z^0 + \frac{2q}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[\cos s_k\zeta_0 - \cos s_k(\zeta_0+b)\right]S_z(x,s_k)\cos s_k\zeta, \tag{7.7.6}$$

which is statically equivalent to zero at the end faces $\zeta = \pm L$ of the cylinder. Here

$$S_z(x,\beta) = \frac{1}{\psi(\beta)}\left\{I_0(\beta x)\left[3\beta I_0(\beta) - 2(2-\nu)I_1(\beta) - \beta^2 I_1(\beta)\right] + \right.$$
$$\left.\beta x I_1(\beta x)\left[\beta I_0(\beta) - I_1(\beta)\right]\right\} \tag{7.7.7}$$

and $\beta = s_k = k\pi/L$ in eqs. (7.7.6) and (7.7.7).

In the table below, which was calculated by Filon for the values

$$L = \frac{\pi}{2}, \quad b = \zeta_0 = \frac{L}{3} = \frac{\pi}{6},$$

the normal stress is presented in the different sections of the cylinder. Here σ_m denotes the mean value for $\zeta = \zeta_0$, i.e. $\sigma_m = P/\pi a^2 = 2qb$. Let us notice that the mean value of this stress is equal to zero for $\zeta > \zeta_0 + b$.

ζ/L	$x=0$	$x=0,2$	$x=0,4$	$x=0,6$	$x=1$
0	0,689	0,719	0,810	0,962	1,117
0,1	0,673	0,700	0,786	0,937	1,163
0,2	0,631	0,652	0,720	0,859	1,334
0,3	0,582	0,594	0,637	0,737	2,022
0,4	0,539	0,545	0,565	0,617	1,368

Table 5.3 for ratio σ_z/σ_m

The last row shows the distribution of stress σ_z in the loaded part of the cylinder. It can be concluded from the table that the distribution of the normal stress σ_z becomes more homogeneous with increasing distance from the place of loading. However even in the mid-section ($\zeta = 0$) this stress varies from 69% to 122% of the mean value. This is explained by the fact that in the considered case the cylinder is loaded over a considerable part (one third) of the lateral surface and the St.-Venant principle is no longer valid.

5.7.8 Homogeneous solutions

Our consideration is restricted to the case of axially symmetric deformation of the solid cylinder without load on the side surface $x = 1$. Let us take the homogeneous system of linear equations (7.7.4)

$$\left.\begin{aligned} C\left\{(3-2\nu)\beta I_0(\beta) - \left[4(1-\nu)+\beta^2\right] I_1(\beta)\right\} + \\ D\beta\left[I_1(\beta) - \beta I_0(\beta)\right] = 0, \\ C\left[\beta I_0(\beta) - 2(1-\nu) I_1(\beta)\right] + D\beta I_1(\beta) = 0, \end{aligned}\right\} \tag{7.8.1}$$

determining coefficients C and D of the solutions

$$b_r = C I_1(\beta x) e^{i\beta\zeta}, \quad b_0 = D I_0(\beta x) e^{i\beta\zeta}. \tag{7.8.2}$$

This system may have non-trivial solutions for the values of β for which the determinant (7.6.11) of the system vanishes

$$\psi(\beta) = \beta^2\left[I_0^2(\beta) - I_1^2(\beta)\right] - 2(1-\nu) I_1^2(\beta). \tag{7.8.3}$$

The power series for $\psi(\beta)$ can be obtained by using the following formula for the product of Bessel functions

$$I_m(\beta) I_n(\beta) = \left(\frac{1}{2}\beta\right)^{m+n} \sum_{s=0}^{\infty} \frac{(m+n+2s)!}{s!\,(m+s)!\,(n+s)!\,(m+n+s)!} \left(\frac{1}{2}\beta\right)^{2s}.$$

The result is as follows

$$\psi(\beta) = \beta^2 \sum_{s=0}^{\infty} \left[1 - (1-\nu)\frac{2s+1}{(s+1)(s+2)}\right] \frac{(2s)!}{(s+1)!\,(s!)^3} \left(\frac{1}{2}\beta\right)^{2s}, \tag{7.8.4}$$

implying that for $\nu < 1/2$ the value in the brackets is positive for any integer s. Hence $\psi(\beta)$ has no roots for real values of β except for the evident double zero root. It has no pure imaginary roots which can be proved by putting $\beta = i\mu$ and grouping the terms of the series into pairs, it then turns out that $\psi(i\mu) < 0$.

Thus, all roots of function $\psi(\beta)$ are complex-valued. However this function is even and the coefficients of its expansion (7.8.4) into the series are real-valued. For this reason, the roots are split into four groups:

$$\beta_s = \gamma_s + i\delta_s,\ \beta_s^* = -\gamma_s + i\delta_s,\ \bar\beta_s = \gamma_s - i\delta_s,\ \bar\beta_s^* = -\gamma_s - i\delta_s,$$
$$(\gamma_s > 0,\ \delta_s > 0). \tag{7.8.5}$$

The values of the first three roots lying in the first quadrant of the plane β and some functions of these roots are collected in Table 5.4, the calculation being carried out for $\nu = 0,25$.

s	β_s	$I_0(\beta_s)$	$I_1(\beta_s)$
1	$1,367 + 2,698i$	$-0,4695 + 0,7269i$	$-0,5453 + 0,7233i$
2	$1,558 + 6,060i$	$0,4853 - 0,5576i$	$0,4937 - 0,5794i$
3	$1,818 + 9,320i$	$-0,567 + 0,562i$	$-0,568 + 0,563i$
s	$\lambda_s = \dfrac{I_0(\beta_s)}{I_1(\beta_s)}$	$\lambda_s\beta_s$	$\psi'(\beta_s)$
1	$0,9528 - 0,0692i$	$1,489 + 2,476i$	$2,85 - 1,48i$
2	$0,9712 + 0,0102i$	$1,451 + 5,901i$	$8,22 - 0,84i$
3	$0,997 + 0,000i$	$1,812 + 9,29i$	$12,06 - 1,56i$

Table 5.4

As $|\beta_3| = 9.496$ the asymptotic formulae for the roots provide one with sufficient accuracy for $n > 3$ (with the accuracy to the terms of order $n^{-1}\ln n$)

$$\beta_n \sim n\pi i + \frac{1}{2}\ln 4n\pi - i\left\{\frac{\ln 4n\pi}{4\pi n} - \frac{1}{2\pi n}\left[\frac{1}{4} - 2(1-\nu)\right]\right\} \tag{7.8.6}$$

and the functions of them

$$\left.\begin{aligned} I_0(\beta_n) &\sim \frac{e^{\beta_n}}{\sqrt{2\pi\beta_n}}\left\{1 - \frac{1}{8\beta_n} + \left[\frac{5}{128} + \frac{1}{2}(1-\nu)\right]\frac{1}{\beta_n^2}\right\},\\ I_1(\beta_n) &\sim \frac{e^{\beta_n}}{\sqrt{2\pi\beta_n}}\left\{1 - \frac{1}{8\beta_n} + \left[\frac{5}{128} - \frac{1}{2}(1-\nu)\right]\frac{1}{\beta_n^2}\right\},\end{aligned}\right\} \tag{7.8.7}$$

$$\lambda_n \sim 1 + \frac{1-\nu}{\beta_n^2},\quad \psi'(\beta_n) \sim 2\beta_n I_1^2(\beta_n)\left(1 - \frac{2(1-\nu)}{\beta_n}\right). \tag{7.8.8}$$

Let us also notice that for the adopted denotation

$$\lambda = \frac{I_0(\beta)}{I_1(\beta)} \tag{7.8.9}$$

equation (7.8.3) is reset as follows

$$\beta^2\left(\lambda^2 - 1\right) = 2(1-\nu). \tag{7.8.10}$$

Returning to equations (7.8.1) we find the dependence between the constants C and D corresponding to the root β_s

$$D^{(s)} = \left[\frac{2(1-\nu)}{\beta_s} - \lambda_s\right] C^{(s)} = \left[\beta_s\left(\lambda_s^2 - 1\right) - \lambda_s\right] C^{(s)}.$$

The constants $C^{(s)}$ remain undetermined. Further we introduce the notation

$$C^{(s)} = \frac{L_s}{\beta_s I_1(\beta_s)} = \frac{M_s + iN_s}{\beta_s I_1(\beta_s)},$$

allowing us to write down the expressions for the displacements for each root in the form

$$u_s = L_s u_s(x, \beta_s) e^{i\beta_s \zeta}, \quad w_s = L_s w_s(x, \beta_s) e^{i\beta_s \zeta}, \tag{7.8.11}$$

where

$$\left.\begin{aligned} \frac{1}{a} u_s(x, \beta_s) &= \frac{1}{\beta_s I_1(\beta_s)} \{-\beta_s x I_0(\beta_s x) + [2(1-\nu) + \lambda_s \beta_s] I_1(\beta_s x)\}, \\ \frac{1}{a} w_s(x, \beta_s) &= -\frac{i}{\beta_s I_1(\beta_s)} \{\beta_s x I_1(\beta_s x) + [2(1-\nu) - \lambda_s \beta_s] I_0(\beta_s x)\}. \end{aligned}\right\} \tag{7.8.12}$$

The stresses are written in the following way

$$\left.\begin{aligned} \frac{\sigma_r^s}{2G} &= L_s \sigma_r^s(x, \beta_s) e^{i\beta_s \zeta}, & \frac{\sigma_\varphi^s}{2G} &= L_s \sigma_\varphi^s(x, \beta_s) e^{i\beta_s \zeta}, \\ \frac{\sigma_z^s}{2G} &= L_s \sigma_z^s(x, \beta_s) e^{i\beta_s \zeta}, & \frac{\tau_{rz}^s}{2G} &= L_s \tau_{rz}^s(x, \beta_s) e^{i\beta_s \zeta}, \end{aligned}\right\} \tag{7.8.13}$$

with

$$\left.\begin{aligned} I_1(\beta_s) \sigma_r^s(x, \beta_s) &= I_0(\beta_s x)(1 + \beta_s \lambda_s) - \\ &\qquad \frac{I_1(\beta_s x)}{\beta_s x}\left[2(1-\nu) + \beta_s \lambda_s + \beta_s^2 x^2\right], \\ I_1(\beta_s) \sigma_\varphi^s(x, \beta_s) &= -(1-2\nu) I_0(\beta_s x) + \frac{I_1(\beta_s x)}{\beta_s x}[2(1-\nu) + \beta_s \lambda_s], \\ I_1(\beta_s) \sigma_z^s(x, \beta_s) &= (2 - \beta_s \lambda_s) I_0(\beta_s x) + \beta_s x I_1(\beta_s x), \\ I_1(\beta_s) \tau_{rz}^s(x, \beta_s) &= -i[\beta_s x I_0(\beta_s x) - \beta_s \lambda_s I_1(\beta_s x)]. \end{aligned}\right\} \tag{7.8.14}$$

Referring to eqs. (7.8.9) and (7.8.10) it is easy to prove that the complex-valued functions $\sigma_r^s(x, \beta_s)$ and $\tau_{rz}^s(x, \beta_s)$ are zero on the surface of the cylinder $x = 1$. Superscripts r and i denote respectively the real and imaginary parts.

Thus, we have constructed a system of "homogeneous solutions" of the equilibrium equations for an elastic cylinder, that is the solutions ensuring that surface $x = 1$ is free from loading. The stresses corresponding to these solutions are statically equivalent to zero in any cross-section of the cylinder. It follows immediately from static reasoning and is confirmed by the calculation

$$\int_0^1 \sigma_z^s (x, \beta_s) x dx = \\ = \frac{1}{I_1(\beta_s)} \left[(2 - \beta_s \lambda_s) \int_0^1 I_0(\beta_s x) x dx + \beta_s \int_0^1 I_1(\beta_s x) x^2 dx \right] = 0$$

along with eq. (7.8.9) and the relationship

$$\int x^n I_{n-1}(x) dx = x^n I_n(x), \quad I_2(x) = I_0(x) - \frac{2I_1(x)}{x}.$$

Separating the real part in eq. (7.8.11) we obtain

$$\left.\begin{aligned} u_s^r &= \left\{ M_s \left[u^{(s,r)} \cos \gamma_s \zeta - u^{(s,i)} \sin \gamma_s \zeta \right] - \right. \\ &\qquad \left. N_s \left[u^{(s,r)} \sin \gamma_s \zeta + u^{(s,i)} \cos \gamma_s \zeta \right] \right\} e^{-\delta_s \zeta}, \\ w_s^r &= \left\{ M_s \left[w^{(s,r)} \cos \gamma_s \zeta - w^{(s,i)} \sin \gamma_s \zeta \right] - \right. \\ &\qquad \left. N_s \left[w^{(s,r)} \sin \gamma_s \zeta + w^{(s,i)} \cos \gamma_s \zeta \right] \right\} e^{-\delta_s \zeta}. \end{aligned}\right\} \tag{7.8.15}$$

Taking the imaginary part of u_s and w_s, we obtain expressions which differ from the above only by the sign. Thus, for each root β_s in the first quadrant of the plane β we have two particular homogeneous solutions corresponding to the independent constants M_s and N_s. The factor $e^{-\delta_s \zeta}$ in eq. (7.8.15) indicates that these solutions decrease exponentially from the edge $\zeta = 0$ of the cylinder. The rate of decrease increases with the number of the solution, for instance $\delta_1 = 2.698$ and $\delta_3 = 9.320$. The solutions decreasing from the edge $\zeta = L$ are obtained by replacing the factor $e^{-i\beta\zeta}$ by $e^{-i\beta_1\zeta}, \zeta_1 = L - \zeta$ and changing the signs of w and τ_{rz}. Using the roots lying on other quadrants of the plane β does not lead to new solutions. Thus, for each β_s we obtain four independent particular solutions, among them two solutions decrease with increasing distance from the end $\zeta = 0$ and the other two decrease with increasing distance from the end $\zeta_1 = 0$.

5.7.9 *Boundary conditions on the end faces*

The homogeneous solutions introduced in Subsection 5.7.8 can be used for approximate fulfillment of the boundary conditions on the end faces of the cylinder since superimposing does not change the loading condition on the side face of the cylinder.

We can limit our consideration to the case of an half-infinite cylinder since the perturbation of the state of stress due to the unsatisfied boundary conditions at one end face is negligibly small near the other end. This assumption is admitted even for a cylinder with $L \approx 2$ and is justified by the exponential decreasing the homogeneous solutions.

The expressions for the $s-th$ pair of the real-valued homogeneous solutions for the normal and shear stresses at the end $\zeta = 0$ are

$$M_s \sigma_z^{(s,r)} - N_s \sigma_z^{(s,i)}, \quad M_s \tau_{rz}^{(s,r)} - N_s \tau_{rz}^{(s,i)}, \tag{7.9.1}$$

where $\sigma_z^{(s,r)}, \tau_{rz}^{(s,r)}$ and $\sigma_z^{(s,i)}, \tau_{rz}^{(s,i)}$ denote respectively the real and imaginary parts of functions $\sigma_z^s(x,s), \tau_{rz}^s(x,s)$ given by eq. (7.8.14).

The boundary conditions at $\zeta = 0$ are set in the form

$$\zeta = 0: \quad \sigma_z = F(x), \quad \tau_{rz} = -\Phi(x), \tag{7.9.2}$$

where functions $F(x)$ and $\Phi(x)$ are prescribed by the law of loading on the end face. Here $F(x) > 0$ under tension and $\Phi(x) > 0$ if the shear stresses have the direction of increasing x. The distribution of the normal stresses over the end face is assumed to be statically equivalent to zero

$$\int_0^1 xF(x)\,dx = 0, \tag{7.9.3}$$

since removing the force of tension Z (the principal vector of the normal stresses) requires only superimposing the elementary solution $\sigma_z = Z/\pi a^2$.

Hence, the problem is reduced to expansion of two given functions in the series

$$\left.\begin{aligned} F(x) &= \sum_{s=1}^{\infty} \left[M_s \sigma_z^{(s,r)} - N_s \sigma_z^{(s,i)}\right], \\ \Phi(x) &= -\sum_{s=1}^{\infty} \left[M_s \tau_{rz}^{(s,r)} - N_s \tau_{rz}^{(s,i)}\right]. \end{aligned}\right\} \tag{7.9.4}$$

Restricting ourselves by the approximate solution we keep a finite number n of terms in the right hand sides of eq. (7.9.4) and introduce a quadratic deviation over the area

$$\begin{aligned} &\Psi(M_1, M_2, \ldots, M_n; N_1, N_2, \ldots, N_n) = \\ &= \int_0^1 \left\{ \left[F_x - \sum_{s=1}^{n} \left(M_s \sigma_z^{(s,r)} - N_s \sigma_z^{(s,i)}\right)\right]^2 + \right. \\ &\qquad \left. \left[\Phi(x) + \sum_{s=1}^{n} \left(M_s \tau_{rz}^{(s,r)} - N_s \tau_{rz}^{(s,i)}\right)\right]^2 \right\} x\,dx. \end{aligned} \tag{7.9.5}$$

The coefficients M_s, N_s are determined by the condition of minimum quadratic deviation which leads to the system of $2n$ linear equations

$$\frac{\partial \Psi}{\partial M_k} = 0, \quad \frac{\partial \Psi}{\partial N_k} = 0, \quad k = 1, 2, \dots, n, \tag{7.9.6}$$

or in expanded form

$$\sum_{s=1}^{n} (A_{sk} M_s + B_{sk} N_s) = \varphi_k, \quad \sum_{s=1}^{n} (B_{sk} M_s + C_{sk} N_s) = \psi_k, \tag{7.9.7}$$

where

$$\left.\begin{aligned} A_{sk} &= \int_0^1 \left(\sigma_z^{(s,r)} \sigma_z^{(k,r)} + \tau_{rz}^{(s,r)} \tau_{rz}^{(k,r)}\right) x dx = A_{ks}, \\ B_{sk} &= -\int_0^1 \left(\sigma_z^{(s,i)} \sigma_z^{(k,r)} + \tau_{rz}^{(s,i)} \tau_{rz}^{(k,r)}\right) x dx, \\ C_{sk} &= \int_0^1 \left(\sigma_z^{(s,i)} \sigma_z^{(k,i)} + \tau_{rz}^{(s,i)} \tau_{rz}^{(k,i)}\right) x dx = C_{ks} \end{aligned}\right\} \tag{7.9.8}$$

and

$$\varphi_k - i\psi_k = \int_0^1 \left[F(x) \sigma_z^k (x, \beta_k) - \Phi(x) \tau_{rz}^k (x, \beta_k)\right] x dx. \tag{7.9.9}$$

Given Poisson's ratio, coefficients A_{sk}, B_{sk}, C_{sk} are calculated only once. For $\nu = 1/4$ they are as follows

$$\left.\begin{aligned} A_{sk} - iB_{sk} &= \frac{1}{2} \left[J_+ (\beta_s, \beta_k) + J_- (\beta_s, \bar{\beta}_k)\right], \\ C_{sk} + iB_{sk} &= \frac{1}{2} \left[-J_+ (\beta_s, \beta_k) + J_- (\bar{\beta}_s, \beta_k)\right], \end{aligned}\right\} \tag{7.9.10}$$

hence

$$\left.\begin{aligned} J_+ (\beta_s, \beta_k) &= \frac{4}{(\beta_s - \beta_k)^2} \left[\frac{3}{4} - \frac{\beta_s \beta_k (\beta_s \lambda_k - \beta_k \lambda_s)}{\beta_s^2 - \beta_k^2}\right], \\ J_- (\beta_s, \beta_k) &= \frac{4}{(\beta_s - \beta_k)^2} \left[-\frac{3}{4} + \frac{\beta_s \beta_k (\beta_s \lambda_k - \beta_k \lambda_s)}{\beta_s^2 - \beta_k^2}\right] \end{aligned}\right\} \tag{7.9.11}$$

and for $s = k$

$$\left.\begin{aligned} J_+ (\beta_s, \beta_s) &= -\frac{2}{3} \beta_s \lambda_s + 2,667 - 4,5 \frac{\lambda_s}{\beta_s} + \frac{5,25}{\beta_s^2}, \\ J_- (\beta_s, \beta_s) &= 1 - \frac{3}{2} \frac{\beta_s}{\lambda_s} + \frac{3}{\beta_s^2}. \end{aligned}\right\} \tag{7.9.12}$$

Using Table 5.4 the calculation for $n = 2$ yields the following system of four equations

$$\left.\begin{aligned} 1,007M_1 - 0,2571M_2 + 0,401N_1 - 0,05085N_2 &= \varphi_1, \\ -0,2571M_1 + 3,051M_2 - 0,525N_1 + 1,662N_2 &= \varphi_2, \\ 0,401M_1 - 0,525M_2 + 0,2915N_1 - 0,2678N_2 &= \psi_1, \\ -0,05085M_1 + 1,662M_2 - 0,2678N_1 + 1,650N_2 &= \psi_2, \end{aligned}\right\} \quad (7.9.13)$$

whose solution is as follows

$$\left.\begin{aligned} M_1 &= 2,391\varphi_1 - 0,4256\varphi_2 - 4,222\psi_1 - 0,1829\psi_2, \\ M_2 &= -0,4256\varphi_1 + 0,9730\varphi_2 + 1,675\psi_1 - 0,7215\psi_2, \\ N_1 &= -4,222\varphi_1 + 1,675\varphi_2 + 12,44\psi_1 + 0,2018\psi_2, \\ N_2 &= -0,1829\varphi_1 - 0,7215\varphi_2 + 0,2018\psi_1 + 1,360\psi_2. \end{aligned}\right\} \quad (7.9.14)$$

Determining coefficients M_s, N_s in terms of φ_s, ψ_s we find the real-valued homogeneous solutions for $s = 1, 2$ by eq. (7.8.14). The numerical values of functions $I_0(\beta_s x), I_1(\beta_s x)$ and the corresponding values of stresses $\sigma_z^{(s,r)} + i\sigma_z^{(s,i)}$, $\tau_{rz}^{(s,r)} + i\tau_{rz}^{(s,i)}$ and displacements are presented in Tables 5.5-5.8.

x	$I_0(\beta_1 x)$	$I_1(\beta_1 x)$	$I_0(\beta_2 x)$	$I_1(\beta_2 x)$
0	1	0	0	0
0,2	0,9452+0,01780i	0,1230+0,2675i	0,6776+0,1584i	0,0794+0,5193i
0,3	0,8752+0,1551i	0,1600+0,3960i	0,331+0,280i	0,0040+0,627i
0,6	0,4760+0,5078i	0,0785+0,7162i	-0,589+0,0727i	-0,439+0,0599i
0,8	0,04951+0,7017i	-0,1704+0,8090i	-0,3618-0,4973i	-0,2326-0,5937i
1	-0,4695+0,7269i	-0,5453+0,7233i	0,4853-0,5576i	0,4937-0,5794i

Table 5.5

x	$\sigma_r^{(1,r)} + i\sigma_r^{(1,i)}$	$\sigma_r^{(2,r)} + i\sigma_r^{(2,i)}$	$\sigma_\varphi^{(1,r)} + i\sigma_\varphi^{(2,i)}$	$\sigma_\varphi^{(2,r)} + i\sigma_\varphi^{(2,i)}$
0	0,4302-1,669i	-2,119+2,490i	0,4302-1,669i	-2,119+2,490i
0,2	0,403-1,517i	-1,247+1,658i	0,4547-1,638i	-1,959+2,784i
0,3	0,368-1,309i	-0,34-0,02i	0,485-1,562i	-1,75+2,024i
0,6	0,172-0,485i	1,58-2,233i	0,6053-1,185i	-0,602-0,244i
0,8	0,014-0,081i	0,721-0,746i	0,676-0,843i	0,2103-0,6687i
1	0	0	0,7004-0,4769i	0,5456-0,2271i

Table 5.6

x	$\sigma_z^{(1,r)}+i\sigma_z^{(1,i)}$	$\sigma_z^{(2,r)}+i\sigma_z^{(2,r)}$	$\tau_{rz}^{(1,r)}+i\tau_{rz}^{(1,i)}$	$\tau_{rz}^{(2,r)}+i\tau_{rz}^{(2,i)}$
0	-2,552+1,195i	6,369-4,479i	0	0
0,2	-2,273+0,952i	4,252-2,410i	-0,5007+0,6171i	2,669-2,880i
0,3	-1,966+0,685i	2,092-0,581i	-0,7186+0,8357i	3,17-3,04i
0,6	-0,450-0,221i	-2,42+1,17i	-1,030+0,8790i	0,54+0,28i
0,8	0,776-0,351i	-0,456-0,551i	-0,7628+0,4486i	-1,325+0,889i
1	1,681+0,303i	2,151+0,335i	0	0

Table 5.7

x	$u^{(1,r)}+iu^{(1,i)}$	$u^{(2,r)}+iu^{(2,r)}$	$w^{(1,r)}+iw^{(1,i)}$	$w^{(2,r)}+iw^{(2,i)}$
0	0	0	0,8906-0,1552i	-1,118+0,5443i
0,2	0,1478-0,2394i	-0,4335+0,4759i	0,7953-0,1139i	-0,7377+0,2753i
0,3	0,2124-0,3376i	-0,524+0,522i	0,6819-0,0707i	0,1756+0,0306i
0,6	0,3240-0,4836i	-0,190+0,011i	-0,358+0,051i	0,388-0,137i
0,8	0,3059-0,4706i	0,0927-0,2208i	0,1568-0,0441i	0,0794+0,0462i
1	0,2241-0,4422i	0,0600-0,2322i	-0,3050-0,2799i	-0,2448-0,1173i

Table 5.8

5.7.10 Generalised orthogonality

The difficulty of fulfilling the boundary conditions on the end faces of the cylinder resides in the necessity of simultaneous representation of two independent functions by series (7.9.4) in terms of the non-orthogonal system of "homogeneous solutions", i.e. the solutions ensuring the free side surface of the cylinder.

These solutions possess the property of "generalised orthogonality". It can be used for exact fulfillment of one of the boundary conditions, that is the arbitrariness admitting the possibility of approximately satisfying the second condition is retained.

We introduce into consideration the following functions

$$\left.\begin{aligned}\rho_s &= \rho\left(\beta_s x\right) = \beta_s\left(\lambda_s^2-1\right) I_0\left(\beta_s x\right),\\ \varepsilon_s &= \varepsilon\left(\beta_s x\right) = \frac{1}{\beta_s}\left[\beta_s x I_1\left(\beta_s x\right)-\beta_s\lambda_s I_0\left(\beta_s x\right)\right]\end{aligned}\right\} \tag{7.10.1}$$

and their derivatives with respect to argument $\beta_s x$

$$\left.\begin{aligned}\rho_s' &= \rho'\left(\beta_s x\right) = \beta_s\left(\lambda_s^2-1\right) I_1\left(\beta_s x\right),\\ \varepsilon_s' &= \varepsilon'\left(\beta_s x\right) = \frac{1}{\beta_s}\left[\beta_s x I_0\left(\beta_s x\right)-\beta_s\lambda_s I_1\left(\beta_s x\right)\right].\end{aligned}\right\} \tag{7.10.2}$$

Here, as before, β_s are the roots of the transcendental equation

$$2\left(1-\nu\right) = \beta_s\left(\lambda_s^2-1\right), \quad \lambda_s = \frac{I_0\left(\beta_s\right)}{I_1\left(\beta_s\right)}. \tag{7.10.3}$$

The homogeneous solutions of the problem of the cylinder are presented by the series

$$\left.\begin{aligned} u(x,\zeta) &= a\sum_s \frac{C_s}{\beta_s}(\rho_s' - \varepsilon_s')\cos\beta_s\zeta, \\ w(x,\zeta) &= a\sum_s \frac{C_s}{\beta_s}(\rho_s + \varepsilon_s)\sin\beta_s\zeta, \end{aligned}\right\} \tag{7.10.4}$$

where for each root $s = 1, 2, \ldots$ one obtains four terms for the roots β_s from eq. (7.8.5). Under an appropriate choice of constants C_s, series (7.10.4) describe the real-valued functions.

Using eq. (7.10.4) we find the stresses

$$\left.\begin{aligned} \sigma_r &= -2G\sum_s C_s\left(\varepsilon_s + \frac{\rho_s' - \varepsilon_s'}{\beta_s x}\right)\cos\beta_s\zeta, \\ \tau_{rz} &= 2G\sum_s C_s\varepsilon_s'\sin\beta_s\zeta, \\ \sigma_\varphi &= 2G\sum_s C_s\left(\frac{\nu\rho_s}{1-\nu} + \frac{\rho_s' - \varepsilon_s'}{\beta_s x}\right)\cos\beta_s\zeta, \\ \sigma_z &= 2G\sum_s C_s\left(\varepsilon_s + \frac{\rho_s}{1-\nu}\right)\cos\beta_s\zeta. \end{aligned}\right\} \tag{7.10.5}$$

The introduced functions ε_s' and ρ_s' have the property of generalised orthogonality (P.A. Schiff, 1883)

$$\int_0^1 (\varepsilon_k'\rho_s' + \varepsilon_s'\rho_k')\,xdx = 0 \quad (s \neq k). \tag{7.10.6}$$

This can be proved directly

$$\begin{aligned} \int_0^1 x(\varepsilon_k'\rho_s' + \varepsilon_s'\rho_k')\,dx = \int_0^1 \Big\{x^2\big[\beta_s(\lambda_s^2 - 1)I_0(\beta_k x)I_1(\beta_s x) + {}\\ \beta_k(\lambda_k^2 - 1)I_0(\beta_s x)I_1(\beta_k x)\big] - {}\\ x\left[\lambda_k\beta_s(\lambda_s^2 - 1) + \lambda_s\beta_k(\lambda_k^2 - 1)\right]I_1(\beta_k x)I_1(\beta_s x)\Big\}\,dx. \end{aligned}$$

The integrals are estimated as follows

$$\int_0^1 I_1(\beta_s x)I_1(\beta_k x)\,xdx =$$

$$= \begin{cases} I_1(\beta_s x)I_1(\beta_k x)\dfrac{\beta_s\lambda_s - \beta_k\lambda_k}{\beta_s^2 - \beta_k^2} & (s \neq k), \\ \dfrac{1}{2}\left[I_1^2(\beta_s) - I_0^2(\beta_s)\right] + \dfrac{1}{\beta_s}I_0(\beta_s)I_1(\beta_k) & (s = k), \end{cases} \tag{7.10.7}$$

$$\int_0^1 x^2 I_0(\beta_s x) I_1(\beta_k x)\,dx =$$

$$= \begin{cases} \dfrac{I_1(\beta_s) I_1(\beta_k)}{\beta_s^2 - \beta_k^2}\left[\beta_s - \beta_k \lambda_s \lambda_k + 2\beta_k \dfrac{\beta_s \lambda_k - \beta_k \lambda_s}{\beta_s^2 - \beta_k^2}\right] & (s \neq k), \\ \dfrac{1}{2\beta_s} I_1^2(\beta_s) & (s = k), \end{cases} \tag{7.10.8}$$

and are substituted into eq. (7.10.6). At the same time

$$2\int_0^1 \rho_k' \varepsilon_k' x dx = N_k = I_1^2(\beta_k)\left(\lambda_k^2 - 1\right)\left[1 + \beta_k \lambda_k\left(\lambda_k^2 - 1\right) - 2\lambda_k^2\right]. \tag{7.10.9}$$

P.A. Schiff and later P.F. Papkovich (1941) indicated the possibility of a simultaneous representation of two independent functions $F_1(x)$ and $F_2(x)$ in the form of the series in terms of the functions having the property of generalised orthogonality. Being applied to functions ε_s', ρ_s' these representations are put in the following form

$$F_1(x) = \sum_s D_s \varepsilon_s', \quad F_2(x) = \sum_s D_s \rho_s'. \tag{7.10.10}$$

The coinciding coefficients D_s of these series are determined with the help of the property of generalised orthogonality

$$\int_0^1 \left[\rho_k' F_1(x) + \varepsilon_k' F_2(x)\right] x dx = \sum_s D_s \int_0^1 \left(\rho_k' \varepsilon_s' + \varepsilon_k' \rho_s'\right) x dx = D_k N_k. \tag{7.10.11}$$

For example, by putting $F_2(x) = 0$ we have

$$N_s D_s = \int_0^1 \rho_s' F_1(x)\, x dx, \quad F_1(x) = \sum_s D_s \varepsilon_s', \quad \sum_s D_s \rho_s' = 0.$$

The convergence of the series and the study of the class of functions $F_1(x), F_2(x)$, for which the joint expansions of the type (7.10.10) in terms of the generalised functions are possible, are given by G.A. Grinberg (1951) for the example of bending of plates.

Function $2GF_1(x)$ can be taken as being equal to the shear stress on the end face $\zeta = L$, i.e.

$$(\tau_{xz})_{\zeta = L} = 2G \sum_s C_s \varepsilon_s' \sin \beta_s L = 2GF_1(x). \tag{7.10.12}$$

Then, by virtue of eq. (7.10.11)

$$C_s = \frac{1}{N_s \sin \beta_s L} \int_0^1 \left[\rho_s' F_1(x) + \varepsilon_s' F_2(x)\right] x dx. \tag{7.10.13}$$

The arbitrariness in prescribing function $F_2(x)$ can be used for an approximate fulfillment of another boundary condition. Adopting the following representation

$$F_2(x) = \sum_{k=1}^{n} a_k \varphi_k(x), \tag{7.10.14}$$

we obtain that coefficients C_r are linear functions of parameters a_k

$$C_r = \frac{1}{N_r \sin \beta_r L} \left[\int_0^1 x \rho_r' F_1(x) dx + \sum_{k=1}^{n} a_k \int_0^1 x \varepsilon_r' \varphi_k(x) dx \right]. \tag{7.10.15}$$

These parameters can be determined, for example, from the condition of minimum quadratic deviation of the normal stress σ_z from the required value (7.10.5)

$$\Psi(a_1, a_2, \ldots, a_n) = \\ = \frac{1}{2} \int_0^1 x \left[\left(\frac{\sigma_z}{2G}\right)_{\zeta=L} - \sum_s C_s \left(\varepsilon_s + \frac{\rho_s}{1-\nu}\right) \cos \beta_s L \right]^2 dx = \min. \tag{7.10.16}$$

Thus we arrive at the system of n linear equations ($k = 1, 2, \ldots n$)

$$\frac{\partial \Psi}{\partial a_k} = -\int_0^1 \left\{ x \left[\left(\frac{\sigma_z}{2G}\right)_{\zeta=L} - \sum_s C_s \left(\varepsilon_s + \frac{\rho_s}{1-\nu}\right) \cos \beta_s L \right] \times \right. \\ \left. \sum_r \frac{\partial C_r}{\partial a_k} \left(\varepsilon_r + \frac{\rho_r}{1-\nu}\right) \cos \beta_r L \right\} dx = 0. \tag{7.10.17}$$

Taking, for instance,

$$F_2(x) = a_1 x + a_2 x^3 + \ldots + a_n x^{2n-1}, \quad \varphi_k(x) = x^{2k-1},$$

we have

$$N_r \sin \beta_r L \frac{\partial C_r}{\partial a_k} = \int_0^1 t^{2k} \varepsilon_r' dt = \int_0^1 t^{2k} \left[t I_0(\beta_r t) - \lambda_r I_1(\beta_r t)\right] dt.$$

The integrals in this formula are estimated in the following way

$$\left.\begin{aligned}
&\int\limits_0^x t^{2k+1} I_0(t)\,dt = x^{2k+1} I_1(x) - 2kx^{2k} I_0(x) + 4k^2 \int\limits_0^x t^{2k-1} I_0(t)\,dt, \\
&\int\limits_0^x t^{2k} I_1(t)\,dt = x^{2k} I_0(x) - 2kx^{2k-1} I_1(x) + 4k(k-1) \int\limits_0^x t^{2k-2} I_1(t)\,dt, \\
&\int\limits_0^x x I_0(t)\,dt = x I_1(x), \quad \int\limits_0^x I_1(t)\,dt = I_0(x) - 1.
\end{aligned}\right\} \tag{7.10.18}$$

Let us recall that the system of stresses σ_z which are statically equivalent to zero is presented in the form of eq. (7.10.5)

$$\int\limits_0^1 x\sigma_z dx = 0, \quad \int\limits_0^1 x\left(\varepsilon_s + \frac{\rho_s}{1-\nu}\right) dx = 0.$$

In the case of a cylinder extended by an axial concentrated forces Q one should take

$$\zeta = \pm L: \quad \sigma_z = -\frac{Q}{2\pi a^2} + \begin{cases} \dfrac{Q}{\pi a^2 \rho^2}, & x < \rho, \\ 0, & x > \rho, \end{cases} \qquad \rho \to 0, \quad \tau_{rz}|_{\zeta=L} = 0.$$

Hence

$$\begin{aligned}
\int\limits_0^1 \sigma_z|_{\zeta=L}\left(\varepsilon_r + \frac{\rho_r}{1-\nu}\right) x dx &= \frac{Q}{\pi a^2}\left[\frac{1}{\rho^2}\int\limits_0^\rho \left(\varepsilon_r + \frac{\rho_r}{1-\nu}\right) x dx\right]_{\rho\to 0} \\
&= \frac{Q}{2\pi a^2 \beta_r}(2 - \lambda_r\beta_r),
\end{aligned}$$

and the system of linear equations (7.10.17) takes the form ($k = 1, 2, \ldots, n$)

$$\begin{aligned}
\sum_s \sum_r C_s \frac{\partial C_r}{\partial a_k} \cos\beta_s L \cos\beta_r L \int\limits_0^1 x dx \left(\varepsilon_r + \frac{\rho_r}{1-\nu}\right)\left(\varepsilon_s + \frac{\rho_s}{1-\nu}\right) = \\
= \frac{Q}{4\pi G a^2} \sum_r \frac{1}{\beta_r}(2 - \lambda_r\beta_r) \frac{\partial C_r}{\partial a_k} \cos\beta_r L,
\end{aligned} \tag{7.10.19}$$

where

$$C_s = \frac{1}{N_s \sin \beta_s L} \sum_{k=1}^{n} a_k \int_0^1 x \varepsilon'_s \varphi_k(x)\, dx,$$

$$\frac{\partial C_r}{\partial a_k} = \frac{1}{N_r \sin \beta_r L} \int_0^1 x \varepsilon'_r \varphi_k(x)\, dx.$$

Estimation of the integrals is carried out by formulae (7.10.7)-(7.10.9) and by differentiating with respect to β_s, β_r. The difficulties are caused by calculating the double series in system (7.10.9).

6

Saint-Venant's problem

6.1 The state of stress

6.1.1 Statement of Saint-Venant's problem

A prismatic rod is the body obtained by translating a plane figure S along a straight line which is perpendicular to the plane of the figure. In this case the plane figure S presents the cross-section of the rod. The axis Oz of the rod is the straight line which is the locus of the centres of inertia of the cross-sections whereas axes Ox and Oy lying in the cross-sectional plane are directed along the principal axes of inertia of the cross-section. The origin O of the system of axes Oxy lies in a cross-section (in the cross-section $z=$ const). The cross-sections $z=0$ and $z=l$ are referred to as the end faces, their centres of inertia being respectively denoted as O^- and O^+. Let I_x and I_y designate the moments of inertia of the cross-section about the corresponding axis of this cross-section and S denote its cross-sectional area. Then

$$\left.\begin{array}{l} S=\iint\limits_S do, \quad \iint\limits_S x do=0, \quad \iint\limits_S y do=0, \\ I_x=\iint\limits_S y^2 do, \quad I_y=\iint\limits_S x^2 do, \quad \iint\limits_S xy do=0 \quad (do=dxdy), \end{array}\right\} \qquad (1.1.1)$$

for all $z \subset [0,l]$.

The Saint-Venant problem deals with the state of stress in a prismatic rod loaded by distributed forces on the end faces, the lateral surface being free of load.

The boundary conditions are set in the following form: on the end faces

$$\left.\begin{array}{llll} z=0 & -\tau_{zx}=X_z^-(x,y)\,, & -\tau_{yz}=Y_z^-(x,y)\,, & -\sigma_z=Z_z^-(x,y)\,, \\ z=l & \tau_{zx}=X_z^+(x,y)\,, & \tau_{yz}=Y_z^+(x,y)\,, & \sigma_z=Z_z^+(x,y)\,, \end{array}\right\} \tag{1.1.2}$$

where $X_z^{\mp}, Y_z^{\mp}, Z_z^{\mp}$ denote the projections of the surface forces on the coordinate axes, whilst on the lateral surface we have

$$\sigma_x n_x + \tau_{xy} n_y = 0, \quad \tau_{xy} n_x + \sigma_y n_y = 0, \tag{1.1.3}$$

$$\tau_{xz} n_x + \tau_{yz} n_y = 0. \tag{1.1.4}$$

Here $\mathbf{n}$ stands for the unit vector of the external normal to both the lateral surface and contour Γ of the cross-section, so that

$$n_x = \frac{dy}{ds}, \quad n_y = -\frac{dx}{ds}, \quad n_z = 0, \tag{1.1.5}$$

where

$$x = x(s)\,, \quad y = y(s)\,, \quad z = \text{const} \tag{1.1.6}$$

is the equation of contour Γ and s denotes the arc of the contour.

6.1.2 Integral equations of equilibrium

Let the projections on the coordinate axes of the principal vector and the principal moment $\mathbf{m}^{(O^+)}$ about the cross-sectional centre of inertia O^+ of the surface forces on the right end ($z = l$) be denoted by P, Q, R and m_x, m_y, m_z, respectively. Then

$$P = \iint\limits_S X_z^+(x,y)\,do, \quad Q = \iint\limits_S Y_z^+(x,y)\,do, \quad R = \iint\limits_S Z_z^+(x,y)\,do, \tag{1.2.1}$$

$$\left.\begin{array}{l} m_x = \displaystyle\iint\limits_S yZ_z^+(x,y)\,do, \; m_y = -\iint\limits_S xZ_z^+(x,y)\,do, \\ m_z = \displaystyle\iint\limits_S \left[xY_z^+(x,y) - yX_z^+(x,y)\right] do. \end{array}\right\} \tag{1.2.2}$$

The forces P and Q are referred to as the transverse forces whereas R is called the axial force. Further, m_x and m_y are termed bending moments whilst m_z is called the torque.

Let us obtain the conditions of equilibrium of the left part of the rod $[z, l]$. The stresses acting on the left end face present a system of distributed surface forces with the projections $-\tau_{zx}, -\tau_{yz}, -\sigma_z$ on the coordinate axes.

Since the lateral surface is free and mass forces are absent, six equations of statics can be written down as follows: three equations of the projections of the forces

$$\iint\limits_S \tau_{zx} do = P, \quad \iint\limits_S \tau_{yz} do = Q, \quad \iint\limits_S \sigma_z do = R \tag{1.2.3}$$

and three equations for the moments about axes Ox, Oy, Oz of cross-section z

$$\left.\begin{aligned} &\iint\limits_S y\sigma_z do = m_x - (l-z)\,Q, \\ -&\iint\limits_S x\sigma_z do = m_y + (l-z)\,P, \quad \iint\limits_S (x\tau_{yz} - y\tau_{zx})\,do = m_z. \end{aligned}\right\} \tag{1.2.4}$$

When $z = 0$ one can replace $\tau_{zx}, \tau_{yz}, \sigma_z$ by their expressions from eq. (1.1.2) and obtain the equilibrium conditions for the external forces, these conditions being assumed to be satisfied.

6.1.3 Main assumptions

The integral equations of equilibrium (1.2.3) and (1.2.4) can be satisfied by assuming that τ_{zx} and τ_{yz} do not depend on z whereas σ_z is a linear function of $(l-z)$

$$\tau_{zx} = \tau_{zx}(x,y), \quad \tau_{yz} = \tau_{yz}(x,y), \quad \sigma_z = \sigma_z^0(x,y) + (l-z)\,\sigma_z^1(x,y). \tag{1.3.1}$$

Of course, these assumptions are not the consequence of the above mentioned equations, but further progress in solving the problem of the equilibrium of the rod is possible only under these assumptions.

Two of the three equations of statics in the volume are written down in the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{1.3.2}$$

A consequence of these equations as well as the boundary conditions (1.1.3) on the lateral surface of the rod is the acceptance of the second set of assumptions admitted in the statement of Saint-Venant's problem

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \sigma_y = 0. \tag{1.3.3}$$

Assumptions (1.3.1) and (1.3.3) comprise the essence of the "semi-inverse method of Saint-Venant", that is, some stresses (or displacements) are prescribed and then the equations defining the remaining unknown variables become easier to deal with. Evidently, these assumptions force one to abandon the exact solution to the boundary-value problem. For instance, in Saint-Venant's problem the boundary conditions on the end faces (1.1.2) are not exactly fulfilled and they are replaced by the integral relationships (1.2.3) and (1.2.4). The acceptance of this replacement is substantiated by Saint-Venant's principle, see Subsection 4.2.8.

Due to the efforts of the founders of the theory of elasticity (Lamé, Kelvin, Boussinesq, Cerruti and others) the rigorous solutions to some boundary-value problems of elasticity theory for the regions bounded by the surfaces prescribed by a single parameter (sphere, half-space) have been obtained. Investigations aimed at exact solutions have been carried out more recently and their number increases. However the celebrated memoirs by Saint-Venant "On torsion of prisms" and "On bending of prisms", in which "the semi-inverse method" and Saint-Venant's principle were suggested, should be proclaimed as the origin of elasticity theory as an applied discipline.

6.1.4 *Normal stress σ_z in Saint-Venant's problem*

This stress can be determined in the general form for a rod of any cross-section. It is not sufficient to have the equations of statics. When solving the problem in terms of stresses, it is necessary to use the Beltrami-Michell dependences, eq. (1.5.9) of Chapter 4. In these dependences, the sum of the normal stresses σ is replaced by σ_z, the latter being a linear function of $(l - z)$

$$\sigma = \sigma_z = \sigma_z^0(x, y) + (l - z)\,\sigma_z^1(x, y). \tag{1.4.1}$$

It also follows from eq. (1.3.1) that all the second derivatives of the sought functions with respect to z vanish, so that the Laplace operator contains only derivatives with respect to x and y

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.4.2}$$

From three Beltrami-Michell equations for $\sigma_x, \sigma_y, \tau_{xy}$ we obtain

$$\frac{\partial^2 \sigma}{\partial x^2} = 0, \quad \frac{\partial^2 \sigma}{\partial y^2} = 0, \quad \frac{\partial^2 \sigma}{\partial x \partial y} = 0, \tag{1.4.3}$$

and the equation for σ_z, due to eq. (1.4.1), is satisfied identically. Thus, σ is proved to be a linear function of x and y

$$\sigma = \sigma_z = a_1 x + a_2 y + a_0 + (l - z)(b_1 x + b_2 y + b_0). \tag{1.4.4}$$

Turning to the geometric and static relationships (1.1.1), (1.2.3) and (1.2.4) it is easy to obtain

$$\left.\begin{array}{lll} a_0 = \dfrac{R}{S}, & a_1 = -\dfrac{m_y}{I_y}, & a_2 = \dfrac{m_x}{I_x}, \\ b_0 = 0, & b_1 = -\dfrac{P}{I_y}, & b_2 = -\dfrac{Q}{I_x}, \end{array}\right\} \tag{1.4.5}$$

and thus

$$\sigma_z = \frac{R}{S} - \frac{1}{I_y}\left[m_y + P(l-z)\right]x + \frac{1}{I_x}\left[m_x - Q(l-z)\right]y. \tag{1.4.6}$$

This is the law governing the distribution of the normal stresses in an elastic rod loaded by the axial tensile force R and bent by the moments m_x, m_y and the transverse forces, all applied to the end face. The quantities

$$M_x = m_x - Q(l-z), \quad M_y = m_y + P(l-z) \tag{1.4.7}$$

are the bending moments in the cross-section z. In the framework of the elementary theory of bending of beams it is adopted that

$$\sigma_z = \frac{R}{S} - \frac{M_y}{I_y}x + \frac{M_x}{I_x}y, \tag{1.4.8}$$

and this distribution of the normal stress σ_z is also valid in the case of the arbitrary bending moments M_x and M_y acting on the lateral surface.

6.1.5 Shear stresses τ_{xz} and τ_{yz}

These stresses are determined from the third equation of statics and the remaining two Beltrami-Michell equations

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} = -\left(\frac{P}{I_y}x + \frac{Q}{I_x}y\right), \tag{1.5.1}$$

$$\nabla^2 \tau_{xz} = -\frac{1}{1+\nu}\frac{P}{I_y}, \quad \nabla^2 \tau_{yz} = -\frac{1}{1+\nu}\frac{Q}{I_x}. \tag{1.5.2}$$

They are considered together with the boundary condition (1.1.4) on the lateral surface or the equivalent on contour Γ of region S

$$\text{on } \Gamma: \quad \tau_{xz}n_x + \tau_{yz}n_y = 0, \tag{1.5.3}$$

and the remaining integral conditions (1.2.3) and (1.2.4)

$$\iint\limits_S \tau_{xz}do = P, \quad \iint\limits_S \tau_{yz}do = Q, \quad \iint\limits_S (x\tau_{yz} - y\tau_{zx})\, do = m_z. \tag{1.5.4}$$

Let us notice that any solution of the static equations in the volume, eq. (1.5.1), and on the surface, eq. (1.5.3), satisfies the first two conditions in eq. (1.5.4). Indeed, multiplying by x, integrating over the area S of the cross-section and referring to eqs. (1.1.1) and (1.5.1) we have

$$\iint\limits_S x\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}\right) do =$$

$$= \iint\limits_S \left(\frac{\partial}{\partial x} x\tau_{xz}+\frac{\partial}{\partial y} x\tau_{yz}\right) do - \iint\limits_S \tau_{xz} do = -\frac{P}{I_y}\iint\limits_S x^2 do = -P.$$

By virtue of eq. (1.5.3) and the rule of transforming a surface integral into an integral over a contour, we obtain

$$\iint\limits_S \left(\frac{\partial}{\partial x} x\tau_{xz}+\frac{\partial}{\partial y} x\tau_{yz}\right) do = \oint\limits_\Gamma x\left(\tau_{xz}n_x+\tau_{yz}n_y\right) ds = 0,$$

which leads to the first relationship (1.5.4). The second relationship is derived analogously.

We also notice that eqs. (1.5.1) and (1.5.3) are consistent with each other which follows from the relationship

$$\iint\limits_S \left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}\right) do = \oint\limits_\Gamma \left(\tau_{xz}n_x+\tau_{yz}n_y\right) ds$$

$$= -\frac{P}{I_y}\iint\limits_S x do - \frac{Q}{I_x}\iint\limits_S y do = 0.$$

It is proved below that differential equations (1.5.1), (1.5.2) and the boundary conditions determine uniquely the shear stresses τ_{xz} and τ_{yz} for the prescribed value of torque m_z. As one can see, the fulfillment of the conditions on the end faces, eq. (1.1.2), is not required here. It can be said that the solution of the problem in the form stated by Saint-Venant is the rigorous solution only under the condition that the normal stresses on the end faces obey exactly the law (1.4.6) and the shear stresses are distributed according to the law obtained from the solution of the problems stated above. However the obtained solutions form a system of surface forces which is statically equivalent to any prescribed distribution of the surface forces $X_z^{\mp}, Y_z^{\mp}, Z_z^{\mp}$ on the end faces. Saint-Venant's principle answers the question to what extent the replacement of one problem by another is acceptable. This principle was mentioned in Subsection 5.7.6 when stating the problem for the equilibrium of an elastic circular cylinder. This principle was formulated and discussed in Subsections 4.2.4 and 5.1.14. Being applied to Saint-Venant's problem this principle reduces to the statement that the statically equivalent systems of forces, distributed over the end faces comprising a small part of the total surface of a sufficiently long rod, produce the states of stress in the rod's body which are essentially different in the regions near the ends and practically coincident at a sufficient distance from the ends. For verification we can refer to the examples of the exponentially decreasing stresses with increasing distance from the end loaded by a system of forces which is statically equivalent to zero in the problems of torsion of the circular cylinder (Subsection 5.7.9) and the homogeneous solutions (Subsections 5.7.8 and 5.7.9). However one can find counter-examples, among them the case of torsion of a rod with a thin-walled open section, for instance trough-like ($\sqsupset$) and Z-like (Z) and similar sections. In these examples, the influence of the system of forces which is statically equivalent to zero propagates along the rod to a considerable distance. Saint-Venant's principle is not universal but it remains an indispensable means of approaching the overwhelming majority of problems of elasticity theory.

6.2 Reduction to the Laplace and Poisson equations

6.2.1 Introducing the stress function

Let us introduce into consideration two functions $G\chi(x, y)$ and $G\alpha\Phi(x, y)$, where G denotes the shear modulus and α is a constant which will be determined below. Function χ is assumed to satisfy the Laplace equation

$$\nabla^2\chi = \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 0. \tag{2.1.1}$$

The equation of statics (1.5.1) can be satisfied identically by assuming

$$\left.\begin{aligned} \tau_{xz} &= G\alpha\frac{\partial\Phi}{\partial y} + G\left[\frac{\partial\chi}{\partial x} - \frac{1}{2G(1+\nu)}\left(\frac{P}{I_y}x^2 + 2\nu\frac{Q}{I_x}xy\right)\right], \\ \tau_{yz} &= -G\alpha\frac{\partial\Phi}{\partial x} + G\left[\frac{\partial\chi}{\partial y} - \frac{1}{2G(1+\nu)}\left(2\nu\frac{P}{I_y}xy + \frac{Q}{I_x}y^2\right)\right]. \end{aligned}\right\} \tag{2.1.2}$$

Then

$$\nabla^2\tau_{xz} = G\alpha\frac{\partial}{\partial y}\nabla^2\Phi - \frac{1}{1+\nu}\frac{P}{I_y}, \quad \nabla^2\tau_{yz} = -G\alpha\frac{\partial}{\partial x}\nabla^2\Phi - \frac{1}{1+\nu}\frac{Q}{I_x},$$

and substitution into the Beltrami-Michell equation, eq. (1.5.9) of Chapter 4, leads to the equalities

$$\frac{\partial}{\partial y}\nabla^2\Phi = 0, \quad \frac{\partial}{\partial x}\nabla^2\Phi = 0, \quad \nabla^2\Phi = \text{const}.$$

The latter constant is assumed to be equal to -2 which does not affect the generality, since the expressions for the stresses in terms of Φ already contain the constant-valued parameter α. Thus, Φ is determined from the Poisson equation

$$\nabla^2 \Phi = -2. \tag{2.1.3}$$

Taking into account eq. (1.1.5) we can write the boundary condition (1.5.3) in the form

$$\alpha \frac{\partial \Phi}{\partial s} + \left[\frac{\partial \chi}{\partial n} - \frac{P}{EI_y} \left(x^2 n_x + 2\nu xy n_y \right) - \frac{Q}{EI_x} \left(2\nu xy n_x + y^2 n_y \right) \right] = 0,$$

where $E = 2(1+\nu) G$ is Young's modulus and, as usual,

$$\frac{\partial \Phi}{\partial s} = \frac{\partial \Phi}{\partial x} \frac{dx}{ds} + \frac{\partial \Phi}{\partial y} \frac{dy}{ds}, \quad \frac{\partial \chi}{\partial n} = \frac{\partial \chi}{\partial x} n_x + \frac{\partial \chi}{\partial y} n_y.$$

Thus, functions Φ and χ can be subjected to the boundary conditions

$$\text{on } \Gamma: \quad \frac{\partial \Phi}{\partial s} = 0, \tag{2.1.4}$$

$$\text{on } \Gamma: \quad \frac{\partial \chi}{\partial n} = \frac{P}{EI_y} \left(x^2 n_x + 2\nu xy n_y \right) + \frac{Q}{EI_x} \left(2\nu xy n_x + y^2 n_y \right). \tag{2.1.5}$$

The harmonic function χ is determined by means of prescribing the normal derivative on the contour of region S. This is a classical Neumann problem which has a solution since

$$\oint_\Gamma \frac{\partial \chi}{\partial n} ds = \frac{P}{EI_y} \oint_\Gamma \left(x^2 n_x + 2\nu xy n_y \right) ds + \frac{Q}{EI_x} \oint_\Gamma \left(2\nu xy n_x + y^2 n_y \right) ds$$
$$= 2(1+\nu) \left(\frac{P}{EI_y} \iint_S x do + \frac{Q}{EI_x} \iint_S y do \right) = 0.$$

The harmonic function χ can be conveniently presented by a sum of two harmonic functions

$$\chi = \frac{P}{EI_y} \chi_1 + \frac{Q}{EI_x} \chi_2, \quad \nabla^2 \chi_1 = 0, \quad \nabla^2 \chi_2 = 0, \tag{2.1.6}$$

determined by the boundary conditions

$$\frac{\partial \chi_1}{\partial n} = x^2 n_x + 2\nu xy n_y, \quad \frac{\partial \chi_2}{\partial n} = 2\nu xy n_x + y^2 n_y. \tag{2.1.7}$$

As χ_1 and χ_2 are harmonic functions one can introduce two functions of the complex argument $\zeta = x + iy$

$$f_1(\zeta) = \chi_1 + i\vartheta_1, \quad f_2(\zeta) = \chi_2 + i\vartheta_2. \tag{2.1.8}$$

Functions ϑ_k relate to χ_k by means of the well-known Cauchy-Riemann relationships

$$\frac{\partial \chi_k}{\partial x} = \frac{\partial \vartheta_k}{\partial y}, \quad \frac{\partial \chi_k}{\partial y} = -\frac{\partial \vartheta_k}{\partial x}$$

or

$$\frac{\partial \chi_k}{\partial n} = \frac{\partial \vartheta_k}{\partial s}, \quad \frac{\partial \chi_k}{\partial s} = -\frac{\partial \vartheta_k}{\partial n} \quad (k = 1, 2) \tag{2.1.9}$$

and are determined by quadratures in terms of χ_k up to an additive constant.

Instead of the stress function Φ satisfying Poisson's equation, we introduce the following function

$$\psi(x, y) = \Phi + \frac{1}{2}\left(x^2 + y^2\right). \tag{2.1.10}$$

Due to eq. (2.1.3) this function is harmonic in S

$$\nabla^2 \psi = 0, \tag{2.1.11}$$

and is considered, in what follows, as the imaginary part of the function of the complex variable $F(\zeta)$ whose real part is denoted as $\varphi(x, y)$. Hence

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad \text{or} \quad \frac{\partial \varphi}{\partial n} = \frac{\partial \psi}{\partial s}, \quad \frac{\partial \varphi}{\partial s} = -\frac{\partial \psi}{\partial n}, \tag{2.1.12}$$

so that

$$\varphi + i\psi = F(\zeta). \tag{2.1.13}$$

The boundary condition (2.1.4) is now presented in the form

$$\frac{\partial \psi}{\partial s} = \frac{\partial \varphi}{\partial n} = x\frac{dx}{ds} + y\frac{dy}{ds} = -xn_y + yn_x. \tag{2.1.14}$$

A slightly different form of the equations for the shear stresses (2.1.2) is as follows

$$\left.\begin{aligned} \frac{1}{G}\tau_{xz} &= \alpha\frac{\partial \Phi}{\partial y} + a\left(\frac{\partial \chi_1}{\partial x} - x^2\right) + b\left(\frac{\partial \chi_2}{\partial x} - 2\nu xy\right), \\ \frac{1}{G}\tau_{yz} &= -\alpha\frac{\partial \Phi}{\partial x} + a\left(\frac{\partial \chi_1}{\partial y} - 2\nu xy\right) + b\left(\frac{\partial \chi_2}{\partial y} - y^2\right). \end{aligned}\right\} \tag{2.1.15}$$

Here and in what follows we introduce, for brevity, the notation

$$a = \frac{P}{EI_y}, \quad b = \frac{Q}{EI_x}. \tag{2.1.16}$$

6.2.2 Displacements in Saint-Venant's problem

Using the basic assumptions of Saint-Venant's problem and the generalised Hooke law we can write down the following equalities

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\nu}{E}\sigma_z, \; \varepsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu}{E}\sigma_z, \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \; \frac{\partial w}{\partial z} = \frac{\sigma_z}{E}. \tag{2.2.1}$$

Replacing σ_z by means of eq. (1.4.6) we obtain

$$\left.\begin{aligned} u &= -\nu\left(\frac{R}{ES}x - \frac{m_y}{2EI_y}x^2 + \frac{m_x}{EI_x}xy\right) + \\ &\qquad \nu(l-z)\left(\frac{1}{2}ax^2 + bxy\right) + U(y,z), \\ v &= -\nu\left(\frac{R}{ES}y - \frac{m_y}{EI_y}xy + \frac{m_x}{2EI_x}y^2\right) + \\ &\qquad \nu(l-z)\left(axy + \frac{1}{2}by^2\right) + V(z,x), \\ w &= \frac{R}{ES}z + (l-z)\left(\frac{m_y}{EI_y}x - \frac{m_x}{EI_x}y\right) + \\ &\qquad \frac{1}{2}(l-z)^2(ax+by) + W(x,y), \end{aligned}\right\} \tag{2.2.2}$$

where U, V, W are some functions playing the role of "the integration constants". By means of the third relationship (2.2.1) we have

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left\{\frac{\partial U}{\partial y} + \frac{\nu y}{EI_y}\left[m_y + P(l-z)\right]\right\} + \\ \left\{\frac{\partial V}{\partial x} - \frac{\nu x}{EI_x}\left[m_x - Q(l-z)\right]\right\} = 0.$$

The expressions in the braces depend only on z otherwise the latter equation would relate the independent variables x, y, z. Denoting the expressions in the braces by $-Z_0(z)$ and $Z_0(z)$ respectively we obtain

$$\left.\begin{aligned} U &= -\frac{\nu y^2}{2EI_y}\left[m_y + P(l-z)\right] - yZ_0(z) + Z_1(z), \\ V &= \frac{\nu x^2}{2EI_x}\left[m_x - Q(l-z)\right] + xZ_0(z) + Z_2(z). \end{aligned}\right\} \tag{2.2.3}$$

Using the two relationships of the generalised Hooke's law which have not been utilised

$$\frac{\partial w}{\partial x} = \frac{1}{G}\tau_{xz} - \frac{\partial u}{\partial z}, \quad \frac{\partial w}{\partial y} = \frac{1}{G}\tau_{yz} - \frac{\partial v}{\partial z} \tag{2.2.4}$$

and referring to eq. (2.1.15) yields

$$
\left.
\begin{aligned}
\frac{\partial W}{\partial x} &= \alpha \frac{\partial \Phi}{\partial y} + a\left(\frac{\partial \chi_1}{\partial x} - x^2\right) + b\left(\frac{\partial \chi_2}{\partial x} - 2\nu xy\right) + \nu bxy + \\
&\quad \frac{\nu a}{2}\left(x^2 - y^2\right) + yZ_0' - \left[Z_1' + \frac{m_y}{EI_y}(l-z) + \frac{P}{2EI_y}(l-z)^2\right], \\
\frac{\partial W}{\partial y} &= -\alpha \frac{\partial \Phi}{\partial x} + a\left(\frac{\partial \chi_1}{\partial y} - 2\nu xy\right) + b\left(\frac{\partial \chi_2}{\partial y} - y^2\right) + \nu axy - \\
&\quad \frac{\nu b}{2}\left(x^2 - y^2\right) - xZ_0' - \left[Z_2' - \frac{m_x}{EI_x}(l-z) + \frac{Q}{2EI_x}(l-z)^2\right].
\end{aligned}
\right\} \tag{2.2.5}
$$

The expressions in the brackets must be constant-valued since W does not depend on z. They are denoted as $-\omega_y^0$ and ω_y^0, respectively. Then

$$
\left.
\begin{aligned}
Z_1(z) &= \frac{1}{2}\frac{m_y}{EI_y}(l-z)^2 + \frac{1}{6}\frac{P}{EI_y}(l-z)^3 + u_0 + \omega_y^0 z, \\
Z_2(z) &= -\frac{1}{2}\frac{m_x}{EI_y}(l-z)^2 + \frac{1}{6}\frac{Q}{EI_x}(l-z)^3 + v_0 - \omega_x^0 z,
\end{aligned}
\right\} \tag{2.2.6}
$$

where u_0 and v_0 are the integration constants. It follows from eq. (2.2.5) that $Z_0'(z)$ is constant, thus the integrability condition written in the form

$$
\frac{\partial}{\partial y}\frac{\partial W}{\partial x} - \frac{\partial}{\partial x}\frac{\partial W}{\partial y} = 0, \quad \alpha \nabla^2 \Phi + 2Z_0'(z) = 0,
$$

and eq. (2.1.3) yield that

$$
Z_0'(z) = \alpha, \quad Z_0 = \alpha z + \omega_z^0, \quad \omega_z^0 = \mathrm{const}\,. \tag{2.2.7}
$$

Referring to eqs. (2.1.10) and (2.1.12) we can perform the following replacement

$$
\begin{aligned}
\alpha\frac{\partial \Phi}{\partial y} + yZ_0'(z) &= \alpha\left(\frac{\partial \Phi}{\partial y} + y\right) = \alpha\frac{\partial \psi}{\partial y} = \alpha\frac{\partial \varphi}{\partial x}, \\
-\alpha\frac{\partial \Phi}{\partial x} - xZ_0'(z) &= -\alpha\left(\frac{\partial \Phi}{\partial x} + x\right) = -\alpha\frac{\partial \psi}{\partial x} = \alpha\frac{\partial \varphi}{\partial y}
\end{aligned}
$$

in eq. (2.2.5). Integrating relationships (2.2.5) yields

$$
\begin{aligned}
W(x,y) &= \alpha\varphi(x,y) + a\left[\chi_1 - \frac{1}{6}(2-\nu)x^3 - \frac{1}{2}\nu xy^2\right] + \\
&\quad b\left[\chi_2 - \frac{1}{6}(2-\nu)y^3 - \frac{1}{2}\nu x^2 y\right] + w_0 + \omega_x^0 y - \omega_y^0 x.
\end{aligned} \tag{2.2.8}
$$

It remains to insert the obtained expressions (2.2.3), (2.26)-(2.2.8) into the equalities (2.2.2). The result is the following expressions for the displacements

$$
\left.
\begin{aligned}
u = {} & -\alpha yz - \frac{\nu R}{ES}x + \frac{\nu}{2EI_y}\left[m_y + P(l-z)\right]\left(x^2 - y^2\right) - \\
& \frac{\nu}{EI_x}\left[m_x - Q(l-z)\right]xy + \frac{1}{2}\frac{m_y}{EI_y}(l-z)^2 + \\
& \qquad \frac{1}{6}\frac{P}{EI_y}(l-z)^3 + u_0 + \omega_y^0 z - \omega_z^0 y, \\
v = {} & \alpha zx - \frac{\nu R}{ES}y + \frac{\nu}{EI_y}\left[m_y + P(l-z)\right]xy + \\
& \frac{\nu}{2EI_x}\left[m_x - Q(l-z)\right]\left(x^2 - y^2\right) - \frac{1}{2}\frac{m_x}{EI_x}(l-z)^2 + \\
& \qquad \frac{1}{6}\frac{Q}{EI_x}(l-z)^3 + v_0 + \omega_z^0 x - \omega_x^0 z, \\
w = {} & \alpha\varphi(x,y) + \frac{P}{EI_y}\left[\chi_1 - \frac{1}{6}(2-\nu)x^3 - \frac{1}{2}\nu xy^2\right] + \\
& \frac{Q}{EI_x}\left[\chi_2 - \frac{1}{6}(2-\nu)y^3 - \frac{1}{2}\nu x^2 y\right] + \frac{R}{ES}z + \\
& \qquad \frac{x(l-z)}{EI_y}\left[m_y + \frac{1}{2}P(l-z)\right] - \\
& \frac{y(l-z)}{EI_x}\left[m_x - \frac{1}{2}Q(l-z)\right] + w_0 + \omega_x^0 y - \omega_y^0 x.
\end{aligned}
\right\} \tag{2.2.9}
$$

The six constants $u_0, v_0, w_0, \omega_x^0, \omega_y^0, \omega_z^0$ determine the rigid body displacement.

The expressions for the projections of the linear vector of rotation $\boldsymbol{\omega}$ are as follows

$$
\left.
\begin{aligned}
\omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\gamma_{yz} - \frac{\partial v}{\partial z} = \frac{\tau_{yz}}{2G} - \frac{\partial v}{\partial z}, \\
\omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\frac{1}{2}\gamma_{zx} + \frac{\partial u}{\partial z} = -\frac{\tau_{zx}}{2G} + \frac{\partial u}{\partial z},
\end{aligned}
\right\} \tag{2.2.10}
$$

$$
\begin{aligned}
\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \alpha z + \omega_z^0 + \frac{\nu}{EI_y}\left[m_y + P(l-z)\right]y + \\
\frac{\nu}{EI_x}\left[m_x - Q(l-z)\right]x.
\end{aligned} \tag{2.2.11}
$$

The latter equation yields the value of the angle of rotation about the rod axis related to the length unit

$$
\frac{\partial \omega_z}{\partial z} = \alpha - \frac{\nu P}{EI_y}y + \frac{\nu Q}{EI_x}x. \tag{2.2.12}
$$

Integrating over the cross-sectional area and taking into account eq. (1.1.1) we obtain

$$\alpha = \frac{1}{S}\frac{\partial}{\partial z}\iint\limits_S \omega_z do. \tag{2.2.13}$$

This equality provides one with a kinematic interpretation of parameter α, i.e. it is an average twist angle of the cross-section related to the length unit.

6.2.3 Elastic line

The displacements of the points of the rod axis, i.e. its elastic line, can be obtained by setting $x = y = 0$ in eq. (2.2.9). Recalling that the harmonic functions χ_1, χ_2, φ are determined by the solution of Neumann's problem up to an additive constant we can adopt

$$\chi_1(0,0) = 0, \quad \chi_2(0,0) = 0, \quad \varphi(0,0) = 0. \tag{2.3.1}$$

Then we obtain the following equations for the elastic line

$$\left.\begin{aligned}
u(z) &= \frac{1}{2EI_y}\left[m_y(l-z)^2 + \frac{1}{3}P(l-z)^3\right] + u_0 + \omega_y^0 z,\\
v(z) &= \frac{1}{2EI_x}\left[-m_x(l-z)^2 + \frac{1}{3}Q(l-z)^3\right] + v_0 - \omega_x^0 z,\\
w(z) &= \frac{R}{ES}z + w_0.
\end{aligned}\right\} \tag{2.3.2}$$

These are the formulae for the elementary theory of bending and tension of rods. The appearing constants should be determined by means of the boundary conditions on the left end of the rod. Let us assume that the centre of inertia does not move, then

$$\left.\begin{aligned}
u(0) = 0: \quad u(z) &= -\frac{m_y l^2}{2EI_y}\left(2\frac{z}{l} - \frac{z^2}{l^2}\right) - \\
&\quad \frac{Pl^3}{6EI_y}\left(3\frac{z}{l} - 3\frac{z^2}{l^2} + \frac{z^3}{l^3}\right) + \omega_y^0 z,\\
v(0) = 0: \quad v(z) &= \frac{m_x l^2}{2EI_x}\left(2\frac{z}{l} - \frac{z^2}{l^2}\right) - \\
&\quad \frac{Ql^3}{EI_x}\left(3\frac{z}{l} - 3\frac{z^2}{l^2} + \frac{z^3}{l^3}\right) - \omega_x^0 z,\\
w(0) = 0: \quad w(z) &= \frac{R}{ES}z
\end{aligned}\right\} \tag{2.3.3}$$

and furthermore

$$\left.\begin{aligned}
\frac{\partial u}{\partial z} &= -\frac{m_y l}{EI_y}\left(1 - \frac{z}{l}\right) - \frac{Pl^2}{2EI_y}\left(1 - 2\frac{z}{l} + \frac{z^2}{l^2}\right) + \omega_y^0,\\
\frac{\partial v}{\partial z} &= \frac{m_x l}{EI_x}\left(1 - \frac{z}{l}\right) - \frac{Ql^2}{2EI_y}\left(1 - 2\frac{z}{l} + \frac{z^2}{l^2}\right) - \omega_x^0.
\end{aligned}\right\} \tag{2.3.4}$$

Let the end $z=0$ of the rod be "fixed". The Saint-Venant solution with three constants $\omega_x^0, \omega_y^0, \omega_z^0$ to be determined allows us to treat the concept of "being fixed" in two ways. The first way is adopted in the elementary theory: it is assumed that fixing admits no rotation of the tangent to the elastic line of the rod at the fixing point, that is

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0, \quad \left(\frac{\partial v}{\partial z}\right)_{z=0} = 0. \tag{2.3.5}$$

These equations determine ω_x^0 and ω_y^0 and, by eqs. (2.3.3) and (2.3.4) we obtain the equations for the elastic line and the displacements of the end of the rod axis

$$\left.\begin{aligned} u(z) &= \frac{m_y z^2}{2EI_y} + \frac{Plz^2}{6EI_y}\left(3 - \frac{z}{l}\right), \quad & u(l) &= \frac{m_y l^2}{2EI_y} + \frac{Pl^3}{3EI_y} = f_x, \\ v(z) &= -\frac{m_x z^2}{2EI_x} + \frac{Qlz^2}{6EI_x}\left(3 - \frac{z}{l}\right), \quad & v(l) &= -\frac{m_x l^2}{2EI_x} + \frac{Ql^3}{3EI_x} = f_y, \end{aligned}\right\} \tag{2.3.6}$$

which are well-known in the elementary theory of bending.

The second treatment of "being fixed" prohibits axial displacement of the elements dx, dy which lie in the cross-section $z=0$ and are adjacent to the centre of inertia of this cross-section

$$\text{for } x=0,\ y=0,\ z=0: \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0. \tag{2.3.7}$$

Due to eq. (2.2.4) these conditions can be written down in the form

$$\left(\frac{\partial u}{\partial z}\right)_0 = \omega_y^0 - \frac{m_y l}{EI_y} - \frac{Pl^2}{2EI_y} = \frac{\tau_{zx}^0}{G}, \left(\frac{\partial v}{\partial z}\right)_0 = -\omega_x^0 + \frac{m_y l}{EI_x} - \frac{Ql^2}{2EI_x} = \frac{\tau_{yz}^0}{G} \tag{2.3.8}$$

where τ_{zx}^0 and τ_{yz}^0 are the shear stresses at the centre of inertia of the cross-section. The equations for the elastic line need to be completed by the terms

$$\frac{1}{G}\tau_{xz}^0 z, \quad \frac{1}{G}\tau_{yz}^0 z, \tag{2.3.9}$$

and the displacements, accounting for the corrections due to the shear, are now as follows

$$f_x^* = f_x + \frac{\tau_{zx}^0}{G} l, \quad f_y^* = f_y + \frac{\tau_{yz}^0}{G} l. \tag{2.3.10}$$

6.2.4 Classification of Saint-Venant's problems

The solution of Saint-Venant's problem in the general statement depends upon six values which are the three projections P, R, Q and the three moments m_x, m_y, m_z. Each of the six particular problems corresponds to the action of a single force factor. Three cases, namely the action of the axial force R and the moments m_x, m_y, are elementary since no shear stresses appear and thus solving the boundary-value problem is not needed.

The solution to the problem of tension by the axial force is given by the formulae

$$\sigma_z = \frac{R}{S}, \quad u = -\frac{\nu R}{ES}x, \quad v = -\frac{\nu R}{ES}y, \quad w = \frac{R}{ES}z, \tag{2.4.1}$$

contained in the general relationships (1.4.6) and (2.2.9). Along with eq. (2.3.6) these provide us with the solution to the bending by moment m_x

$$\left.\begin{aligned} \sigma_z &= \frac{m_x}{I_x}y; & u &= -\frac{\nu m_x}{EI_x}xy, \\ v &= \frac{\nu m_x}{2EI_x}\left(x^2 - y^2\right) - \frac{m_x (l-z)^2}{2EI_x}, & w &= -\frac{m_x}{EI_x}y(l-z) \end{aligned}\right\} \tag{2.4.2}$$

and by moment m_y

$$\left.\begin{aligned} \sigma_z &= -\frac{m_y}{I_y}x; \quad u = \frac{\nu m_y}{2EI_y}\left(x^2 - y^2\right) + \frac{m_y (l-z)^2}{2EI_y}, \\ v &= \frac{\nu m_y}{EI_y}xy, \quad w = \frac{m_y}{EI_y}x(l-z). \end{aligned}\right\} \tag{2.4.3}$$

In the problem of torsion the torque is the only non-vanishing force factor. The normal stress σ_z is absent and according to eq. (2.1.2) the shear stresses are expressed in terms of the stress function Φ determined by Poisson's equation (2.1.3) subjected to the boundary condition (2.1.4)

$$\tau_{xz} = G\alpha\frac{\partial \Phi}{\partial y}, \quad \tau_{yz} = -G\alpha\frac{\partial \Phi}{\partial x}; \quad \nabla^2\Phi = -2; \quad \text{on } \Gamma: \ \frac{\partial \Phi}{\partial s} = 0. \tag{2.4.4}$$

Here parameter α implies the twist angle of a unit length rather than "the average twist angle" since P and Q vanish in formula (2.2.12). This parameter is determined from eq. (1.5.4) in terms of the torque. The displacements are obtained by means of formulae (2.2.9) and are equal to

$$u = -\alpha yz, \quad v = \alpha zx, \quad w = \alpha\varphi(x, y). \tag{2.4.5}$$

It follows from these relations that the cross-section of the rod rotates about the rod axis and does not remain plane, i.e. its points moves along the rod axis. Discovery of this fact is one of the most important achievements of

Saint-Venant's theory. The harmonic function $\varphi(x,y)$ is the solution of Neumann's problem (2.1.14) and, by virtue of eq. (2.4.5), is single-valued in S. Let us notice that the search for this function and the stress function Φ has no relation to the problem of bending by forces P or Q.

In the problem of bending by force P the normal and shear stresses are determined by formulae (1.4.6), (2.1.15) and (2.1.9):

$$\left.\begin{aligned}
\sigma_z &= -\frac{P}{I_y}(l-z)\,x, \\
\frac{1}{G}\tau_{xz} &= \alpha\frac{\partial\Phi}{\partial y} + \frac{P}{EI_y}\left(\frac{\partial\chi_1}{\partial x} - x^2\right) = \alpha\frac{\partial\Phi}{\partial y} + \frac{P}{EI_y}\left(\frac{\partial\vartheta_1}{\partial y} - x^2\right), \\
\frac{1}{G}\tau_{yz} &= -\alpha\frac{\partial\Phi}{\partial x} + \frac{P}{EI_y}\left(\frac{\partial\chi_1}{\partial y} - 2\nu xy\right) \\
&= -\alpha\frac{\partial\Phi}{\partial x} - \frac{P}{EI_y}\left(\frac{\partial\vartheta_1}{\partial x} + 2\nu xy\right),
\end{aligned}\right\} \tag{2.4.6}$$

and the displacements are as follows

$$\left.\begin{aligned}
u &= -\alpha yz + \frac{\nu P}{2EI_y}\left(x^2-y^2\right)(l-z) + \frac{Plz^2}{6EI_y}\left(3-\frac{z}{l}\right) + \left[\frac{\tau^0_{xz}}{G}z\right], \\
v &= \alpha zx + \frac{\nu P}{EI_y}xy\,(l-z) + \left[\frac{\tau^0_{yz}}{G}z\right], \\
w &= \alpha\varphi(x,y) + \frac{P}{EI_y}\left[\chi_1 - \frac{1}{6}(2-\nu)\,x^3 - \frac{1}{2}\nu xy^2\right] + \\
&\qquad \frac{Pl^2x}{EI_y}\left(\frac{z^2}{2l^2} - \frac{z}{l}\right) - \left[\frac{1}{G}\left(\tau^0_{xz}x + \tau^0_{yz}y\right)\right].
\end{aligned}\right\} \tag{2.4.7}$$

In these formulae α denotes "the average twist angle" which is also obtained by means of the torque

$$m_z = \iint\limits_S \left(x\tau_{yz} - y\tau_{zx}\right)do$$

due to the shear stresses caused by force P. According to eq. (2.4.6), these shear stresses should be expressed in terms of two stress functions Φ and χ_1 or Φ and ϑ_1. Function Φ is determined by the boundary-value problem (2.4.4) whereas χ_1 or ϑ_1 are obtained from the following boundary-value problems

$$\nabla^2\chi_1 = 0, \quad \nabla^2\vartheta_1 = 0; \quad \text{on } \Gamma: \frac{\partial\chi_1}{\partial n} = \frac{\partial\vartheta_1}{\partial s} = x^2 n_x + 2\nu xy n_y, \tag{2.4.8}$$

see eqs. (2.1.6), (2.1.7) and (2.1.9). Let us notice that the latter equality in eq. (2.4.7) requires that the following sum

$$\alpha\varphi(x,y) + \frac{P}{EI_y}\chi_1 \tag{2.4.9}$$

is single-valued.

Clearly, the results of the problem of bending by force Q are analogous to eqs. (2.4.6)-(2.4.9). A closer examination of the separate problems, namely tension, bending by a moment, torsion and bending by a force, is given below, see Subsection 6.2.7 as well as Sections 6.3 and 6.4.

6.2.5 Determination of parameter α

The problem is considered under the assumption of the simultaneous action of the torque and the transverse forces. By eqs. (1.5.4), (2.1.15) and (2.1.9) we have

$$\frac{1}{G}m_z = \frac{1}{G}\iint\limits_S (x\tau_{yz} - y\tau_{zx})\, do = -\alpha \iint\limits_S \left(x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y}\right) do -$$
$$a\left[\iint\limits_S \left(x\frac{\partial \vartheta_1}{\partial x} + y\frac{\partial \vartheta_1}{\partial y}\right) do - (1-2\nu)\iint\limits_S x^2 y\, do\right] -$$
$$b\left[\iint\limits_S \left(x\frac{\partial \vartheta_2}{\partial x} + y\frac{\partial \vartheta_2}{\partial y}\right) do + (1-2\nu)\iint\limits_S xy^2 do\right]. \quad (2.5.1)$$

Let us restrict the consideration to the case of the simply connected region S. The stress function which, by eq. (2.4.4) is constant on contour Γ can be taken as being equal to zero on it:

$$\text{on } \Gamma: \quad \Phi = 0. \qquad (2.5.2)$$

Then

$$-\iint\limits_S \left(x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y}\right) do = -\iint\limits_S \left(\frac{\partial}{\partial x}x\Phi + \frac{\partial}{\partial y}y\Phi\right) do + 2\iint\limits_S \Phi do$$
$$= -\oint\limits_\Gamma \Phi\,(xn_x + yn_y)\, ds + 2\iint\limits_S \Phi do = 2\iint\limits_S \Phi do.$$

The value

$$C = 2\iint\limits_S \Phi do \qquad (2.5.3)$$

is referred to as the geometric rigidity at torsion. Its definition for multiple connected regions is given in Subsection 6.3.4.

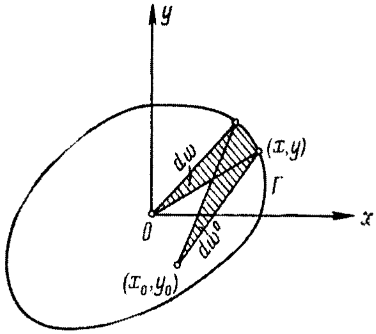

FIGURE 6.1.

By analogy one can transform the following integral

$$-\iint_S \left(x\frac{\partial \vartheta_k}{\partial x} + y\frac{\partial \vartheta_k}{\partial y} \right) do = 2\iint_S \vartheta_k do - \oint_\Gamma \vartheta_k (xn_x + yn_y)\, ds$$
$$= 2\iint_S \vartheta_k do - 2\oint_\Gamma \vartheta_k d\omega, \qquad (2.5.4)$$

where $d\omega = \frac{1}{2}(xn_x + yn_y)\, ds = \frac{1}{2}(xdy - ydx)$ denotes the area of the sector formed by the position vectors of two infinitesimally close points on Γ, these vectors originating from point O, Fig. 6.1. In the present chapter $\oint$ designates the integral over the closed contour in a counterclockwise direction. In traversing the contour, the region S lies on the left. The area of the closed curve

$$S = \omega = \oint_\Gamma d\omega = \frac{1}{2}\oint_\Gamma (xdy - ydx)$$

does not depend on the position of the origin. Indeed, having moved the origin of the coordinate system to point x_0, y_0 we have

$$\frac{1}{2}\oint_\Gamma (x'dy' - y'dx') = \frac{1}{2}\oint_\Gamma (xdy - ydx) - \frac{1}{2}\left(x_0\oint_\Gamma dy - y_0\oint_\Gamma dx \right) = \omega,$$

since the integrals $\oint_\Gamma dx$ and $\oint_\Gamma dy$ are equal to zero.

The further transformation of eq. (2.5.4) suggested by V.V. Novozhilov utilises Green's formula

$$\iint_S \left(\Phi\nabla^2\vartheta_k - \vartheta_k\nabla^2\Phi \right) do = \oint_\Gamma \left(\Phi\frac{\partial \vartheta_k}{\partial n} + \vartheta_k\frac{\partial \Phi}{\partial n} \right) ds.$$

Using it and eqs. (2.1.3), (2.5.2), (2.1.10) and (2.1.12) yields

$$2\iint_S \vartheta_k do = -\oint_\Gamma \vartheta_k \frac{\partial \psi}{\partial n} ds + 2\oint_\Gamma \vartheta_k d\omega = \oint_\Gamma \vartheta_k \frac{\partial \varphi}{\partial s} ds + 2\oint_\Gamma \vartheta_k d\omega,$$

and thus

$$2\left(\iint_S \vartheta_k do - \oint_\Gamma \vartheta_k d\omega\right) = \oint_\Gamma \vartheta_k \frac{\partial \varphi}{\partial s} ds = -\oint_\Gamma \varphi \frac{\partial \vartheta_k}{\partial s} ds, \qquad (2.5.5)$$

since the integration by parts produces no non-integral term (φ and ϑ_k being harmonic functions are single-valued in the simply connected region).

Referring now to eqs. (2.4.8) and (2.1.10) we have

$$\oint_\Gamma \varphi \frac{\partial \vartheta_1}{\partial s} ds = \oint_\Gamma \varphi\left(x^2 n_x + 2\nu xy n_y\right) ds = \iint_S \left(\frac{\partial}{\partial x} x^2\varphi + 2\nu x \frac{\partial}{\partial y} y\varphi\right) do$$

$$= 2(1+\nu)\iint_S x\varphi do + \iint_S \left(x^2 \frac{\partial \varphi}{\partial x} + 2\nu xy \frac{\partial \varphi}{\partial y}\right) do$$

and furthermore

$$\iint_S \left(x^2 \frac{\partial \varphi}{\partial x} + 2\nu xy \frac{\partial \varphi}{\partial y}\right) do = \iint_S \left(x^2 \frac{\partial \psi}{\partial y} - 2\nu xy \frac{\partial \psi}{\partial x}\right) do$$

$$= (1-2\nu)\iint_S x^2 y do + \iint_S \left(x^2 \frac{\partial \Phi}{\partial y} - 2\nu xy \frac{\partial \Phi}{\partial x}\right) do.$$

It remains only to notice that due to eq. (2.5.2)

$$\iint_S x^2 \frac{\partial \Phi}{\partial y} do = \oint_\Gamma \Phi x^2 n_y ds = 0,$$

$$-\oint_\Gamma xy \frac{\partial \Phi}{\partial x} do = -\iint_S y \frac{\partial}{\partial x} x\Phi do + \iint_S y\Phi do$$

$$= -\oint_\Gamma xy\Phi n_x ds + \iint_S y\Phi do = \iint_S y\Phi do.$$

Inserting these relationships in eq. (2.5.4) we obtain

$$\iint_S \left(x \frac{\partial \vartheta_1}{\partial x} + y \frac{\partial \vartheta_1}{\partial y}\right) do = \oint_\Gamma \varphi \frac{\partial \vartheta_1}{\partial s} ds$$

$$= 2(1+\nu)\iint_S x\varphi do + 2\nu \iint_S y\Phi do + (1-2\nu)\iint_S x^2 y do \qquad (2.5.6)$$

and by analogy

$$\iint\limits_S \left(x\frac{\partial \vartheta_2}{\partial x} + y\frac{\partial \vartheta_2}{\partial y} \right) do = \oint\limits_\Gamma \varphi \frac{\partial \vartheta_2}{\partial s} ds$$
$$= 2(1+\nu) \iint\limits_S y\varphi do - 2\nu \iint\limits_S x\Phi do - (1-2\nu) \iint\limits_S xy^2 do.$$

Returning now to the original relationship (2.5.1) we arrive at the equation determining the average twist angle α

$$m_z = GC\alpha - \frac{P}{I_y}\left(\frac{\nu}{1+\nu} \iint\limits_S y\Phi do + \iint\limits_S x\varphi do \right) +$$
$$\frac{Q}{I_x}\left(\frac{\nu}{1+\nu} \iint\limits_S x\Phi do - \iint\limits_S y\varphi do \right). \quad (2.5.7)$$

6.2.6 Centre of rigidity

Let x^* and y^* denote the values

$$\left. \begin{aligned} x^* &= \frac{1}{I_x}\left(\frac{\nu}{1+\nu} \iint\limits_S x\Phi do - \iint\limits_S y\varphi do \right), \\ y^* &= \frac{1}{I_y}\left(\frac{\nu}{1+\nu} \iint\limits_S y\Phi do + \iint\limits_S x\varphi do \right), \end{aligned} \right\} \quad (2.6.1)$$

having the dimension of length. Their calculation requires only the solution of the boundary value problem of torsion. Formula (2.5.7) is reset as follows

$$GC\alpha = m_z - (x^*Q - y^*P). \quad (2.6.2)$$

The system of shear stresses τ_{zx}, τ_{yx} in any cross-section is statically equivalent to the force $\mathbf{F} = \mathbf{i}_1 P + \mathbf{i}_2 Q$ passing through the origin of the coordinate system (i.e. the centre of inertia O of the cross-section) and the moment m_z, see Fig. 6.2. It is known from the statics that such a system (m_z and $\mathbf{F}$ at point O) is statically equivalent to a single force $\mathbf{F}^* = \mathbf{F}$ having the line of action L described by the equation

$$m_z = xQ - yP. \quad (2.6.3)$$

Equation (2.6.2) is now rewritten in the form

$$GC\alpha = (x - x_*)Q - (y - y_*)P. \quad (2.6.4)$$

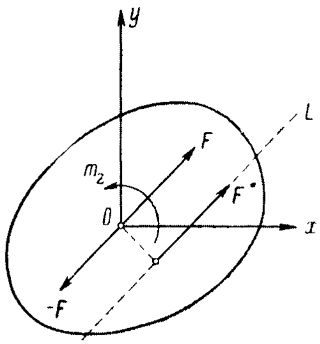

FIGURE 6.2.

Hence, $\alpha = 0$ if the straight line L passes through the point with the coordinates (x^*, y^*)

$$x = x_*, \quad y = y_* : \quad \alpha = 0. \tag{2.6.5}$$

Hence the average twist angle α is equal to zero, that is, bending is not accompanied by torsion if the line of action of force $\mathbf{F}^*$ passes through point $C(x^*, y^*)$ referred to as the centre of rigidity (or the centre of bending), Fig. 6.3a.

In the case of $m_z = 0$ the system of shear stresses τ_{zx}, τ_{yx} is statically equivalent to the force $\mathbf{F}$ with the line of action L_0 passing through the centre of inertia, then

$$GC\alpha = -x_*Q + y_*P, \tag{2.6.6}$$

see Fig. 6.3b. Generally speaking, in this case $\alpha \neq 0$, i.e. the force applied in the centre of inertia produces deformation accompanied by an average

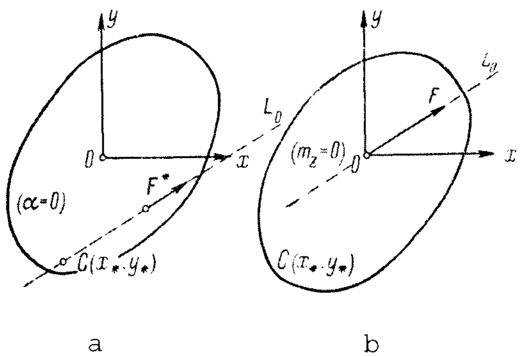

FIGURE 6.3.

twist. The exception is the case when L_0 passes simultaneously through the centre of rigidity.

When the cross-section is symmetric about a line, this line is one of the principal axes of inertia, say, axis Ox. Then the stress function Φ and the function of warping φ are respectively even and odd with respect to variable y

$$\Phi(x,y) = \Phi(x,-y), \quad \varphi(x,y) = -\varphi(x,-y).$$

It is easy to see from eq. (2.6.1) that in this case $y_* = 0$, that is, the centre of rigidity of the cross-section with a symmetry axis lies on this axis. When the cross-section has two axes of symmetry, the centre of rigidity is coincident with the point of intersection of these axes, i.e. with the centre of inertia O. In this particular case, considering the problem of bending does not require the problem of torsion to be solved.

6.2.7 Elementary solutions

As mentioned in Subsection 6.2.4 the elementary cases are concerned with the loading by the axial force R and the bending moments m_x, m_y. In these cases the only nontrivial stress is the normal stress σ_z and for this reason there is no need to consider the boundary value problems related to the shear stresses τ_{zx}, τ_{zy}.

The solution of the problem of tension is given by formulae (2.4.1), i.e. the cross-section translates along the rod axis and is subjected to an affine transformation in its plane. The relation between the initial (x,y,z) and actual coordinates (x_1, y_1, z_1) of the particle of the rod is as follows

$$x_1 = \left(1 - \frac{\nu R}{ES}\right) x, \quad y_1 = \left(1 - \frac{\nu R}{ES}\right) y, \quad z_1 = \left(1 + \frac{R}{ES}\right) z. \tag{2.7.1}$$

Under bending by moment m_y in plane zx these relations take, by virtue of eq. (2.4.3), a more complicated form

$$\left.\begin{array}{ll} x_1 = x + \dfrac{\nu}{2c}\left(x^2 - y^2\right) + \dfrac{\zeta^2}{2c}, & y_1 = y + \dfrac{\nu}{c} xy, \\ \zeta_1 = \zeta\left(l - \dfrac{x}{c}\right) & (\zeta = l - z), \end{array}\right\} \tag{2.7.2}$$

where $\dfrac{1}{c} = \dfrac{m_y}{EI_y}$ denotes the curvature of the elastic line. Further consideration implies that only the terms linear in ratios $x/c, y/c, z/c$ need to be kept for analysis of the distortion of the cross-section. The unit vectors of the external normal and the tangent to contour Γ of the cross-section $\zeta =$ const of the rod are denoted respectively as $\mathbf{n}$ and $\boldsymbol{\tau}$, the vectors $\mathbf{n}, \boldsymbol{\tau}, \mathbf{i}_3$ coinciding with the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. Putting equations (1.1.6) for this contour in the vectorial form $\mathbf{r} = \mathbf{r}(s)$ we have

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{ds}, \quad \frac{d\boldsymbol{\tau}}{ds} = \mp\frac{\mathbf{n}}{\rho}, \quad \frac{d\mathbf{n}}{ds} = \pm\frac{\boldsymbol{\tau}}{\rho}, \tag{2.7.3}$$

where $1/\rho$ denotes the curvature of curve Γ and the upper (lower) sign is taken if the curve is presented to the concave (convex) side to the origin of the coordinate system which is the centre of inertia of the cross-section.

Equation (2.7.2) contains formulae for the surface of the bent rod, where $s = q^1, \zeta = q^2$ play the part of the Gaussian coordinates. Denoting the position vector of the point on this surface by $\mathbf{R}\left(q^1, q^2\right)$, we construct the expressions for the base vectors on the surface

$$\left.\begin{aligned} \mathbf{R}_1 &= \frac{\partial \mathbf{R}}{\partial s} = \boldsymbol{\tau}\left(1 + \frac{\nu}{c}x\right) - \mathbf{n}\frac{\nu}{c}y - \frac{\zeta}{c}\frac{dx}{ds}\mathbf{i}_3, \\ \mathbf{R}_2 &= \frac{\partial \mathbf{R}}{\partial \zeta} = \mathbf{i}_3\left(1 - \frac{x}{c}\right) + \mathbf{i}_1\frac{\zeta}{c}, \end{aligned}\right\} \tag{2.7.4}$$

so that with the suggested accuracy

$$g_{11} = \mathbf{R}_1 \cdot \mathbf{R}_1 = 1 + \frac{2\nu}{c}x, \quad g_{22} = \mathbf{R}_2 \cdot \mathbf{R}_2 = 1 - \frac{2x}{c}, \quad g_{12} = \mathbf{R}_1 \cdot \mathbf{R}_2 = 0.$$

The unit vector $\mathbf{m}$ of the normal to the surface is as follows

$$\mathbf{m} = \frac{1}{\sqrt{g_{11}g_{22}}}\mathbf{R}_1 \times \mathbf{R}_2 = \mathbf{n} + \boldsymbol{\tau}\frac{\nu y}{c} - \mathbf{i}_3\frac{\zeta}{c}\frac{dy}{ds}. \tag{2.7.5}$$

Referring to eq. (2.7.3) we have

$$\left.\begin{aligned} \mathbf{R}_{11} &= \frac{\partial^2 \mathbf{R}}{\partial s^2} = \mp\frac{1}{\rho}\mathbf{n}\left(1 + \frac{\nu x}{c} \pm \frac{\nu\rho}{c}\frac{dy}{ds}\right) \pm \\ &\qquad \boldsymbol{\tau}\frac{\nu}{c}\left(\frac{dx}{ds} \mp \frac{y}{\rho}\right) - \frac{d^2x}{ds^2}\frac{\zeta}{c}\mathbf{i}_3, \\ \mathbf{R}_{22} &= \frac{\partial^2 \mathbf{R}}{\partial \zeta^2} = \frac{1}{c}\mathbf{i}_1, \quad \mathbf{R}_{12} = \frac{\partial^2 \mathbf{R}}{\partial \zeta \partial s} = -\mathbf{i}_3\frac{1}{c}\frac{dx}{ds}, \end{aligned}\right\} \tag{2.7.6}$$

and the coefficients of the second quadratic form for the surface are equal to

$$\left.\begin{aligned} b_{11} &= \mathbf{R}_{11} \cdot \mathbf{m} = \mp\frac{1}{\rho}\left(1 + \frac{\nu x}{c} \pm \frac{\nu\rho}{c}\frac{dy}{ds}\right), \\ b_{22} &= \mathbf{R}_{22} \cdot \mathbf{m} = -\frac{1}{c}\frac{dy}{ds}, \quad b_{12} = \mathbf{R}_{12} \cdot \mathbf{m} = 0. \end{aligned}\right\} \tag{2.7.7}$$

With the adopted accuracy, the coordinate lines turn out to be the curvature lines $(b_{12} = 0, g_{12} = 0)$. The vector on the normal curvature of the surface is determined by the equality

$$\tilde{k}\mathbf{m} = \frac{b_{11}ds^2 + 2b_{12}dsd\zeta + b_{22}d\zeta^2}{g_{11}ds^2 + 2g_{12}dsd\zeta + g_{22}d\zeta^2}\mathbf{m},$$

thus, the curvatures of the curvature lines (the principal normal sections $\zeta = \text{const}$ and $s = \text{const}$) are equal to

$$\tilde{k}_1 = \frac{1}{R_1} = \mp\frac{1}{\rho}\left(1 - \frac{\nu x}{c}\right) - \frac{\nu}{c}\frac{dy}{ds}, \quad \tilde{k}_2 = \frac{1}{c}\frac{dy}{ds}. \tag{2.7.8}$$

The curvature centres lie on the normal $\mathbf{m}$ to the surface. The first and the second formulae determine respectively the curvature of contour Γ^* of the deformed cross-section and that of the fiber $s = \text{const}$ on the rod surface.

For a circular cross-section of radius a we obtain, by eq. (2.7.8), that

$$x = a\cos\frac{s}{a}, \quad y = a\sin\frac{s}{a}, \quad \tilde{k}_1 = -\frac{1}{a}, \quad \tilde{k}_2 = -\frac{1}{c}\cos\frac{s}{a},$$

and the vectorial form of the equation for contour Γ^* is as follows

$$\mathbf{r}_1 = \mathbf{r} + \frac{\nu a^2}{2c}\left(\mathbf{i}_1\cos\frac{2s}{a} + \mathbf{i}_2\sin\frac{2s}{a}\right) + \mathbf{i}_1\frac{\zeta^2}{2c}, \quad \zeta_1 = \zeta\left(1 - \frac{a}{c}\cos\frac{s}{a}\right).$$

Within the accuracy adopted in this calculation, the curvatures of the contours Γ and Γ^* coincide, however the centre of the curvature moves from the centre of the circle $\mathbf{r}_0$ to point C

$$\mathbf{r}^*_{C_1} = -a\mathbf{m} = \mathbf{r}_0 - \frac{a^2}{c}\left(\nu\sin\frac{s}{a}\boldsymbol{\tau} - \frac{\zeta}{a}\cos\frac{s}{a}\mathbf{i}_3\right).$$

For the fibers $s = 0$ and $s = a\pi$ we have $\mathbf{m} = \pm\left(\mathbf{i}_1 - \frac{\zeta}{c}\mathbf{i}_3\right)$. Their centre of curvature C_2 is coincident with the centre of curvature C of the elastic line of the rod $\mathbf{r}^*_{C_2} = c\mathbf{i}_1 - \zeta\mathbf{i}_3$. The curvature of fibres $s = \frac{\pi}{2}a$ and $s = \frac{3\pi}{2}a$ is zero.

Let us also consider the case in which the contour possesses a straight line segment described by the equations

$$x = x_0 - s\cos\alpha, \quad y = s\sin\alpha \quad \left(0 \le x \le \frac{x_0}{\cos\alpha}\right).$$

The curvatures are as follows

$$\tilde{k}_1 = -\frac{\nu}{c}\sin\alpha, \quad \tilde{k}_2 = \frac{1}{c}\sin\alpha$$

and have the opposite signs. The flat part is said to become an anticlastic surface. The ratio of the curvatures is equal to Poisson's ratio ν and this fact was used for the experimental determination of ν by means of the interference bands observed by the transmission of light through the plate parallel to the flat lateral surface of the bent rod.

Let us consider the case of a rod with a rectangular cross-section. Sides $x = \pm a, \zeta = \zeta_0$ become (approximately) parabolas, which are convex toward the positive direction of axis x. The curvatures are $-\nu/c$ and $1/c$, the centres of the parabolas' curvature lie on the negative axis $\mathbf{i}_1$ at point c/ν whilst the curvature centre of the bent fiber $x = a, y = 0$ is at point $c\mathbf{i}_1$ which is the centre of curvature of the elastic line.

For the adopted approximation the lateral faces $y = \pm b$ remain flat

$$y_1 = \pm b\left(1 + \frac{\nu}{c}x\right), \quad \zeta_1 = \zeta_0\left(1 - \frac{x}{c}\right).$$

6.3 The problem of torsion

6.3.1 Statement of the problem

The torsion problem is a special case of the general Saint-Venant problem of the state of stress of a prismatic rod loaded on the end faces. The statement of the problem was given in Sections 6.1 and 6.2. However the importance and detailed development of this case suggests an independent study.

In the problem of torsion, the integral equations on the end faces $z = 0$ and $z = l$ reduce to the requirement

$$\iint_S (x\tau_{yz} - y\tau_{zx})\,do = m_z, \tag{3.1.1}$$

whereas all remaining conditions (1.2.3), (1.2.4) are homogeneous, i.e. $P = Q = R = 0, m_x = m_y = 0$. This allows one to keep the basic assumptions (1.3.3) of the semi-inverse method and additionally take

$$\sigma_z = 0. \tag{3.1.2}$$

Stresses τ_{xz} and τ_{yz} do not depend on z and, by virtue of eq. (1.5.11), are determined with the help of the remaining equations of statics in the volume and on the lateral surface

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \quad \tau_{xz}n_x + \tau_{yz}n_y = 0, \tag{3.1.3}$$

as well as condition (3.1.1) on the end faces and Beltrami's equations. As normal stresses are absent, the latter reduce to the following two equations

$$\nabla^2\tau_{xz} = 0, \quad \nabla^2\tau_{yz} = 0, \tag{3.1.4}$$

where here, and throughout the present chapter, ∇^2 denotes the plane Laplace operator (1.4.7).

The equation of statics is identically satisfied provided that the shear stresses are presented in terms of function Φ referred to as the stress function

$$\tau_{xz} = G\alpha\frac{\partial \Phi}{\partial y}, \quad \tau_{yz} = -G\alpha\frac{\partial \Phi}{\partial x}, \tag{3.1.5}$$

where α is a constant and G denotes the shear modulus. The equation of statics on the lateral surface reduces to the condition on the contour Γ of the cross-section

$$\text{on } \Gamma: \quad \frac{\partial \Phi}{\partial y} n_x - \frac{\partial \Phi}{\partial x} n_y = \frac{\partial \Phi}{\partial y}\frac{dy}{ds} + \frac{\partial \Phi}{\partial x}\frac{dx}{ds} = \frac{\partial \Phi}{\partial s} = 0. \tag{3.1.6}$$

Beltrami's equations (3.1.4) are now set in the form

$$\frac{\partial}{\partial y}\nabla^2 \Phi = 0, \quad \frac{\partial}{\partial x}\nabla^2 \Phi = 0 \tag{3.1.7}$$

and imply that $\nabla^2 \Phi$ is constant in region S. This constant can be taken as being arbitrary since the expressions for the stresses contain another constant-valued parameter α. It is adopted that

$$\nabla^2 \Phi = -2. \tag{3.1.8}$$

It remains to express condition (3.1.1) in terms of the stress function

$$-G\alpha \iint\limits_S \left(x\frac{\partial \Phi}{\partial x} + y\frac{\partial \Phi}{\partial y} \right) do = m_z. \tag{3.1.9}$$

Let us introduce the "vector of shear stresses"

$$\boldsymbol{\tau} = \tau_{xz}\mathbf{i}_1 + \tau_{yz}\mathbf{i}_2.$$

By eq. (3.1.5) we have for an arbitrary contour L

$$\boldsymbol{\tau} \cdot \mathbf{n} = \tau_n = \tau_{xz} n_x + \tau_{yz} n_y = G\alpha \frac{\partial \Phi}{\partial s}, \tag{3.1.10}$$

that is, $\tau_n = 0$ if contour L is one of the curves of the family

$$\Phi(x, y) = \text{const} = B. \tag{3.1.11}$$

At any point of the cross-section, vector $\boldsymbol{\tau}$ has the direction of the tangent vector to the curve $\Phi = \text{const}$ passing through this point. The projection of vector $\boldsymbol{\tau}$ on the tangent to L is equal to

$$\tau_{xz}\frac{dx}{ds} + \tau_{yz}\frac{dy}{ds} = -G\alpha \left(\frac{\partial \Phi}{\partial y} n_y + \frac{\partial \Phi}{\partial x} n_x \right) = -G\alpha \frac{\partial \Phi}{\partial n}. \tag{3.1.12}$$

If L belongs to the family (3.1.11), then $\tau_n = 0$ and the absolute values of this projection and vector $\boldsymbol{\tau}$ coincide and are as follows

$$\tau = G\alpha \left| \frac{\partial \Phi}{\partial n} \right|. \tag{3.1.13}$$

Thus, at the places of the cross-section where the curves of family (3.1.10) are approaching each other (i.e. the distance δn between the adjacent curves B = const and $B + \delta B$ = const decrease), a concentration of the shear stresses is observed. It can be said that the density of the curves (the trajectories of the shear stresses) serves as a measure of the value of these stresses.

The shear stress τ achieves its maximum on the contour of the region. The proof is based on the positiveness of the Laplace operator $\nabla^2\tau^2$. We have, cf. eqs. (3.1.4) and (B.4.20).

$$\frac{1}{2}\nabla^2\tau^2 = \frac{1}{2}\nabla^2\left(\tau_{xz}^2 + \tau_{yz}^2\right) = \left|\nabla\tau_{xz}\right|^2 + \left|\nabla\tau_{yz}\right|^2 > 0. \tag{3.1.14}$$

It we assume that the maximum of τ is achieved at point M of the region, then in the vicinity σ of this point and on the small circle γ with centre M we have

$$\frac{\partial\tau^2}{\partial n} < 0, \quad \iint\limits_{\sigma} \nabla^2\tau^2 do = \oint\limits_{\gamma} \frac{\partial\tau^2}{\partial n} ds < 0,$$

which is in conflict with eq. (3.1.14).

In the majority of cases the maximum of the shear stress is achieved at the point of the boundary L closest to the centre of inertia of the cross-section. However there exist exceptions which Saint-Venant discovered.

6.3.2 Displacements

Among six components of the stress tensor, the following four are equal to zero

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0, \ \varepsilon_y = \frac{\partial v}{\partial y} = 0, \ \varepsilon_z = \frac{\partial w}{\partial z} = 0, \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{3.2.1}$$

which is the result of the absence of normal stresses and shear stress τ_{xy}. The non-vanishing shears γ_{zx} and γ_{yz}, by eq. (3.1.5), can be presented in the form

$$\left.\begin{aligned} \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G}\tau_{zx} = \alpha\frac{\partial\Phi}{\partial y}, \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{G}\tau_{yx} = -\alpha\frac{\partial\Phi}{\partial x}. \end{aligned}\right\} \tag{3.2.2}$$

By virtue of eq. (3.1.8) and the fourth condition in eq. (3.2.1) we have

$$\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 2\alpha; \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\alpha z + 2f(x,y);$$
$$\frac{\partial u}{\partial y} = -\alpha z - f; \qquad \frac{\partial v}{\partial x} = \alpha z + f.$$

From the first and the second conditions (3.2.1) it is easy to conclude that $f = \mathrm{const} = \omega_z^0$. Hence,

$$u = -\alpha yz - \omega_z^0 y + u_0, \quad v = \alpha xz + \omega_z^0 x + v_0.$$

The terms describing the rigid-body rotation and translation are omitted in the following. The formulae

$$u = -\alpha yz, \quad v = \alpha xz \tag{3.2.3}$$

determine the displacement due to rotation of the cross-section $z = \mathrm{const}$ about its axis through the angle αz. The constant-valued parameter α (the twist angle) represents the angle of the relative rotation of two cross-sections which are spaced at unit length along the rod axis. The cross-sections do not remain flat since the axial displacement w is not zero, i.e. the cross-section warps. The only exception to this is for a rod with a circular cross-section.

Let us return to eq. (3.2.2) and insert u, v from eq. (3.2.3). The result is

$$\frac{\partial w}{\partial x} = \alpha\left(\frac{\partial \Phi}{\partial y} + y\right) = \alpha\frac{\partial \psi}{\partial y}, \quad \frac{\partial w}{\partial y} = -\alpha\left(\frac{\partial \Phi}{\partial x} + x\right) = -\alpha\frac{\partial \psi}{\partial x}. \tag{3.2.4}$$

Here the function

$$\psi(x, y) = \Phi + \frac{1}{2}\left(x^2 + y^2\right), \tag{3.2.5}$$

is introduced with an accuracy to an additive constant. As follows from eq. (3.1.8), $\psi(x, y)$ is harmonic in S and the conditions (3.2.4) relating function $\alpha\psi$ to w are the Cauchy-Riemann conditions. For this reason, w is also a harmonic function, and $w + i\alpha\psi$ is a function of the complex variable $x + iy$. Recalling denotation (2.1.13) we have

$$\nabla^2\psi = 0, \quad \nabla^2 w = 0, \quad w = \alpha\varphi, \quad \varphi + i\psi = F(x + iy). \tag{3.2.6}$$

Function F is referred to as the complex-valued function of torsion.

Using eq. (3.1.6) it is easy to obtain the boundary conditions for the harmonic functions ψ and φ

$$\text{on } \Gamma: \quad \frac{\partial \psi}{\partial s} = \frac{\partial \varphi}{\partial n} = yn_x - xn_y = y\frac{dy}{ds} + x\frac{dx}{ds} = \frac{1}{2}\left(x^2 + y^2\right). \tag{3.2.7}$$

Determination of warping $w(x, y)$ is reduced to Neumann's problem, which involves determining the harmonic function in terms of the normal derivative prescribed on the contour. This problem has a solution which is determined with an accuracy to an additive constant, because the necessary and sufficient conditions of its existence

$$\oint_\Gamma \frac{\partial \varphi}{\partial n} ds = \oint_\Gamma (ydy + xdx) = 0 \tag{3.2.8}$$

hold true. The independence φ of z follows from the third condition (3.2.1).

6.3.3 Theorem on the circulation of shear stresses

A prismatic rod is assumed to have hollow cavities such that the cross-section S of the rod is a multiple connected region. Its contour Γ consists of the external contour Γ_0 and the internal non-contacting contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ bounding the external regions $S_1, S_2, \ldots, S_n$, see Fig. 6.4. Let $\mathbf{n}$ denote the unit vector of the outward normal to Γ_0 (i.e. directed into S) and $\mathbf{n}_k$ denote the unit vector of the outward normal to Γ_k (i.e. directed into S_k).

According to eq. (3.1.6) the stress function Φ is constant, i.e. it is equal to $C_0, C_1, \ldots, C_n$ on each of these contours. One of these constants can be arbitrarily prescribed, neither stresses nor warping being affected. Let us take $C_k = 0$, then

$$\Phi = 0 \quad \text{on } \Gamma_0, \qquad \Phi = C_k \quad \text{on } \Gamma_k, \qquad (k = 1, 2, \ldots, n). \tag{3.3.1}$$

The constants C_k are not known in advance and their determination presents a difficult part of the problem. The solution is given by the theorem on the circulation of shear stresses.

Let $\Phi_0(x, y)$ denote the solution of the Poisson differential equation (3.1.8)

$$\nabla^2 \Phi_0 = -2 \tag{3.3.2}$$

subjected to the boundary conditions

$$\Phi = 0 \text{ on } \Gamma_0, \Gamma_1, \ldots, \Gamma_n. \tag{3.3.3}$$

Let $\Phi_k(x, y)$ denote a function which is harmonic in S, equal to 1 on Γ_k and zero on the all others contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}, \Gamma_{k+1}, \ldots, \Gamma_n$

$$\nabla^2 \Phi_k = 0, \quad \Phi_k = 0, \quad \text{on } \Gamma_0, \qquad \Phi_k = \delta_{sk} \quad \text{on } \Gamma_s. \tag{3.3.4}$$

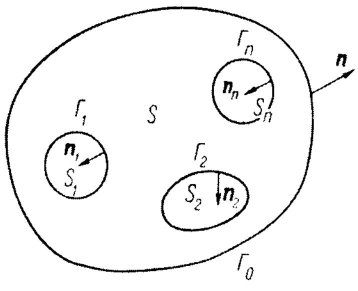

FIGURE 6.4.

These conditions determine all functions Φ_0, Φ_k in S and the stress function $\Phi(x, y)$ can be presented in terms of these functions in the following way

$$\Phi(x,y) = \Phi_0(x,y) + \sum_{k=1}^{n} C_k \Phi_k(x,y). \tag{3.3.5}$$

We introduce the circulation K of the shear stresses calculated along any closed contour L in region S

$$K = \oint_L (\tau_{xz} dx + \tau_{yz} dy) = G\alpha \oint_L \left(\frac{\partial \Phi}{\partial y} dx - \frac{\partial \Phi}{\partial x} dy \right) = -G\alpha \oint_L \frac{\partial \Phi}{\partial m} ds,$$

where $\mathbf{m}$ denotes the unit vector of the outward normal to area Ω bounded by L. On the other hand, using eqs. (3.1.5) and (3.2.4) we have

$$K = G \oint_L \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \right) + G\alpha \oint_L (x dy - y dx) = 2G\alpha\Omega,$$

since the first integral vanishes as the warping w is a single-valued function in S whereas the second integral is equal to the double area bounded by contour L. We thus arrive at the theorem on circulation of shear stresses

$$-\oint_L \frac{\partial \Phi}{\partial m} ds = 2\Omega. \tag{3.3.6}$$

Being applied to contour Γ_t, which is the boundary of cavity S_t, this formula leads to the relationship

$$\oint_{\Gamma_t} \frac{\partial \Phi}{\partial n_t} ds = 2S_t \qquad (t = 1, 2, \dots, n), \tag{3.3.7}$$

since vector $\mathbf{n}_t$ is directed into S_t and traversing contour Γ_t has its usual meaning, that is, the region S_t lies on the left. Substituting eq. (3.3.5) for Φ we arrive at the system of n equations for the unknowns C_k

$$\oint_{\Gamma_t} \frac{\partial \Phi_0}{\partial n_t} ds + \sum_{k=1}^{n} C_k \oint_{\Gamma_t} \frac{\partial \Phi_k}{\partial n_t} ds = 2S_t \qquad (t = 1, 2, \dots, n). \tag{3.3.8}$$

Applying eq. (3.3.6) to Γ_0 yields

$$\oint_{\Gamma_0} \frac{\partial \Phi_0}{\partial n} ds + \sum_{k=1}^{n} C_k \oint_{\Gamma_0} \frac{\partial \Phi_k}{\partial n} ds = -2S_* = -2\left(S + \sum_{t=1}^{n} S_t \right),$$

where the expression in parentheses is the area bounded by contour Γ_0. However this equation is a result of system (3.3.8). Indeed, replacing $S_1, S_2, \ldots, S_n$ by means of eq. (3.3.8) we obtain

$$\oint\limits_{\Gamma_0} \frac{\partial \Phi_0}{\partial n} ds + \sum_{t=1}^{n} \oint\limits_{\Gamma_0} \frac{\partial \Phi_0}{\partial n_t} ds + 2S + \sum_{k=1}^{n} C_k \left(\oint\limits_{\Gamma_0} \frac{\partial \Phi_k}{\partial n} ds + \sum_{t=1}^{n} \oint\limits_{\Gamma_t} \frac{\partial \Phi_k}{\partial n_t} ds \right) = 0. \quad (3.3.9)$$

Let Γ_* denote the set of contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ bounding region S and $\mathbf{n}_*$ denote the unit vector of the normal to Γ_* outward to S. Utilising Green's formula yields

$$\oint\limits_{\Gamma_0} \frac{\partial \Phi_k}{\partial n} ds + \sum_{t=1}^{n} \oint\limits_{\Gamma_t} \frac{\partial \Phi_k}{\partial n_t} ds = \oint\limits_{\Gamma_*} \frac{\partial \Phi_k}{\partial n_*} ds = \iint\limits_S \nabla^2 \Phi_k do = 0,$$

and applying this formula to Φ_0 we obtain

$$\oint\limits_{\Gamma_*} \frac{\partial \Phi_0}{\partial n_*} ds = \iint\limits_S \nabla^2 \Phi do = -2 \iint\limits_S do = -2S,$$

which is required.

Searching constants C_k is thus reduced to the system of linear equations (3.3.8)

$$\sum_{k=1}^{n} B_{tk} C_k = B_t, \quad B_{tk} = \oint\limits_{\Gamma_t} \frac{\partial \Phi_k}{\partial n_t} ds, \quad B_t = 2S_t - \oint\limits_{\Gamma_t} \frac{\partial \Phi_0}{\partial n_t} ds, \qquad (3.3.10)$$

the symmetry of matrix $\|B_{ik}\|$ being proved in Subsection 6.3.5. The obtained system of equations (3.3.10) will be discussed in Subsection 6.3.17.

6.3.4 Torsional rigidity

Rearranging formula (3.1.9) for the torque yields

$$\begin{aligned} m_z &= -G\alpha \iint\limits_S \left(x \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} \right) do \\ &= -G\alpha \iint\limits_S \left(\frac{\partial}{\partial x} x\Phi + \frac{\partial}{\partial y} y\Phi \right) do + 2G\alpha \iint\limits_S \Phi do \\ &= -G\alpha \oint\limits_{\Gamma_*} (x n_{*x} + y n_{*y}) \Phi ds + 2G\alpha \iint\limits_S \Phi do. \end{aligned}$$

Using eqs. (3.3.3)-(3.3.5) we have

$$\oint_{\Gamma_*} (xn_{*x} + yn_{*y})\, \Phi ds = \sum_{t=1}^{n} C_t \oint_{\Gamma_t} (xn_{tx} + yn_{ty})\, ds = -2 \sum_{t=1}^{n} C_t S_t,$$

since vector $\mathbf{n}_t$ is directed into S_t. Hence

$$m_z = 2G\alpha \left(\iint_S \Phi do + \sum_{t=1}^{n} C_t S_t \right) = G\alpha C, \tag{3.4.1}$$

where the value

$$C = 2 \left(\iint_S \Phi do + \sum_{t=1}^{n} C_t S_t \right) \tag{3.4.2}$$

having the dimension of L^4 (dimension of the moment of inertia of the area) is referred to as the geometric torsional rigidity. For the simply connected region

$$C = 2 \iint_S \Phi do, \tag{3.4.3}$$

cf. eq. (2.5.3). The geometric torsional rigidity is positive which immediately follows from the reasonings of energy, see Clapeyron's formula, eq. (3.3.3) of Chapter 2. Indeed, let us take a part of the rod $z_1 - z_2 = l$, take into account that its lateral surface is free whereas the cross-sections S_1 and S_2 are loaded by the surface forces

$$\mathbf{F}_1 = \tau_{xz}\mathbf{i}_1 + \tau_{yz}\mathbf{i}_2, \quad \mathbf{F}_2 = -\left(\tau_{xz}\mathbf{i}_1 + \tau_{yz}\mathbf{i}_2\right)$$

and undergo the displacements

$$u_1 = -\alpha y z_1, \quad v_1 = \alpha x z_1, \quad u_1 = -\alpha y z_2, \quad v_2 = \alpha x z_2,$$

see eq. (3.2.3). Then, by virtue of eqs. (3.2.3) and (3.4.1), we have

$$\begin{aligned} 2a &= \iint_{S_1} (\tau_{xz} u_1 + \tau_{yz} v_1)\, do - \iint_{S_2} (\tau_{xz} u_2 + \tau_{yz} v_2)\, do \\ &= l\alpha \iint_S (x\tau_{yz} - y\tau_{xz})\, do. \end{aligned}$$

Thus

$$2a = CG\alpha^2 l \tag{3.4.4}$$

and $C > 0$ since $a > 0$.

6.3.5 The membrane analogy of Prandtl (1904)

It is known that the problem of equilibrium of the membrane fixed on the external contour Γ_0 and loaded by the surface load p reduces to the Poisson boundary-value problem

$$\nabla^2\zeta(x,y) = -\frac{p}{T}; \qquad \zeta = 0 \quad \text{on } \Gamma_0. \tag{3.5.1}$$

Here $\zeta(x,y)$ and T denote respectively the membrane deflection and the tension intensity, the latter being constant. Assuming p to be constant and using eq. (3.5.1) we obtain

$$\iint\limits_{\Omega} \nabla^2\zeta do = \oint\limits_{L} \frac{\partial\zeta}{\partial m} ds = -\frac{p}{T}\Omega \tag{3.5.2}$$

which is the equilibrium equation of an arbitrary part Ω of the membrane bounded by contour L and $\mathbf{m}$ denotes the vector of the external normal to L.

Let us imagine that the membrane fixed on contour Γ_0 carries the rigid discs $S_1, S_2, \ldots, S_n$ bounded by contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and the discs can move only in translation in the direction of the membrane deflection, see Fig. 6.5. Region S occupied by the membrane material is bounded by the set of contours Γ_* which consists of the external contour Γ_0 and internal contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. The problem of equilibrium reduces to solving, in $n-$connected region S, the boundary value problem

$$\nabla^2\zeta(x,y) = -\frac{p}{T}; \qquad \zeta = 0 \quad \text{on } \Gamma_0, \qquad \zeta = \gamma_k \quad \text{on } \Gamma_k, \tag{3.5.3}$$

where γ_k are *a priori* unknown constants. This problem is seen to be analogous to the problem of torsion. The solution of problem (3.5.3), similar to problem (3.3.5) is presented in the form

$$\zeta(x,y) = \zeta_0(x,y) + \sum_{k=1}^{n} \gamma_k\zeta_k(x,y), \tag{3.5.4}$$

where

$$\left.\begin{aligned} &\nabla^2\zeta_0 = -\frac{p}{T}, \quad \zeta_0 = 0 \qquad \text{on } \Gamma_*, \\ &\nabla^2\zeta_k = 0, \qquad \zeta_k = \delta_{ks} \quad \text{on } \Gamma_s, \quad \zeta_k = 0 \quad \text{on } \Gamma_0. \end{aligned}\right\} \tag{3.5.5}$$

The problem of torsion becomes fully identical to the problem of the membrane equilibrium if one adopts

$$\Phi(x,y) = 2\frac{T}{p}\zeta(x,y), \quad C_k = 2\frac{T}{p}\gamma_k. \tag{3.5.6}$$

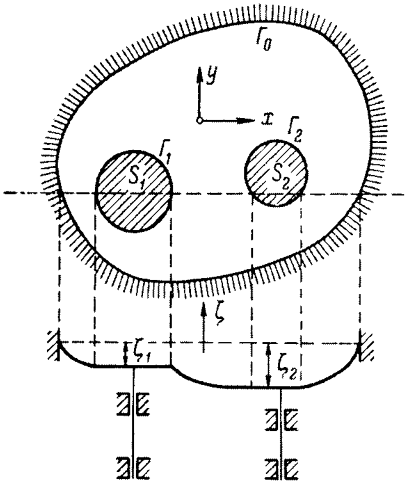

FIGURE 6.5.

The results of the problem of torsion obtained in Subsections 6.3.1-6.3.4 become transparent after comparing to the membrane. For instance, the horizontals of the membrane surface correspond to a family of the trajectories of the shear stresses $\Phi(x,y) = \text{const}$. The horizontals become more dense at the locations of rapid change in the surface, that is, these are the locations of the stress concentration in the problem of torsion.

The volume V is bounded by the curved surface of the membrane and the "plateaus" due to the discs. For this reason, the geometric torsional rigidity is proportional to this volume

$$V = \iint\limits_S \zeta(x,y)\, do + \sum_{t=1}^{n} S_t \gamma_t = \frac{p}{2T}\left(\iint\limits_S \Phi do + \sum_{t=1}^{n} C_t S_t\right) = \frac{p}{4T} C.$$

The theorem of the circulation of shear stresses (3.3.6) now has a simple explanation, because it takes the form of equation (3.5.2) for equilibrium of part Ω of the membrane. Indeed, using eq. (3.5.2) we obtain

$$\oint\limits_L \frac{\partial \zeta}{\partial m} ds = \frac{p}{2T} \oint\limits_L \frac{\partial \Phi}{\partial m} ds = -\frac{p\Omega}{T}, \quad \oint\limits_L \frac{\partial \Phi}{\partial m} ds = -2\Omega,$$

which is required.

Applying the reciprocity theorem it is easy to establish the symmetry of matrix $\|B_{tk}\|$ of the coefficients B_{tk} of the system of equations (3.3.10).

For the membrane, it is necessary to prove the following equality

$$\oint_{\Gamma_k} \frac{\partial \zeta_t}{\partial n_k} ds = \oint_{\Gamma t} \frac{\partial \zeta_k}{\partial n_t} ds. \tag{3.5.7}$$

Let us consider the case of absence of the surface load ($p = 0$), disc S_k is subjected to displacement $\gamma_k = 1$ whereas the remaining discs do not move. Then the deflection $\zeta_k(x, y)$ of membrane S is given by the solution of the boundary value problem (3.5.5). This state of the membrane with the discs will be called "state k". Then the resultant of the tension forces on contour Γ_t of disc S_t is proportional to $\oint_{\Gamma t} \frac{\partial \zeta_k}{\partial n_t} ds$. Equality (3.5.7) follows from the reciprocity theorem applied to states k and t. For another proof of the symmetry of matrix $\|B_{tk}\|$ see Subsection 6.3.17.

6.3.6 *Torsion of a rod with elliptic cross-section*

The contour of the cross-section is the ellipse

$$\Gamma: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \tag{3.6.1}$$

and the stress function Φ vanishing on L_0 can be taken in the form

$$\Phi = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

It remains to find the constant A such that the Poisson equation (3.1.8) is satisfied

$$\nabla^2 \Phi = -2A\frac{a^2 + b^2}{a^2 b^2} = -2, \quad \Phi = \frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{3.6.2}$$

Now we obtain

$$\tau_{xz} = -\frac{2G\alpha a^2}{a^2 + b^2} y, \quad \tau_{yz} = \frac{2G\alpha b^2}{a^2 + b^2} x. \tag{3.6.3}$$

The maximum shear stress is observed on the contour at the ends of the minor semi-axis, i.e. at the points of contour Γ which are nearest to the centre of the cross-section. For $a > b$ it is as follows

$$\tau_{\max} = \frac{2G\alpha a^2 b}{a^2 + b^2}. \tag{3.6.4}$$

The trajectories of the shear stresses Φ = const are presented by the family of ellipses similar to the contour ellipse Γ which are thickened in the region near the end of semi-axis b. The vector of the shear stress has the

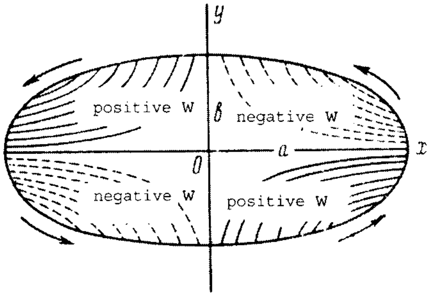

FIGURE 6.6.

direction of the tangent to the ellipse passing through the considered point in the direction of rotation from axis x to axis y, see the signs in formulae (3.6.3).

Further we have

$$\begin{aligned}\psi &= \Phi + \frac{1}{2}\left(x^2+y^2\right) = \frac{a^2b^2}{a^2+b^2} + \frac{a^2-b^2}{2\left(a^2+b^2\right)}\left(x^2-y^2\right) \\ &= \frac{a^2b^2}{a^2+b^2} + \frac{a^2-b^2}{2\left(a^2+b^2\right)} \operatorname{Im} i\left(x+iy\right)^2 . \end{aligned} \tag{3.6.5}$$

Using eq. (3.2.6) and ignoring the additive constant we obtain the warping

$$w = \alpha \frac{a^2-b^2}{2\left(a^2+b^2\right)} \operatorname{Re} i\left(x+iy\right)^2 = -\alpha \frac{a^2-b^2}{a^2+b^2} xy. \tag{3.6.6}$$

The curves $w = \text{const}$ determining the surface of the cross-section are the families of hyperbolas $xy = \text{const}$ as shown in Fig. 6.6 reproduced from the memoir by Saint-Venant.

By eq. (3.4.3) the geometric torsional rigidity is equal to

$$C = \frac{2a^2b^2}{a^2+b^2} \iint\limits_S \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) do = \frac{\pi a^3 b^3}{a^2+b^2} = \frac{4I_x I_y}{I_x+I_y} = \frac{4I_x I_y}{I_p}, \tag{3.6.7}$$

where $I_x = \frac{1}{4}\pi ab^3$ and $I_y = \frac{1}{4}\pi a^3 b$ are the moments of inertia of the cross-section whilst I_p denotes the polar moment of inertia.

In the case of a circular cross-section, warping is absent and the torsional rigidity is equal to the polar moment of inertia. Saint-Venant was the first who pointed out the fallacy of identifying the geometric torsional rigidity with the polar moment of inertia (Coulomb) for a rod with any cross-sections but circular.

6.3.7 *Inequalities for the torsional rigidity*

In what follows, the torsional rigidity for a rod with an arbitrary simple connected cross-section is compared with the rigidity of circular and elliptic rods. To this aim, we use eqs. (3.2.5), (2.5.5) and represent the torsional rigidity in the form

$$C = 2\iint\limits_S \Phi do = 2\iint\limits_S \psi do - I_p = 2\oint\limits_\Gamma \psi d\omega + \oint\limits_\Gamma \psi \frac{\partial \varphi}{\partial s} ds - I_p.$$

Turning now to eq. (2.1.10) and recalling that $\Phi = 0$ on the border we have

$$\begin{aligned} 2\oint\limits_\Gamma \psi d\omega &= \frac{1}{2}\oint\limits_\Gamma \left(x^2+y^2\right)\left(xn_x+yn_y\right) ds \\ &= \frac{1}{2}\iint\limits_S \left(3x^2+3y^2+x^2+y^2\right) do = 2I_p, \end{aligned}$$

so that by eq. (2.1.12)

$$C = I_p - \oint\limits_\Gamma \psi \frac{\partial \psi}{\partial n} ds. \tag{3.7.1}$$

On the other hand

$$\oint\limits_\Gamma \psi \frac{\partial \psi}{\partial n} ds = \frac{1}{2}\oint\limits_\Gamma \frac{\partial \psi^2}{\partial n} ds = \frac{1}{2}\iint\limits_S \nabla^2 \psi^2 do = \iint\limits_S \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] do \tag{3.7.2}$$

as $\nabla^2\psi^2 = 2\nabla\psi \cdot \nabla\psi$. This proves that the rigidity of the rod is less than its polar moment of inertia

$$C < I_p, \tag{3.7.3}$$

except for the case of the circular rod ($\psi = \text{const}$) for which $C = I_p$. A more substantial theorem has also been proved: for a given cross-sectional area, the rod of the circular cross-section has the maximum rigidity (Polya, 1948).

A more accurate estimate follows from comparing with a certain cross-section S_0 for which the solution of the torsion problem is known rather than with the circular cross-section. Assuming

$$\psi(x,y) = \psi_0(x,y) + f(x,y), \tag{3.7.4}$$

where $\psi_0(x,y)$ is the solution for S_0, i.e. a function which is harmonic in S_0 however not satisfying the boundary condition on Γ (since the boundary condition of the torsion problem is fulfilled on Γ_0). The values of ψ_0 and $\partial\psi_0/\partial n$ are given on Γ as this function is given in region S. Along with $\psi_0(x,y)$ the correcting harmonic function $f(x,y)$ determines the solution $\psi(x,y)$ of the torsion problem in S, with ψ being equal to $\frac{1}{2}(x^2+y^2)$ on Γ.

Equation (3.7.1) now takes the form

$$\oint_\Gamma \frac{\partial\psi}{\partial n}\psi ds = \oint_\Gamma \left(\frac{\partial\psi_0}{\partial n}+\frac{\partial f}{\partial n}\right)(\psi_0+f)\,ds$$
$$= \oint_\Gamma \left(\frac{\partial\psi_0}{\partial n}\psi_0+\frac{\partial\psi_0}{\partial n}f+\frac{\partial f}{\partial n}\psi_0\right)ds+\oint_\Gamma \frac{\partial f}{\partial n}f ds.$$

As f and ψ_0 are harmonic functions, Green's formula, Subsection 6.2.5, yields

$$\oint_\Gamma \frac{\partial f}{\partial n}\psi_0 ds = \oint_\Gamma f\frac{\partial\psi_0}{\partial n}ds = \oint_\Gamma \psi\frac{\partial\psi_0}{\partial n}ds - \oint_\Gamma \psi_0\frac{\partial\psi_0}{\partial n}ds.$$

Referring to formulae (3.7.1) and (3.7.2) and using the expression for ψ on the boundary we obtain

$$C = I_p + \iint_S \nabla\psi_0\cdot\nabla\psi_0 do - \oint_\Gamma \frac{\partial\psi_0}{\partial n}\left(x^2+y^2\right)do - \oint_\Gamma \nabla f\cdot\nabla f do. \tag{3.7.5}$$

The Nikolai inequality (1916) suggests the following expression for ψ_0

$$\psi_0 = \frac{2}{\sqrt{\pi}}\frac{(I_xI_y)^{3/4}}{I_p}+\frac{I_x-I_y}{2I_p}\left(x^2-y^2\right), \tag{3.7.6}$$

where I_x, I_y and $I_p = I_x + I_y$ are the moments of inertia and the polar moment of inertia of area S, respectively. The above formula presents the solution to the problem of torsion for a rod with an elliptic cross-section S_0. It can be easily proved by substituting the parameters of the ellipse (see Subsection 6.3.6) and leads to solution (3.6.5). Now we have

$$\iint_S \nabla\psi_0\cdot\nabla\psi_0 do = \frac{(I_y-I_x)^2}{I_p^2}\iint_S \left(x^2-y^2\right)do = \frac{(I_y-I_x)^2}{I_p},$$

$$\oint_\Gamma \frac{\partial\psi_0}{\partial n}\left(x^2+y^2\right)ds = \frac{I_y-I_x}{I_p}\oint_\Gamma (xn_x-yn_y)\left(x^2+y^2\right)ds = \frac{2(I_y-I_x)^2}{I_p},$$

and inserting into eq. (3.7.5) results in the inequality by E.L. Nikolai (1916)

$$C = I_p - \frac{(I_y-I_x)^2}{I_p} - \iint_S \nabla f\cdot\nabla f do = \frac{4I_xI_y}{I_p} - \iint_S \nabla f\cdot\nabla f do \le \frac{4I_xI_y}{I_p}, \tag{3.7.7}$$

an equality sign occurring for a rod with an elliptic cross-section. Evidently, this estimate is more accurate than that in eq. (3.7.3) because the right hand side in inequality (3.7.7) is less than I_p.

6.3.8 Torsion of a rod having a rectangular cross-section

For the last hundred years or so, since the classical memoir by Saint-Venant, the problem of rod torsion has been and still remains the subject of numerous investigations. The results accumulated within this period can hardly be comprehended and a wide variety of exact and approximate methods of mathematical physics have been used to construct solutions. It is worth mentioning the backward influence, that is, the problem of torsion served as an example on which these methods were developed and the feasibility of the efficient application was tested. In what follows, a few solutions for particular regions are obtained.

Let us begin with the problem of torsion of an infinite strip $-\infty \le x \le \infty, |y| \le b$. In this simple case the stress function does not depend upon x and is the solution of the boundary value problem

$$\nabla^2\Phi = \frac{d^2\Phi}{dy^2} = -2; \qquad y = \pm b: \quad \Phi = 0.$$

The solution is

$$\Phi = b^2 - y^2, \tag{3.8.1}$$

so that

$$\tau_{xz} = -2G\alpha y, \quad \tau_{yz} = 0 \tag{3.8.2}$$

and the torsional rigidity for the part $-a \le x \le a$ of the strip is equal to

$$C = 2\int\limits_{-a}^{a} dx \int\limits_{-b}^{b} \left(b^2 - y^2\right) dy = \frac{16}{3}ab^3. \tag{3.8.3}$$

The solution of the boundary value problem for the rectangular region $|x| \le a, |y| \le b$

$$\nabla^2\Phi = -2, \quad x = \pm a, \quad y = \pm b: \quad \Phi = 0 \tag{3.8.4}$$

will be sought as the sum of solution (3.8.1) and the correcting harmonic function $f(x, y)$

$$\Phi = b^2 - y^2 + f(x, y). \tag{3.8.5}$$

under the assumption $a \ge b$. Then

$$\nabla^2 f = 0; \quad x = \pm a : f(\pm a, y) = y^2 - b^2; \quad y = \pm b : f(x, \pm b) = 0. \tag{3.8.6}$$

A particular solution of Laplace equation for the rectangular region which satisfies the second condition in eq. (3.8.6) and is even with respect to x and y is as follows

$$A_k \cos\frac{2k+1}{2}\frac{\pi y}{b}\cosh\frac{2k+1}{2}\frac{\pi x}{b} \qquad (k=0,1,2,\dots).$$

For this reason, expanding $f(x,y)$ in the series

$$f(x,y)=\sum_{k=1}^{\infty} A_k \cos\frac{2k+1}{2}\frac{\pi y}{b}\cosh\frac{2k+1}{2}\frac{\pi y}{b} \tag{3.8.7}$$

it is necessary to impose the following condition

$$f(\pm a,y)=\sum_{k=0}^{\infty} A_k \cos\frac{2k+1}{2}\frac{\pi y}{b}\cosh\frac{2k+1}{2}\frac{\pi y}{b}=y^2-b^2.$$

The coefficients A_k can be determined by multiplying both parts of this equation by $\cos\frac{2k+1}{2}\frac{\pi y}{b}$ and integrating with respect to y over $(-b,b)$. The result is

$$A_s \cosh\frac{(2s+1)\pi a}{2b}=\frac{32}{\pi^3}\frac{(-1)^{s+1}}{(2s+1)^3}b^2$$

and the series

$$\frac{32}{\pi^3}b^2\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)^3}\frac{(2k+1)}{2b}\pi y$$

is a Fourier series for the even continuous function in the interval $(-2b,2b)$, this function being the parabola y^2-b^2 for $b\le y\le 2b$ and the parabola $b^2-(2b-y)^2$ for $0\le y\le b$. Thus in segment $[-2b,2b]$ its sum is equal to y^2-b^2 which is required. Thus we have arrived at the solution for the stress function

$$\Phi=b^2-y^2+\frac{32}{\pi^3}b^2\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{(2s+1)^3}\frac{\cosh\frac{(2s+1)\pi x}{2b}}{\cosh\frac{(2s+1)\pi a}{2b}}\cos\frac{(2s+1)\pi y}{2b}. \tag{3.8.8}$$

The geometric torsional rigidity is equal to

$$C=2\int_{-a}^{a}dx\int_{-b}^{b}\Phi dy=\frac{16}{3}\pi ab^3\left[1-\frac{192}{\pi^5}\frac{b}{a}\sum_{k=0}^{\infty}\frac{\tanh\frac{(2k+1)\pi a}{2b}}{(2k+1)^5}\right]=$$
$$=C_0 f_1\left(\frac{a}{b}\right), \tag{3.8.9}$$

where C_0 denotes the rigidity of the infinite strip (3.8.3). The maximum shear stress occurs at the points of the contour nearest to the centre of the rectangular and is equal to

$$\tau_{\max} = \left|\tau_{xz}\right|_{\substack{x=0\\ y=\pm b}} = \tag{3.8.10}$$

$$= 2G\alpha b\left[1 - \frac{8}{\pi^2}\sum_{s=0}^{\infty}\frac{1}{(2s+1)^2\cosh\frac{(2s+1)\pi a}{2b}}\right] = \tau^0_{\max} f_2\left(\frac{a}{b}\right).$$

Parameters C_0 and $\tau^0_{\max}$ are calculated for the strip. The following table is due to Saint-Venant and presents the values of f_1 and f_2.

a/b	1,00	1,25	1,50	2	3	4	5	10	∞
$f_1(a/b)$	0,420	0,514	0,584	0,681	0,783	0,838	0,870	0,932	1
$f_2(a/b)$	0,675	0,775	0,85	0,93	0,985	0,995	1	1	1

Table 6.1

6.3.9 *Closed-form solutions*

A wealth of closed-form solutions were obtained by Saint-Venant. One prescribes a harmonic function $\psi(x,y)$ and looks for the contours on which

$$\Phi = \psi(x,y) - \frac{1}{2}\left(x^2+y^2\right) + \text{const} = 0. \tag{3.9.1}$$

Let us consider two examples.

1. *Equilateral triangle.* We take the third-order harmonic polynomial

$$\Phi = A\left(x^3 - 3xy^2\right) - \frac{1}{2}\left(x^2+y^2\right) + D.$$

For $A = -\dfrac{1}{6a}, D = \dfrac{2}{3}a^2$ we have

$$\begin{aligned}\Phi &= -\frac{1}{6a}\left(x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3\right)\\ &= \frac{1}{6a}(a-x)\left(x+2a+y\sqrt{3}\right)\left(x+2a-y\sqrt{3}\right),\end{aligned} \tag{3.9.2}$$

i.e. Φ vanishes on the following straight lines

$$x - a = 0, \quad x + 2a \pm y\sqrt{3} = 0,$$

forming an equilateral triangle with height $3a$. Figure 6.7 shows this triangle and the family of curves $\Phi = \text{const}$.

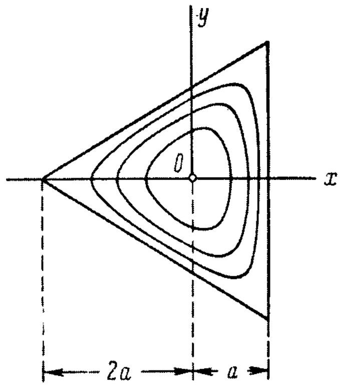

FIGURE 6.7.

2. *Circular rod with a circular longitudinal groove* (Weber), Fig. 6.8. In polar coordinates, function Φ is given by

$$\Phi = Ar\cos\theta + B\frac{\cos\theta}{r} - \frac{1}{2}r^2 + D. \tag{3.9.3}$$

Let us take the origin of the polar coordinate system at the centre of the groove. The equation for the contour of the circle of radius a with a circular groove of radius b is as follows

$$\left(b^2 - r^2\right)(r - 2a\cos\theta) = r\left(b^2 - 2ab^2\frac{\cos\theta}{r} + 2ar\cos\theta - r^2\right). \tag{3.9.4}$$

Hence the function

$$\Phi = \frac{1}{2}\left(b^2 - 2ab^2\frac{\cos\theta}{r} + 2ar\cos\theta - r^2\right) \tag{3.9.4}$$

having the form required by eq. (3.9.3) is the solution of the problem of torsion for the considered region. The shear stress at the points of axis x is equal to

$$\tau_{yz}\big|_{y=0} = -Ga\left.\frac{\partial\Phi}{\partial x}\right|_{\theta=0} = -G\alpha\left.\frac{\partial\Phi}{\partial r}\right|_{\theta=0} = G\alpha\left(r - \frac{ab^2}{r^2} - a\right),$$

i.e. at the middle of the groove $(r = b)$ and at the end of the diameter opposite to the origin of the coordinate system $(r = 2a - b)$ the stresses are respectively given by

$$\tau_{yz}^{(1)} = G\alpha\,(b - 2a), \quad \tau_{yz}^{(2)} = G\alpha\left(a - \frac{b^2}{4a}\right).$$

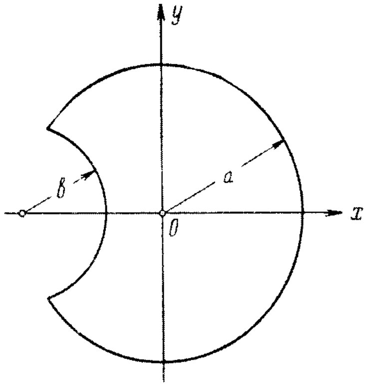

FIGURE 6.8.

For $a \gg b$ we have

$$\left|\tau_{yz}^{(1)}\right| = 2G\alpha a, \quad \tau_{yz}^{(2)} = G\alpha a. \tag{3.9.5}$$

The presence of a vanishingly small groove causes a stress concentration, namely, the shear stresses in the middle of the groove turn out to be twice as much as the stress $G\alpha a$ for the circular shaft.

6.3.10 Double connected region

The cross-section of the twisted rod is an annular region S bounded by the external contour Γ_0 and internal contour Γ_1, the regions in Γ_0 and Γ_1 being denoted respectively as S_0 and S_1, so that $S = S_0 - S_1$. The conformal transformation of the circular ring σ in the plane $\zeta = \rho e^{i\vartheta}$ into S is assumed to be given. The function carrying out this transformation is given in σ by a Laurent series

$$z = x + iy = \omega(\zeta) = \sum_{n=-\infty}^{\infty} (\alpha_n + i\beta_n)\zeta^n = \sum_{n=-\infty}^{\infty} \rho^n (\alpha_n + i\beta_n) e^{in\vartheta}. \tag{3.10.1}$$

The radii of the circles γ_0 and γ_1 transformed into Γ_0 and Γ_1 are denoted as ρ_0 and ρ_1 respectively.

The stress function Φ solving the torsion problem for region S is given by series (3.3.5)

$$\Phi = \Phi_0(x, y) + C_1 \Phi_1(x, y), \tag{3.10.2}$$

where, due to eqs. (3.3.2)-(3.34)

$$\nabla^2 \Phi_0 = -2; \qquad \Phi_0 = 0 \quad \text{on } \Gamma_0,\ \Gamma_1; \tag{3.10.3}$$

$$\nabla^2 \Phi_1 = 0; \qquad \Phi_1 = 0 \quad \text{on } \Gamma_0; \qquad \Phi_1 = 1 \quad \text{on } \Gamma_1. \tag{3.10.4}$$

It follows immediately from conditions (3.10.4) that

$$\Phi_1 = \frac{1}{\ln\dfrac{\rho_1}{\rho_0}} \ln\frac{\rho}{\rho_0}. \tag{3.10.5}$$

Assuming that the stress function $\Phi_*(x,y)$ for the solid rod (i.e. for region S) is known we put $\Phi_0(x,y)$ in the form

$$\Phi_0(x,y) = \Phi_*(x,y) - \sum_{k=1}^{\infty}\left(\frac{\rho^k}{\rho_0^k} - \frac{\rho_0^k}{\rho^k}\right)(b_k\cos k\vartheta + b_k'\sin k\vartheta) - \frac{b_0}{\ln\dfrac{\rho_1}{\rho_0}}\ln\frac{\rho}{\rho_0}. \tag{3.10.6}$$

Indeed, $\Delta\Phi_0 = -2$ and Φ_0 is equal to zero on Γ_0. It remains only to subject the constants b_k, b_k', b_0 to condition (3.10.3) on Γ_1. Let

$$\begin{aligned}\chi(\vartheta) &= \Phi_*(x,y)|_{\Gamma_1} \\ &= \Phi_*\left(\sum_{n=-\infty}^{\infty}\rho_1^n(\alpha_n\cos n\vartheta - \beta_n\sin n\vartheta),\ \sum_{n=-\infty}^{\infty}\rho_1^n(\alpha_n\sin n\vartheta + \beta_n\cos n\vartheta)\right)\end{aligned}$$

designate $\Phi_*(x,y)$ on Γ_1 (on γ_1). It is a $2\pi-$periodic function of ϑ which can be presented by the trigonometric series

$$\chi(\vartheta) = \chi_0 + \sum_{k=1}^{\infty}(p_k\cos k\vartheta + q_k\sin k\vartheta), \tag{3.10.7}$$

and the unknowns b_k, b_k', b_0 can be expressed in terms of the coefficients of this series. The solution (3.10.2) now can be set in the form

$$\begin{aligned}\Phi(x,y) = (C_1 - \chi_0)\frac{\ln\dfrac{\rho}{\rho_0}}{\ln\dfrac{\rho_0}{\rho_1}} &+ \\ \Phi_*(x,y) - \sum_{k=1}^{\infty}\frac{\rho_1^k}{\rho^k}\frac{\rho^{2k}-\rho_0^{2k}}{\rho_1^{2k}-\rho_0^{2k}}&(p_k\cos k\vartheta + q_k\sin k\vartheta).\end{aligned} \tag{3.10.8}$$

Should the first term be preserved, the expression (3.2.6) for the complex function of torsion will contain a term proportional to $\ln\zeta$ and the warping $w(x,y)$ will no longer be single-valued. This yields C_1

$$C_1 = \chi_0 = \frac{1}{2\pi}\int_0^{2\pi}\chi(\vartheta)\,d\vartheta = \frac{1}{2\pi}\int_0^{2\pi}\Phi_*(x,y)|_{\Gamma_1}\,d\vartheta. \tag{3.10.9}$$

Let us now proceed to calculating the geometric rigidity. By virtue of eq. (3.4.2) for the double-connected region we have

$$C = 2\chi_0 S_1 + 2\iint\limits_S \Phi(x,y)\,do = 2\chi_0 S_1 + C_* - 2\iint\limits_{S_1} \Phi_*(x,y)\,do -$$

$$2\sum_{k=1}^{\infty} \frac{\rho_1^k}{\rho_1^{2k}-\rho_0^{2k}} \int\limits_{\rho_1}^{\rho_0} \left(\rho^k - \frac{\rho_0^{2k}}{\rho^k}\right) \rho d\rho \int\limits_0^{2\pi} (p_k \cos k\vartheta + q_k \sin k\vartheta) \left|\omega'(\zeta)\right|^2 d\vartheta. \tag{3.10.10}$$

Here C_* denotes the geometric rigidity of the solid rod (of area S_0 bounded by contour Γ_0) and the area element $d\sigma$ is replaced by the product of the area element $\rho d\rho d\vartheta$ and $\left|\omega'(\zeta)\right|^2$, while integrating over the area of the circular ring σ.

The projection τ_s of vector $\boldsymbol{\tau}$ on the tangent to the trajectory of the shear stress $\Phi = \text{const}$ is given by eq. (3.1.12), i.e.

$$\tau_s = -G\alpha \left[\frac{\partial \Phi_*}{\partial n} - \frac{1}{\left|\omega'(\zeta)\right|} \sum_{k=1}^{\infty} \frac{k\rho_1^k}{\rho_1^{2k}-\rho_0^{2k}} \frac{\rho^{2k}+\rho_0^{2k}}{\rho^{k+1}} (p_k \cos k\vartheta + q_k \sin k\vartheta)\right] \tag{3.10.11}$$

as $\delta n = \left|\omega'(\zeta)\right| \delta\rho$. In particular, on contours Γ_0 and Γ_1 belonging to the family $\Phi = \text{const}$, we have

$$\begin{aligned} (\tau_s)_0 &= -G\alpha \left[\frac{\partial \Phi_*}{\partial n} - \frac{2}{\left|\omega'(\zeta)\right|} \sum_{k=1}^{\infty} \frac{k}{\rho_0} \frac{\rho_1^k \rho_0^k}{\rho_1^{2k}-\rho_0^{2k}} (p_k \cos k\vartheta + q_k \sin k\vartheta)\right]_{\Gamma_0}, \\ (\tau_s)_1 &= -G\alpha \left[\frac{\partial \Phi_*}{\partial n} - \frac{1}{\left|\omega'(\zeta)\right|} \sum_{k=1}^{\infty} \frac{k}{\rho_1} \frac{\rho_1^{2k}+\rho_0^{2k}}{\rho_1^{2k}-\rho_0^{2k}} (p_k \cos k\vartheta + q_k \sin k\vartheta)\right]_{\Gamma_1}. \end{aligned} \tag{3.10.12}$$

The magnitude τ of the vector of shear stress is equal to the absolute value of expression (3.10.11).

6.3.11 Elliptic ring

The cross-section S is a ring-shaped region bounded from the inside and outside by the ellipses Γ_0 and Γ_1 which have identical focal points. The conformal transformation (3.10.1) of the ring σ into this region is carried out by the formula

$$z = x + iy = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) = R\left[\left(\rho + \frac{m}{\rho}\right)\cos\vartheta + i\left(\rho - \frac{m}{\rho}\right)\sin\vartheta\right] \tag{3.11.1}$$

where the constants R and m are determined in terms of the semi-axes (a_0, b_0) and (a_1, b_1) of ellipses Γ_0 and Γ_1

$$\left.\begin{aligned} R &= \frac{a_0+b_0}{2} = \frac{a_1+b_1}{2\rho_1}, \quad m = \frac{a_0-b_0}{a_0+b_0} = \frac{a_1-b_1}{a_1+b_1}\rho_1^2; \\ \rho_1 &= \frac{a_1+b_1}{a_0+b_0} = \frac{a_0-b_0}{a_1-b_1}. \end{aligned}\right\} \tag{3.11.2}$$

Here the internal and external radii γ_0 and γ_1 of the circles bounding the ring σ are taken to be equal to $\rho_0 = 1, \rho_1 < 1$, thus $0 < m \leq \rho_1 \leq \rho \leq 1. R > 0$ and the family of ellipses $\rho = \text{const}$ consists of the cofocal ellipses with the distance $2c = \sqrt{a^2 - b^2} = 4R\sqrt{m}$ between the foci.

The stress function $\Phi_*(x, y)$ for the solid ellipse (3.6.2) takes the form

$$\Phi_*(x,y) = \frac{R^2}{2(1+m^2)}\left[\left(1-m^2\right)^2 - (1-m)^2\frac{x^2}{R^2} - (1+m)^2\frac{y^2}{R^2}\right], \tag{3.11.3}$$

and its value on ellipse Γ_1 is given by

$$\Phi_*\left(R\left(\rho_1 + \frac{m}{\rho_1}\right)\cos\vartheta, R\left(\rho_1 - \frac{m}{\rho_1}\right)\sin\vartheta\right) = \chi(\vartheta) =$$
$$= R^2\left(1-\rho_1^2\right)\frac{\rho_1^2 - m^2}{\rho_1^2}\left(\frac{1}{2} - \frac{m}{1+m^2}\cos 2\vartheta\right).$$

Hence, by virtue of eqs. (3.10.9) and (3.10.8)

$$C_1 = \frac{1}{2}R^2\left(1-\rho_1^2\right)\frac{\rho_1^2 - m^2}{\rho_1^2}, \tag{3.11.4}$$

$$\Phi(x,y) = \Phi_*(x,y) + \frac{mR^2}{1+m^2}\frac{\rho_1^2 - m^2}{\rho_1^2 + 1}\left(\frac{1}{\rho^2} - \rho^2\right)\cos 2\vartheta. \tag{3.11.5}$$

The geometric torsional rigidity is due to eq. (3.10.10)

$$C = C_* + R^2\left(1-\rho_1^2\right)\frac{\rho_1^2 - m^2}{\rho_1^2}S_1 - 2\iint\limits_{S_1}\Phi_*(x,y)\,do +$$
$$2\frac{mR^2}{1+m^2}\frac{\rho_1^2 - m^2}{\rho_1^2 + 1}\int\limits_{\rho_1}^{1}\left(\frac{1}{\rho^2} - \rho^2\right)d\rho\int\limits_0^{2\pi}\left|\omega'(\zeta)\right|^2\cos 2\vartheta d\vartheta.$$

Here, by eq. (3.11.3)

$$\iint\limits_{S_1}\Phi_*(x,y)\,do = \frac{R^2}{2(1+m^2)}\left[\left(1-m^2\right)^2 S_1 - \right.$$
$$\left. (1-m)^2\frac{I_y^1}{R^2} - (1+m)^2\frac{I_x^1}{R^2}\right],$$

where I_x^1 and I_y^1 denote the moments of inertia of region S_1

$$S_1 = \pi R^2 \frac{\rho_1^4 - m^2}{\rho_1^2}, \quad I_x^1 = \frac{1}{4} R^2 S_1 \left(\rho_1 - \frac{m}{\rho_1}\right)^2, \quad I_y^1 = \frac{1}{4} R^2 S_1 \left(\rho_1 + \frac{m}{\rho_1}\right)^2.$$

Noticing that

$$\begin{aligned} |\omega'(\zeta)|^2 &= \omega'(\zeta)\,\bar{\omega}'(\bar{\zeta}) = R^2 \left(1 - \frac{m}{\zeta^2}\right)\left(1 - \frac{m}{\bar{\zeta}^2}\right) \\ &= R^2 \left(1 + \frac{m^2}{\rho^4} - \frac{2m}{\rho^2}\cos 2\vartheta\right) \qquad \left(\bar{\zeta} = x - iy = \rho e^{-i\vartheta}\right), \end{aligned}$$

we obtain the following expression

$$C = C_* - \left\{ R^2 S_1 \left[\frac{m\left(1 - m^2\right)}{1 + m^2} + \frac{1}{2}\left(\rho_1 - \frac{m}{\rho_1}\right)^2 \right] + 2\pi R^4 \frac{m^2}{1 + m^2} \frac{\rho_1^2 - m^2}{\rho_1^2} \frac{\left(1 - \rho_1^2\right)^2}{1 + \rho_1^2} \right\}. \tag{3.11.6}$$

The value in the braces describes the decrease in the rigidity due to cavity S_1.

6.3.12 *Eccentric ring*

The cross-section of a twisted rod is an annular region S of plane z bounded by the external circle Γ_0 of radius r_0 and the internal circle Γ_1 of radius r_1. The distance between the centres of the circles is denoted as e. The function performing the conformal transformation of the annular ring σ in the plane $\zeta = \rho e^{i\vartheta}$ into region S has the form

$$z = x + iy = \omega(\zeta) = \frac{\zeta}{1 - a\zeta} = \frac{\rho(\cos\vartheta - a\rho) + i\rho\sin\vartheta}{1 + a^2\rho^2 - 2a\rho\cos\vartheta}, \tag{3.12.1}$$

where a denotes a real-valued constant and $a\rho < 1$ in region σ $(\rho_1 \le \rho \le \rho_0)$.

The circles γ_0 and γ_1 of the radii ρ_0 and ρ_1, i.e. the internal and external boundaries of ring σ, respectively, are transformed into the circles Γ_0 and Γ_1 of region S. The abscissas of the points where these circles intersect axis x are equal to

$$x_1^0 = \frac{\rho_0}{1 - a\rho_0}, \quad x_2^0 = -\frac{\rho_0}{1 + a\rho_0}; \quad x_1^1 = \frac{\rho_1}{1 - a\rho_1}, \quad x_2^1 = -\frac{\rho_1}{1 + a\rho_1},$$

see Fig. 6.9. Therefore, the abscissas of the centres of circles Γ_0 and Γ_1, their radii and the distance between their centres are as follows

$$\left.\begin{aligned} c_0 &= \frac{1}{2}\left(x_1^0 + x_2^0\right) = \frac{a\rho_0^2}{1 - a^2\rho_0^2}, \\ c_1 &= \frac{1}{2}\left(x_1^1 + x_2^1\right) = \frac{a\rho_1^2}{1 - a^2\rho_1^2}, \end{aligned}\right\} \tag{3.12.2}$$

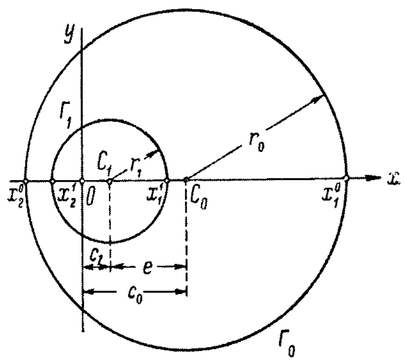

FIGURE 6.9.

$$\left.\begin{aligned} r_0 &= \frac{1}{2}\left(x_1^0 - x_2^0\right) = \frac{\rho_0}{1 - a^2\rho_0^2}, \; r_1 = \frac{\rho_1}{1 - a^2\rho_1^2}; \\ e &= c_0 - c_1 = a\left[r_0^2 - r_0^1 - \left(c_0^2 - c_1^2\right)\right]. \end{aligned}\right\} \tag{3.12.3}$$

Three equations (3.12.3) allow one to determine the parameters ρ_0, ρ_1, a in terms of the geometric dimensions of the cross-section S

$$\left.\begin{aligned} \rho_0 = \frac{2r_0}{\sqrt{1 + 4a^2r_0^2} + 1}, \quad \rho_1 = \frac{2r_1}{\sqrt{1 + 4a^2r_1^2} + 1}, \\ a = e\left[(r_0 - r_1 + e)(r_0 + r_1 - e)(r_0 + r_1 + e)(r_0 - r_1 - e)\right]^{-1/2}. \end{aligned}\right\} \tag{3.12.4}$$

The equations for circles Γ_0 and Γ_1 are written down as follows

$$\left.\begin{aligned} r_0^2 - (x - c_0)^2 - y^2 = 0, \\ r_1^2 - (x - c_1)^2 - y^2 = 0. \end{aligned}\right\} \tag{3.12.5}$$

The stress function for a solid circular rod of cross-section S_0 is set in the form

$$\Phi_*(x, y) = \frac{1}{2}\left[r_0^2 - (x - c_0)^2 - y^2\right], \tag{3.12.6}$$

compare eq. (3.6.2) for $a = b$, and, by virtue of eqs. (3.12.5) and (3.12.3), its value on Γ_1 is given by

$$\begin{aligned} \Phi_*|_{\Gamma_1} &= \frac{1}{2}\left\{\left[r_0^2 - (x - c_0)^2 - y^2\right] - \left[r_1^2 - (x - c_1)^2 - y^2\right]\right\} \\ &= \frac{1}{2}\frac{e}{a} + ex = \frac{1}{2}\frac{e}{a} + \frac{e}{a}\sum_{k=1}^{\infty}(a\rho_1)^k \cos k\vartheta. \end{aligned} \tag{3.12.7}$$

Here we used an expansion for $x|_{\Gamma_1}$ in a trigonometric series obtained with the help of eq. (3.12.1). Using eqs. (3.10.8) and (3.10.9) we have

$$\Phi(x,y) = \Phi_*(x,y) + e\sum_{k=1}^{\infty}\frac{\alpha^k a^{k-1}}{1-\alpha^k}\left(\rho^k - \frac{\rho_0^{2k}}{\rho^k}\right)\cos k\vartheta, \tag{3.12.8}$$

where $\alpha = \rho_1^2/\rho_0^2$ and $C_1 = e/2a$.

The geometric rigidity is given by eq. (3.10.10) where $C_* = I_p$, then

$$\begin{aligned} 2C_1S_1 - 2\iint\limits_{S_1}\Phi_* do &= \pi r_1^2\left[\frac{e}{a} - \left(r_0^2 - c_0^2\right) + \left(\frac{r_1^2}{2} + c_1^2\right) - 2c_0c_1\right] \\ &= -\frac{\pi r_1^4}{2} - 2c_1 e\pi r_1^2. \end{aligned}$$

Noticing that

$$\begin{aligned} |\omega'(\zeta)|^2 &= \left(1 - 2a\rho\cos\vartheta + a^2\rho^2\right)^{-2} \\ &= \frac{1+a^2\rho^2}{(1-a^2\rho^2)^3} + 2\sum_{k=1}^{\infty}\frac{(a\rho)^k}{(1-a^2\rho^2)^2}\left(\frac{1+a^2\rho^2}{1-a^2\rho^2} + k\right)\cos k\vartheta, \end{aligned}$$

we obtain

$$\begin{aligned} a^k &= \int\limits_{\rho_1}^{\rho}\left(\rho^k - \frac{\rho_0^{2k}}{\rho^k}\right)\rho d\rho\int\limits_0^{2\pi}|\omega'(\zeta)|^2\cos k\vartheta d\vartheta \\ &= 2\pi\int\limits_{\rho_1}^{\rho_0}\left[(a\rho)^{2k} - (a\rho_0)^{2k}\right]\left(\frac{1+a^2\rho^2}{1-a^2\rho^2} + k\right)\frac{\rho d\rho}{(1-a^2\rho^2)^2}. \end{aligned}$$

By introducing a new variable $q = 1 - a^2\rho^2$, we can easily estimate the integral

$$\begin{aligned} &\int\limits_{\rho_1}^{\rho_0}\left[(a\rho)^{2k} - (a\rho_0)^{2k}\right]\left(\frac{1+a^2\rho^2}{1-a^2\rho^2} + k\right)\frac{\rho d\rho}{(1-a^2\rho^2)^2} = \\ &\qquad = \frac{1}{2a^2}\left[(a\rho_0)^{2k}\int\limits_{q_1}^{q_0}\left(k - 1 + \frac{2}{q}\right)\frac{dq}{q^2} + \int\limits_{q_1}^{q_0} d\frac{(1-q)^{k+1}}{q^2}\right]. \end{aligned}$$

Thus we obtain

$$2e\sum_{k=1}^{\infty}\frac{\alpha^k a^{k-1}}{1-\alpha^k}\int_{\rho_1}^{\rho_0}\left(\rho^k-\frac{\rho_0^{2k}}{\rho^k}\right)\rho d\rho\int_0^{2\pi}\left|\omega'(\zeta)\right|^2\cos k\vartheta d\vartheta =$$

$$= 2\pi\frac{e}{a^3}\sum_{k=1}^{\infty}\left\{\frac{(a\rho_1)^{2k}}{1-\alpha^k}\left[(k+1)\left(\frac{1-q_1}{q_1}-\frac{1-q_0}{q_0}\right)+\left(\frac{1-q_1}{q_1}\right)^2-\right.\right.$$
$$\left.\left.\left(\frac{1-q_0}{q_0}\right)^2\right]+\frac{\alpha^k}{1-\alpha^k}\left[\frac{(1-q_0)^{k+1}}{q_0^2}-\frac{(1-q_1)^2}{q_1^2}\right]\right\}.$$

Inserting the expressions for q_0, q_1 and utilising relationships (3.12.2) and (3.12.3) for the geometric parameters we arrive at the following expression for the geometric rigidity

$$C = C_* - \left[\frac{\pi r_1^4}{2}+2\pi e^2 r_1^2+2\pi\frac{e^2}{a^2}\sum_{k=1}^{\infty}k\frac{\alpha^k (a\rho_1)^{2k}}{1-\alpha^k}\right]. \tag{3.12.9}$$

The series in this equation can be transformed to the form suggested by N.I. Muskhelishvili

$$C = C_* - \left[\frac{\pi r_1^4}{2}+2\pi e^2 r_1^2+2\pi e^2\rho_1^2\sum_{s=1}^{\infty}\frac{\alpha^s}{\left(1-a_1^2\rho_1^2\alpha^s\right)^2}\right]. \tag{3.12.10}$$

6.3.13 Variational determination of the stress function

Following the assumptions of the semi-inverse method of Saint-Venant it is necessary to consider all the relationships of the problem of torsion as being prescribed if they do not depend upon the varied stress function. In particular, the expressions for the displacements u and v on the end faces $z=0$ and $z=l$ follow from eq. (3.3.2) and are given by

$$\left.\begin{array}{ll} z=0: & u=v=0; \\ z=l: & u=-\alpha l y,\ v=\alpha l x. \end{array}\right\} \tag{3.13.1}$$

According to the principle of minimum complementary work, Subsection 4.2.5, the state of stress in the solid differs from all statically admissible states of stress (satisfying the static equations in the volume and on the surface) in that it renders a minimum to functional Ψ, which is the complementary work. In the problem of torsion, the only nonvanishing stresses are shear stresses τ_{xz} and τ_{yz}, hence Ψ takes the form

$$\Psi = \frac{1}{2G}\int_0^l dz\iint_{S_1}\left(\tau_{xz}^2+\tau_{yz}^2\right)do-\iint_{S_l}\left(u\tau_{xz}+v\tau_{yz}\right)do. \tag{3.13.2}$$

Here the surface integral is over the part of the border where the displacements are prescribed. In the considered case, this part of the surface reduces to surface S_l of the end face $z = l$ because of the zero displacement on the end face $z = 0$. The system of stresses τ_{xz} and τ_{yz} given by formulae (3.1.5) identically satisfies the static equation in volume, eq. (3.1.3), which allows us to put expression (3.13.2) in the form

$$\Psi = \frac{1}{2} G l \alpha^2 \iint\limits_S \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 + 2 \left(x \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} \right) \right] do. \quad (3.13.3)$$

In addition to this, the static equation on the lateral surface must hold, that is,

$$\text{on } \Gamma: \quad \tau_{xz} n_x + \tau_{yz} n_y = G\alpha \frac{\partial \Phi}{\partial s} = 0. \quad (3.13.4)$$

In other words, on the contours bounding the cross-section S function Φ minimising functional Ψ must satisfy conditions (3.3.1). Repeating the above calculation we have

$$\iint\limits_S \left(x \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} \right) do = -2 \iint\limits_S \Phi do - 2 \sum_{k=1}^{n} C_k S_k,$$

and the minimised functional is set in the following form

$$J = \frac{1}{2} \iint\limits_S \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 - 4\Phi \right] do - 2 \sum_{k=1}^{n} C_k S_k, \quad (3.13.5)$$

where the unnecessary constant multiplier is omitted. The variation of the integral of the sum of squares of the first derivatives is presented as follows

$$\delta \frac{1}{2} \iint\limits_S \left[\left(\frac{\partial \Phi}{\partial x} \right)^2 + \left(\frac{\partial \Phi}{\partial y} \right)^2 \right] do = \iint\limits_S \left(\frac{\partial \Phi}{\partial x} \frac{\partial \delta \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \delta \Phi}{\partial y} \right) do$$
$$= - \iint\limits_S \nabla^2 \Phi \delta \Phi do + \oint\limits_{\Gamma_*} \frac{\partial \Phi}{\partial n_*} \delta \Phi ds,$$

where Γ_* denotes the set of contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ bounding S and $\mathbf{n}_*$ is the vector of the outward normal to S (directed outs from S on Γ_0 and into regions S_k on contours Γ_k).

As mentioned above, function Φ minimising functional J must vanish on Γ_0 and be equal to *a priori* unknown constant values C_k on Γ_k. This implies that

$$\delta \Phi = 0 \quad \text{on } \Gamma_0;$$
$$\delta \Phi = \delta C_k \quad \text{on } \Gamma_k \qquad (k = 1, 2, \ldots, n),$$

where δC_k are arbitrary. We thus arrive at the relationship

$$\delta J = -\iint\limits_S \delta\Phi\left(\nabla^2\Phi + 2\right) do = \sum_{k=1}^{n} \delta C_k \left(\oint\limits_{\Gamma_k} \frac{\partial \Phi}{\partial n_k} ds - 2S_k\right) = 0, \tag{3.13.6}$$

where $\mathbf{n}_k$ is the normal to Γ_k directed into S. Hence, provided that the comparison functions Φ satisfying conditions (3.3.1) are prescribed, the variational problem (3.13.5) is equivalent to the boundary value problem for Poisson's equation (3.1.8) stated earlier. On each of the contours Γ_k the solution of the variational problem satisfies the condition of the theorem on circulation ensuring that the warping $w(x, y)$ is single-valued. Equation (3.13.6) is the variational equation for Galerkin's approach, Subsection 4.2.4.

It is easy to obtain the minimum value of functional J. To this end, it is sufficient to reset eq. (3.13.5) in the form

$$J = \frac{1}{2}\left\{\iint\limits_S \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 - 2\Phi\right] do - 2\sum_{k=1}^{n} C_k S_k\right\} - \left(\iint\limits_S \Phi do + \sum_{k=1}^{n} C_k S_k\right)$$

and notice that the expression in the braces is equal to zero. In order to prove this, it is sufficient to perform the following transformation

$$\begin{aligned}\iint\limits_S \left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right] do &= -\iint\limits_S \Phi\nabla^2\Phi do + \oint\limits_{\Gamma_*} \frac{\partial \Phi}{\partial n_*}\Phi ds \\ &= 2\left(\iint\limits_S \Phi do + \sum_{k=1}^{n} C_k S_k\right).\end{aligned}$$

Hence, due to eq. (3.4.2)

$$J = -\left(\iint\limits_S \Phi do + \sum_{k=1}^{n} C_k S_k\right) = -\frac{1}{2}C, \tag{3.13.7}$$

that is, the minimum is equal to half of the geometric rigidity with a minus sign. If we satisfy approximately the condition of the minimum with the help of function Φ^* and calculating the corresponding geometric rigidity we have

$$\left.\begin{aligned} &J^* = -\frac{1}{2}C^*, \\ &J = -\frac{C}{2} < J^* = -\frac{1}{2}C^*, \\ &C > C^*, \end{aligned}\right\} \tag{3.13.8}$$

i.e. C^* yields the lower bound of the geometric rigidity.

The second way of the variational statement of the problem of torsion is based on the principle of minimum potential energy, Subsection 4.2.2. In Chapter 4 this functional was denoted by Φ which designates the stress function in the present chapter. For this reason, the minimised functional is denoted here as Q and is written down in the form of eq. (2.1.3) of Chapter 4

$$Q = \frac{1}{2}Gl \iint\limits_S \left(\gamma_{yz}^2 + \gamma_{zx}^2\right) do - \iint\limits_{S^l+S^0} \left(uF_x + vF_y\right) do, \tag{3.13.9}$$

where $S^l + S^0$ denotes the surfaces of the end faces where the external forces are prescribed. The displacements are determined by means of the basic assumptions of the solution of Saint-Venant's problem, thus, they are not allowed to vary and that is why the second term in expression (3.13.9) is omitted. On the other hand,

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \alpha\left(\frac{\partial \varphi}{\partial x} - y\right), \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \alpha\left(\frac{\partial \varphi}{\partial y} + x\right),$$

hence ignoring the multiplier $Gl\alpha^2$ we arrive at the problem of minimising the functional

$$\begin{aligned} J_1 &= \frac{1}{2}\iint\limits_S \left[\left(\frac{\partial \varphi}{\partial x} - y\right)^2 + \left(\frac{\partial \varphi}{\partial y} + x\right)^2\right] do \qquad (3.13.10) \\ &= \frac{1}{2}\iint\limits_S \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] do + \iint\limits_S \left(x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x}\right) do + \frac{1}{2}I_p. \end{aligned}$$

Its variation is equal to

$$\begin{aligned} \delta J_1 &= \iint\limits_S \left[\left(\frac{\partial \varphi}{\partial x} - y\right)\frac{\partial \delta\varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial y} + x\right)\frac{\partial \delta\varphi}{\partial y}\right] do \\ &= -\iint\limits_S \delta\varphi \nabla^2 \varphi do + \oint\limits_{\Gamma_*} \delta\varphi\left(\frac{\partial \varphi}{\partial n} + xn_y - yn_x\right) ds, \qquad (3.13.11) \end{aligned}$$

and Euler's equations for this variational problem reduce to Neumann's boundary value problem for Laplace equation

$$\text{in } S: \quad \nabla^2 \varphi = 0; \quad \text{on } \Gamma_*: \quad \frac{\partial \varphi}{\partial n} = y n_x - x n_y. \tag{3.13.12}$$

It follows from the expression for δJ_1 that function $\varphi(x, y)$ minimising functional J_1 satisfies the boundary condition (3.13.12). Therefore, this condition can be ignored while choosing $\varphi(x, y)$ for the approximate solution to the problem.

The geometric torsional rigidity can be presented in terms of function $\varphi(x, y)$ in the form

$$C = \frac{m_z}{G\alpha} = \frac{1}{G\alpha} \iint_S (x\tau_{yz} - y\tau_{zx})\, do \tag{3.13.13}$$
$$= \iint_S \left[\left(\frac{\partial \varphi}{\partial y} + x \right) x - \left(\frac{\partial \varphi}{\partial x} - y \right) y \right] do = I_p + \iint_S \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) do.$$

Referring to eqs. (3.7.1), (3.7.2) and (3.2.4) we have

$$C = I_p - \iint_S \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] do = I_p - \iint_S \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] do.$$

Returning to eq. (3.13.10) we obtain

$$J_1 = \frac{1}{2} (I_p - C) + C - I_p + \frac{1}{2} I_p = \frac{1}{2} C, \tag{3.13.14}$$

that is, the minimum of functional J_1 is equal to half of the geometric rigidity. Hence, calculating rigidity C^{**} by means of function $\varphi^{**}(x, y)$, which minimises functional J_1 approximately, we arrive at the upper bound for the geometric rigidity

$$J_1^{**} = \frac{1}{2} C^{**} > J_1 = \frac{1}{2} C, \quad C^{**} > C. \tag{3.13.15}$$

Clearly, formulae (3.13.7) and (3.13.4) are the results of the general relationships (2.2.4) and (2.5.9) of Chapter 5 applied to the problem of torsion.

6.3.14 Approximate solution to the problem of torsion

In what follows we consider two examples of applying Galerkin's approach to solving the problems of torsions of the rods with rectangular and trapezoidal cross-sections.

In the case of a single-valued region the expression for variation of the minimised functional J is given by

$$\delta J = \iint\limits_S \left(\nabla^2 \Phi + 2\right) \delta \Phi do. \tag{3.14.1}$$

1. Rectangle. Following the modification of Galerkin's approach suggested by L.V. Kantorovich we take

$$\Phi = \left(b^2 - y^2\right) X(x), \quad X(\pm a) = 0, \tag{3.14.2}$$

which satisfies the boundary conditions on sides $x = \pm a, y = \pm b$ of the rectangle. The dependence of the sought solution on y is thus prescribed whereas the dependence on x will be obtained from condition (3.14.1). The solution is expected to be rather accurate for $a > b$ since it contains the solution for an infinite strip corrected by accounting for the boundary conditions at $x = \pm a$.

Now

$$\nabla^2 \Phi = -2X(x) + \left(b^2 - y^2\right) X''(x),$$
$$\delta \Phi = \left(b^2 - y^2\right) \delta X(x)$$

and condition (3.14.1) reduces to the form

$$\int\limits_{-a}^{a} \delta X(x)\, dx \int\limits_{-b}^{b} \left[\left(b^2 - y^2\right) X''(x) - 2X(x) + 2\right] \left(b^2 - y^2\right) dy = 0.$$

Integrating with respect to y and ignoring the constant multiplier we arrive at the relationship

$$\int\limits_{-a}^{a} \delta X(x) \left(X'' - \frac{5}{2b^2} X + \frac{5}{2b^2}\right) dx = 0. \tag{3.14.3}$$

It must hold for arbitrary $\delta X(x)$, hence

$$X'' - \frac{5}{2b^2} X + \frac{5}{2b^2} = 0.$$

The solution of this differential equation subjected to the boundary condition (3.14.2) is

$$X(x) = 1 - \frac{\cosh \sqrt{\frac{5}{2}} \frac{x}{b}}{\cosh \sqrt{\frac{5}{2}} \frac{a}{b}}. \tag{3.14.4}$$

The torsional rigidity is equal to

$$C=2\int_{-b}^{b}\left(b^2-y^2\right)dy\int_{-a}^{a}X(x)\,dx=\frac{16}{3}ab^3\left(1-\sqrt{\frac{2}{5}}\frac{b}{a}\tanh\sqrt{\frac{5}{2}}\frac{a}{b}\right). \tag{3.14.5}$$

For a square ($a=b$) we obtain the solution coinciding with the exact solution $C=0.419\frac{16}{3}b^4$ (see Subsection 6.3.8) to three significant figures. A lower level accuracy is achieved for the maximum shear stress, the error is about 5% for a square. This is a general feature of all approximate methods, namely that the obtained values of the integral characteristics are more accurate than the local values of the derivatives of the sought function. It is relatively simple to estimate the volume of the surface and it is more difficult to obtain the details of the surface.

Clearly, the suggested solution can be made more accurate by assuming that $\Phi(x,y)$ depends upon several sought functions, for example, by assuming the following

$$\Phi(x,y)=\left(b^2-y^2\right)\left[X_0(x)+y^2X_1(x)+y^4X_2(x)\right],\quad X_k(\pm a)=0.$$

Functions $X_k(x)$ are determined from a system of linear differential equations with the above boundary conditions, the number of equations in the system being equal to the number of unknown functions. It is evident that the calculation becomes more difficult.

The Kantorovich method implies that the boundary value problem for the partial differential equation (Poisson equation) is replaced by a boundary value problem for the ordinary differential equation. One can avoid solving differential equations at all and reduce the problem to a system of linear differential equations by prescribing the form of the solution. For instance, we take for a rectangle

$$\Phi=\left(b^2-y^2\right)\left(a^2-x^2\right)\left(c_0+c_1x^2+c_2y^2+c_3x^2y^2+\ldots\right).$$

Then

$$\delta\Phi=\left(b^2-y^2\right)\left(a^2-x^2\right)\left(\delta c_0+x^2\delta c_1+y^2\delta c_2+x^2y^2\delta c_3+\ldots\right)$$

and the arbitrariness of variations δc_k leads to the above system of linear equations.

In the case of a rectangle, keeping only a single constant c_0 we have

$$\int_{-a}^{a}dx\int_{-a}^{a}dy\left(\nabla^2\Phi+2\right)\delta\Phi=$$

$$8\int_0^a dx\int_0^a dy\left(a^2-x^2\right)\left(a^2-y^2\right)\left[1-c_0\left(a^2-x^2\right)-c_0\left(a^2-y^2\right)\right]\delta c_0=0$$

and after the calculation and omitting the constant multiplier, we arrive at the relationship

$$\left(1-\frac{8}{5}a^2c_0\right)\delta c_0=0.$$

Because of the arbitrariness of variation δc_0 we obtain

$$c_0=\frac{5}{8a^2},\quad \Phi=\frac{5}{8}a^2\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right).$$

The errors for the geometric rigidity and the maximum shear stress are -1.2% and -6.2%, respectively. Taking two coefficients $(c_0, c_1=c_2)$, a system of two linear equations for their determination is obtained. The solution has the form

$$\Phi=a^2\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right)\left[\frac{1295}{2216}+\frac{525}{4432}\left(x^2+y^2\right)\right]$$

and the errors for the geometric rigidity and the maximum shear stress are -0.2% and -4.3%, respectively.

Kantorovich's method yields nearly the same accuracy as the first approximation. This is to be expected as the solution (minimum of the functional) was sought within a more general class of functions since function $X(x)$ was determined by the solution of the constructed variational problem rather than being prescribed in advance.

2. Trapezoidal cross-section. Evidently, the successful application of Kantorovich's method is due to the feasibility of integrating the obtained differential equation. Another example is the case of a trapezoidal cross-section bounded by the lines

$$x=a,\quad x=a+h=b,\quad y\pm x\tan\alpha=0.$$

The boundary conditions are satisfied when one takes

$$\Phi=\left(y^2-x^2\tan^2\alpha\right)X(x),\quad X(a)=0,\quad X(b)=0.$$

Calculation yields

$$\int\limits_a^{a+h}\delta X dx\int\limits_{-x\tan\alpha}^{x\tan\alpha}\left(\nabla^2\Phi+2\right)\left(y^2-x^2\tan^2\alpha\right)dy=$$

$$=\int\limits_a^{a+h}\delta X(x)\left[\frac{16}{15}x^5X''(x)\tan^5\alpha+\frac{16}{3}x^4X'(x)\tan^5\alpha+\right.$$

$$\left.\frac{8}{3}x^3X(x)\tan^3\alpha\left(\tan^2\alpha-1\right)-\frac{8}{3}x^3\tan^3\alpha\right]dx$$

and leads to the differential equation of Euler's type

$$x^2 X''(x) + 5xX'(x) + \frac{5}{2}\left(1 - \cot^2\alpha\right) X(x) = \frac{5}{2}\cot^2\alpha$$

which is integrable by quadratures. The solution is as follows

$$X(x) = \frac{\cos^2\alpha}{\cos 2\alpha}\left[\frac{\left(a^{\gamma_1} - b^{\gamma_1}\right) x^{\gamma_2} - \left(a^{\gamma_2} - b^{\gamma_2}\right) x^{\gamma_1}}{a^{\gamma_1} b^{\gamma_2} - b^{\gamma_1} a^{\gamma_2}} - 1\right],$$

where γ_1 and γ_2 are the roots of the characteristic equation

$$\gamma^2 + 4\gamma - \frac{5}{2}\frac{\cos 2\alpha}{\sin^2\alpha} = 0.$$

For a equilateral triangle ($a = 0, \alpha = 5\pi/6$) we obtain $\gamma_1 = 1, \gamma_2 = -5$ and the second root should be neglected since function $\Phi(x, y)$ must be bounded. At the same time, the first condition (at the vertex of the triangle) is neglected as well. Then we arrive at the solution

$$\Phi = \frac{3}{2}\left(y^2 - \frac{1}{3}x^2\right)\left(\frac{x}{b} - 1\right),$$

which is the exact solution, cf. eq. (3.9.2), where b denotes the height of the triangle.

The case of an isosceles right-angled triangle ($\alpha = 45°$) results in the differential equation

$$x^2 X''(x) + 5xX'(x) = \frac{5}{2}.$$

The corresponding characteristic equation has the root $\gamma_1 = 0$ and the neglected root $\gamma_2 = -4$, whilst the particular solution is sought in the form of $C \ln x$. Then we obtain

$$\Phi = \frac{5}{8}\left(y^2 - x^2\right)\ln\frac{x}{b}.$$

The geometric rigidity obtained by means of this solution is equal to

$$C = 2\iint_S \Phi do = \frac{5}{48}b^4 = 0,104b^4,$$

whereas the exact solution obtained by L.S. Leibenzon is given by the series

$$C = b^4\left(\frac{1}{3} - \frac{64}{\pi^5}\sum_{k=1}^{\infty}\frac{1}{k^5}\frac{1 + \cosh k\pi}{\sinh k\pi}\right)$$

and leads to the same numerical result. The maximum shear stress is in the middle of the hypothenuse and is equal to

$$\tau_{\max} = G\alpha \left| \frac{\partial \Phi}{\partial x} \right|_{\substack{x=b \\ y=0}} = \frac{5}{8} G\alpha b = 0.625 G\alpha b,$$

whereas the exact solution yields the numerical multiplier of 0.652.

More accurate calculations indicate that the values of the geometric rigidity obtained by varying J lead to a lower bound, as was stated in Subsection 6.3.13.

6.3.15 Oblong profiles

In the following we use the coordinate system with a supporting curve Γ^*. The position of point M^* on Γ^* is given by the curvilinear abscissa σ measured from the point M_0^*, see Fig. 6.10. The position vector of point M^* on the supporting curve, the unit vector of the tangent to this curve at point M^* and the unit vector of the normal (opposite to the principal normal) are denoted by $\mathbf{r}_0(\sigma)$, $\mathbf{t}$ and $\mathbf{n}$ respectively, so that

$$\mathbf{t} = \frac{d\mathbf{r}_0}{d\sigma}, \quad \frac{d\mathbf{t}}{d\sigma} = -\frac{\mathbf{n}}{\rho}, \quad \frac{d\mathbf{n}}{d\sigma} = \frac{\mathbf{t}}{\rho}, \tag{3.15.1}$$

where $1/\rho$ denotes the curvature.

The position of any point M of the profile of the cross-section is given by the curvilinear coordinates σ and ζ, where ζ is counted along the normal to the supporting curve, see Fig. 6.10. The position vector $\mathbf{r}$ of this point is as follows

$$\mathbf{r} = \mathbf{r}_0(\sigma) + \mathbf{n}\zeta, \tag{3.15.2}$$

such that, due to eq. (3.15.1),

$$d\mathbf{r} = \mathbf{t}\left(1 + \frac{\zeta}{\rho}\right) d\sigma + \mathbf{n} d\zeta, \; ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \left(1 + \frac{\zeta}{\rho}\right)^2 d\sigma^2 + d\zeta^2. \tag{3.15.3}$$

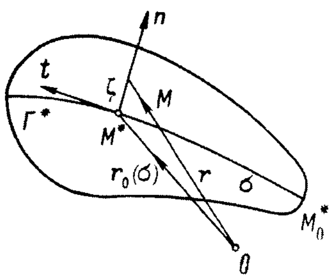

FIGURE 6.10.

The Lamé coefficients and the area element in the coordinate system σ, ζ are equal to

$$H_\sigma = 1 + \frac{\zeta}{\rho}, \quad H_\zeta = 1, \quad do = \left(1 + \frac{\zeta}{\rho}\right) d\sigma d\zeta. \tag{3.15.4}$$

The Laplace operator and the gradient over the scalar are set in the form, cf. eqs. (C.5.5) and (C.3.8)

$$\left.\begin{aligned} \nabla^2 \Phi &= \frac{1}{1+\frac{\zeta}{\rho}} \left[\frac{\partial}{\partial \sigma} \frac{1}{1+\frac{\zeta}{\rho}} \frac{\partial \Phi}{\partial \sigma} + \frac{\partial}{\partial \zeta} \left(1 + \frac{\zeta}{\rho}\right) \frac{\partial \Phi}{\partial \zeta} \right], \\ \nabla \Phi &= \frac{1}{1+\frac{\zeta}{\rho}} \mathbf{t} \frac{\partial \Phi}{\partial \sigma} + \mathbf{n} \frac{\partial \Phi}{\partial \zeta}. \end{aligned}\right\} \tag{3.15.5}$$

The functional (3.13.5) to be minimised is presented in the form

$$J = \frac{1}{2} \int_{\sigma_1}^{\sigma_2} d\sigma \int_{\delta_1(\sigma)}^{\delta_2(\sigma)} \left[\frac{1}{1+\frac{\zeta}{\rho}} \left(\frac{\partial \Phi}{\partial \sigma}\right)^2 + \left(1 + \frac{\zeta}{\rho}\right) \left(\frac{\partial \Phi}{\partial \zeta}\right)^2 - 4 \left(1 + \frac{\zeta}{\rho}\right) \Phi \right] d\zeta, \tag{3.15.6}$$

where it is assumed that

$$\sigma_1 \le \sigma \le \sigma_2, \quad \delta_1(\sigma) \le \zeta \le \delta_2(\sigma).$$

1. Sector of a thin circular ring. For a thin sector with mid-radius ρ, central angle 2α and thickness 2δ we have

$$-\alpha\rho \le \sigma \le \alpha\rho, \quad \delta < \zeta < \delta.$$

Following Kantorovich's method we take

$$\Phi = \left(\delta^2 - \zeta^2\right) f(\sigma), \quad f(\pm\alpha\rho) = 0.$$

Then

$$J = \frac{1}{2} \int_{-\alpha\rho}^{\alpha\rho} d\sigma \int_{-\delta}^{\delta} \left[\frac{\left(\delta^2 - \zeta^2\right)^2}{1+\frac{\zeta}{\rho}} f'^2(\sigma) + 4\left(1 + \frac{\zeta}{\rho}\right) f^2(\sigma) \zeta^2 - \right.$$
$$\left. 4\left(1 + \frac{\zeta}{\rho}\right) f(\sigma) \left(\delta^2 - \zeta^2\right) \right] d\zeta =$$
$$= \frac{16}{15} \delta^5 \int_0^{\alpha\rho} \left(f'^2 + \frac{5}{2\delta^2} f^2 - 5 \frac{f}{\delta^2} \right) d\sigma = \frac{16}{15} \delta^5 \int_0^{\alpha\rho} L(f)\, d\sigma,$$

where terms of order $(\delta/\rho)^2$ and higher are omitted. We arrive then at the variational problem, for which Euler's equation is set in the form

$$\frac{d}{d\sigma}\frac{\partial L}{\partial f'} - \frac{\partial L}{\partial f} = 2\left(f'' - \frac{5}{2\delta^2}f + \frac{5}{2\delta^2}\right) = 0,$$

thus

$$f = 1 - \frac{\cosh\sqrt{\frac{5}{2}\frac{\sigma}{\delta}}}{\cosh\sqrt{\frac{5}{2}\frac{\alpha\rho}{\delta}}}, \quad \Phi = \left(\delta^2 - \zeta^2\right)\left(1 - \frac{\cosh\sqrt{\frac{5}{2}\frac{\sigma}{\delta}}}{\cosh\sqrt{\frac{5}{2}\frac{\alpha\rho}{\delta}}}\right).$$

It is clear that under the adopted approximation (the curvature is neglected) the obtained solution repeats solution (3.14.4). According to eq. (3.14.5) the geometric rigidity is as follows

$$C = \frac{16}{3}\alpha\rho\delta^3\left(1 - \sqrt{\frac{5}{2}}\frac{\delta}{\alpha\rho}\tanh\sqrt{\frac{5}{2}\frac{\alpha\rho}{\delta}}\right).$$

For $\alpha = \pi$, i.e. for a cut circular ring, we obtain

$$C = \frac{16}{3}\pi\rho\delta^3\left(1 - \sqrt{\frac{2}{5}}\frac{\delta}{\pi\rho}\tanh\sqrt{\frac{5}{2}\frac{\pi\rho}{\delta}}\right) \approx \frac{16}{3}\pi\rho\delta^3\left(1 - \sqrt{\frac{2}{5}}\frac{\delta}{\pi\rho}\right),$$

whereas the rigidity of the uncut circular ring is equal to

$$C_* = \frac{1}{2}\pi\left[(\rho+\delta)^4 - (\rho-\delta)^4\right] = 4\pi\left(\rho^2+\delta^2\right)\rho\delta \approx 4\pi\rho^3\delta,$$

therefore

$$\frac{C}{C_*} = \frac{4}{3}\frac{\delta^2}{\rho^2}\left(1 - \sqrt{\frac{2}{5}}\frac{\delta}{\pi\rho}\right), \qquad C \ll C_*.$$

2. *Symmetric aerofoil.* The region is taken as being symmetric about axis x and bounded by the curves having the equations

$$y = a\psi\left(\frac{x}{b}\right), \quad y = -a\psi\left(\frac{x}{b}\right), \tag{3.15.7}$$

see Fig. 6.11. These curves are tangent to axis y at the origin of the coordinate system and intersect axis x at points O and b

$$\psi(0) = 0, \quad \psi(1) = 0, \quad \psi'(0) = \infty.$$

It is also assumed that the derivative $\psi'(t)$ is continuous and is zero only at $t = t_*$ in the interval $(0,1)$. The values of b and $2a\psi(t_*)$ determine

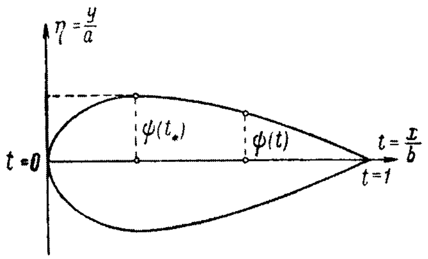

FIGURE 6.11.

respectively the chord and the width of the profile. The profile is assumed to be thin, that is $a \ll b$. The above conditions are met by the curves

$$\psi(t) = t^m (1 - t^p)^q,$$

where $o < m, 1, p > 0, q > 0, t_* = (1 + pq/m)^{-1/p}$. For example, a semi-cubic parabola is prescribed by $p = q = 1, m = 1/2$ whereas $p = 1, q = m = 1/2$ prescribes an ellipse with the major and minor axes b and a and the centre at the point $(b/2, 0)$.

The supporting curve is axis x and the integral to be minimised can be presented in the form

$$J = \frac{1}{2\lambda} \int_0^1 dt \int_{-\psi(t)}^{\psi(t)} \left[\lambda^2 \left(\frac{\partial \Phi}{\partial t} \right)^2 + \left(\frac{\partial \Phi}{\partial \eta} \right)^2 - 4a^2 \Phi \right] d\eta \quad \left(\eta = \frac{y}{a} \right),$$

where $\lambda = a/b$ is a small parameter. Let us limit our consideration to the following form of function Φ vanishing on the contour of the region

$$\Phi = Aa^2 \left(\psi^2 - \eta^2 \right), \tag{3.15.8}$$

where A is a constant to be determined. We arrive at the relationship

$$J(A) = \frac{4}{\lambda} a^4 \int_0^1 dt \int_0^{\psi(t)} \left[A^2 \lambda^2 \psi^2 \psi'^2 + A^2 \eta^2 - A \left(\psi^2 - \eta^2 \right) \right] d\eta$$

and the condition of minimum $J'(A) = 0$ yields

$$A = \frac{1}{1 + 3\lambda^2 \varepsilon}, \quad \varepsilon = \frac{\int_0^1 \psi^3 \psi'^2 \, dt}{\int_0^1 \psi^3 dt}. \tag{3.15.9}$$

The lower bound for the geometric rigidity leads to the expression

$$C^- = 4ab \int_0^1 dt \int_0^{\psi(t)} \Phi d\eta = \frac{8}{3} a^3 b A \int_0^1 \psi^3 dt. \tag{3.15.10}$$

The upper bound C^+ can be obtained by minimising integral (3.13.10)

$$J_1 = \frac{1}{\lambda} \int_0^1 dt \int_0^{\psi(t)} \left[\lambda^2 \left(\frac{\partial \varphi}{\partial t} - ab\eta \right)^2 + \left(\frac{\partial \varphi}{\partial \eta} + abt \right)^2 \right] d\eta. \tag{3.15.11}$$

The minimising function is taken in the following form

$$\varphi = Bxy + Cby = ab\left(B\eta t + C\eta\right),$$

where the linear term corresponds to placing the origin of the coordinate system in the *a priori* unknown point $x_0 = -b\frac{C}{B}$ on axis x. Parameters B and C are obtained from the condition of the minimum. A cumbersome calculation carried out in the paper by L.S. Leibenzon[1] results in the following rigidity for the semi-cubic parabola

$$C^+ = \frac{256}{3465} \frac{a^3 b}{1 + \frac{7}{11}\lambda^2}, \tag{3.15.12}$$

whilst

$$C^- = \frac{256}{3465} \frac{a^3 b}{1 + \frac{11}{13}\lambda^2}, \quad \frac{C^-}{C^+} = \frac{1 + \frac{7}{11}\lambda^2}{1 + \frac{11}{13}\lambda^2} \approx 1 - 0,21\lambda^2.$$

This calculation is sufficiently accurate for a long and narrow profile.

6.3.16 *Torsion of a thin-walled tube*

This profile is bounded by the external Γ_0 and internal Γ_1 contours. The contour Γ^* lying in the middle between Γ_0 and Γ_1 is taken as a supporting curve, such that in the curvilinear coordinates the equations for contours σ and ζ can be set in the form

$$\Gamma_0: \quad \zeta = \frac{1}{2}\delta(\sigma); \qquad \Gamma_1: \quad \zeta = -\frac{1}{2}\delta(\sigma),$$

[1] Leibenzon L.S. Variational methods of solving problems of the theory of elasticity (in Russian). Collection of works, Publishers of the USSR Academy of Sciences, 1951, pp. 324-356.

where $\delta(\sigma)$ denotes the thickness of the wall of the tube, Fig. 6.12. The stress function is given in the form

$$\Phi = \frac{C_1}{2}\left(1 - \frac{2\zeta}{\delta(\sigma)}\right), \tag{3.16.1}$$

that is, it vanishes on Γ_0 and is equal to an *a priori* unknown constant C_1 on Γ_1. This form of Φ is acceptable for a small thickness of the wall. From a perspective of the membrane analogy, contour Γ_1 carries a disc subjected to the vertical displacement C_1, contour Γ_0 is fixed while the membrane is located in the thin annular slot between these contours. The form (3.16.1) for Φ is justified by neglecting the curvature of the surface of the curved membrane in the transverse direction of the slot.

Due to eqs. (3.13.5) and (3.1.5.6) the minimising functional J is set in the form

$$J = \frac{1}{2}\oint_{\Gamma^*} d\sigma \int_{-\delta/2}^{\delta/2} \left[\left(1+\frac{\xi}{\rho}\right)^{-1}\left(\frac{\partial\Phi}{\partial\sigma}\right)^2 + \left(1+\frac{\xi}{\rho}\right)\left(\frac{\partial\Phi}{\partial\xi}\right)^2 - 4\left(1+\frac{\xi}{\rho}\right)\Phi\right] d\xi - 2C_1S_1. \tag{3.16.2}$$

Let s^- and s^+ denote the areas of the ring-shaped regions between the contours (Γ_1, Γ^*) and (Γ^*, Γ_0) respectively. Clearly, $s^- + s^+ = S$ which is the area of the contour. Then we have

$$s^- = \oint_{\Gamma^*} d\sigma \int_{-\delta/2}^{0}\left(1+\frac{\xi}{\rho}\right) d\xi = \frac{1}{2}\oint_{\Gamma^*} \delta\, d\sigma - \frac{1}{8}\oint_{\Gamma^*} \frac{\delta^2}{\rho} d\sigma,$$

$$s^+ = \frac{1}{2}\oint_{\Gamma^*} \delta\, d\sigma + \frac{1}{8}\oint_{\Gamma^*} \frac{\delta^2}{\rho} d\sigma,$$

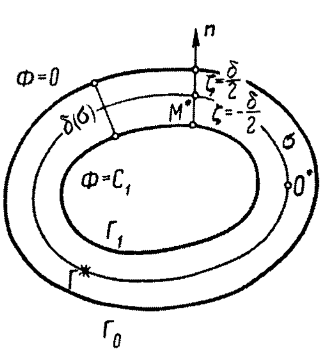

FIGURE 6.12.

therefore

$$2C_1S_1 + 2\oint_{\Gamma^*} d\sigma \int_{-\delta/2}^{\delta/2} \left(1+\frac{\xi}{\rho}\right)\Phi d\xi = 2C_1S_1^* + \frac{1}{12}C_1\oint_{\Gamma^*}\frac{\delta^2}{\rho}d\sigma,$$

where $S_1^* = S_1 + s^+$ denotes the area bounded by the supporting contour Γ^*. Calculation yields

$$J = \frac{1}{2}C_1^2\left[\oint_{\Gamma^*}\frac{d\sigma}{\delta} + \oint_{\Gamma^*}\frac{\delta'^2}{\delta^4}\left(\rho^3\ln\frac{1+\dfrac{\delta}{2\rho}}{1-\dfrac{\delta}{2\rho}} - \rho^2\delta\right)d\sigma\right] - 2C_1\left(S_1^* + \frac{1}{24}\oint_{\Gamma^*}\frac{\delta^2}{\rho}d\sigma\right)$$

or when neglecting the terms of the order of $(\delta/\rho)^2$

$$J = \frac{1}{2}C_1^2\oint_{\Gamma^*}\frac{1}{\delta}\left(1+\frac{1}{12}\delta'^2\right)d\sigma - 2C_1\left(S_1^* + \frac{1}{24}\oint_{\Gamma^*}\frac{\delta^2}{\rho}d\sigma\right). \tag{3.16.3}$$

The condition of minimum $\partial J/\partial C_1 = 0$ yields

$$C_1 = 2\frac{S_1^* + \dfrac{1}{24}\oint_{\Gamma^*}\dfrac{\delta^2}{\rho}d\sigma}{\oint_{\Gamma^*}\dfrac{1}{\delta}\left(1+\dfrac{\delta'^2}{12}\right)d\sigma}. \tag{3.16.4}$$

By virtue of eq. (3.13.8) the expression for the geometric rigidity (the lower bound) is presented in the form

$$C = 4\frac{\left(S_1^* + \dfrac{1}{24}\oint_{\Gamma^*}\dfrac{\delta^2}{\rho}d\sigma\right)^2}{\oint_{\Gamma^*}\dfrac{1}{\delta}\left(1+\dfrac{\delta'^2}{12}\right)d\sigma}. \tag{3.16.5}$$

Under the assumption of a smooth change in the wall thickness, that is $\delta'(\sigma) \ll 1$, and a small curvature, these formulae are written down in a simplified form

$$J = \frac{1}{2}C_1^2\oint_{\Gamma^*}\frac{d\sigma}{\delta} - 2C_1S_1^*, \quad C = \frac{4S_1^{*2}}{\gamma}, \quad \gamma = \oint_{\Gamma^*}\frac{d\sigma}{\delta}, \tag{3.16.6}$$

which corresponds to prescribing functional J in the following form

$$J = \frac{1}{2} \oint_{\Gamma^*} d\sigma \int_{-\delta/2}^{\delta/2} \left[\left(\frac{\partial \Phi}{\partial \xi} \right)^2 - 4\Phi \right] d\xi - 2C_1 S_1$$

$$= \frac{1}{2} \oint_{\Gamma^*} d\sigma \int_{-\delta/2}^{\delta/2} \left(\frac{\partial \Phi}{\partial \xi} \right)^2 d\xi - 2C_1 S_1^* \tag{3.16.7}$$

and neglecting the difference $s^- - s^+$. It is necessary to add that the original linear dependence of the stress function Φ on ζ, eq. (3.16.1), calls into question the applicability of the refinements (3.16.3) and (3.16.5). For example, prescribing Φ in the form

$$\Phi = \frac{1}{2} C_1 \left(1 - \frac{2\zeta}{\delta} \right) + B \left(1 - \frac{4\zeta^2}{\delta^2} \right),$$

such that $\Phi = 0$ on Γ_0 and $\Phi = C_1$ on Γ_1, and determining the constants C_1 and B from the conditions of the minimum of functional J, eq. (3.16.2), one obtains the expression for the geometric rigidity C which differs from eq. (3.16.5)[2].

6.3.17 *Multiple-connected regions*

The stress function is assumed to be presented in the form (3.3.5) where $\Phi_0, \Phi_1, \ldots, \Phi_n$ are the solutions of the boundary-value problems (3.3.2)-(3.3.4). The system (3.3.10) of linear equations serves for searching the unknown values of C_k of the stress function on the internal contours Γ_k. The same system is obtained by searching the minimum of integral J with respect to constants C_k. By eqs. (3.13.5) and (3.3.5) we have

$$J = \frac{1}{2} \iint_S \left[\left(\frac{\partial \Phi_0}{\partial x} + \sum_{k=1}^n C_k \frac{\partial \Phi_k}{\partial x} \right)^2 + \left(\frac{\partial \Phi_0}{\partial y} + \sum_{k=1}^n C_k \frac{\partial \Phi_k}{\partial y} \right)^2 - 4 \left(\Phi_0 + \sum_{k=1}^n C_k \Phi_k \right) \right] do - 2 \sum_{k=1}^n C_k S_k, \tag{3.17.1}$$

[2]The solution for which Φ is determined by the solution of Poisson's differential equation $\nabla^2 \Phi + 2 = 0$ in the form of a series in terms of powers of parameter δ/l (l is the length of contour Γ_*) is derived in Chapter 7 of the monograph by N.Kh. Arutyunyan and B.L. Abramyan, see Bibliography to Chapter 6.

and the minimum conditions are set in the form

$$\frac{\partial J}{\partial C_s} = \sum_{k=1}^{n} C_k \iint\limits_S \left(\frac{\partial \Phi_k}{\partial x} \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_k}{\partial y} \frac{\partial \Phi_s}{\partial y} \right) do - \\ 2 \left\{ S_s + \iint\limits_S \left[\Phi_s - \frac{1}{2} \left(\frac{\partial \Phi_0}{\partial x} \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_0}{\partial y} \frac{\partial \Phi_s}{\partial y} \right) \right] do \right\} = 0$$

or

$$\sum_{k=1}^{n} C_k B_{ks} - B_s = 0, \quad s = 1, 2, \ldots, n, \tag{3.17.2}$$

where

$$B_{ks} = \iint\limits_S \left(\frac{\partial \Phi_k}{\partial x} \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_k}{\partial y} \frac{\partial \Phi_s}{\partial y} \right) do = \oint\limits_{\Gamma^*} \Phi_k \frac{\partial \Phi_s}{\partial n} ds = \oint\limits_{\Gamma^*} \frac{\partial \Phi_s}{\partial n_k} ds, \tag{3.17.3}$$

$$\begin{aligned} B_s &= 2 \left\{ S_s + \iint\limits_S \left[\Phi_s - \frac{1}{2} \left(\frac{\partial \Phi_0}{\partial x} \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_0}{\partial y} \frac{\partial \Phi_s}{\partial y} \right) \right] do \right\} \\ &= 2 \left\{ S_s + \iint\limits_S \left[\Phi_s - \frac{1}{2} \left(\frac{\partial}{\partial x} \Phi_s \frac{\partial \Phi_0}{\partial x} + \frac{\partial}{\partial y} \Phi_s \frac{\partial \Phi_0}{\partial y} - \Phi_s \nabla^2 \Phi_0 \right) \right] do \right\} \\ &= 2S_s - \oint\limits_{\Gamma^*} \Phi_s \frac{\partial \Phi_0}{\partial n} ds = 2S_s - \oint\limits_{\Gamma_s} \frac{\partial \Phi_0}{\partial n} ds. \end{aligned} \tag{3.17.4}$$

These transformations use the definitions of functions $\Phi_0, \Phi_1, \ldots, \Phi_n$ by relationships (3.3.2)-(3.3.4). We arrive then to formulae (3.3.10), the symmetry of matrix $\|B_{ks}\|$ requiring no proof, cf. Subsection 6.3.5.

It is easy to prove that the determinant of this matrix is not equal to zero (it is positive). Indeed, writing the expression for the stress function $\Phi(x, y)$ in the form

$$\Phi(x, y) = \sum_{k=0}^{n} C_k \Phi_k (x, y)$$

(instead of eq. (3.3.5)) we have

$$\begin{aligned} \frac{1}{2} \iint\limits_S \left(\tau_{xz}^2 + \tau_{yz}^2 \right) do &= \frac{1}{2} G^1 \alpha^2 \sum_{k=0}^{n} \sum_{s=0}^{n} C_k C_s \iint\limits_S \left(\frac{\partial \Phi_k}{\partial x} \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_k}{\partial y} \frac{\partial \Phi_s}{\partial y} \right) do \\ &= \frac{1}{2} G^2 \alpha^2 \sum_{k=0}^{n} \sum_{s=0}^{n} B_{ks} C_k C_s. \end{aligned}$$

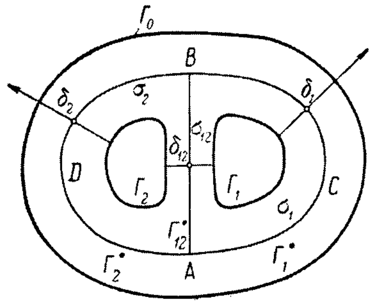

FIGURE 6.13.

This is a positive definite quadratic form of variables $C_0, C_1, \ldots, C_n$. By Sylvester's theorem, the determinant of the coefficients and all the diagonal minor determinants are positive, among them the determinant

$$|B_{ks}| \quad (k, s = 1, 2, \ldots, n).$$

By way of an example let us consider a triple-connected profile which is a thin-walled tube with a cross-piece. The supporting curves $\Gamma_1^*, \Gamma_{12}^*, \Gamma_2^*$ are the arcs ACB, BA, BDA with the corresponding abscissas $\sigma_1, \sigma_{12}, \sigma_2$, Fig. 6.13. The external contour Γ_0 and the contours of the openings Γ_1, Γ_2 are given by the following functions of coordinate ζ

$$\left.\begin{array}{llllll} \Gamma_0: & \zeta = \dfrac{\delta_1}{2} & \text{for} \quad \sigma = \sigma_1, & \zeta = \dfrac{\delta_2}{2} & \text{for} \quad \sigma = \sigma_2, \\ \Gamma_1: & \zeta = -\dfrac{\delta_1}{2} & \text{for} \quad \sigma = \sigma_1, & \zeta = \dfrac{\delta_{12}}{2} & \text{for} \quad \sigma = \sigma_{12}, \\ \Gamma_2: & \zeta = -\dfrac{\delta_2}{2} & \text{for} \quad \sigma = \sigma_2 & \zeta = -\dfrac{\delta_{12}}{2} & \text{for} \quad \sigma = \sigma_{12}. \end{array}\right\} \qquad (3.17.5)$$

It allows one to suggest the following form for the stress function

$$\Phi(\sigma, \zeta) = \begin{cases} \dfrac{1}{2} C_1 \left(1 - \dfrac{2\zeta}{\delta_1}\right) & \text{for} \quad \sigma = \sigma_1, \\ \dfrac{1}{2} C_1 \left(1 + \dfrac{2\zeta}{\delta_{12}}\right) + \dfrac{1}{2} C_2 \left(1 - \dfrac{2\zeta}{\delta_{12}}\right) & \text{for} \quad \sigma = \sigma_{12}, \\ \dfrac{1}{2} C_2 \left(1 - \dfrac{2\zeta}{\delta_2}\right) & \text{for} \quad \sigma = \sigma_2. \end{cases} \qquad (3.17.6)$$

The integral over the area S of the profile is the sum of three integrals

$$\int_{\Gamma_1^*} d\sigma_1 \int_{-\delta_1/2}^{\delta_1/2} (\ldots)\, d\zeta + \int_{\Gamma_{12}^*} d\sigma_{12} \int_{-\delta_{12}/2}^{\delta_{12}/2} (\ldots)\, d\zeta + \int_{\Gamma_2^*} d\sigma_2 \int_{-\delta_2/2}^{\delta_2/2} (\ldots)\, d\zeta,$$

and functional J, by eq. (3.16.7), can be set as follows

$$J = \frac{1}{2}\left[C_1^2 \int\limits_{\Gamma_1^*} \frac{d\sigma_1}{\delta_1} + \left(C_1 - C_2\right)^2 \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}} + C_2^2 \int\limits_{\Gamma_1^*} \frac{d\sigma_2}{\delta_2}\right] - 2C_1 S_1^* - 2C_2 S_2^*. \tag{3.17.7}$$

The constants C_1 and C_2 are obtained from the following system of two equations

$$\left.\begin{aligned} \frac{\partial J}{\partial C_1} &= C_1\left(\int\limits_{\Gamma_1^*} \frac{d\sigma_1}{\delta_1} + \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}}\right) - C_2 \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}} - 2S_1^* = 0, \\ \frac{\partial J}{\partial C_2} &= -C_1 \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}} + C_2\left(\int\limits_{\Gamma_2^*} \frac{d\sigma_2}{\delta_2} + \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}}\right) - 2S_2^* = 0. \end{aligned}\right\} \tag{3.17.8}$$

Introducing the notation

$$\int\limits_{\Gamma_1^*} \frac{d\sigma_1}{\delta_1} = \gamma_1, \quad \int\limits_{\Gamma_{12}^*} \frac{d\sigma_{12}}{\delta_{12}} = \gamma_{12}, \quad \int\limits_{\Gamma_2^*} \frac{d\sigma_2}{\delta_2} = \gamma_2$$

we obtain

$$\left.\begin{aligned} C_1 &= \frac{2}{\Delta}\left[S_1^*\left(\gamma_{12} + \gamma_2\right) + S_2^* \gamma_{12}\right], \\ C_2 &= \frac{2}{\Delta}\left[S_1^* \gamma_{12} + S_2^*\left(\gamma_{12} + \gamma_1\right)\right], \\ \Delta &= \left(\gamma_1 + \gamma_2\right)\gamma_{12} + \gamma_1\gamma_2. \end{aligned}\right\} \tag{3.17.9}$$

The geometric rigidity of the profile is given by

$$C = 2C_1 S_1^* + 2C_2 S_2^* = \frac{4}{\Delta}\left[\left(\gamma_{12} + \gamma_2\right) S_1^{*^2} + 2\gamma_{12} S_1^* S_2^* + \left(\gamma_{12} + \gamma_1\right) S_2^{*^2}\right]. \tag{3.17.10}$$

The shear stresses in the tube wall and in the cross-piece are determined from the relationships

$$\tau = G\alpha\left|\frac{\partial\Phi}{\partial\zeta}\right| = \begin{cases} \dfrac{C_1}{\delta_1} = \dfrac{2}{\delta_1\Delta}\left[S_1^*\left(\gamma_{12} + \gamma_2\right) + S_2^*\gamma_{12}\right] & \text{on} \quad \Gamma_1, \\ \dfrac{\left|C_1 - C_2\right|}{\delta_{12}} = \dfrac{2}{\delta_{12}\Delta}\left|S_1^*\gamma_2 - S_2^*\gamma_1\right| & \text{on} \quad \Gamma_{12}, \\ \dfrac{C_2}{\delta_2} = \dfrac{2}{\delta_2\Delta}\left[S_1^*\gamma_{12} + S_2^*\left(\gamma_{12} + \gamma_1\right)\right] & \text{on} \quad \Gamma_2. \end{cases} \tag{3.17.11}$$

If the profile is symmetric, i.e. $\gamma_1 = \gamma_2 = \gamma, S_1^* = S_2^* = S^*$, the stress in the cross-piece is absent and removing the cross-piece does not affect the geometric rigidity. This also follows from formulae (3.17.10) and (3.16.6), however in the latter formula S^* and γ need to be replaced by $2S^*$ and 2γ respectively.

The explained method for approximate calculation can be generalised easily to the profiles of arbitrary connectivity and appears to be more simple than the approaches based on the theorem of circulation of shear stresses.

6.4 Bending by force

6.4.1 Stresses

Saint-Venant's solution of the bending problem implies that the only non-vanishing components of the stress tensor are σ_z, τ_{zx} and τ_{yz}. The normal stress σ_z is given by eq. (1.4.6)

$$\sigma_z = -\left(\frac{P}{I_y}x + \frac{Q}{I_x}y\right)(l - z), \tag{4.1.1}$$

and the equations of statics in the volume and on the surface (i.e. on the contour of the cross-section of the rod) are written in the form

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\left(\frac{P}{I_y}x + \frac{Q}{I_x}y\right), \quad \tau_{zx}n_x + \tau_{yz}n_y = 0, \tag{4.1.2}$$

cf. eqs. (1.5.1) and (1.5.3).

The stress distribution described by these relationships is equivalent to the bending moments

$$M_x = -Q(l - z), \quad M_y = P(l - z) \tag{4.1.3}$$

in the cross-section z and the transverse forces equal to P and Q in any cross-section. The latter condition is satisfied by any statically admissible (i.e. satisfying the equations of statics (4.1.2)) system of stresses τ_{zx}, τ_{yz}. The torque due to these stresses is equal to

$$m_z = \iint\limits_S (x\tau_{yz} - y\tau_{zx})\, do = aQ - bP, \tag{4.1.4}$$

where (a, b) is the point on the line L of action of force $\mathbf{i}_i P + \mathbf{i}_2 Q$. The torque vanishes if this line passes through the centre of inertia O of the cross-section (the origin of the coordinate system), however, in general, the bending is accompanied by torsion ($\alpha \neq 0$). The torsion is absent ($\alpha = 0$)

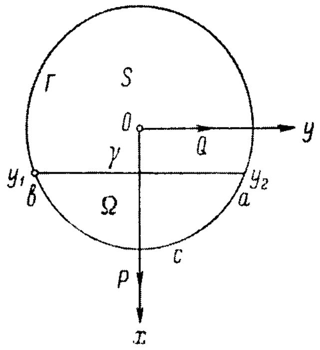

FIGURE 6.14.

when line L passes through the centre of rigidity x^*, y^*, see eq. (2.6.11), then $m_z \neq 0$. The case $m_z = 0, \alpha = 0$ takes place under the condition that L is the line linking the centre of inertia and the centre of rigidity. For example, this is the case for the cross-section which has a symmetry axis coinciding with the line of action of the force or the cross-section with two symmetry axes and the action line passing through the centre of inertia.

Determining the shear stresses requires solving the boundary-value problems stated in Subsection 6.2.1. However the mean value of the shear stresses can be obtained using only the static equations (4.1.2). Indeed, let us consider area Ω which is a part of S, see Fig. 6.14. Area Ω is bounded by contour $abca$ constructed from arc bca of the cross-sectional contour Γ and arc γ (ab) lying in S. By eq. (4.1.2)

$$\int\limits_{abca} (\tau_{zx} n_x + \tau_{yz} n_y)\, ds = \iint\limits_{\Omega} \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right) do$$
$$= -\left(\frac{P}{I_y} \iint\limits_{\Omega} x do + \frac{Q}{I_x} \iint\limits_{\Omega} y do \right).$$

On the other hand

$$\int\limits_{abca} (\tau_{zx} n_x + \tau_{yz} n_y)\, ds = \int\limits_{\gamma} (\tau_{zx} n_x + \tau_{yz} n_y)\, ds,$$

as the integrand is equal to zero on $\Gamma\,(bca)$. Hence

$$\int\limits_{\gamma} (\tau_{zx} n_x + \tau_{yz} n_y)\, ds = -\left(\frac{P}{I_y} \iint\limits_{\Omega} x do + \frac{Q}{I_x} \iint\limits_{\Omega} y do \right). \qquad (4.1.5)$$

In particular, cutting area Ω of the straight line γ parallel to axis y, Fig. 6.14, we have

$$n_x = -1, \qquad n_y = 0, \qquad ds = -dy$$
$$\iint_{\Omega} x do = x_c \Omega, \quad \iint_{\Omega} y do = y_c \Omega; \quad b = y_2 - y_1,$$

where x_c and y_c denote the coordinates of the centre of inertia of area Ω and b is the length of γ. Using eq. (4.1.5) we arrive at the relationship

$$\int_{y_1}^{y_2} \tau_{zx} dy = b \left(\tau_{zx}\right)_m = \Omega \left(\frac{P}{I_y} x_c + \frac{Q}{I_x} y_c \right)$$

obtained by Tricomi in 1933. Let us notice in passing that Tricomi's proof does take into account that formula (4.1.6) follows from a pure static reasoning. The latter equation can be found in textbooks on strength of materials for the case $y_c = 0$, i.e. if axis x is the symmetry axis of the cross-section. It expresses the fact that the sum of projections of the forces acting on volume $(l - z)\,\Omega$ (cut by a plane parallel to yz) on axis x is zero.

6.4.2 Bending of a rod with elliptic cross-section

Because of the symmetry it is sufficient to consider the case of the force parallel to axis x and to make the line of action of this force coincident with axis x, otherwise the problem of bending is superimposed by the problem of torsion, see Subsection 6.3.6. According to eqs. (2.1.1), (2.1.6) and (2.1.7) the distribution of the shear stresses is given by

$$\tau_{zx} = \frac{P}{2(1+\nu) I_y} \left(\frac{\partial \chi}{\partial x} - x^2 \right), \quad \tau_{yz} = \frac{P}{2(1+\nu) I_y} \left(\frac{\partial \chi}{\partial y} - 2\nu xy \right), \tag{4.2.1}$$

where the stress function χ is determined by the solution of Neumann's problem for Laplace's equation

$$\text{in } S: \quad \nabla^2 \chi = 0; \quad \text{on } \Gamma: \quad \frac{\partial \chi}{\partial n} = x^2 n_x + 2\nu xy n_y. \tag{4.2.2}$$

On the contour of the ellipse (3.6.1)

$$\frac{x dx}{a^2} + \frac{y dy}{b^2} = 0, \quad -\frac{x}{a^2} n_y + \frac{y}{b^2} n_x = 0. \tag{4.2.3}$$

It is easy to understand that boundary condition (4.2.2) is satisfied by prescribing the harmonic function (which must be odd with respect to x) in the form

$$\chi = \frac{1}{3} A \left(x^3 - 3xy^2\right) + Bx. \tag{4.2.4}$$

Indeed, in this case

$$\frac{\partial \chi}{\partial n} = A\left[\left(x^2 - y^2\right) n_x - 2xyn_y\right] + Bn_x$$
$$= A\left[x^2 - \left(1 + 2\frac{a^2}{b^2}\right) y^2\right] n_x + Bn_x$$

and condition (4.2.2) can be set in the form

$$\text{on } \Gamma: \; A\left[x^2 - \left(1 + 2\frac{a^2}{b^2}\right) y^2\right] + B = x^2 + 2\nu \frac{a^2}{b^2} y^2 + \lambda \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

Onto the right hand side we have added the term vanishing on the ellipse. The three constants λ, A, B are obtained from the equations

$$A = 1 - \frac{\lambda}{a^2}, \quad A\left(1 + 2\frac{a^2}{b^2}\right) = \frac{\lambda}{b^2} - 2\nu \frac{a^2}{b^2}, \quad \lambda = B,$$

such that from eq. (4.2.4) we have

$$\chi = \frac{(1-2\nu) a^2}{3\left(3a^2 + b^2\right)} \left(x^3 - 3xy^2\right) + \frac{b^2 + 2a^2 (1+\nu)}{3a^2 + b^2} a^2 x \tag{4.2.5}$$

and the shear stresses are given by the formulae

$$\left.\begin{aligned} \tau_{zx} &= \frac{P}{2(1+\nu) I_y} \frac{2(1+\nu) a^2 + b^2}{3a^2 + b^2} \left[a^2 - x^2 - \frac{(1-2\nu) a^2 y^2}{2(1+\nu) a^2 + b^2}\right], \\ \tau_{yz} &= -\frac{a^2 (1+\nu) + \nu b^2}{(1+\nu)\left(3a^2 + b^2\right)} \frac{P}{I_y} xy. \end{aligned}\right\} \tag{4.2.6}$$

The stress τ_{zx} on the semi-axis $x = 0$ obeys a parabolic law

$$\left(\tau_{zx}\right)_{x=0} = \frac{Pa^2}{2(1+\nu) I_y} \frac{2(1+\nu) a^2 + b^2}{3a^2 + b^2} \left(1 - \frac{1-2\nu}{2(1+\nu) a^2 + b^2} y^2\right). \tag{4.2.7}$$

The mean value of this stress is as follows

$$\tau_m = \frac{1}{2b} \int_{-b}^{b} \left(\tau_{zx}\right)_{x=0} dy = \frac{Pa^2}{3I_y} = \frac{4}{3}\frac{P}{S} \quad \left(I_y = \frac{\pi}{4} a^3 b, \; S = \pi ab\right),$$

and, as expected, is independent of Poisson's ratio. However the deviation from the mean value depends both on ν and the ratio of the semi-axes of the ellipse and it can be considerable. For instance, for $\nu = 0,25$ this deviation is

$$\frac{1}{\tau_m} \left(\tau_{\max} - \tau_m\right) = \frac{1}{\tau_m} \left(\tau_{zx}\right)_{\substack{x=0 \\ y=0}} - 1 = 0,20 \frac{b^2}{3a^2 + b^2}$$

and can reach 20% for $b \gg a$.

6.4.3 The stress function of S.P. Timoshenko

Introducing this stress function instead of χ simplifies solution of the problem of bending of the symmetric simple-connected profile loaded by a force perpendicular to the symmetry axis. Let ϑ denote the harmonic function related to χ by the Cauchy-Riemann conditions

$$\frac{\partial \chi}{\partial x} = \frac{\partial \vartheta}{\partial y}, \quad \frac{\partial \chi}{\partial y} = -\frac{\partial \vartheta}{\partial x}.$$

We introduce function F in the following way

$$2F(1+\nu) = \vartheta + \nu x^2 y - 2(1+\nu) G(y), \tag{4.3.1}$$

where $G(y)$ is defined in what follows. Then

$$\nabla^2 F = \frac{\nu}{1+\nu} y - G''(y), \tag{4.3.2}$$

and expressions (4.2.1) for stresses and boundary condition (4.1.2) are set in the form

$$\tau_{xz} = \frac{P}{2(1+\nu) I_y}\left(\frac{\partial \vartheta}{\partial y} - x^2\right) = \frac{P}{I_y}\left[\frac{\partial F}{\partial y} - \frac{1}{2}x^2 + G'(y)\right], \tag{4.3.3}$$

$$\left.\begin{aligned} \tau_{yz} &= \frac{P}{2(1+\nu) I_y}\left(-\frac{\partial \vartheta}{\partial x} - 2\nu xy\right) = -\frac{P}{I_y}\frac{\partial F}{\partial x}, \\ \frac{\partial F}{\partial y} n_x - \frac{\partial F}{\partial x} n_y &= \frac{\partial F}{\partial s} = \left[\frac{1}{2}x^2 - G'(y)\right]\frac{dy}{ds}. \end{aligned}\right\} \tag{4.3.4}$$

The equation for the contour, which is symmetric about axis y, can be presented in the form $x^2 = 2f(y)$. Hence it is sufficient to take $G'(y) = f(y)$ and one arrives at the boundary-value problem for Poisson's equation

$$\text{in } S: \quad \nabla^2 F = \frac{\nu}{1+\nu} y - f'(y); \quad \text{on } \Gamma: \quad F = 0. \tag{4.3.5}$$

This boundary-value problem is equivalent to the problem of bending the membrane fixed on the contour provided that the load is given by the function on the right hand side of equation (4.3.5). The boundary condition remains the same if, in addition to the arcs symmetric to axis y, the contour contains the straight lines $y = \text{const}$ parallel to axis x ($dy/ds = 0$ on these lines).

6.4.4 Rectangular cross-section

Let the length of the sides parallel to axes x and y be denoted by $2a$ and $2b$ respectively. It is now sufficient to take $f(y) = \frac{1}{2}a^2$, then by eq. (4.3.5)

$$\nabla^2 F = \frac{\nu}{1+\nu} y; \quad F = 0 \quad \text{for} \quad x = \pm a,\ y = \pm b. \tag{4.4.1}$$

The stresses are given by formulae (4.3.3)

$$\tau_{xz} = \frac{P}{2I_y}\left(a^2 - x^2\right) + \frac{P}{I_y}\frac{\partial F}{\partial y}, \quad \tau_{yz} = -\frac{P}{I_y}\frac{\partial F}{\partial x}. \tag{4.4.2}$$

The solution is sought as a series satisfying the boundary conditions on the sides $y = \pm b$

$$F = \frac{\nu}{1+\nu}\sum_{n=1}^{\infty} X_n(x)\sin\frac{n\pi y}{b}.$$

In order to determine the unknown functions $X_n(x)$ we have the relationship

$$\sum_{n=1}^{\infty}\left(X_n'' - \frac{n^2\pi^2}{b^2}X_n\right)\sin\frac{n\pi y}{b} = \frac{2b}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi y}{b},$$

where the right hand side presents a trigonometric series for a periodic function equal to y for $-b < y < b$. The solution of the obtained boundary-value problem

$$X_n'' - \frac{n^2\pi^2}{b^2}X_n = \frac{2b}{n\pi}(-1)^{n+1}, \quad X_n(\pm a) = 0$$

which is odd with respect to x is set in the form

$$X_n(x) = \frac{2b^3}{\pi^3 n^3}(-1)^n\left(1 - \frac{\cosh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}}\right).$$

Hence

$$F = \frac{\nu}{1+\nu}\frac{2b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(1 - \frac{\cosh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}}\right)\sin\frac{n\pi y}{b}, \tag{4.4.3}$$

and taking into account that

$$\frac{2b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi y}{b} = \frac{1}{6}\left(y^3 - b^2 y\right), \quad |y| \ll b,$$

we arrive at the following expressions for the stresses $\left(I_y = \frac{4}{3}a^3 b = \frac{1}{3}a^2 S\right)$

$$\left.\begin{aligned}
\tau_{zx} &= \frac{3P}{2S}\left[\left(1 - \frac{x^2}{a^2}\right) - \frac{\nu}{1+\nu}\frac{b^2}{a^2}\left(\frac{1}{3} - \frac{y^2}{b^2} + \right.\right.\\
&\qquad \left.\left.\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}}\cos\frac{n\pi y}{b}\right)\right]\\
\tau_{yz} &= \frac{6P}{\pi^2 S}\frac{\nu}{1+\nu}\frac{b^2}{a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\sinh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}}\sin\frac{n\pi y}{b}.
\end{aligned}\right\} \tag{4.4.4}$$

The elementary solution obtained by formula (4.1.6) yields the parabolic distribution of the shear stress

$$\tau'_{zx} = \frac{3P}{2S}\left(1 - \frac{x^2}{a^2}\right). \tag{4.4.5}$$

The maximum of the stress τ_{zx} is on axis y at the points $y = 0, y = \pm b$ of the maximum change in the surface of the membrane

$$\left.\begin{aligned}
(\tau_{zx})_{\substack{x=0\\y=0}} &= \frac{3P}{2S}\left[1 - \frac{\nu}{1+\nu}\frac{b^2}{a^2}\left(\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{1}{\cosh\frac{n\pi a}{b}}\right)\right] \\
&= \frac{3P}{2S}f_1\left(\frac{a}{b}\right), \\
(\tau_{zx})_{\substack{x=0\\y=\pm b}} &= \frac{3P}{2S}\left[1 + \frac{\nu}{1+\nu}\frac{b^2}{a^2}\left(\frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{1}{\cosh\frac{n\pi a}{b}}\right)\right] \\
&= \frac{3P}{2S}f_2\left(\frac{a}{b}\right).
\end{aligned}\right\} \tag{4.4.6}$$

The book by S.P. Timoshenko and J. Goodier contains a small table of values of these functions for $\nu = 1/4$

$\frac{a}{b}$	2	1	1/2	1/4
$f_1\left(\frac{a}{b}\right)$, eq. (4.4.6)	0,983	0,940	0,856	0,805
$f_2\left(\frac{a}{b}\right)$, eq. (4.4.6)	1,033	1,126	1,396	1,988
$f_2\left(\frac{a}{b}\right)$, eq. (4.7.4)	1,065	1,146	1,424	2,064

Table 6.2

It follows from this table that the elementary theory is in good agreement with the exact theory for $a/b \geq 2$ and deviates considerably even for $a/b \leq 1/2$.

In the problem of the bending of a long thin strip $(b \gg a)$, as a first approximation, one can neglect the boundary conditions on the short sides $y = \pm b$. In the framework of the membrane analogy it corresponds to the assumption that the deflection (as well as the membrane loading) is a linear function of y. Then by virtue of eqs. (4.4.1) and (4.4.2)

$$\frac{\partial^2 F}{\partial x^2} = \frac{\nu}{1+\nu}y, \quad F = \frac{\nu}{2(1+\nu)}y\left(x^2 - a^2\right), \tag{4.4.7}$$

$$\tau_{zx} = \frac{1}{1+\nu}\frac{P}{2I_y}\left(a^2 - x^2\right), \quad \tau_{yz} = -\frac{P}{I_y}\frac{\nu}{1+\nu}xy \tag{4.4.8}$$

that is, the shear stresses in the middle of the strip are $1+\nu$ times smaller than those from the elementary theory. For $\nu = 1/4$ they are 80% of the latter which is in agreement with the data of the above table already $b = 4a$. Of course, this solution is unacceptable for $y = \pm b$ however it can be used for searching for the exact solution of the differential equation (4.4.1) in the following form

$$F = \frac{\nu}{2(1+\nu)} y \left(x^2 - a^2\right) + F_*. \tag{4.4.9}$$

Then F_* is determined from the boundary value

$$\nabla^2 F_* = 0, \quad F_*(\pm a, y) = 0, \quad F_*(x, \pm b) = \pm \frac{\nu b}{2(1+\nu)} \left(a^2 - x^2\right),$$

whose solution is sought in the form

$$F_* = \frac{\nu b}{1+\nu} \frac{16a^3}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \frac{\sinh \frac{(2k+1)\pi y}{2a}}{\sinh \frac{(2k+1)\pi b}{2a}} \cos \frac{(2k+1)\pi x}{2a}, \tag{4.4.10}$$

see Subsection 6.3.8.

The stresses are as follows

$$\left.\begin{aligned}
\tau_{xz} &= \frac{3P}{2S} \frac{1}{1+\nu} \left[\left(1 - \frac{x^2}{a^2}\right) + \right. \\
&\quad \left. \frac{16}{\pi^2} \frac{\nu b}{a} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{\sinh \frac{(2k+1)\pi y}{2a}}{\sinh \frac{(2k+1)\pi b}{2a}} \cos \frac{(2k+1)\pi x}{2a} \right], \\
\tau_{yz} &= \frac{3P}{2S} \frac{\nu}{1+\nu} \left[-\frac{2xy}{a^2} + \right. \\
&\quad \left. \frac{16}{\pi^2} \frac{b}{a} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{\sinh \frac{(2k+1)\pi y}{2a}}{\sinh \frac{(2k+1)\pi b}{2a}} \sin \frac{(2k+1)\pi x}{2a} \right],
\end{aligned}\right\} \tag{4.4.11}$$

and for $b > 15a$ the maximum stresses are the horizontal stresses τ_{yz} (ignored by the elementary theory) on sides $x = \pm a$ at points $y = \pm\eta$ near the corners of the rectangle. For example,

$$\text{for } b = 15a \quad \frac{2S}{3P} \left(\tau_{xz}\right)_{\substack{x=0 \\ y=b}} = 5,255, \quad \frac{2S}{3P} \left(\tau_{yz}\right)_{\substack{x=a \\ y=\eta}} = 5,202; \; \eta = 0,875b;$$

$$\text{for } b = 25a \quad \frac{2S}{3P} \left(\tau_{xz}\right)_{\substack{x=0 \\ y=b}} = 8,255, \quad \frac{2S}{3P} \left(\tau_{yz}\right)_{\substack{x=a \\ y=\eta}} = 9,233; \; \eta = 0,917b.$$

6.4.5 Variational statement of the problem of bending

Let us begin with the representation of the shear stresses τ_{zx}, τ_{yz} in terms of the stress function F

$$\tau_{zx} = \frac{P}{I_y}\left[\frac{\partial F}{\partial y} - \frac{1}{2}x^2 + f(y)\right], \quad \tau_{yz} = -\frac{P}{I_y}\frac{\partial F}{\partial x}, \tag{4.5.1}$$

where f and f_x, f_y, f_z denote the surface force and its projections on the corresponding axes. The static equation (4.1.2) in the volume is then satisfied identically however the static equation on the surface (on contour Γ of the cross-section)

$$\text{on } \Gamma: \ f_z = \tau_{zx}n_x + \tau_{yz}n_y = \frac{P}{I_y}\left\{\frac{\partial F}{\partial s} - \left[\frac{1}{2}x^2 - f(y)\right]n_x\right\} = 0 \tag{4.5.2}$$

does not hold identically. For this reason, aiming at the principle of minimum complementary work, which uses a comparison with the statically admissible states of stress, it is necessary to introduce only the functions satisfying condition (4.5.2) on Γ, that is

$$\text{on} \quad \Gamma: \ \delta\frac{\partial F}{\partial s} = \frac{\partial}{\partial s}\delta F = 0, \quad \delta F = \text{const} = \delta F_\Gamma. \tag{4.5.3}$$

The principle of minimum potential energy implies that the difference in variations of the strain energy expressed in terms of the stresses and the work of variations of the surface force $\iint_O (u\delta f_x + v\delta f_y + w\delta f_z)\, do$ is zero.

By condition (4.5.3) this expression is equal to zero on the lateral surface. Stress σ_z is given by expression (4.1.1) and thus is not varied in the volume and on the end faces, hence $\delta\sigma_z = 0$. The values of the projections u, v of the displacement vector are given by formulae (2.2.9) and are independent of the choice of F, thus $\delta u = 0, \delta v = 0$. We have

$$u(l) - u(0) = -\alpha l y - \frac{\nu P l}{2EI_y}\left(x^2 - y^2\right) - \frac{1}{6}\frac{Pl^3}{EI_y} + \omega_y^0 l,$$
$$v(l) - v(0) = \alpha l x - \frac{\nu P l}{EI_y}xy - \omega_x^0 l$$

and accounting for $f_x = \mp\tau_{zx}, f_y = \mp\tau_{zy}$ at $z = 0$ and $z = l$ we can present the work of the variation of the surface forces in the form of variation of the integral over the surface of the end faces

$$\delta \iint_S \{[u(l) - u(0)]\,\tau_{zx} + [v(l) - v(0)]\,\tau_{yz}\}\,do =$$

$$-l\alpha \frac{P}{I_y}\delta \iint_S \left(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right) do - \frac{\nu P^2 l}{2EI_y^2}\delta \iint_S \left[\left(x^2 - y^2\right)\frac{\partial F}{\partial y} - 2xy\frac{\partial F}{\partial x}\right] do$$

$$-\frac{1}{6}\frac{P^2 l^3}{EI_y^2}\delta \iint_S \frac{\partial F}{\partial y} do + \frac{lP}{I_y}\delta \iint_S \left(\omega_y^0 \frac{\partial F}{\partial y} + \omega_x^0 \frac{\partial F}{\partial x}\right) do. \quad (4.5.4)$$

The terms whose variation is zero, i.e. the products of $u(0) - u(l)$ and the terms in τ_{zx} which do not depend on F, are not written down here. The terms in the third line in eq. (4.5.4) are transformed into the contour integrals vanishing by virtue of eq. (4.5.3).

The variation of the strain energy is set in the form

$$\delta\left[\frac{1}{2E}\int_0^l dz \iint_S \sigma_z^2 do + \frac{1}{2G}\iint_S \left(\tau_{xz}^2 + \tau_{yz}^2\right) do\right] =$$

$$= \frac{lP^2}{2GI_y^2}\left\{\delta \iint_S \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right] do - \delta \iint_S \left[x^2 - 2f(y)\right]\frac{\partial F}{\partial y} do\right\}, \quad (4.5.5)$$

and the principle of minimum complementary work leads to the problem of minimising the integral

$$J_2 = \frac{1}{2}\iint_S \left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 - \left[x^2 - 2f(y)\right]\frac{\partial F}{\partial y} + \right.$$
$$\left. \frac{\nu}{2(1+\nu)}\left[\left(x^2 - y^2\right)\frac{\partial F}{\partial y} - 2xy\frac{\partial F}{\partial x}\right] + 2\alpha G\frac{I_y}{P}\left(x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right)\right\} do, \quad (4.5.6)$$

the choice of the minimising function F is subjected to the boundary condition (4.5.2).

Varying this integral and transforming the integrals over S into the contour integrals we arrive at the expression

$$\delta J_2 = -\iint_S \delta F\left(\nabla^2 F - \frac{\nu y}{1+\nu} + f'(y) + 2\alpha G\frac{I_y}{P}\right) do +$$
$$\delta F_\Gamma\left[\oint_\Gamma \frac{\partial F}{\partial n} ds + \oint_\Gamma f(y)\,n_y ds + \alpha G\frac{I_y}{P}\oint_\Gamma (x n_x + y n_y)\,ds\right], \quad (4.5.7)$$

since the remaining contour integrals are zero, for instance

$$\oint_{\Gamma} \left[\left(x^2 - y^2\right) n_y - 2xyn_x \right] ds = -4 \iint_S ydo = 0.$$

We arrive then at the Poisson differential equation

$$\text{in } S: \ \nabla^2 F - \frac{\nu}{1+\nu} y + f'(y) + 2\alpha G \frac{I_y}{P} = 0, \tag{4.5.8}$$

which is coincident with eq. (4.3.5) for $\alpha = 0$. It follows from this that

$$\begin{aligned} \iint_S \nabla^2 F do = \oint_{\Gamma} \frac{\partial F}{\partial n} ds = \frac{\nu}{1+\nu} \iint_S ydo - \iint_S f'(y)\, do - 2\alpha G \frac{I_y}{P} S \\ = - \oint_{\Gamma} f(y)\, n_y ds - \alpha G \frac{I_y}{P} \oint_{\Gamma} (xn_x + yn_y)\, ds, \end{aligned}$$

that is, the expression in the parentheses in eq. (4.5.7) is zero.

6.4.6 The centre of rigidity

In what follows we consider the cross-sections which are symmetric about axis y and loaded by force P parallel to axis x.

To this point, we have used the coordinate system Oxy with origin at the centre of inertia O of the rod. In the problem of bending it is preferable to use a more general description in order to have a more simple form of the equation for the contour of the region. The axes of the new system $O'\xi\eta$ are parallel to those of the old system and the origin lies on the axis of symmetry at point $O'(0, y_0')$, such that

$$\xi = x, \quad \eta = y - y_0'. \tag{4.6.1}$$

The equation $y = b$ for the line of action of the force, the coordinates of the centre of rigidity ξ^*, η^* and the equation for the contour in this system of axes are as follows

$$\left.\begin{array}{l} \eta = b - y_0' = b'; \quad \xi^* = x^* = 0, \quad \eta^* = y^* - y_0'; \\ \xi = x = \pm\theta(\eta), \quad \left[\theta^2(\eta) = 2f(y)\right]. \end{array}\right\} \tag{4.6.2}$$

It is assumed in the following that the line of action of the force passes through the centre of rigidity, i.e. $b' = \eta^*$ and $\alpha = 0$. The variational equation of Galerkin's approach is set in the form

$$\iint_S \delta F \left[\nabla^2 F - \frac{\nu}{1+\nu} (\eta + y_0') + \frac{1}{2} \frac{d\theta^2(\eta)}{d\eta} \right] do = 0. \tag{4.6.3}$$

By eqs. (4.5.1) and (1.2.4) the torque m_z is given by the relationship

$$m_z = -bP = -\left(y_0' + b'\right)P = -\left(y_0' + \eta^*\right)P = \iint\limits_S \left[x\tau_{zy} - \left(\eta + y_0'\right)\tau_{zx}\right] do$$
$$= -\frac{P}{I_y}\iint\limits_S \left(x\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial \eta}\right) do - y_0' P + \frac{1}{2}\frac{P}{I_y}\iint\limits_S \eta\left(x^2 - \theta^2(\eta)\right) do.$$

Transforming the integrals we obtain

$$-\iint\limits_S \left(x\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial \eta}\right) do = 2\iint\limits_S F do - 2\oint\limits_\Gamma F d\omega = 2\iint\limits_S F do,$$

as, according to eqs. (4.5.2) and (4.6.2), we can take for the simple connected regions that

$$\text{on } \Gamma: \ F = 0. \tag{4.6.4}$$

Next,

$$\iint\limits_S \eta\left[x^2 - \theta^2(\eta)\right] do = \int\limits_{\eta_1}^{\eta_2} \eta d\eta \int\limits_{-\theta(\eta)}^{-\theta(\eta)} \left[x^2 - \theta^2(\eta)\right] dx = -\frac{4}{3}\int\limits_{\eta_1}^{\eta_2} \eta\theta^3(\eta)\, d\eta.$$

Setting the expression for the moment of inertia I_y in the form

$$I_y = \iint\limits_S x^2 do = \int\limits_{\eta_1}^{\eta_2} d\eta \int\limits_{-\theta(\eta)}^{-\theta(\eta)} x^2 dx = \frac{2}{3}\int\limits_{\eta_1}^{\eta_2} \theta^3(\eta)\, d\eta,$$

we arrive at the following expression for the coordinate of the centre of rigidity (W. Dunkan (1933), L.S. Leibenzon (1933)) in terms of the stress function F

$$\eta^* = \left[\int\limits_{\eta_1}^{\eta_2} \theta^3(\eta)\, d\eta\right]^{-1} \left[\int\limits_{\eta_1}^{\eta_2} \eta\theta^3(\eta)\, d\eta - 3\iint\limits_S F do\right]. \tag{4.6.5}$$

If the profile is symmetric, the expression for the coordinates of the centre of rigidity (2.6.1) can be transformed to the form containing only function

Φ. Indeed, referring to eqs. (2.1.10), (2.1.12) we have

$$\begin{aligned}\iint_S x\varphi do &= \iint_S \frac{\partial}{\partial x}\frac{x^2}{2}\varphi do - \iint_S \frac{x^2}{2}\frac{\partial \varphi}{\partial x} do \\
&= \frac{1}{2}\oint_\Gamma \theta^2(\eta)\varphi n_x ds - \frac{1}{2}\iint_S x^2 \frac{\partial \varphi}{\partial x} do = \frac{1}{2}\iint_S \left[\theta^2(\eta) - x^2\right]\frac{\partial \varphi}{\partial x} do \\
&= \frac{1}{2}\iint_S \left[\theta^2(\eta) - x^2\right]\left(\frac{\partial \Phi}{\partial y} + y\right) do \\
&= \frac{1}{2}\iint_S y\left(\theta^2 - x^2\right) do + \frac{1}{2}\iint_S \left\{\frac{\partial}{\partial y}\Phi\left[\theta^2(\eta) - x^2\right] - \frac{\partial \theta^2}{\partial y}\Phi\right\} do \\
&= -\frac{1}{2}\iint_S \frac{\partial \theta^2}{\partial y}\Phi do + \frac{2}{3}\int_{y_1}^{y_2} y\theta^3(\eta)\, dy,\end{aligned}$$

since $\Phi = 0$ on Γ. We then arrive at the formula (G.Yu. Dzhanelidze, 1963)

$$y^* = \left[\int_{\eta_1}^{\eta_2} \theta^3(\eta)\, d\eta\right]^{-1}\left[\int_{y_1}^{y_2} y\theta^3(\eta)\, dy + \frac{3}{2}\iint_S \left(\frac{\nu}{1+\nu}y - \frac{1}{2}\frac{\partial \theta^2}{\partial \eta}\right)\Phi do\right] \tag{4.6.6}$$

or in the coordinate system $O'\xi\eta$

$$\eta^* = \left[\int_{\eta_1}^{\eta_2} \theta^3(\eta)\, d\eta\right]^{-1}\left[\int_{\eta_1}^{\eta_2} \eta\theta^3 d\eta + \frac{3}{2}\iint_S \left(\frac{\nu}{1+\nu}\eta - \frac{1}{2}\frac{\partial \theta^2}{\partial \eta}\right)\Phi do + \frac{3}{4}\frac{\nu}{1+\nu}y_0' C\right], \tag{4.6.7}$$

with C denoting the geometric torsional rigidity.

6.4.7 Approximate solutions

We consider the problem of bending of the symmetric profile by a force with a line of action passing through the centre of rigidity, the profile being bounded by the curves $x = \pm\theta(\eta) = \pm c\eta^m$ and the straight lines $\eta = b_1, \eta = b_2$. In accordance with Kantorovich's method, Subsection 6.3.14, the solution of the variational equation (4.6.3) satisfying the boundary condition (4.6.3) is taken in the form

$$F = \left(c^2\eta^{2m} - x^2\right)\omega(\eta), \quad \omega(b_1) = \omega(b_2) = 0. \tag{4.7.1}$$

We arrive at the relationship

$$\int_{b_1}^{b_2} d\eta\delta\omega(\eta) \int_{-c\eta^m}^{c\eta^m} \left(c^2\eta^{2m} - x^2\right) \left[\nabla^2 \left(c^2\eta^{2m} - x^2\right) \omega(\eta) + mc^2\eta^{2m-1} - \frac{\nu}{1+\nu}\left(\eta + y_0'\right)\right] dx = 0.$$

Integrating over x and using the arbitrariness of variation $\delta\omega$ results in the ordinary differential equation

$$L(\omega) = -\frac{5}{4c^2\eta^{2m}} \left[mc^2\eta^{2m-1} - \frac{\nu}{1+\nu}\left(\eta + y_0'\right)\right], \tag{4.7.2}$$

where $L(\omega)$ denotes the differential operator

$$L(\omega) = \omega'' + \frac{5m}{\eta}\omega' + \frac{5}{2}\left(\frac{2m-1}{\eta^2}m - \frac{1}{c^2\eta^{2m}}\right)\omega. \tag{4.7.3}$$

The solution is expressed in terms of Bessel functions for any m. We will consider the simple cases with elementary integration.

1. Rectangle $(-a \le x \le a, -b \le y \le b,)$. In this case $m = 0, c = a, y_0' = 0, y = \eta$ and the solution is put in the form

$$F = \frac{1}{2}\frac{\nu b}{1+\nu}\left(a^2 - x^2\right)\left(\frac{\sinh\sqrt{\frac{5}{2}}\frac{\eta}{b}}{\sinh\sqrt{\frac{5}{2}}\frac{a}{b}} - \frac{\eta}{b}\right) \tag{4.7.4}$$

instead of the above exact solution, eqs. (4.4.9) and (4.4.10). The shear stresses τ_{zx} for $x = 0, y = b$ obtained by means of eq. (4.7.4) are set in the form of eq. (4.4.6) and collected in Table 6.2.

2. Trapezoidal cross-section. The case of $m = 1$ corresponds to the trapezoidal cross-section considered in Subsection 6.3.14 for the problem of torsion and reduces to the differential equation of Euler's type, eq. (4.7.2), which is integrated by quadratures. Let us study two cases.

i) Isosceles right-angles triangle $(b_1 = 0, b_2 = h, m = 1, c = 1)$

$$F = -\frac{5}{8}\frac{\nu}{1+\nu}\left(\eta^2 - x^2\right)\left[\frac{1}{3}h\ln\frac{\eta}{h} + \frac{2}{5\nu}(\eta - h)\right]. \tag{4.7.5}$$

ii) Equilateral triangle $(b_1 = 0, b_2 = h, m = 1, c = \tan 30^\circ)$

$$F = \frac{1}{2}\frac{\nu}{1+\nu}\left(\frac{1}{3}\eta^2 - x^2\right)\left[h - \eta + \frac{5}{12}\left(\frac{1}{\nu} - 2\right)\eta\ln\frac{\eta}{h}\right]. \tag{4.7.6}$$

In both cases the origin of the coordinate system lies in the vertex of the triangle. For $\nu = 1/2$ formula (4.7.6) yields the exact solution

$$F = \frac{1}{6}\left(\frac{\eta^2}{3} - x^2\right)(h - \eta),$$

and it is interesting to notice that the centres of rigidity and inertia of the cross-section are coincident. It is easy to prove by eq. (4.6.5) that $\eta^* = \frac{2}{3}h$.

3. Segment of parabola $x^2 = \frac{h^2}{4b}\eta$ *bounded by chord* $\eta = b$ *of length* h. Differential equation (4.7.2) in which $m = 1/2, c^2 = h^2/4b, y_0' = -\frac{3}{5}b$ is integrated in terms of elementary functions (Bessel functions with the "half-integer" index). The result is

$$F = \left(\frac{h^2}{4b}\eta - x^2\right)\frac{\nu b}{2(1+\nu)}\left[\frac{3}{5} + \frac{2}{5\lambda}\frac{2+\nu}{\nu} - \frac{\eta}{b} + \right.$$
$$\left.\left(\frac{2}{5} - \frac{5}{2\lambda}\frac{2+\nu}{\nu}\right)\left(\frac{b}{\eta}\right)^{3/2}\frac{\sinh\sqrt{\frac{\lambda\eta}{b}} - \sqrt{\frac{\lambda\eta}{b}}\cosh\sqrt{\frac{\lambda\eta}{b}}}{\sinh\sqrt{\lambda} - \sqrt{\lambda}\cosh\sqrt{\lambda}}\right], \tag{4.7.7}$$

where $\lambda = 40b^2/h^2$ and the expression for the coordinate of the centre of rigidity (4.6.5) is as follows

$$\eta^* = \frac{5}{7} + \frac{8}{35}\frac{\nu}{1+\nu} - \frac{4}{5}\frac{\nu}{1+\nu}f_1(\lambda) - \frac{2+\nu}{7(1+\nu)}f_2(\lambda),$$
$$f_1(\lambda) = \frac{5}{\sqrt{\lambda}\coth\sqrt{\lambda} - 1} - \frac{15}{\lambda}, \quad f_2(\lambda) = \frac{35}{\lambda}\left[1 - f_1(\lambda)\right].$$

Calculation by means of these formulae yields ($\nu = 1/4$)

$\lambda = 0$	0,4	2,5	4,9	∞
$\frac{\eta^*}{b} = 0,343$	0,350	0,375	0,418	0,760

6.4.8 Aerofoil profile

In accordance with the notation of Subsection 6.4.6 the equations for the contour are set in the form

$$x = \pm a\theta\left(\frac{\eta}{b}\right) = \pm a g(t), \quad g(t) = t^m (1 - t^p)^q, \tag{4.8.1}$$

cf. Subsection 6.3.15. Then by eqs. (3.15.8) and (3.15.9)

$$\Phi = Aa^2\left(g^2 - \xi^2\right), \quad A = \frac{1}{1 + 3\lambda^2\varepsilon}, \quad \varepsilon = \frac{\int_0^1 g^3 g'^2\, dt}{\int_0^1 g^3 dt}, \tag{4.8.2}$$

where $x = a\xi, \lambda = a/b$.

As the stress function is chosen we can find the coordinate of the centre of rigidity with the help of eq. (4.6.7). In order to present the result in a compact form we introduce the notation

$$\eta_G^* = b\frac{\int_0^1 tg^3(t)\,dt}{\int_0^1 g^3(t)\,dt}, \quad \eta_0' = -y_0' = b\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}. \tag{4.8.3}$$

The parameter η_G^* was suggested by Griffith and denotes an approximate expression for the coordinate of the centre of bending. Next η_0' is the coordinate of the centre of inertia of the cross-section in the coordinate system $O'\xi\eta$. Clearly, it is opposite in sign to coordinate y_0' of the origin O' of this coordinate system relative to axes Oxy with the origin at the centre of inertia.

Taking into account that

$$\frac{1}{2}\iint_S \frac{dg^2(t)}{dt}\left(g^2 - \zeta^2\right)do = \int_0^1 \frac{dg^2}{dt}dt\int_0^g\left(g^2 - \xi^2\right)d\xi = \frac{4}{3}\int_0^1 g'g^4dt = 0, \tag{4.8.4}$$

as $g(0) = 0, g(1) = 0$ and using expression (3.15.10) for the geometric rigidity and eq. (4.6.7) we arrive at the following expression for the centre of rigidity for the aerofoil profile

$$\eta^* = \eta_G^* + \frac{2\nu}{1+\nu}\left(\eta_G^* - \eta_0'\right)\frac{1}{1+3\lambda^2\varepsilon} \tag{4.8.5}$$

obtained first by Leibenzon in 1933. The multiplier $\left(1+3\lambda^2\varepsilon\right)^{-1}$ must be omitted when the profile is oblong and thin.

Let us proceed to the problem of bending the aerofoil. To this end, the stress function vanishing on contour (4.8.1) is determined from the variational equation (4.6.3). Rewriting this equation in the form

$$\int_0^1 dt\int_0^{g(t)}\left[\left(\lambda^2\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}\right)F_1 - \frac{\nu}{1+\nu}\left(t - \frac{\eta_0'}{b}\right) + \lambda^2\frac{dg^2(t)}{dt}\right]\delta F_1 d\xi = 0, \tag{4.8.6}$$

where $F = F_1 a^2 b$ and varying only B we have

$$F_1 = B\left[g^2(t) - \xi^2\right], \quad \delta F_1 = \left[g^2(t) - \xi^2\right]\delta B.$$

Then B is determined by the equation

$$B = \left[\int_0^1 dt \int_0^{g(t)} \left(g^2 - \xi^2\right) \left(\lambda^2 \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}\right) \left(g^2 - \xi^2\right) d\xi \right]^{-1} \times$$

$$\left[\frac{\nu}{1+\nu} \int_0^1 dt \int_0^{g(t)} \left(t - \frac{\eta_0'}{b}\right) \left(g^2 - \xi^2\right) d\xi \right],$$

the last term in eq. (4.8.6) vanishes, cf. eq. (4.8.4). Using eq. (4.8.3) we obtain

$$F = -\frac{1}{2b} \frac{\nu}{1+\nu} \frac{\eta_G^* - \eta_0'}{1 + 3\lambda^2 \varepsilon} \left(g^2 - \xi^2\right), \tag{4.8.7}$$

and formula (4.6.5) yields the above expression (4.8.5) for the coordinate of the centre of bending.

6.5 Michell's problem

6.5.1 *Statement of the problem*

This problem was first studied by Michell in 1900 and is a natural continuation of Saint-Venant's problem. We consider the state of stress in a prismatic rod uniformly loaded on the lateral surface. The boundary conditions (1.1.3) and (1.1.4) of Saint-Venant's problem are written down in the form

$$\text{on } \Gamma \quad \left\{ \begin{array}{l} \sigma_x n_x + \tau_{xy} n_y = F_x(s), \\ \tau_{yx} n_x + \sigma_y n_y = F_y(s), \\ \tau_{zx} n_x + \tau_{yz} n_y = F_z(s), \end{array} \right. \tag{5.1.1}$$

where Γ denotes the contour of the cross-section and s is the arc measured along it.

As Saint-Venant's principle is used and the solution of Saint-Venant's problem is assumed to be known one can, without loss of generality, consider the right end $x = l$ to be free of loads. In terms of the adopted notation, eqs. (1.2.3) and (1.2.4), this condition is set as follows

$$z = l: \quad P = 0, \quad Q = 0, \quad R = 0; \quad m_x = 0, \quad m_y = 0, \quad m_z = 0. \tag{5.1.2}$$

We introduce into consideration the following integral values: the projections of the principal vector of the forces distributed along contour Γ of any cross-section z

$$q_x = \oint_{\Gamma} F_x ds, \quad q_y = \oint_{\Gamma} F_y ds, \quad q_z = \oint_{\Gamma} F_z ds \tag{5.1.3}$$

and the principal moments about axes $O_z xyz$ with the origin in this cross-section

$$\mu_x = \oint_{\Gamma} y F_z ds, \quad \mu_y = -\oint_{\Gamma} x F_z ds, \quad \mu_z = \oint_{\Gamma} (x F_y - y F_x) ds. \tag{5.1.4}$$

As above, it is taken that O_z is the centre of inertia of the rod, x, y are the principal axes of inertia at this point and z is the longitudinal axis of the rod.

The three integral equations of equilibrium for the part of the rod $[z, l]$, expressing that the principal vector of the external forces is zero, are as follows

$$\left.\begin{array}{c} (l-z) q_x - \iint\limits_S \tau_{zx} do = 0, \quad (l-z) q_y - \iint\limits_S \tau_{yz} do = 0, \\ (l-z) q_z - \iint\limits_S \sigma_z do = 0. \end{array}\right\} \tag{5.1.5}$$

The moments about the axes of the system $O_z xyz$ distributed over the lateral surface of this part of the rod are equal to

$$\oint_{\Gamma} ds \int_z^l \left[y F_z - (\zeta - z) F_y \right] d\zeta = (l-z) \mu_x - \frac{1}{2} (l-z)^2 q_y,$$

$$\oint_{\Gamma} ds \int_z^l \left[y F_z - (\zeta - z) F_y \right] d\zeta = (l-z) \mu_x - \frac{1}{2} (l-z)^2 q_y,$$

$$\oint_{\Gamma} ds \int_z^l (x F_y - y F_x) d\zeta = (l-z) \mu_z,$$

and thus the remaining three integral equations of equilibrium have the form

$$\left.\begin{array}{c} (l-z) \mu_x - \frac{1}{2} (l-z)^2 q_y - \iint\limits_S y \sigma_z do = 0, \\ (l-z) \mu_y + \frac{1}{2} (l-z)^2 q_x + \iint\limits_S x \sigma_z do = 0, \\ (l-z) \mu_z - \iint\limits_S (x \tau_{yz} - y \tau_{zx}) do = 0. \end{array}\right\} \tag{5.1.6}$$

Following the idea of the semi-inverse Saint-Venant method we satisfy these equations by assuming that τ_{zx} and τ_{yz} are linear in z while σ_z is a quadratic function of z

$$\left.\begin{aligned} &\tau_{zx} = \tau^0_{zx} + (l-z)\,\tau^1_{zx}, \;\; \tau_{yz} = \tau^0_{yz} + (l-z)\,\tau^1_{yz}, \\ &\sigma_z = \sigma^0_z + (l-z)\,\sigma^1_z + \frac{1}{2}(l-z)^2\,\sigma^{(2)}_z, \end{aligned}\right\} \tag{5.1.7}$$

where the seven functions $\tau^0_{zx}, \dots, \sigma^{(2)}_z$ must satisfy the following integral conditions

$$\iint\limits_S \tau^0_{zx} do = 0, \quad \iint\limits_S \tau^0_{yz} do = 0, \quad \iint\limits_S \left(x\tau^0_{yz} - y\tau^0_{zx}\right) do = 0, \tag{5.1.8}$$

$$\iint\limits_S \tau^1_{zx} do = q_x, \quad \iint\limits_S \tau^1_{yz} do = q_y, \quad \iint\limits_S \left(x\tau^1_{yz} - y\tau^1_{zx}\right) do = \mu_z, \tag{5.1.9}$$

$$\iint\limits_S \sigma^0_z do = 0, \quad \iint\limits_S y\sigma^0_z do = 0, \quad \iint\limits_S x\sigma^0_z do = 0, \tag{5.1.10}$$

$$\iint\limits_S \sigma^1_z do = q_z, \quad \iint\limits_S y\sigma^1_z do = \mu_x, \quad \iint\limits_S x\sigma^1_z do = -\mu_y, \tag{5.1.11}$$

$$\iint\limits_S \sigma^{(2)}_z do = 0, \quad \iint\limits_S y\sigma^{(2)}_z do = -q_y, \quad \iint\limits_S x\sigma^{(2)}_z do = -q_x. \tag{5.1.12}$$

Let us proceed now to the equations of statics in the volume. Inserting expressions (5.1.7) into these equations leads to the systems of equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \tau^1_{xz}, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_x}{\partial y} = \tau^1_{yz}, \tag{5.1.13}$$

$$\frac{\partial \tau^0_{zx}}{\partial x} + \frac{\partial \tau^0_{yz}}{\partial y} = \sigma^1_z, \tag{5.1.14}$$

$$\frac{\partial \tau^1_{zx}}{\partial x} + \frac{\partial \tau^1_{yz}}{\partial y} = \sigma^{(2)}_z. \tag{5.1.15}$$

These need to be subjected to the appropriate boundary conditions. For system (5.1.13)

$$\text{on } \Gamma: \quad \sigma_x n_x + \tau_{xy} n_y = F_x(s), \quad \tau_{xy} n_x + \sigma_y n_y = F_y(s), \tag{5.1.16}$$

and for systems (5.1.14), (5.1.15)

$$\text{on } \Gamma: \; \tau^0_{zx} n_x + \tau^0_{yz} n_y = F_z(s), \tag{5.1.17}$$

$$\text{on } \Gamma: \; \tau^1_{zx} n_x + \tau^1_{yz} n_y = 0. \tag{5.1.18}$$

6.5.2 *Distribution of normal stresses*

Let us consider now Beltrami's dependences. Keeping the notation ∇^2 for the plane Laplace operator and representing the first invariant of the stress tensor in the form

$$\sigma = \sigma^0 + (l - z)\,\sigma_z^1 + \frac{1}{2}\,(l - z)^2\,\sigma_z^{(2)}, \quad \sigma^0 = \sigma_x + \sigma_y + \sigma_z^0, \tag{5.2.1}$$

we can satisfy all these dependences by assuming

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\sigma^0}{\partial x^2} = 0,\ \nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\sigma^0}{\partial y^2} = 0,\ \nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\sigma^0}{\partial x\partial y} = 0, \tag{5.2.2}$$

$$\frac{\partial^2\sigma_z^1}{\partial x^2} = \frac{\partial^2\sigma_z^1}{\partial y^2} = \frac{\partial^2\sigma_z^1}{\partial x\partial y} = 0, \quad \frac{\partial^2\sigma_z^{(2)}}{\partial x^2} = \frac{\partial^2\sigma_z^{(2)}}{\partial y^2} = \frac{\partial^2\sigma_z^{(2)}}{\partial x\partial y} = 0, \tag{5.2.3}$$

$$\nabla^2\tau_{zx}^0 = \frac{1}{1+\nu}\frac{\partial\sigma_z^1}{\partial x}, \quad \nabla^2\tau_{yz}^0 = \frac{1}{1+\nu}\frac{\partial\sigma_z^1}{\partial y}, \tag{5.2.4}$$

$$\nabla^2\tau_{zx}^1 = \frac{1}{1+\nu}\frac{\partial\sigma_z^{(2)}}{\partial x}, \quad \nabla^2\tau_{yz}^1 = \frac{1}{1+\nu}\frac{\partial\sigma_z^{(2)}}{\partial y}, \tag{5.2.5}$$

$$\nabla^2\sigma_z^0 + \frac{2+\nu}{1+\nu}\sigma_z^{(2)} = 0, \quad \nabla^2\sigma_z^1 = 0, \quad \nabla^2\sigma_z^{(2)} = 0. \tag{5.2.6}$$

The second and third relationships are identical to eq. (5.2.3) which suggests that σ_z^1 and $\sigma_z^{(2)}$ are linear functions of x, y. The particular form of these functions is given by the integral equations (5.1.11) and (5.1.12)

$$\sigma_z^1(x, y) = \frac{q_z}{S} - \frac{\mu_y}{I_y}x + \frac{\mu_x}{I_x}y, \tag{5.2.7}$$

$$\sigma_z^{(2)}(x, y) = -\left(\frac{q_x}{I_y}x + \frac{q_y}{I_x}y\right). \tag{5.2.8}$$

Hence, the distribution of the normal stresses over the cross-section is given by the relationship

$$\sigma_z = \sigma_z^0 + (l - z)\left(\frac{q_z}{S} - \frac{\mu_y}{I_y}x + \frac{\mu_x}{I_x}y\right) - \frac{1}{2}(l - z)^2\left(\frac{q_x}{I_y}x + \frac{q_y}{I_x}y\right), \tag{5.2.9}$$

and the differential equations (5.2.4) and (5.2.5) take the form

$$\nabla^2\tau_{zx}^0 = -\frac{1}{1+\nu}\frac{\mu_y}{I_y}, \quad \nabla^2\tau_{yz}^0 = \frac{1}{1+\nu}\frac{\mu_x}{I_x}, \tag{5.2.10}$$

$$\nabla^2\tau_{zx}^1 = -\frac{1}{1+\nu}\frac{q_x}{I_y}, \quad \nabla^2\tau_{yz}^1 = -\frac{1}{1+\nu}\frac{q_y}{I_x}. \tag{5.2.11}$$

It follows from the above-said and Subsection 6.5.1 that Michell's problem is split into three problems. Two of them are "autonomous" in that constructing their solutions does not require solving other problems.

The first problem is to determine the stresses $\tau_{zx}^{(1)}, \tau_{yz}^{(1)}$. It is identical to Saint-Venant's problem of bending by force and torsion of the rod. It follows from the identity of the system of equations (5.1.9), (5.1.15), (5.1.18) and (5.2.11) with the system (1.5.4), (1.5.1), (1.5.3), (1.5.2) after replacing q_x, q_y, μ_z by P, Q, m_z respectively.

The second problem is to determine the stresses τ_{zx}^0, τ_{yz}^0 appearing due to the tensile surface forces $F_z(s)$. These forces result in the force factors q_z, μ_x, μ_y appearing in the system of equations (5.1.8), (5.1.14), (5.1.17), (5.2.10) for these stresses. This system can be reduced to the boundary-value problem of Saint-Venant and another boundary-value problem for Laplace equations, see Subsection 6.5.3. The third problem is to determine $\sigma_x, \sigma_y, \tau_{xy}$ (σ^0 is obtained in passing) and is more difficult. It is reduced to the plane problem of the theory of elasticity, Chapter 7, whose statement requires the solution of the above two problems.

6.5.3 Tension of the rod

As mentioned above, the longitudinal surface forces F_z results in the normal stress

$$(l-z)\,\sigma_z^1 = (l-z)\left(\frac{q_z}{S} - \frac{\mu_y}{I_y}x + \frac{\mu_x}{I_x}y\right) \tag{5.3.1}$$

and the shear stresses τ_{zx}^0, τ_{yz}^0 described by the following system of equations

$$\frac{\partial \tau_{zx}^0}{\partial x} + \frac{\partial \tau_{yz}^0}{\partial y} = \frac{q_z}{S} - \frac{\mu_y}{I_y}x + \frac{\mu_x}{I_x}y, \tag{5.3.2}$$

$$\nabla^2 \tau_{zx}^0 = -\frac{1}{1+\nu}\frac{\mu_y}{I_y}, \quad \nabla^2 \tau_{yz}^0 = \frac{1}{1+\nu}\frac{\mu_x}{I_x}, \tag{5.3.3}$$

$$\tau_{zx}^0 n_x + \tau_{yz}^0 n_y = F_z(s), \tag{5.3.4}$$

$$\iint_S \tau_{zx}^0 do = 0, \quad \iint_S \tau_{yz}^0 do = 0, \quad \iint_S \left(x\tau_{yz}^0 - y\tau_{zx}^0\right) do = 0. \tag{5.3.5}$$

It is easy to prove that any solution of the static equations in the volume and on the surface, eqs. (5.3.2) and (5.3.4), identically satisfies the first two integral conditions (5.3.5). Indeed, by eq. (5.3.2)

$$-\mu_y = \iint_S x\left(\frac{\partial \tau_{zx}^0}{\partial x} + \frac{\partial \tau_{yz}^0}{\partial y}\right) do = \oint_\Gamma x\left(\tau_{zx}^0 n_x + \tau_{yz}^0 n_y\right) ds - \iint_S \tau_{zx}^0 do,$$

such that by eqs. (5.3.4) and (5.1.4)

$$-\mu_y = \oint\limits_{\Gamma} xF_z ds - \iint\limits_{S} \tau^0_{zx} do, \qquad \iint\limits_{S} \tau^0_{xz} do = 0.$$

The second relation (5.3.5) is proved by analogy. Besides

$$q_z = \iint\limits_{S} \left(\frac{\partial \tau^0_{zx}}{\partial x} + \frac{\partial \tau^0_{yz}}{\partial y} \right) do = \oint\limits_{\Gamma} \left(\tau^0_{zx} n_x + \tau^0_{yz} n_y \right) ds = \oint\limits_{\Gamma} F_z ds,$$

which corresponds to eq. (5.1.3).

The static equations and Beltrami's dependences (5.3.3) can be satisfied by assuming

$$\left.\begin{aligned} \tau^0_{zx} &= G\alpha_0 \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi^0}{\partial y} + \frac{q_z}{2S} x - \frac{1}{2(1+\nu)} \left(\frac{\mu_y}{I_y} x^2 - 2\nu \frac{\mu_x}{I_x} xy \right), \\ \tau^0_{yz} &= -G\alpha_0 \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi^0}{\partial x} + \frac{q_z}{2S} y + \frac{1}{2(1+\nu)} \left(-2\nu \frac{\mu_y}{I_y} xy + \frac{\mu_x}{I_x} y^2 \right). \end{aligned}\right\} \tag{5.3.6}$$

Here Φ is the stress function solving the problem of torsion for region S

$$\text{in } S: \quad \nabla^2 \Phi = -2; \qquad \text{on } \Gamma: \quad \frac{\partial \Phi}{\partial s} = 0, \tag{5.3.7}$$

and Ψ^0 is the solution of Laplace equation subjected to the boundary condition

$$\begin{aligned} \text{on } \Gamma: \quad \frac{\partial \Psi^0}{\partial s} = & \frac{1}{2(1+\nu)} \left[\frac{\mu_y}{I_y} \left(x^2 n_x + 2\nu xy n_y \right) - \frac{\mu_x}{I_x} \left(2\nu xy n_x + y^2 n_y \right) \right] \\ & + F_z(s) - \frac{q_z}{S} \frac{d\omega}{ds} \qquad \left[d\omega = \frac{1}{2} (x dy - y dx) \right], \end{aligned} \tag{5.3.8}$$

where $d\omega$ denotes (analogous to Subsection 6.2.5) the elementary sectorial area. Accounting for eq. (5.1.3) it is easy to prove that the introduced function Ψ^0 is uniquely defined on Γ, since

$$\oint\limits_{\Gamma} \frac{\partial \Psi^0}{\partial s} ds = 0. \tag{5.3.9}$$

One can present Ψ^0 as a sum of three terms

$$\Psi^0 = \frac{1}{2(1+\nu)} \left(\frac{\mu_y}{I_y} \vartheta_1 - \frac{\mu_x}{I_x} \vartheta_2 \right) + \vartheta_3 \tag{5.3.10}$$

which are the solutions of the boundary-value problems

$$\left.\begin{aligned} &\text{in } S: \nabla^2\vartheta_1 = 0; \qquad \text{on } \Gamma: \frac{\partial \vartheta_1}{\partial s} = x^2 n_x + 2\nu xy n_y, \\ &\text{in } S: \nabla^2\vartheta_2 = 0; \qquad \text{on } \Gamma: \frac{\partial \vartheta_2}{\partial s} = 2\nu xy n_x + y^2 n_y, \end{aligned}\right\} \tag{5.3.11}$$

$$\text{in } S: \nabla^2\vartheta_3 = 0; \qquad \text{on } \Gamma: \frac{\partial \vartheta_3}{\partial s} = F_z(s) - \frac{q_z}{S}\frac{d\omega}{ds}. \tag{5.3.12}$$

Functions ϑ_1, ϑ_2 were obtained earlier, see Saint-Venant's problem of bending by force, eqs. (2.1.7)-(2.1.9), where P and Q should be replaced by μ_y and $-\mu_x$ respectively. The boundary-value problem for ϑ_3 was not encountered earlier.

The torque μ_z^{01} corresponding to the stress function Φ and the components ϑ_1, ϑ_2 of function Ψ was determined in Subsection 6.2.5 and is equal to

$$\mu_z^{01} = GC\alpha_0 - \mu_y y^* - \mu_x x^*,$$

where x^*, y^* are the coordinates of the centre of rigidity. The component ϑ_3 determines the torque by means of eq. (2.5.5), that is

$$\begin{aligned} \mu_z^{02} &= -\iint\limits_S \left(x\frac{\partial \vartheta_3}{\partial x} + y\frac{\partial \vartheta_3}{\partial y}\right) do = -2\oint\limits_\Gamma \vartheta d\omega + 2\iint\limits_S \vartheta_3 do = -\oint\limits_\Gamma \varphi \frac{\partial \vartheta_3}{\partial s} ds \\ &= -\oint\limits_\Gamma \left(F_z(s) - \frac{q_z}{S}\frac{d\omega}{ds}\right)\varphi ds = -\oint\limits_\Gamma F_z(s)\varphi ds + \frac{q_z}{S}\iint\limits_S \varphi do. \end{aligned}$$

The choice of constant α_0 must obey the third condition (5.3.5). It reduces to the requirement $\mu_z^{01} + \mu_z^{01} = 0$ and is set in the form

$$GC\alpha_0 = \mu_x x^* + \mu_y y^* + \oint\limits_\Gamma F_z(s)\varphi ds - \frac{q_z}{S}\iint\limits_S \varphi do. \tag{5.3.13}$$

It is easy to determine the displacements in terms of function ϑ_3

$$\left.\begin{aligned} u^{(02)} &= -\frac{\nu q_z}{ES}(l - z)x, \quad v^{(02)} = -\frac{\nu q_z}{ES}(l - z)y, \\ w^{(02)} &= \frac{q_z}{2ES}\left[x^2 + y^2 - (l - z)^2\right] + \frac{1}{G}\chi_3, \end{aligned}\right\} \tag{5.3.14}$$

where χ_3 is the harmonic function related to ϑ_3 by Cauchy-Riemann's conditions (2.1.9). It is also necessary to add the displacement obtained by formulae (2.2.9) where one should take $R = 0, m_x = m_y = 0$, replace P, Q by $\mu_y, -\mu_x$ and the constant α by means of eq. (5.3.13).

Tension of the rod by forces of constant intensity

This solution was obtained by L.M. Zubov. When the tensile force acting on the contour of the cross-section has a constant intensity ($F_z = \text{const}$) and the centres of gravity of area S and contour Γ coincide, then we have

$$\mu_x = 0, \quad \mu_y = 0, \quad q_z = LF_z,$$

where L denotes the perimeter of the contour. The boundary condition (5.3.12) is set in the form

$$\text{on } \Gamma : \frac{\partial \vartheta_3}{\partial s} = LF_z \left(\frac{1}{L} - \frac{1}{S} \frac{d\omega}{ds} \right).$$

In order to simplify the notion for a simple connected region, instead of ϑ_3 we introduce the harmonic function

$$U = \frac{1}{LF_z} \vartheta_3$$

and arrive at Dirichlet's problem

$$\nabla^2 U = 0 \text{ in } S; \quad U_\Gamma = \frac{s}{L} - \frac{\omega}{S}. \tag{5.3.15}$$

According to eqs. (5.3.6) and (5.3.1) the corresponding stresses are given by the formulae

$$\left.\begin{aligned} \tau^{0\prime}_{zx} &= LF_z \left(\frac{\partial U}{\partial y} + \frac{1}{2} \frac{x}{S} \right), \\ \tau^{0\prime}_{yz} &= LF_z \left(-\frac{\partial U}{\partial x} + \frac{1}{2} \frac{y}{S} \right), \\ (l - z)\, \sigma^1_z &= \frac{LF_z}{S} (l - z). \end{aligned}\right\} \tag{5.3.16}$$

If the rod of a rectangular cross-section ($|x| < a, |y|, b$) is considered and the arcs are measured from the point $x = a, y = 0$ the boundary condition is reduced to the form

$$U(\pm a, y) = \pm \frac{b - a}{8(a + b)} \frac{y}{b}, \quad U(x, \pm b) = \pm \frac{b - a}{8(a + b)} \frac{x}{a},$$

and is satisfied by the following harmonic function

$$U(x, y) = \frac{b - a}{8(a + b)} \frac{xy}{ab} = \frac{2(b - a)}{SL} xy.$$

In particular, $U = 0$ for a square and this result is valid for any regular polygon and for the circle. Indeed, boundary condition (5.3.15) takes the form

$$U_\Gamma = \frac{s}{L} - \frac{hs}{hL} = 0,$$

where h denotes the apothem.

The problem becomes more difficult for a rod with an elliptic cross-section. Introducing the elliptic coordinates α, β

$$x = c\cosh\alpha\cos\beta, \quad y = c\sinh\alpha\sin\beta \quad \left(c = \sqrt{a^2 - b^2}\right),$$

and denoting the value of α on the contour of the cross-section of the ellipse with the semi-axes a, b by α_0 we have

$$c\cosh\alpha_0 = a,\ c\sinh\alpha_0 = b,\ d\omega = \frac{1}{2}\left(x_\Gamma dy_\Gamma - y_\Gamma dx_\Gamma\right) = \frac{1}{2}ab d\beta,\ \omega = S\frac{\beta}{2\pi}.$$

The classical representation of the arc of the ellipse has the form

$$\frac{s}{L} = \frac{\int\limits_0^\beta \sqrt{\cosh^2\alpha_0 - \cos^2\beta}\,d\beta}{\int\limits_0^{2\pi} \sqrt{\cosh^2\alpha_0 - \cos^2\beta}\,d\beta} = \frac{1}{4E(k)}\left[E(k) - E\left(\frac{\pi}{2} - \beta, k\right)\right],$$

where $E\left(\frac{\pi}{2} - \beta, k\right)$ denotes the elliptic integral of the second kind in Legendre's normal form with modulus $k = (\cosh\alpha_0)^{-1}$ and $E(k)$ denotes the complete elliptic integral of the second kind.

Function $E\left(\frac{\pi}{2} - \beta, k\right)$ can be expanded into a trigonometric series[3]

$$E\left(\frac{\pi}{2} - \beta, k\right) = B_0\left(\frac{\pi}{2} - \beta\right) + B_1\sin 2\beta + B_2\sin 4\beta + B_3\sin 6\beta + \dots,$$

where

$$B_0 = \frac{2E(k)}{\pi}, \quad B_n = \frac{1}{n}\sum_{m=0}^{\infty}\frac{(m+1)\dots(m+n)}{(m+n+1)\dots(m+2n)} \times \\ \left[\frac{1\cdot 3\cdot 5\dots(2m+2n-1)}{2\cdot 4\cdot 6\dots(2m+2n)}\right]^2 \frac{k^{2m+2n}}{2m+2n+1}.$$

Boundary condition (5.3.15) is now presented by the periodic function

$$U_\Gamma = \frac{1}{4E(k)}\left(B_1\sin 2\beta + B_2\sin 4\beta + B_3\sin 6\beta + \dots\right),$$

and the solution of the problem is the following series

$$U = -\frac{1}{4E(k)}\sum_{n=1}^{\infty} B_n \frac{\sinh 2n\alpha}{\sinh 2n\alpha_0}\sin 2n\beta.$$

[3] Zhuravsky A.M. Handbook of elliptic functions (in Russian). Publishers of the USSR Academy of Sciences, 1941.

Indeed, each term of this series is proportional to the imaginary part of function $\cosh 2n\,(\alpha + i\beta)$ and thus satisfies Laplace equation. It is continuous together with the derivatives with respect to α and β in the whole ellipse, including the passage through the cut between the foci.

6.5.4 Shear stresses τ^1_{zx}, τ^1_{yz}

Replacing P, Q, m_z in the solution of Subsection 6.2.1 by q_x, q_y, μ_z we arrive at the expressions

$$\left.\begin{aligned} \tau^1_{zx} &= G\alpha_1 \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi^1}{\partial y} - \frac{1}{2\,(1+\nu)} \left(\frac{q_x}{I_y} x^2 + 2\nu \frac{q_y}{I_x} xy \right), \\ \tau^1_{yz} &= -G\alpha_1 \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi^1}{\partial x} - \frac{1}{2\,(1+\nu)} \left(2\nu \frac{q_x}{I_y} xy + \frac{q_y}{I_x} y^2 \right), \end{aligned}\right\} \tag{5.4.1}$$

where

$$\Psi^1 = \frac{1}{2\,(1+\nu)} \left(\frac{q_x}{I_y} \vartheta_1 + \frac{q_y}{I_x} \vartheta_2 \right), \tag{5.4.2}$$

and ϑ_1, ϑ_2 are the solutions of the boundary-value problems (5.3.11). The constant-valued parameter α is determined by the condition

$$CG\alpha_1 = \mu_z - (x^* q_y - y^* q_x)\,. \tag{5.4.3}$$

6.5.5 Stresses $\sigma_x, \sigma_y, \tau_{xy}$

We must consider the static equations in the volume (5.1.13), on the surface (5.1.16), dependences (5.2.2) for σ_z, relationships (5.2.6) and the integral conditions (5.1.10).

Using eq. (5.4.1) and replacing ϑ_1, ϑ_2 by the harmonic functions ψ_1, ψ_1 (see eq. (2.1.9)) and introducing function φ by means of relationships (2.1.10), (2.1.12) we can rewrite the first set of equations in the following form

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= \frac{\partial \Lambda}{\partial x} - G\alpha_1 y - \frac{1}{2\,(1+\nu)} \left(\frac{q_x}{I_y} x^2 + 2\nu \frac{q_y}{I_x} xy \right), \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= \frac{\partial \Lambda}{\partial y} + G\alpha_1 x - \frac{1}{2\,(1+\nu)} \left(2\nu \frac{q_x}{I_y} xy + \frac{q_y}{I_x} y^2 \right), \\ \Lambda &= G\alpha_1 \varphi + \frac{1}{2\,(1+\nu)} \left(\frac{q_x}{I_y} \chi_1 + \frac{q_y}{I_x} \chi_2 \right). \end{aligned}\right\} \tag{5.5.1}$$

These equations are satisfied if we take

$$\left.\begin{aligned}\sigma_x &= -\frac{1}{6(1+\nu)}\left[\frac{q_x}{I_y}x^3+\nu\frac{q_y}{I_x}\left(3x^2y-y^3\right)\right]+\Lambda+\frac{\partial^2 U}{\partial y^2},\\ \sigma_y &= -\frac{1}{6(1+\nu)}\left[\frac{q_y}{I_x}y^3+\nu\frac{q_x}{I_y}\left(3xy^2-x^3\right)\right]+\Lambda+\frac{\partial^2 U}{\partial x^2},\\ \tau_{xy} &= \frac{1}{2}G\alpha_1\left(x^2-y^2\right)-\frac{\partial^2 U}{\partial x\partial y},\end{aligned}\right\}\tag{5.5.2}$$

where we introduced Airy's stress function which is well-known in the plane problem of the theory of elasticity.

Inserting these expressions into Beltrami's dependences and rearranging the result we arrive at the relationships

$$\left.\begin{aligned}\frac{2}{1+\nu}\frac{q_x}{I_y}x+\frac{\nu}{1+\nu}\frac{q_y}{I_x}y-(1+\nu)\nabla^4 U &= \frac{\partial^2}{\partial x^2}\left(\sigma_z^0+2\Lambda-\nu\nabla^2 U\right),\\ \frac{2}{1+\nu}\frac{q_y}{I_x}y+\frac{\nu}{1+\nu}\frac{q_x}{I_y}x-(1+\nu)\nabla^4 U &= \frac{\partial^2}{\partial y^2}\left(\sigma_z^0+2\Lambda-\nu\nabla^2 U\right),\\ \frac{\nu}{1+\nu}\left(\frac{q_x}{I_y}y+\frac{q_y}{I_x}x\right) &= \frac{\partial^2}{\partial x\partial y}\left(\sigma_z^0+2\Lambda-\nu\nabla^2 U\right).\end{aligned}\right\}\tag{5.5.3}$$

where $\nabla^4 = \nabla^2\nabla^2$ denotes the biharmonic operator which is the plane Laplace operator applied twice. By eqs. (5.2.6) and (5.2.8)

$$\nabla^2\sigma_z^0 = \frac{2+\nu}{1+\nu}\left(\frac{q_x}{I_y}x+\frac{q_y}{I_x}y\right).\tag{5.5.4}$$

Adding the first and second equations in (5.5.3) and taking into account that Λ is a harmonic function we arrive at the biharmonic differential equation for Airy's function

$$\nabla^4 U = \frac{\partial^4 U}{\partial x^4}+2\frac{\partial^4 U}{\partial x^2\partial y^2}+\frac{\partial^4 U}{\partial y^4}=0.\tag{5.5.5}$$

Accounting for the relationships

$$\frac{\partial^2 U}{\partial y^2}n_x-\frac{\partial^2 U}{\partial x\partial y}n_y=\frac{d}{ds}\frac{\partial U}{\partial y},\qquad -\frac{\partial^2 U}{\partial x\partial y}n_x+\frac{\partial^2 U}{\partial x^2}n_y=-\frac{d}{ds}\frac{\partial U}{\partial x}$$

we can set the boundary conditions for this function as follows

$$\text{on }\Gamma:\quad \frac{\partial U}{\partial y}=\int_0^s X\,ds,\qquad \frac{\partial U}{\partial x}=-\int_0^s Y\,ds,\tag{5.5.6}$$

where, due to eqs. (5.1.16) and (5.5.2)

$$\left.\begin{aligned}
X &= F_x(s) + \frac{1}{6(1+\nu)}\left[\frac{q_x}{I_y}x^3 + \nu\frac{q_y}{I_x}\left(3x^2y - y^3\right)\right]n_x - \\
&\qquad \frac{1}{2}G\alpha_1\left(x^2 - y^2\right)n_y - \Lambda n_x, \\
Y &= F_y(s) + \frac{1}{6(1+\nu)}\left[\nu\frac{q_x}{I_y}\left(3xy^2 - x^3\right) + \frac{q_y}{I_x}y^3\right]n_y - \\
&\qquad \frac{1}{2}G\alpha_1\left(x^2 - y^2\right)n_x - \Lambda n_y.
\end{aligned}\right\} \tag{5.5.7}$$

It is easy to prove that for the simple connected region

$$\oint_\Gamma X ds = 0, \qquad \oint_\Gamma Y ds = 0, \tag{5.5.8}$$

which ensures that the derivatives of Airy's function are single-valued on the contour. Indeed, using eq. (5.1.3) and transforming the contour integrals we have

$$\oint_\Gamma X ds = q_x + \frac{1}{2(1+\nu)}q_x - \iint_S \frac{\partial \Lambda}{\partial x}do.$$

By virtue of eqs. (2.1.9), (2.1.11), (2.1.12) and (2.1.4)

$$\begin{aligned}
\iint_S \frac{\partial \Lambda}{\partial x}do &= G\alpha_1 \iint_S \frac{\partial \Phi}{\partial x}do + \frac{1}{2(1+\nu)}\left(\frac{q_x}{I_y}\iint_S \frac{\partial \vartheta_1}{\partial y}do + \frac{q_y}{I_x}\iint_S \frac{\partial \vartheta_2}{\partial y}do\right) \\
&= G\alpha_1 \oint_\Gamma n_y \Phi ds + \frac{1}{2(1+\nu)}\left(\frac{q_x}{I_y}\oint_\Gamma \vartheta_1 n_y ds + \frac{q_y}{I_x}\oint_\Gamma \vartheta_2 n_y ds\right) \\
&= \frac{1}{2(1+\nu)}\left(\frac{q_x}{I_y}\oint_\Gamma x\frac{\partial \vartheta_1}{\partial s}ds - \frac{q_y}{I_x}\oint_\Gamma y\frac{\partial \vartheta_2}{\partial s}ds\right),
\end{aligned}$$

and then referring to eq. (5.3.11) we obtain

$$\iint_S \frac{\partial \Lambda}{\partial x}do = \frac{3+2\nu}{2(1+\nu)}q_x, \tag{5.5.9}$$

which is required. The second equality in eq. (5.5.8) is proved by analogy. It is also evident that this follows from pure static reasoning, cf. Subsection 6.1.5.

6.5.6 Determining σ_z^0

Returning to eq. (5.5.3) and removing $\nabla^2 U$ by means of eq. (5.5.2) we arrive at the relationship

$$\sigma_z^0 = -2(1+\nu)\Lambda + \nu(\sigma_x + \sigma_y) + \frac{1}{6}(2-\nu)\left(\frac{q_x}{I_y}x^3 + \frac{q_y}{I_x}y^3\right) + \\ \frac{1}{2}\nu\left(\frac{q_x}{I_y}xy^2 + \frac{q_y}{I_x}x^2 y\right) + c_0 + c_1 x + c_2 y, \quad (5.6.1)$$

where the constants c_0, c_1, c_2 are obtained from conditions (5.1.10) expressing that the system of stresses σ_z^0 is statically equivalent to zero. Determining these constants requires obtaining the moments of the stresses of the first and second order

$$\iint\limits_S (\sigma_x + \sigma_y)\, do, \quad \iint\limits_S x(\sigma_x + \sigma_y)\, do, \quad \iint\limits_S y(\sigma_x + \sigma_y)\, do.$$

This is achieved by using the formulae of Subsections 1.4.3 and 1.4.4 for two static equations in (5.1.13) where $-\tau_{xz}^1, -\tau_{yz}^1$ play the part of the volume forces whilst F_x, F_y are the surface forces. Then

$$\iint\limits_S (\sigma_x + \sigma_y)\, do = \oint\limits_\Gamma (xF_x + yF_y)\, ds - \iint\limits_S \left(x\tau_{xz}^1 + y\tau_{yz}^1\right) do,$$

$$\iint\limits_S x(\sigma_x + \sigma_y)\, do = \frac{1}{2}\oint\limits_\Gamma \left[\left(x^2 - y^2\right)F_x + 2xyF_y\right] ds - \\ \frac{1}{2}\iint\limits_S \left[\left(x^2 - y^2\right)\tau_{zx}^1 + 2xy\tau_{yz}^1\right] do \text{ etc.}$$

Using Subsection 1.4.10 one can determine the expressions for the moments of the second order for stresses τ_{xz}^1, τ_{yz}^1 only partly, i.e. in the combinations of the sort

$$3\iint\limits_S x^2\tau_{zx}^1 do = \frac{q_x}{I_y}\iint\limits_S x^4 do + \frac{q_y}{I_x}\iint\limits_S x^3 y\, do,$$

$$\iint\limits_S \left(y^2\tau_{zx}^1 + 2xy\tau_{yz}^1\right) do = \frac{q_x}{I_y}\iint\limits_S x^2y^2 do + \frac{q_y}{I_x}\iint\limits_S xy^3 do.$$

The integrals

$$\iint\limits_S y^2\tau_{zx}^1 do, \quad \iint\limits_S x^2\tau_{yz}^1 do$$

can not be determined in terms of the loads when only the equations of statics are used.

6.5.7 *Bending of a heavy rod*

Let us consider a heavy horizontal rod whose end $z = l$ is free. Axis x is directed along the downward vertical. Hence the only nontrivial component of the volume force is $\rho K_z = \gamma$ where γ denotes the weight of the unit volume. The particular solution of the equilibrium equations can be taken in the form of $\sigma'_x = -\gamma x$ and the corresponding distribution of the surface forces on the lateral surface is given by

$$F'_x = -\gamma x n_x, \quad F'_y = 0, \quad F'_z = 0.$$

The lateral surface is, however, free of load, hence the taken particular solution should be superimposed by the solution of Michell's problem with

$$F_x = \gamma x n_x, \quad F_y = 0, \quad F_z = 0. \tag{5.7.1}$$

Among the six integral characteristics (5.1.3), (5.1.4) the only non-vanishing one is

$$q_x = \gamma \oint x n_x ds = \gamma S. \tag{5.7.2}$$

For this reason, using eq. (5.2.9) we have

$$\sigma_z = \sigma_z^0 + \frac{1}{2}(l-z)^2 \sigma_z^{(2)} = \sigma_z^0 - \frac{1}{2}(l-z)^2 \frac{\gamma S}{I_y} x, \tag{5.7.3}$$

$$\tau_{zx} = (l-z)\tau^1_{zx}, \quad \tau_{yz} = (l-z)\tau'_{yz}. \tag{5.7.4}$$

Hence we deal with the problem of bending by force, whose solution is given by the formulae of Subsection 6.5.4, and the plane problem yielding stresses $\sigma_x, \sigma_y, \tau_{xy}$ and the vanishing component σ_z^0 of stress σ_z.

In the particular case of the circular rod of radius a we utilise eqs. (4.2.5), (5.5.1) and (4.2.6) to obtain

$$\left.\begin{aligned}
\chi_1 &= \frac{1-2\nu}{12}\left(x^3 - 3xy^2\right) + \frac{3+2\nu}{4}a^2 x,\\
\Lambda &= \frac{\gamma S}{2(1+\nu) I_y}\chi_1 = \frac{2\gamma}{a^2(1+\nu)}\chi_1,\\
\tau^1_{zx} &= \frac{\nu}{2(1+\nu)a^2}\left[(3+2\nu)\left(a^2-x^2\right) - (1-2\nu)y^2\right],\\
\tau^1_{yz} &= -\frac{\gamma}{a^2}\frac{1+2\nu}{1+\nu}xy.
\end{aligned}\right\} \tag{5.7.5}$$

Using eq. (5.5.2) yields

$$\left.\begin{aligned}
\sigma_x &= \frac{\gamma x}{2a^2(1+\nu)}\left[(3+2\nu)\left(a^2 - \frac{1}{3}x^2\right) - (1-2\nu)y^2\right] + \frac{\partial^2 U}{\partial y^2},\\
\sigma_y &= \frac{\gamma x}{2a^2(1+\nu)}\left[\frac{1}{3}(1+2\nu)\left(x^2-3y^2\right) + (3+2\nu)a^2\right] + \frac{\partial^2 U}{\partial x^2},\\
\tau_{xy} &= -\frac{\partial^2 U}{\partial x \partial y},
\end{aligned}\right\} \tag{5.7.6}$$

the stress function U being the solution of the biharmonic boundary-value problem (5.5.6) with the following values of X and Y on the contour of the region $x^2 + y^2 - a^2 = 0$

$$X = \frac{\nu\gamma x}{a^2(1+\nu)}\left(\frac{4}{3}x^2 - a^2\right)n_x, \quad Y = -\frac{\gamma x}{a^2(1+\nu)}\left[\frac{2}{3}(1+2\nu)x^2 + a^2\right]n_y.$$

It is easy to prove that any harmonic function multiplied by x or y or $x^2 + y^2$ satisfies the biharmonic equation. The inverse statement is also true: any biharmonic function can be represented in one of the following forms

$$f_1 + xf_2, \quad f_1 + yf_2, \quad f_1 + \left(x^2 + y^2\right)f_2, \tag{5.7.7}$$

where f_1 and f_2 are some harmonic functions. For the circular region, the cosines n_x, n_y of the angle between the normal and the coordinate axes are proportional to x, y respectively. Hence setting the boundary conditions in the form

$$\begin{aligned} \frac{\partial^2 U}{\partial y^2}x - \frac{\partial^2 U}{\partial x \partial y}y &= \frac{\nu\gamma x^2}{a^2(1+\nu)}\left(\frac{4}{3}x^2 - a^2\right), \\ -\frac{\partial^2 U}{\partial x \partial y}x + \frac{\partial^2 U}{\partial x^2}y &= -\frac{\gamma xy}{a^2(1+\nu)}\left[\frac{2}{3}(1+2\nu)x^2 + a^2\right], \end{aligned} \tag{5.7.8}$$

it is sufficient to look for U as a polynomial of order not higher than five. By virtue of the above-said this polynomial is presented by a sum of a harmonic polynomial of fifth order and the product of a harmonic polynomial of fourth order and x (or y). This sum can be added by any polynomial of third order which is clearly always biharmonic. It turns out to be sufficient to take

$$U = \frac{\gamma}{a^2(1+\nu)}\left[A\left(x^5 - 10x^3y^2 + 5xy^4\right) + Bx\left(x^4 - 6x^2y^2 + y^4\right) + Ca^2x^3 + Da^2xy^2\right], \tag{5.7.9}$$

the above harmonic polynomials being equal to the real parts of $(x+iy)^5$ and $(x+iy)^4$. Substituting eq. (5.7.9) into the boundary conditions (5.7.8) leads to a system of five equations, one of them being the result of the others. Hence, it is sufficient to introduce four constants. We obtain

$$U = \frac{\gamma}{24a^2(1+\nu)}\left[\frac{1}{5}(1-\nu)\left(x^5 - 10x^3y^2 + 5xy^4\right) - \frac{1}{2}\left(x^5 - 6x^3y^2 + xy^4\right) - (5+2\nu)a^2x^3 - (1-2\nu)a^2xy^2\right]$$

and by eq. (5.7.6) the expressions for the sought stresses are as follows

$$\left.\begin{aligned}
\sigma_x &= \frac{\gamma}{12a^2(1+\nu)}\left[(5+2\nu)xa^2 - 3(1-2\nu)xy^2 - (5+2\nu)x^3\right],\\
\tau_{yx} &= -\frac{\gamma}{12a^2(1+\nu)}\left[-(1-2\nu)ya^2 + (1-2\nu)y^3 + 3(1+2\nu)x^2y\right],\\
\sigma_y &= \frac{\gamma}{12a^2(1+\nu)}\left[3(1+2\nu)xa^2 - 3(1+2\nu)xy^2 - (1-2\nu)x^3\right],
\end{aligned}\right\} \tag{5.7.10}$$

where stress $\sigma_x' = -\gamma x$ was also taken into account.

Due to eq. (5.6.1) the distribution of stress σ_z^0 over the cross-section is set in the form

$$\sigma_z^0 = -\frac{2+\nu}{3(1+\nu)}\gamma x\left[1 - \frac{3}{2a^2}\left(x^2+y^2\right)\right], \tag{5.7.11}$$

where c_1 is determined from condition (5.1.10).

At the points $x = \pm a, y = 0$ of the cross-section the stress σ_z, due to eq. (5.7.3) is equal to

$$\sigma_z = \mp 2\gamma a\left[\left(\frac{l-z}{a}\right)^2 - \frac{2+\nu}{12(1+\nu)}\right]. \tag{5.7.12}$$

The second terms yields the correction which does not appear in the framework of the elementary theory.

The problem of bending a heavy rod with an elliptic cross-section can be studied by analogy. The problem of bending of such a rod (without accounting for the weight) was considered in Subsection 6.4.2. The stress function U should be prescribed in the form of eq. (5.7.9) and in the boundary condition (5.7.8) it is necessary to take into account that n_x, n_y are proportional to $x/a^2, y/b^2$ respectively.

6.5.8 *Mean values of stresses*

The mean values of stresses $\tau_{zx}, \tau_{yz}, \sigma_x, \sigma_y$ on the straight lines parallel to the coordinate axes can be found using static reasoning, i.e. without solving the boundary-value problems.

Referring to the systems of equations (5.1.14), (5.1.17), (5.2.7) and (5.1.15), (5.1.18), (5.2.8) and repeating the derivation of Subsection 6.4.1 we obtain

$$\int_{y_1}^{y_2} \tau_{zx}dy = b\left(\tau_{zx}\right)_m = -q_z\frac{\Omega(x)}{S} + \int_{\Gamma_1} F_z ds + \\ \Omega(x)\left\{\frac{x_c}{I_y}\left[q_x(l-z)+\mu_y\right] + \frac{y_c}{I_x}\left[q_y(l-z)-\mu_x\right]\right\}, \tag{5.8.1}$$

where Γ_1 is the arc bca of contour Γ, see Fig. 6.14. The expression for the mean value of τ_{yz} on the straight line parallel to axis x is written down by analogy.

We will also need the value of the mean value τ_{zx}^1 on the area $\Omega(x)$. We have

$$\iint_{\Omega} x\left(\frac{\partial \tau_{zx}^1}{\partial x}+\frac{\partial \tau_{yz}^1}{\partial y}\right) do=-\left(\frac{q_x}{I_y} j_x+\frac{q_y}{I_x} j_{xy}\right),$$

where j_x and j_{xy} denote respectively the moment of inertia about axis x and the product of inertia of this area. On the other side, taking into account eq. (5.1.18) we have

$$\begin{aligned}\iint_{\Omega} x\left(\frac{\partial \tau_{zx}^1}{\partial x}+\frac{\partial \tau_{yz}^1}{\partial y}\right) do &= \int_{\gamma} x\left(\tau_{zx}^1 n_x+\tau_{yz}^1 n_y\right) ds-\iint_{\Omega(x)} \tau_{yz}^1 do \\ &= -x\int_{y_1}^{y_2} \tau_{zx}^1 dy-\iint_{\Omega(x)} \tau_{yz}^1 do,\end{aligned}$$

and referring to eq. (5.8.1) we obtain

$$\iint_{\Omega(x)} \tau_{zx}^1 do=\frac{q_x}{I_y}\left(j_y-x x_c \Omega(x)\right)+\frac{q_y}{I_x}\left(j_{xy}-x y_c \Omega(x)\right). \tag{5.8.2}$$

By eqs. (5.1.13) and (5.1.16) we have

$$\iint_{\Omega(x)}\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\right) do=\int_{\Gamma_1} F_x ds-\int_{y_1}^{y_2} \sigma_x dy=\iint_{\Omega(x)} \tau_{yz}^1 do,$$

hence applying eq. (5.8.2) we arrive at the same expression for the mean value of stress σ_x on the line γ

$$\begin{aligned}&\int_{y_1}^{y_2} \sigma_x dy=b(x)\left(\sigma_x\right)_m= \\ &\qquad=\int_{\Gamma_1} F_x ds-\frac{q_x}{I_y}\left(j_y-x x_c \Omega(x)\right)-\frac{q_y}{I_x}\left(j_{xy}-x y_c \Omega(x)\right).\end{aligned} \tag{5.8.3}$$

If the cross-section is symmetric about axis x one should put $y_c=0, j_{xy}=0$ in formulae (5.8.1)-(5.8.3).

6.5.9 On Almansi's problem

In 1901 Almansi studied the problem of the state of stress in a prismatic rod whose lateral surface is loaded by forces which are polynomials in the axial coordinate z

$$\left.\begin{aligned} \sigma_x n_x + \tau_{xy} n_y &= F_x = \sum_{k=0}^{n} a_k(s) \frac{(l-z)^k}{k!}, \\ \tau_{xy} n_x + \sigma_y n_y &= F_y = \sum_{k=0}^{n} b_k(s) \frac{(l-z)^k}{k!}, \\ \tau_{zx} n_x + \tau_{yz} n_y &= F_z = \sum_{k=0}^{n} c_k(s) \frac{(l-z)^k}{k!}. \end{aligned}\right\} \tag{5.9.1}$$

This problem is a natural continuation of the problems of Saint-Venant and Michell and its solution can be reduced to a consequent solving these problems.

7

The plane problem of the theory of elasticity

7.1 Statement of the plane problems of theory of elasticity

7.1.1 Plane strain

The name "plane problem" is assigned to the extensive and well developed area at elasticity theory. The plane problem is concerned with the plane strain problem and the plane stress problem. Though these problems are different, they are unified by the mathematical method of their solution.

In the plane strain problem, one considers the particular solution of the equations of elasticity theory satisfying the following assumptions: displacements u, v are independent of coordinate $x_3 = z$ whereas w is independent of x, y and is a linear function of z, i.e.

$$u = u(x, y), \quad v = v(x, y), \quad w = ez + w_0. \tag{1.1.1}$$

An evident consequence of these assumptions is that the stresses τ_{zx}, τ_{yz} vanish, i.e.

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0, \quad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0, \tag{1.1.2}$$

and that the remaining components $\sigma_x, \sigma_y, \tau_{xy}, \sigma_z$ of the stress tensor are independent of coordinate z.

Under these conditions, the generalised Hooke law is set as follows

$$\left.\begin{array}{ll}\sigma_x = 2\mu\left[\dfrac{\nu}{1-2\nu}(\vartheta_1+e)+\dfrac{\partial u}{\partial x}\right], & \sigma_y = 2\mu\left[\dfrac{\nu}{1-2\nu}(\vartheta_1+e)+\dfrac{\partial v}{\partial y}\right],\\ \tau_{xy} = \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right), & \sigma_z = 2\mu\left(\dfrac{\nu}{1-2\nu}\vartheta_1+\dfrac{1-\nu}{1-2\nu}e\right).\end{array}\right\} \tag{1.1.3}$$

Here

$$\vartheta_1 = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = \frac{1-2\nu}{2\mu}(\sigma_x+\sigma_y) - 2\nu e, \tag{1.1.4}$$

the latter equation being the result of the first two relationships of Hooke's law (1.1.3). Inserting ϑ_1, eq. (1.1.4), into eq. (1.1.3) we can express σ_z in the form

$$\sigma_z = \nu(\sigma_x+\sigma_y)+Ee, \quad E = 2\mu(1+\nu). \tag{1.1.5}$$

Hence, the problem of determining σ_z becomes of secondary importance and the main issue is to determine the plane field of stresses $\sigma_x, \sigma_y, \tau_{xy}$.

The plane strain occurs in a prismatic body of an infinite length loaded by the surface and volume forces which are perpendicular to axis z and whose intensity does not depend on z. All of the cross-sections are under the same conditions which justifies prescribing displacements in the form (1.1.1). A plane strain is realised approximately in the mid-part of the body of finite length. The dependence of the state of stress on z is taken into account in the statement of the problems of Michell and Almansi (Section 6.5) and is reduced to superposition of Saint-Venant's problem and the plane problem.

In the volume, the equations of statics in the plane problem is written down in the form

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\rho K_x = 0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\rho K_y = 0, \tag{1.1.6}$$

whilst on the surface, or equivalently on contour Γ of the cross-section, we have

$$\sigma_x n_x + \tau_{xy} n_y = F_x, \quad \tau_{xy} n_x + \sigma_y n_y = F_y. \tag{1.1.7}$$

Let us consider any part of the body cut by two cross-sections. The external volume and surface forces applied to this part must be in equilibrium. This leads to the equilibrium equations in which the principal vector is zero

$$\iint\limits_S \rho K_x do + \oint\limits_\Gamma F_x ds = 0, \quad \iint\limits_S \rho K_y do + \oint\limits_\Gamma F_y ds = 0 \tag{1.1.8}$$

and the principal moment about axis z is also zero

$$\iint_S \rho\left(xK_y - yK_x\right) do + \oint_\Gamma \left(xF_y - yF_x\right) ds = 0. \qquad (1.1.9)$$

Here S denotes the area of the cross-section of the body and the traversing a contour Γ is in the counterclockwise direction.

Equations (1.1.8) can be easily obtained from the static equations (1.1.6), (1.1.7) with the help of the transformation

$$\iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho K_x\right) do = \oint_\Gamma \left(\sigma_x n_x + \tau_{xy} n_y\right) ds + \iint_S \rho K_x do$$
$$= \oint_\Gamma F_x ds + \iint_S \rho K_x do = 0.$$

By analogy one arrives at the equations for the moments (1.1.9)

$$\iint_S \left[x\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho K_y\right) - y\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho K_x\right)\right] do =$$
$$= \oint_\Gamma \left(xF_y - yF_x\right) ds + \iint_S \rho\left(xK_y - yK_x\right) do = 0.$$

The same method allows us to find the first and second moments of stresses σ_x, σ_y. We have

$$\iint_S x\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho K_x\right) do = \oint_\Gamma xF_x ds +$$
$$\iint_S \rho x K_x do - \iint_S \sigma_x do = 0$$

$$\iint_S x^2\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho K_x\right) do = \oint_\Gamma x^2 F_x ds +$$
$$\iint_S \rho x^2 K_x do - 2\iint_S x\sigma_x do = 0,$$

$$\iint_S xy\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho K_x\right) do = \oint_\Gamma xyF_x ds +$$
$$\iint_S \rho xy K_x do - \iint_S \left(y\sigma_x + y\tau_{xy}\right) do = 0,$$

$$\iint_S x^2\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho K_y\right) do = \oint_\Gamma x^2 F_y ds +$$
$$\iint_S \rho x^2 K_y do - 2\iint_S x\tau_{xy} do = 0.$$

These equations and the analogous equalities yield

$$\left.\begin{aligned}
&\iint\limits_S (\sigma_x+\sigma_y)\,do = \oint\limits_\Gamma (xF_y+yF_x)\,ds + \iint\limits_S \rho\,(xK_y-yK_x)\,do,\\
&\iint\limits_S x\,(\sigma_x+\sigma_y)\,do = \frac{1}{2}\oint\limits_\Gamma \left[\left(x^2-y^2\right)F_x+2xyF_y\right]ds+\\
&\qquad\qquad\qquad\qquad \iint\limits_S \rho\left[\left(x^2-y^2\right)K_x+2xyK_y\right]do,\\
&\iint\limits_S y\,(\sigma_x+\sigma_y)\,do = \frac{1}{2}\oint\limits_\Gamma \left[2xyF_x+\left(y^2-x^2\right)F_y\right]ds+\\
&\qquad\qquad\qquad\qquad \iint\limits_S \rho\left[2xyK_x+\left(y^2-x^2\right)K_y\right]do.
\end{aligned}\right\}
\tag{1.1.10}$$

These expressions allow us to determine the principal vector and the moment of stress σ_z in the cross-section of the body in terms of the given volume and surface forces

$$\left.\begin{aligned}
R &= \iint\limits_S \sigma_z do = \nu \iint\limits_S (\sigma_x+\sigma_y)\,do + ESe,\\
m_x &= \iint\limits_S y\sigma_z do = \nu \iint\limits_S y\,(\sigma_x+\sigma_y)\,do + ESey_0,\\
m_y &= -\iint\limits_S x\sigma_z do = -\nu \iint\limits_S x\,(\sigma_x+\sigma_y)\,do - ESex_0,
\end{aligned}\right\}
\tag{1.1.11}$$

where x_0, y_0 denote the coordinate of the centre of gravity of the cross-section.

If the end faces are fixed in such a way that no axial force appears, then $R=0$, and this determines the constant e introduced by formula (1.1.1). If any axial displacement is prohibited, then $e=0$.

Formulae (1.1.0) and (1.1.11) determine the force factors $-R, -m_x, -m_y$ of the simple Saint-Venant's problems (tension and bending by a moment). These solutions should be imposed on the solution of the plane strain of the prismatic body with unloaded end faces.

7.1.2 Airy' stress function

It is shown in Subsection 1.1.6 that the homogeneous equations of statics are satisfied by expressing stresses $\sigma_x, \sigma_y, \tau_{xy}$ in terms of a single stress function, see eq. (1.6.11) of Chapter 1. This result is easily obtained directly. Indeed, each of two homogeneous static equations

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \quad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{1.2.1}$$

holds identically with the help of the functions

$$\sigma_x = \frac{\partial \varphi_1}{\partial y}, \quad \tau_{yx} = -\frac{\partial \varphi_1}{\partial x}; \quad \tau_{xy} = \frac{\partial \varphi_2}{\partial y}, \quad \sigma_y = -\frac{\partial \varphi_2}{\partial x}$$

and the condition $\tau_{yx} = \tau_{xy}$ leads to an equation analogous to eq. (1.2.1)

$$\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} = 0.$$

Introducing into consideration a new function $U(x, y)$ referred to as the Airy function (Airy, 1862) we have

$$\varphi_1 = \frac{\partial U}{\partial y}, \quad \varphi_2 = -\frac{\partial U}{\partial x}$$

and arrive at the basic relationships

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}. \tag{1.2.2}$$

Of course, they can be proved easily by inserting into eq. (1.2.1). Let $\sigma_x^0, \tau_{xy}^0, \sigma_y^0$ denote the particular solutions of the static equations (1.1.6) due to the volume forces, i.e. the solutions which are independent of z. Then we have

$$\left.\begin{array}{ll} \sigma_x = \sigma_x^0 + \dfrac{\partial^2 U}{\partial y^2}, & \tau_{xy} = \tau_{xy}^0 - \dfrac{\partial^2 U}{\partial x \partial y}, \\ \sigma_y = \sigma_y^0 + \dfrac{\partial^2 U}{\partial x^2}, & \sigma_z = \nu\left(\sigma_x^0 + \sigma_y^0\right) + \nu \nabla^2 U + Ee. \end{array}\right\} \tag{1.2.3}$$

Here and in what follows ∇^2 denote the Laplacian with respect to two variables x, y.

7.1.3 Differential equation for the stress function

In the following we assume that the particular solution $\sigma_x^0, \tau_{xy}^0, \sigma_y^0$ is not only statically possible in the volume but it also satisfies Beltrami's dependences in which σ_z^0 is determined by relationship (1.1.15) and $\tau_{xz}^0 = \tau_{zx}^0 = 0$. Omitting here and in what follows the constant e we obtain the following expression for the sum of the normal stresses

$$\sigma^0 = \sigma_x^0 + \sigma_y^0 + \sigma_z^0 = (1 + \nu)\left(\sigma_x^0 + \sigma_y^0\right). \tag{1.3.1}$$

It enables us to set Beltrami's dependences, eq. (1.5.7) of Chapter 4, for the particular solution in the form

$$\left.\begin{aligned}
&\nabla^2\sigma_x^0 + \frac{\partial^2}{\partial x^2}\left(\sigma_x^0 + \sigma_y^0\right) + 2\rho\frac{\partial K_x}{\partial x} + \rho\frac{\nu}{1-\nu}\operatorname{div}\mathbf{K} = 0,\\
&\nabla^2\sigma_y^0 + \frac{\partial^2}{\partial y^2}\left(\sigma_x^0 + \sigma_y^0\right) + 2\rho\frac{\partial K_y}{\partial y} + \rho\frac{\nu}{1-\nu}\operatorname{div}\mathbf{K} = 0,\\
&\nabla^2\tau_{xy}^0 + \frac{\partial^2}{\partial x\partial y}\left(\sigma_x^0 + \sigma_y^0\right) + \rho\left(\frac{\partial K_x}{\partial y} + \frac{\partial K_y}{\partial x}\right) = 0,\\
&\nabla^2\left(\sigma_x^0 + \sigma_y^0\right) + \frac{\rho}{1-\nu}\operatorname{div}\mathbf{K} = 0\\
&\qquad\qquad\qquad\qquad\left(\operatorname{div}\mathbf{K} = \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y}\right),
\end{aligned}\right\} \tag{1.3.2}$$

where the last equation is the consequence of the two first equations.

It remains to require that Airy's function satisfies homogeneous Beltrami's dependences in which

$$\sigma = (1+\nu)\nabla^2 U. \tag{1.3.3}$$

This yields the conditions

$$\begin{aligned}
&\nabla^2\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2}{\partial x^2}\nabla^2 U = 0, &\quad &\nabla^2\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2}{\partial y^2}\nabla^2 U = 0,\\
&-\nabla^2\frac{\partial^2 U}{\partial x\partial y} + \frac{\partial^2}{\partial x\partial y}\nabla^2 U = 0, &\quad &\nu\nabla^2\nabla^2 U = 0,
\end{aligned}$$

leading to the single equation

$$\nabla^2\nabla^2 U = 0. \tag{1.3.4}$$

Thus, Airy's stress function satisfies this differential equation of fourth order referred to the biharmonic equation. It is homogeneous provided that the particular solution satisfies the static equations in the volume and Beltrami's dependences.

7.1.4 Plane stress

In this case the volume and surface forces are perpendicular to axis z, i.e. $K_z = 0, F_z = 0$. The particular solution that corresponds to the volume forces and is feasible in the elastic body is assumed to be given. For this reason, the volume forces are not considered in what follows.

We consider the state of stress which ensures that the stresses vanish in the planes perpendicular to axis z

$$\tau_{xz} = 0, \quad \tau_{yz} = 0, \quad \sigma_z = 0. \tag{1.4.1}$$

This state of stress is referred to as plane stress. Clearly, the homogeneous static equations in the volume are written down in the form (1.2.1) and can be satisfied by introducing Airy's functions

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}. \tag{1.4.2}$$

It does not mean that this function and in turn the stresses are independent of z. Indeed, Beltrami's dependences should be written down in the form

$$\left.\begin{array}{l}
\nabla^2 \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^4 U}{\partial z^2 \partial y^2} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial x^2}\nabla^2 U = 0, \\
\nabla^2 \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^4 U}{\partial z^2 \partial x^2} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial y^2}\nabla^2 U = 0, \\
-\nabla^2 \dfrac{\partial^2 U}{\partial x \partial y} - \dfrac{\partial^4 U}{\partial z^2 \partial x \partial y} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial x \partial y}\nabla^2 U = 0, \\
\qquad \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial y \partial z}\nabla^2 U = 0, \\
\qquad \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial z \partial x}\nabla^2 U = 0, \\
\qquad \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial z^2}\nabla^2 U = 0.
\end{array}\right\} \tag{1.4.3}$$

Adding the first and second equations and accounting for the sixth equation it is easy to see that the stress function is biharmonic with respect to variables x, y

$$\nabla^2 \nabla^2 U = 0 \quad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{1.4.4}$$

The last three equations (1.4.3) yield

$$\frac{\partial}{\partial z}\nabla^2 U = c, \quad \nabla^2 U = cz + a(x, y), \tag{1.4.5}$$

where c is a constant and $a(x, y)$ is a biharmonic function. Let us introduce into consideration a biharmonic function $U_1(x, y)$ whose Laplacian is equal to $a(x, y)$

$$\nabla^2 U_1(x, y) = a(x, y).$$

By virtue of eq. (1.4.5) we obtain

$$U(x, y, z) = \frac{1}{4} cz\left(x^2 + y^2\right) + U_1(x, y) + b(x, y, z), \tag{1.4.6}$$

where b is a biharmonic function with respect to x, y. Inserting into the first three equations (1.4.3) yields

$$\frac{\partial^2 a}{\partial y^2} + \frac{1}{1+\nu}\frac{\partial^2 a}{\partial x^2} + \frac{\partial^4 b}{\partial z^2 \partial y^2} = 0,$$
$$\frac{\partial^2 a}{\partial x^2} + \frac{1}{1+\nu}\frac{\partial^2 a}{\partial y^2} + \frac{\partial^4 b}{\partial z^2 \partial x^2} = 0,$$
$$-\frac{\partial^2 a}{\partial x \partial y}\frac{\nu}{1+\nu} - \frac{\partial^4 b}{\partial z^2 \partial x \partial y} = 0.$$

Taking into account that $\partial^2 a/\partial x^2 = -\partial^2 a/\partial y^2$ we obtain from these equations that

$$\frac{\partial^2 b}{\partial z^2} = -\frac{\nu}{1+\nu} a(x, y) + m''(z)\, x + n''(z)\, y + p''(z),$$

where m'', n'', p'' are arbitrary functions of z. This equation yields

$$b(x, y, z) = -\frac{\nu z^2}{2(1+\nu)} a(x, y) + z U_2(x, y) + U_3(x, y) + \\ m(z)\, x + n(z)\, y + p(z),$$

where U_2, U_3 are harmonic functions. Substitution into eq. (1.4.6) leads to the expression

$$U(x, y, z) = \frac{1}{4} cz\left(x^2 + y^2\right) + U_1(x, y) - \frac{\nu z^2}{2(1+\nu)} \nabla^2 U_1(x, y) + \\ z U_2(x, y) + U_3(x, y) + m(z)\, x + n(z)\, y + p(z).$$

Here $U_3(x, y)$ can be included into the harmonic function $U_1(x, y)$ and the terms

$$m(z)\, x + n(z)\, y + p(z)$$

can be omitted and this does not affect the state of stress. Finally we arrive at the following expression for the stress function

$$U(x, y, z) = U_1(x, y) - \frac{\nu z^2}{2(1+\nu)} \nabla^2 U_1(x, y) + z U_2(x, y) + \frac{1}{4} cz\left(x^2 + y^2\right), \tag{1.4.7}$$

where U_1 and U_2 are respectively biharmonic and harmonic functions

$$\nabla^2 \nabla^2 U_1(x, y) = 0, \quad \nabla^2 U_2(x, y) = 0. \tag{1.4.8}$$

Thus we have determined a general class of the states of stress satisfying conditions (1.4.1), the static equations (1.2.1) and Beltrami's dependences (1.4.3).

The stresses are proved to be quadratic functions of z. Therefore, the plane stress can be realised in a solid only under the condition that the forces on the lateral surface obey this law.

7.1.5 The generalised plane stress

Let $2h$ denote the dimension of the body along axis z. Instead of U_1 we introduce into consideration another biharmonic function $\Phi(x, y)$

$$U_1 = \Phi + \frac{\nu}{2(1+\nu)} \frac{h^2}{3} \nabla^2 \Phi, \quad \nabla^2 U_1 = \nabla^2 \Phi. \tag{1.5.1}$$

Assuming additionally that the state of stress in the body is symmetric about the body's mid-plane $z = 0$ so that $U_2 = 0, c = 0$ we can rewrite relationship (1.4.7) in the form

$$U = \Phi + \frac{\nu}{2(1+\nu)} \left(\frac{h^2}{3} - z^2 \right) \nabla^2 \Phi. \tag{1.5.2}$$

The stresses determined by formulae (1.4.2) are as follows

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 \Phi}{\partial^2 y} + \frac{\nu}{2(1+\nu)} \left(\frac{h^2}{3} - z^2 \right) \frac{\partial^2 \nabla^2 \Phi}{\partial y^2}, \\ \sigma_y &= \frac{\partial^2 \Phi}{\partial^2 x} + \frac{\nu}{2(1+\nu)} \left(\frac{h^2}{3} - z^2 \right) \frac{\partial^2 \nabla^2 \Phi}{\partial x^2}, \\ \tau_{xy} &= -\frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\nu}{2(1+\nu)} \left(\frac{h^2}{3} - z^2 \right) \frac{\partial^2 \nabla^2 \Phi}{\partial x \partial y}. \end{aligned}\right\} \tag{1.5.3}$$

It is assumed in what follows that the body is a plate of thickness $2h$ which is small compared with the other dimensions. It allows one to take that, with a sufficient accuracy, the state of stress can be described by the stresses averaged over the thickness of the plate

$$\bar{\sigma}_x = \frac{1}{2h} \int_{-h}^{h} \sigma_x dz, \quad \bar{\tau}_{xy} = \frac{1}{2h} \int_{-h}^{h} \tau_{xy} dz, \quad \bar{\sigma}_y = \frac{1}{2h} \int_{-h}^{h} \sigma_y dz. \tag{1.5.4}$$

Noticing that

$$\int_{-h}^{h} \left(\frac{h^2}{3} - z^2 \right) dx = 0,$$

we arrive at the following expressions for the mean values

$$\bar{\sigma}_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad \bar{\tau}_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}, \quad \bar{\sigma}_y = \frac{\partial^2 \Phi}{\partial x^2}. \tag{1.5.5}$$

These formulae determine the so-called generalised plane stress. It is clear that the surface forces should also be averaged over the thickness of the plate

$$\bar{F}_x(s) = \frac{1}{2h} \int_{-h}^{h} F_x(z, s)\, dz, \quad \bar{F}_y(s) = \frac{1}{2h} \int_{-h}^{h} F_y(z, s)\, dz. \tag{1.5.6}$$

When the stresses $\sigma_x, \tau_{xy}, \sigma_y$ are absent, the generalised Hooke law is set in the form

$$\left.\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x} = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right), \quad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right), \\ &\varepsilon_z = \frac{\partial w}{\partial z} = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right), \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu}\tau_{xy}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \\ &\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \end{aligned}\right\} \tag{1.5.7}$$

and the equations

$$\frac{\partial \bar{u}}{\partial x} = \frac{1}{E}\left(\bar{\sigma}_x - \nu\bar{\sigma}_y\right), \quad \frac{\partial \bar{v}}{\partial y} = \frac{1}{E}\left(\bar{\sigma}_y - \nu\bar{\sigma}_x\right), \quad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = \frac{1}{\mu}\bar{\tau}_{xy} \tag{1.5.8}$$

serve for determining displacement u, v averaged over the thickness. Using the third equation in (1.5.7) and eq. (1.5.4) we obtain the difference in the value of $w(x, y, \pm h)$ on the upper and lower sides of the plate, that is the change in the plate's thickness

$$w(x, y, h) - w(x, y, -h) = -\frac{2\nu h}{1-\nu}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = -\frac{2\nu h}{1-\nu}\bar{\vartheta}. \tag{1.5.9}$$

The remaining equations (1.5.7) determine the differences in the values of u, v on the sides of the plate in terms of the derivatives of the mean value of $\bar{w}(x, y)$ with respect to x and y, the value of $\bar{w}(x, y)$ remaining undetermined.

7.1.6 The plane problem

In the following while denoting the quantities of the generalised plane stress averaged over the thickness we omit the bars and designate the stress function as U instead of Φ. The formulae of Subsection 7.1.5 are written down in the form

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \tag{1.6.1}$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} = \frac{1}{2\mu}\left(\sigma_x - \frac{\nu}{1+\nu}\sigma\right), \quad \frac{\partial v}{\partial y} = \frac{1}{2\mu}\left(\sigma_y - \frac{\nu}{1+\nu}\sigma\right), \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\mu}\tau_{xy}. \end{aligned}\right\} \tag{1.6.2}$$

In the case of the plane strain, formulae (1.6.1) remain valid. However, by virtue of eqs. (1.1.3) and (1.1.4), the generalised Hooke law for $e = 0$ is set in the form

$$\frac{\partial u}{\partial x}=\frac{1}{2\mu}\left(\sigma_x-\nu\sigma\right), \quad \frac{\partial v}{\partial y}=\frac{1}{2\mu}\left(\sigma_y-\nu\sigma\right), \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{1}{\mu}\tau_{xy}, \quad (1.6.3)$$

where here, as well as in eq. (1.6.2)

$$\sigma_x+\sigma_y=\sigma. \quad (1.6.4)$$

Comparison suggests that having the solution of the plane strain problem at the disposal one can obtain the solution of the corresponding plane stress problem (i.e. for the same volume forces and the same boundary conditions) by replacing

$$\nu \text{ by } \frac{\nu}{1+\nu}. \quad (1.6.5)$$

In both cases one deals with the same biharmonic boundary-value problem referred to as the plane problem of the theory of elasticity and sometimes the plane elasticity.

Evidently, the essence of these problems is different. In the first case one deals with the state of a sufficiently long prismatic body whereas in the second case the main issue is the averaged state of stress in a thin plate. In the first case, strain ε_z is absent however stress σ_z is present, whereas in the second case $\sigma_z=0$ however the thickness of the plate changes ($\varepsilon_z\neq 0$).

The forthcoming analysis of plane regions is concerned with the case of plane strain unless otherwise stated. The word "generalised" is omitted while considering the generalised plane stress.

In what follows the insignificant dimension along axis z is taken to be equal to the unit length.

Let us notice that rule (1.6.5) becomes more complicated in the case of thermal stresses as Hooke's law contains new terms depending on ν, see Subsection 7.5.8.

7.1.7 Displacements in the plane problem

Determining displacements u, v reduces to integration of the system of equations (1.6.3) in which the stresses are replaced by their expressions (1.6.1) in terms of biharmonic stress function $U(x,y)$. This system of three equations for two unknown functions is integrable since Beltrami's dependences (which are equivalent to the continuity conditions) are satisfied.

Let us denote the harmonic function $\nabla^2 U$ and its complex conjugate by s and t, respectively

$$\nabla^2 U=s;\quad \frac{\partial s}{\partial x}=\frac{\partial t}{\partial y},\quad \frac{\partial s}{\partial y}=-\frac{\partial t}{\partial x};\quad s+it=f(z),\quad z=x+iy \quad (1.7.1)$$

and consider the function of the complex variable

$$S(x,y) + iT(x,y) = F(z) = \int^{z} f(z)\,dz \tag{1.7.2}$$

introduced as the indefinite integral of $f(z)$. Following the notion of N.I. Muskhelishvili $F(z)$ will be replaced below by $4\varphi(z)$.

Let us notice that t is determined by s up to a real-valued additive constant denoted by ω_0. Thus $F(z)$ is determined up to an additive linear function

$$i\omega_0 z + u_0 + iv_0 = (-\omega_0 y + u_0) + i(\omega_0 x + v_0). \tag{1.7.3}$$

Using eq. (1.7.2) we obtain

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial z} = f(z) = s + it = \frac{\partial S}{\partial x} + i\frac{\partial T}{\partial x},\\
\frac{\partial F}{\partial y} &= i\frac{\partial F}{\partial z} = if(z) = is - t = \frac{\partial S}{\partial y} + i\frac{\partial T}{\partial y}
\end{aligned}$$

and thus

$$\frac{\partial S}{\partial x} = \frac{\partial T}{\partial y} = s, \qquad \frac{\partial S}{\partial y} = -\frac{\partial T}{\partial x} = -t. \tag{1.7.4}$$

This allows us to replace eq. (1.6.3) by the following equations

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2\mu}\left(\frac{\partial^2 U}{\partial y^2} - \nu\nabla^2 U\right) = \frac{1}{2\mu}\left[-\frac{\partial^2 U}{\partial x^2} + (1-\nu)\nabla^2 U\right]\\
&= \frac{1}{2}\frac{\partial}{\partial x}\left[-\frac{\partial U}{\partial x} + (1-\nu)S\right],\\
\frac{\partial v}{\partial y} &= \frac{1}{2\mu}\left(\frac{\partial^2 U}{\partial x^2} - \nu\nabla^2 U\right) = \frac{1}{2\mu}\frac{\partial}{\partial y}\left[-\frac{\partial U}{\partial y} + (1-\nu)T\right],\\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= -\frac{1}{\mu}\frac{\partial^2 U}{\partial x \partial y}.
\end{aligned}\right\} \tag{1.7.5}$$

Hence we have

$$u = \frac{1}{2\mu}\left[(1-\nu)S - \frac{\partial U}{\partial x}\right] + f_1(y), \; v = \frac{1}{2\mu}\left[(1-\nu)T - \frac{\partial U}{\partial y}\right] + f_2(x),$$

and insertion into the third equation in (1.7.5) and taking into account eq. (1.7.4) yields

$$f_1'(y) + f_2'(x) = 0, \quad f_1'(y) = -\omega_0, \quad f_2'(x) = \omega_0 = \text{const},$$

since the sum of the functions of x and y can vanish only if they are constant-valued and of opposite sign. Thus we arrive at the sought expressions for the displacements

$$\left.\begin{aligned} u &= \frac{1}{2\mu}\left[(1-\nu) S - \frac{\partial U}{\partial x}\right] + u_0 - \omega_0 y, \\ v &= \frac{1}{2\mu}\left[(1-\nu) T - \frac{\partial U}{\partial y}\right] + v_0 + \omega_0 x. \end{aligned}\right\} \tag{1.7.6}$$

Here the terms of the type, described by eq. (1.7.3), represent the displacements of a rigid plane body in its plane, (u_0, v_0) denote the projections of a point of this body and ω_0 is the small angle of turn about axis x_3. In the plane stress ν is replaced according to the rule (1.6.5), that is

$$\left.\begin{aligned} u &= \frac{1}{E}\left[S - (1+\nu)\frac{\partial U}{\partial x}\right] + u_0 - \omega_0 y, \\ v &= \frac{1}{2\mu}\left[T - (1+\nu)\frac{\partial U}{\partial y}\right] + v_0 + \omega_0 x. \end{aligned}\right\} \tag{1.7.7}$$

7.1.8 The principal vector and the principal moment

Let us consider an arc l in the plane field of the stress tensor

$$\hat{T} = \mathbf{i}_1\mathbf{i}_1\sigma_x + \mathbf{i}_2\mathbf{i}_2\sigma_y + (\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)\,\tau_{xy}\,. \tag{1.8.1}$$

The unit vectors of the normal $\mathbf{n}$ and the tangent $\mathbf{t}$ (in the direction of increasing s) to arc l are taken as being axes x, y of the Cartesian coordinate system. Then the product

$$\mathbf{n}\cdot\hat{T} = \mathbf{F} \tag{1.8.2}$$

describes the distribution of the surface forces acting from the side of the medium "over" the curve (i.e. in the part of the plane where vector $\mathbf{n}$ is directed to) on the medium "under" the curve. Using eqs. (1.8.1) and (1.8.2) we have

$$\left.\begin{aligned} F_x &= \mathbf{n}\cdot\hat{T}\cdot\mathbf{i}_1 = \sigma_x n_x + \tau_{xy} n_y \\ &\qquad = \frac{\partial^2 U}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 U}{\partial x \partial y}\frac{dx}{ds} = \frac{d}{ds}\frac{\partial U}{\partial y}, \\ F_y &= \mathbf{n}\cdot\hat{T}\cdot\mathbf{i}_2 = \tau_{xy} n_x + \sigma_y n_y \\ &\qquad = -\left(\frac{\partial^2 U}{\partial x \partial y}\frac{dy}{ds} + \frac{\partial^2 U}{\partial x^2}\frac{dx}{ds}\right) = -\frac{d}{ds}\frac{\partial U}{\partial x}. \end{aligned}\right\} \tag{1.8.3}$$

From these equations we obtain the expressions for the projections of the principal vector of these forces distributed along arc l from the initial point

$s = 0$ to the considered point $M(s)$

$$\int_0^s F_x ds = P = \frac{\partial U}{\partial y}, \quad \int_0^s F_y ds = Q = -\frac{\partial U}{\partial x}. \tag{1.8.4}$$

The principal vector of these forces about the origin of the coordinate system (about axis x_3) is equal to

$$m^O = \mathbf{i}_3 \cdot \int_0^s (\mathbf{i}_1 x + \mathbf{i}_2 y) \times \mathbf{n} \cdot \hat{T} ds = -\int_0^s \left(x d\frac{\partial U}{\partial x} + y d\frac{\partial U}{\partial y} \right)$$
$$= -\int_0^s d\left(x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} - U \right),$$

or

$$m^O = U + xQ - yP. \tag{1.8.5}$$

The insignificant additive integration constants are omitted in these formulae. Should one retain these terms, there appears a linear function of the coordinate which does not affect the stresses.

Formula (1.8.5) can be written down in the form

$$U = m^O - xQ + yP = m^M, \tag{1.8.6}$$

where m^M denotes the principal moment about point $M(s)$ of force $\mathbf{F}$ along arc l (from the initial point M_0 to actual point M).

7.1.9 Orthogonal curvilinear coordinates

Denoting the unit vectors of the tangents to the coordinate lines $[q^1], [q^2]$ by $\mathbf{e}_1, \mathbf{e}_2$ we can set eq. (1.8.1) in the form

$$\hat{T} = \sigma_1 \mathbf{e}_1\mathbf{e}_1 + \sigma_2 \mathbf{e}_2\mathbf{e}_2 + (\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)\tau_{12}. \tag{1.9.1}$$

Referring to eq. (1.6.10) of Chapter 1 and eqs. (C.3.9), (B.4.14)

$$\hat{T} = \operatorname{inc} \mathbf{i}_3\mathbf{i}_3 U = \nabla \times [\nabla \times \mathbf{i}_3\mathbf{i}_3 U]^* \qquad \left(\nabla = \frac{\mathbf{e}_1}{H_1}\frac{\partial}{\partial q^1} + \frac{\mathbf{e}_2}{H_2}\frac{\partial}{\partial q^2} \right),$$

and carrying out calculations using formulae (C.4.8) we obtain

$$\left.\begin{aligned}
\sigma_1 &= \frac{1}{H_2}\frac{\partial}{\partial q^2}\frac{\partial U}{H_2 \partial q^2} + \frac{\partial \ln H_2}{H_1 \partial q^1}\frac{\partial U}{H_1 \partial q^1}, \\
\sigma_2 &= \frac{1}{H_1}\frac{\partial}{\partial q^1}\frac{\partial U}{H_1 \partial q^1} + \frac{\partial \ln H_1}{H_2 \partial q^2}\frac{\partial U}{H_2 \partial q^2}, \\
\tau_{12} &= -\frac{1}{H_1 H_2}\frac{\partial^2 U}{\partial q^1 \partial q^2} + \frac{\partial \ln H_1}{H_2 \partial q^2}\frac{\partial U}{H_1 \partial q^1} + \frac{\partial \ln H_2}{H_1 \partial q^1}\frac{\partial U}{H_2 \partial q^2}.
\end{aligned}\right\} \tag{1.9.2}$$

The consequence of these relationships is the expression for the sum of the normal stresses determining the Laplacian over U

$$\sigma_1+\sigma_2=\frac{1}{H_1H_2}\left(\frac{\partial}{\partial q^1}\frac{H_2}{H_1}\frac{\partial U}{\partial q^1}+\frac{\partial}{\partial q^2}\frac{H_1}{H_2}\frac{\partial U}{\partial q^2}\right).$$

The equilibrium equations are written down with the help of eq. (1.9.2) of Chapter 4. It can be proved that they are identically satisfied by stresses (1.9.2) when the volume forces are absent. The formulae expressing the relations between the components of stress tensor and the displacement vector have the form of eq. (C.5.9)

$$\left.\begin{array}{l}\varepsilon_{11}=\dfrac{\partial u_1}{H_1\partial q^1}+\dfrac{u_2}{H_2}\dfrac{\partial\ln H_1}{\partial q^2},\quad \varepsilon_{22}=\dfrac{\partial u_2}{H_2\partial q^2}+\dfrac{u_1}{H_1}\dfrac{\partial\ln H_2}{\partial q^1},\\ \gamma_{12}=\dfrac{\partial u_1}{H_2\partial q^2}+\dfrac{\partial u_2}{H_1\partial q^1}-\dfrac{u_1}{H_2}\dfrac{\partial\ln H_1}{\partial q^2}-\dfrac{u_2}{H_1}\dfrac{\partial\ln H_2}{\partial q^1}.\end{array}\right\}\qquad(1.9.4)$$

7.1.10 *Polar coordinates in the plane*

In these coordinates denoted by r,θ the equilibrium equations take the form, see eq. (1.9.4) of Chapter 4

$$\left.\begin{array}{l}\dfrac{\partial\sigma_r}{\partial r}+\dfrac{\sigma_r-\sigma_\theta}{r}+\dfrac{\partial\tau_{r\theta}}{r\partial\theta}+\rho K_r=0,\\ \dfrac{\partial\sigma_\theta}{r\partial\theta}+\dfrac{\partial\tau_{r\theta}}{\partial r}+\dfrac{2\tau_{r\theta}}{r}+\rho K_\theta=0.\end{array}\right\}\qquad(1.10.1)$$

One can obtain these equations directly by considering the forces acting on the element of the medium bounded by the radii $\theta,\theta+d\theta$ and the circles $r,r+dr$. The expressions for the stresses in terms of the stress function which identically satisfy the homogeneous equilibrium equations are presented in the form

$$\sigma_r=\frac{\partial^2U}{r^2\partial\theta^2}+\frac{1}{r}\frac{\partial U}{\partial r},\quad \sigma_\theta=\frac{\partial^2U}{\partial r^2},\quad \tau_{r\theta}=-\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial U}{\partial\theta}.\qquad(1.10.2)$$

The components of the strain tensor are as follows

$$\varepsilon_r=\frac{\partial u_r}{\partial r},\quad \varepsilon_\theta=\frac{\partial u_\theta}{r\partial\theta}+\frac{u_r}{r},\quad \gamma_{r\theta}=\frac{\partial u_\theta}{\partial r}+\frac{\partial u_r}{r\partial\theta}-\frac{u_\theta}{r}.\qquad(1.10.3)$$

7.1.11 *Representing the biharmonic function*

Clearly, any harmonic function is also biharmonic. It is straightforward to verify that the functions

$$xf_1,\quad yf_2,\quad \left(x^2+y^2\right)f_3,$$

where $f_i(x,y)$ denotes a harmonic function, satisfy the biharmonic equation. To this end, it is sufficient to recall the expression for the Laplacian of the product

$$\nabla^2 \varphi\psi = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi.$$

Then, for example,

$$\begin{aligned}\nabla^2\left(x^2+y^2\right)f_3 &= \left(x^2+y^2\right)\nabla^2 f_3 + f_3\nabla^2\left(x^2+y^2\right) + 4\left(x\frac{\partial f_3}{\partial x} + y\frac{\partial f_3}{\partial y}\right) \\ &= 4\left[f_3 + \left(x\frac{\partial f_3}{\partial x} + y\frac{\partial f_3}{\partial y}\right)\right]\end{aligned}$$

and furthermore

$$\nabla^4\left(x^2+y^2\right)f_3 = 4\nabla^2\left(x\frac{\partial f_3}{\partial x} + y\frac{\partial f_3}{\partial y}\right) = 8\nabla^2 f_3 = 0,$$

which is required.

It follows from the above-said that the functions

$$xf_1 + f_4,\ yf_2 + f_5,\ \left(x^2+y^2\right)f_3 + f_6;\ \nabla^2 f_i = 0 \quad (i = 1,\dots,6), \tag{1.11.1}$$

are biharmonic. The inverse statement, namely that any biharmonic function can be presented in the above form, is proved below, see Remark in Subsection 7.1.14.

For instance, let us show that biharmonic function $xf_1 + yf_2$ can be presented in the first of the above types. To this aim it is sufficient to introduce the harmonic function f_3 related to f_2 by the Cauchy-Riemann conditions

$$\frac{\partial f_3}{\partial x} = \frac{\partial f_2}{\partial y}, \quad \frac{\partial f_3}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

Denoting $f_4 = yf_2 - xf_3$ and $f_1 + f_3 = f_1^*$ where f_4 and f_1^* are harmonic functions we have

$$xf_1 + yf_2 = x\left(f_1 + f_3\right) + \left(yf_2 - xf_3\right) = xf_1^* + f_4,$$

which is required.

While solving the plane problems in the Cartesian coordinates one often takes the harmonic functions in the form of homogeneous harmonic polynomials equal to the real (Re) or imaginary (Im) parts of powers of $z = x + iy$

$$\operatorname{Re} z^n, \quad \operatorname{Im} z^n. \tag{1.11.2}$$

For this reason the biharmonic polynomials of power $(n+1)$ are represented in the form

$$C_1 x \operatorname{Re} z^n + C_2 \operatorname{Re} z^{n+1}, \quad C_3 y \operatorname{Im} z^n + C_4 \operatorname{Im} z^{n+1},$$
$$C_5 \left(x^2+y^2\right) \operatorname{Re} z^{n-1} + C_6 \operatorname{Re} z^{n+1}$$

etc. Evidently, any polynomial of order lower than four is harmonic.

When polar coordinates are used one can replace the complex variable z by the expression in terms of the modulus and the argument and arrive at the following representation of the biharmonic functions

$$C_1 r^{n+2} \begin{matrix}\cos\\ \sin\end{matrix} n\theta + C_2 r^n \begin{matrix}\cos\\ \sin\end{matrix} n\theta, \quad \frac{C_3}{r^{n-2}} \begin{matrix}\cos\\ \sin\end{matrix} n\theta + \frac{C_4}{r^n} \begin{matrix}\cos\\ \sin\end{matrix} n\theta.$$

It stands to reason that the above-said does not exhaust the diversity of solutions to the biharmonic equation.

7.1.12 *Introducing a complex variable*

Instead of Cartesian coordinates we take the following independent coordinates

$$z = x + iy, \quad \bar{z} = x - iy \quad \left[x = \frac{1}{2}(z+\bar{z}),\ y = \frac{1}{2i}(z-\bar{z})\right]. \tag{1.12.1}$$

Here and throughout the book a bar over the letter implies the complex conjugate of the complex value. For example, notion $\bar{\varphi}(\bar{z})$ means that not only z is replaced by its complex conjugate $\bar{z}$ but also all of the complex values in $\varphi(z)$, for instance the coefficients of Taylor series for this function, are replaced by their complex conjugates. According to this rule, $\overline{\varphi(\bar{z})}$ denotes $\bar{\varphi}(z)$ and so on.

Considering $U(x,y)$ as a function of $z, \bar{z}$ we arrive at the rule of differentiation

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)U, \quad \frac{\partial U}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1.12.2}$$

and vice versa

$$\frac{\partial U}{\partial x} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)U, \quad \frac{\partial U}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)U. \tag{1.12.3}$$

The consequence of these formulae are the expressions for the second derivatives, Laplacian and the biharmonic operator

$$\left.\begin{aligned}
\frac{\partial^2 U}{\partial x^2} &= \left(\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right) U, \\
\frac{\partial^2 U}{\partial y^2} &= -\left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right) U, \\
\frac{\partial^2 U}{\partial x \partial y} &= i\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2}\right) U, \quad \nabla^2 U = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}, \\
\nabla^4 U &= 16\frac{\partial^4 U}{\partial z^2 \partial \bar{z}^2}.
\end{aligned}\right\} \tag{1.12.4}$$

7.1.13 Transforming the formulae of the plane problem

The result of formulae (1.12.4) is the following expressions for the stress vectors in the planes perpendicular to the axes

$$\sigma_x + i\tau_{xy} = 2\left(\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right) U, \quad \tau_{xy} + i\sigma_y = 2i\left(\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right) U. \tag{1.13.1}$$

Here and in what follows the vectors in the plane are given by the complex-valued quantities. The consequence of the latter formulae yielding the compact expressions for the stresses is the formulae of Kolosov

$$\sigma_x + \sigma_y = \nabla^2 U = 4\frac{\partial^2 U}{\partial z \partial \bar{z}}, \quad \sigma_y - \sigma_x + 2i\tau_{xy} = 4\frac{\partial^2 U}{\partial z^2}. \tag{1.13.2}$$

The unit vector of the normal to arc l can be presented by the complex number

$$n = n_x + in_y = \frac{dy}{ds} - i\frac{dx}{ds} = -i\frac{dz}{ds}, \tag{1.13.3}$$

and the stress vector in the plane with normal n is represented as follows

$$\begin{aligned}
F_x + iF_y = \mathbf{n} \cdot \hat{T} &= (\sigma_x n_x + \tau_{xy} n_y) + i(\tau_{xy} n_x + \sigma_y n_y) \\
&= \frac{1}{2}\left[(\sigma_x + i\tau_{xy})(n + \bar{n}) + (\sigma_y - i\tau_{xy})(n - \bar{n})\right] \\
&= \frac{1}{2}\left[(\sigma_x + \sigma_y) n + (\sigma_x - \sigma_y + 2i\tau_{xy}) \bar{n}\right].
\end{aligned}$$

Referring to eq. (1.13.2) one can set this result in the form

$$\begin{aligned}
\mathbf{n} \cdot \hat{T} &= 2\left(\frac{\partial^2 U}{\partial z \partial \bar{z}} n - \frac{\partial^2 U}{\partial \bar{z}^2} \bar{n}\right) \\
&= -2i\left(\frac{\partial^2 U}{\partial z \partial \bar{z}} \frac{dz}{ds} + \frac{\partial^2 U}{\partial \bar{z}^2} \frac{d\bar{z}}{ds}\right) = -2i\frac{d}{ds}\frac{\partial U}{\partial \bar{z}}.
\end{aligned} \tag{1.13.4}$$

Therefore we arrive at the following representation of the principal vector of the surface forces on arc l corresponding to formulae (1.8.4)

$$P + iQ = \int_0^s \mathbf{n} \cdot \hat{T} ds = -2i \frac{\partial U}{\partial \bar{z}} = \frac{\partial U}{\partial y} - i \frac{\partial U}{\partial x}. \tag{1.13.5}$$

According to eqs. (1.8.5) and (1.13.5) the principal moment of these forces on arc l about the coordinate origin is written as

$$m^O = U - \left(z \frac{\partial U}{\partial z} + \bar{z} \frac{\partial U}{\partial \bar{z}} \right). \tag{1.13.6}$$

In polar coordinates, the stress vectors on the arc of the circle and along the radii are presented by the expressions

$$\mathbf{e}_r \cdot \hat{T} = \mathbf{e}_r \sigma_r + \mathbf{e}_\theta \tau_{r\theta}, \quad \mathbf{e}_\theta \cdot \hat{T} = \mathbf{e}_r \tau_{r\theta} + \mathbf{e}_\theta \sigma_\theta.$$

The vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are given by the complex numbers

$$e^{i\theta}, \ ie^{i\theta},$$

respectively. Returning therefore to eq. (1.13.4) we arrive at the following expressions for the stress vectors

$$\left(\sigma_r + i\tau_{r\theta}\right) e^{i\theta} = 2 \left(\frac{\partial^2 U}{\partial z \partial \bar{z}} e^{i\theta} - \frac{\partial^2 U}{\partial \bar{z}^2} e^{-i\theta} \right),$$
$$\left(\sigma_\theta - i\tau_{r\theta}\right) e^{i\theta} = 2 \left(\frac{\partial^2 U}{\partial z \partial \bar{z}} e^{i\theta} + \frac{\partial^2 U}{\partial \bar{z}^2} e^{-i\theta} \right)$$

or

$$\sigma_r + i\tau_{r\theta} = 2 \left(\frac{\partial^2 U}{\partial z \partial \bar{z}} - \frac{\partial^2 U}{\partial \bar{z}^2} e^{-2i\theta} \right),$$
$$\sigma_\theta - i\tau_{r\theta} = 2 \left(\frac{\partial^2 U}{\partial z \partial \bar{z}} + \frac{\partial^2 U}{\partial \bar{z}^2} e^{-2i\theta} \right). \tag{1.13.7}$$

This leads to the following representations for Kolosov's formulae

$$\sigma_r + \sigma_\theta = 4 \frac{\partial^2 U}{\partial z \partial \bar{z}}, \ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 4 \frac{\partial^2 U}{\partial z^2} e^{2i\theta} = \left(\sigma_y - \sigma_x + 2i\tau_{xy}\right) e^{2i\theta}. \tag{1.13.8}$$

Evidently, they can be obtained by using the expressions for the components of the stress tensor in axes rotated through angle θ about the original Cartesian system.

Let us close these formal transformations by the formula for the displacement vector. To this end, we refer to eqs. (1.12.2), (1.7.6) and introduce the notion

$$s + it = \nabla^2 U + it = 4\varphi'(z), \quad F(z) = S + iT = 4\varphi(z). \tag{1.13.9}$$

The result is

$$2\mu\left(u+iv\right)=4\left(1-\nu\right)\varphi\left(z\right)-2\frac{\partial U}{\partial \bar{z}}+2\mu\left(u_0+iv_0+i\omega_0 z\right) \qquad (1.13.10)$$

or in polar coordinates

$$2\mu\left(u_r+iu_\theta\right)=e^{-i\theta}\left[4\left(1-\nu\right)\varphi\left(z\right)-2\frac{\partial U}{\partial \bar{z}}\right]+2\mu e^{-i\theta}\left(u_0+iv_0\right)+2\mu i\omega_0 r. \qquad (1.13.11)$$

7.1.14 Goursat's formula

This formula provides one with the representation of a biharmonic function in terms of two functions of the complex variable. The basic relationships are eqs. (1.12.4) and (1.13.9)

$$\nabla^2 U=4\frac{\partial^2 U}{\partial z\partial \bar{z}},\quad \nabla^2 U+it=4\varphi'\left(z\right),\quad \nabla^2 U-it=4\bar{\varphi}'\left(\bar{z}\right).$$

From these equations we obtain

$$\frac{1}{2}\nabla^2 U=2\frac{\partial^2 U}{\partial z\partial \bar{z}}=\varphi'\left(z\right)+\bar{\varphi}'\left(\bar{z}\right), \qquad (1.14.1)$$

and the integration over z introduces an additive function of $\bar{z}$ denoted $\bar{\chi}'\left(\bar{z}\right)$

$$2\frac{\partial U}{\partial \bar{z}}=\varphi\left(z\right)+z\bar{\varphi}'\left(\bar{z}\right)+\bar{\chi}'\left(\bar{z}\right).$$

One further integration leads to the required representation

$$2U=\bar{z}\varphi\left(z\right)+z\bar{\varphi}\left(\bar{z}\right)+\bar{\chi}\left(\bar{z}\right)+\chi\left(z\right). \qquad (1.14.2)$$

Integrating over $\bar{z}$ results in an additive function of z, namely $\chi\left(z\right)$, since U is a real-valued function.

All of the formulae of the previous subsection are easily expressed in terms of functions $\varphi\left(z\right),\chi\left(z\right)$. We will use the short-hand notion

$$\varphi'\left(z\right)=\Phi\left(z\right),\quad \chi'\left(z\right)=\psi\left(z\right),\quad \psi'\left(z\right)=\Psi\left(z\right). \qquad (1.14.3)$$

We arrive then at the following relationships for the stresses

$$\left.\begin{aligned}\sigma_x+\sigma_y&=2\left[\varphi'\left(z\right)+\bar{\varphi}'\left(\bar{z}\right)\right]=2\left[\Phi\left(z\right)+\bar{\Phi}\left(z\right)\right]=4\operatorname{Re}\Phi\left(z\right),\\ \sigma_y-\sigma_x+2i\tau_{xy}&=2\left[\bar{z}\varphi''\left(z\right)+\psi'\left(z\right)\right]=2\left[\bar{z}\Phi'\left(z\right)+\Psi\left(z\right)\right]\end{aligned}\right\} \qquad (1.14.4)$$

and the displacement vector

$$2\mu\left(u+iv\right)=\left(3-4\nu\right)\varphi\left(z\right)-z\bar{\varphi}'\left(\bar{z}\right)-\bar{\psi}\left(\bar{z}\right)+2\mu\left(u_0+iv_0+i\omega_0 z\right). \tag{1.14.5}$$

Let us also notice the formulae

$$\left.\begin{aligned}\sigma_x+i\tau_{xy}&=\Phi\left(z\right)+\bar{\Phi}\left(\bar{z}\right)-z\bar{\Phi}'\left(\bar{z}\right)-\bar{\Psi}\left(\bar{z}\right),\\ \sigma_y+i\tau_{xy}&=\Phi\left(z\right)+\bar{\Phi}\left(\bar{z}\right)+\bar{z}\Phi'\left(\bar{z}\right)+\Psi\left(\bar{z}\right),\end{aligned}\right\} \tag{1.14.6}$$

$$\left.\begin{aligned}P+iQ&=-i\left[\varphi\left(z\right)+z\bar{\varphi}'\left(\bar{z}\right)+\bar{\psi}\left(\bar{z}\right)\right],\\ m^O&=\operatorname{Re}\left[\chi\left(z\right)-z\psi\left(z\right)-z\bar{z}'\varphi'\left(z\right)\right],\end{aligned}\right\} \tag{1.14.7}$$

as well as the relationship

$$2\mu\left(u+iv\right)=4\left(1-\nu\right)\varphi\left(z\right)-i\left(P+iQ\right)+2\mu\left(u_0+iv_0+i\omega_0 z\right). \tag{1.14.8}$$

The representations in polar coordinates take the form

$$\left.\begin{aligned}&\sigma_r+i\tau_{r\theta}=\Phi\left(z\right)+\bar{\Phi}\left(\bar{z}\right)-\bar{z}\bar{\Phi}'\left(\bar{z}\right)-\frac{\bar{z}}{z}\bar{\Psi}\left(\bar{z}\right),\\ &\sigma_r-i\tau_{r\theta}=\Phi\left(z\right)+\bar{\Phi}\left(\bar{z}\right)+\bar{z}\bar{\Phi}'\left(\bar{z}\right)+\frac{\bar{z}}{z}\bar{\Psi}\left(\bar{z}\right),\quad\left(e^{-2i\theta}=\frac{\bar{z}}{z}\right),\\ &\sigma_r+\sigma_\theta=4\operatorname{Re}\Phi\left(z\right),\quad \sigma_\theta-\sigma_r+2i\tau_{r\theta}=2\left[z\Phi'\left(z\right)+\frac{z}{\bar{z}}\Psi\left(z\right)\right].\end{aligned}\right\} \tag{1.14.9}$$

Function $\chi\left(z\right)$ appears only in the expression for moment m^O and is often redundant. Thus determining the stress function is often unnecessary. The state of stress and the displacement in the plane problem are completely determined by two functions of the complex variable $\varphi\left(z\right),\psi\left(z\right)$ and their derivatives. It was N.I. Muskhelishvili who systematically applied these functions to solving the boundary-value problems of the plane theory of elasticity and for this reason they are termed Muskhelishvili's functions.

The analytic character of these functions will be given in Section 7.5 whilst Sections 7.2-7.4 are devoted to solving the problems which do not require application of the theory of functions of complex variable.

The projection of the vector of rotation $\boldsymbol{\omega}$ on axis x_3 is called turn ε and is given by

$$\begin{aligned}\mathbf{i}_3\cdot\boldsymbol{\omega}=\varepsilon&=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}\left[\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)v-i\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)u\right]\\ &=-\frac{i}{2}\left[\frac{\partial}{\partial z}\left(v+iu\right)-\frac{\partial}{\partial\bar{z}}\left(v-iu\right)\right].\end{aligned}$$

By eq. (1.14.5)

$$\mu\varepsilon=-\left(1-\nu\right)i\left[\Phi\left(z\right)-\bar{\Phi}\left(\bar{z}\right)\right]+\mu\omega_0. \tag{1.14.10}$$

Remark. Setting in Goursat's formula

$$\varphi(z) = p + iq, \quad \operatorname{Re}\chi(z) = r,$$

we obtain

$$U = xp + yq + r,$$

and this general representation of the biharmonic function is transformed to one of the forms

$$U = 2xp + (yq - xp + r) = 2yq + (xp - yq + r),$$

where the quantities in parentheses are harmonic functions. Putting in Goursat's formula

$$\varphi(z) = z\varphi_1(z), \quad \operatorname{Re}\varphi_1(z) = p_1$$

we have

$$U = \frac{1}{2} z\bar{z}\left[\varphi_1(z) + \bar{\varphi}_1(\bar{z})\right] + r = \left(x^2 + y^2\right) p_1 + r.$$

This confirms that any biharmonic function can be presented in one of the forms (1.11.1).

7.1.15 Translation of the coordinate origin

Let $\Phi_1(z_1)$ and $\Psi_1(z_1)$ denote the new values of Muskhelishvili's function under the parallel translation of the coordinate system into to point $z = c$

$$z = z_1 + c, \quad z_1 = x_1 + iy_1. \tag{1.15.1}$$

The components of the stress tensor remain unchanged. Hence, due to eq. (1.14.4) we have

$$\Phi(z) + \bar{\Phi}(\bar{z}) = \Phi_1(z_1) + \bar{\Phi}_1(\bar{z}_1), \ \bar{z}\Phi'(z) + \Psi(z) = \bar{z}_1\Phi'(z_1) + \Psi_1(z_1). \tag{1.15.2}$$

Hence

$$\left[\Phi_1(z_1) - \Phi_1(z_1 + c)\right] + \left[\bar{\Phi}_1(\bar{z}_1) - \bar{\Phi}_1(\bar{z}_1 + \bar{c})\right] = 0,$$

and this relation can be satisfied by taking

$$\Phi_1(z_1) - \Phi_1(z_1 + c) = i\alpha,$$

where α denotes a real-valued constant. It can be set to zero, as adding a pure imaginary constant to $\Phi_1(z_1)$ does not affect the stresses and results only in the term corresponding to rigid body motion in the expression for vector $u = iv$. Thus,

$$\Phi_1(z_1) = \Phi_1(z_1 + c), \quad \varphi_1(z_1) = \varphi_1(z_1 + c). \tag{1.15.3}$$

Turning now to the second expression in eq. (1.15.2) we have

$$\Psi_1(z_1) = \Psi_1(z_1 + c) + (\bar{z}_1 + \bar{c})\,\Phi'(z_1 + c) - \bar{z}_1\Phi_1'(z_1),$$

hence, by virtue of eq. (1.1.5.3)

$$\Psi_1(z_1) = \Psi_1(z_1 + c) + \bar{c}\Phi'(z_1 + c), \quad \psi_1(z_1) = \psi(z_1 + c) + \bar{c}\Phi(z_1 + c). \tag{1.15.4}$$

7.2 Beam and bar with a circular axis

7.2.1 Statement of the plane problem for beam and bar

The generalised plane stress for a rectangular strip of length l and height $2b$ $(0 \le x \le l, -b \le y \le b)$ is considered and it is assumed that $2b \ll l$. According to Saint-Venant's principle the boundary conditions exactly hold only on the long sides of the rectangular region and the surface forces on the short sides $(x = 0, x = l)$ can be replaced by a statically equivalent distribution, i.e. axial and transverse forces P and Q respectively and a bending moment μ. The cross-section of the beam is a rectangle of thickness h and height $2b$ with $h \ll b$ which allows one to reduce the problem to the stresses and displacements averaged over the thickness of the beam. The adopted statement is also applicable to the problem of plane strain in a plate (theoretically) unbounded in the direction of axis x_3 provided that the loads on faces $y = \pm b, x = 0, x = l$ are independent of x_3. The size along axis x_3 will not appear in the following and therefore it can be taken as being of unit length. The formulae for the problem for a plate can be obtained from the corresponding formulae for the beam by replacing

$$\nu \quad \text{by} \quad \nu_1 = \frac{\nu}{1-\nu}. \tag{2.1.1}$$

As the value of the shear modulus retains under this replacement we have

$$2\mu = \frac{E}{1+\nu} = \frac{E_1}{1+\nu_1} = E_1(1-\nu), \quad E_1 = \frac{E}{1-\nu^2}. \tag{2.1.2}$$

In what follows, the longitudinal sides $y = b$ and $y = -b$ of the beam are referred to as the upper and lower sides, respectively, whilst the cross-sections $x = 0$ and $x = l$ are respectively called the left and right end faces.

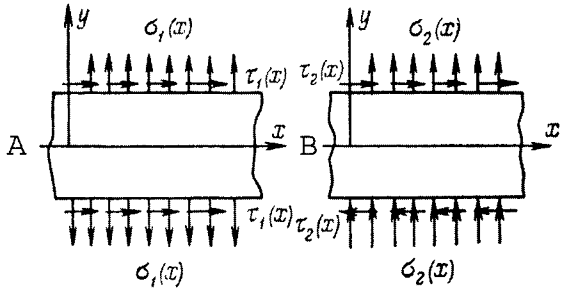

FIGURE 7.1.

The conditions for the sides of the beam are written as follows

$$\left.\begin{array}{lll} y=b: & \sigma_y = Y^+(x), & \tau_{xy} = X^+(x), \\ y=-b: & \sigma_y = -Y^-(x), & \tau_{xy} = -X^-(x), \end{array}\right\} \tag{2.1.3}$$

where functions $X^{\pm}(x), Y^{\pm}(x)$ are given for $0 < x < l$. Following the conventional practice of structural mechanics we split the load into symmetric and skew-symmetric parts and introduce the functions

$$\left.\begin{array}{ll} \tau_1(x) = \frac{1}{2}\left[X^+(x) + X^-(x)\right], & \tau_2(x) = \frac{1}{2}\left[X^+(x) - X^-(x)\right], \\ \sigma_1(x) = \frac{1}{2}\left[Y^+(x) - Y^-(x)\right], & \sigma_2(x) = \frac{1}{2}\left[Y^+(x) + Y^-(x)\right]. \end{array}\right\} \tag{2.1.4}$$

This allows us to split the problem into two problems, denoted in what follows as problems A and B

$$\left.\begin{array}{llll} \mathrm{A} & y=\pm b: & \sigma_y = \sigma_1(x), & \tau_{xy} = \pm\tau_1(x), \\ \mathrm{B} & y=\pm b: & \sigma_y = \pm\sigma_2(x), & \tau_{xy} = \tau_2(x). \end{array}\right\} \tag{2.1.5}$$

It is easy to verify that the superposition of these solutions leads to the problem with the prescribed boundary conditions.

In problem A, the normal surface forces on the upper and lower sides have opposite directions (tension along axis y for $\sigma_1 > 0$) whereas the shear stresses have coincident directions (tension along axis x for $\tau_1 > 0$). In problem B, the normal surface forces on the upper and lower sides have coincident directions (upwards, i.e. along axis y for $\sigma_2 > 0$) whereas the shear stresses have opposite directions ($\tau_2 > 0$ at $y = b$). In problem A, displacements u and v are respectively even and odd with respect to y and in problem B u and v are respectively odd and even with respect to y. Problem A will be referred to as the problem of tension while Problem B will be termed the problem of bending, Fig. 7.1.

The system of surface forces on the part $(0, x)$ of the beam is statically equivalent to the longitudinal and transverse forces

$$\left.\begin{aligned} P(x) &= \int_0^x \left[X^+(x) + X^-(x)\right] dx = 2\int_0^x \tau_1(x)\, dx, \\ Q(x) &= \int_0^x \left[Y^+(x) + Y^-(x)\right] dx = 2\int_0^x \sigma_2(x)\, dx \end{aligned}\right\} \tag{2.1.6}$$

and the bending moment about point $(x, 0)$

$$\mu(x) = -\int_0^x (x-\xi)\left[Y^+(\xi) + Y^-(\xi)\right] d\xi - b\int_0^x \left[X^+(\xi) - X^-(\xi)\right] d\xi$$

$$= -2\int_0^x (x-\xi)\,\sigma_2(\xi)\, d\xi - 2b\int_0^x \tau_2(\xi)\, d\xi. \tag{2.1.7}$$

In particular, on the right end

$$\left.\begin{aligned} x = l:\quad P(l) = 2\int_0^l \tau_1(x)\, dx, \quad Q(l) = 2\int_0^l \sigma_2(x)\, dx, \\ \mu(l) = -lQ + 2\int_0^l \xi\sigma_2(\xi)\, d\xi - 2b\int_0^l \tau_2(x)\, dx, \end{aligned}\right\} \tag{2.1.8}$$

that is, in order to make this end free of load in the sense of Saint-Venant forces $-P(l), -Q(l)$ and moment $-\mu(l)$ are needed.

7.2.2 Plane Saint-Venant's problem

It is assumed that sides $y = \pm b$ are free of load whilst the surface forces on the right end are statically equivalent to the axial force P, transverse force Q and bending moment μ.

In the problem of tension by a longitudinal force $\sigma_y = 0, \tau_{xy} = 0$ and the non-zero stress σ_x is uniformly distributed over the height of the strip. Airy's function is simple and given by

$$U_A = \frac{P}{4b} y^2. \tag{2.2.1}$$

A positive (i.e. rotating from axis x to axis y) bending moment μ causes only stresses σ_x, compressive on the upper part of the strip $(y > 0)$ and linearly distributed over the height. Referring, for example, to eq. (1.4.6) of Chapter 6 we have

$$\sigma_x = -\frac{\mu}{I} y, \quad \tau_{xy} = 0, \quad \sigma_y = 0, \quad I = \frac{2}{3} b^3,$$

where I denotes the moment of area inertia of the cross-section about axis x_3 (for a unit size of the thickness). Therefore

$$U_B^{(1)} = -\frac{\mu}{6I}y^3 = -\frac{\mu}{4b^3}y^3. \tag{2.2.2}$$

The problem of bending due to the transverse force Q applied at the right end is more difficult. In this case

$$\sigma_x = -\frac{Q(l-x)}{I}y, \quad \frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} = -\frac{Q}{I}y, \quad \frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau_{xy}}{\partial x}$$

and one can satisfy these equations and the boundary conditions of zero stresses τ_{xy}, σ_y by taking

$$\tau_{xy} = \frac{Q}{2I}\left(b^2 - y^2\right), \quad \sigma_y = 0.$$

These stresses are due to the following stress function

$$U_B^{(2)} = \frac{Q}{2I}(l-x)\,y\left(b^2 - \frac{y^2}{3}\right). \tag{2.2.3}$$

Referring to formulae of Section 7.1 we have

$$s = \nabla^2 U_B^{(2)} = \sigma_x = -\frac{Q}{I}(l-x)\,y, \quad s + it = \frac{Q}{I}i\left(lz - \frac{1}{2}z^2\right),$$

that is

$$S + iT = \frac{Q}{2I}i\left(lz^2 - \frac{z^3}{3}\right),$$

$$S = \frac{Q}{2I}\left(-2lxy + x^2y - \frac{1}{3}y^3\right), \quad T = \frac{Q}{2I}\left[l\left(x^2 - y^2\right) - \frac{x^3}{3} + xy^2\right]$$

and the displacements are as follows

$$\left.\begin{aligned} EIu &= \frac{Q}{2}\left[(l+\nu)\,y\left(b^2 - \frac{y^2}{3}\right) - 2lxy + x^2y - \frac{1}{3}y^3\right] + EI\left(u_0 - \omega_0 y\right), \\ EIv &= \frac{Q}{2}\left[-(l+\nu)(l-x)\left(b^2 - y^2\right) + l\left(x^2 - y^2\right) - \frac{x^3}{3} + xy^2\right] + \\ &\qquad\qquad EI\left(v_0 + \omega_0 x\right). \end{aligned}\right\} \tag{2.2.4}$$

Constants u_0, v_0 are obtained from the conditions that the displacements at point $(0,0)$ vanish. Following Subsection 6.2.2, in order to determine ω_0 we assume

$$\left(\frac{\partial v}{\partial x}\right)_{00} = 0 \quad \text{or} \quad \left(\frac{\partial u}{\partial y}\right)_{00} = 0.$$

The first case corresponds to fixing element dx on the beam axis at point $(0, 0)$ whereas the second case describes fixing element dy in the cross-section of the beam. The equation for the elastic line obtained by means of the second equation in (2.2.4) for $y = 0$ is as follows

$$v(x, 0) = \frac{Q}{2EI}\left(lx^2 - \frac{x^3}{3}\right) + \left[\frac{\tau_{xy}^0}{\mu}x\right], \tag{2.2.5}$$

where the term in the brackets is added for the second way of fixing and accounts for the influence of the shear stresses on the deflection. The value of this correction can be characterised by the ratio of it to the end deflection obtained by the first way of fixing

$$\frac{\tau_{xy}^0}{\mu} : \frac{Ql^3}{3EI} = 3(1+\nu)\frac{b^2}{l^2}. \tag{2.2.6}$$

As mentioned above, the applicability of Saint-Venant's principle assumes the smallness of ratio b/l. The correction in the deflection due to shear stresses is proportional to the square of ratio b/l and presents a standard order for the correction term in the technical theory of beams.

7.2.3 Operator representation of solutions

This simple notion for the solution of problem of mathematical physics for strip $|y| \le b$ is explained in what follows for the Laplace equation

$$\nabla^2 F = \partial^2 F + \frac{\partial^2 F}{\partial y^2} = 0 \quad \left(\partial = \frac{\partial}{\partial x}\right). \tag{2.3.1}$$

The solution is sought as a series in y

$$F = \sum_{s=0}^{\infty} y^{2s} f_{2s}(x) + \sum_{s=0}^{\infty} y^{2s+1} f'_{2s}(x). \tag{2.3.2}$$

Here $f'_{2s}(x)$ does not denote a derivative of $f_{2s}(x)$ with respect to x as the differentiation with respect to x is denoted by ∂. Substituting eq. (2.3.2) into eq. (2.3.1) we obtain after a rearrangement

$$\sum_{s=0}^{\infty} y^{2s}\left[\partial^2 f_{2s}(x) + (2s+1)(2s+2) f_{2s+2}(x)\right] +$$
$$\sum_{s=0}^{\infty} y^{2s+1}\left[\partial^2 f'_{2s}(x) + (2s+2)(2s+3) f'_{2s+2}(x)\right] = 0.$$

Equating the coefficients associated with y^{2s}, y^{2s+1} for $s = 0, 1, 2, \ldots$ we arrive at the following chain of equalities

$$\begin{array}{ll}
\partial^2 f_0(x) + 1 \cdot 2 f_2(x) = 0, & \partial^2 f_0'(x) + 2 \cdot 3 f_2'(x) = 0, \\
\partial^2 f_2(x) + 3 \cdot 4 f_4(x) = 0, & \partial^2 f_2'(x) + 4 \cdot 5 f_4'(x) = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\partial^2 f_{2s}(x) + (2s+1)(2s+2) f_{2s+2}(x) = 0, & \\
& \partial^2 f_{2s}'(x) + (2s+2)(2s+3) f_{2s+2}'(x) = 0,
\end{array}$$

enabling us to express all of these coefficients of series (2.3.2) in terms of $f_0(x)$ and $f_0'(x)$, i.e. in terms of the sought solution $F(x, y)$ and its first derivative with respect to y on the straight line $y = 0$ (the strip axis)

$$f_{2s}(x) = \frac{(-1)^s}{(2s)!} \partial^{2s} f_0(x), \quad f_{2s}'(x) = \frac{(-1)^s}{(2s+1)!} \partial^{2s} f_0'(x). \tag{2.3.3}$$

Now

$$F(x, y) = \sum_{s=0}^{\infty} \left[\frac{(-1)^s}{(2s)!} (y\partial)^{2s} f_0(x) + \frac{(-1)^s}{(2s+1)!} \frac{(y\partial)^{2s+1}}{\partial} f_0'(x) \right], \tag{2.3.4}$$

however this result can be set in another form

$$F(x, y) = \cos y\partial f_0(x) + \frac{\sin y\partial}{\partial} f_0'(x). \tag{2.3.5}$$

This result has a clear meaning: $\cos y\partial$ and $\sin y\partial$ should be replaced by the power series in ∂^2 and letter ∂ should have the original meaning of the operator of differentiation with respect to x over functions $f_0(x)$ and $f_0'(x)$.

It is however easy to obtain solution (2.3.5) without series. We can write Laplace equation (2.3.1) in the form of an ordinary differential equation with respect to the independent variable y in which ∂ is considered, for the time being, as a number

$$F'' + \partial^2 F = 0. \tag{2.3.6}$$

The solution of this equation subjected to the initial conditions

$$\text{at } \ y = 0 \quad F = f_0(x), \quad F' = f_0'(x), \tag{2.3.7}$$

is set in the form of eq. (2.3.5) and has the above interpretation. The intermediate calculations are carried out for representation (2.3.5) rather than directly for the series. For example, the mean value of $F(x, y)$ averaged over "the height of the strip" and the normal derivative of F at $y = b$ can be written down as follows

$$\frac{1}{2b} \int_{-b}^{b} F(x, y)\, dy = \frac{\sin b\partial}{b\partial} f_0(x),$$

$$\left. \frac{\partial F}{\partial y} \right|_{y=b} = -\partial \sin b\partial f_0(x) + \cos b\partial f_0'(x) \text{ etc.}$$

There is no need to take care of the convergence of series (2.3.4) presented in the form (2.3.5). The latter provides us with a formal way of constructing the solution which can be expressed in any other form.

7.2.4 Stress function for the strip problem

Following Subsection 7.2.3 we set the biharmonic equation for Airy's function in the form

$$U^{IV} + 2\partial^2 U'' + \partial^4 U = 0. \tag{2.4.1}$$

Four initial conditions are needed, namely the values of U and its derivatives of the first, second and third order at $y = 0$. For the sake of simplicity of the forthcoming formulae we will seek U as a sum of even and odd functions of y which corresponds to splitting the problem into problems A and B. The evaluations will be carried out in parallel. The initial conditions are put in the form

$$\left.\begin{array}{llllll} \mathrm{A} & y=0: & U = f_0(x), & U' = 0, & U'' = f_0''(x), & U''' = 0 \\ \mathrm{B} & y=0: & U = 0, & U' = f_0'(x), & U'' = 0, & U''' = f_0'''(x), \end{array}\right\} \tag{2.4.2}$$

and the corresponding solutions of the "ordinary" differential equation (2.4.1) are

$$\left.\begin{array}{ll} \mathrm{A} & U(x,y) = \cos y\partial f_0(x) + \dfrac{1}{2} y \dfrac{\sin y\partial}{y\partial}\left[f_0''(x) + \partial^2 f_0(x)\right], \\ & U(x,y) = \dfrac{\sin y\partial}{\partial} f_0'(x) + \\ \mathrm{B} & \dfrac{1}{2\partial^2}\left(\dfrac{\sin y\partial}{\partial} - y\cos y\partial\right)\left[f_0'''(x) + \partial^2 f_0'(x)\right], \end{array}\right\} \tag{2.4.3}$$

so that

$$\left.\begin{array}{ll} \mathrm{A} & \nabla^2 U = \cos y\partial\left[f_0''(x) + \partial^2 f_0(x)\right], \\ \mathrm{B} & \nabla^2 U = \dfrac{\sin y\partial}{\partial}\left[f_0'''(x) + \partial^2 f_0'(x)\right]. \end{array}\right\} \tag{2.4.4}$$

The boundary conditions (2.1.5) can be set in the form

$$\begin{array}{llll} \mathrm{A} & y = \pm b: & \sigma_y = \partial^2 U = \sigma_1(x), & \tau_{xy} = -\partial\dfrac{\partial U}{\partial y} = \pm\tau_1(x), \\ \mathrm{B} & y = \pm b: & \sigma_y = \partial^2 U = \pm\sigma_2(x), & \tau_{xy} = -\partial\dfrac{\partial U}{\partial y} = \tau_2(x) \end{array}$$

and under the notion

$$m_k(x) = \int_0^x (x-\xi)\,\sigma_k(\xi)\,d\xi, \quad t_k(x) = \int_0^x \tau_k(\xi)\,d\xi, \quad k = 1, 2, \tag{2.4.5}$$

they determine the stress function and its normal derivative at $y = \pm b$

$$\left.\begin{array}{llll} \mathrm{A} & y=\pm b: & U=m_1(x), & \dfrac{\partial U}{\partial y}=\mp t_1(x), \\ \mathrm{B} & y=\pm b: & U=\pm m_2(x), & \dfrac{\partial U}{\partial y}=-t_2(x). \end{array}\right\} \tag{2.4.6}$$

Using these conditions we determine the unknown functions in solution (2.4.3) and then the solutions themselves

$$\begin{aligned} \mathrm{A}\quad U(x,y) &= \frac{1}{F_A(2b\partial)}\left[\left(\frac{y}{b}\sin y\partial\sin b\partial+\cos y\partial\cos b\partial+\right.\right. && (2.4.7\mathrm{A}) \\ &\left.\left.+\cos y\partial\frac{\sin b\partial}{b\partial}\right)m_1(x)-\left(\frac{y}{b}\frac{\sin y\partial}{\partial}\cos b\partial-\cos y\partial\frac{\sin b\partial}{\partial}\right)t_1(x)\right], \\ \mathrm{B}\quad U(x,y) &= \frac{1}{F_B(2b\partial)}\left[\left(\frac{y}{b}\cos y\partial\cos b\partial+\sin y\partial\sin b\partial-\right.\right. && (2.4.7\mathrm{B}) \\ &\left.\left.-\frac{\sin y\partial}{b\partial}\cos b\partial\right)m_2(x)+\left(y\cos y\partial\frac{\sin b\partial}{b\partial}-\frac{\sin y\partial}{\partial}\cos b\partial\right)t_2(x)\right], \end{aligned}$$

where

$$F_A(2b\partial)=1+\frac{\sin 2b\partial}{2b\partial},\quad F_B(2b\partial)=1-\frac{\sin 2b\partial}{2b\partial}. \tag{2.4.8}$$

The power series in ∂ in the expressions in brackets in problems A and B begin respectively with terms of first and second order. Thus, functions $U(x,y)$ in both problems are presented by the series containing only positive powers of ∂ in addition to the constant terms.

The displacements obtained by formulae (1.7.7) are equal to

$$\left.\begin{aligned} 2\mu u &= \frac{1}{F_A(2b\partial)}\left[\partial\left(-\cos y\partial\cos b\partial-\frac{y}{b}\sin y\partial\sin b\partial+\right.\right. \\ &\qquad\left.\frac{1-\nu}{1+\nu}\cos y\partial\frac{\sin b\partial}{b\partial}\right)m_1(x)+ \\ &\left.\left(-\cos y\partial\sin b\partial+\frac{y}{b}\sin y\partial\cos b\partial-\frac{2}{1+\nu}\cos y\partial\frac{\cos b\partial}{b\partial}\right)t_1(x)\right], \\ 2\mu v &= \frac{1}{F_A(2b\partial)}\left[\partial\left(\sin y\partial\cos b\partial-\frac{y}{b}\cos y\partial\sin b\partial+\right.\right. \\ &\qquad\left.\frac{2}{1+\nu}\sin y\partial\frac{\sin b\partial}{b\partial}\right)m_1(x)+ \\ &\left.\left(\sin y\partial\sin b\partial+\frac{y}{b}\cos y\partial\cos b\partial-\frac{1-\nu}{1+\nu}\frac{\sin y\partial}{b\partial}\cos b\partial\right)t_1(x)\right], \end{aligned}\right\} \tag{2.4.9A}$$

$$\left.\begin{aligned}
2\mu u &= \frac{1}{F_B(2b\partial)}\left[-\partial\left(\frac{y}{b}\cos y\partial\cos b\partial + \sin y\partial\sin b\partial + \right.\right.\\
&\qquad\left.\frac{1-\nu}{1+\nu}\sin y\partial\frac{\cos b\partial}{b\partial}\right)m_2(x) + \\
&\left.\left(-\frac{y}{b}\cos y\partial\sin b\partial + \sin y\partial\cos b\partial - \frac{2}{1+\nu}\sin y\partial\frac{\sin b\partial}{b\partial}\right)t_2(x)\right],\\
2\mu v &= \frac{1}{F_B(2b\partial)}\left[\partial\left(\frac{y}{b}\sin y\partial\cos b\partial - \cos y\partial\sin b\partial + \right.\right.\\
&\qquad\left.\frac{2}{1+\nu}\cos y\partial\frac{\cos b\partial}{b\partial}\right)m_2(x) + \\
&\left.\left(\frac{y}{b}\sin y\partial\sin b\partial + \cos y\partial\cos b\partial + \frac{1-\nu}{1+\nu}\cos y\partial\frac{\sin b\partial}{b\partial}\right)t_2(x)\right].
\end{aligned}\right\} \tag{2.4.9B}$$

These formulae contain the terms whose expansions in series also have negative powers. This can be predicted since determining the displacements requires further integrating expressions for the surface forces.

The expressions for the stresses are easily obtained by differentiating the stress function (2.4.7). Omitting the cumbersome formulae we notice only that, in the theory of beams, the representation of the key quantities turn out to be rather simple. For example, the normal stresses $\sigma_x(x, \pm b)$ on the longitudinal sides of the beam are as follows

$$\left.\begin{aligned}
&\mathrm{A}\quad \sigma_x(x,\pm b) = -\frac{F_B(2b\partial)}{F_A(2b\partial)}\sigma_1(x) - \frac{2\cos^2 b\partial}{F_A(2b\partial)}\frac{1}{b}t_1(x),\\
&\mathrm{B}\quad \sigma_x(x,\pm b) = \mp\frac{F_A(2b\partial)}{F_B(2b\partial)}\sigma_2(x) \mp \frac{2\sin^2 b\partial}{F_B(2b\partial)}\frac{1}{b}t_2(x).
\end{aligned}\right\} \tag{2.4.10}$$

The shear stress $\tau_{xy}(x, 0)$ on the axis of the beam is given by

$$\left.\begin{aligned}
&\mathrm{A}\quad \tau_{xy}(x,0) = 0,\\
&\mathrm{B}\quad \tau_{xy}(x,0) = \frac{1}{F_B(2b\partial)}\left[-\sin b\partial\sigma_2(x) + \right.\\
&\qquad\qquad\left.\left(\cos b\partial - \frac{\sin b\partial}{b\partial}\right)\tau_2(x)\right].
\end{aligned}\right\} \tag{2.4.11}$$

The equation for the elastic line is set in the form

$$\left.\begin{aligned}
&\mathrm{A}\quad v(x,0) = 0,\\
&\mathrm{B}\quad 2\mu v(x,0) = \frac{\partial}{F_B(2b\partial)}\left[\left(-\sin b\partial + \frac{2}{1+\nu}\frac{\cos b\partial}{b\partial}\right)m_2(x) + \right.\\
&\qquad\left.\left(\cos b\partial + \frac{1-\nu}{1+\nu}\frac{\sin b\partial}{b\partial}\right)t_2(x)\right] + 2\mu(\omega_0 x + v_0).
\end{aligned}\right\} \tag{2.4.12}$$

Seemingly, these formulae can be applicable because the functions describing the surface forces are infinitely differentiable. However it will be shown below that this restriction can be removed. The formal representations derived can gain a certain meaning for the piecewise continuous surface forces and even for concentrated forces and moments.

Let us also notice the representations for operators F_A and F_B and the inverse operators

$$\left.\begin{aligned} F_A(2b\partial) &= 2\left(1-\frac{1}{3}b^2\partial^2+\frac{1}{15}b^4\partial^4-\frac{2}{315}b^6\partial^6+\ldots\right),\\ \frac{1}{F_A(2b\partial)} &= \frac{1}{2}\left(1+\frac{1}{3}b^2\partial^2+\frac{2}{45}b^4\partial^4-\frac{1}{2635}b^6\partial^6+\ldots\right), \end{aligned}\right\} \quad (2.4.13A)$$

$$\left.\begin{aligned} F_B(2b\partial) &= \frac{2}{3}b^2\partial^2\left(1-\frac{1}{5}b^2\partial^2+\frac{2}{105}b^4\partial^4-\frac{1}{945}b^6\partial^6+\ldots\right),\\ \frac{1}{F_B(2b\partial)} &= \frac{3}{2b^2\partial^2}\left(1+\frac{1}{5}b^2\partial^2+\frac{11}{525}b^4\partial^4+\frac{34}{23625}b^6\partial^6+\ldots\right), \end{aligned}\right\} \quad (2.4.13B)$$

which will be used in what follows.

The multiplier ∂^{-2} in the expression for F_B^{-1} makes the problem of bending more difficult in comparison with the problem of tension.

7.2.5 The elementary theory of beams

Keeping in the power series in ∂ only the constant term the expressions for the stress function (2.4.7) are presented in the form

$$\left.\begin{aligned} &\mathrm{A}\quad U^0(x,y) = m_1(x)+\frac{1}{2b}\left(b^2-y^2\right)t_1(x),\\ &\mathrm{B}\quad U^0(x,y) = \frac{y}{I}\left[\left(b^2-\frac{1}{3}y^2\right)m_2(x)+\frac{1}{3}\left(b^2-y^2\right)bt_2(x)\right], \end{aligned}\right\} \quad (2.5.1)$$

and the stresses are as follows

$$\left.\begin{aligned} \sigma_x &= -\frac{t_1(x)}{b}-\frac{2y}{I}\left[m_2(x)+bt_2(x)\right],\\ \sigma_y &= \sigma_1(x)+\frac{1}{2}\left(b^2-y^2\right)\frac{1}{b}\tau_1'(x)+\frac{y}{I}\left[\left(b^2-\frac{1}{3}y^2\right)\sigma_2(x)+\right.\\ &\qquad\left.\frac{1}{3}\left(b^2-y^2\right)b\tau_2'(x)\right],\\ \tau_{xy} &= \frac{y}{b}\tau_1(x)-\frac{1}{I}\left[\left(b^2-y^2\right)m_2'(x)+\left(\frac{1}{3}b^2-y^2\right)b\tau_2(x)\right], \end{aligned}\right\} \quad (2.5.2)$$

where, as previously, $I=\frac{2}{3}b^3$. Evidently, these expressions satisfy the static equations in the volume and the boundary conditions on the longitudinal sides of the beam. However Beltrami's dependences are not satisfied as the stress functions (2.5.1) are not biharmonic for arbitrary surface forces. In the obtained solution, the end $x = 0$ is free in Saint-Venant's sense, that is, the longitudinal and transverse forces and the bending moments are equal to zero.

The distribution of σ_x and the shear stresses due to the normal load given by eq. (2.5.2) correspond to those in the elementary theory of beams whereas the normal stress σ_y are not taken into account within this theory.

As stress functions (2.5.1) are not biharmonic, the displacements should be determined by the first term in the expansion in the series in ∂ obtained from the general representations (2.4.9). We arrive then at the expressions

$$\left.\begin{aligned} Eu &= -\nu m_1'(x) - \frac{1}{b}\int_0^x t_1(\xi)\,d\xi + \frac{y}{I}\int_0^x \mu^0(\xi)\,d\xi + u_0 - \omega_0 y, \\ Ev &= y\sigma_1(x) + \nu\frac{y}{b}t_1(x) - \frac{1}{I}\int_0^x (x-\xi)\,\mu^0(\xi)\,d\xi + v_0 + \omega_0 y, \end{aligned}\right\} \tag{2.5.3}$$

where

$$\mu^0(x) = -2\left[m_2(x) + bt_2(x)\right] \tag{2.5.4}$$

presents the bending moment about point $(x, 0)$ caused by the surface forces distributed over part $[0, x]$ of the beam.

The equation for the elastic line of the beam takes the form

$$EI\frac{d^2}{dx^2}v(x,0) = -\mu^0(x). \tag{2.5.5}$$

Thus, the elementary theory is completely contained in the first term of the expansion of the rigorous solution. Further terms with the derivatives of the functions describing the load are the corrections added to the elementary theory one after another. The order of these corrections compared to the principal terms of the elementary theory is proportional to the sequential powers of ratio b^2/l^2. They are essential for relatively short beams and if the load rapidly changes along the beam. It can also be mentioned that these corrections describe the system of stresses which are statically equivalent to zero since the static equations in the volume and on the surface have already been satisfied in the first approximation (2.5.1).

7.2.6 Polynomial load (Mesnager, 1901)

When the surface forces are prescribed by polynomials of order n the higher order of the polynomials on the right hand sides of the expressions in eq. (2.4.7) is equal to $n+2$. The expansions of these expressions in power series in ∂ are truncated and the stress function is presented by an automatically obtained polynomial of x, y.

For example, under an uniform normal load $Y^+(x) = \text{const}$

$$\sigma_1 = \sigma_2 = \frac{1}{2} Y^+(x), \quad m_1(x) = m_2(x) = \frac{1}{4} Y^+ x^2$$

and in the expansion of problem A it is necessary to keep only the terms with ∂^2 whilst in the problem B the terms with ∂^4 should also be retained as formulae (2.4.7B) contain factor ∂^{-2}. The powers of the terms in the series increases by two, thus the evaluation is carried out to the above-mentioned ∂^2 and ∂^4. When the load obeys a quadratic or cubic law it is necessary to keep the fourth and sixth power in problems A and B, respectively, and so on.

Calculation for a uniform or a linear normal load leads to the following expressions for the stress function

$$\left.\begin{aligned} &\text{A} \quad U_A(x,y) = U_A^0(x,y), \\ &\text{B} \quad U_B(x,y) = U_B^0(x,y) + \frac{y}{30I}\left(b^2 - y^2\right)^2 \sigma_2(x), \end{aligned}\right\} \tag{2.6.1}$$

as the terms with ∂^2 vanish in problem A. Expressions for $U_A^0(x,y)$ and $U_B^0(x,y)$ are given by formulae (2.5.1). The stresses imposed on solutions (2.5.2) are equal to

$$\left.\begin{aligned} \sigma_x^{(1)} &= \frac{2}{15I} y\left(5y^2 - 3b^2\right)\sigma_2(x), \\ \sigma_y^{(1)} &= \frac{1}{30I}\left(b^2 - y^2\right)^2 y\sigma_2''(x), \\ \tau_{xy}^{(1)} &= -\frac{1}{30I}\left(b^4 - 6b^2y^2 + 5y^4\right)\sigma_2'(x), \end{aligned}\right\} \tag{2.6.2}$$

where $\sigma_2''(x) = 0$ and $\sigma_2'(x)$ is constant under a linear load and vanishes under a uniform load.

The equation for the elastic line constructed by formulae (2.4.12) contains, in addition to the term from the elementary theory (proportional to the double integral over the bending moment $\mu^0(\xi)$) the terms proportional to this moment and the second derivative of $\sigma_2(x)$

$$EIv(x,0) = -\int_0^x (x-\xi)\,\mu^0(\xi)\,d\xi - \frac{b^2}{5}(8+5\nu)\,m_2(x) + \\ \frac{1}{5}b^4\left(\frac{22}{105} - \frac{3+4\nu}{4}\right)\sigma_2(x) + v_0 + \omega_0 x. \tag{2.6.3}$$

However the term proportional to $\sigma_2(x)$ is included into the expression for the rigid body displacement.

In practice, the present formulae are applicable to any law of the polynomial load as the corrections are of the order of b^4/l^4 and higher. Their account assumes applicability of Saint-Venant's principle and is hardly appropriate.

Having the solution for the case of the linear load we can limit our consideration only to loads which are statically equivalent to zero. Indeed, given a law of load one can determine the load

$$\tilde{f}(x) = f(x) - \frac{1}{l}\int_0^l f(\xi)\,d\xi - \frac{3}{l}\left(l - \frac{2x}{l}\right)\int_0^l \left(1 - \frac{2\xi}{l}\right) f(\xi)\,d\xi, \quad (2.6.4)$$

which satisfies the requirement for the load to be statically equivalent to zero

$$\int_0^l \tilde{f}(x)\,dx = 0, \qquad \int_0^l x\tilde{f}(x)\,dx = 0 \quad (2.6.5)$$

and differ from $f(x)$ in the load which is linear in x

$$f_*(x) = -\frac{1}{l}\left[\int_0^l f(\xi)\,d\xi + 3\left(1 - \frac{2x}{l}\right)\int_0^l \left(1 - \frac{2\xi}{l}\right) f(\xi)\,d\xi\right]. \quad (2.6.6)$$

The solution for this particular load is known. One can limit consideration to the loads with the vanishing principal vector, to this end, it is sufficient to keep two terms on the right hand side of eq. (2.6.4).

7.2.7 *Sinusoidal load, solutions of Ribiere (1898) and Filon (1903)*

Relationships of Subsection 7.2.4 can be set in the general form

$$\Phi(x, y) = Q(y, \partial) f(x), \quad (2.7.1)$$

where $\Phi(x, y)$ is a sought quantity (for example the stress function, a stress, a displacement), $f(x)$ is determined by the load whereas $Q(y, \partial)$ is a prescribed function of operator ∂ and depends on y. When the load has the form of a sine or cosine function

$$f(x) = \begin{cases} \sin \alpha x \\ \cos \alpha x \end{cases}, \quad \partial^2 f(x) = -\alpha^2 f(x), \ldots, \partial^{2s} f(x) = \left(-\alpha^2\right)^s f(x)$$

relation (2.7.1) can be set in the explicit form

$$\Phi(x, y) = Q(y, \alpha i) f(x), \quad (2.7.2)$$

without resorting to expansion in a series. For instance,

$$\cos y\partial f(x) = \cos y\alpha i f(x) = \cosh \alpha y f(x), \qquad \frac{\sin y\partial}{\partial} f(x) = \frac{\sinh \alpha y}{\alpha} f(x)$$

etc.

Given a trigonometric series for the load

$$f(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \alpha_k x + b_k \sin \alpha_k x \right), \quad \alpha_k = 2\frac{k\pi}{l}, \tag{2.7.3}$$

where

$$a_k = \frac{2}{l} \int_0^l f(x) \cos \alpha_k x dx, \quad b_k = \frac{2}{l} \int_0^l f(x) \sin \alpha_k x dx, \tag{2.7.4}$$

we have

$$\Phi(x,y) = \Phi_*(x,y) + \sum_{k=1}^{\infty} Q(y, \alpha_k i) \left(a_k \cos \alpha_k x + b_k \sin \alpha_k x \right), \tag{2.7.5}$$

where $\Phi_*(x,y)$ denotes the solution corresponding to the constant term of series (2.7.3) which is the mean value of $f(x)$ averaged over the length of the beam. It is constructed by the rule of Subsection 7.2.6.

For example, for the normal load only the upper side of the beam $y = b$

$$Y^+(x) = 2\sigma_1(x) = 2\sigma_2(x) = \sum_{k=1}^{\infty} \left(a_k \cos \alpha_k x + b_k \sin \alpha_k x \right) \tag{2.7.6}$$

the stress function, by eq. (2.4.7), is presented by the series

$$\left.\begin{aligned} U_A(x,y) &= -b \sum_{k=1}^{\infty} \frac{a_k \cos \alpha_k x + b_k \sin \alpha_k x}{\alpha_k \left(2b\alpha_k + \sinh 2b\alpha_k \right)} \left(-\frac{y}{b} \sinh \alpha_k y \sinh \alpha_k b + \right. \\ &\left. \cosh \alpha_k y \cosh \alpha_k b + \cosh \alpha_k y \frac{\sinh \alpha_k b}{\alpha_k b} \right) + U_A^*(x,y), \\ U_B(x,y) &= -b \sum_{k=1}^{\infty} \frac{a_k \cos \alpha_k x + b_k \sin \alpha_k x}{\alpha_k \left(2b\alpha_k - \sinh 2b\alpha_k \right)} \left(\frac{y}{b} \cosh \alpha_k y \cosh \alpha_k b - \right. \\ &\left. \sinh \alpha_k y \sinh \alpha_k b - \frac{\sinh \alpha_k y}{\alpha_k b} \cosh \alpha_k b \right) + U_B^*(x,y). \end{aligned}\right\} \tag{2.7.7}$$

Here $U_A^*(x,y)$ and $U_B^*(x,y)$ are the solutions of a similar form to (2.6.1) and correspond to the terms constant and linear in x in the expansion of moments $m_k(x)$ in the trigonometric series

$$m_1(x) = m_2(x) = \frac{1}{2} \left[\sum_{k=1}^{\infty} \left(\frac{a_k}{\alpha_k^2} + x \frac{b_k}{\alpha_k} \right) - \sum_{k=1}^{\infty} \frac{1}{\alpha_k^2} \left(a_k \cos \alpha_k x + b_k \sin \alpha_k x \right) \right]. \tag{2.7.8}$$

The convergence of series (2.7.7) and the stresses obtained in terms of these series is ensured by term $\sinh 2b\alpha_k$ in the denominator. The convergence is also ensured on the loaded side $y = b$ of the beam excluding the points of discontinuity in the load.

Example. A beam of length $2l$ is loaded by a "triangular" load which is even with respect to x

$$Y^+(x) = 2\sigma_1(x) = 2\sigma_2(x) = \frac{1}{2}q\left(1 - \frac{2|x|}{l}\right), \quad |x| \leq l.$$

The trigonometric series for the load has the form

$$Y^+(x) = \frac{4q}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^2}, \quad |x| < l, \quad \alpha_k = (2k-1)\frac{\pi}{l},$$

and due to eq. (2.7.8)

$$m_1(x) = m_2(x) = \frac{2q}{l^2}\left(\sum_{k=1}^{\infty}\frac{1}{\alpha_k^4} - \sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^4}\right) = \frac{ql^2}{48} - \frac{2q}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^4},$$

where we used the well-known relationship

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$$

The normal stresses on the sides of the beam obtained by eq. (2.4.10) are as follows

$$\sigma_x(x, \pm b) =$$
$$= \frac{2q}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^4}\left(\frac{\sinh 2b\alpha_k - 2b\alpha_k}{\sinh 2b\alpha_k + 2b\alpha_k} \pm \frac{\sinh 2b\alpha_k + 2b\alpha_k}{\sinh 2b\alpha_k - 2b\alpha_k}\right) \mp \frac{b}{I}\frac{ql^2}{24},$$

where the constant term is given by eq. (2.5.2).

These formulae are transformed to the form

$$\left.\begin{aligned} \sigma_x(x, b) &= Y^+(x) + \frac{32qb^2}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\sinh^2 2b\alpha_k - 4b^2\alpha_k^2} - \frac{b}{I}\frac{ql^2}{24}, \\ \sigma_x(x, -b) &= -\frac{16qb}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x \sinh 2b\alpha_k}{\left(\sinh^2 2b\alpha_k - 4b^2\alpha_k^2\right)\alpha_k} + \frac{b}{I}\frac{ql^2}{24}. \end{aligned}\right\} \qquad (2.7.9)$$

Also here one can see the difference in the absolute values of stress σ_x on the loaded and unloaded sides of the beam.

In the case of a very thin beam ($b \ll l$) the coefficients of series (2.7.9) are replaced by the expansions in power series in $b\alpha_k$

$$\left(\sinh^2 2b\alpha_k - 4b^2\alpha_k^2\right)^{-1} = \frac{3}{16b^4\alpha_k^4}\left(1 - \frac{8}{15}\alpha_k^4 b^2 + \ldots\right),$$

$$\frac{\sinh 2b\alpha_k}{\alpha_k}\left(\sinh^2 2b\alpha_k - 4b^2\alpha_k^2\right)^{-1} = \frac{3}{8b^4\alpha_k^4}\left(1 + \frac{2}{15}\alpha_k^4 b^2\right),$$

and formulae (2.7.9) are set as follows

$$\sigma_x(x, b) = Y^+(x) - \frac{16q}{5b^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^2} + \frac{6q}{l^2b^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^4} - \frac{b}{I}\frac{ql^2}{24},$$

$$\sigma_x(x, -b) = -\frac{4q}{5b^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^2} - \frac{6q}{l^2b^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^4} + \frac{b}{I}\frac{ql^2}{24}$$

or in the equivalent form

$$\sigma_x(x, \pm b) = \pm\left[\frac{1}{5}Y^+(x) + \frac{b}{I}\mu^0(x)\right] \tag{2.7.10}$$

if we remember the representation of $Y^+(x)$ in the form of a trigonometric series. Here we have denoted

$$\mu^0(x) = \frac{4q}{l^2}\sum_{k=1}^{\infty}\frac{\cos\alpha_k x}{\alpha_k^2} - \frac{ql^2}{24} = -2m_2(x).$$

According to eq. (2.5.4) this is the bending moment about point $(|x|, 0)$ on the beam axis. Using eq. (2.4.5) it can be presented, for $0 < x < l$, in the form

$$\mu^0(x) = -\frac{1}{2}q\int_0^x (x - \xi)\left(1 - \frac{2\xi}{l}\right)d\xi = -\frac{1}{4}q\left(x^2 - \frac{2}{3}\frac{x^3}{l}\right),$$

and it is easy to verify that function $\mu^0(x)$, even with respect to x, has the above trigonometric series for $|x| < l$.

In the approximate solution (2.7.10) the above-mentioned difference in the absolute values of $\sigma_x(x, \pm b)$ disappears. It is easy to understand that (2.7.10) corresponds to the solution to the problem of a beam of length l with a free left end ($x = 0$) and loaded by a linear load

$$Y^+(x) = \frac{1}{2}q\left(1 - \frac{2x}{l}\right) \qquad 0 < x < l.$$

Let us note that due to the symmetry of the loading, the left ($x < 0$) and right ($x > 0$) parts of the beam of length $2l$, the transverse force and the bending moment are absent in the middle $x = 0$ of the beam. The elementary theory of beams allows one to remove the left part of the beam ($-l < x < 0$) and reduce the problem to considering only the right part. The rigorous (2.7.9) solution differs from the approximate (2.7.10) solution is that the action of the left part on the right one transmitted via cross-section $x = 0$ is determined by the state of stress in this cross-section rather than only statically equivalent characteristics.

7.2.8 Concentrated force (Karman and Seewald, 1927)

Function $f(x)$ describing the load can be determined not by a series but a Fourier integral

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[f_c(\lambda) \cos\lambda x + f_s(\lambda) \sin\lambda x \right] d\lambda.$$

Here $f_c(\lambda), f_s(\lambda)$ denote the sine and cosine Fourier transforms

$$f_c(\lambda) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) \cos u\lambda du, \qquad f_s(\lambda) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) \sin u\lambda du.$$

The solution of eq. (2.7.1) is also presented by the Fourier integral

$$\Phi(x,y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[\Phi_c(\lambda, y) \cos\lambda x + \Phi_s(\lambda, y) \sin\lambda x \right] d\lambda,$$

where $\Phi_c(\lambda, y), \Phi_s(\lambda, y)$ are the Fourier transforms

$$\Phi_c(\lambda, y) = Q(\lambda i, y) f_c(\lambda), \quad \Phi_s(\lambda, y) = Q(\lambda i, y) f_s(\lambda).$$

In what follows we consider the problem of the beam supported on ends $x = \pm l$ and loaded normally to its longitudinal side $y = b$ by force Q concentrated at point $(0, b)$ and the reaction forces $(-Q/2)$ on ends $(\pm l, b)$. For this load, the bending moment $\mu(x)$ in the cross-section x has a "triangular" form

$$\mu(x) = \begin{cases} \frac{1}{2} Ql \left(1 - \frac{|x|}{l}\right), & |x| \le l, \\ 0, & |x| > l. \end{cases} \tag{2.8.1}$$

The Fourier integral for this function, even with respect to x, is set as follows

$$\mu(x) = \frac{Ql}{\pi} \int_0^{\infty} \cos\lambda x d\lambda \int_0^l \left(1 - \frac{u}{l}\right) \cos\lambda u du = \frac{Q}{\pi} \int_0^{\infty} \frac{\cos\lambda x}{\lambda^2} (1 - \cos\lambda l) d\lambda.$$

Using eq. (2.4.7), taking into account that only for normal load $m_1 = m_2 = -\frac{1}{2}\mu$ and replacing ∂ by $i\lambda$ we can represent the stress function in the form

$$U(x,y) = -\frac{Q}{2\pi}\left\{\int_0^\infty \frac{1-\cos\lambda l}{\lambda^2\left(1+\dfrac{\sinh 2b\lambda}{2b\lambda}\right)}\left(-\frac{y}{b}\sinh y\lambda \sinh b\lambda + \cosh y\lambda \cosh b\lambda + \cosh y\lambda \frac{\sinh b\lambda}{b\lambda}\right)\cos\lambda x d\lambda + \int_0^\infty \left[\frac{1}{1-\dfrac{\sinh 2b\lambda}{2b\lambda}}\left(\frac{y}{b}\cosh y\lambda\cosh b\lambda - \sinh y\lambda \sinh b\lambda - \frac{\sinh y\lambda}{b\lambda}\cosh b\lambda\right) - \frac{y}{I}\left(b^2-\frac{y^2}{3}\right)\right]\frac{1-\cos\lambda l}{\lambda^2}\cos\lambda x d\lambda\right\} - \frac{\mu(x)}{2I}y\left(b^2-\frac{y^2}{3}\right). \tag{2.8.2}$$

The expansion of the integrand in the second integral in the power series in λ begins with the term of the second order.

The normal stress σ_x is given by the following expression

$$\sigma_x = \frac{y}{I}\mu(x) - \frac{Q}{2\pi}\left\{\int_0^\infty \frac{1-\cos\lambda l}{1+\dfrac{\sinh 2b\lambda}{2b\lambda}}\left(\cosh y\lambda\cosh b\lambda - \frac{y}{b}\sinh\lambda y \sinh \lambda b - \cosh y\lambda\frac{\sinh b\lambda}{b\lambda}\right)\cos\lambda x d\lambda + \int_0^\infty \left[\frac{1}{1-\dfrac{\sinh 2b\lambda}{2b\lambda}}\left(\frac{y}{b}\cosh y\lambda \cosh b\lambda - \sinh y\lambda\sinh b\lambda + \frac{\sinh y\lambda}{b\lambda}\cosh b\lambda\right) + \frac{2y}{I\lambda^2}\right](1-\cos\lambda l)\cos\lambda x d\lambda\right\}. \tag{2.8.3}$$

This results in the normal stress σ_x on the beam axis

$$\sigma_x(x,0) = -\frac{Q}{\pi}\int_0^\infty \frac{b\lambda\cosh b\lambda - \sinh b\lambda}{2b\lambda + \sinh 2b\lambda}(1-\cos\lambda l)\cos\lambda x d\lambda \tag{2.8.4}$$

which is not taken into account in the elementary theory of beam and the stresses of the longitudinal sides

$$\sigma_x(x, \pm b) = \pm \frac{b}{I}\mu(x) -$$
$$\frac{2Q}{\pi}\int_0^\infty \left[\left(\frac{2b\lambda - \sinh 2b\lambda}{2b\lambda + \sinh 2b\lambda} \pm \frac{2b\lambda + \sinh 2b\lambda}{2b\lambda - \sinh 2b\lambda}\right) \pm \frac{2b}{I\lambda^2}\right](1 - \cos\lambda l)\cos\lambda x d\lambda.$$

On the side $y = -b$ this stress is determined by the convergent integral

$$\sigma_x(x, -b) = -\frac{2Q}{\pi}\int_0^\infty \left(\frac{2b\lambda\sinh 2b\lambda}{\sinh^2 2b\lambda - 4b^2\lambda^2} - \frac{b}{2I\lambda^2}\right) \times$$
$$(1 - \cos\lambda l)\cos\lambda x d\lambda - \frac{b}{I}\mu(x), \quad (2.8.5)$$

whereas convergence is lost on the side $y = b$ of the beam loaded by the concentrated forces. The obtained expression is easily reduced to the form

$$\sigma_x(x, b) = \frac{b}{I}\mu(x) + \frac{Q}{\pi}\int_0^\infty (1 - \cos\lambda l)\cos\lambda x d\lambda +$$
$$\frac{Q}{\pi}\int_0^\infty \left(\frac{8b^2\lambda^2}{\sinh^2 2b\lambda - 4b^2\lambda^2} - \frac{3}{2b^2\lambda^2}\right)(1 - \cos\lambda l)\cos\lambda x d\lambda. \quad (2.8.6)$$

Noticing that

$$\cos\lambda x\cos\lambda l = \frac{1}{2}\left[\cos\lambda(l + x) + \cos\lambda(l - x)\right], \quad (2.8.7)$$

and recalling the Fourier representation of the delta-function

$$\delta(x) = \frac{1}{\pi}\int_0^\infty \cos\lambda x d\lambda,$$

we arrive at the expression

$$\sigma_x(x, b) = \frac{b}{I}\mu(x) + Q\delta(x) - \frac{1}{2}Q\delta(l + x) - \frac{1}{2}Q\delta(l - x) +$$
$$\frac{Q}{\pi}\int_0^\infty \left(\frac{8b^2\lambda^2}{\sinh^2 2b\lambda - 4b^2\lambda^2} - \frac{3}{2b^2\lambda^2}\right)\left[\cos\lambda x - \frac{1}{2}\cos\lambda(l + x) -\right.$$
$$\left.\frac{1}{2}\cos\lambda(l - x)\right] d\lambda. \quad (2.8.8)$$

The expression

$$\delta(y-\xi)+\frac{1}{\pi}\int_0^\infty\left(\frac{8b^2\lambda^2}{\sinh^2 2b\lambda-4b^2\lambda^2}-\frac{3}{2b^2\lambda^2}\right)\cos\lambda(x-\xi)\,d\lambda= \\ =\delta(x-\xi)+\psi(x-\xi)$$

should be called the influence function of the unit force at point(ξ,b). The terms correcting the value $\sigma_x(x,b)$ from the elementary theory determine the action of forces Q at point $(0,b)$ and $-Q/2$ at points $(-l,b)$ and (l,b).

The influence function allows one to obtain the expressions for the terms correcting the elementary theory in the case of the distributed load. For instance, for the load

$$q(x)=\begin{cases} q_0, & |x|<a, \\ 0, & a<|x|<l, \end{cases}$$

we have

$$\int_{-a}^{a} q(\xi)\,\delta(x-\xi)\,d\xi=q(x),\quad \int_{-a}^{a} q(\xi)\cos\lambda(x-\xi)\,d\xi=2q_0\frac{\sin\lambda a}{\lambda}\cos\lambda x,$$

and the normal stress $\sigma_x(x,b)$ is as follows

$$\sigma_x(x,b)=\frac{b}{I}\mu(x)+\frac{2q_0}{\pi}\int_0^\infty\left(\frac{8b^2\lambda^2}{\sinh^2 2b\lambda-4b^2\lambda^2}-\frac{3}{2b^2\lambda^2}\right)\frac{\sin\lambda a}{\lambda}\cos\lambda x d\lambda \\ -q_0 a\left[\delta(x+l)+\delta(x-l)+\psi(x+l)+\psi(x-l)\right].$$

Here $\mu(x)$ denotes the bending moment in the simply supported beam under the considered load.

The plots of the normal σ_x,σ_y and shear τ_{xy} stresses sue to a concentrated force at $y=\pm b/2,0,\pm b$ versus x are shown in Seewald's paper[1] and reproduced in "Theory of elasticity" by Timoshenko. It is natural that perturbations of the stresses obtained by means of the elementary theory extend distances compared to the beam thickness $2b$. The difference between the stress $\sigma_x(x,b)$ and the corresponding stress due to the elementary theory practically vanish even for $x=3b$.

[1] Seewald, F. Die Spannungen und Formänderungen von Balken mit rechteckigem Querschnitt. Abhandlungen aus dem aerodynamischen Institut Aachen, Heft 7, 1927.

The equation for the elastic line is constructed by means of eq. (2.4.12)

$$v(x,0) = -\frac{Q}{2\pi bE} \int_0^\infty \frac{1-\cos\lambda l}{\lambda^2\left(1-\dfrac{\sinh 2b\lambda}{2b\lambda}\right)} \times \\ [(1+\nu)\, b\lambda \sinh b\lambda + 2\cosh b\lambda] \cos\lambda x d\lambda. \quad (2.8.9)$$

Introducing into consideration the elementary solution (2.6.3)

$$v(x,0) = \frac{Ql^3}{6EI}\left(1-\frac{3x^2}{l^2}+\frac{|x|^3}{l^3}\right) + \frac{Ql}{2E}\left(1-\frac{|x|}{l}\right)\frac{3(8+5\nu)}{20b}, \quad (2.8.10)$$

and subtracting its Fourier integral (2.8.9) we obtain the correction $v_*(x,0)$ to the elementary theory

$$v_*(x,0) = -\frac{Q}{2\pi bE}\int_0^\infty \left\{\frac{1}{1-\dfrac{\sinh 2b\lambda}{2b\lambda}}[(1+\nu)\, b\lambda\sinh b\lambda + 2\cosh b\lambda] + \\ \frac{3}{\lambda^2 b^2} + \frac{3}{10}(8+5\nu)\right\}\frac{1-\cos\lambda l}{\lambda^2}\cos\lambda x d\lambda + v_*^0. \quad (2.8.11)$$

The constant v_*^0 should be determined by the condition $v_*(x,l) = 0$. The terms introduced into eq. (2.8.11) are removed together with the first two terms of the expansion of the following expression

$$\frac{1}{1-\dfrac{\sinh 2b\lambda}{2b\lambda}}[(1+\nu)\, b\lambda\sinh b\lambda + 2\cosh b\lambda]$$

in the series. This ensures the convergence of integral (2.8.11) for small λ.

Comparing formulae (2.8.2) and (2.8.7) it is easy to conclude that the terms in integral (2.8.11) with the factor $\cos\lambda x \cos\lambda l$ take into account the action of forces $-Q/2$ at points $x = \pm l$. Removing this factor and replacing x by $x - l$ we arrive at the expression

$$v_*(x-\xi,0) = -\frac{Q}{2\pi bE}\int_0^\infty \left\{\frac{1}{1-\dfrac{\sinh 2b\lambda}{2b\lambda}}[(1+\nu)\, b\lambda\sinh b\lambda + 2\cosh b\lambda] + \\ \frac{3}{\lambda^2 b^2} + \frac{3}{10}(8+5\nu)\right\}\frac{1}{\lambda^2}\cos\lambda(x-\xi)\, d\lambda, \quad (2.8.12)$$

determining the correction to the deflection due to the concentrated force Q at point (ξ, b) obtained from the elementary theory.

7.2.9 *Bar with a circular axis loaded on the end faces (Golovin, 1881)*

A circular bar (an arch) bounded by the concentric circles of radii r_0, r_1 $(r_0 < r_1)$ and the parts of the straight lines $\theta = 0, \theta = \theta_0$ respectively on the left and right ends is considered, Fig. 7.2. The surface forces on the lateral sides are absent, i.e.

$$\begin{aligned} r = r_0: \quad \sigma_r = 0, \quad \tau_{r\theta} = 0; \\ r = r_1: \quad \sigma_r = 0, \quad \tau_{r\theta} = 0. \end{aligned} \tag{2.9.1}$$

The forces on the right end are statically equivalent to the longitudinal $P(\theta_0)$ and transverse $Q(\theta_0)$ forces and the bending moment $m^O(\theta_0)$ about the centre O of the circles

$$\theta = \theta_0: \quad \int_{r_0}^{r_1} \sigma_\theta dr = P(\theta_0), \quad \int_{r_0}^{r_1} \tau_{r\theta} dr = Q(\theta_0), \quad \int_{r_0}^{r_1} r\sigma_\theta dr = m^O(\theta_0). \tag{2.9.2}$$

If follows from the equilibrium of the bar that the surface forces distributed over the left end face are statically equivalent to forces $P(0), Q(0)$ and the moment $m^O(0)$ given by the formulae

$$\theta = \theta_0: \quad \left. \begin{aligned} P(0) &= P(\theta_0)\cos\theta_0 + Q(\theta_0)\sin\theta_0, \\ Q(0) &= -P(\theta_0)\sin\theta_0 + Q(\theta_0)\cos\theta_0, \\ m^O(0) &= m^O(\theta_0) = m^O. \end{aligned} \right\} \tag{2.9.3}$$

The tensile and transverse forces are positive if they are caused by the positive normal and shear stresses respectively, and the positive bending moment is due to the positive normal stress σ_θ on the upper side of the bar.

Provided that only the bending moment acts the integral equations of equilibrium can be satisfied by assuming that the stresses in the bar are

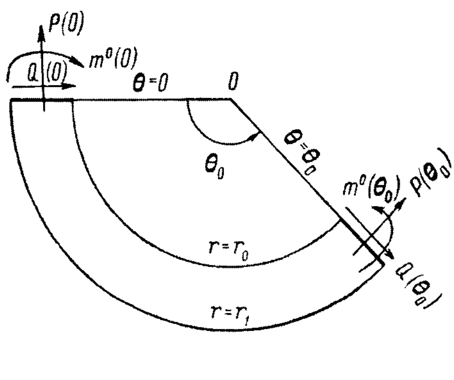

FIGURE 7.2.

independent of θ. The general expression for a biharmonic function which is independent of θ and the product of such a function and r^2 has the form

$$U = A + B\ln r + r^2 (A_1 + B_1 \ln r), \tag{2.9.4}$$

where constant A can be evidently dropped out. Referring to (1.10.2) we can write the boundary conditions (2.9.1) as follows

$$\left.\begin{aligned} \left(\frac{\partial U}{\partial r}\right)_{r=r_1} &= \frac{B}{r_1} + 2A_1 r_1 + B_1 r_1 (2\ln r_1 + 1) = 0, \\ \left(\frac{\partial U}{\partial r}\right)_{r=r_0} &= \frac{B}{r_0} + 2A_1 r_0 + B_1 r_0 (2\ln r_0 + 1) = 0. \end{aligned}\right\} \tag{2.9.5}$$

Conditions (2.9.2) on the end face are given by

$$\begin{aligned} \int_{r_0}^{r_1} \sigma_\theta dr &= \int_{r_0}^{r_1} \frac{\partial^2 U}{\partial r^2} dr = \left(\frac{\partial U}{\partial r}\right)_{r=r_1} - \left(\frac{\partial U}{\partial r}\right)_{r=r_0} = 0, \\ \int_{r_0}^{r_1} r\sigma_\theta dr &= r_1 \left(\frac{\partial U}{\partial r}\right)_{r=r_1} - r_0 \left(\frac{\partial U}{\partial r}\right)_0 - (U_1 - U_0) = m^O. \end{aligned}$$

The first condition needs no attention inasmuch as conditions (2.9.2) hold whereas the second condition yields the third equation

$$B\ln\frac{r_1}{r_0} + A_1 \left(r_1^2 - r_0^2\right) + B_1 \left(r_1^2 \ln r_1 - r_0^2 \ln r_0\right) = m^O. \tag{2.9.6}$$

Three equations (2.9.5), (2.9.6) determine constants B, A_1, B_1. The result is

$$\left.\begin{aligned} \sigma_r &= \frac{4m^O}{N}\left(\frac{r_0^2 r_1^2}{r^2}\ln\frac{r_1}{r_0} + r_1^2 \ln\frac{r}{r_1} - r_0^2 \ln\frac{r}{r_0}\right), \\ \sigma_\theta &= \frac{4m^O}{N}\left[-\frac{r_0^2 r_1^2}{r^2}\ln\frac{r_1}{r_0} + r_1^2 \ln\frac{r}{r_1} - r_0^2 \ln\frac{r}{r_0} + \left(r_1^2 - r_0^2\right)\right], \\ \tau_{r\theta} &= 0, \end{aligned}\right\} \tag{2.9.7}$$

where

$$N = \left(r_1^2 - r_0^2\right)^2 - 4r_1^2 r_0^2 \left(\ln\frac{r_1}{r_0}\right)^2. \tag{2.9.8}$$

The expression for the stress function is set in the form

$$\begin{aligned} U = -\frac{2m^O}{N}\Bigg[& r_1^2 \left(r^2 \ln\frac{r_1}{r} - r_0^2 \ln\frac{r_1}{r_0}\ln r\right) - \\ & r_0^2 \left(r^2 \ln\frac{r_0}{r} - r_1^2 \ln\frac{r_0}{r_1}\ln r\right) + \frac{1}{2}r^2\left(r_1^2 - r_0^2\right)\Bigg]. \end{aligned} \tag{2.9.9}$$

The stress function in the problem of the axial and transverse forces is sought in the form $P(\theta_0)\cos(\theta_0-\theta)F(r)$ and $Q(\theta_0)\sin(\theta_0-\theta)F(r)$, respectively. These products are the biharmonic functions if we take

$$F(r)=\frac{A}{r}+Br^3+Cr\ln r.$$

The stresses are now given by the formulae

$$\left.\begin{array}{l}
\sigma_r=\dfrac{1}{r}\dfrac{\partial U}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial\theta^2}=\left\{\begin{array}{l}P(\theta_0)\cos(\theta_0-\theta)\\ Q(\theta_0)\sin(\theta_0-\theta)\end{array}\right.\dfrac{d}{dr}\dfrac{F(r)}{r},\\
\tau_{r\theta}=\dfrac{1}{r^2}\dfrac{\partial U}{\partial\theta}-\dfrac{1}{r}\dfrac{\partial^2 U}{\partial r\partial\theta}=\left\{\begin{array}{l}-P(\theta_0)\sin(\theta_0-\theta)\\ \quad Q(\theta_0)\cos(\theta_0-\theta)\end{array}\right.\dfrac{d}{dr}\dfrac{F(r)}{r},\\
\sigma_\theta=\dfrac{\partial^2 U}{\partial r^2}=\left\{\begin{array}{l}P(\theta_0)\cos(\theta_0-\theta)\\ Q(\theta_0)\sin(\theta_0-\theta)\end{array}\right.\dfrac{d^2F(r)}{dr^2}.
\end{array}\right\}\qquad(2.9.10)$$

Boundary conditions (2.9.1) are thus satisfied if the following two relations

$$\left(\frac{d}{dr}\frac{F(r)}{r}\right)_{r=r_0}=0,\quad \frac{d}{dr}\left(\frac{F(r)}{r}\right)_{r=r_1}=0\qquad(2.9.11)$$

hold. Then

$$\int_{r_0}^{r_1} r\frac{d^2F}{dr^2}dr=\left(r\frac{dF}{dr}-F\right)\Big|_{r_0}^{r_1}=\left[r^2\frac{d}{dr}\frac{F(r)}{r}\right]\Big|_{r_0}^{r_1},$$

and if conditions (2.9.11) hold, the bending moment m^O is identically equal to zero.

Turning to the remaining conditions (2.9.2) we have

$$\theta=\theta_0:\quad P(\theta_0)=P(\theta_0)\int_{r_0}^{r_1}\frac{d^2F}{dr^2}dr=P(\theta_0)\frac{dF}{dr}\Big|_{r_0}^{r_1},$$

$$\theta=\theta_0:\quad Q(\theta_0)=Q(\theta_0)\int_{r_0}^{r_1}\frac{d}{dr}\frac{F}{r}dr=Q(\theta_0)\frac{F}{r}\Big|_{r_0}^{r_1}=Q(\theta_0)\frac{dF}{dr}\Big|_{r_0}^{r_1},$$

where the latter equality is the result of the boundary conditions (2.9.11). Thus, in both problems the missing third equation is reduced to the same form

$$\frac{1}{r}F(r)\Big|_{r_0}^{r_1}=1.\qquad(2.9.12)$$

From three equations (2.9.11) and (2.9.12) we can determine the constants A, B, C. Function U solving both problems (on axial and transverse forces) is represented as follows

$$U = \frac{1}{2N_1} \left[P(\theta_0) \cos(\theta_0 - \theta) + Q(\theta_0) \sin(\theta_0 - \theta) \right] \times \\ \left(2r \ln r + \frac{r_0^2 r_1^2}{r_0^2 + r_1^2} \frac{1}{r} - \frac{r^3}{r_0^2 + r_1^2} \right),$$

$$N_1 = \ln \frac{r_1}{r_0} - \frac{r_1^2 - r_0^2}{r_1^2 + r_0^2}, \tag{2.9.13}$$

and the stresses take the form

$$\left.\begin{aligned}
\sigma_r &= \frac{1}{N_1} \left[P(\theta_0) \cos(\theta_0 - \theta) + Q(\theta_0) \sin(\theta_0 - \theta) \right] \times \\
&\qquad \left(\frac{1}{r} - \frac{r_0^2 r_1^2}{r_0^2 + r_1^2} \frac{1}{r^3} - \frac{r}{r_0^2 + r_1^2} \right), \\
\sigma_\theta &= \frac{1}{N_1} \left[P(\theta_0) \cos(\theta_0 - \theta) + Q(\theta_0) \sin(\theta_0 - \theta) \right] \times \\
&\qquad \left(\frac{1}{r} + \frac{r_0^2 r_1^2}{r_0^2 + r_1^2} \frac{1}{r^3} - \frac{3r}{r_0^2 + r_1^2} \right), \\
\tau_{r\theta} &= \frac{1}{N_1} \left[-P(\theta_0) \sin(\theta_0 - \theta) + Q(\theta_0) \cos(\theta_0 - \theta) \right] \times \\
&\qquad \left(\frac{1}{r} - \frac{r_0^2 r_1^2}{r_0^2 + r_1^2} \frac{1}{r^3} - \frac{r}{r_0^2 + r_1^2} \right).
\end{aligned}\right\} \tag{2.9.14}$$

The comparison of stresses in Golovin's problem and those in the elementary theory of a curved bar is performed in detail in the book by Timoshenko.

In polar coordinates, the displacement vector, given by eq. (1.7.7), is

$$u_r + i u_\theta = \frac{1}{E} \left[(S + iT) e^{-i\theta} - (1 + \nu) \left(\frac{\partial U}{\partial r} + i \frac{\partial U}{r \partial \theta} \right) \right] + \\ (u_0 + i v_0) e^{-i\theta} + i \omega_0 r, \tag{2.9.15}$$

where $S + iT$ is determined as shown in Subsection 7.1.7.

An evident calculation using these formulae yields: in the first problem

$$\left.\begin{aligned}
E u_r &= -\frac{4m^O}{N} r \left[(1 - \nu) \left(r_1^2 \ln \frac{r_1}{r} - r_0^2 \ln \frac{r_0}{r} \right) + \right. \\
&\qquad \left. r_1^2 - r_0^2 + (1 + \nu) \frac{r_0^2 r_1^2}{r^2} \ln \frac{r_1}{r_0} \right], \\
E u_\theta &= \frac{8m^O}{N} r \theta \left(r_1^2 - r_0^2 \right)
\end{aligned}\right\} \tag{2.9.16}$$

and in the second problem

$$E\left(u_r + iu_\theta\right) = \frac{2}{N_1}\left[\left(P - iQ\right)e^{i(\theta_0-\theta)}\left(\ln r + i\theta\right) - \right.$$
$$\left.\left(P + iQ\right)e^{-i(\theta_0-\theta)}\frac{r^2}{r_1^2 + r_0^2}\right] - \frac{1+\nu}{2N_1}\left[\left(P - iQ\right)e^{i(\theta_0-\theta)}\left(2\ln r + 1 - \right.\right.$$
$$\left.\left.\frac{2r^2}{r_0^2 - r_1^2}\right) + \left(P + iQ\right)e^{-i(\theta_0-\theta)}\left(1 - \frac{r_0^2 r_1^2}{r_0^2 + r_1^2}\frac{1}{r^2} - \frac{r^2}{r_0^2 + r_1^2}\right)\right], \qquad (2.9.17)$$

where $P = P(\theta_0), Q = Q(\theta_0)$ and the terms describing the rigid body displacement of the figure in its plane are omitted.

Clearly, the solutions obtained are rigorous in the framework of Saint-Venant's principle. In general, the state of stress differs from the obtained one in local perturbations in the vicinity of the end faces.

7.2.10 *Loading the circular bar on the surface*

The method of solving the problem of the beam with a straight axis, suggested in Subsections 7.2.3-7.2.8, can be applied to the case of the circular bar. Indeed, writing the Laplace equation in polar coordinates in the form of an "ordinary" differential equation of Euler's type

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + \partial_\theta^2\frac{1}{r^2}F = 0, \quad \partial_\theta = \frac{\partial}{\partial\theta},$$

we can present its solution in the form

$$F = \cos\left(\ln r\partial_\theta\right) f_1(\theta) + \frac{\sin\left(\ln r\partial_\theta\right)}{\partial_\theta} f_2(\theta).$$

By analogy with eq. (2.3.4), this result can be treated as a representation of the series

$$F = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{(2k)!}\left(\ln r\right)^{2k}\partial_\theta^{2k} f_1(\theta) + \frac{(-1)^k}{(2k+1)!}\left(\ln r\right)^{2k+1}\partial_\theta^{2k} f_2(\theta)\right].$$

The biharmonic function is presented by a sum of a harmonic function and the product of a harmonic function and r^2. Introducing, instead of r, a new independent variable t

$$e^t = \frac{r}{r_0} \quad \left(0 \le t \le t_1 = \ln\frac{r_1}{r_0}\right), \qquad (2.10.1)$$

we obtain another representation of the stress function

$$U = \cos t\partial_\theta f_1(\theta) + \frac{\sin t\partial_\theta}{\partial_\theta} f_2(\theta) + e^{2t}\left[\cos t\partial_\theta f_3(\theta) + \frac{\sin t\partial_\theta}{\partial_\theta} f_4(\theta)\right]. \qquad (2.10.2)$$

In what follows, for shortage, we consider only the normal loading of the bar on surface $r = r_1$, that is

$$t = 0: \quad \sigma_r = 0,\ \tau_{r\theta} = 0; \quad t = t_1: \quad \sigma_r = f(\theta),\ \tau_{r\theta} = 0. \tag{2.10.3}$$

It determines functions $f_i(\theta)$ in eq. (2.10.2). We have

$$\sigma_r = \frac{e^{-2t}}{r_0^2}\left(\frac{\partial}{\partial t} + \partial_\theta^2\right)U, \quad \sigma_\theta = \frac{e^{-2t}}{r_0^2}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}\right)U,$$
$$\tau_{r\theta} = -\frac{e^{-2t}}{r_0^2}\left(\frac{\partial}{\partial t} - 1\right)\partial_\theta U. \tag{2.10.4}$$

The boundary conditions on surface $t = 0$ are satisfied if we take

$$U = \left(1 - e^{2t}\right)\left(\cos t\partial_\theta f_1 + \frac{\sin t\partial_\theta}{\partial_\theta} f_2\right) + 2e^{2t}\frac{\sin t\partial_\theta}{\partial_\theta} f_1(\theta). \tag{2.10.5}$$

Then

$$\left.\begin{aligned}
r_0^2\sigma_r = {} & \left[\left(3 - e^{-2t}\right)\partial_\theta \sin t\partial_\theta + 4\frac{\sin t\partial_\theta}{\partial_\theta} - \left(1 - e^{-2t}\right)\partial_\theta^2 \cos t\partial_\theta\right] f_1(\theta) + \\
& \left[\left(e^{-2t} - 1\right)\left(\cos t\partial_\theta + \partial_\theta \sin t\partial_\theta\right) - 2\frac{\sin t\partial_\theta}{\partial_\theta}\right] f_2(\theta), \\
r_0^2\tau_{r\theta} = {} & \left[\left(e^{-2t} - 1\right)\left(\partial_\theta \sin t\partial_\theta + \cos t\partial_\theta\right) - 2\frac{\sin t\partial_\theta}{\partial_\theta}\right] \partial_\theta f_1(\theta) + \\
& \left[\left(e^{-2t} + 1\right)\frac{\sin t\partial_\theta}{\partial_\theta} - \left(e^{-2t} - 1\right)\cos t\partial_\theta\right] \partial_\theta f_2(\theta), \\
r_0^2\sigma_\theta = {} & \left[\left(e^{-2t} + 1\right)\partial_\theta \sin t\partial_\theta + 4\left(\frac{\sin t\partial_\theta}{\partial_\theta} + \cos t\partial_\theta\right) + \right. \\
& \left. \left(1 - e^{2t}\right)\partial_\theta^2 \cos t\partial_\theta\right] f_1(\theta) - \\
& \left[\left(e^{-2t} - 1\right)\left(\cos t\partial_\theta + \partial_\theta \sin t\partial_\theta\right) + 2\left(\frac{\sin t\partial_\theta}{\partial_\theta} + 2\cos t\partial_\theta\right)\right] f_2(\theta), \\
r_0^2\nabla^2 U = {} & 4\left(\partial_\theta \sin t\partial_\theta + 2\frac{\sin t\partial_\theta}{\partial_\theta} + \cos t\partial_\theta\right) f_1(\theta) - \\
& 4\left(\frac{\sin t\partial_\theta}{\partial_\theta} + \cos t\partial_\theta\right) f_2(\theta).
\end{aligned}\right\} \tag{2.10.6}$$

Requiring satisfying the boundary conditions on surface $t = t_1$ $(r = r_1)$ we find $f_1(\theta), f_2(\theta)$. Inserting into the expressions for the stress function leads to the equality

$$U = \left[\left(1 - e^{2t}\right)\left(1 - e^{-2t_1}\right)\partial_\theta^2 \cos(t_1 - t)\partial_\theta + \right.$$
$$2\left(1 + e^{-2(t_1 - t)}\right)\sin t\partial_\theta \sin t_1\partial_\theta - \left(e^{2t} - e^{-2t_1}\right)\partial_\theta \sin(t_1 - t)\partial_\theta +$$
$$\left.\left(1 - e^{-2(t_1 - t)}\right)\partial_\theta \sin(t_1 + t)\partial_\theta\right]\frac{r_0^2 f(\theta)}{\left(1 + \partial_\theta^2\right)\Delta},$$
$$\Delta = 4e^{-2t_1}\sin^2 t_1\partial_\theta - \left(e^{-2t_1} - 1\right)^2\partial_\theta^2$$

which can be rewritten as follows

$$U = \frac{e^{t_1+t}}{\sin^2 t_1\partial_\theta - \partial_\theta^2 \sinh^2 t_1} \Big[\cosh(t_1-t)\sin t\partial_\theta \sin t_1\partial_\theta - \partial_\theta^2 \sinh t \sinh t_1 \cos(t_1-t)\,\partial_\theta - \frac{1}{2}\partial_\theta \sinh(t_1+t)\sin(t_1-t)\,\partial_\theta + \frac{1}{2}\partial_\theta \sinh(t_1-t)\sin(t_1+t)\,\partial_\theta\Big] r_0^2 F(\theta). \quad (2.10.7)$$

Here $F(\theta)$ is given by

$$F(\theta) = \frac{f(\theta)}{1+\partial_\theta^2} = \int_0^\theta f(\xi)\sin(\theta-\xi)\,d\xi + C_1\cos\theta + C_2\sin\theta. \quad (2.10.8)$$

One can easily verify the equalities

$$t=0: \quad U = \frac{\partial U}{\partial t} = 0; \quad t=t_1: \quad U = \frac{\partial U}{\partial t} = r_0 e^{2t_1}\frac{f(\theta)}{1+\partial_\theta^2},$$

confirming, by eq. (2.10.4), that all boundary conditions hold.

When the lateral surfaces are free, that is $f(\theta)=0$, then by eq. (2.10.8)

$$\partial_\theta^{2n} F(\theta) = i^{2n}(C_1\cos\theta + C_2\sin\theta)$$

and, by virtue of eq. (2.10.7), the solution takes the form

$$\begin{aligned} U &= U|_{\partial_\theta\to i} \quad (2.10.9)\\ &= \frac{e^{t+t_1}}{t_1\cosh t_1 - \sinh t_1}\left[t\cosh t_1 - \sinh t\cosh(t_1-t)\right](C_1\cos\theta + C_2\sin\theta), \end{aligned}$$

which is just another form of eq. (2.9.13) determining the stress function in the case of the bar loaded on the end faces.

When the load is a linear function of θ we have

$$f(\theta) = f_0 + f_1\theta, \quad F(\theta) = f_0 + f_1\theta,$$

because the terms in $F(\theta)$ proportional to $\cos\theta, \sin\theta$ determine only the solution of the type (2.10.9) which is omitted in what follows. The solution of the problem is obtained by keeping only the free term in the expansion of U in the power series in ∂_θ^2

$$U = r_0^2\frac{e^{t+t_1}}{t_1^2 - \sinh^2 t_1}\Big[tt_1\cosh(t_1-t) - \sinh t\sinh t_1 - \frac{1}{2}(t_1-t)\sinh(t_1-t) + \frac{1}{2}(t_1+t)\sinh(t_1-t)\Big](f_0+f_1\theta). \quad (2.10.10)$$

Obtaining the solution for the case of the load

$$f(\theta) = f_n \cos n\theta + g_n \sin n\theta, \quad F(\theta) = \frac{f_n \cos n\theta + g_n \sin n\theta}{1 - n^2} \qquad (n \neq \pm 1)$$

presents no problem. As explained in Subsection 7.2.7, it is sufficient to replace operator ∂_θ by $\pm ni$. Then

$$U = U|_{\partial_\theta \to \pm ni} = r_0^2 \frac{e^{t+t_1}}{n^2 \cosh^2 t_1 - \sinh^2 nt_1} \left[\cosh(t_1 - t) \sinh nt \sinh nt_1 \right.$$
$$- n^2 \cosh n(t_1 - t) \sinh t \sinh t_1 - \frac{1}{2} n \sinh(t_1 + t) \sinh n(t_1 - t) +$$
$$\left. \frac{1}{2} n \sinh(t_1 - t) \sinh n(t_1 + t) \right] \frac{f_n \cos n\theta + g_n \sin n\theta}{n^2 - 1}. \qquad (2.10.11)$$

7.2.11 Cosinusoidal load

The "resonant" case $n = 1$ excluded above from the consideration requires special attention. We have

$$f(\theta) = f_0 \cos\theta, \quad F(\theta) = \frac{1}{2} f_0 \theta \sin\theta.$$

Now

$$\partial_\theta^2 F = -\frac{1}{2} (\theta \sin\theta - 2\cos\theta) f_0, \quad \partial_\theta^4 F = \frac{1}{2} (\theta \sin\theta - 4\cos\theta) f_0$$

and, in general,

$$\partial_\theta^2 F = (-1)^k \frac{1}{2} (\theta \sin\theta - 2k \cos\theta) f_0.$$

Denoting therefore the multiplier-operator associated with $F(\theta)$ in eq. (2.10.7) by $\Phi\left(t, \partial_\theta^2\right)$ and putting it in the form

$$\Phi\left(t, \partial_\theta^2\right) = \Phi_0(t) + \Phi_1(t) \partial_\theta^2 + \Phi_2(t) \partial_\theta^4 + \ldots \qquad (2.11.1)$$

we have

$$U = \Phi\left(t, \partial_\theta^2\right) F(\theta) = \frac{1}{2} f_0 \left[\Phi_0(t) - \Phi_1(t) + \Phi_2(t) - \ldots\right] \theta \sin\theta +$$
$$\frac{1}{2} f_0 \left[2\Phi_1(t) - 4\Phi_2(t) + 6\Phi_3(t) - \ldots\right] \cos\theta.$$

Referring to eq. (2.11.1) we can write the latter result in the form

$$U = \frac{1}{2} f_0 \left[\Phi\left(t, \partial_\theta^2\right) \theta \sin\theta + \frac{1}{\partial_\theta} \frac{\partial \Phi\left(t, \partial_\theta^2\right)}{\partial \partial_\theta} \cos\theta \right]_{\partial_\theta \to i}. \qquad (2.11.2)$$

The result of the calculation by means of this formula is presented by a sum of three terms

$$U_1 = r_0^2 f_0 \frac{e^{t+t_1}}{2(t_1 \cosh t_1 - \sinh t_1)} [t_1 \cosh t_1 - \sinh t \cosh(t_1 - t)] \theta \sin\theta, \tag{2.11.3}$$

$$\left.\begin{aligned} U_2 = -r_0^2 f_0 &\frac{e^{t+t_1}}{4 \sinh t_1 (t_1 \cosh t_1 - \sinh t_1)} [t t_1 \cosh 2t_1 - \\ &\frac{1}{2} t_1 \sinh 2t + \frac{1}{2} t \sinh 2t_1 - \sinh t \sinh t_1 \cosh(t_1 - t) - \\ &2(t_1 - t) \sinh t \sinh t_1 \sinh(t_1 - t)] \cos\theta, \\ U_3 = r_0^2 f_0 &\frac{t_1^2 \cosh 2t_1 - \sinh^2 t_1}{2 \sinh t_1 (t_1 \cosh t_1 - \sinh t_1)^2} \times \\ &e^{t+t_1} [t \cosh t_1 - \sinh t \cosh(t_1 - t)] \cos\theta, \end{aligned}\right\} \tag{2.11.4}$$

where U_3 determines the loading on the end faces of the bar and can be omitted, see eq. (2.10.9).

Term U_1 satisfies all boundary conditions which can be easily proved by means of eq. (2.10.4)

$$\begin{aligned} \sigma_r^1 &= f_0 \frac{e^{t_1 - t}}{t_1 \cosh t_1 - \sinh t_1} \Big\{ [t \cosh t_1 - \sinh t \cosh(t_1 - t)] \cos\theta + \\ &\qquad \frac{1}{2} [\cosh t_1 - \cosh(2t - t_1)] \theta \sin\theta \Big\}, \\ \tau_{r\theta}^1 &= -f_0 \frac{e^{t_1 - t}}{t_1 \cosh t_1 - \sinh t_1} [\cosh t_1 - \cosh(2t - t_1)] \frac{1}{2} (\sin\theta + \theta\cos\theta). \end{aligned}$$

This is however not the solution to the problem as U_1 is not a biharmonic function. Function U_2 is not biharmonic as well, and the stresses $\sigma_r^{(2)}, \tau_{r\theta}^{(2)}$ obtained in terms of this function are as follows

$$\left.\begin{matrix} \sigma_r^{(2)} \\ \tau_{r\theta}^{(2)} \end{matrix}\right\} = -f_0 \frac{e^{t_1 - t}}{2 \sinh t_1 (t_1 \cosh t_1 - \sinh t_1)} [t_1 \sinh(t_1 + t) \sinh(t_1 - t)$$
$$- (t_1 - t) \sinh t_1 \sinh(t_1 - 2t) + 2 \sinh t_1 \sinh t \sinh(t_1 - t)] \left\{ \begin{matrix} \cos\theta, \\ \sin\theta. \end{matrix} \right.$$

The solution is the sum

$$U = U_1 + U_2, \tag{2.11.5}$$

and it can be checked easily by direct calculation that U is a biharmonic function.

7.2.12 Homogeneous solutions

A refinement of the solutions to the problems on a rectangular strip and a circular bar which are based on Saint-Venant's principle can be achieved by imposing the "homogeneous" solutions, i.e. the solutions corresponding to the free longitudinal sides of the strip $y = \pm b$ (or free lateral surfaces $r = r_0, r = r_1$ of the beam). Such solutions have already been used in the problem of the circular cylinder, Subsection 5.7.8, for refining the boundary conditions on the end faces. A similar construction is performed here for the rectangular strip and it can also be repeated for a circular bar.

Taking the stress function of the sort

$$\left.\begin{array}{ll} \text{A} & U(x,y) = \cos y\partial f_0(x) + \dfrac{1}{2} y \dfrac{\sin y\partial}{\partial}\left[f_0''(x) + \partial^2 f_0(x)\right], \\ \text{B} & U(x,y) = \dfrac{\sin y\partial}{\partial} f_0'(x) + \\ & \dfrac{1}{2\partial^2}\left(\dfrac{\sin y\partial}{\partial} - y\cos y\partial\right)\left[f_0'''(x) + \partial^2 f_0'(x)\right] \end{array}\right\} \tag{2.12.1}$$

and requiring the stresses σ_y, τ_{xy} to be zero on the edges $y = \pm b$ of the strip we arrive at the system of equations

$$\left.\begin{array}{ll} \text{A} & \dfrac{1}{2} b\partial \sin b\partial \left(f_0'' + \partial^2 f_0\right) + \cos b\partial\partial^2 f_0 = 0, \\ & \dfrac{1}{2}\left(\dfrac{\sin b\partial}{b\partial} + \cos b\partial\right)\left(f_0'' + \partial^2 f_0\right) + \dfrac{\sin b\partial}{b\partial}\partial^2 f_0 = 0, \\ \text{B} & \dfrac{1}{2}\left(\dfrac{\sin b\partial}{\partial} - b\cos b\partial\right)\left(f_0''' + \partial^2 f_0'\right) + \partial \sin b\partial f_0' = 0, \\ & \dfrac{1}{2} b \sin b\partial \left(f_0''' + \partial^2 f_0'\right) + \partial \cos b\partial f_0' = 0. \end{array}\right\} \tag{2.12.2}$$

In what follows we consider the exponential prescribing functions f_0, f_0', f_0'' and f_0''' which implies that the differentiation is equivalent to the multiplication by a constant factor

$$\frac{d}{dx} e^{kx} = \partial e^{kx} = k e^{kx}, \quad \frac{d^n}{dx^n} e^{kx} = \partial^n e^{kx} = k^n e^{kx}.$$

In the system of equations (2.12.1) one should replace ∂ by k and take the functions

$$f_0, \quad f_0'' + \partial^2 f_0; \quad f_0', \quad f_0''' + k^2 f_0' \tag{2.12.3}$$

to be proportional to e^{kx}. The nontrivial solutions of these systems exists for the values of parameter $bk = \gamma$ for which their determinant vanishes. This leads to the transcendental equations

$$\left.\begin{array}{ll} \text{A} & \Delta_A(2\gamma) = 2\gamma + \sin 2\gamma = 0; \\ \text{B} & \Delta_B(2\gamma) = 2\gamma - \sin 2\gamma = 0 \end{array}\right\} \tag{2.12.4}$$

and enables one to represent expression (2.12.3) in the form

$$
\begin{aligned}
f_0 &= -b^2 \frac{\sin\gamma}{\gamma} e^{\gamma x/b}, & f_0'' + k^2 f_0 &= 2\cos\gamma e^{\gamma x/b}, \\
f_0' &= -b\left(\frac{\sin\gamma}{\gamma} - \cos\gamma\right) e^{\gamma x/b}, & f_0''' + k^2 f_0 &= 2\frac{\gamma}{b}\sin\gamma e^{\gamma x/b}.
\end{aligned}
$$

Thus we have arrived at the following stress functions

$$
\left.
\begin{aligned}
&\mathrm{A} \quad U(x,y) = \left(-\frac{\sin\gamma}{\gamma}\cos\gamma\eta + \eta\cos\gamma\frac{\sin\gamma\eta}{\gamma}\right) b^2 e^{\gamma\xi}, \\
&\mathrm{B} \quad U(x,y) = \left(\cos\gamma\frac{\sin\gamma n}{\gamma} - \eta\frac{\sin\gamma}{\gamma}\cos\gamma\eta\right) b^2 e^{\gamma\xi},
\end{aligned}
\right\} \qquad (2.12.5)
$$

where $b\eta = y, b\xi = x$. These stress functions ensure that the edges $y = \pm b$ are free of load and now it is easy to obtain the formulae for the stresses. The system of stresses in any cross-section $\xi = \text{const}$ of the bar are statically equivalent to zero. It is sufficient to check that in problem A the longitudinal force vanishes

$$
\int_{-b}^{b} \sigma_x dy = \frac{1}{b}\int_{-b}^{b} \frac{\partial^2 U}{\partial\eta^2} d\eta = \frac{1}{b}\frac{\partial U}{\partial\eta}\bigg|_{-1}^{1} = \frac{b}{\gamma} e^{\gamma\xi} \Delta_A(2\gamma) = 0,
$$

and in problem B the transverse force and the bending moment are zero

$$
\begin{aligned}
\int_{-b}^{b} \tau_{xy} dy &= -\frac{1}{b}\int_{-1}^{1} \frac{\partial^2 U}{\partial\xi\partial\eta} d\eta = -\frac{1}{b}\frac{\partial U}{\partial\xi}\bigg|_{-1}^{1} = 0, \\
\int_{-b}^{b} y\sigma_x dy &= -\frac{1}{b}\int_{-1}^{1} \eta\frac{\partial^2 U}{\partial\eta^2} d\eta = \left(\eta\frac{\partial U}{\partial\eta} - U\right)\bigg|_{-1}^{1} = 0.
\end{aligned}
$$

All roots of equation (2.12.4), except the trivial ones, are complex-valued. They lie in four quadrants of the complex plane γ and are symmetric about the origin of the coordinate system, i.e. if γ is a root then $-\gamma, \pm\bar{\gamma}$ are also the roots. The stress functions given by formulae (2.12.5) are complex-valued, however they can be utilised for constructing the real-valued stress functions. In this way we arrive at the homogeneous solutions which are statically equivalent to zero and ensure that the longitudinal sides of the strip are free.

Table 7.1 displays twice the double values of the first five roots of eq. (2.12.4) lying in the first quadrant of plane γ, $x_k = \alpha_k + i\beta_k = 2\gamma_k$. The real part of the roots increase rapidly with root number, hence the stresses proportional to $e^{\frac{1}{2}\alpha\xi}$ decrease very rapidly with the distance from edge of the strip. Therefore it is necessary to take the roots of the second and first quadrants for constructing the solutions for edges $x = 0$ and $x = l$ respectively. This substantiates the admissibility of Saint-Venant's principle for the narrow and long strip.

The roots of equation $\Delta_A(x) = x + \sin x = 0$	The roots of equation $\Delta_B(x) = x - \sin x = 0$
$\alpha_1 = 4,21239$ $\beta_1 = 2,25072$	$\alpha_1 = 7,45761$ $\beta_1 = 13,89995$
$\alpha_2 = 10,74253$ $\beta_2 = 3,10314$	$\alpha_2 = 3,35220$ $\beta_2 = 20,23871$
$\alpha_3 = 17,07336$ $\beta_3 = 3,55108$	$\alpha_3 = 3,35220$ $\beta_3 = 3,71676$
$\alpha_4 = 23,39835$ $\beta_4 = 3,85880$	$\alpha_4 = 26,55454$ $\beta_4 = 3,98314$
$\alpha_5 = 29,70811$ $\beta_5 = 4,09337$	$\alpha_5 = 32,85974$ $\beta_5 = 4,19325$

Table 7.1

Several methods of using the homogeneous solutions for satisfying the boundary conditions on the transverse sides of the strip were suggested. None of these yields a rigorous solution to this problem, that is, the solution does not exactly satisfy the boundary conditions on each end face. The simplest way if satisfying the boundary conditions "on average" is described in Subsection 5.7.9. Satisfying the boundary conditions in some, *a priori* taken points is made difficult as the homogeneous solutions have alternating signs, namely, the higher the number of the root, the more frequently the sign alters for the corresponding solution.

7.3 Elastic plane and half-plane

7.3.1 Concentrated force and concentrated moment in elastic plane

A concentrated force with projections X, Y on the coordinate axes is assumed to be applied at the origin of the coordinate system. The state of stress in the unbounded plane due to this force is sought, therefore the issue is the construction of an analogue of Kelvin-Somigliana's tensor (Subsection 4.3.5) in the plane problem. Using the complex variable, i.e. formulae (1.14.7) and (1.14.5), is the shortest way to the goal.

The principal vector of stresses on any closed contour C enclosing point $z = 0$ must evidently be in equilibrium with the applied force. This reasoning leads to the first condition of the problem

$$\Delta_C (P + iQ) + X + iY = 0, \tag{3.1.1}$$

where here and in what follows the symbol $\Delta_C f$ denotes the change in function f under the traversing closed contour in such a way that the region enclosed by the contour lies on the left. It is evident that $\Delta_C f = 0$ if function f is single-valued. In the problem considered the displacement vector is required to be single-valued, i.e.

$$\Delta_C (u + iv) = 0. \tag{3.1.2}$$

As will be seen later, these conditions are sufficient. When they hold, it guarantees that the stresses are single-valued and the principal moment m^O of the stresses on C vanishes.

By virtue of eq. (1.4.7) we have

$$\Delta_C (P + iQ) = -i\Delta_C \left[\varphi (z) + z\bar{\varphi}' (\bar{z}) + \bar{\psi} (\bar{z})\right]. \tag{3.1.3}$$

It is easy to comprehend that for functions of the type

$$\varphi (z) = (\alpha + i\beta) \ln z, \quad \psi (z) = \left(\alpha' + i\beta'\right) \ln z, \tag{3.1.4}$$

where $(\alpha + i\beta)$ and $\left(\alpha' + i\beta'\right)$ are constant, this value has a constant value which is independent of the choice of contour C enclosing the coordinate origin.

Indeed, then

$$\Delta_C\varphi (z) = 2\pi i (\alpha + i\beta), \quad \Delta_C\bar{\varphi}' (\bar{z}) = 0, \quad \Delta_C\bar{\psi} (\bar{z}) = -2\pi i \left(\alpha' - i\beta'\right) \tag{3.1.5}$$

and, by virtue of eqs. (3.1.1), (3.12), (1.14.5), we arrive at the system of equations

$$\left.\begin{array}{l} -i \left[2\pi i (\alpha + i\beta) - 2\pi i \left(\alpha' - i\beta'\right)\right] + X + iY = 0, \\ 2\pi i (3 - 4\nu) (\alpha + i\beta) + 2\pi i \left(\alpha' - i\beta'\right) = 0. \end{array}\right\} \tag{3.1.6}$$

If follows from this system that

$$\alpha + i\beta = -\frac{X + iY}{8\pi (1 - \nu)}, \quad \alpha' - i\beta' = (3 - 4\nu) \frac{X + iY}{8\pi (1 - \nu)}, \tag{3.1.7}$$

thus,

$$\varphi (z) = -\frac{X + iY}{8\pi (1 - \nu)} \ln z, \quad \psi (z) = (3 - 4\nu) \frac{X - iY}{8\pi (1 - \nu)} \ln z. \tag{3.1.8}$$

The stresses expressed in terms of functions $\Phi(z), \Psi(z)$ by formulae (1.14.9) are single-valued and are given by

$$\sigma_r + \sigma_\theta = -\frac{X\cos\theta + Y\sin\theta}{2\pi(1-\nu)r}, \quad \sigma_\theta - \sigma_r + 2i\tau_{r\theta} =$$
$$= \frac{1}{\pi r}(X\cos\theta + Y\sin\theta) + i\frac{1-2\nu}{2\pi(1-\nu)r}(X\sin\theta - Y\cos\theta). \quad (3.1.9)$$

With accuracy up to an immaterial term, linear in z, we have

$$U = \frac{1}{16\pi(1-\nu)}\left[-(X+iY)\bar{z}\ln z - (X-iY)z\ln\bar{z} + \right.$$
$$\left.(3-4\nu)(X-iY)z\ln z + (3-4\nu)(X+iY)\bar{z}\ln\bar{z}\right], \quad (3.1.10)$$

and one can easily prove the single-valuedness of expression (1.13.6)

$$\Delta_C\left[U - \left(z\frac{\partial U}{\partial z} + \bar{z}\frac{\partial U}{\partial \bar{z}}\right)\right] = \Delta_C m^O = 0. \quad (3.1.11)$$

The displacement vector is determined by the equality

$$2\mu(u+iv) = \frac{1}{4\pi(1-\nu)}\left[-(3-4\nu)(X+iY)\ln r + \frac{1}{2}(X-iY)e^{2i\theta}\right]. \quad (3.1.12)$$

By analogy, one can consider the action of a concentrated moment. Condition (3.1.2) is retained whilst the static equation (3.1.1) is replaced by the following equation

$$M^O + m^O = M^O +$$
$$\Delta_C\frac{1}{2}\left[\chi(z) + \bar{\chi}(\bar{z}) - z\chi'(z) - \bar{z}\bar{\chi}'(\bar{z}) - z\bar{z}\varphi'(z) - z\bar{z}\bar{\varphi}'(\bar{z})\right] = 0. \quad (3.1.13)$$

Here M^O denotes the moment concentrated at the origin of the elastic plane and m^O is the principal moment of the stresses on any closed contour C enclosing point $z = 0$.

Equation (3.1.3) can be satisfied by assuming

$$\chi(z) = (\alpha'' + i\beta'')\ln z, \quad \psi(z) = \chi'(z) = \frac{\alpha'' + i\beta''}{z}, \quad \varphi(z) = 0.$$

Condition (3.1.2) then holds as $u + iv$ is a single-valued function of the coordinates. We obtain

$$M^O + \frac{1}{2}2\pi i\left(\alpha'' + i\beta'' - \alpha'' + i\beta''\right) = 0,$$

that is α'' remains undetermined and $\beta'' = \frac{1}{2\pi}M^O$. Finally

$$\chi = \frac{iM^O}{2\pi}\ln z, \quad \psi(z) = \frac{iM^O}{2\pi z}, \quad \Psi(z) = -\frac{iM^O}{2\pi z^2}. \quad (3.1.14)$$

7.3.2 Flamant's problem (1892)

The action of a concentrated force normal to the border $y = 0$ of the elastic half-plane $y > 0$ is considered. This problem is analogous to Boussinesq's problem (Subsection 5.2.2) for the half-plane.

Axis Oy is directed into the half-plane and the force K_2 is applied at the origin of the coordinate system Oxy and has the direction of axis Oy. The point $x = -\infty, y = 0$ is taken as the starting point for traversing the border $y = 0$, so that the region $y > 0$ lies on the left. According to formulae (1.8.4), (1.8.6) the boundary conditions can be set in the form

$$y = 0: \quad \frac{\partial U}{\partial y} = 0, \quad U = m^M = \begin{cases} 0, & x < 0, \\ -xK_2, & x > 0. \end{cases} \tag{3.2.1}$$

The biharmonic stress function is naturally sought in the form

$$U = f_1(x, y) + y f_2(x, y), \tag{3.2.2}$$

where f_1, f_2 are functions harmonic in the half-plane $y > 0$, see also eq. (1.11.1). By the first condition (3.2.1) we have

$$y = 0: \quad \frac{\partial U}{\partial y} = \frac{\partial f_1}{\partial y} + f_2 = 0, \quad \frac{\partial f_1}{\partial y} = -f_2. \tag{3.2.3}$$

If the boundary values of two biharmonic functions are equal to each other, these functions are then equal in the whole region. This means that

$$y \geq 0: \quad \frac{\partial f_1}{\partial y} = -f_2, \quad U(x, y) = f_1(x, y) - y \frac{\partial f_1(x, y)}{\partial y}. \tag{3.2.4}$$

Here we have obtained a general representation for the stress function in the half-plane $y > 0$ if there is no load tangent to the border $(y = 0 : \tau_{xy} = 0)$.

Proceeding to construction of the stress function of the type of (3.2.4), let us consider the harmonic function

$$1 - \frac{1}{\pi} \arctan \frac{y}{x} = 1 - \frac{\theta}{\pi} = \operatorname{Re}\left(1 - \frac{1}{\pi i} \ln z\right) = \operatorname{Re} \Phi_0(z).$$

On the border $y = 0$ of the region $y > 0$ it takes the form

$$y = 0: \quad 1 - \frac{\theta}{\pi} = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

and now it is easy to verify that the harmonic function

$$-K_2 \operatorname{Re} z\Phi_0(z) = -K_2 \left[x\left(1 - \frac{\theta}{\pi}\right) - y\frac{\ln r}{\pi}\right] = f_1(x, y) \tag{3.2.5}$$

satisfies the boundary condition

$$f_1(x, 0) = \begin{cases} 0, & x < 0, \\ -K_2 x, & x > 0, \end{cases}$$

which is required due to eq. (3.2.1). By eq. (3.2.4) we find

$$U = f_1(x, y) - y\frac{\partial f_1}{\partial y} = -K_2 x\left(1 - \frac{\theta}{\pi}\right) - \frac{K_2}{\pi}y.$$

The terms which are linear in coordinates x, y are immaterial and can be neglected. This allows us to present the solution in the form

$$U = \frac{1}{\pi}K_2 x\theta = \frac{1}{\pi}K_2 r\theta\cos\theta. \tag{3.2.6}$$

The stresses are easily obtained by formulae (1.10.2). The only non-vanishing components are

$$\sigma_r = -\frac{2K_2}{\pi}\frac{\sin\theta}{r} = -\frac{2K_2}{\pi}\frac{\cos\psi}{r} \quad \left(\psi = \frac{\pi}{2} - \theta,\ -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\right), \tag{3.2.7}$$

where ψ denotes the angle measured from axis Oy (the direction of the force) to axis Ox. Therefore, on the planes perpendicular to the position vector $\mathbf{r}$ there are only compressive normal stresses whereas on the planes along $\mathbf{r}$ there are no stresses at all. The stress at the point where the force is applied is infinitely great and it is explained by the fact that the concentrated force is thought as a limiting case of the force distributed over a small area.

The lines of the equal normal stresses are the curves

$$\sigma_r = -\frac{2K_2}{\pi r}\cos\psi = \sigma_r^0 = \text{const}, \quad r = d\cos\psi \quad \left(d = -\frac{2K_2}{\pi\sigma_r^0}\right). \tag{3.2.8}$$

They are the circles of diameter d tangent to the border at the point where the force is applied, Fig. 7.3. The maximum shear stress is known to be equal to half the difference in the principal normal stresses, so that in the plane stress case ($\sigma_z = 0$)

$$\tau_{\max} = \frac{1}{2}\sigma_r.$$

Therefore, the lines $\sigma_r = \text{const}$ are simultaneously the lines $\tau_{\max} = \text{const}$. The optical method allows one to observe and photograph the lines $\tau_{\max} = \text{const}$ in thin transparent stressed models. Near the points of applying the concentrated forces these lines are circles, indeed.

The components of the stress tensor in the Cartesian coordinate system are given by the equalities

$$\left.\begin{array}{l} \sigma_x = \sigma_r\cos^2\theta = -\dfrac{2K_2}{\pi}\dfrac{yx^2}{r^4}, \quad \sigma_y = \sigma_r\sin^2\theta = -\dfrac{2K_2}{\pi}\dfrac{y^3}{r^4}, \\ \tau_{xy} = -\dfrac{2K_2}{\pi}\dfrac{xy^2}{r^4}. \end{array}\right\} \tag{3.2.9}$$

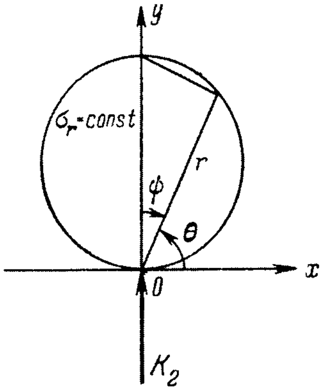

FIGURE 7.3.

For the plane stress, the displacement vector obtained in terms of the stress function (3.2.6) with the help of eqs. (1.7.7), (1.7.2) has the following projections

$$\left.\begin{aligned} u &= \frac{K_2}{\pi E}\left[(1-\nu)\,\theta + \frac{1}{2}(1+\nu)\sin 2\theta\right] + u_0 - \omega_0 y, \\ v &= -\frac{K_2}{\pi E}\left[2\ln r + (1+\nu)\cos^2\theta\right] + v_0 + \omega_0 x. \end{aligned}\right\} \tag{3.2.10}$$

In the problem of plane strain the displacements are as follows

$$\left.\begin{aligned} u &= \frac{K_2}{2\mu\pi}\left[(1-2\nu)\,\theta + \frac{1}{2}\sin 2\theta\right] + u_0 - \omega_0 y, \\ v &= -\frac{K_2}{2\mu\pi}\left[2(1-\nu)\ln r + \cos^2\theta\right] + v_0 + \omega_0 x. \end{aligned}\right\} \tag{3.2.11}$$

In particular, on axis x we have $r = |x|$ and omitting the constant term and the displacement due to rotation we obtain

$$v(x,0) = K_2\beta \ln\frac{1}{|x|}, \tag{3.2.12}$$

where

$$\beta = \frac{2}{\pi E}$$

in the problem of plane stress and

$$\beta = \frac{1-\nu}{\pi\mu}$$

in the case of plane strain.

7.3.3 General case of normal loading

The solution of Subsection 7.3.2 is easily generalised to the case of any normal loading on the surface of the elastic half-plane

$$y = 0: \quad \sigma_y = -q(x), \quad \tau_{xy} = 0. \tag{3.3.1}$$

Indeed, the state of stress due to a unit force concentrated not at the coordinate origin but at point $x = \xi$ is determined by the stress function

$$U = \frac{x-\xi}{\pi} \arctan \frac{y}{x-\xi},$$

see eq. (3.2.6). Summing the actions of the elementary loads $q(\xi)\, d\xi$ over the $[x_0, x_1]$ of the boundary we arrive at the following expression for the stress function

$$U = \frac{1}{\pi} \int_{x_0}^{x_1} (x-\xi) \arctan \frac{y}{x-\xi} q(\xi)\, d\xi = \frac{1}{\pi} \int_{x_0}^{x_1} (x-\xi)\, \theta_\xi q(\xi)\, d\xi. \tag{3.3.2}$$

In particular, let $q(\xi) = q_0 = \text{const}$, then placing the coordinate origin in the middle of the load we have

$$U = \frac{q_0}{\pi} \int_{-a}^{a} (x-\xi)\, \theta_\xi d\xi.$$

Let $\mathbf{r}$ denote the position vector of the point of observation $M(x, y)$ with the origin at the point $(\xi, 0)$ and $\mathbf{r}_0, \mathbf{r}_1$ denote the position vector of M for $\xi = \pm a$, that is, their origins lie at the ends of the load. Then, see Fig. 7.4

$$x - \xi = r\cos\theta_\xi, \qquad y = r\sin\theta_\xi,$$
$$x - a = r_1 \cos\theta_1, \quad x + a = r_0 \cos\theta_0$$

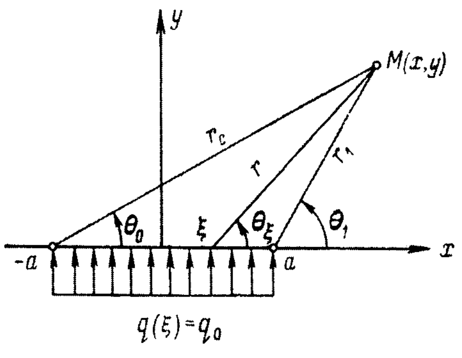

FIGURE 7.4.

and further

$$x - \xi = y \cot \theta_\xi, \quad d\xi = -y d \cot \theta_\xi,$$
$$U = -\frac{q_0 y^2}{2\pi} \int_{\theta_0}^{\theta_1} \theta_\xi d \cot^2 \theta_\xi.$$

Integrating by parts we find

$$U = -\frac{q_0 y^2}{2\pi} \left[\theta_1 \left(\cot^2 \theta_1 + 1\right) - \theta_0 \left(\cot^2 \theta_0 + 1\right) - \left(\cot \theta_1 - \cot \theta_0\right)\right],$$

and after a rearrangement we have

$$U = -\frac{q_0}{2\pi} \left(r_1^2 \theta_1 - r_0^2 \theta_0\right) + \frac{q_0}{\pi} a y, \tag{3.3.3}$$

where the term linear in y can be omitted.

The state of stress can be determined by the sum of states calculated by formulae (1.10.2) in the system of polar coordinates with the centres at the ends of the load. Denoting the system of unit vectors for these coordinate systems as $\{\mathbf{e}_r^0, \mathbf{e}_\theta^0\}$, $\{\mathbf{e}_r^1, \mathbf{e}_\theta^1\}$ we have

$$\hat{T} = -\frac{q_0}{\pi} \left[\left(\mathbf{e}_r^1 \mathbf{e}_r^1 + \mathbf{e}_\theta^1 \mathbf{e}_\theta^1\right) \theta_1 - \left(\mathbf{e}_r^0 \mathbf{e}_r^0 + \mathbf{e}_\theta^0 \mathbf{e}_\theta^0\right) \theta_0\right] + \frac{q_0}{2\pi} \left[\left(\mathbf{e}_r^1 \mathbf{e}_\theta^1 + \mathbf{e}_\theta^1 \mathbf{e}_r^1\right) - \left(\mathbf{e}_r^0 \mathbf{e}_\theta^0 + \mathbf{e}_\theta^0 \mathbf{e}_r^0\right)\right].$$

The relation between these systems is given by the evident equalities

$$\mathbf{e}_r^1 = \mathbf{e}_r^0 \cos\alpha + \mathbf{e}_\theta^0 \sin\alpha, \quad \mathbf{e}_\theta^1 = -\mathbf{e}_r^0 \sin\alpha + \mathbf{e}_\theta^0 \cos\alpha, \quad \alpha = \theta_1 - \theta_0.$$

Then we obtain

$$\mathbf{e}_r^1 \mathbf{e}_r^1 + \mathbf{e}_\theta^1 \mathbf{e}_\theta^1 = \mathbf{e}_r^0 \mathbf{e}_r^0 + \mathbf{e}_\theta^0 \mathbf{e}_\theta^0,$$
$$\mathbf{e}_r^1 \mathbf{e}_\theta^1 + \mathbf{e}_\theta^1 \mathbf{e}_r^1 = \left(\mathbf{e}_\theta^0 \mathbf{e}_\theta^0 - \mathbf{e}_r^0 \mathbf{e}_r^0\right) \sin 2\alpha + \left(\mathbf{e}_r^0 \mathbf{e}_\theta^0 + \mathbf{e}_\theta^0 \mathbf{e}_r^0\right) \cos 2\alpha,$$

and the expression for the stress tensor in the system $\{\mathbf{e}_r^0, \mathbf{e}_\theta^0\}$ as follows

$$\hat{T} = -\frac{q_0}{\pi} \left[\frac{1}{2} (2\alpha + \sin 2\alpha) \mathbf{e}_r^0 \mathbf{e}_r^0 + \frac{1}{2} (2\alpha - \sin 2\alpha) \mathbf{e}_\theta^0 \mathbf{e}_\theta^0 + \left(\mathbf{e}_r^0 \mathbf{e}_\theta^0 + \mathbf{e}_\theta^0 \mathbf{e}_r^0\right) \sin^2 \alpha\right].$$

The principal stresses are determined by eq. (2.1.5) of Chapter 1

$$\begin{vmatrix} -\frac{q_0}{2\pi} (2\alpha + \sin 2\alpha) - \sigma & -\frac{q_0}{\pi} \sin^2 \alpha \\ -\frac{q_0}{\pi} \sin^2 \alpha & -\frac{q_0}{2\pi} (2\alpha - \sin 2\alpha) - \sigma \end{vmatrix} = 0$$

and are equal to

$$\sigma_1 = -\frac{q_0}{\pi}(\alpha + \sin\alpha), \quad \sigma_2 = -\frac{q_0}{\pi}(\alpha - \sin\alpha),$$

and the principal axes are directed at angles $\frac{1}{2}\alpha$ and $\frac{1}{2}(\pi + \alpha)$ to axis Ox.

On axis x (the boundary of the half-plane) we have: $\alpha = 0$ outside and $\alpha = \pi$ inside the range of the load. This is confirmed by the absence of shear stresses $\tau_{r\theta_0} = \tau_{xy}$ on the whole boundary. The normal stresses σ_x, σ_y on the boundary are equal to each other; they vanish outside and are equal to $-q_0$ inside the range of the load.

7.3.4 Loading by a force directed along the boundary

In this case the boundary conditions are set by means of eqs. (1.8.4), (1.8.5) in the form

$$y = 0: \quad U = 0, \quad \frac{\partial U}{\partial y} = \int_{-\infty}^{x} p_0(x)\,dx = -\int_{-\infty}^{x} \tau_{xy} dx = P(x). \qquad (3.4.1)$$

Turning to eq. (3.2.2) we find that harmonic function f_1 is equal to zero on the boundary, thus, it identically vanishes in the region $y > 0$. Harmonic function f_2 is determined by the solution of Dirichlet's problem for the half-plane

$$y = 0: \quad f_2(x, 0) = P(x),$$

having the form

$$f_2(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{p(\xi)\,d\xi}{(x-\xi)^2 + y^2}. \qquad (3.4.2)$$

In the case of a force K_1 concentrated at the origin of the coordinate system we have

$$P(x) = \begin{cases} 0, & x < 0, \\ K_1, & x > 0, \end{cases}$$

and by virtue of eq. (3.4.2)

$$f_2(x, y) = \frac{K_1}{\pi}\int_0^{\infty} \frac{y\,d\xi}{(x-\xi)^2 + y^2} = \frac{K_1}{\pi}\int_0^{\infty} \frac{\partial\theta_\xi}{\partial\xi}d\xi$$
$$= \frac{K_1}{\pi}(\pi - \theta) = \frac{K_1}{\pi}\left(\pi - \arctan\frac{y}{x}\right),$$

as $\theta_\xi|_{\xi\to\infty} = \pi$, $\theta_\xi|_{\xi\to 0} = \theta$. Using eq. (3.2.2) and omitting the term linear in y we obtain

$$U(x,y) = -\frac{K_1}{\pi} r\theta \sin\theta. \tag{3.4.3}$$

As in Flamant's problem, the only non-vanishing stress is

$$\sigma_r = -\frac{2K_1}{\pi r}\cos\theta. \tag{3.4.4}$$

Since angle θ is measured from the direction of the force (axis Oy) both cases (normal and tangential loading by the concentrated force) are formulated in the framework of the same statement. In the general case of the concentrated force directed at angle γ to axis Oy

$$\mathbf{K} = K_1\mathbf{i}_1 + K_2\mathbf{i}_2 = K(\mathbf{i}_1\cos\gamma + \mathbf{i}_2\sin\gamma)$$

we have, referring to formulae (3.4.4), (3.2.7) that

$$\sigma_r = -\frac{2K}{\pi r}(\cos\theta\cos\gamma + \sin\theta\sin\gamma) = -\frac{2K}{\pi r}\cos(\theta-\gamma) = -\frac{2K}{\pi r}\cos\psi, \tag{3.4.5}$$

where ψ denotes the angle measured from the direction of the force. This formula includes the above cases as particular ones.

When the surface forces are uniformly distributed over the part $-a \le x \le a$ of the boundary $y = 0$, then

$$P(x) = \begin{cases} 0, & -\infty < x \le -a, \\ q(x+a) & -a \le x \le a, \\ 2qa & a \le x < \infty, \end{cases}$$

and by eq. (3.4.2)

$$f_2(x,y) = \frac{qy}{\pi}\int_{-a}^{a}\frac{\xi + a}{(x-\xi)^2 + y^2}d\xi + \frac{2qa}{\pi}y\int_{a}^{\infty}\frac{d\xi}{(x-\xi)^2+y^2}.$$

A simple calculation leads to the following expressions for the stress function

$$U(x,y) = \frac{qy}{\pi}\left[(x-a)\theta_1 - (x+a)\theta_0 - y\ln\frac{r_0}{r_1}\right], \tag{3.4.6}$$

where the notion of Subsection 7.3.3 is used and the linear term is dropped out.

7.3.5 The plane contact problem

We consider the problem of the state of stress in the elastic half-plane caused by a rigid smooth die pressed against the half-plane boundary on the part $(-a, a)$.

It is assumed that the shear stresses are absent over the entire boundary whereas the normal stresses are absent outside the loaded part. On the loaded part, the displacement is prescribed

$$v = v(x, 0) = f(x), \quad -a < x < a, \tag{3.5.1}$$

where $f(x)$ is determined by the form of the die surface contacting with the plane. Evidently, the law of distribution of the normal stress on this part

$$\sigma_y = -q(x)$$

presents the main unknown of the problem. The force pressing the die against the plane is prescribed

$$Q = \int_{-a}^{a} q(x)\, dx. \tag{3.5.2}$$

Referring to eq. (3.2.12) we can represent the displacement by the integral

$$v(x, 0) = \beta \int_{-a}^{a} q(\xi) \ln \frac{1}{|x - \xi|} d\xi = f(x), \tag{3.5.3}$$

and the problem reduces to searching the unknown function $q(\xi)$ subjected to condition (3.5.2) from the above integral equation of the first kind. The solution of integral equation (3.5.3) can also be obtained by reducing to Riemann's boundary-value problem[2].

We introduce into consideration the logarithmic single layer potential distributed over the range $(-a, a)$ with the unknown density $q(x)$

$$\omega(x, y) = \int_{-a}^{a} q(\xi) \ln \frac{1}{r} d\xi, \quad r = \sqrt{(x - \xi)^2 + y^2}. \tag{3.5.4}$$

It is known that it is a harmonic function continuous everywhere in plane Oxy. Its normal derivative experiences a jump under a passage from the "lower" side of the layer to its "upper" side. The limiting values of the layer are given by the equalities

$$\left.\frac{\partial \omega}{\partial y}\right|_{y \to \pm 0} = \begin{cases} \mp \pi q(x), & |x| < a, \\ 0 & |x| > a, \end{cases} \tag{3.5.5}$$

[2]see Gakhov, F.D. Boundary value problems (in Russian). Fizmatgiz, Moscow, 1963.

where $q(x)$ is assumed to be continuous at point x.

The behaviour of the logarithmic simple layer potential as $r \to \infty$ is described by the relationship

$$\omega_\infty(x,y) = \ln\frac{1}{r_0}\int_{-a}^{a} q(\xi)\,d\xi = Q\ln\frac{1}{r_0}, \quad r_0 = \sqrt{x^2+y^2} \to \infty, \tag{3.5.6}$$

following directly from definition (3.5.4) and condition (3.5.2). Inversely, any harmonic function possessing the listed properties (continuity, the character of the discontinuity of the normal derivative, behaviour at infinity) is a logarithmic simple layer potential and can be presented by integral (3.5.4).

Referring to eqs. (3.5.4) and (3.5.3) we have

$$\omega(x,0) = \frac{1}{\beta} f(x). \tag{3.5.7}$$

The problem is thus reduced to searching, in plane Oxy, the harmonic function which takes the prescribed value on the range $(-a,a)$ of axis Ox and satisfying condition (3.5.6) at infinity. Having the solution to this problem, we can find the distribution law of the pressure over the contact by means of relationship (3.5.5).

It is easy to relate function $\omega(x,y)$ to the harmonic function $f_1(x,y)$ introduced in Subsection 7.3.2 and use eq. (3.2.4) for obtaining the stress function. Indeed, by eqs. (3.2.4), (3.3.2) and (3.5.4) we have

$$\nabla^2 U = -2\frac{\partial^2 f_1}{\partial y^2} = -\frac{2}{\pi}\int_{-a}^{a} q(\xi)\frac{y}{r^2}d\xi = \frac{2}{\pi}\int_{-a}^{a} q(\xi)\frac{\partial}{\partial y}\frac{1}{r}d\xi = \frac{2}{\pi}\frac{\partial\omega}{\partial y},$$

so that

$$\frac{\partial f_1}{\partial y} = -\frac{1}{\pi}\omega. \tag{3.5.8}$$

This allows one to express the stresses in terms of potential $\omega(x,y)$

$$\left.\begin{aligned}
\sigma_x &= \frac{\partial^2 U}{\partial y^2} = -\left(\frac{\partial^2 f_1}{\partial y^2} + y\frac{\partial^3 f_1}{\partial y^3}\right) = \frac{1}{\pi}\left(\frac{\partial\omega}{\partial y} + y\frac{\partial^2\omega}{\partial y^2}\right),\\
\sigma_y &= \frac{\partial^2 U}{\partial x^2} = -\left(\frac{\partial^2 f_1}{\partial y^2} - y\frac{\partial^3 f_1}{\partial y^3}\right) = \frac{1}{\pi}\left(\frac{\partial\omega}{\partial y} - y\frac{\partial^2\omega}{\partial y^2}\right),\\
\tau_{xy} &= -\frac{\partial^2 U}{\partial x\partial y} = y\frac{\partial^3 f_1}{\partial x\partial y^2} = -\frac{1}{\pi}y\frac{\partial^2\omega}{\partial x\partial y}.
\end{aligned}\right\} \tag{3.5.9}$$

This function and the function

$$\vartheta = -\int_{-a}^{a} q(\xi)\,\theta_\xi d\xi, \quad \frac{\partial\omega}{\partial x} = \frac{\partial\vartheta}{\partial y}, \quad \frac{\partial\omega}{\partial y} = -\frac{\partial\vartheta}{\partial x} \quad \left(\theta_\xi = \arctan\frac{y}{x-\xi}\right) \tag{3.5.10}$$

related to $\omega(x,y)$ by the Cauchy-Riemann relations are used for obtaining the displacements in the generalised plane stress

$$\left.\begin{aligned} u &= -\frac{1}{\pi E}\left[(1-\nu)\vartheta + (1+\nu)y\frac{d\omega}{dx}\right] + u_0 - \gamma_0 y, \\ v &= \frac{1}{\pi E}\left[2\omega - (1+\nu)y\frac{d\omega}{dy}\right] + v_0 + \gamma_0 x \end{aligned}\right\} \tag{3.5.11}$$

with the corresponding replacement the elastic constants in the case of the plane strain.

It follows from formulae (3.5.9) that when only the normal forces are applied to the boundary, the normal stresses σ_x, σ_y are equal to each other which has already been pointed out in Subsection 7.3.3 in the case of uniform loading the part of the boundary.

7.3.6 *Constructing potential ω*

The interior of the unit circle $\xi = \rho e^{i\varphi}$ is transformed by means of a conformal mapping into plane Oxy cut on the part $|x| < a$. This transformation has the form

$$z = x + iy = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right), \quad x = \frac{a}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi, \quad y = \frac{a}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi. \tag{3.6.1}$$

The circles $\rho = \rho_0 < 1$ of plane ξ are transformed into ellipses with the semi-axes $\frac{a}{2}\left(\rho_0 + \frac{1}{\rho_0}\right)$ and $\frac{a}{2}\left|\rho_0 - \frac{1}{\rho_0}\right|$ in the plane z with the foci at points $\pm a$. For a circle $\rho_0 = 1$ the ellipse degenerates into a segment $(-a, a)$ of plane z. The lower side $(y \to -0)$ traversed in the direction from a to $-a$ corresponds to the upper semicircle $\rho_0 = 1\,(0 \le \varphi \le \pi)$ whereas the upper side $(y \to +0)$ corresponds to the lower semicircle $\rho_0 = 1\,(\pi \le \varphi \le 2\pi)$. An infinite point $(x, y \to \infty)$ of plane z is mapped into the origin of the coordinates $(\rho = 0)$ in plane $\xi = 0$.

The sought potential $\omega(x,y)$ is transformed into function $\omega_*(\rho,\varphi)$ which is also harmonic as transformation (3.6.1) is conformal. On the "ellipse" $\rho_0 = 1$

$$x = a\cos\varphi, \quad y = 0,$$

and the boundary condition (3.5.7) for function ω_* is set in the form

$$\omega_*(1,\varphi) = \frac{1}{\beta} f(a\cos\varphi). \tag{3.6.2}$$

Noticing that for $(x,y) \to \infty$

$$x^2 + y^2 \to \frac{a^2}{4\rho^2}, \quad \ln\frac{1}{r_0} \to \ln\frac{2\rho}{a},$$

we can transform the condition at infinity (3.5.6) to the form

$$\omega_* (0, \varphi) - Q \ln \frac{2\rho}{a} \to 0, \quad \rho \to 0. \tag{3.6.3}$$

Introducing then, instead of $\omega_* (\rho, \varphi)$, the following harmonic function

$$\Omega (\rho, \varphi) = \omega_* (\rho, \varphi) - Q \ln \frac{2\rho}{a}, \tag{3.6.4}$$

we arrive at the problem of obtaining this function from the conditions

$$\Omega (1, \varphi) = \frac{1}{\beta} f (a \cos \varphi) - Q \ln \frac{2}{a}, \quad \Omega (0, \varphi) = 0. \tag{3.6.5}$$

Its solution is presented by the Poisson integral

$$\Omega (\rho, \varphi) = \frac{1}{2\pi\beta} \int_0^{2\pi} \frac{\left(1 - \rho^2\right) f (a \cos \psi)}{1 + \rho^2 - 2\rho \cos (\psi - \varphi)} d\psi - Q \ln \frac{2}{a}, \tag{3.6.6}$$

a consequence of the second condition in eq. (3.6.5) being the requirements

$$Q = \frac{1}{2\pi\beta \ln \frac{2}{a}} \int_0^{2\pi} f (a \cos \psi) \, d\psi = \frac{1}{\pi\beta \ln \frac{2}{a}} \int_{-a}^{a} \frac{f (\xi)}{\sqrt{a^2 - \xi^2}} d\xi. \tag{3.6.7}$$

Recalling the representation of Poisson's kernel in the form of a trigonometric series

$$\frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos (\psi - \varphi)} = 1 + 2 \sum_{k=1}^{\infty} \rho^k \cos k (\psi - \varphi)$$

and noticing that $f (a \cos \varphi)$ is even with respect to φ (i.e. the same on the upper and lower sides of the cut) we have

$$\Omega (\rho, \varphi) = \sum_{k=1}^{\infty} \alpha_k \rho^k \cos k\varphi, \quad \omega_* (\rho, \varphi) = Q \ln \frac{2\rho}{a} + \sum_{k=1}^{\infty} \rho^k \alpha_k \cos k\varphi. \tag{3.6.8}$$

Here α_k denotes the Fourier coefficients

$$\alpha_k = \frac{1}{\pi\beta} \int_0^{2\pi} f (a \cos \psi) \cos k\psi d\psi. \tag{3.6.9}$$

It remains to construct the expression for the distribution $q (x)$ of the surface force over the surface of the contact. We have

$$\frac{\partial \omega_*}{\partial \rho} = \frac{\partial \omega}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial \omega}{\partial y} \frac{\partial y}{\partial \rho} = \frac{a}{2} \left[\frac{\partial \omega}{\partial x} \left(1 - \frac{1}{\rho^2}\right) \cos \varphi + \frac{\partial \omega}{\partial y} \left(1 + \frac{1}{\rho^2}\right) \sin \varphi \right]$$

and referring to eqs. (3.5.5), (3.6.8) we find

$$\left.\frac{\partial \omega_*}{\partial \rho}\right|_{\rho\to 1} = a\left.\frac{\partial \omega}{\partial y}\right|_{y\to\pm 0} \sin\varphi = \mp a\pi q\left(a\cos\varphi\right)\sin\varphi.$$

As $\sin\varphi \lesseqgtr 0$ at $y \to \pm 0$ we arrive at the formal representation of $q(x)$ in the following form

$$q(x) = q(a\cos\varphi) = \frac{1}{a\pi\left|\sin\varphi\right|}\left(Q + \left.\frac{\partial \Omega}{\partial \rho}\right|_{\rho\to 1}\right) = \frac{Q + \sum\limits_{k=1}^{\infty} k\alpha_k \cos k\varphi}{\pi a\left|\sin\varphi\right|}. \tag{3.6.10}$$

The series on the right hand side is the limiting value of the real part of the function

$$\sum_{k=1}^{\infty} k\alpha_k \rho^k \left(\cos k\varphi + i\sin k\varphi\right)$$

(analytical in the circle $|\xi| < 1$) at $|\xi| = 1$. The limiting value of the imaginary part is equal to

$$\sum_{k=1}^{\infty} k\alpha_k \rho^k \sin k\varphi = -\frac{1}{\beta}\frac{df\left(a\cos\varphi\right)}{d\varphi},$$

which follows from definition (3.6.9) of coefficients α_k. Referring to Hilbert's inversion formulae[3], we have

$$\sum_{k=1}^{\infty} k\alpha_k \rho^k \cos k\varphi = -\frac{1}{2\pi\beta}\int_0^{2\pi} \frac{df\left(a\cos\psi\right)}{d\psi}\cot\frac{\psi-\varphi}{2}d\psi,$$

so that another representation of $q(x)$ is

$$q(x) = \frac{1}{\pi a\left|\sin\varphi\right|}\left[Q - \frac{1}{2\pi\beta}\int_0^{2\pi} \frac{df\left(a\cos\psi\right)}{d\psi}\cot\frac{\psi-\varphi}{2}d\psi\right]. \tag{3.6.11}$$

Splitting the integration interval into two parts: $(0,\pi)$ and $(\pi,2\pi)$ and replacing the integration variable ψ in the second integral by $2\pi - \psi$ we

[3]see, for example, Gakhov, F.D. Boundary value problems (in Russian). Fizmatgiz, Moscow, 1963.

can transform the integral in eq. (3.6.11) to the form

$$\int_0^\pi \frac{df(a\cos\psi)}{d\psi}\left(\cot\frac{\psi-\varphi}{2}+\cot\frac{\psi+\varphi}{2}\right)d\psi =$$

$$= 2\int_0^\pi \frac{df(a\cos\psi)}{d\psi}\frac{\sin\psi}{\cos\varphi-\cos\psi}d\psi.$$

Returning to the variables $x = a\cos\varphi, \xi = a\cos\psi$ we obtain

$$q(x) = \frac{1}{\pi\sqrt{a^2-x^2}}\left[Q - \frac{1}{\pi\beta}\int_{-a}^{a}\frac{f'(\xi)\sqrt{a^2-\xi^2}}{\xi-x}d\xi\right]. \tag{3.6.12}$$

The integral on the right hand side is understood here in the sense of the principal value

$$\int_{-a}^{a}\frac{f'(\xi)\sqrt{a^2-\xi^2}}{\xi-x}d\xi = \lim_{\varepsilon\to 0}\left[\int_{-a}^{x-\varepsilon}\frac{f'(\xi)\sqrt{a^2-\xi^2}}{\xi-x}d\xi + \int_{x+\varepsilon}^{a}\frac{f'(\xi)\sqrt{a^2-\xi^2}}{\xi-x}d\xi\right]$$

and a sufficient condition for its existence is Hölder's condition

$$|f'(x') - f'(x)| < M|x'-x|^\alpha \quad (0 < \alpha \le 1).$$

At the points where this condition does not hold, $q(x)$ is unbounded. For example, at the corner point $x = 0$ of the die bounded by the curve $y = f(|x|)$, the derivative is $y' = f'(|x|)\operatorname{sign} x$, Hölder's condition is not satisfied and the pressure is infinite.

7.3.7 A plane die

In this simple case $f(x) = f_0 = \text{const}$ and according to eq. (3.6.12) the distribution of the contact pressure is given by the formula (Sadowsky, 1928)

$$q(x) = \frac{Q}{\pi\sqrt{a^2-x^2}}. \tag{3.7.1}$$

The expression for the potential ω determined by eq. (5.5.8) is rather simple

$$\omega = Q\ln\frac{2\rho}{a}. \tag{3.7.2}$$

Displacement v on axis Ox given by formula (3.5.11) can be set as follows

$$v = \frac{2Q}{\pi E}\ln\frac{2\rho}{a} + v_0 = \frac{2Q}{\pi E}\left(\ln\frac{2\rho}{a} - \ln\frac{2}{a}\right) + \text{const} = \frac{2Q}{\pi E}\ln\rho + f_0. \tag{3.7.3}$$

Similar to the three-dimensional problem of a plane die (Subsection 5.6.3) the pressure is unbounded on the edge of the contact area. In contrast to the above problem the displacement of the plane die can be determined only up to an additive constant. This is explained by the fact that in the three-dimensional problem the displacement vector at infinity is equal to zero whereas in the plane problem it unboundedly increases without bounds due to a logarithmic law.

7.3.8 Die with a parabolic profile

Referring to eq. (6.1.11) of Chapter 5

$$f(x) = \delta - Ax^2 = \delta - Aa^2\cos^2\varphi = \delta - \frac{1}{2}Aa^2 - \frac{1}{2}Aa^2\cos 2\varphi, \qquad (3.8.1)$$

so that

$$\left.\begin{aligned} Q &= \frac{\delta - \frac{1}{2}Aa^2}{\beta \ln \frac{2}{a}}, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{1}{2\beta}Aa^2, \\ q(x) &= \frac{1}{\pi a\left|\sin\varphi\right|}\left(Q - \frac{1}{\beta}Aa^2\cos 2\varphi\right) \\ &= \frac{1}{\pi\sqrt{a^2 - x^2}}\left(Q + \frac{A}{\beta}a^2 - 2\frac{A}{\beta}x^2\right). \end{aligned}\right\} \qquad (3.8.2)$$

The unknown width $2a$ of the contact area is determined such that the contact pressure is finite on the edges. The result is

$$Q = \frac{A}{\beta}a^2, \quad a = \sqrt{\frac{Q\beta}{A}} = \sqrt{\frac{2Q}{\pi EA}}, \qquad (3.8.3)$$

and the expression for the contact pressure takes the form

$$q(x) = \frac{2Q}{\pi a^2}\sqrt{a^2 - x^2}. \qquad (3.8.4)$$

7.3.9 Concentrated force in the elastic half-plane

The problem of action of the concentrated force $X + iY$ on the elastic half-plane was considered in Subsection 7.3.1. For the plane strain the solution is expressed in terms of the stress function given by eq. (3.1.10) under the assumption that the force is applied at the origin of the coordinate system $z = 0$. When the force is applied at point η_0 of axis Oy it is sufficient to replace z by $z - i\eta_0$.

Let us consider the action of two forces: force $X + iY$ at point $i\eta_0$ and force $X - iY$ at point $(-i\eta_0)$. The stress function U_* describing the state of the elastic half-plane can be obtained by superimposing the functions of the sort (3.1.10). Up to an additive constant the derivative of U_* with respect to $\bar{z}$ is as follows

$$4\frac{\partial U_*}{\partial \bar{z}} = \frac{1}{4\pi(1-\nu)}\Big[-(X+iY)\ln(z-i\eta_0) - (X-iY)\ln(z+i\eta_0) - \\ (X-iY)\frac{z-i\eta_0}{\bar{z}+i\eta_0} - (X+iY)\frac{z+i\eta_0}{\bar{z}-i\eta_0} + \\ (3-4\nu)(X+iY)\ln(\bar{z}+i\eta_0) + (3-4\nu)(X-iY)\ln(\bar{z}-i\eta_0)\Big]. \quad (3.9.1)$$

In the problem considered, we seek the state of stress in the elastic half-plane $y > 0$ provided that a concentrated force $X + iY$ is applied at point $i\eta_0$ and the boundary is free. Denoting the stress function by U we have

$$U = U_* + U_{**}. \quad (3.9.2)$$

The boundary conditions determining U_{**} are set as follows

$$y = 0, \; z = \bar{z} = x: \quad \frac{\partial U}{\partial \bar{z}} = \frac{\partial U_*}{\partial \bar{z}} + \frac{\partial U_{**}}{\partial \bar{z}} = 0 \quad (3.9.3)$$

or in expanded form

$$4\frac{\partial U_{**}}{\partial \bar{z}}\Big|_{\bar{z}=z=x} = -\frac{1}{\pi(1-\nu)}\left\{X\left[(1-2\nu)\ln\sqrt{x^2+\eta_0^2} - \frac{x^2-\eta_0^2}{2(x^2+\eta_0^2)}\right] - \\ iY\left[(1-\nu)\ln\frac{x-i\eta_0}{x+i\eta_0} + i\frac{x\eta_0}{x^2+\eta_0^2}\right]\right\}.$$

Introducing angle θ_0

$$\cos\theta_0 = \frac{x}{\sqrt{x^2+\eta_0^2}}, \quad \sin\theta_0 = \frac{\eta_0}{\sqrt{x^2+\eta_0^2}}, \quad \theta_0 = \arctan\frac{\eta_0}{x} \;\; (0 \le \theta_0 \le \pi),$$

we can rewrite this relation in another form

$$\left(\frac{\partial U_{**}}{\partial x} + i\frac{\partial U_{**}}{\partial y}\right)\Big|_{y=0} = -\frac{1}{2\pi(1+\nu)}\left\{X\left[(1-2\nu)\ln r_0 - \frac{1}{2}\cos 2\theta_0\right] + \\ Y\left[-2(1-\nu)\theta_0 + \eta_0\frac{\cos\theta_0}{r_0}\right]\right\}, \quad (3.9.4)$$

where $r_0 = \sqrt{x^2+\eta_0^2}$. The quantity on the right hand side is real-valued and the derivative with respect to y of the sought stress function is equal to zero on the boundary. It could be expected as the shear stress τ_{xy} on axis Ox vanishes because of the symmetry of loading the half-plane by forces $X \pm iY$ at points $\pm i\eta_0$. Function U_{**} corresponds to the state of stress due to the loading normal to the boundary and it can be sought in the form (3.2.4)

$$U_{**} = f(x,y) - y\frac{\partial f}{\partial y}, \quad \left.\frac{\partial U_*}{\partial x}\right|_{y=0} = \left.\frac{\partial f}{\partial x}\right|_{y=0}, \tag{3.9.5}$$

where $f(x,y)$ is a function harmonic in the upper half-plane where $\partial f/\partial y$ is the solution of Dirichlet's problem. A part of the solution can be written down easily because the following functions

$$\ln r_0, \quad \theta_0, \quad \frac{\cos\theta_0}{r_0}$$

are the values at $y=0$ of the functions

$$\ln r = \ln\sqrt{x^2 + (y+\eta_0)^2}, \quad \theta = \arctan\frac{y+\eta_0}{x}, \quad \frac{\cos\theta}{r} = \frac{x}{x^2+(y+\eta_0)^2},$$

which are harmonic for $y>0$. The above functions are in turn the derivatives, with respect to x, of the following functions

$$\ln r = \frac{\partial}{\partial x}\left[x\ln r - (y+\eta_0)\,\theta - x\right], \quad \theta = \frac{\partial}{\partial x}\left[x\theta + (y+\eta_0)\ln r\right],$$
$$\frac{\cos\theta}{r} = \frac{\partial}{\partial x}\ln r.$$

At the same time

$$\cos 2\theta_0 = \frac{x^2-\eta_0^2}{x^2+\eta_0^2} = 1 - \frac{2\eta_0^2}{x^2+\eta_0^2} = \frac{\partial}{\partial x}\left(x + 2\eta_0\theta_0\right) = \frac{\partial}{\partial x}\left.\left(x+2\eta_0\theta\right)\right|_{y=0}.$$

We arrive at the following representation of function $f(x,y)$ harmonic for $y>0$

$$f(x,y) = -\frac{1}{2\pi(1-\nu)}\Big\{X\Big[(1-2\nu)\left\{x\ln r - (y+\eta_0)\,\theta - x\right\} -$$
$$\frac{1}{2}\left(x+2\eta_0\theta\right)\Big] + Y\left[-2(1-\nu)\left\{x\theta + (y+\eta_0)\ln r\right\} + \eta_0\ln r\right]\Big\}.$$

Using eq. (3.9.5) and omitting the linear terms we find

$$U_{**}(x,y) = -\frac{1}{2\pi(1-\nu)}\left\{X\left[(1-2\nu)\left(x\ln r - \eta_0\theta\right) - \eta_0\left(\theta - \frac{xy}{r^2}\right)\right] + \right.$$
$$\left. Y\left[-2(1-\nu)\left(x\theta + \eta_0\ln r\right) + \eta_0\left(\ln r - \frac{y(y+\eta_0)}{r^2}\right)\right]\right\}. \tag{3.9.6}$$

The solution of the problem is thus presented in the form of a sum of two stress functions where U_1 and U_2 describe the action of the horizontal and vertical force, respectively. Then we have

$$2\frac{\partial U_1}{\partial \bar{z}} = \frac{X}{2\pi(1-\nu)}\left[\frac{1}{4}(3-4\nu)\ln\left(\bar{z}^2+\eta_0^2\right) - \frac{1}{4}\ln\left(z^2+\eta_0^2\right) - \right.$$
$$\frac{z\bar{z}-\eta_0^2}{2\left(\bar{z}^2+\eta_0^2\right)} - (1-2\nu)\ln r - (1-2\nu)(x-i\eta_0)\frac{z+i\eta_0}{r^2}$$
$$\left. -\eta_0\frac{2y+\eta_0}{r^2} + 2\eta_0(z+i\eta_0)\frac{xy}{r^4}\right], \qquad (3.9.7)$$

$$2\frac{\partial U_2}{\partial \bar{z}} = \frac{Y}{2\pi(1-\nu)}\left[\frac{i}{4}(3-4\nu)\ln\frac{\bar{z}+i\eta_0}{\bar{z}-i\eta_0} + \frac{i}{4}\ln\frac{z+i\eta_0}{z-i\eta_0} + \eta_0\frac{x}{\bar{z}^2+\eta_0^2} + \right.$$
$$2(1-\nu)\theta + 2(1-\nu)i\frac{x-i\eta_0}{r^2}(z+i\eta_0) -$$
$$\left. \frac{\eta_0\bar{z}}{r^2} - 2\eta_0(z+i\eta_0)\frac{y(y+\eta_0)}{r^4}\right], \qquad (3.9.8)$$

These functions which are biharmonic in the upper half-plane have the required singularity at point $z = i\eta_0$ where force $X + iY$ is applied and take the constant value on the boundary $y = 0$. The stresses are calculated by formulae (1.13.2).

7.4 Elastic wedge

7.4.1 Concentrated force in the vertex of the wedge

We consider an infinite region of a wedge form which is bounded by two half-lines $y = \pm\tan\alpha$. The angle of the wedge is 2α, axis Ox is directed into the region and the origin of the coordinate system (the vertex of the wedge) is taken as being the origin of the polar coordinate system (r,θ), i.e. $-\alpha \le \theta \le \alpha$. The projection of the force applied at the wedge vertex on axes Ox, Oy are denoted by X, Y. The faces of the wedge are assumed to be free, i.e.

$$\theta = \pm\alpha: \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0. \qquad (4.1.1)$$

These conditions can be satisfied by assuming that they hold in the whole region rather than only on the boundary

$$|\theta| \le \alpha: \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0. \qquad (4.1.2)$$

Under these conditions, stress σ_r is a function harmonic in the region since

$$\nabla^2(\sigma_r + \sigma_\theta) = \nabla^2\sigma_r = \nabla^2\nabla^2 U = 0. \qquad (4.1.3)$$

The static equations

$$\int_{-a}^{a} \sigma_r \cos\theta r d\theta + X = 0, \quad \int_{-a}^{a} \sigma_r \sin\theta r d\theta + Y = 0 \tag{4.1.4}$$

express the conditions of equilibrium for the part of the wedge bounded by a circle of radius r and are required to hold for any r. These relations can be satisfied by assuming σ_r to be inversely proportional to r. As σ_r is a harmonic function one should take

$$\sigma_r = A\frac{\cos\theta}{r} + B\frac{\sin\theta}{r}. \tag{4.1.5}$$

No other harmonic function satisfying condition (4.1.4) exists. Determining the constants A and B using these conditions we find

$$\sigma_r = -\frac{2}{r}\left(\frac{X\cos\theta}{2\alpha + \sin 2\alpha} + \frac{Y\sin\theta}{2\alpha - \sin 2\alpha}\right). \tag{4.1.6}$$

As the roots of the denominators are complex-valued (except $\alpha = 0$), hence stress σ_r is finite everywhere for $r \neq 0$ and tends to zero as $r \to \infty$. The stress function is presented in the form

$$U = r\theta\left(-\frac{X\sin\theta}{2\alpha + \sin 2\alpha} + \frac{Y\cos\theta}{2\alpha - \sin 2\alpha}\right) = \theta\left(-\frac{yX}{2\alpha + \sin 2\alpha} + \frac{xY}{2\alpha - \sin 2\alpha}\right). \tag{4.1.7}$$

The displacement vector for the plane stress is obtained in terms of the above stress function

$$\left.\begin{aligned} Eu = &-\frac{X}{2\alpha + \sin 2\alpha}\left[2\ln r + (1+\nu)\frac{y^2}{r^2}\right] + \\ &\frac{Y}{2\alpha - \sin 2\alpha}\left[(1-\nu)\theta + (1+\nu)\frac{xy}{r^2}\right] + u_0 - \omega_0 y, \\ Ev = &\frac{X}{2\alpha + \sin 2\alpha}\left[(1+\nu)\frac{xy}{r^2} - (1-\nu)\theta\right] - \\ &\frac{Y}{2\alpha - \sin 2\alpha}\left[2\ln r + (1+\nu)\frac{x^2}{r^2}\right] + v_0 + \omega_0 x. \end{aligned}\right\} \tag{4.1.8}$$

For $\alpha = \pi/2$ we return to the case of the half-plane loaded by a concentrated force, the case $\alpha = \pi$ yields the solution of the problem for the stress in the plane cut along the negative axis Ox provided that the force is applied at the vertex of the cut.

Aiming at comparing the solution with the elementary theory of bending of the beam we take in the expressions for the components of the stress tensor (for $X = 0$)

$$r = \frac{x}{\cos\theta}, \quad I = \frac{2}{3}x^3\tan^3\alpha = \frac{2}{3}h^3,$$

where I denotes the moment of inertia of the cross-section, $x = \text{const}$, $2h \cdot 1$ is its area. Then we arrive at the formulae

$$\sigma_x = -\frac{2Y}{2\alpha - \sin 2\alpha} \frac{x^2 y}{r^4} = -\frac{4Y \tan^3 \alpha}{2\alpha - \sin 2\alpha} \frac{xy}{3I} \cos^4 \theta,$$
$$\sigma_y = -\frac{2Y}{2\alpha - \sin 2\alpha} \frac{y^3}{r^4} = -\frac{4Y \tan^3 \alpha}{2\alpha - \sin 2\alpha} \frac{y^3}{3Ix} \cos^4 \theta,$$
$$\tau_{xy} = -\frac{2Y}{2\alpha - \sin 2\alpha} \frac{xy^2}{r^4} = -\frac{4Y \tan^3 \alpha}{2\alpha - \sin 2\alpha} \frac{y^2}{3I} \cos^4 \theta$$

and for a sufficiently small α we can take

$$2\alpha - \sin 2\alpha \approx \frac{4}{3}\alpha^3, \quad \left(\frac{\tan \alpha}{\alpha}\right)^3 \cos^4 \theta \approx 1,$$

which leads to the following result

$$\sigma_x \approx -\frac{Yxy}{I}, \quad \tau_{xy} \approx \frac{Yy^2}{I} = -\frac{3}{2}\frac{Yy^2}{h^3}.$$

Under this approximation the expressions for the normal stresses are coincident with those due to the elementary theory, however

$$(\tau_{xy})_{y=\pm h} = -3\frac{Y}{2h} = -3\,(\tau_{xy})_m$$

is twice the maximum stress from the elementary theory.

7.4.2 Mellin's integral transform in the problem of a wedge

For the sake of simplifying the notion in what follows we consider loading the wedge only by the surface forces normal to wedge's faces. The cases of the symmetric and skew-symmetric loads (problem A and B) are studied separately

$$\left.\begin{array}{lll} \mathrm{A} & \theta = \pm\alpha: & \sigma_\theta = q_1(r), \quad \tau_{r\theta} = 0, \\ \mathrm{B} & \theta = \pm\alpha: & \sigma_\theta = \pm q_2(r), \quad \tau_{r\theta} = 0. \end{array}\right\} \tag{4.2.1}$$

The biharmonic stress functions satisfying the condition of absence of the shear stresses on the wedge's faces are presented in the form

$$\left.\begin{array}{ll} \mathrm{A} & U_1(\rho,\theta) = C_1\rho^{-s}\left[(s+2)\sin(s+2)\alpha\cos s\theta - s\sin s\alpha\cos(s+2)\theta\right], \\ \mathrm{B} & U_2(\rho,\theta) = C_2\rho^{-s}\left[(s+2)\cos(s+2)\alpha\sin s\theta - s\cos s\alpha\sin(s+2)\theta\right], \end{array}\right\} \tag{4.2.2}$$

where $\rho = r/r_0$ and r_0 is some linear dimension of length. The normal stress σ_θ is given by the formula

$$\text{A and B} \quad r_0^2\rho^2\sigma_\theta^{(k)} = s(s+1)U_k(\rho_0,\theta) \quad (k = 1, 2). \tag{4.2.3}$$

Let us recall that the Mellin transform of function $f(t)$ given in the range $(0, \infty)$ is defined as the integral

$$\bar{f}(s) = \int_0^\infty f(x)\, x^{s-1} dx, \tag{4.2.4}$$

where s is a complex number. Under the assumption that $f(x)$ satisfies Dirichlet's condition in any finite range from $(0, \infty)$ we have the following formal representation

$$f(x) = \frac{1}{2\pi i} \int_L \bar{f}(s)\, x^{-s} ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s)\, x^{-s} ds, \tag{4.2.5}$$

where L is a straight line parallel to the imaginary axis of the complex plane s. Without going into detail we note that if the integral

$$\int_0^\infty |f(x)|\, x^{s_*-1} dx \tag{4.2.6}$$

converges, then line L lies in the strip left on the left side of the line $\operatorname{Re} s = s$ and on the right side of the singular point s_{**} closest to s_*. The value of the integral (4.2.5) does not depend on the choice of c in the range (s_{**}, s_*).

Referring now to the representations (4.2.2) let us take, according to eq. (4.2.5), the stress functions in the form

$$\text{A and B} \quad U_k(\rho, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{U}_k(s, \theta)\, \rho^{-s} ds, \tag{4.2.7}$$

where $\bar{U}_k(s, \theta)$ denotes the Mellin transforms

$$\left.\begin{array}{ll} \text{A} & \bar{U}_1(s, \theta) = C_1(s)\left[(s+2)\sin(s+2)\alpha\cos s\theta - s\sin s\alpha\cos(s+2)\theta\right], \\ \text{B} & \bar{U}_2(s, \theta) = C_2(s)\left[(s+2)\cos(s+2)\alpha\sin s\theta - s\cos s\alpha\sin(s+2)\theta\right]. \end{array}\right\} \tag{4.2.8}$$

Using eq. (4.2.3), the Mellin transforms of functions $r^2\sigma_\theta^{(k)}$ are equal to $s(s+1)\bar{U}_k(s, \theta)$, that is

$$\text{A and B} \quad r^2\sigma_\theta^{(k)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s(s+1)\, \bar{U}_k(s, \theta)\, \rho^{-s} ds. \tag{4.2.9}$$

At the same time, by virtue of eqs. (4.2.1) and (4.2.5)

$$\left.\begin{aligned} &\text{A}\quad r^2\sigma_\theta^{(1)}\Big|_{\theta=\pm\alpha} = r_0^2\rho^2 q_1(\rho) = \frac{r_0^2}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \bar{q}_1(s)\,\rho^{-s}ds, \\ &\text{B}\quad r^2\sigma_\theta^{(2)}\Big|_{\theta=\pm\alpha} = \pm r_0^2\rho^2 q_2(\rho) = \pm\frac{r_0^2}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \bar{q}_2(s)\,\rho^{-s}ds, \end{aligned}\right\} \tag{4.2.10}$$

where $\bar{q}_k(s)$ are the transforms of functions $\rho^2 q_k(\rho)$ which are equal to

$$\text{A and B}\quad \bar{q}_k(s) = \int\limits_0^\infty \rho^{s+1} q_k(\rho)\,d\rho \qquad (k=1,2), \tag{4.2.11}$$

see eq. (4.2.4). By eq. (4.2.1) we have

$$\left.\begin{aligned} &\text{A}\quad \begin{aligned} &C_1(s) = r_0^2\frac{\bar{q}_1(s)}{s(s+1)G_1(s,\alpha)}, \\ &G_1(s,\alpha) = (s+1)\sin 2\alpha + \sin 2(s+1)\alpha, \end{aligned} \\ &\text{B}\quad \begin{aligned} &C_2(s) = r_0^2\frac{\bar{q}_2(s)}{s(s+1)G_2(s,\alpha)}, \\ &G_2(s,\alpha) = -(s+1)\sin 2\alpha + \sin 2(s+1)\alpha, \end{aligned} \end{aligned}\right\} \tag{4.2.12}$$

that is, the stress functions satisfying the boundary conditions are written down in the form

$$\left.\begin{aligned} &\text{A}\quad U_1(\rho,\theta) = \frac{r_0^2}{2\pi i}\times \\ &\int\limits_{c-i\infty}^{c+i\infty} \frac{\bar{q}_1(s)}{s(s+1)}\frac{(s+2)\sin(s+2)\alpha\cos s\theta - s\sin s\alpha\cos(s+2)\theta}{(s+1)\sin 2\alpha + \sin 2(s+1)\alpha}\rho^{-s}ds, \\ &\text{B}\quad U_2(\rho,\theta) = \frac{r_0^2}{2\pi i}\times \\ &\int\limits_{c-i\infty}^{c+i\infty} \frac{\bar{q}_2(s)}{s(s+1)}\frac{(s+2)\cos(s+2)\alpha\sin s\theta - s\cos s\alpha\sin(s+2)\theta}{-(s+1)\sin 2\alpha + \sin 2(s+1)\alpha}\rho^{-s}ds, \end{aligned}\right\} \tag{4.2.13}$$

It is assumed that $U_k(\rho,\theta)$ increases as $\rho\to\infty$ not faster than ρ (this condition holds for example in the problem of the concentrated force at the vertex of the wedge, see eq. (4.1.7)), then integral (4.2.6) converges for $s_* < -1$ and line L lies in the strip $(\operatorname{Re} s_1, -1)$ where s_1 denotes the root of function $G_k(s,\alpha)$ lying on the left side of -1 which has the maximum real part.

Evaluating integrals (4.2.13) is carried out separately for $\rho < 1$ and $\rho > 1$. In the first case, a line L is completed by a semicircle of radius

$|s| = R$. Provided that Jordan's lemma is applicable, the integral over the semicircle tends to zero at $R \to \infty$ and the evaluated integrals are equal to the products of $2\pi i$ and the sum of the residues at all poles of the integrand lying on the left side of L. The same reasoning for $\rho > 1$ suggests that the integral is equal to the product of $(-2\pi i)$ and the sum of the residues lying on the right side of L (the traversed region now lies on the right).

The terms $U_k^{-1}(\rho, \theta), U_k^0(\rho, \theta)$ in these sums correspond to the poles $s = -1$ and $s = 0$ are of a special interest. Introducing the notion

$$\begin{aligned} A \quad f_1(s, \alpha) &= \frac{1}{G_1(s, \alpha)} \left[(s+2)\sin(s+2)\alpha\cos s\theta - s\sin s\alpha\cos(s+2)\theta\right] \\ B \quad f_2(s, \alpha) &= \frac{1}{G_2(s, \alpha)} \left[(s+2)\cos(s+2)\alpha\sin s\theta - s\cos s\alpha\sin(s+2)\theta\right] \end{aligned} \tag{4.2.14}$$

we obtain by means of l'Hospital's rule that

$$\left.\begin{aligned} \text{A} \quad f_1(s, \alpha)|_{s\to -1} &= 2\frac{(\sin\alpha + \alpha\cos\alpha)\cos\theta + \theta\sin\theta\sin\alpha}{2\alpha + \sin 2\alpha}, \\ \text{B} \quad f_2(s, \alpha)|_{s\to -1} &= 2\frac{(\alpha\sin\alpha - \cos\alpha)\sin\theta + \theta\cos\theta\cos\alpha}{2\alpha - \sin 2\alpha}. \end{aligned}\right\} \tag{4.2.15}$$

Recalling eq. (4.2.11) we obtain the equalities

$$\text{A and B} \quad -2r_0\bar{q}_1(-1)\sin\alpha = X, \quad 2r_0\bar{q}_2(-1)\cos\alpha = Y \tag{4.2.16}$$

where X and Y are the projections of the principal vector of the surface forces on the wedge faces on axes Ox and Oy respectively. Omitting the insignificant terms which are linear in $x = r\cos\theta$ and $y = r\sin\theta$ we obtain

$$\text{A and B} \quad U_1^{-1}(\rho, \theta) = -y\theta\frac{X}{2\alpha + \sin 2\alpha}, \quad U_2^{-1}(\rho, \theta) = x\theta\frac{Y}{2\alpha - \sin 2\alpha} \tag{4.2.17}$$

which is in full agreement with formulae (4.1.7). It follows from these relations that, if the sums $U_k(\rho, \theta)$ have no terms increasing faster than ρ as $\rho \to \infty$, the stress in the wedge at $\rho \to \infty$ is coincident with that in the case of loading the wedge by force (X, Y) at the vertex.

By analogy we have

$$\text{A and B} \quad f_1(0, \alpha) = 1, \quad f_2(s, \alpha)|_{s\to 0} = \frac{\sin 2\theta - 2\theta\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}, \tag{4.2.18}$$

and thus $U_1^0(\rho, \theta) = \text{const}$ whereas if

$$\left.\begin{aligned} \text{B} \quad G_2'(0, \alpha) = -\sin 2\alpha + 2\alpha\cos 2\alpha \neq 0, \quad \tan 2\alpha \neq 2\alpha, \\ 2\alpha \neq 2\alpha_* = 4,493 \approx 257^\circ, \end{aligned}\right\} \tag{4.2.19}$$

then we have

B
$$\begin{aligned} U_2^0(\rho,\theta) &= -r_0^2\bar{q}_2(0)\, f_2(s,\alpha)\big|_{s\to 0} \\ &= -\frac{1}{2}m^O\frac{\sin 2\theta - 2\theta\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha} \quad (\alpha \neq \alpha_*). \end{aligned} \tag{4.2.20}$$

Here by eq. (4.2.11)

$$m^O = 2r_0^2\int_0^\infty q_2(\rho)\,\rho d\rho \tag{4.2.21}$$

is the principal moment of the surface forces on the faces about the vertex of the wedge. For $\alpha = \alpha_*$ one obtains a double root $s = 0$ of function $G_2(s,\alpha)$ and there is no need to demonstrate the result of this tiresome calculation in this special case.

As will be shown in Subsection 7.4.3, solution $U_2^0(\rho,\theta)$ can not be directly obtained by considering the problem of the wedge loaded by a concentrated moment m^O at the vertex. It follows from the above-said that it presents the principal term of $U_2^0(\rho,\theta)$ at infinity ($\rho\to\infty$) if function $G_2(s,\alpha)$ has no roots in the strip $(-1,0)$. It will be proved that it takes place for $\alpha < \alpha_*$ whereas for $\alpha > \alpha_*$ function $G_2(s,\alpha)$ gains an additional real-valued negative root

B
$$-\frac{1}{2} < s = \lambda(\alpha) < 0. \tag{4.2.22}$$

The corresponding term $U_2^{(\lambda)}(\rho,\theta)$ in the stress function

B
$$U_2^{(\lambda)}(\rho,\theta) = \frac{\bar{q}_2(\lambda)}{\lambda(\lambda+1)}\times \frac{(2+\lambda)\cos(\lambda+2)\alpha\sin\lambda\theta - \lambda\cos\lambda\alpha\sin(\lambda+2)\theta}{\sin 2\alpha - 2\alpha\cos 2(\lambda+1)\alpha}\left(\frac{r}{r_0}\right)^{|\lambda|} \tag{4.2.23}$$

presents the main terms in the expansion of the stress function. Its expression depends on the distribution of the surface forces $q_2(\rho)$ on the sides of the wedge and can not be determined in terms of the bending moment m^O. In eq. (4.2.23) we denoted

$$\bar{q}_2^*(\lambda) = r_0^2\int_0^\infty \rho^{\lambda+1}q_2(\rho)\,d\rho.$$

7.4.3 Concentrated moment at the vertex of the wedge

Boundary condition (4.1.1)

$$\theta = \pm\alpha: \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0 \tag{4.3.1}$$

is completed by the equilibrium equations for the part of the wedge cut by an arc of arbitrary radius

$$\left.\begin{aligned} X &= -\int_{-a}^{a} (\sigma_r \cos\theta - \tau_{r\theta}\sin\theta)\, r d\theta = 0, \\ Y &= -\int_{-a}^{a} (\sigma_r \sin\theta + \tau_{r\theta}\cos\theta)\, r d\theta = 0, \\ &\int_{-a}^{a} \tau_{r\theta} r^2 d\theta = -m^O. \end{aligned}\right\} \tag{4.3.2}$$

These conditions allows one to seek the solution to the problem by assuming that normal stress σ_θ is absent in the whole region

$$|\theta| \le \alpha: \quad \sigma_\theta = \frac{\partial^2 U}{\partial r^2} = 0. \tag{4.3.3}$$

The stress function is odd with respect to θ and linear in r. The odd biharmonic function which is proportional to r is $C\theta r\cos\theta$ (except for the trivial function $r\sin\theta$) yielding the solution of the problem of bending the wedge by the force concentrated at the vertex. For this reason, the stress function in this problem should be taken as depending only on θ. Such a function satisfying boundary condition (4.3.1) is

$$U = \frac{1}{2} C \left(\sin 2\theta - 2\theta\cos 2\alpha\right). \tag{4.3.4}$$

Expressing stresses $\sigma_r, \tau_{r\theta}$ in terms of these functions it is easy to check that the first two equations of statics (4.3.2) are satisfied identically whilst the third equation leads to the relationship

$$\int_{-a}^{a} \tau_{r\theta} r^2 d\theta = C\left(\sin 2\alpha - 2\alpha\cos 2\alpha\right) = -m^O,$$

which under the assumption that $\alpha \ne \alpha_*$ yields the following solution

$$U = -\frac{1}{2} m^O \frac{\sin 2\theta - 2\theta\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha} \quad (\alpha \ne \alpha_*), \tag{4.3.5}$$

(Carothers, 1912; Inglis, 1922) and the corresponding stresses

$$\sigma_r = \frac{2m^O}{r^2}\frac{\sin 2\theta}{\sin 2\alpha - 2\alpha\cos 2\alpha}, \quad \tau_{r\theta} = \frac{m^O}{r^2}\frac{\cos 2\alpha - \cos 2\theta}{\sin 2\alpha - 2\alpha\cos 2\alpha}. \tag{4.3.6}$$

The result seems paradoxical, namely the solution is absent for a single value of wedge angle $\alpha = \alpha_*$.

The explanation of the paradox requires refining the concept of the bending moment. It is natural to accept the following definition: the faces of the wedge are assumed to be loaded in the vicinity of the wedge by the following skew-symmetric normal load

$$\theta = \pm\alpha; \quad \sigma_\theta = \pm q_2(r) \quad (0 \le r \le r_0); \quad \tau_{r\theta} = 0 \quad (0 < r < \infty)$$

such, that its principal vector vanishes and the principal moment about the vertex is prescribed

$$\int_0^{r_0} q_2(r)\, dr = 0, \quad 2\int_0^{r_0} q_2(r)\, r dr = m^O. \tag{4.3.7}$$

If these equalities hold at $r_0 \to 0, q_2(r) \to \infty$ then it is said that the bending moment m^O is applied at the wedge vertex. Of course, this definition can be generalised to other types of loading on the wedge near the vertex.

The stress function $U_2(\rho, \theta)$ for the problem of skew-symmetric normal loading the wedge is given by expression (4.2.13) where the principal term of representation (4.2.20) at infinity for $\alpha < \alpha_*$ is exactly the solution (4.3.5). For $\alpha > \alpha_*$ the principal term of function $U_2(\rho, \theta)$ at infinity given by eq. (4.2.23) and the corresponding stresses depend on the distribution of the surface forces on the part $(0, r_0)$ rather than the moment only. The term of the type (4.3.5) appears in $U_2(\rho, \theta)$ also for $\alpha > \alpha_*$ but it is not the principal term. The corresponding stresses at $\rho \to \infty$ is of the order of r^{-2} whilst the principal term yields stresses of the order of $r^{-2+|\lambda|}$.

The integral equation of statics (4.3.2) was used for the statement of the problem in the present subsection. This excludes from consideration the stresses presented by those terms in the series for $U_2(\rho, \theta)$ which are different from (4.3.5). Their presence should be related to the singularities which are statically equivalent to zero (vanishing principal vector and principal moment) at the corner point. Neglecting these terms when they are caused by the load distributed over a small part of the boundary is typical for the solutions dealing with the classical formulation of Saint-Venant's principle. It is allowed if the corresponding stresses decrease faster with the growth of the distance from the load than the stresses due to the moment of these forces.

Returning to the problem of the present subsection it is necessary to accept that the Carothers-Inglis solution (4.3.5) is applicable only for $\alpha < \alpha_*$ whereas for $\alpha > \alpha_*$ the very statement of the problem of loading the wedge by the moment concentrated at the vertex is meaningless. The solution for $\alpha > \alpha_*$ depends on the law of distribution of the load on the parts $(0, r_0)$ of the wedge faces and is not reducible to the action of a single moment.

7.4.4 Loading the side faces

The calculation by formulae (4.2.13) can be continued in two ways: the residue theorem and the direct evaluation of the integrals.

The first way requires the roots of functions (4.2.12) which are obtained from the equations

$$\left.\begin{array}{ll} \mathrm{A} & \Delta_A(z) = \dfrac{\sin 2\alpha}{2\alpha} + \dfrac{\sin z}{z} = 0, \\ \mathrm{B} & \Delta_B(z) = \dfrac{\sin 2\alpha}{2\alpha} - \dfrac{\sin z}{z} \quad (z = 2(s+1)\alpha). \end{array}\right\} \tag{4.4.1}$$

Roots z_k lie symmetrically in the four quadrants of plane z

$$z_k = \pm a_k \pm i b_k, \quad s_k + 1 = \frac{1}{2\alpha}(\pm a_k \pm i b_k). \tag{4.4.2}$$

Table 7.2 shows the values of these roots for some values of α.

k	α							
	$\pi/8$		$\pi/4$		$3\pi/8$		$5\pi/8$	
	a_k	b_k	a_k	b_k	a_k	b_k	a_k	b_k
1	4,234	2,137	4,303	1,758	4,442	0,8501	2,645	0
2	10,72	2,996	10,75	2,641	10,82	1,863	7,694	0,8702
3	17,08	3,445	17,10	3,095	17,14	2,331	14,02	1,585
4	23,40	3,753	23,42	3,404	23,45	2,646	20,32	1,976
5	29,71	3,951	29,72	3,640	29,75	2,884	26,62	2,2553

Table 7.2A. Roots $z_k = a_k + ib_k$ of equation $\Delta_A(z) = \dfrac{\sin 2\alpha}{2\alpha} + \dfrac{\sin z}{z} = 0$

k	α							
	$\pi/8$		$\pi/4$		$3\pi/8$		$5\pi/8$	
	a_k	b_k	a_k	b_k	a_k	b_k	a_k	b_k
1	0,7854	0	1,571	0	2,356	0	3,927	0
2	7,511	2,660	7,553	2,300	7,641	1,497	5,114	0
3	13,91	3,246	13,93	2,894	13,98	2,125	10,86	1,302
4	20,24	3,611	20,26	3,262	20,30	2,501	17,17	1,801
5	26,56	3,878	26,57	3,529	26,60	2,772	23,47	2,124

Table 7.2B. Roots $z_k = a_k + ib_k$ of equation $\Delta_B(z) = \dfrac{\sin 2\alpha}{2\alpha} - \dfrac{\sin z}{z} = 0$

The first root $a_1 = 2\alpha, b_1 = 0$ of Table 7.2B corresponds to the root $s = 0$ of function $G_2(s,\alpha)$. The root $s = -1, z = 0$ is lost while passing to the equations in (4.4.1). The asymptotic values of the roots (for large k) are given by

$$\left.\begin{array}{ll} \mathrm{A} & a_k \approx \left(2k - \dfrac{1}{2}\right)\pi, \quad b_k \approx \ln\left(\dfrac{\sin 2\alpha}{2\alpha} a_k\right), \\ \mathrm{B} & a_k \approx \left(2k + \dfrac{1}{2}\right)\pi, \quad b_k \approx \ln\left(\dfrac{\sin 2\alpha}{2\alpha} a_k\right). \end{array}\right\} \tag{4.4.3}$$

The general representation of formula (4.2.13) can be set in the form

$$f(r,\theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{f}(s,\theta)\, \rho^{-s} ds, \tag{4.4.4}$$

where $c = -1-\varepsilon$, where ε is sufficiently small. Applying the residue theorem leads to different analytic representations of solutions $f(r,\theta)$ for $\rho < 1$ and $\rho > 1$. When $\rho < 1$, in other words, in the region adjacent to the wedge's vertex the calculation is performed at the poles lying to the left of the straight line $s = -1$

$$s_k^- = -\left[1 + \frac{1}{2\alpha}(a_k \pm i b_k)\right], \tag{4.4.5}$$

whereas the continuation of the solution in the region $\rho > 1$ describing the behaviour of the solution at infinity is constructed by means of the poles lying on the right of this line

$$s_k^+ = -1 + \frac{1}{2\alpha}(a_k \pm i b_k). \tag{4.4.6}$$

The stress functions for $\rho < 1$ are then presented by the series of the type

$$f(r,\theta) = \sum_k \rho^{1+a_k/2\alpha}\left[A_k(\theta)\, \rho^{ib_k/2\alpha} + \bar{A}_k(\theta)\, \rho^{-ib_k/2\alpha}\right], \tag{4.4.7}$$

and the stresses at the vertex of the wedge tend either to zero or infinity depending upon the sign of the inequality

$$\frac{a_k}{2\alpha} - 1 \gtrless 0, \quad a_k \gtrless 2\alpha. \tag{4.4.8}$$

The analysis of the roots shows that under symmetric loading (equation $\Delta_A(z) = 0$) the above inequality has the upper sign for $0 < 2\alpha < \pi$ whereas for $\pi < 2\alpha < 2\pi$ there is a real-valued root smaller than 2α. In the skew-symmetric case (equation $\Delta_B(z) = 0$) the change in the inequality sign occurs at $2\alpha = 2\alpha_*$ given by eq. (4.2.19) and the stresses at the wedge's vertex tends to zero for $2\alpha < 2\alpha_*$ and to infinity for $2\alpha > 2\alpha_*$, respectively.

For example, in the case of a symmetric load for $\alpha = \pi/4$ we have $\alpha_1 = 4,303 > \pi/2$ whilst for $\alpha = 5\pi/8$ we have $\alpha_1 = 2,303 > 1,25\pi$. That is, in the first case the stresses at the vertex vanish whilst in the second case they are unbounded. In the case of a skew-symmetric load for the above values of α we have respectively $\alpha_2 = 7,553 > \pi/2, \alpha_2 = 5,114 > 1,25\pi$ and the stresses are absent whereas for $\alpha_* < \alpha$ there is a root $\alpha_2 < 2\alpha$.

According to eq. (4.4.6) the stress function for $\rho > 1$ is given by the series

$$f(r,\theta) = \sum_k \rho^{1-a_k/2\alpha}\left[B_k(\theta)\rho^{ib_k/2\alpha} + \bar{B}_k(\theta)\rho^{-ib_k/2\alpha}\right], \qquad (4.4.9)$$

where among the roots there is $z = a = 0$ whereas for a skew-symmetric load there is also a root $a_1 = 2\alpha$. The stresses at infinity are of the order $\rho^{(1+a_k/2\alpha)}$. The principal term of these series under both symmetric and skew-symmetric load is due to the root $z = a\,(s = -1)$ and this is the solution (4.2.17) determined by the principal vector of the surface forces. For a symmetric load, the stresses due to the next term of series (4.4.9) decreases, as $\rho \to \infty$, essentially faster than ρ^{-1}. For a skew-symmetric load and vanishing the principal vector of the load, the stresses due to the principal moment (root a_1) are of the order of ρ^{-2} however for $\alpha > \alpha_*$ this term is not the principal one.

Further consideration is carried out for the case of symmetric loading by the pressure p uniformly distributed over the parts $0 < r < r_0, \theta = \pm\alpha$. Using eq. (4.2.11) we have

$$\left.\begin{aligned} q_1(\rho) &= -p_0\delta_0(1-\rho), \\ \delta_0(1-\rho) &= \begin{cases} 1, & 0<\rho<1 \\ 0, & \rho>1, \end{cases} \\ \bar{q}_1(\rho) &= -p_0\int_0^1 \rho^{s+1}ds = -\frac{p_0}{s+2}\,. \end{aligned}\right\} \qquad (4.4.10)$$

Returning to formula (4.2.13A) we obtain the following expression for the normal stress σ_r

$$\sigma_r = -\frac{p_0}{2\pi i}\times \qquad (4.4.11)$$

$$\int_{c-i\infty}^{c+i\infty} ds \frac{(s+4)\sin s\alpha\cos(s+2)\theta - (s+2)\sin(s+2)\alpha\cos s\theta}{[(s+1)\sin 2\alpha + \sin 2(s+1)\alpha](s+2)}\rho^{-2-s}.$$

It simplifies considerably for $\theta = \pm\alpha$, indeed, transforming the numerator to the form

$$(s+4)\sin s\alpha\cos(s+2)\alpha - (s+2)\sin(s+2)\alpha\cos s\alpha =$$
$$= (s+1)\sin 2\alpha - 2(s+2)\sin 2\alpha + \sin 2(s+1)\alpha$$

and referring to eq. (4.4.10) we obtain

$$\sigma_r\big|_{\theta=\pm\alpha} = -p_0\delta_0(1-\rho) + \frac{p_0\sin 2\alpha}{i\pi}\int\limits_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\frac{\rho^{-z/2\alpha-1}dz}{z\sin 2\alpha + 2\alpha\sin z}, \tag{4.4.12}$$

where the integration path is a straight line which is parallel to the imaginary axis of the plane $z = 2(s+1)\alpha$, lies on the left to this axis and is infinitesimally close to it ($\varepsilon \to +0$).

For $\rho > 1$ the integral is equal to the product of $2\pi i$ and the sum of the residues at the poles

$$z_k = -a_k + ib_k, \quad \bar{z}_k = -a_k - ib_k \quad (a_k > 0)$$

in the left half-plane, hence

$$\sigma_r\big|_{\theta=\pm\alpha} = -p_0 + 2p_0\sin 2\alpha \times \tag{4.4.13}$$
$$\sum_{k=1}^{\infty}\rho^{-a_k/2\alpha-1}\left[\frac{e^{ib_k/2\alpha\times\ln\rho}}{\sin 2\alpha + 2\alpha\cos z_k} + \frac{e^{-ib_k/2\alpha\times\ln\rho}}{\sin 2\alpha + 2\alpha\cos \bar{z}_k}\right].$$

For $\rho > 1$ the poles of the integrand in the right half-plane

$$z_k = a_k + ib_k, \quad \bar{z}_k = a_k - ib_k$$

includes the pole at zero. The sum of the residues is multiplied by $-2\pi i$. We obtain

$$\rho > 1: \ \sigma_r\big|_{\theta=\pm\alpha} = -\frac{2p_0r_0}{r}\frac{\sin 2\alpha}{2\alpha+\sin 2\alpha} - \tag{4.4.14}$$
$$2p_0\sin 2\alpha\sum_{k=1}^{\infty}\rho^{-(a_k/2\alpha+1)}\left[\frac{e^{-ib_k/2\alpha\times\ln\rho}}{\sin 2\alpha + 2\alpha\cos z_k} + \frac{e^{ib_k/2\alpha\times\ln\rho}}{\sin 2\alpha + 2\alpha\cos \bar{z}_k}\right].$$

The series converges rapidly as the numbers a_k increase rapidly with the growth of k.

The integral in formula (4.4.12) can be presented in a real-valued form by splitting it into the integrals over the imaginary axis from $-\infty$ to 0, the semicircle of radius $\varepsilon \to 0$ on the left to the origin of the coordinate system and the imaginary axis from 0 to ∞. Then

$$\sigma_r\big|_{\theta=\pm\alpha} = -p_0\delta_0(1-\rho) -$$
$$\frac{r_0p_0\sin 2\alpha}{r}\left[\frac{1}{2\alpha+\sin 2\alpha} + \frac{2}{\pi}\int\limits_0^{\infty}\frac{\sin(u\ln\rho)\,du}{u\sin 2\alpha + \sinh 2\alpha u}\right]. \tag{4.4.15}$$

The integral can be evaluated numerically. The difficulties associated with this are caused by considerable oscillations of the function $\sin(u\ln\rho)$ for small and large values of ρ.

7.5 Boundary-value problems of the plane theory of elasticity

7.5.1 *Classification of regions*

Let the part of the region occupied by the material and the remaining part be denoted L and R respectively. The consideration will be limited to the following cases: (i) a simply-connected finite region with an opening, (ii) an infinite region with an opening and (iii) a double-connected ring-shaped region. In the first case the boundary is a smooth closed contour Γ which has no corner points and does not cross itself; in the second case such a contour bounds L from inside whilst an infinite point $z = \infty$ is assumed to belongs to the boundary as well; and in the third case the boundary Γ breaks down into two contours: external Γ_0 and internal Γ_1. By traversing the boundary in the positive direction the region L must lie on the left. In other words, the traversing a finite simply-connected region is in the counterclockwise direction, whereas for a double-connected region traversing Γ_0 is counterclockwise and traversing Γ_1 is clockwise. According to this rule, the integral along the contour of the region is presented as follows

$$\text{(i)} \oint_{\Gamma}, \quad \text{(ii)} \oint_{\Gamma}, \quad \text{(iii)} \oint_{\Gamma} = \oint_{\Gamma_0} + \oint_{\Gamma_1}. \tag{5.1.1}$$

In the following we consider two simple boundary-value problems. They determine the stresses in L in terms of either the displacement vector on Γ (the first one) or the surface forces prescribed on Γ (the second one). The solution is based on an evident assumption that both the stresses and the displacements (in the case of no distortion) in L and Γ are single-valued. Except for the points where the force singularities are applied, these solutions are continuous and have derivatives of any order since they are solutions of the equations of the elliptic type. This imposes certain restrictions onto Muskhelishvili's functions $\varphi(z)$ and $\psi(z)$, namely that the stresses and the displacements must satisfy the above requirements. In particular, functions $\varphi(z)$ and $\psi(z)$ are holomorphic in the simple-connected region. In two remaining cases it is possible to split the solution into two parts: holomorphic parts $\varphi_*(z), \psi_*(z)$ and the logarithmic (multivalued) parts.

7.5.2 *Boundary-value problems for the simply-connected finite region*

Let the conformal transformation

$$z = \omega(\zeta) \tag{5.2.1}$$

map the interior of the unit circle $|\zeta| < 1$ onto region L and let $\zeta = \sigma = e^{i\theta}$ denote the value of ζ on the circle $|\zeta| = 1$ designated by γ. Without loss of generality we can assume that $\omega(0) = 0$, i.e. the centre of the circle corresponds to the origin of the coordinate system $z = 0$ in region L. The assumption

$$\omega'(\zeta) \neq 0, \quad |\zeta| \leq 1, \tag{5.2.2}$$

ensures (for $|\zeta| < 1$) a unique solution of eq. (5.2.1) for ζ in L. Holding condition (5.2.2) on γ is guaranteed by the absence of the corner points on Γ (the boundary of region L). Hence

$$z = \omega(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^n \quad (c_1 = \omega'(0) \neq 0), \tag{5.2.3}$$

where c_1 can be taken as being a real-valued number. Taking the inverse of series (5.2.3) we arrive at the representation $\zeta = \zeta(z)$ in the form of a power series convergent in L.

Muskhelishvili's functions $\varphi(z), \psi(z)$ are holomorphic in L and can be presented in γ be the holomorphic functions

$$\varphi(\omega(\zeta)) = \varphi_1(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n, \quad \psi(\omega(\zeta)) = \psi_1(\zeta) = \sum_{n=0}^{\infty} A'_n \zeta^n. \tag{5.2.4}$$

This establishes that functions $\varphi(z), \psi(z)$ can be represented by the power series in z

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} a'_n z^n \tag{5.2.5}$$

convergent in L.

The complex number defining the unit vector of the outward normal to Γ is denoted as follows

$$n = e^{i\alpha} = \cos\alpha + i\sin\alpha = \frac{dy}{ds} - i\frac{dx}{ds} = -i\frac{dz}{ds} = -i\omega'(\sigma)\frac{d\sigma}{ds}.$$

We have on Γ

$$ds = |dz| = |\omega'(\sigma)|\,|d\sigma| = \sqrt{\omega'(\sigma)\,\bar{\omega}'(\bar{\sigma})}\,|d\sigma|$$

and on γ

$$d\sigma = i\sigma d\theta, \quad |d\sigma| = d\theta, \quad \bar{\sigma} = \frac{1}{\sigma}. \tag{5.2.6}$$

Hence

$$n = -i\frac{dz}{ds} = \sigma\sqrt{\frac{\omega'(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}}, \quad \bar{n} = \frac{d\bar{z}}{ds} = \frac{1}{\sigma}\sqrt{\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\omega'(\sigma)}}. \tag{5.2.7}$$

Under transformation (5.2.1) the functions $\Phi(z), \Psi(z)$ take the form

$$\Phi(z) = \varphi'(z) = \frac{\varphi_1'(\zeta)}{\omega'(\zeta)}, \quad \Psi(z) = \psi'(z) = \frac{\psi_1'(\zeta)}{\omega'(\zeta)}. \tag{5.2.8}$$

The vector of the surface force F on Γ

$$F = F_x + iF_y = (F_n + iF_t)\, n, \quad F_n + iF_t = \bar{n}(F_x + iF_y), \tag{5.2.9}$$

where F_n, F_t denote the projections of F on the normal and the tangent to Γ, can be written, by virtue of eqs. (1.13.4) and (1.14.4), in the form

$$\left.\begin{aligned} F_x + iF_y &= \left\{\left[\Phi(z) + \bar{\Phi}(\bar{z})\right] n - \left[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\right]\bar{n}\right\}_\Gamma, \\ F_n + iF_t &= \left\{\Phi(z) + \bar{\Phi}(\bar{z}) - \left[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\right]\bar{n}^2\right\}_\Gamma. \end{aligned}\right\} \tag{5.2.10}$$

In particular, when Γ is a circle, we return to formulae (1.14.9).

The principal vector of the surface forces on arc l of contour Γ is given by formula (1.14.7)

$$\begin{aligned} -Q + iP &= \left[\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z})\right]_\Gamma \\ &= \varphi_1(\sigma) + \frac{\omega(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}\bar{\varphi}_1'\left(\frac{1}{\sigma}\right) + \bar{\psi}_1\left(\frac{1}{\sigma}\right), \end{aligned} \tag{5.2.11}$$

where the left hand side is determined up to an additive constant. Another form of the condition on Γ in the second boundary-value problem is given by formula (5.2.10)

$$\begin{aligned} F_n + iF_t &= \left\{\Phi(z) + \bar{\Phi}(\bar{z}) - \bar{n}^2\left[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\right]\right\}_\Gamma \\ &= \left\{\frac{\varphi_1'(\zeta)}{\omega'(\zeta)} + \frac{\bar{\varphi}'\left(\frac{1}{\sigma}\right)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)} - \frac{1}{\sigma^2\omega'(\sigma)}\left[\frac{\omega(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}\bar{\varphi}_1''\left(\frac{1}{\sigma}\right) - \right.\right. \\ &\qquad \left.\left. \omega(\sigma)\frac{\bar{\omega}''\left(\frac{1}{\sigma}\right)}{\bar{\omega}'^2\left(\frac{1}{\sigma}\right)}\bar{\varphi}_1'\left(\frac{1}{\sigma}\right) + \bar{\psi}_1'\left(\frac{1}{\sigma}\right)\right]\right\}. \end{aligned} \tag{5.2.12}$$

We introduce the notion

$$\Phi(z) = \Phi_1(\zeta) = \frac{\varphi_1'(\zeta)}{\omega'(\zeta)}, \quad \Psi(z) = \Psi_1(\zeta) = \frac{\psi_1'(\zeta)}{\omega'(\zeta)}. \tag{5.2.13}$$

By eq. (5.2.2) these functions are holomorphic in γ. It is evident that

$$\Phi'(z) = \frac{\Phi_1'(\zeta)}{\omega'(\zeta)} = \frac{\bar{\varphi}_1''(\zeta)}{\omega'^2(\zeta)} - \frac{\varphi_1'(\zeta)}{\omega'^3(\zeta)}\omega''(\zeta), \tag{5.2.14}$$

which allows one to rewrite eq. (5.2.12) as follows

$$F_n + iF_t = \Phi_1(\sigma) + \bar{\Phi}_1\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2\omega'(\sigma)}\left[\omega(\sigma)\bar{\Phi}_1'\left(\frac{1}{\sigma}\right) + \bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\Psi}_1\left(\frac{1}{\sigma}\right)\right]. \quad (5.2.15)$$

The condition on Γ of the first boundary-value problem is set in the form

$$\left[(3-4\nu)\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})\right]_\Gamma = \left[(3-4\nu)\varphi_1(\sigma) - \frac{\omega(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}\bar{\varphi}_1'\left(\frac{1}{\sigma}\right) - \bar{\psi}_1\left(\frac{1}{\sigma}\right)\right] = 2\mu(u+iv)_\Gamma. \quad (5.2.16)$$

Due to eq. (5.2.11)

$$2\mu(u+iv)_\Gamma + i(P+iQ)_\Gamma = 4(1-\nu)\varphi(z)|_\Gamma, \quad (5.2.17)$$

that is, if $\varphi(z)|_\Gamma$ is obtained from the first or second boundary-value problem, one can immediately obtain $(P+iQ)_\Gamma$ (or $(u+iv)_\Gamma$) in terms of vector $u+iv$ (or $P+iQ$) prescribed on the contour.

By definition, the functions which are holomorphic in a region are single-valued in it. For this reason, the solutions of the boundary-value problems, represented in terms of Muskhelishvili's functions in a simply-connected finite region, ensures that the stresses and the displacements are single-valued. It can be concluded from formulae (5.2.11) and (5.2.16) that the single-valuedness of functions $\varphi(z), \psi(z), \chi(z)$ leads to a zero principal vector and a zero principal moment of the surface forces on Γ (as well as on any closed contour in L). Conversely, the condition that the surface forces are statically equivalent to zero guarantees the single-valuedness of the above functions and, thus, the existence of the solution.

7.5.3 Definiteness of Muskhelishvili's functions

According to eq. (1.14.4) the sum of the normal stresses determines the real part of function $\Phi(z)$. The imaginary part is determined up to an additive constant C and thus function $\Phi(z)$ is determined up to an imaginary constant iC and $\varphi(z)$ is determined up to the term

$$iCz + \alpha + i\beta$$

which is linear in z. The second formula in eq. (1.14.4) completely determines $\Psi(z)$ whereas $\psi(z)$ is determined in terms of $\Psi(z)$ up to the constant term

$$\alpha' + i\beta'.$$

Thus, replacing

$$\varphi(z) \text{ by } \varphi(z) + iCz + \alpha + i\beta, \quad \psi(z) \text{ by } \psi(z) + \alpha' + i\beta' \tag{5.3.1}$$

the stresses do not change whilst the displacement vector, using eq. (5.2.16), gains the term

$$\frac{1}{2\mu}\left[4(1-\nu)Ciz + (3-4\nu)(\alpha+i\beta) - (\alpha' - i\beta')\right], \tag{5.3.2}$$

which corresponds to a rigid-body displacement of the figure in its plane. Thus, by means of a proper choice of $C, \alpha, \beta, \alpha', \beta'$, we can take

$$\varphi(0) = 0, \quad \operatorname{Im}\varphi'(0) = 0, \quad \psi(0) = 0, \tag{5.3.3}$$

which uniquely determines functions $\varphi(z), \psi(z)$.

The structure of expression (5.3.2) suggests that when solving the first boundary-value problem one of functions $\varphi(z), \psi(z)$ can be subjected to the condition

$$\varphi(0) = 0 \qquad \text{or} \qquad \psi(0) = 0. \tag{5.3.4}$$

7.5.4 Infinite region with an opening

The conformal transformation of the exterior of the unit circle (region $|\zeta| > 1$) on region L which is an infinite plane bounded from inside by the closed smooth contour Γ is given by the function

$$z = \omega(\zeta) = c_0\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots, \tag{5.4.1}$$

the infinite point of plane ζ being mapped onto the infinite point in plane z. We assume that the condition

$$\omega'(\zeta) \neq 0, \quad |\zeta| \geq 1 \tag{5.4.2}$$

holds, which ensures that eq. (5.4.1) is uniquely resolved for ζ. Constant c_0 can be taken as being real-valued.

Referring to eq. (3.1.8) we introduce the functions

$$\varphi_0(z) = -\frac{X+iY}{8\pi(1-\nu)}\ln z, \quad \psi_0(z) = (3-4\nu)\frac{X-iY}{8\pi(1-\nu)}\ln z, \tag{5.4.3}$$

which determine the state of stress in the infinite plane loaded by force $X+iY$ in the origin of the coordinate system (i.e. at the point which does not belong to the considered region L). It is evident that this force is equal to the principal vector of the external forces on any closed contour enclosing point $z = 0$, in particular on the contour of the opening Γ. Let us recall that the displacement vector obtained in terms of functions $\varphi_0(z), \psi_0(z)$ is single-valued.

Hence, introducing functions $\varphi_*(z), \psi_*(z)$ which are single-valued in L, assuming

$$\varphi(z) = \varphi_0(z) + \varphi_*(z), \quad \psi(z) = \psi_0(z) + \psi_*(z) \tag{5.4.4}$$

and referring to eq. (5.2.11) we can put the condition on Γ for the second boundary-value problem in the form

$$i\left[\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z})\right]_\Gamma = P_* + iQ_*, \tag{5.4.5}$$

where $P_* + iQ_*$ denotes the principal vector of the external surface forces distributed along arc l. The change of sign is due to the fact that in the above formula $P+iQ$ denotes the principal vector of the stresses which are distributed along arc l and acting from the side of the elastic medium.

Another form of condition (5.4.5) is

$$\begin{aligned}\varphi_*(z) + z\bar{\varphi}_*'(\bar{z}) + \bar{\psi}_*(\bar{z}) = Q_* - iP_* + \\ \frac{1}{8\pi(1-\nu)}\left\{(X+iY)\left[\ln z - (3-4\nu)\ln\bar{z}\right] + (X-iY)\frac{z}{\bar{z}}\right\}.\end{aligned} \tag{5.4.6}$$

The problem is reduced to determination of functions $\varphi_*(z), \psi_*(z)$ which are single-valued in L and requires the condition at infinity to be prescribed. It follows from the structure of Kolosov-Muskhelishvili's formulae, eq. (1.14.4), that the stresses obtained in terms of $\varphi_0(z), \psi_0(z)$ are equal to zero at infinity, i.e. we deal with $\varphi_*(z), \psi_*(z)$. It is easy to conclude from these formulae that the positive powers of z, i.e. from z to z^n inclusive, result in the stresses growing as $|z|^{n-1}$ at infinity. Hence the representations

$$\left.\begin{aligned}\varphi_*(z) &= a_0 + ib_0 + (a_1+ib_1)z + \frac{1}{2}(a_2+ib_2)z^2 + \varphi_{**}(z), \\ \psi_*(z) &= a_0' + ib_0' + (a_1'+ib_1')z + \frac{1}{2}(a_2'+ib_2')z^2 + \psi_{**}(z),\end{aligned}\right\} \tag{5.4.7}$$

with holomorphic and vanishing at infinity functions

$$\varphi_{**}(z) = \sum_{k=1}^{\infty}\frac{\alpha_k + i\beta_k}{z^k}, \quad \psi_{**}(z) = \sum_{k=1}^{\infty}\frac{\alpha_k' + i\beta_k'}{z^k}, \tag{5.4.8}$$

lead to the stresses

$$\left.\begin{aligned}\frac{1}{2}\left(\sigma_x^\infty + \sigma_y^\infty\right) &= 2a_1 + (a_2+ib_2)z + (a_2-ib_2)\bar{z}, \\ \frac{1}{2}\left(\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty\right) &= (a_2+ib_2)\bar{z} + a_1' + ib_1' + (a_2+ib_2)z,\end{aligned}\right\} \tag{5.4.9}$$

which are linear in x, y at infinity and the following displacement vector

$$\begin{aligned}2\mu\left(u^{\infty}+iv^{\infty}\right) = {} & 2\left(1-2\nu\right)a_1 z-\left(a_1'-ib_1'\right)\bar{z}-\left(a_2-ib_2\right)z\bar{z}+\\ & \frac{1}{2}\left(3-4\nu\right)\left(a_2+ib_2\right)z^2-\frac{1}{2}\left(a_2'-ib_2'\right)\bar{z}^2+\\ & \left[4\left(1-\nu\right)ib_1 z+\left(3-4\nu\right)\left(a_0+ib_0\right)-\left(a_0'-ib_0'\right)\right]. \qquad (5.4.10)\end{aligned}$$

Following Subsection 7.5.3 we put

$$a_0+ib_0=0, \quad b_1=0, \quad a_0'+ib_0'=0, \qquad (5.4.11)$$

which does not affect the stresses and eliminates a rigid-body displacement from eq. (5.4.10).

The expressions obtained result in stresses linear in x, y at infinity. In the forthcoming notion we assume that the state of stress is homogeneous at infinity, then

$$a_2+ib_2=0,\ a_2'+ib_2'=0,$$

and the remaining coefficients a_1, a_1', b_1' can be expressed in terms of the principal stresses at infinity $\sigma_1^{\infty}, \sigma_2^{\infty}$ and the angle of the principal axis of tensor $\hat{T}^{\infty}$ to axis Ox. By eq. (5.4.9)

$$\sigma_1^{\infty}+\sigma_2^{\infty}=4a_1,\ \sigma_2^{\infty}-\sigma_1^{\infty}=\left(\sigma_y^{\infty}-\sigma_x^{\infty}+2i\tau_{xy}^{\infty}\right)e^{2i\alpha}=2\left(a_1'+ib_1'\right)e^{2i\alpha} \qquad (5.4.12)$$

and returning to eq.(5.4.7) we obtain

$$\varphi_*\left(z\right)=\frac{1}{4}\left(\sigma_1^{\infty}+\sigma_2^{\infty}\right)z+\varphi_{**}\left(z\right),\ \psi_*\left(z\right)=\frac{1}{2}\left(\sigma_2^{\infty}-\sigma_1^{\infty}\right)e^{-2i\alpha}z+\psi_{**}\left(z\right). \qquad (5.4.13)$$

By eq. (1.14.10) the constant b_1 can be expressed in terms of the rotation at infinity

$$b_1=\frac{\mu}{2\left(1-\nu\right)}\varepsilon^{\infty}. \qquad (5.4.14)$$

Functions $\Phi\left(z\right), \Psi\left(z\right)$ are single-valued in L and their expressions have the form

$$\left.\begin{aligned}\Phi\left(z\right) &= \frac{1}{4}\left(\sigma_1^{\infty}+\sigma_2^{\infty}\right)+\frac{i\mu\varepsilon^{\infty}}{2\left(1-\nu\right)}-\frac{X+iY}{8\pi\left(1-\nu\right)}\frac{1}{z}+\Phi_{**}\left(z\right),\\ \Psi\left(z\right) &= \frac{1}{2}\left(\sigma_2^{\infty}-\sigma_1^{\infty}\right)e^{-2i\alpha}+\left(3-4\nu\right)\frac{X-iY}{8\pi\left(1-\nu\right)}\frac{1}{z}+\Psi_{**}\left(z\right),\end{aligned}\right\} \qquad (5.4.15)$$

where the series for $\Phi_{**}\left(z\right), \Psi_{**}\left(z\right)$ begin with the z^{-2} terms.

By eq. (3.1.14), the imaginary part of the coefficients associated with z^{-2} in the expression for $\Psi_{**}(z)$ is determined by the principal moment of the external forces on Γ. The condition on Γ in the form analogous to eq. (5.2.10) is written down in terms of $\Phi(z), \Psi(z)$

$$F_x + iF_y = \left\{\left[\Phi(z) + \bar{\Phi}(\bar{z})\right] n_* - \left[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\right] \bar{n}_*\right\}_\Gamma, \tag{5.4.16}$$

where n_* denotes the vector of the normal external to L (i.e. directed into the opening) and $F_x + iF_y$ denotes the surface forces on Γ. In the particular case of the free contour of the opening and the homogeneous state of stress at infinity the boundary conditions (5.4.6) or (6.4.16) take the form

$$\left.\begin{aligned} &\varphi_{**}(z) + z\bar{\varphi}'_{**}(\bar{z}) + \bar{\psi}'_{**}(\bar{z}) = \\ &\qquad = -\left[\frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right) z + \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right) \bar{z} e^{2i\alpha}\right], \\ &\Phi_{**}(z) + \bar{\Phi}_{**}(\bar{z}) + \frac{d\bar{z}}{dz}\left[z\bar{\Phi}'_{**}(\bar{z}) + \bar{\Psi}_{**}(\bar{z})\right] = \\ &\qquad = -\frac{1}{2}\left[\left(\sigma_1^\infty + \sigma_2^\infty\right) + \left(\sigma_2^\infty - \sigma_1^\infty\right)\frac{d\bar{z}}{dz} e^{2i\alpha}\right]. \end{aligned}\right\} \tag{5.4.17}$$

The literature on the solution of the stated boundary-value problem related to determining the stress contraction near the opening is extensive, see Subsections 7.8.1-7.8.3.

The expression for the displacement vector is set in the form

$$\begin{aligned} 2\mu(u + iv) = {} & \left[(3 - 4\nu)\varphi_{**}(z) - z\bar{\varphi}'_{**}(\bar{z}) - \bar{\psi}_{**}(\bar{z})\right] + 2\mu i z \varepsilon^\infty + \\ & \frac{1}{2}\left[(1 - 2\nu)\left(\sigma_1^\infty + \sigma_2^\infty\right) z - \left(\sigma_2^\infty - \sigma_1^\infty\right) \bar{z} e^{2i\alpha}\right] - \\ & \frac{3 - 4\nu}{8\pi(1 - \nu)}(X + iY)\ln z\bar{z} + \frac{X - iY}{8\pi(1 - \nu)}\frac{z}{\bar{z}}, \end{aligned} \tag{5.4.18}$$

and this equality is the condition on Γ of the first boundary-value problem. It is deficient to prescribe only the displacement vector on Γ. It is also necessary to prescribe the principal vector of the surface forces on Γ. The requirement of vanishing the displacement at infinity which can be stated in the three-dimensional problem in the case of zero stresses at infinity (see for example Subsection 4.3.5) holds true only if $X + iY = 0$.

Let us notice that by introducing functions $\varphi_1(\zeta), \psi_1(\zeta), \Phi_1(\zeta), \Psi_1(\zeta)$ by the formulae, similar to eqs. (5.2.8), (5.2.13), we arrive at the boundary-value problems on the circle γ of the opening. They reduce to searching functions $\varphi_{1**}(\zeta), \psi_{1**}(\zeta)$ which are holomorphic at infinity. While using variable ζ one can replace $\ln z$ by $\ln\zeta$ since the second term in the equality

$$\ln z = \ln\zeta + \ln\left(c_0 + \frac{c_1}{\zeta^2} + \frac{c_2}{\zeta^3} + \ldots\right)$$

can be understood as the part of functions $\varphi_1(\zeta), \psi_1(\zeta)$ which are holomorphic at infinity.

7.5.5 Double-connected region. Distortion

In the case of a double-connected region bounded by the interior Γ_1 and exterior Γ_0 contours each of the functions $\varphi(z), \psi(z)$ can be represented as a sum

$$\varphi(z) = \varphi_1(z) + \varphi_2(z), \quad \psi(z) = \psi_1(z) + \psi_2(z),$$

where $\varphi_1(z), \psi_1(z)$ are holomorphic in the finite region bounded by the exterior Γ_0 contour Γ_0 and $\varphi_2(z), \psi_2(z)$ are given by

$$\varphi_2(z) = -\frac{X+iY}{8\pi(1-\nu)} \ln z + \varphi_{2*}(z),$$
$$\psi_2(z) = \frac{3-4\nu}{8\pi(1-\nu)} (X - iY) \ln z + \psi_{2*}(z),$$

where $\varphi_2(z), \psi_2(z)$ are holomorphic outside of Γ_1 and $X+iY$ denotes the principal vector of the surface forces on Γ_1. Thus, $\varphi(z), \psi(z)$ are represented in the form

$$\varphi(z) = -\frac{X+iY}{8\pi(1-\nu)} \ln z + \varphi_*(z),$$
$$\psi(z) = \frac{3-4\nu}{8\pi(1-\nu)} (X - iY) \ln z + \psi_*(z) \tag{5.5.1}$$

where the expansions $\varphi_*(z), \psi_*(z)$ contain both positive and negative powers of z

$$\varphi_*(z) = \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k) z^k, \; \psi_*(z) = \sum_{k=-\infty}^{\infty} (\alpha'_k + i\beta'_k) z^k. \tag{5.5.2}$$

The boundary conditions are given on Γ_0, Γ_1, see Subsections 7.5.4 and 7.5.2.

When determining the state of stress in the double-connected region subjected to the distortion without load, functions $\varphi(z), \psi(z)$ should be determined by the conditions

$$\left.\begin{aligned}
&\Delta_{\Gamma_*}\left[\varphi(z) + \bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z})\right] = 0, \\
&2\mu\Delta_{\Gamma_*}(u + iv) = \Delta_{\Gamma_*}\left[(3-4\nu)\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})\right] \\
&\qquad\qquad = 2\mu\left[(c_1 + ic_2) + ib_3 z\right].
\end{aligned}\right\} \tag{5.5.3}$$

The first condition means that the principal vector of the stresses vanishes on any closed contour Γ_* which can not be reduced to a point by a continuous mapping in L. The second condition describes a jump in the displacement vector given by the translational and rotational distortions (constants c_1, c_2, b_3 , see Subsection 2.2.4).

These conditions can be satisfied by adopting

$$\varphi(z) = Az \ln z + (\gamma + i\delta) \ln z + \varphi_* (z), \quad \psi(z) = \left(\gamma' + i\delta'\right) \ln z + \psi_* (z), \tag{5.5.4}$$

where $\varphi_* (z), \psi_* (z)$ are single-valued in L and thus can be represented in the form of eq. (5.5.2). Indeed,

$$\Delta_{\Gamma_*} \varphi(z) = 2\pi i \left[zA + (\gamma + i\delta)\right], \quad \Delta_{\Gamma_*} z\bar{\varphi}'(\bar{z}) = -2\pi i z\bar{A},$$
$$\Delta_{\Gamma_*} \bar{\psi}(\bar{z}) = -2\pi i \left(\gamma' - i\delta'\right)$$

and conditions (5.5.3) are reduced to the equations

$$z\left(A - \bar{A}\right) + (\gamma + i\delta) - \left(\gamma' - i\delta'\right) = 0;$$
$$2\pi i \left[(3 - 4\nu)(zA + \gamma + i\delta) + z\bar{A} + \gamma' - i\delta'\right] = 2\mu\left(c + ic_2 + ib_3 z\right).$$

These yield

$$A = \bar{A}, \quad \operatorname{Im} A = 0; \quad \gamma + i\delta = \gamma' - i\delta';$$
$$4\pi(1 - \nu) A = \mu b_3, \quad 4\pi(1 - \nu)(\gamma + i\delta) = -\mu i (c_1 + ic_2),$$

such that

$$\left.\begin{aligned} \varphi(z) &= \frac{\mu b_2}{4\pi(1-\nu)} z \ln z - \frac{\mu i (c_1 + ic_2)}{4\pi(1-\nu)} \ln z + \varphi_*(z), \\ \psi(z) &= \frac{\mu i (c_1 - ic_2)}{4\pi(1-\nu)} \ln z + \psi_*(z). \end{aligned}\right\} \tag{5.5.5}$$

Functions $\varphi_* (z), \psi_* (z)$ are determined in terms of the boundary conditions of free contours Γ_0, Γ_1.

7.5.6 Representing the stress function in the double-connected region (Michell)

In the general case of the distortion and loading on contours Γ_0, Γ_1 functions $\varphi(z), \psi(z)$ in the double-connected region are as follows

$$\left.\begin{aligned} \varphi(z) &= \frac{\mu b_3}{4\pi(1-\nu)} z \ln z - \\ &\qquad \frac{1}{8\pi(1-\nu)} \left[(X + iY) + 2i\mu(c_1 + ic_2)\right] \ln z + \varphi_*(z), \\ \psi(z) &= \frac{1}{8\pi(1-\nu)} \left[(3 - 4\nu)(X - iY) + 2i\mu(c_1 - ic_2)\right] \ln z + \psi_*(z), \end{aligned}\right\} \tag{5.6.1}$$

where $\varphi_*(z), \psi_*(z)$ are given by eq. (5.5.2). By virtue of eq. (1.14.3) we have

$$\chi(z) = \frac{1}{8\pi(1-\nu)}\left[(3-4\nu)(X-iY) + 2i\mu(c_1 - ic_2)\right] z\ln z + \left(\alpha'_{-1} + i\beta'_{-1}\right)\ln z + \chi_*(z), \quad (5.6.2)$$

where $\chi_*(z)$ is holomorphic in L and is determined up to an additive constant $\alpha + i\beta$. Using eq. (1.14.2) we arrive at the following representation for Airy's function

$$U = \frac{\theta}{2\pi}\left(xY - yX - 2\pi\beta'_{-1}\right) + \left[\frac{\mu b_3}{4\pi(1-\nu)} r^2 + \frac{\mu(c_2 x - c_1 y)}{2\pi(1-\nu)} + \alpha'_{-1}\right]\ln r + \frac{1-2\nu}{4\pi(1-\nu)}(xX + yY)\ln r + U_*(r,\theta), \quad (5.6.3)$$

where $U_*(r,\theta)$ is determined by the single-valued parts of functions $\varphi(z)$ and $\chi(z)$

$$U_*(r,\theta) = \alpha + \alpha_0 x + \beta_0 y + \alpha_1 r^2 + r^2\sum_{k=1}^{\infty} r^k\left(\alpha_{k+1}\cos k\theta - \beta_{k+1}\sin k\theta\right) + \sum_{k=0}^{\infty}\frac{1}{r^k}\left[\alpha_{-(k+1)}\cos(k+2)\theta + \beta_{-(k+1)}\sin(k+2)\theta\right] + \sum_{k=1}^{\infty}\frac{r^k}{k}\left(\alpha'_{k-1}\cos k\theta - \beta'_{k-1}\sin k\theta\right) - \sum_{k=1}^{\infty}\frac{1}{kr^k}\left(\alpha'_{-(k+1)}\cos k\theta + \beta'_{-(k+1)}\sin k\theta\right). \quad (5.6.4)$$

By eqs. (5.6.3) and (3.1.14)

$$\Delta_{\Gamma_*} U = xY - yX - 2\pi\beta'_{-1} = -M^O + xY - yX, \quad \beta'_{-1} = \frac{M^O}{2\pi}, \quad (5.6.5)$$

where M^O is the principal moment, about the coordinate origin, of the external surface forces on contour Γ_1 bounding region L from inside.

The terms in the stress function which are linear in x, y, not affecting the stresses and entering a constant term in the expression for the displacement vector are as follows

$$\alpha + (\alpha_0 + \alpha'_0)x + (\beta_0 - \beta'_0)y. \quad (5.6.6)$$

Omitting them, we obtain Michell's form for Airy's stress function

$$U=\frac{\theta}{2\pi}\left(xY-yX-M^O\right)+\frac{1-2\nu}{4\pi(1-\nu)}(xX+yY)\ln r+$$
$$\frac{\mu}{2\pi(1-\nu)}\left(\frac{1}{2}b_3r^2+c_2x-c_1y\right)\ln r+\alpha_1r^2+\alpha'_{-1}\ln r+$$
$$r^3\left(C_1\cos\theta+S_1\sin\theta\right)+\sum_{n=2}^{\infty}r^n\left[\left(a_n+C_nr^2\right)\cos n\theta+\right.$$
$$\left.\left(b_n+S_nr^2\right)\sin n\theta\right]+\frac{1}{r}\left(\alpha'_1\cos\theta+b'_1\sin\theta\right)+$$
$$\sum_{n=2}^{\infty}\frac{1}{r^n}\left[\left(a'_n+C'_nr^2\right)\cos n\theta+\left(b'_n+S'_nr^2\right)\sin n\theta\right]. \tag{5.6.7}$$

7.5.7 Thermal stresses. Plane strain

The temperature $\theta(x,y)$ is assumed to be independent of coordinate x_3. Considering the case of the plane strain ($\varepsilon_3=0$) and referring to Hooke's law in the form of eq. (1.1.4.2) of Chapter 4 we have

$$\left.\begin{aligned}&\sigma_z-\frac{\nu}{1+\nu}\left(\sigma_x+\sigma_y+\sigma_z\right)+2\mu\alpha\theta=0,\\&\sigma_z=\nu\left(\sigma_x+\sigma_y\right)-2\mu(1+\nu)\theta.\end{aligned}\right\} \tag{5.7.1}$$

Therefore

$$\sigma=\sigma_x+\sigma_y+\sigma_z=(1+\nu)\left(\sigma_1-2\mu\alpha\theta\right),\quad \sigma_1=\sigma_x+\sigma_y \tag{5.7.2}$$

and the non-zero components of the strain tensor are presented in the form

$$\left.\begin{aligned}&2\mu\varepsilon_x=\sigma_x-\nu\sigma_1+2\mu(1+\nu)\alpha\theta,\\&2\mu\varepsilon_y=\sigma_y-\nu\sigma_1+2\mu(1+\nu)\alpha\theta,\quad \mu\gamma_{xy}=\tau_{xy}.\end{aligned}\right\} \tag{5.7.3}$$

There is no doubt that the static relationships (1.2.2) expressing the stresses in terms of Airy's stress function as well as Kolosov's formulae (1.13.2) remain valid. However the stress function is, in general, no longer biharmonic, i.e. it can not be represented in Goursat's form. The Kolosov-Muskhelishvili relationships, Subsection 7.1.14, need to be supplemented with an additional term.

Turning now to Beltrami's dependences, eq. (1.14.13) of Chapter 4, and taking into account eq. (5.7.2) we have

$$\left.\begin{aligned}&\nabla^2\sigma_x+\frac{\partial^2\sigma_1}{\partial x^2}=-2\mu\alpha\frac{1+\nu}{1-\nu}\nabla^2\theta,\\&\nabla^2\sigma_y+\frac{\partial^2\sigma_1}{\partial y^2}=-2\mu\alpha\frac{1+\nu}{1-\nu}\nabla^2\theta,\quad \nabla^2\tau_{xy}+\frac{\partial^2\sigma_1}{\partial x\partial y}=0,\\&\nu\nabla^2\sigma_1-2\mu\alpha(1+\nu)\nabla^2\theta=-2\mu\alpha\frac{1+\nu}{1-\nu}\nabla^2\theta.\end{aligned}\right\} \tag{5.7.4}$$

Substituting $\nabla^2 U$ for σ_1 in the latter equation yields

$$\nabla^2\nabla^2 U = -2\mu\alpha\frac{1+\nu}{1-\nu}\nabla^2\theta, \tag{5.7.5}$$

and it can be proved easily that the remaining relations (5.7.4) hold true. By eq. (5.7.5) we have

$$\nabla^2 U = 4\frac{\partial^2 U}{\partial z\partial\bar{z}} = -2\mu\alpha\frac{1+\nu}{1-\nu}\theta + s\left(x,y\right), \quad \nabla^2 s = 0,$$

where the right hand side contains an additive harmonic function s. According to Subsection 7.1.14 it is denoted as

$$s = 4\,\mathrm{Re}\,\varphi'\left(z\right) = 2\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right].$$

Thus,

$$\frac{\partial^2 U}{\partial z\partial\bar{z}} = \frac{1}{2}\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right] - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\theta\left(z,\bar{z}\right). \tag{5.7.6}$$

We arrive at the following representation for the stress function

$$U = \frac{1}{2}\left[\bar{z}\varphi\left(z\right) + z\bar{\varphi}\left(\bar{z}\right) + \chi\left(z\right) + \bar{\chi}\left(\bar{z}\right)\right] - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\int^{z} d\zeta\int^{\bar{z}}\theta\left(\zeta,\bar{\zeta}\right)d\bar{\zeta}, \tag{5.7.7}$$

and the Kolosov-Muskhelishvili formulae (1.14.4) take the form

$$\left.\begin{aligned}\sigma_1 = \sigma_x + \sigma_y = 2\left[\Phi\left(z\right) + \bar{\Phi}\left(\bar{z}\right)\right] - 2\mu\alpha\frac{1+\nu}{1-\nu}\theta,\\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\Phi'\left(z\right) + \Psi\left(z\right)\right] - 2\mu\alpha\frac{1+\nu}{1-\nu}\int^{\bar{z}}\frac{\partial\theta\left(\zeta,\bar{\zeta}\right)}{\partial z}d\bar{\zeta}.\end{aligned}\right\} \tag{5.7.8}$$

The first formula in eq. (5.7.3) can be represented as follows

$$\begin{aligned}2\mu\frac{\partial u}{\partial x} &= -\sigma_y + \left(1-\nu\right)\sigma_1 + 2\mu\left(1+\nu\right)\alpha\theta\\ &= -\frac{\partial^2 U}{\partial x^2} + 2\left(1-\nu\right)\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right) - \mu\alpha\frac{1+\nu}{1-\nu}\theta\right] + 2\mu\alpha\left(1+\nu\right)\alpha\theta\\ &= -\frac{\partial^2 U}{\partial x^2} + 2\left(1-\nu\right)\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right].\end{aligned}$$

The second formulae is transformed by analogy, to give

$$\begin{aligned}2\mu\frac{\partial u}{\partial x} &= -\frac{\partial^2 U}{\partial x^2} + 2\left(1-\nu\right)\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right],\\ 2\mu\frac{\partial v}{\partial y} &= -\frac{\partial^2 U}{\partial y^2} + 2\left(1-\nu\right)\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right].\end{aligned}$$

Using the rules of differentiation (1.12.3) we obtain

$$\varphi'(z) + \bar{\varphi}'(\bar{z}) = \frac{\partial}{\partial x}[\varphi(z) + \bar{\varphi}(\bar{z})] = -i\frac{\partial}{\partial y}[\varphi(z) - \bar{\varphi}(\bar{z})],$$

and we can take

$$\left.\begin{aligned} 2\mu u &= -\frac{\partial U}{\partial x} + 2(1-\nu)[\varphi(z) + \bar{\varphi}(\bar{z})], \\ 2\mu v &= -\frac{\partial U}{\partial y} - 2i(1-\nu)[\varphi(z) - \bar{\varphi}(\bar{z})] \end{aligned}\right\} \tag{5.7.9}$$

and furthermore

$$\begin{aligned} 2\mu(u+iv) &= -2\frac{\partial U}{\partial \bar{z}} + 4(1-\nu)\varphi(z) \qquad (5.7.10) \\ &= (3-4\nu)\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) + \mu\alpha\frac{1+\nu}{1-\nu}\int^{z}\theta(\zeta,\bar{z})\,d\zeta. \end{aligned}$$

It is not difficult to prove that the third relationship in eq. (5.7.3) holds true. By eq. (5.7.9) we have

$$\begin{aligned} &2\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \\ &\quad = -2\frac{\partial^2 U}{\partial x \partial y} + 2(1-\nu)\left\{\frac{\partial}{\partial y}[\varphi(z) + \bar{\varphi}(\bar{z})] - i\frac{\partial}{\partial x}[\varphi(z) - \bar{\varphi}(\bar{z})]\right\}, \end{aligned}$$

where the expression in the braces vanishes identically, which is required. Clearly, formula (5.7.10) determines the displacement vector up to a plane rigid-body displacement.

7.5.8 Plane stress

By definition (1.4.1)

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0, \tag{5.8.1}$$

and the problem is complicated by the dependence of the stresses, displacements and temperature on x_3. As before, the Laplacian with respect to two variables is denoted by ∇^2 thus the Laplacian with respect to three variables is

$$\nabla_3^2 = \nabla^2 + \frac{\partial^2}{\partial x_3^2}.$$

The expressions (1.2.2) for stresses in terms of the stress function remain valid however the latter depends also on x_3. Therefore, Beltrami's dependences, eq. (1.14.13) of Chapter 4, with the thermal terms are written now in the form

$$\left.\begin{aligned}
&\nabla^2 \frac{\partial^2 U}{\partial y^2} + \frac{\partial^4 U}{\partial x_3^2 \partial y^2} + \frac{1}{1+\nu}\frac{\partial^2 \nabla^2 U}{\partial x^2} = -2\mu\alpha\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{1+\nu}{1-\nu}\nabla_3^2\theta\right),\\
&\nabla^2 \frac{\partial^2 U}{\partial x^2} + \frac{\partial^4 U}{\partial x_3^2 \partial y^2} + \frac{1}{1+\nu}\frac{\partial^2 \nabla^2 U}{\partial y^2} = -2\mu\alpha\left(\frac{\partial^2 \theta}{\partial y^2} + \frac{1+\nu}{1-\nu}\nabla_3^2\theta\right),\\
&\frac{1}{1+\nu}\frac{\partial^2 \nabla^2 U}{\partial x_3^2} = -2\mu\alpha\left(\frac{\partial^2 \theta}{\partial x_3^2} + \frac{1+\nu}{1-\nu}\nabla_3^2\theta\right) =\\
&\qquad\qquad = -2\mu\alpha\left(\frac{2}{1-\nu}\frac{\partial^2 \theta}{\partial x_3^2} + \frac{1+\nu}{1-\nu}\nabla^2\theta\right),\\
&\frac{\partial^2}{\partial x \partial y}\left(-\nabla_3^2 U + \frac{\nabla^2 U}{1+\nu} + 2\mu\alpha\theta\right) = 0,\\
&\frac{\partial^2}{\partial x \partial x_3}\left[\nabla^2 U + 2\mu\alpha\left(1+\nu\right)\theta\right] = 0,\\
&\frac{\partial^2}{\partial y \partial x_3}\left[\nabla^2 U + 2\mu\alpha\left(1+\nu\right)\theta\right] = 0.
\end{aligned}\right\} \tag{5.8.2}$$

Adding the first two equations and making use of the third one we arrive at the relation containing no derivatives with respect to x_3

$$\nabla^2\nabla^2 U + 2\mu\alpha\left(1+\nu\right)\nabla^2\theta = 0, \tag{5.8.3}$$

which allows us to present $\nabla^2 U$ is the form

$$\nabla^2 U = -2\mu\alpha\left(1+\nu\right)\theta + s\left(x,y\right), \quad \nabla^2 s = 0, \quad s = 2\left[\varphi'\left(z\right) + \bar{\varphi}'\left(\bar{z}\right)\right]. \tag{5.8.4}$$

The fifth and sixth dependences in eq. (5.8.2) are then satisfied and the third can be reduced to the form

$$\nabla_3^2\theta = 0, \tag{5.8.5}$$

that is, the plane stress can be realised under a stationary temperature field.

The remaining (first, second, fourth) Beltrami's dependences reduce to the form

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial y^2}\left(\nabla_3^2 U - \frac{s\left(x,y\right)}{1+\nu}\right) = 0, \quad \frac{\partial^2}{\partial x^2}\left(\nabla_3^2 U - \frac{s\left(x,y\right)}{1+\nu}\right) = 0,\\
&\qquad\qquad \frac{\partial^2}{\partial x \partial y}\left(\nabla_3^2 U - \frac{s\left(x,y\right)}{1+\nu}\right) = 0.
\end{aligned}\right\} \tag{5.8.6}$$

This allows us to view U as the solution of the three-dimensional Poisson equation

$$\nabla_3^2 U\left(x,y,x_3\right) = \frac{s\left(x,y\right)}{1+\nu}. \tag{5.8.7}$$

The displacements are determined from the equations of generalised Hooke's law

$$2\mu\frac{\partial u}{\partial x}=\sigma_x-\frac{\nu}{1+\nu}\sigma+2\mu\alpha\theta,\quad 2\mu\frac{\partial v}{\partial y}=\sigma_y-\frac{\nu}{1+\nu}\sigma+2\mu\alpha\theta,$$
$$\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\tau_{xy}\quad(\sigma=\sigma_x+\sigma_y).$$

Repeating the rearrangement of Subsection 7.5.7 we obtain from these equations

$$2\mu(u+iv)=-2\frac{\partial U}{\partial\bar z}+\frac{4}{1+\nu}\varphi(z).\tag{5.8.8}$$

In order to find displacement w we begin with the equalities

$$2\mu\frac{\partial w}{\partial x_3}=-\frac{\nu}{1+\nu}\sigma+2\mu\alpha\theta,\quad\frac{\partial w}{\partial x}=-\frac{\partial u}{\partial x_3},\quad\frac{\partial w}{\partial y}=-\frac{\partial v}{\partial x_3}$$

which are an evident consequence of eq. (5.8.1). With the help of eqs. (5.8.4), (5.8.7) and (5.8.8) they can be lead to the form

$$2\mu\frac{\partial w}{\partial x_3}=\frac{\partial^2 U}{\partial x_3^2},\quad 2\mu\frac{\partial w}{\partial\bar z}=\frac{\partial^2 U}{\partial\bar z\partial x_3}.$$

Hence

$$2\mu w=\frac{\partial U}{\partial x_3}.\tag{5.8.9}$$

The generalised plane stress which is approximately realised in the thin plate deals with the mean values of stresses, stress function and displacements. Keeping the above denotation for the mean values we obtain from eq. (5.8.4)

$$U=\frac{1}{2}\left[\bar z\varphi(z)+z\bar\varphi(\bar z)+\chi(z)+\bar\chi(\bar z)-\mu\alpha(1+\nu)\int^{z}d\zeta\int^{\bar z}\theta(\zeta,\bar\zeta)\,d\bar\zeta\right]\tag{5.8.10}$$

which is analogous to eq. (5.7.7). The modification in eq. (5.7.8) reduces to replacing

$$\alpha\frac{1+\nu}{1-\nu}\quad\text{by}\quad\alpha(1+\nu).\tag{5.8.11}$$

The displacement vector is now given by the equality

$$2\mu(u+iv)=\frac{3-\nu}{1+\nu}\varphi(z)-z\bar\varphi'(\bar z)-\bar\psi(\bar z)+\mu\alpha(1+\nu)\int^{z}\theta(\zeta,\bar z)\,d\zeta,\tag{5.8.12}$$

differing from eq. (5.7.10) in the replacement (1.6.5)

$$\nu \quad \text{by} \quad \frac{\nu}{1+\nu}. \tag{5.8.13}$$

in addition to replacement (5.8.11).

7.5.9 Stationary temperature distribution

Let us consider the case of a stationary thermal regime. The temperature in the case of the plane strain (or the mean temperature in the generalised plane stress) is the harmonic function of the coordinates

$$\nabla^2 \theta = 4 \frac{\partial^2 \theta}{\partial z \partial \bar{z}} = 0. \tag{5.9.1}$$

Under this condition, the stress function, by virtue of eqs. (5.7.6) and (5.8.4), is biharmonic and the Laplacian over it is a harmonic function.

As a harmonic single-valued function, the temperature is presented in the double-connected region L in the following form

$$\theta = \sum_{n=0}^{\infty} r^n \left(\theta_n \cos n\theta + g_n \sin n\theta\right) + \theta_0' \ln r + \frac{1}{r} \left(\theta_1' \cos\theta + g_1' \sin\theta\right) + \sum_{n=2}^{\infty} \frac{1}{r^n} \left(\theta_n' \cos n\theta + g_n' \sin n\theta\right). \tag{5.9.2}$$

The function of the complex variable whose real part is θ is given by the equality

$$\Theta'(z) = \theta + ig = \sum_{n=0}^{\infty} \left(\theta_n - ig_n\right) z^n + \theta_n' \ln z + \left(\theta_1' + ig_1'\right) \frac{1}{z} + \sum_{n=2}^{\infty} \frac{\theta_n' + ig_n'}{z^n} \tag{5.9.3}$$

so that

$$\Theta(z) = \theta_0' z \ln z + \left(\theta_1' + ig_1'\right) \ln z + \Theta_*(z), \tag{5.9.4}$$

where the term $-\theta_0' z$ is included into $\Theta_*(z)$. Here and in what follows an asterisk denotes the single-valued (holomorphic) part of the function of the complex variable.

In the following we deal with the plane strain. Putting θ in the form

$$\theta = \frac{1}{2} \left[\Theta'(z) + \bar{\Theta}'(\bar{z})\right], \tag{5.9.5}$$

we arrive, by means of eqs. (5.7.7), (5.7.8) and (5.7.10), to the formulae

$$
\left.
\begin{aligned}
U &= \frac{1}{2}\left\{\bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z}) - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\left[\bar{z}\Theta(z) + z\bar{\Theta}(\bar{z})\right]\right\}, \\
\sigma_x + \sigma_y &= 2\left\{\Phi(z) + \bar{\Phi}(\bar{z}) - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\left[\Theta'(z) + \bar{\Theta}'(\bar{z})\right]\right\}, \\
\sigma_y - \sigma_x + 2i\tau_{xy} &= 2\left[\bar{z}\Phi'(z) + \Psi(z) - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\bar{z}\Theta''(z)\right], \\
2\mu(u+iv) &= (3-4\nu)\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) + \\
&\qquad \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\left[\Theta(z) - z\bar{\Theta}'(\bar{z})\right].
\end{aligned}
\right\}
\tag{5.9.6}
$$

The expediency of considering the function

$$
\varphi_0(z) = \varphi(z) - \frac{1}{2}\mu\alpha\frac{1+\nu}{1-\nu}\Theta(z) \tag{5.9.7}
$$

becomes evident. With the help of this function the above system can be set in the form

$$
\left.
\begin{aligned}
U &= \frac{1}{2}\left[\bar{z}\varphi_0(z) + z\bar{\varphi}_0(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z})\right], \\
\sigma_x + \sigma_y &= 2\left[\Phi_0(z) + \bar{\Phi}_0(\bar{z})\right], \\
\sigma_y - \sigma_x + 2i\tau_{xy} &= 2\left[\bar{z}\Phi_0'(z) + \Psi(z)\right],
\end{aligned}
\right\}
\tag{5.9.8}
$$

$$
2\mu(u+iv) = (3-4\nu)\varphi_0(z) - z\bar{\varphi}_0'(\bar{z}) - \bar{\psi}(\bar{z}) + 2\mu\alpha(1+\nu)\Theta(z). \tag{5.9.9}
$$

Under these denotations the expression for the principal vector of the stresses on any arc l in region L is given, due to eq. (1.14.7), by the formula

$$
-Q + iP = \varphi_0(z) + z\bar{\varphi}_0'(\bar{z}) + \bar{\psi}(\bar{z}), \tag{5.9.10}
$$

and if contour Γ is free, then the boundary condition on Γ is homogeneous

$$
\left[\varphi_0(z) + z\bar{\varphi}_0'(\bar{z}) + \bar{\psi}(\bar{z})\right]_\Gamma = 0. \tag{5.9.11}
$$

Therefore, if function $\Theta(z)$ is single-valued

$$
\theta_0' = 0, \quad \theta_1' = 0, \quad g_1' = 0, \tag{5.9.12}
$$

which is guaranteed for a simply connected finite region L without thermal sources, then the solution of the problem is

$$
\varphi_0(z) = 0, \quad \psi(z) = 0. \tag{5.9.13}
$$

Then, due to eq. (5.9.8), stresses $\sigma_x, \sigma_y, \tau_{xy}$ are equal to zero

$$\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = 0, \tag{5.9.14}$$

and by virtue of eq. (5.9.9) the single-valued vector of displacement is as follows

$$u + iv = (1 + \nu)\,\alpha\Theta(z). \tag{5.9.15}$$

Stress σ_z is obtained from the condition $\varepsilon_z = 0$ by means of the generalised Hooke law

$$\sigma_z = -2\mu(1 + \nu)\,\alpha\theta = -E\alpha\theta. \tag{5.9.16}$$

Let us consider the case of the unloaded double-connected region when conditions (5.9.12) do not hold. The trivial solution (5.9.13) is not suitable as the corresponding displacement vector is not single-valued. The requirement for the single-valuedness and the static condition of zero principal vector of stresses on any contour Γ in L which can not be mapped to a point by any continuous transformation results exactly in the relations which were used in Subsection 7.5.5 for establishing multivaluedness of functions $\varphi(z), \psi(z)$ due to the distortion. The distortion's constants are equal in value and opposite in sign to the constants determining the character of the multivaluedness of the function on the right hand side of eq. (5.9.15). Referring to eqs. (5.5.3) and (5.9.4) we have

$$\begin{aligned} c_1 + ic_2 + ib_3 z &= -(1 + \nu)\,\alpha\Delta_{\Gamma_*}\Theta(z) \\ &= -2\pi i\alpha(1 + \nu)\left(\theta_0' z + \theta_1' + ig_1'\right), \end{aligned} \tag{5.9.17}$$

so that

$$c_1 + ic_2 = 2\pi\alpha(1 + \nu)\left(g_1' - i\theta_1'\right), \quad b_3 = -2\pi\alpha(1 + \nu)\,\theta_0'. \tag{5.9.18}$$

By eq. (5.5.5) we have

$$\left.\begin{aligned} \varphi_0(z) &= -\frac{1}{2}\mu\alpha\frac{1 + \nu}{1 - \nu}\left[\theta_0' z \ln z + \left(\theta_1' + ig_1'\right)\ln z\right] + \varphi_{0_*}(z), \\ \psi(z) &= -\frac{1}{2}\mu\alpha\frac{1 + \nu}{1 - \nu}\left(\theta_1' - ig_1'\right)\ln z + \psi_*(z). \end{aligned}\right\} \tag{5.9.19}$$

The second boundary-value problem for the unloaded contour reduces to searching the holomorphic (and thus single-valued) functions in L in terms of the conditions on Γ_0 and Γ_1

$$\begin{aligned} &\varphi_{0_*}(z) + z\bar{\varphi}_{0_*}'(\bar{z}) + \bar{\psi}_*(\bar{z}) = \\ &= \frac{1}{2}\mu\alpha\frac{1 + \nu}{1 - \nu}\left[2\left(\theta_0' z + \theta_1' + ig_1'\right)\ln r + \theta_0' z + \left(\theta_1' - ig_1'\right)\frac{z}{\bar{z}}\right], \end{aligned} \tag{5.9.20}$$

where $r = |z|$.

Determining thermal stresses $\sigma_x, \sigma_y, \tau_{xy}$ requires only the logarithmic term and the term proportional to r^{-1} in the expression (5.9.2) for the temperature. Prescribing the temperature is used for the displacement vector and σ_z. By virtue of eq. (5.9.9) we have

$$2\mu (u + iv) = (3 - 4\nu) \varphi_{0_*}(z) - z\bar{\varphi}'_{0_*}(\bar{z}) - \bar{\psi}_4(\bar{z}) + 2\mu\alpha (1 + \nu) \Theta_*(z) + \frac{1}{2}\mu\alpha \frac{1+\nu}{1-\nu} \left[2 \left(\theta'_0 z + \theta'_1 + i g'_1 \right) \ln r + \theta'_0 z + \left(\theta'_1 - i g'_1 \right) \frac{z}{\bar{z}} \right], \quad (5.9.21)$$

and the comparison with eq. (5.9.20) yields that on the free contours

$$\Gamma_1, \Gamma_0: \quad 2\mu (u + iv) = 4 (1 - \nu) \varphi_{0_*}(z) + 2\mu\alpha (1 + \nu) \Theta_*(z). \quad (5.9.22)$$

This generalises equality (5.9.15) to the case of a double-connected region. In region L eq. (5.9.11) can be set in the form

$$2\mu (u + iv) = 4 (1 - \nu) \varphi_{0_*}(z) + 2\mu (1 + \nu) \Theta_*(z) - i (P + iQ). \quad (5.9.23)$$

Hence

$$\Delta_{\Gamma_*} (P + iQ) = 0, \quad (5.9.24)$$

that is, the principal vector of the stresses on any closed contour in L is identically equal to zero.

The statement of the first boundary-value problem is based on relation (5.9.21). For example, in the case of the elastic body placed in a rigid and non-smooth casing with a heat insulator we have

$$(u + iv)_\Gamma = 0. \quad (5.9.25)$$

By eq. (5.9.23) the distribution of the surface forces due to the casing is given by

$$(P + iQ)_\Gamma = -i \left[4 (1 - \nu) \varphi_{0_*}(z) + 2\mu (1 + \nu) \Theta_*(z) \right]_\Gamma, \quad (5.9.26)$$

where $\varphi_{0_*}(z)$ is obtained from the solution of the boundary-value problem (5.9.25).

7.5.10 Cauchy's theorem and Cauchy's integral

Let us consider first the case of L—region inside a simple closed contour, the value of z on the contour being denoted as t.

Let $f(t)$ denote the value of function $f(z)$ which is analytical on Γ and continuous up to the boundary, then Cauchy's theorem

$$\oint_\Gamma f(t)\, dt = 0 \quad (5.10.1)$$

and Cauchy's integral formula

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{t-z}dt=\begin{cases} f(z), & z\subset L,\\ 0, & z\subset R,\end{cases} \tag{5.10.2}$$

hold, where R denotes the region external to L.

Let L be an infinite region outside of Γ and L' denote the double-connected region bounded by Γ from the inside and circle C of a sufficiently large radius r from the outside. Let $f(t)$ denote function $f(z)$ which is holomorphic in L (and thus in L') on both Γ and C. Applying Cauchy's integral formula in L' yields

$$\frac{1}{2\pi i}\left(\oint_{\Gamma}\frac{f(z)}{t-z}dt+\oint_{C}\frac{f(z)}{t-z}dt\right)=\begin{cases} f(z), & z\subset L',\\ 0, & z\subset R,\end{cases}$$

where R is a region external to L', i.e. it lies in Γ.

On the other side

$$I=\frac{1}{2\pi i}\oint_{C}\frac{f(t)\,dt}{t-z}=\frac{1}{2\pi i}\oint_{C}\frac{f\left(re^{i\theta}\right)id\theta}{1-\dfrac{z}{re^{i\theta}}},\qquad \lim_{r\to\infty}I=f(\infty),$$

that is,

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(t)}{t-z}dt=\begin{cases} f(z)-f(\infty), & z\subset L,\\ -f(\infty), & z\subset R.\end{cases} \tag{5.10.3}$$

Now let $f(z)$ be a function which is holomorphic in L inside of Γ everywhere, but pole $z=a$, and the principal part of the expansion of this function at the pole be equal to

$$g(z)=\sum_{s=1}^{n}\frac{A_s}{(z-a)^s}\qquad (a\subset L). \tag{5.10.4}$$

As $g(z)$ is holomorphic in $R-$region (outside of Γ) and $g(\infty)=0$ we obtain by means of eq. (5.10.3) that

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{g(t)}{t-z}dt=\begin{cases} 0, & z\subset L,\\ -g(z), & z\subset R.\end{cases}$$

Taking into account that $f(z)-g(z)$ is holomorphic in L and referring to eq. (5.10.2) we have

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(t)}{t-z}dt=\begin{cases} f(z)-g(z), & z\subset L,\\ -g(z), & z\subset R.\end{cases} \tag{5.10.5}$$

It remains to consider the case in which L is external with respect to Γ and $f(z)$ is holomorphic in L everywhere except for an infinite point where this function has a pole with the principal part

$$g(z) = a_0 + a_1 z + \ldots + a_n z^n, \tag{5.10.6}$$

i.e. $g(z)$ is holomorphic in R.

Then

$$\frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{g(t)}{t-z} dt = \begin{cases} 0, & z \subset L, \\ -g(z), & z \subset R, \end{cases}$$

so that

$$\frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{f(t)}{t-z} dt = \begin{cases} f(z) - g(z), & z \subset L, \\ -g(z), & z \subset R, \end{cases} \tag{5.10.7}$$

as function $f(z) - g(z)$ is holomorphic in L and equal to zero at infinity. Let us notice that the formulae in eqs. (5.10.5) and (5.10.7) have the same structure

$$\frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{f(t)}{t-z} dt = \begin{cases} f(z) - g(z), & z \subset L, \\ -g(z), & z \subset R, \end{cases} \tag{5.10.8}$$

where region L lies on the left hand side while traversing contour Γ.

7.5.11 Integrals of Cauchy's type. The Sokhotsky-Plemelj formula

Let $\varphi(t)$ be prescribed on Γ, restricting ourselves by the assumption that the integral

$$\oint\limits_{\Gamma} |\varphi(t)| \, dt \tag{5.11.1}$$

is bounded, we can prove that the functions

$$\Phi^L(z) = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{\varphi(t)\, dt}{t-z} \quad (z \subset L), \quad \Phi^R(z) = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{\varphi(t)\, dt}{t-z} \quad (z \subset R), \tag{5.11.2}$$

referred to as integrals of Cauchy's type are holomorphic in L and R, respectively. These equalities define the function

$$\Phi(z) = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{\varphi(t)}{t-z} dt \tag{5.11.3}$$

in the whole plane z except for Γ. It is necessary to distinguish the limiting values of $\Phi(z)$ on contour Γ obtained by approaching from the inside and outside

$$\lim_{L\supset z\to t_0} \Phi(z) = \Phi^L(t_0), \quad \lim_{R\supset z\to t_0} \Phi(z) = \Phi^R(t_0) \tag{5.11.4}$$

from the "direct value" defined by the principal value of the integral of Cauchy's type

$$\Phi(t_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{\varphi(t)}{t-t_0}dt. \tag{5.11.5}$$

The relation between these values is provided by the Sokhotsky-Plemelj formula

$$\left.\begin{aligned}\Phi^L(t_0) &= \frac{1}{2}\varphi(t_0) + \frac{1}{2\pi i}\oint_\Gamma \frac{\varphi(t)}{t-t_0}dt,\\ \Phi^R(t_0) &= -\frac{1}{2}\varphi(t_0) + \frac{1}{2\pi i}\oint_\Gamma \frac{\varphi(t)}{t-t_0}dt.\end{aligned}\right\} \tag{5.11.6}$$

They can also be set in the form

$$\Phi^L(t_0) - \Phi^R(t_0) = \varphi(t_0), \quad \Phi^L(t_0) + \Phi^R(t_0) = \frac{1}{\pi i}\oint_\Gamma \frac{\varphi(t)}{t-t_0}dt. \tag{5.11.7}$$

Given that function $\Phi(z)$ is holomorphic in the whole plane, excluding contour Γ, and is equal to zero at infinity as well as the difference $\varphi(t)$ between the values of $\Phi(z)$ is prescribed, then by the first formula in eq. (5.11.7) we have

$$\Phi(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\varphi(t)}{t-z}dt. \tag{5.11.8}$$

The uniqueness of this solution is easily proved by referring to Liouville's theorem which states that if a function is holomorphic in the whole plane then this function is a constant. If it is known in advance that $\Phi(z)$ at infinity grows not faster than z^n and has a pole at point $z = a \subset L$ then

$$\Phi(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\varphi(t)}{t-z}dt + P_n(z) + \sum_{s=1}^{m} \frac{\alpha_s}{(z-a)^s}, \tag{5.11.9}$$

where $P_n(z)$ is a polynomials of degree n and α_s are constants.

7.6 Regions with a circular boundary

7.6.1 Round disc loaded by concentrated forces

The concentrated forces (the normal and tangential forces denoted respectively by R_s and T_s) are applied at points Q_s lying on circle Γ of the disc of radius r_0 and having the angular coordinates θ_s. Each of these points is assumed to be the origin of the local coordinate system $Q_s x_s y_s$, the axes $Q_s x_s, Q_s y_s$ being made coincident with the directions of the unit base vectors $\mathbf{e}_r^s, \mathbf{e}_\theta^s$ of the polar coordinate system r, θ with the origin at the centre of disc C, Fig. 7.5. It is also assumed that forces R_s, T_s have respectively the directions $\mathbf{e}_r^s, \mathbf{e}_\theta^s$ provided that $R_s > 0, T_s > 0$. Let r_s, ψ_s denote the polar coordinates of the observation point $M(r, \theta)$ with the origin at point Q_s and the polar axis $\mathbf{e}_r^s$. The system of forces R_s, T_s is assumed to be statically equipollent to zero, i.e.

$$\sum_{s=1}^{n} (R_s \cos\theta_s - T_s \sin\theta_s) = 0,$$

$$\sum_{s=1}^{n} (R_s \sin\theta_s + T_s \cos\theta_s) = 0, \quad \sum_{s=1}^{n} r_0 T_s = 0,$$

or equivalently

$$\sum_{s=1}^{n} (R_s + iT_s)\, e^{i\theta_s} = 0, \quad \sum_{s=1}^{n} T_s = 0. \tag{6.1.1}$$

Let

$$R_s = K_s \cos\alpha_s, \quad T_s = K_s \sin\alpha_s,$$

where α_s designates the angle between force K_s (the resultant of forces R_s, T_s) applied at point Q_s and direction $\mathbf{e}_r^s$. Referring to formula (3.4.5)

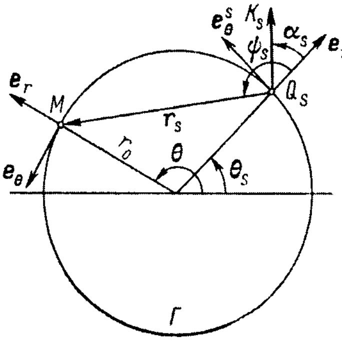

FIGURE 7.5.

we introduce into consideration the stress function

$$U_* = -\sum_{s=1}^{n} \frac{1}{\pi} K_s r_s \psi_s \sin(\psi_s - \alpha_s) = -\frac{1}{\pi} \sum_{s=1}^{n} r_s \psi_s (R_s \sin \psi_s - T_s \cos \psi_s), \tag{6.1.2}$$

describing the stresses due to the singularities corresponding to the concentrated forces at point Q_s. In the problem of the disc, the stress function is sought in the form

$$U = U_* + U_{**}, \tag{6.1.3}$$

where the correcting function U_{**} is determined by the condition

$$\sigma_r^{**} = -\sigma_r^*, \quad \tau_{r\theta}^{**} = -\tau_{r\theta}^*, \tag{6.1.4}$$

expressing that the disc boundary is free at points different from Q_s.

The state of stress caused by force K_s at the source point Q_s reduces to the single stress σ_{r_s} on the surface perpendicular to $\mathbf{r}_s$

$$\sigma_{r_s} = -\frac{2K_s}{\pi r_s} \cos(\psi_s - \alpha_s) = -\frac{2}{\pi r_s} (R_s \cos \psi_s + T_s \sin \psi_s).$$

On the circle Γ, Fig. 7.5,

$$r_s = 2r_0 \sin \frac{\theta - \theta_s}{2}, \quad \psi_s = \frac{1}{2}\pi + \frac{\theta - \theta_s}{2},$$

so that

$$\text{on} \quad \Gamma: \quad \sigma_{r_s} = \frac{1}{\pi r_0 \sin \frac{\theta - \theta_s}{2}} \left(R_s \sin \frac{\theta - \theta_s}{2} - T_s \cos \frac{\theta - \theta_s}{2} \right).$$

In polar coordinates, the components $\sigma_r^s, \tau_{r\theta}^s$ of the stress tensor are given by

$$\text{on} \quad \Gamma: \quad \left\{ \begin{aligned} \sigma_r^s &= \sigma_{r_s} \sin^2 \frac{\theta - \theta_s}{2} \\ &= \frac{1}{\pi r_0} \left(R_s \sin^2 \frac{\theta - \theta_s}{2} - T_s \cos \frac{\theta - \theta_s}{2} \sin \frac{\theta - \theta_s}{2} \right), \\ \tau_{r\theta}^s &= \sigma_{r_s} \sin \frac{\theta - \theta_s}{2} \cos \frac{\theta - \theta_s}{2} = \\ &= \frac{1}{\pi r_0} \left(R_s \sin \frac{\theta - \theta_s}{2} \cos \frac{\theta - \theta_s}{2} - T_s \cos^2 \frac{\theta - \theta_s}{2} \right), \end{aligned} \right.$$

since the angles of vector $\mathbf{r}_s$ to the unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ at the observation point M on Γ are $\pi/2 - \frac{1}{2}(\theta - \theta_s), \frac{1}{2}(\theta - \theta_s)$ respectively. After simple rearrangements and using the equilibrium equation (6.1.1) we have

$$\sigma_r^* + i\tau_{r\theta}^* = \frac{1}{2\pi r_0} \sum_{s=1}^{n} (R_s - iT_s) - \frac{1}{2\pi r_0} \sum_{s=1}^{n} (R_s + iT_s) e^{i\theta_s} = \frac{1}{2\pi r_0} \sum_{s=1}^{n} R_s. \tag{6.1.5}$$

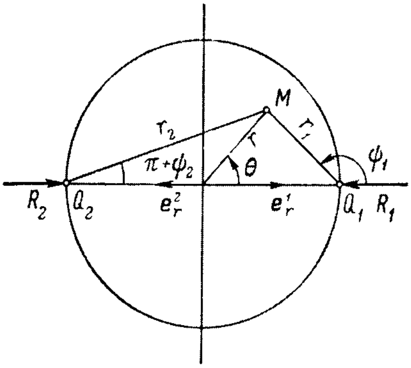

FIGURE 7.6.

Thus we arrive at the unexpectedly simple result, namely, that the correcting tensor determines the axially-symmetric state caused by the normal pressure distributed uniformly on Γ

$$\sigma_r^{**} = -\frac{1}{2\pi r_0}\sum_{s=1}^{n} R_s, \quad \tau_{r\theta}^{**} = 0, \quad U_{**} = -\frac{r^2}{4\pi r_0}\sum_{s=1}^{n} R_s.$$

By virtue of eq. (6.1.3) we have

$$U = -\frac{1}{\pi}\sum_{s=1}^{n} r_s\psi_s\left(R_s\sin\psi_s - T_s\cos\psi_s\right) - \frac{r^2}{4\pi r_0}\sum_{s=1}^{n} R_s, \tag{6.1.6}$$

where on the disc

$$\left.\begin{aligned} r_s &= \left[r^2 + r_0^2 - 2rr_0\cos\left(\theta - \theta_s\right)\right]^{1/2}, \\ \sin\psi_s &= \frac{r}{r_s}\sin\left(\theta - \theta_s\right), \quad \cos\psi_s = \frac{r\cos\left(\theta - \theta_s\right) - r_0}{r_s}. \end{aligned}\right\} \tag{6.1.7}$$

As an example we consider the disc acted on by two concentrated forces applied at the ends of the diameter (Hertz, 1883). Then $R_1 = R_2 = -R$, $T_s = 0$ and by eq. (6.1.6)

$$U = \frac{R}{\pi}\left(r_1\psi_1\sin\psi_1 + r_2\psi_2\sin\psi_2 + \frac{r^2}{2r_0}\right), \tag{6.1.8}$$

where angles ψ_1, ψ_2 are measured from the vectors $\mathbf{e}_r^1, \mathbf{e}_r^2 = -\mathbf{e}_r^1$ at points Q_1, Q_2 directed in opposition to the forces compressing the disc.

Assuming for simplicity that the disc is compressed at the ends of the horizontal diameter, we have, see Fig. 7.6

$$r_1\sin\psi_1 = -r_2\sin\psi_2 = r\sin\theta = \frac{1}{2i}\left(z - \bar{z}\right) = \frac{r_0}{2i}\left(\zeta - \bar{\zeta}\right),$$
$$z_1 = r_0\left(\zeta - 1\right), \quad z_2 = -r_0\left(\zeta + 1\right),$$

and thus

$$e^{2i\psi_1} = \frac{\zeta - 1}{\bar{\zeta} - 1}, \quad e^{2i\psi_2} = \frac{\zeta + 1}{\bar{\zeta} + 1},$$

where ζ denotes a point in the unit circle. Function U can be presented in the form

$$U(z, \bar{z}) = r_0 U(\zeta, \bar{\zeta}) = \frac{1}{2}\frac{r_0 R}{\pi}\left[\zeta\bar{\zeta} + \frac{1}{2}(\zeta - \bar{\zeta})\left(\ln\frac{\zeta + 1}{\bar{\zeta} + 1} + \ln\frac{\bar{\zeta} - 1}{\zeta - 1}\right)\right], \tag{6.1.9}$$

so that

$$\left.\begin{aligned} \varphi(\zeta) &= \frac{1}{2}\frac{R}{\pi}\left(\zeta + \ln\frac{\zeta - 1}{\zeta + 1}\right), \quad \chi(\zeta) = \frac{1}{2}\frac{R}{\pi}\zeta\ln\frac{\zeta + 1}{\zeta - 1}, \\ \psi(\zeta) &= \chi'(\zeta) = \frac{1}{2}\frac{R}{\pi}\left(\ln\frac{\zeta + 1}{\zeta - 1} - \frac{2\zeta}{\zeta^2 - 1}\right). \end{aligned}\right\} \tag{6.1.10}$$

Applying Kolosov-Muskhelishvili's formulae yields the stresses

$$\left.\begin{aligned} \sigma_r + \sigma_\theta &= 2\frac{R}{\pi r_0}\left(1 + \frac{1}{\zeta^2 - 1} + \frac{1}{\bar{\zeta}^2 - 1}\right), \\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} &= 4\frac{R}{\pi r_0}\frac{1 - \zeta\bar{\zeta}}{(\zeta^2 - 1)^2}\frac{\zeta}{\bar{\zeta}}, \end{aligned}\right\} \tag{6.1.11}$$

that is, the stresses are equal to zero at all points on the boundary of the disc, except for points Q_1, Q_2 where the concentrated forces are applied. On the diameter of the disc perpendicular to the line of action of the forces $\xi = 0, \zeta = i\eta$ and formulae (6.1.11) yield

$$\left.\begin{aligned} \sigma_x &= (\sigma_\theta)_{\xi=0} = -\frac{R}{\pi r_0}\frac{1 - \eta^2}{(1 + \eta^2)^2}(3 + \eta^2), \\ \sigma_y &= (\sigma_r)_{\xi=0} = \frac{R}{\pi r_0}\frac{(1 - \eta^2)^2}{(1 + \eta^2)^2}, \quad \tau_{r\theta} = 0. \end{aligned}\right\} \tag{6.1.12}$$

7.6.2 *The general case of loading round disc*

Let us consider the general case of loading the round disc $|z| \leq r_0$ by the normal and tangential surface forces on its boundary

$$|r| = r_0: \quad \sigma_r = f_1(\theta), \quad \tau_{r\theta} = f_2(\theta) \tag{6.2.1}$$

under the assumption that this system of forces is in equilibrium with the concentrated force $X + iY$ applied at the centre and the moment M^O.

The equations of the equilibrium of the disc are as follows

$$\int_0^{2\pi} [f_1(\theta)\cos\theta - f_2(\theta)\sin\theta]\, r_0 d\theta + X = 0,$$

$$\int_0^{2\pi} [f_1(\theta)\sin\theta + f_2(\theta)\cos\theta]\, r_0 d\theta + Y = 0,$$

$$r_0^2 \int_0^{2\pi} f_2(\theta)\, d\theta + M^O = 0$$

or

$$\left.\begin{aligned} & r_0 \int_0^{2\pi} f(\theta)\, e^{i\theta} d\theta + X + iY = 0, \\ & r_0^2 \int_0^{2\pi} \left[f(\theta) - \bar{f}(\theta)\right] d\theta + 2iM^O = 0, \quad [f(\theta) = f_1(\theta) + if_2(\theta)]\,. \end{aligned}\right\} \tag{6.2.2}$$

For the sake of simplicity the problem is considered in the unit circle which results in the change of notation

$$z = r_0\zeta, \quad \zeta = \rho e^{i\theta} = \rho\sigma, \quad \sigma = e^{i\theta}, \quad d\sigma = ie^{i\theta} d\theta = i\sigma d\theta. \tag{6.2.3}$$

The equilibrium equations (6.2.2) reduce to the form

$$\oint_\gamma f(\theta)\, d\sigma + \frac{i}{r_0}(X + iY) = 0, \quad \oint_\gamma \left[f(\theta) - \bar{f}(\theta)\right] \frac{d\sigma}{\sigma} - 2\frac{M^O}{r_0^2} = 0. \tag{6.2.4}$$

Referring to eqs. (5.6.1) and (5.6.5) we seek Muskhelishvili's functions $\varphi(\zeta), \psi(\zeta)$ in the form

$$\left.\begin{aligned} \varphi(\zeta) &= -\frac{X + iY}{8\pi(1-\nu)\, r_0} \ln\zeta + \varphi_*(\zeta), \\ \psi(\zeta) &= \frac{X - iY}{8\pi(1-\nu)\, r_0}(3 - 4\nu)\ln\zeta + \frac{iM^O}{2\pi r_0^2}\frac{1}{\zeta} + \psi_*(\zeta), \end{aligned}\right\} \tag{6.2.5}$$

where $\varphi(\zeta), \psi(\zeta)$ are holomorphic in the unit circle $|\zeta| < 1$. Hence

$$\left.\begin{aligned} \Phi(\zeta) &= -\frac{X + iY}{8\pi(1-\nu)\, r_0}\frac{1}{\zeta} + \Phi_*(\zeta), \\ \Psi(\zeta) &= \frac{X - iY}{8\pi(1-\nu)\, r_0}(3 - 4\nu)\frac{1}{\zeta} - \frac{iM^O}{2\pi r_0^2}\frac{1}{\zeta^2} + \Psi_*(\zeta). \end{aligned}\right\} \tag{6.2.6}$$

By virtue of eq. (5.2.15) the boundary condition on the unit circle γ is written down as follows

$$\zeta=\sigma,\ \bar\zeta=\frac{1}{\sigma}:\ \Phi_*(\sigma)+\bar\Phi_*\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma}\bar\Phi_*'\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma^2}\bar\Psi_*\left(\frac{1}{\sigma}\right)= \tag{6.2.7}$$
$$=f(\theta)+\frac{1}{8\pi(1-\nu)}\left[4(1-\nu)\frac{1}{\sigma}(X+iY)+2\sigma(X-iY)\right]+\frac{iM^O}{2\pi r_0^2}.$$

This condition for the complex conjugated values takes the form

$$\bar\zeta=\frac{1}{\sigma},\quad \zeta=\sigma:\quad \bar\Phi_*\left(\frac{1}{\sigma}\right)+\Phi_*(\sigma)-\sigma\Phi_*'(\sigma)-\sigma^2\Psi_*(\sigma)= \tag{6.2.8}$$
$$=\bar f(\theta)+\frac{1}{8\pi(1-\nu)r_0}\left[4(1-\nu)\sigma(X-iY)+\frac{2}{\sigma}(X+iY)\right]-\frac{iM^O}{2\pi r_0^2}.$$

7.6.3 The method of Cauchy's integrals

The method of Cauchy's integrals for solving boundary-value problems in the plane theory of elasticity has been suggested and developed in detail by N.I. Muskhelishvili. His book contains the rigorous substantiation and numerous applications of this method, and for this reason consideration is limited here to only explaining the basics of calculation.

Multiplying both parts of eq. (6.2.7) by

$$\frac{1}{2\pi i}\frac{d\sigma}{\sigma-\zeta}$$

and integrating around contour γ of the unit circle we have

$$\frac{1}{2\pi i}\left[\oint_\gamma\frac{\Phi_*(\sigma)}{\sigma-\zeta}d\sigma+\oint_\gamma\bar\Phi_*\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta}-\right.$$
$$\left.\oint_\gamma\bar\Phi_*'\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma(\sigma-\zeta)}-\oint_\gamma\bar\Psi_*\left(\frac{1}{\sigma}\right)\frac{d\sigma}{(\sigma-\zeta)\sigma^2}\right]=$$
$$=\frac{1}{2\pi i}\left[\oint_\gamma\frac{f(\theta)}{\sigma-\zeta}d\sigma+\frac{X+iY}{2\pi r_0}\oint_\gamma\frac{d\sigma}{\sigma(\sigma-\zeta)}+\right.$$
$$\left.\frac{X-iY}{4\pi(1-\nu)r_0}\oint_\gamma\frac{\sigma d\sigma}{\sigma-\zeta}+\frac{iM^O}{2\pi r_0^2}\oint_\gamma\frac{d\sigma}{\sigma-\zeta}\right].$$

Referring to Cauchy's integral formulae (5.10.2), (5.10.3) and taking into account that $\bar{\Phi}_* \left(\frac{1}{\sigma}\right)$ is holomorphic for $|\zeta| > 1$ we have

$$\Phi_* (\zeta) + \bar{\Phi}_* (0) = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(\theta)\, d\sigma}{\sigma - \zeta} + \frac{X - iY}{4\pi (1-\nu) r_0} \zeta + \frac{iM^O}{2\pi r_0^2}. \tag{6.3.1}$$

Applying an analogous calculation to condition (6.2.8) yields

$$\begin{aligned} &\bar{\Phi}_* (0) + \Phi_* (\zeta) - \zeta \Phi_*' (\zeta) - \zeta^2 \Psi_* (\zeta) = \\ &\qquad = \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{\bar{f}(\theta)\, d\sigma}{\sigma - \zeta} + \frac{X - iY}{2\pi r_0} \zeta - \frac{iM^O}{2\pi r_0^2}. \end{aligned} \tag{6.3.2}$$

It is also necessary to prove the inverse statement: functions $\Phi_* (\zeta), \Psi_* (\zeta)$ determined by equalities (6.3.1), (6.3.2) in the whole region satisfy the boundary conditions (6.2.7), (6.2.8). The substantiation by means of the theory of potential is presented in the book by Muskhelishvili. Another derivation of these relationships is suggested in Subsections 7.6.13 and 7.6.14 of the present book.

By eq. (6.3.1) we have for $\zeta = 0$

$$\begin{aligned} 2 \operatorname{Re} \Phi_* (0) = \Phi_* (0) + \bar{\Phi}_* (0) &= \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(\theta)}{\sigma} d\sigma + i \frac{M^O}{2\pi r_0^2} \\ &= \frac{1}{2\pi} \int\limits_0^{2\pi} f_1 (\theta)\, d\theta + \frac{i}{2\pi} \left(\int\limits_0^{2\pi} f_2 (\theta)\, d\theta + \frac{M^O}{r_0^2} \right), \end{aligned}$$

and the left part of the latter equality is real if

$$\int\limits_0^{2\pi} f_2 (\theta)\, d\theta + \frac{M^O}{r_0^2} = 0$$

which is the equation for the moment. Hence, taking $\operatorname{Im} \Phi_* (0) = 0$ we have

$$\left.\begin{aligned} \Phi_* (0) = \bar{\Phi}_* (0) &= \frac{1}{4\pi i} \oint\limits_{\gamma} \frac{f(\theta)}{\sigma} d\sigma + \frac{iM^O}{4\pi r_0^2} = \frac{1}{4\pi} \int\limits_0^{2\pi} f_1 (\theta)\, d\theta, \\ \Phi_* (\zeta) &= \frac{1}{2\pi i} \oint\limits_{\gamma} f(\theta) \left(\frac{1}{\sigma - \zeta} - \frac{1}{2\sigma} \right) d\sigma + \frac{iM^O}{4\pi r_0^2} + \frac{X - iY}{4\pi (1-\nu) r_0} \zeta. \end{aligned}\right\} \tag{6.3.3}$$

It remains to obtain $\Psi_*(\zeta)$ with the help of eqs. (6.3.1) and (6.3.2). The result is

$$-\zeta^2\Psi_*(\zeta) = \frac{1}{2\pi i}\oint\limits_\gamma \frac{\bar f(\theta) - f(\theta)}{\sigma - \zeta} d\sigma + \frac{X - iY}{2\pi r_0}\zeta + \frac{\zeta}{2\pi i}\oint\limits_\gamma \frac{f(\theta)}{(\sigma-\zeta)^2} d\sigma - i\frac{M^O}{\pi r_0^2}, \quad (6.3.4)$$

and function $\Psi_*(\zeta)$ obtained is holomorphic in the circle $|\zeta| < 1$ provided that the expansion of $\Psi_*(\zeta)$ in a power series in ζ has no free term and no first order term

$$\frac{1}{2\pi i}\oint\limits_\gamma \left[\bar f(\theta) - f(\theta)\right]\frac{d\sigma}{\sigma} - i\frac{M^O}{\pi r_0^2} = 0, \quad \frac{1}{2\pi i}\oint\limits_\gamma \frac{\bar f(\theta)}{\sigma^2} d\sigma + \frac{X - iY}{2\pi r_0} = 0. \quad (6.3.5)$$

These are the equilibrium equations, one of them being written in the form complex conjugated to eq. (6.2.2). By virtue of eqs. (6.3.4) and (6.3.5) we find

$$\Psi_*(\zeta) = -\frac{1}{2\pi i}\oint\limits_\gamma \left[\frac{f(\theta)}{(\sigma-\zeta)^2\sigma} + \frac{\bar f(\theta)}{\sigma^2(\sigma-\zeta)}\right] d\sigma. \quad (6.3.6)$$

7.6.4 Normal stress σ_θ on the circle

In what follows we assume that the system of surface forces on γ is statically self-equilibrated. Then

$$X + iY = 0, \quad M^O = 0; \quad \int\limits_0^{2\pi} f_2(\theta)\, d\theta = 0, \quad \int\limits_0^{2\pi} f(\theta)\, d\theta = \int\limits_0^{2\pi} f_1(\theta)\, d\theta \quad (6.4.1)$$

and by eq. (6.3.3)

$$\Phi(\zeta) = \frac{1}{2\pi i}\oint\limits_\gamma f(\theta)\frac{d\sigma}{\sigma - \zeta} - \frac{1}{4\pi}\int\limits_0^{2\pi} f_1(\theta)\, d\theta.$$

Under the limiting process $\zeta \to \sigma_1 = e^{i\psi}$ we obtain with the help of Sokhotsky-Plemelj's formulae

$$\Phi(\zeta)|_{\zeta\to\sigma_1} = \frac{1}{2} f(\psi) + \frac{1}{2\pi i}\oint\limits_\gamma f(\theta)\frac{d\sigma}{\sigma - \sigma_1} - \frac{1}{4\pi}\int\limits_0^{2\pi} f_1(\theta)\, d\theta, \quad (6.4.2)$$

where the integral which is understood in the principal value sense is transformed to the form

$$\frac{1}{2\pi i}\oint_{\gamma} f(\theta)\frac{d\sigma}{\sigma-\sigma_1} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{e^{i\theta}d\theta}{e^{i\theta}-e^{i\psi}} =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{1-e^{i(\theta-\psi)}}{2[1-\cos(\theta-\psi)]}d\theta = \frac{1}{4\pi}\int_0^{2\pi} f_1(\theta)\,d\theta - \frac{i}{4\pi}\int_0^{2\pi} f(\theta)\cot\frac{\theta-\psi}{2}d\theta.$$

We arrive at the relationship

$$\Phi(\zeta)|_{\zeta\to\sigma_1} = \frac{1}{2}f(\psi) - \frac{i}{4\pi}\int_0^{2\pi} f(\theta)\cot\frac{\theta-\psi}{2}d\theta, \tag{6.4.3}$$

so that

$$(\sigma_r+\sigma_\theta)|_{|\zeta|=1} = 2\left[\Phi(\zeta)+\bar{\Phi}(\zeta)\right] = 2f_1(\psi) + \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\cot\frac{\theta-\psi}{2}d\theta,$$

or

$$\sigma_\theta|_{r=r_0} = \sigma_r|_{r=r_0} + \frac{1}{\pi}\int_0^{2\pi} \tau_{r\theta}|_{r=r_0}\cot\frac{\theta-\psi}{2}d\theta. \tag{6.4.4}$$

If the shear stresses are absent, the normal stresses on the boundary of the disc are equal to each other.

Using the well-known relations

$$\left.\begin{aligned}
\sin n\psi &= -\frac{1}{2\pi}\int_0^{2\pi}\cos n\theta\cot\frac{\theta-\psi}{2}d\theta,\\
\cos n\psi &= \frac{1}{2\pi}\int_0^{2\pi}\sin n\theta\cot\frac{\theta-\psi}{2}d\theta,\\
\int_0^{2\pi}&\cot\frac{\theta-\psi}{2}d\theta = 0 \quad (n=1,2,\dots)
\end{aligned}\right\} \tag{6.4.5}$$

and representing the distributed shear stresses as a series

$$r=r_0: \quad \tau_{r\theta} = \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta), \tag{6.4.6}$$

we can obtain another representation for formula (6.4.4)

$$r = r_0: \quad \sigma_\theta|_{r=r_0} = \sigma_r|_{r=r_0} + 2\sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta). \qquad (6.4.7)$$

The $n = 1$ term of series (6.4.6) can not be taken independently of the corresponding term ($n = 1$) of the trigonometric series for the normal stress $\sigma_r|_{r=r_0}$. The latter must take the form

$$-a_1 \sin\theta + b_1 \cos\theta,$$

otherwise the conditions for the equilibrium of the surface forces on γ do not hold.

7.6.5 Stresses at the centre of the disc

Under conditions (6.4.1) Kolosov-Muskhelishvili's formulae yield

$$(\sigma_r + \sigma_\theta)_{r=0} = 2\left[\Phi(0) + \bar{\Phi}(0)\right], \quad (\sigma_\theta - \sigma_r + 2i\tau_{r\theta})_{r=0} = 2e^{2i\psi}\Psi(0)$$

and by virtue of eqs. (6.3.3) and (6.36) we obtain

$$\left.\begin{aligned} (\sigma_r + \sigma_\theta)_{r=0} &= \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\, d\theta, \\ (\sigma_\theta - \sigma_r + 2i\tau_{r\theta})_{r=0} &= -\frac{2}{\pi}\int_0^{2\pi} e^{2i(\psi-\theta)} f_1(\theta)\, d\theta, \end{aligned}\right\} \qquad (6.5.1)$$

so that

$$\left.\begin{aligned} \sigma_\theta|_{r=0} &= \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\, d\theta - \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos 2(\psi-\theta)\, d\theta, \\ \sigma_r|_{r=0} &= \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\, d\theta + \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos 2(\psi-\theta)\, d\theta, \\ \tau_{r\theta}|_{r=0} &= -\frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin 2(\psi-\theta)\, d\theta. \end{aligned}\right\} \qquad (6.5.2)$$

It is interesting to note that the shear stresses cause no stresses at the centre of the disc. These stresses are only due to the constant term and the second harmonic in the expansion of the normal surface stress in the trigonometric series.

7.6.6 A statically unbalanced rotating disc

The solution to this problem is provided as an example of applying the general formulae of Subsection 7.6.3. It is assumed that the point O of intersection of the axis of rotation of a homogeneous disc with its middle plane does not coincide with its geometric centre (which is also the centre of gravity). The eccentricity $\overline{OC} = e$ lies on axis Cx of the system of axes Cxy fixed in the disc and rotating together with the angular velocity ω. The coordinates of point O are $x = -e, y = 0$ and the volumetric centrifugal force at point $M\,(x, y)$ are given by the vector

$$\rho \mathbf{K} = \frac{\gamma}{g}\omega^2 (\mathbf{r} + \mathbf{i}_1 e), \quad \mathbf{r} = x\mathbf{i}_1 + y\mathbf{i}_2. \tag{6.6.1}$$

The principal vector and the principal moment of these forces about centre C are as follows

$$\mathbf{F} = \frac{\gamma}{g}\omega^2 \iint\limits_{\Omega} (\mathbf{r} + \mathbf{i}_1 e)\, do = \mathbf{i}_1 e M \omega^2,$$

$$m^C = \mathbf{i}_3 \cdot \iint\limits_{\Omega} \mathbf{r} \times \rho \mathbf{K} do = -\frac{\gamma\omega^2 e}{g} \iint\limits_{\Omega} y do = 0,$$

where $M = \frac{\gamma}{g}\pi r_0^2$ is the mass of the disc. The centrifugal forces are in equilibrium with the reaction force of axis O denoted by $-\mathbf{F}$ and are applied at point $(-e, 0)$. The state of stress in the disc is a superposition of the state due to the centrifugal force

$$\rho \mathbf{K}_0 = \frac{\gamma}{g}\omega^2 \mathbf{r} \tag{6.6.2}$$

and the state due to the homogeneous field of the force

$$\rho \mathbf{K}_1 = \frac{\gamma}{g}\omega^2 e \mathbf{i}_1 \tag{6.6.3}$$

of the statically self-equilibrated reaction force at point O

$$\mathbf{F}_2 = -\mathbf{F} = -M\omega^2 e \mathbf{i}_1. \tag{6.6.4}$$

The solution of the first problem is elementary. The particular solution satisfying the compatibility condition is given by formulae (3.11.6) of Chapter 5 in which ν is replaced due to the rule (1.6.5)

$$\sigma_r' = -\frac{\gamma\omega^2}{8g}(3+\nu) r^2, \quad \sigma_\theta' = -\frac{\gamma\omega^2}{8g}(1+3\nu) r^2, \quad \tau_{r\theta}' = 0. \tag{6.6.5}$$

The correcting axially-symmetric stress function Cr^2 is determined by the condition of zero normal stress σ_r' on the boundary of the disc

$$r = r_0: \quad 2C - \frac{\gamma\omega^2}{8g}(3+\nu) r_0^2 = 0,$$

and the state of stress in the disc rotating about its centre of gravity is given by the formulae

$$\sigma_r^0 = \frac{\gamma\omega^2}{8g}(3+\nu)\left(r_0^2 - r^2\right), \sigma_\theta^0 = \frac{\gamma\omega^2}{8g}\left[(3+\nu)r_0^2 - (1+3\nu)r^2\right], \tau_{r\theta}^0 = 0. \tag{6.6.6}$$

A particular solution of the equations of the theory of elasticity in the homogeneous field (6.6.3) can be taken in the form

$$\sigma_x^{(1)} = -\frac{\gamma}{g}\omega^2 ex = -\frac{Me\omega^2}{\pi r_0^2} r\cos\theta, \quad \sigma_y^{(1)} = 0, \quad \tau_{r\theta}^{(1)} = 0$$

or in polar coordinates

$$\left.\begin{aligned} \sigma_r^{(1)} &= -\frac{Me\omega^2}{\pi r_0^2} r\cos^3\theta, \quad \sigma_\theta^{(1)} = -\frac{Me\omega^2}{\pi r_0^2} r\sin^2\theta\cos\theta, \\ \tau_{r\theta}^{(1)} &= \frac{Me\omega^2}{\pi r_0^2} r\cos^2\theta\sin\theta. \end{aligned}\right\} \tag{6.6.7}$$

The stress function describing the state of stress due to force $\mathbf{F}_2$ is given by eq. (3.1.10)

$$\begin{aligned} U_2(z,\bar{z}) = \frac{Me\omega^2}{16\pi(1-\nu)} \{(\bar{z}+e)\ln(z+e) + (z+e)\ln(\bar{z}+e) - \\ (3-4\nu)\left[(z+e)\ln(z+e) + (\bar{z}+e)\ln(\bar{z}+e)\right]\} \end{aligned} \tag{6.6.8}$$

or by virtue of eq. (1.13.7)

$$\sigma_r^{(2)} + i\tau_{r\theta}^{(2)} = \frac{Me\omega^2}{8\pi(1-\nu)}\left[\frac{1}{z+e} + \frac{1}{\bar{z}+e} + (3-4\nu)\frac{\bar{z}}{z}\frac{1}{\bar{z}+e}\right], \tag{6.6.9}$$

where the notion of the plane strain problem is kept.

Superimposing the states (6.6.7) and (6.6.9) determines the stress on circle γ of the unit circle

$$\begin{aligned} \zeta = \sigma, \quad \bar{\zeta} = \frac{1}{\sigma}: \quad \tilde{\sigma}_r + i\tilde{\tau}_{r\theta} = \frac{M\alpha\omega^2}{4\pi}\left\{-\frac{1}{\sigma}\left(\sigma + \frac{1}{\sigma}\right)^2 + \right. \\ \left. \frac{1}{2(1-\nu)}\left[\frac{1}{\sigma+\alpha} + \frac{\sigma}{1+\alpha\sigma} + (3-4\nu)\frac{1}{\sigma(1+\alpha\sigma)}\right]\right\}, \end{aligned} \tag{6.6.10}$$

where we used the notion of eq. (6.2.3) and introduced the non-dimensional eccentricity $\alpha = e/r_0$.

The problem is thus reduced to determining functions $\Phi(\zeta), \Psi(\zeta)$ such that the stress given by eq. (1.14.9)

$$\sigma_r^* + i\tau_{r\theta}^* = \frac{M\alpha\omega^2}{4\pi}\left[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta}) - \bar{\zeta}\bar{\Phi}'(\bar{\zeta}) - \frac{\zeta}{\bar{\zeta}}\bar{\Psi}(\bar{\zeta})\right] \tag{6.6.11}$$

takes the value (6.6.10) on with γ taking the opposite sign. We arrive at the boundary conditions

$$\left.\begin{aligned}\Phi(\sigma)+\bar{\Phi}\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma}\bar{\Phi}'\left(\frac{1}{\sigma}\right)-\frac{1}{\sigma^2}\bar{\Psi}\left(\frac{1}{\sigma}\right)=f(\sigma),\\ \bar{\Phi}\left(\frac{1}{\sigma}\right)+\Phi(\sigma)-\sigma\Phi'(\sigma)-\sigma^2\Psi(\sigma)=\bar{f}(\sigma).\end{aligned}\right\} \tag{6.6.12}$$

Here

$$f(\sigma)=\frac{1}{\sigma}\left(\sigma+\frac{1}{\sigma}\right)^2-\frac{1}{2(1-\nu)}\left[\frac{1}{\sigma+\alpha}+\frac{\sigma}{1+\alpha\sigma}+(3-4\nu)\frac{1}{\sigma(1+\alpha\sigma)}\right] \tag{6.6.13}$$

and it can be proved easily that the system of surface forces described by this expression is in equilibrium. Indeed, by eq. (6.2.4)

$$\frac{1}{2\pi i}\oint_\gamma f(\sigma)\,d\sigma=\frac{1}{2\pi i}\oint_\gamma\left\{\sigma+\underline{\frac{2}{\sigma}}+\frac{1}{\sigma^3}-\frac{1}{2(1-\nu)}\left[\underline{\frac{1}{\sigma+\alpha}}+\frac{\sigma}{1+\alpha\sigma}+\right.\right.$$
$$\left.\left.(3-4\nu)\underline{\frac{1}{\sigma(1+\alpha\sigma)}}\right]\right\}d\sigma=2-\frac{1}{2(1-\nu)}(1+3-4\nu)=0,$$

where the integrals of the underlined terms do not vanish, however their sum is identically equal to zero.

It remains to calculate functions $\Phi(\zeta),\Psi(\zeta)$. By eq. (6.3.1) we have

$$\frac{1}{2\pi i}\oint_\gamma f(\sigma)\frac{d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\oint_\gamma\left\{\sigma+\underline{\frac{2}{\sigma}}+\frac{1}{\sigma^3}-\frac{1}{2(1-\nu)}\times\right.$$
$$\left.\left[\frac{1}{\sigma+\alpha}+\underline{\frac{\sigma}{1+\alpha\sigma}}+(3-4\nu)\underline{\frac{1}{\sigma(1+\alpha\sigma)}}\right]\right\}\frac{d\sigma}{\sigma-\zeta}.$$

The integrals of the underlined terms do not identically vanish and are evaluated by means of Cauchy's integral formula (5.10.2). The remaining integrals are zero according to the second integral formula (5.10.3).

Then we obtain

$$\frac{1}{2\pi i}\oint_\gamma f(\sigma)\frac{d\sigma}{\sigma-\zeta}=\zeta-\frac{1}{2(1-\nu)}\frac{\zeta}{1+\alpha\zeta}+\frac{3-4\nu}{2(1-\nu)}\frac{\alpha}{1+\alpha\zeta}$$
$$=\Phi(\zeta)+\bar{\Phi}(0),$$

that is,

$$\Phi(\zeta)=\zeta-\frac{1}{2(1-\nu)}\frac{\zeta}{1+\alpha\zeta}+\frac{3-4\nu}{4(1-\nu)}\alpha\frac{1-\alpha\zeta}{1+\alpha\zeta}. \tag{6.6.14}$$

By eq. (6.3.2) we find

$$\Psi(\zeta) = \frac{1}{\zeta^2}\left[\frac{1}{2\pi i}\oint\limits_{\gamma} d\sigma \frac{f(\sigma) - \bar{f}\left(\frac{1}{\sigma}\right)}{\sigma - \zeta} - \zeta\Phi'(\zeta)\right]$$
$$= \frac{\alpha}{2(1-\nu)(1+\alpha\zeta)^2}\left[(3-4\nu)\alpha^2 + 2 + \alpha\zeta\right] - \zeta. \qquad (6.6.15)$$

Calculating the stresses in terms of functions Φ and Ψ presents no difficulty. These stresses should be added with the stresses (6.6.6) due to the centrifugal forces.

7.6.7 The first boundary-value problem for circle

By eq. (5.2.16) the boundary condition on the unit circle γ has the form

$$(3-4\nu)\varphi(\sigma) - \sigma\bar{\varphi}'\left(\frac{1}{\sigma}\right) - \bar{\psi}\left(\frac{1}{\sigma}\right) = 2\mu(u+iv)_\gamma = f(\theta), \qquad (6.7.1)$$

where $f(\theta)$ is a prescribed function and $\varphi(\sigma), \psi(\sigma)$ denote the values of functions $\varphi(\zeta), \psi(\zeta)$ which are holomorphic in the unit circle $|\zeta| < 1$. By virtue of eq. (5.3.4) we can take $\varphi(0) = 0$. The boundary condition, which is a complex conjugate to (6.7.1), has the form

$$(3-4\nu)\bar{\varphi}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\varphi'(\sigma) - \psi(\sigma) = \bar{f}(\theta). \qquad (6.7.2)$$

The power series for $\varphi(\zeta), \psi(\zeta)$ in the unit circle

$$\varphi(\zeta) = \zeta\varphi'(0) + \frac{1}{2}\zeta^2\varphi''(0) + \frac{1}{6}\zeta^3\varphi'''(0) + \dots,$$
$$\psi(\zeta) = \psi(0) + \zeta\psi'(0) + \dots,$$

allows one to put

$$\sigma\bar{\varphi}'\left(\frac{1}{\sigma}\right) = \sigma\bar{\varphi}'(0) + \bar{\varphi}''(0) + \frac{1}{2}\bar{\varphi}'''(0)\frac{1}{\sigma} + \dots,$$
$$\bar{\psi}\left(\frac{1}{\sigma}\right) = \bar{\psi}(0) + \frac{1}{\sigma}\bar{\psi}'(0) + \dots,$$
$$\frac{\varphi'(\sigma)}{\sigma} = \frac{\varphi'(0)}{\sigma} + \varphi''(0) + \frac{1}{2}\varphi'''(0)\sigma + \dots,$$
$$\psi(\sigma) = \psi(0) + \sigma\psi'(0) + \dots$$

Therefore, applying Cauchy's integral, Subsection 7.6.3, and the integral formulae (5.10.2), (5.10.3) leads to the relationships

$$\left.\begin{aligned}
&(3-4\nu)\,\varphi(\zeta)-\zeta\varphi'(0)-\varphi''(0)-\bar{\psi}(0)=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(\theta)}{\sigma-\zeta}d\sigma,\\
&\frac{\varphi'(\zeta)-\varphi'(0)}{\zeta}+\psi(\zeta)=-\frac{1}{2\pi i}\oint_{\gamma}\frac{\bar{f}(\theta)}{\sigma-\zeta}d\sigma.
\end{aligned}\right\} \tag{6.7.3}$$

Equating the free terms in these equalities and the first order terms in the first equality leads to the relations

$$\left.\begin{aligned}
\bar{\varphi}''(0)+\bar{\psi}(0)&=-\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,\\
\varphi''(0)+\psi(0)&=-\frac{1}{2\pi}\int_0^{2\pi} \bar{f}(\theta)\,d\theta,\\
(3-4\nu)\,\varphi'(0)-\bar{\varphi}'(0)&=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,e^{-i\theta}d\theta,
\end{aligned}\right\} \tag{6.7.4}$$

where the second equality is equivalent to the first one. The first equality in eq. (6.7.3) is presented in the form

$$(3-4\nu)\left[\varphi(\zeta)-\zeta\varphi'(0)\right]=\frac{\zeta^2}{2\pi i}\oint_{\gamma}\frac{f(\theta)\,d\sigma}{\sigma^2(\sigma-\zeta)}, \tag{6.7.5}$$

and differentiation yields

$$\frac{1}{\zeta}\left[\varphi'(\zeta)-\varphi'(0)\right]=\frac{1}{2\pi i}\frac{1}{3-4\nu}\oint_{\gamma}\frac{2\sigma-\zeta}{\sigma^2(\sigma-\zeta)^2}f(\theta)\,d\sigma,$$

so that, due to the second equality (6.7.3)

$$\psi(\zeta)=-\frac{1}{2\pi i}\left[\frac{1}{3-4\nu}\oint_{\gamma}\frac{2\sigma-\zeta}{\sigma^2(\sigma-\zeta)^2}f(\theta)\,d\sigma+\oint_{\gamma}\frac{\bar{f}(\theta)\,d\sigma}{\sigma-\zeta}\right]. \tag{6.7.6}$$

Introducing the notation

$$\left.\begin{aligned}
&L(\zeta)=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(\theta)\,d\sigma}{\sigma^2(\sigma-\zeta)^2},\quad M(\zeta)=\frac{1}{2\pi i}\oint_{\gamma}\frac{\bar{f}(\theta)\,d\sigma}{\sigma-\zeta},\\
&Q(\zeta)=2L(\zeta)+\zeta L'(\zeta)=\frac{1}{2\pi i}\oint_{\gamma}\frac{2\sigma-\zeta}{\sigma^2(\sigma-\zeta)^2}f(\theta)\,d\sigma
\end{aligned}\right\} \tag{6.7.7}$$

we can set the displacement vector in the form

$$
\begin{aligned}
2\mu(u+iv) &= (3-4\nu)\varphi(\zeta) - \zeta\bar{\varphi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta}) \\
&= \zeta^2 L(\zeta) + \frac{1-\zeta\bar{\zeta}}{3-4\nu}\bar{Q}(\zeta) + \bar{M}(\zeta) + [(3-4\nu)\varphi'(0) - \bar{\varphi}'(0)]\zeta
\end{aligned}
$$

or, referring to eq. (6.7.4), we have

$$
2\mu(u-iv) = M(\zeta) + \bar{\zeta}^2\bar{L}(\bar{\zeta}) + \frac{1-\zeta\bar{\zeta}}{3-4\nu}Q(\zeta) + \frac{\bar{\zeta}}{2\pi i}\oint_\gamma \bar{f}(\theta)\,d\sigma. \quad (6.7.8)
$$

This result can be checked easily by means of Sokhotsky-Plemelj formula (5.11.6). Indeed, assuming $\zeta \to \sigma_1 = e^{i\psi}$ we have

$$
\left.
\begin{aligned}
M(\zeta)|_{\zeta\to\sigma_1} &= \frac{1}{2}\bar{f}(\psi) + \frac{1}{2\pi}\int_0^{2\pi}\frac{\bar{f}(\theta)\,d\theta}{1-e^{i(\psi-\theta)}}, \\
\zeta^2 L(\zeta)\big|_{\zeta\to\sigma_1} &= \frac{1}{2}f(\psi) + \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2i(\psi-\theta)}}{1-e^{i(\psi-\theta)}}f(\theta)\,d\theta, \\
\bar{\zeta}^2\bar{L}(\bar{\zeta})\big|_{\bar{\zeta}\to\sigma_1} &= \frac{1}{2}\bar{f}(\psi) + \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-2i(\psi-\theta)}}{1-e^{-i(\psi-\theta)}}\bar{f}(\theta)\,d\theta,
\end{aligned}
\right\} \quad (6.7.9)
$$

so that

$$
\left[M(\zeta) + \bar{\zeta}^2\bar{L}(\bar{\zeta})\right]\Big|_{\zeta\to\sigma_1} = \bar{f}(\psi) - \frac{1}{2\pi}\int_0^{2\pi}\bar{f}(\theta)\,e^{-i(\psi-\theta)}d\theta.
$$

Taking into account the equalities

$$
\left(1-\zeta\bar{\zeta}\right)_\gamma = 0, \quad \frac{\bar{\zeta}}{2\pi i}\oint_\gamma \bar{f}(\theta)\,d\sigma = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(\psi-\theta)}\bar{f}(\theta)\,d\theta,
$$

we arrive at the required relationship

$$
2\mu(u-iv)|_\gamma = \bar{f}(\psi).
$$

Let us notice that prescribing $\bar{f}(\theta)$ in the form of a plane rigid body displacement

$$
\bar{f}_*(\theta) = 2\mu\left(u_0 - iv_0 - i\omega_0\frac{r_0}{\sigma}\right)
$$

we obtain by means of eq. (6.7.7)

$$
L(\zeta) = 0, Q(\zeta) = 0, M(\zeta) = 2\mu(u_0 - iv_0), \frac{1}{2\pi i}\oint_\gamma \bar{f}_*(\theta)\,d\sigma = -2\mu i\omega_0 r_0,
$$

that is

$$2\mu\left(u-iv\right)=2\mu\left(u_0-iv_0-i\omega_0r_0\bar{\zeta}\right),$$

which is required.

It remains to determine $\varphi'(0)$. Using the formulae

$$\begin{aligned}2\mu\left[\frac{\partial\left(u-iv\right)}{\partial\bar{z}}+\frac{\partial\left(u+iv\right)}{\partial z}\right]&=2\mu\left(\varepsilon_x+\varepsilon_y\right)\\&=\left(1-2\nu\right)\left(\sigma_x+\sigma_y\right)=\frac{2}{r_0}\left(1-2\nu\right)\left[\varphi'\left(\zeta\right)+\bar{\varphi}'\left(\bar{\zeta}\right)\right]\end{aligned}$$

and referring to eqs. (6.7.5) and (6.7.8) we have

$$\left.\begin{aligned}r_0\left(\sigma_x+\sigma_y\right)&=\frac{2}{3-4\nu}\left[\zeta Q\left(\zeta\right)+\bar{\zeta}\bar{Q}\left(\bar{\zeta}\right)\right]+2\left[\varphi'\left(0\right)+\bar{\varphi}'\left(0\right)\right],\\r_0\left(\sigma_x+\sigma_y\right)&=\frac{1}{1-2\nu}\left[2\bar{\zeta}\bar{L}\left(\bar{\zeta}\right)+\bar{\zeta}^2\bar{L}'\left(\bar{\zeta}\right)-\frac{\zeta Q\left(\zeta\right)}{3-4\nu}+2\zeta L\left(\zeta\right)+\right.\\&\left.\zeta^2L'\left(\zeta\right)-\frac{\bar{\zeta}\bar{Q}\left(\bar{\zeta}\right)}{3-4\nu}+\frac{1}{2\pi}\int_0^{2\pi}\bar{f}\left(\theta\right)\sigma d\theta+\frac{1}{2\pi}\int_0^{2\pi}f\left(\theta\right)\frac{d\theta}{\sigma}\right]\end{aligned}\right\}\tag{6.7.10}$$

or due to eq. (6.7.7)

$$r_0\left(\sigma_x+\sigma_y\right)=\frac{2}{3-4\nu}\left[\zeta Q\left(\zeta\right)+\bar{\zeta}\bar{Q}\left(\bar{\zeta}\right)\right]+\frac{4\mu}{1-2\nu}\frac{1}{2\pi}\int_0^{2\pi}u_r^0d\theta,\tag{6.7.11}$$

where

$$u_r^0=u^0\cos\theta+v^0\sin\theta$$

is the radial displacement on the circle. Hence,

$$\left(1-2\nu\right)\left[\varphi'\left(0\right)+\bar{\varphi}'\left(0\right)\right]=\frac{\mu}{\pi}\int_0^{2\pi}u_r^0d\theta.\tag{6.7.12}$$

Making use of the previously obtained relation (6.7.4)

$$\left(3-4\nu\right)\varphi'\left(0\right)-\bar{\varphi}'\left(0\right)=\frac{\mu}{\pi}\int_0^{2\pi}\left(u_r^0+iu_\theta^0\right)d\theta\tag{6.7.13}$$

and eqs. (6.7.12), (6.7.13) we find

$$\varphi'\left(0\right)=\frac{\mu}{2\pi}\left[\frac{1}{1-2\nu}\int_0^{2\pi}u_r^0d\theta+\frac{i}{2\left(1-\nu\right)}\int_0^{2\pi}u_\theta^0d\theta\right]\tag{6.7.14}$$

as well as the complex conjugate value $\bar{\varphi}'(0)$.

Now eq. (6.7.5) yields

$$\varphi(\zeta) = \frac{\zeta^2}{3-4\nu} L(\zeta) + \frac{\mu}{2\pi}\zeta \left[\frac{1}{1-2\nu} \int_0^{2\pi} u_r^0 d\theta + \frac{i}{2(1-\nu)} \int_0^{2\pi} u_\theta^0 d\theta \right]. \tag{6.7.15}$$

7.6.8 The state of stress

Function $\zeta Q(\zeta)$ can be presented as follows

$$\zeta Q(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{2\sigma\zeta - \zeta^2}{\sigma^2(\sigma-\zeta)^2} f(\theta)\, d\sigma = \frac{1}{2\pi i} \oint_\gamma \frac{f(\theta)}{(\sigma-\zeta)^2} d\sigma - \frac{1}{2\pi i} \oint_\gamma \frac{f(\theta)}{\sigma^2} d\sigma$$

or

$$\zeta Q(\zeta) = N'(\zeta) - \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-i\theta} d\theta, \quad N(\zeta) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\theta)\, d\sigma}{\sigma - \zeta}. \tag{6.8.1}$$

Along with eq. (6.7.14), this equation allows us to present expression (6.7.10) as a sum of the normal stresses

$$r_0(\sigma_x + \sigma_y) = \frac{2}{3-4\nu} \left\{ N'(\zeta) + \bar{N}'(\bar{\zeta}) - \right.$$
$$\left. \frac{1}{2\pi} \int_0^{2\pi} \left[f(\theta) e^{-i\theta} + \bar{f}(\theta) e^{i\theta} \right] d\theta \right\} + \frac{1}{1-2\nu} \frac{1}{2\pi} \int_0^{2\pi} \left[f(\theta) e^{-i\theta} + \bar{f}(\theta) e^{i\theta} \right] d\theta$$

or

$$r_0(\sigma_x + \sigma_y) = \frac{2}{3-4\nu} \operatorname{Re} \left[2N'(\zeta) + \frac{1}{1-2\nu} \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-i\theta} d\theta \right]. \tag{6.8.2}$$

We proceed now to the second formula of Kolosov-Muskhelishvili. We have

$$\varepsilon_x - \varepsilon_y - i\gamma_{xy} = 2\frac{\partial}{\partial z}(u - iv) = \frac{1}{2\mu} e^{-2i\theta} (\sigma_r - \sigma_\theta - 2i\tau_{r\theta}),$$

or

$$r_0(\sigma_r - \sigma_\theta - 2i\tau_{r\theta}) = 2\left[\frac{\zeta}{\bar{\zeta}} M'(\zeta) - \frac{1}{3-4\nu} \zeta Q(\zeta) + \frac{1-\zeta\bar{\zeta}}{3-4\nu} \frac{\zeta}{\bar{\zeta}} Q'(\zeta) \right]. \tag{6.8.3}$$

At the centre of the round region

$$\left.\begin{aligned}
&r_0 \left(\sigma_r + \sigma_\theta\right)\big|_{r=0} = \frac{2\mu}{1-2\nu}\frac{1}{\pi}\int_0^{2\pi} u_r^0 d\theta,\\
&r_0 \left(\sigma_r - \sigma_\theta - 2i\tau_{r\theta}\right)\big|_{r=0} = 2e^{2i\theta}\left[M'(\zeta) + \frac{3}{3-4\nu}L'(\zeta)\right]_{\zeta=0}\\
&= \frac{2\mu}{\pi}e^{2i\theta}\left[\int_0^{2\pi}(u-iv)\big|_{r=r_0} e^{-i\theta}d\theta + \frac{3}{3-4\nu}\int_0^{2\pi}(u+iv)\big|_{r=r_0} e^{-3i\theta}d\theta\right].
\end{aligned}\right\} \tag{6.8.4}$$

In order to obtain the stresses on the circle it is necessary to pass to the limit $\zeta \to \sigma_1 = e^{i\psi}$in the expressions for $N'(\zeta)$ and $M'(\zeta)$, to obtain

$$N'(\zeta) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\theta)\,d\sigma}{(\sigma-\zeta)^2} = -\frac{1}{2\pi i}f(\theta)\frac{\partial}{\partial\sigma}\frac{1}{\sigma-\zeta}d\sigma$$
$$= -\frac{1}{2\pi i}\oint_\gamma \frac{\partial}{\partial\sigma}\frac{f(\theta)}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\oint_\gamma \frac{\partial f(\theta)}{\partial\sigma}\frac{d\sigma}{\sigma-\zeta} = -\frac{1}{2\pi i}\oint_\gamma \frac{f'(\theta)}{\sigma}\frac{d\sigma}{\sigma-\zeta},$$

as the first integral vanishes. Assuming differentiability of the displacements on γ we apply Sokhotsky-Plemelj's formula

$$N'(\zeta)\big|_{\zeta\to\sigma_1} = -\frac{1}{2}if'(\psi)\,e^{-i\psi} - \frac{1}{2\pi}\oint_\gamma \frac{f'(\theta)}{\sigma}\frac{d\sigma}{\sigma-\sigma_1}$$
$$= -\frac{1}{2}e^{-i\psi}\left[if'(\psi) + \frac{1}{2\pi}\int_0^{2\pi} f'(\theta)\cot\frac{\theta-\psi}{2}d\theta\right]. \tag{6.8.5}$$

The sufficient condition for the existence of the above integrals, as well as the forthcoming integrals is given by Hölder's condition for $f'(\theta)$.

Thus,

$$r_0\left(\sigma_r + \sigma_\theta\right)\big|_{r=r_0} = -\frac{2}{3-4\nu}\operatorname{Re}\left\{e^{-i\psi}\left[if'(\psi) + \right.\right.$$
$$\left.\left.\frac{1}{2\pi}\int_0^{2\pi} f'(\theta)\cot\frac{\theta-\psi}{2}d\theta\right] - \frac{1}{1-2\nu}\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,e^{-i\theta}d\theta\right\} \tag{6.8.6}$$

and by analogy

$$r_0\left(\sigma_r - \sigma_\theta - 2i\tau_{r\theta}\right)\big|_{r=r_0} = e^{-i\psi}\left[i\bar f'(\psi) + \frac{1}{2\pi}\int_0^{2\pi}\bar f'(\theta)\cot\frac{\theta-\psi}{2}d\theta\right] +$$
$$\frac{e^{-i\psi}}{3-4\nu}\left[if'(\psi) + \frac{e^{2i\psi}}{2\pi}\int_0^{2\pi} e^{-2i\theta}\cot\frac{\theta-\psi}{2}f'(\theta)\,d\theta + \frac{e^{2i\psi}}{\pi}\int_0^{2\pi} f(\theta)\,e^{-2i\theta}d\theta\right]. \tag{6.8.7}$$

The distribution of the surface forces ensuring the prescribed displacement on the circle $r = r_0$ is given by the vector

$$r_0(\sigma_r - i\tau_{r\theta}) = -\frac{1-2\nu}{3-4\nu} ie^{i\psi}\bar{f}'(\psi) - \frac{1-\nu}{3-4\nu}\frac{e^{i\psi}}{\pi}\int_0^{2\pi}\bar{f}'(\theta)\cot\frac{\theta-\psi}{2}d\theta +$$
$$\frac{1}{2(1-2\nu)}\frac{1}{2\pi}\left(\int_0^{2\pi} f(\theta)e^{-i\theta}d\theta + \frac{1}{3-4\nu}\int_0^{2\pi}\bar{f}(\theta)e^{i\theta}d\theta\right). \quad (6.8.8)$$

For instance, for the pure radial displacement

$$f(\psi) = 2\mu u_r^0(\psi)e^{i\psi},$$

$$\frac{r_0}{2\mu}(\sigma_r - i\tau_{r\theta})_{r=r_0} = -\frac{1-2\nu}{3-4\nu}\left[u_r^0(\psi) + i\frac{du_r^0}{d\psi}\right] +$$
$$\frac{1-\nu}{3-4\nu}\int_0^{2\pi}\left[u_r^0(\theta) - i\frac{du_r^0}{d\theta}\right]e^{-i(\theta-\psi)}\cot\frac{\theta-\psi}{2}d\theta +$$
$$\frac{1-\nu}{(1-2\nu)(3-4\nu)}\frac{1}{\pi}\int_0^{2\pi}u_r^0(\theta)\,d\theta. \quad (6.8.9)$$

The shear stresses are necessary because the radial displacement is changing along the circle. In the case of $u_r^0 = \text{const}$ we obtain, by means of eq. (6.4.5), that

$$\left.\frac{1}{2\mu}r_0\sigma_r\right|_{r=r_0} = \frac{u_r^0}{3-4\nu}\left[-1+2\nu+2-2\nu+\frac{2(1-\nu)}{1-2\nu}\right] = \frac{u_r^0}{1-2\nu},$$
$$\tau_{r\theta}\big|_{r=r_0} = 0, \quad (6.8.10)$$

which can be checked easily by an elementary calculation. It can also be proved that the obtained system of surface forces (6.8.9) is in equilibrium.

7.6.9 Thermal stresses in the disc placed in a rigid casing

Under a stationary temperature distribution $\theta(r,\psi)$ in the disc with the free edge the displacement vector is given by eq. (5.9.15) in which the constants should be replaced according to the rules (5.8.11) and (5.8.13)

$$2\mu(u+iv) = 2\mu\alpha\Theta(z) = 2\mu\alpha\int(\theta+ig)\,dz, \quad (6.9.1)$$

where g is a harmonic function complex conjugated to θ. In the case of the disc placed in a rigid round casing with a thermal insulator the displacement on the edge of the disc is equal to zero. The system of surface forces (the reaction force of the casing) distributed over the edge is given by formula (6.9.11), but with an opposite sign. In formula (6.8.8) we now have

$$f(\psi) = -2\mu\alpha \int (\theta + ig)\, dz \Big|_{r=r_0}, \tag{6.9.2}$$

and Poisson's ration should be replaced according to rule (5.8.13). Limiting the consideration to representation of function θ (which is harmonic in the circle $|z| < r_0$) by the trigonometric series in terms of cosines we have

$$\theta = \sum_{k=0}^{\infty} \theta_k r^k \cos k\psi, \ \theta + ig = \sum_{k=0}^{\infty} \theta_k z^k, \ \Theta(z) = \sum_{k=0}^{\infty} \frac{\theta_k}{k+1} z^{k+1}, \tag{6.9.3}$$

where the integration constant can be omitted. An additive real-valued constant, up to which function g (the complex conjugate to θ) is determined is also immaterial. Indeed, keeping these constants would introduce a rigid body displacement into the displacement vector and thus does not affect the stresses in the disc.

By eq. (6.9.2) we have

$$f(\psi) = -2\mu\alpha r_0 \sum_{k=0}^{\infty} \theta_k \frac{r_0^k}{k+1} e^{i(k+1)\psi},$$

$$f'(\psi) = -2\mu\alpha i r_0 e^{i\psi} \sum_{k=0}^{\infty} \theta_k r_0^k e^{ik\psi} = -2\mu\alpha i r_0 e^{i\psi}\left(\theta^0 + ig^0\right),$$

where θ^0, g^0 denote the values of θ, g on the edge of the disc.

Calculation by formula (6.8.8) leads to the equality

$$\begin{aligned}(\sigma_r - i\tau_{r\theta})\big|_{r=r_0} = -\frac{2\mu\alpha}{3-\nu}\Bigg\{&-(1-\nu)\left(\theta^0 - ig^0\right) + \\ & ie^{i\psi}\frac{1}{\pi}\int_0^{2\pi}\left[\theta^0(\varphi) - ig^0(\varphi)\right] e^{i\varphi}\cot\frac{\varphi-\psi}{2}\, d\varphi + \\ & \frac{1+\nu}{1-\nu}\frac{1}{\pi}\int_0^{2\pi}\theta^0(\varphi)\, d\varphi + i(1+\nu)\frac{1}{2\pi}\int_0^{2\pi} g^0(\varphi)\, d\varphi\Bigg\}\end{aligned} \tag{6.9.4}$$

or using the above representation in the form of the trigonometric series we obtain

$$
\begin{aligned}
(\sigma_r - i\tau_{r\theta})|_{r=r_0} &= -\frac{2\mu\alpha}{3-\nu}\left[(1+\nu)\sum_{k=0}^{\infty}\theta_k r_0^k e^{-ik\psi} + \frac{2(1+\nu)}{1-\nu}\theta_0\right] \\
&= -\frac{2\mu\alpha(1+\nu)}{3-\nu}\sum_{k=0}^{\infty}\theta_k r_0^k e^{-ik\psi} - 2\mu\alpha\frac{1+\nu}{1-\nu}\theta_0.
\end{aligned}
$$

Here θ_0 denotes the constant term in series (6.9.3). This result can also be set in the form

$$
(\sigma_r + i\tau_{r\theta})|_{r=r_0} = -2\mu\alpha\left[\frac{1+\nu}{1-\nu}\theta_0 + \frac{1+\nu}{3-\nu}(\theta - \theta_0 + ig)\Big|_{r=r_0}\right], \qquad (6.9.5)
$$

so that

$$
\left.\begin{aligned}
(\sigma_r)_{r=r_0} &= -2\mu\alpha\left[\frac{1+\nu}{1-\nu}\theta_0 + \frac{1+\nu}{3-\nu}(\theta-\theta_0)_{r=r_0}\right], \\
(\tau_{r\theta})_{r=r_0} &= -2\mu\alpha\frac{1+\nu}{3-\nu}(g)_{r=r_0}.
\end{aligned}\right\} \qquad (6.9.6)
$$

The expression for σ_r which does not depend on the polar angle can be obtained from elementary reasoning.

Stress σ_θ can be obtained for example by means of eq. (6.4.7)

$$
(\sigma_\theta)_{r=r_0} = -2\mu\alpha\left[\frac{1+\nu}{1-\nu}\theta_0 + \frac{3+3\nu}{3-\nu}(\theta-\theta_0)_{r=r_0}\right]. \qquad (6.9.7)
$$

Only distribution of the temperature on the surface of the disc is required for these stresses.

7.6.10 Round opening in an infinite plane

The edge of the opening is assumed to be loaded by the surface forces whose projections on axes $\mathbf{e}_r, \mathbf{e}_\theta$ of the polar coordinate system are denoted as f_r, f_θ. Their principal vector and the principal moment about the centre of the opening are presented by the formulae analogous those in eq. (6.2.4)

$$
\int_0^{2\pi} f(\theta)\, e^{i\theta} r_0 d\theta = -ir_0 \oint_\gamma f(\theta)\, d\sigma = X + iY \quad (f(\theta) = f_r + if_\theta),
$$

$$
\int_0^{2\pi} \frac{r_0^2}{2i}\left[f(\theta) - \bar{f}(\theta)\right] d\theta = -\frac{1}{2}r_0^2 \oint_\gamma \left[f(\theta) - \bar{f}(\theta)\right]\frac{d\sigma}{\sigma} = M^O
$$

or

$$\frac{1}{2\pi i}\oint_{\gamma} f(\theta)\,d\sigma + \frac{1}{2\pi r_0}(X+iY) = 0, \quad \oint_{\gamma}\left[f(\theta) - \bar{f}(\theta)\right]\frac{d\sigma}{\sigma} - \frac{2M^O}{r_0^2} = 0. \tag{6.10.1}$$

The boundary condition on the contour of the opening is set in the form

$$r = r_0, \quad \zeta = \sigma = e^{i\theta}: \quad \sigma_r + i\tau_{r\theta} =$$
$$= \Phi(\sigma) + \bar{\Phi}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\bar{\Phi}'\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2}\bar{\Psi}\left(\frac{1}{\sigma}\right) = -f(\theta), \tag{6.10.2}$$

where a minus sign on the right hand side is due to the fact that the external normal to L–region ($|\zeta| > 1$) is opposite in direction to $\mathbf{e}_r$. The structure of functions $\Phi(\zeta), \Psi(\zeta)$ is as follows

$$\left.\begin{aligned}
\Phi(\zeta) &= \frac{1}{4}\left(\sigma_1^\infty + \sigma_2^\infty\right) + i\mu\frac{\varepsilon^\infty}{2(1-\nu)} - \frac{X+iY}{8\pi(1-\nu)r_0}\frac{1}{\zeta} + \Phi_*(\zeta),\\
\Psi(\zeta) &= \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{-2i\alpha} + \frac{X-iY}{8\pi(1-\nu)r_0}(3-4\nu)\frac{1}{\zeta} + \Psi_*(\zeta),
\end{aligned}\right\} \tag{6.10.3}$$

where the expansion of functions $\Phi_*(\zeta), \Psi_*(\zeta)$, which are holomorphic at infinity, begins with terms ζ^{-2}. The boundary condition determining these functions is now set in the form

$$\zeta = \sigma: \quad \Phi_*(\sigma) + \bar{\Phi}_*\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma}\bar{\Phi}_*'\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2}\bar{\Psi}_*\left(\frac{1}{\sigma}\right) =$$
$$= -f(\theta) - \frac{1}{2}\left(\sigma_2^\infty + \sigma_1^\infty\right) + \frac{1}{2\sigma^2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{2i\alpha} +$$
$$\frac{1}{8\pi(1-\nu)r_0}\left[4(1-\nu)(X+iY)\frac{1}{\sigma} + 2(X-iY)\sigma\right]. \tag{6.10.4}$$

The complex conjugated condition is

$$\bar{\Phi}_*\left(\frac{1}{\sigma}\right) + \Phi_*(\sigma) - \sigma\Phi_*'(\sigma) - \sigma^2\Psi_*(\sigma) =$$
$$= -\bar{f}(\theta) - \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right) + \frac{1}{2}\sigma^2\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{-2i\alpha} +$$
$$+ \frac{1}{8\pi(1-\nu)r_0}\left[4(1-\nu)(X-iY)\sigma + \frac{2}{\sigma}(X+iY)\right]. \tag{6.10.5}$$

In what follows, while applying the method of Cauchy's integral it is necessary to remember that $|\zeta| > 1$. Function $F(\zeta)$ is holomorphic in L ($|\zeta| > 1$) at any point, but infinite where it is given by the polynomial $g_n(\zeta)$

$$F(\zeta) = \sum_{k=1}^{\infty}\frac{\alpha_k + i\beta_k}{\zeta^k} + g_n(\zeta), \quad g_n(\zeta) = a_0 + a_1\zeta + \ldots + a_n\zeta^n,$$

thus, Cauchy's integral formula takes the form

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{F(\sigma)}{\sigma-\zeta}d\sigma = F(\zeta) - g_n(\zeta) \quad (|\zeta|>1). \tag{6.10.6}$$

Then the function

$$F\left(\frac{1}{\zeta}\right) = \sum_{k=1}^{\infty}(\alpha_k + i\beta_k)\zeta^k + a_0 + \left[g_n\left(\frac{1}{\zeta}\right) - a_0\right]$$

is holomorphic for $|\zeta| < 1$ except for the origin of the coordinate system where it has a pole of order n and by eq. (5.10.5)

$$|\zeta|>1: \quad \oint_{\gamma}\frac{F\left(\frac{1}{\sigma}\right)}{\sigma-\zeta}d\sigma = g_n\left(\frac{1}{\zeta}\right) - a_0. \tag{6.10.7}$$

Being guided by these rules and taking into account the form of the expansion of the sought functions in the power series

$$\Phi_*(\zeta) = \frac{\alpha_2 + i\beta_2}{\zeta^2} + \ldots, \qquad \bar{\Phi}_*\left(\frac{1}{\zeta}\right) = (\alpha_2 - i\beta_2)\zeta^2 + \ldots,$$
$$\Phi'_*(\zeta) = -\frac{2(\alpha_2 + i\beta_2)}{\zeta^3} + \ldots, \qquad \Psi_*(\zeta) = \frac{\alpha'_2 + i\beta'_2}{\zeta^2} + \ldots,$$
$$\bar{\Psi}_*\left(\frac{1}{\zeta}\right) = (\alpha'_2 - i\beta'_2)\zeta^2 + \ldots,$$

we multiply both parts of equalities (6.10.4) and (6.10.5) by

$$\frac{1}{2\pi i}\frac{d\sigma}{\sigma-\zeta}$$

and integrate along γ in the direction that L lies on the left. The result is

$$\left.\begin{aligned}
&\Phi_*(\zeta) = -\frac{1}{2\pi i}\oint_{\gamma}\frac{f(\theta)\,d\sigma}{\sigma-\zeta} + \frac{e^{2i\alpha}}{2\zeta^2}\left(\sigma_2^{\infty} - \sigma_1^{\infty}\right) + \frac{1}{2\pi r_0}(X+iY)\frac{1}{\zeta},\\
&\Phi_*(\zeta) - \zeta\Phi'_*(\zeta) - \zeta^2\left[\Psi_*(\zeta) - \frac{\alpha'_2 + i\beta'_2}{\zeta^2}\right] =\\
&\qquad = -\frac{1}{2\pi i}\oint_{\gamma}\frac{\bar{f}(\theta)\,d\sigma}{\sigma-\zeta} + \frac{1}{4\pi(1-\nu)r_0}(X+iY)\frac{1}{\zeta},
\end{aligned}\right\} \tag{6.10.8}$$

so that

$$\Psi_*(\zeta) = \frac{1}{\zeta^2}\left[\frac{1}{2\pi i}\oint_\gamma \frac{\bar f(\theta) - f(\theta)}{\sigma - \zeta}d\sigma + \frac{3}{2\zeta^2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{2i\alpha} + \right.$$

$$\left.\frac{3-4\nu}{4\pi(1-\nu)r_0}(X+iY)\frac{1}{\zeta} + \frac{\zeta}{2\pi i}\oint_\gamma \frac{f(\theta)}{(\sigma-\zeta)^2}d\sigma\right] + \frac{\alpha_2' + i\beta_2'}{\zeta^2}. \quad (6.10.9)$$

It remains to determine $\alpha_2'+i\beta_2'$. To this end, we take boundary condition (6.10.5) and compare the constant terms in both parts, to get

$$\alpha_2' + i\beta_2' = -\frac{1}{2\pi i}\oint_\gamma \bar f(\theta)\frac{d\sigma}{\sigma} + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \bar f(\theta)\,d\theta + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right) = \bar f_0 + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right), \quad (6.10.10)$$

where $\bar f_0$ denotes the free term of the trigonometric series $\bar f(\theta)$.

7.6.11 A uniform loading on the edge of the opening

In this particular case

$$f(\theta) = \frac{X+iY}{2\pi r_0}e^{-i\theta}, \quad \bar f(\theta) = \frac{X-iY}{2\pi r_0}e^{i\theta}, \quad \sigma_1^\infty = \sigma_2^\infty = 0, \quad (6.11.1)$$

and thus

$$|\zeta| > 1: \quad \frac{1}{2\pi i}\oint_\gamma \frac{f(\theta)}{\sigma-\zeta}d\sigma = \frac{X+iY}{2\pi r_0\zeta}, \quad \frac{1}{2\pi i}\oint_\gamma \frac{\bar f(\theta)}{\sigma-\zeta}d\sigma = 0, \quad \bar f_0 = 0.$$

By eqs. (6.10.8)-(6.10.10) and (6.10.3) we have for $\varepsilon^\infty = 0$

$$\left.\begin{aligned}\Phi(\zeta) &= -\frac{X+iY}{8\pi(1-\nu)}\frac{1}{z} \quad (z = r_0\zeta),\\ \Psi(\zeta) &= \frac{3-4\nu}{8\pi(1-\nu)}(X-iY)\frac{1}{z} - \frac{X+iY}{4\pi(1-\nu)}\frac{r_0^2}{z^3}.\end{aligned}\right\} \quad (6.11.2)$$

At $\rho_0 \to 0$ we arrive at the previously obtained solution to the problem of the concentrated force in the plane, Subsection 7.3.1.

7.6.12 Tension of the plane weakened by a round opening

The edge of the opening is assumed to be free, i.e.

$$f(\theta) = 0, \quad X + iY = 0. \quad (6.12.1)$$

From the above relationships we obtain

$$\left.\begin{aligned}\Phi(\zeta) &= \frac{1}{4}\left(\sigma_1^\infty + \sigma_2^\infty\right) + \frac{r_0^2}{2z^2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{2i\alpha},\\ \Psi(\zeta) &= \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{-2i\alpha} + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right)\frac{r_0^2}{z^2} + \frac{3}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right)\frac{r_0^4}{z^4}e^{2i\alpha}\end{aligned}\right\} \tag{6.12.2}$$

and by Kolosov-Muskhelishvili's formulae (1.14.9) we have

$$\begin{aligned}&\sigma_r + \sigma_\theta = \sigma_1^\infty + \sigma_2^\infty + 2\frac{r_0}{r}\left(\sigma_2^\infty - \sigma_1^\infty\right)\cos 2\psi, \quad \sigma_\theta - \sigma_r + 2i\tau_{r\theta} =\\ &= \left(\sigma_1^\infty + \sigma_2^\infty\right)\frac{r_0^2}{r^2} + \left(\sigma_2^\infty - \sigma_1^\infty\right)\left(e^{2i\psi} - 2\frac{r_0^2}{r^2}e^{-2i\psi} + 3\frac{r_0^4}{r^4}e^{-2i\psi}\right),\end{aligned} \tag{6.12.3}$$

where $\psi = \theta - \alpha$ is the angle to the first principal axis.

For example in the problem of tension of the plane weakened by a round opening (Kirsch's problem, 1898) we have

$$\begin{aligned}&\sigma_2^\infty = 0, \quad \sigma_r + \sigma_\theta = \sigma_1^\infty\left(1 - 2\frac{r_0^2}{r^2}\cos 2\psi\right),\\ &\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \sigma_1^\infty\left(\frac{r_0^2}{r^2} - e^{2i\psi} + 2\frac{r_0^2}{r^2}e^{-2i\psi} - 3\frac{r_0^4}{r^4}e^{-2i\psi}\right).\end{aligned} \tag{6.12.4}$$

For $r = r_0$ we obtain $\sigma_r = \tau_{r\theta} = 0$ and

$$\sigma_\theta = \sigma_1^\infty\left(1 - 2\cos 2\psi\right), \quad \left(\sigma_\theta\right)_{\max} = 3\sigma_1^\infty \quad \left(\text{for } \psi = \pm\frac{\pi}{2}\right), \tag{6.12.5}$$

that is, the maximum stress on the contour of the round opening is equal to three times the nominal stress. The stress concentration is of a clear local character, for instance, along the diameter perpendicular to the direction of tension $\left(\psi = \frac{\pi}{2}\right)$

$$\sigma_\theta = \sigma_1^\infty\left(1 + \frac{1}{2}\frac{r_0^2}{r^2} + \frac{3}{2}\frac{r_0^4}{r^4}\right)$$

i.e. $\sigma_\theta = 1{,}074\,\sigma_1^\infty$ for $r = 3r_0$ and $\sigma_\theta = 1{,}022\,\sigma_1^\infty$ for $r = 5r_0$.

Under uniform tension at infinity ($\sigma_1^\infty = \sigma_2^\infty = q$) we have

$$\sigma_\theta = q\left(1 + \frac{r_0^2}{r^2}\right), \ \sigma_r = q\left(1 - \frac{r_0^2}{r^2}\right), \ \tau_{r\theta} = 0, \quad \left(\sigma_\theta\right)_{r=r_0} = 2q. \tag{6.12.6}$$

In the case of a simple shear at infinity ($\sigma_1^\infty = -\sigma_2^\infty = \tau$) we obtain with the help of formulae (6.12.3)

$$\begin{aligned}\sigma_r + \sigma_\theta &= -4\tau\frac{r_0^2}{r^2}\cos 2\psi,\\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} &= -2\tau\left(e^{2i\psi} - 2\frac{r_0^2}{r^2}e^{-2i\psi} + 3\frac{r_0^4}{r^4}e^{-2i\psi}\right).\end{aligned} \tag{6.12.7}$$

that is, the maximum stress on the contour of the opening is equal to four times the nominal stress.

7.6.13 Continuation of $\Phi(z)$

Under the transformation by means of the inverse radii about circle Γ $(r = a)$ a point $M(r, \theta)$ is mapped to point $M^*\left(a^2/r, \theta\right)$; in other words, point $M\left(z = re^{i\theta}\right)$ is mapped to point $M^*\left(a^2/\bar{z} = a^2 r^{-1} e^{i\theta}\right)$ lying on the same straight line with the origin at the centre of Γ. Let us consider an elastic body ($L-$region) bounded by circle Γ, it is a round disc for $|z| < a$ and a plane with a round opening for $|z| > a$. While traversing Γ in the positive direction (counterclockwise for $|z| < a$ and clockwise for $|z| > a$) regions L and R lie to the left and to the right respectively. Let $M(z)$ and $M^*\left(a^2/\bar{z}\right)$ be respectively the points of regions L and R. By eq. (1.14.9)

$$\sigma_r + i\tau_{r\theta} = \Phi(z) + \bar{\Phi}(\bar{z}) - \bar{z}\bar{\Phi}'(\bar{z}) - \frac{\bar{z}}{z}\bar{\Psi}(\bar{z}) \quad (z \subset L). \tag{6.13.1}$$

Functions $\Phi(z), \Psi(z)$ are not determined in $R-$region. In this region z is mapped to $a^2/\bar{z}$ and $\sigma_r + i\tau_{r\theta} = 0$. This allows $\Phi(z)$ to be determined in $R-$region by setting the left hand side of eq. (6.13.1) to zero. Replacing additionally z by $a^2/\bar{z}$ and keeping $\bar{z}$ unchanged we obtain the equality relating functions of $\bar{z}$ only

$$\Phi\left(\frac{a^2}{\bar{z}}\right) = -\bar{\Phi}(\bar{z}) + \bar{z}\bar{\Phi}'(\bar{z}) + \frac{\bar{z}^2}{a^2}\bar{\Psi}(\bar{z}) \quad (z \subset L). \tag{6.13.2}$$

Replacing here z by $a^2/\bar{z}$ we arrive at the equivalent relationships between the functions of z

$$\Phi(z) = -\bar{\Phi}\left(\frac{a^2}{z}\right) + \frac{a^2}{z}\bar{\Phi}'\left(\frac{a^2}{z}\right) + \frac{a^2}{z^2}\bar{\Psi}\left(\frac{a^2}{z}\right) \quad (z \subset R). \tag{6.13.3}$$

This equality determines the continuation of $\Phi(z)$ into $R-$region.

Let $L-$region be the plane with a round opening, then $R-$region is the disc $|z| < a$. Functions $\bar{\Phi}(z), \bar{\Psi}(z)$ which are holomorphic in L are expanded as series in terms of the negative powers of z and the constant terms. Thus, $\bar{\Phi}\left(a^2/z\right), \bar{\Psi}\left(a^2/z\right)$ are series in terms of the positive powers of z and the structure of formula (6.13.3) suggests that $\Phi(z)$ is holomorphic in R everywhere except for the coordinate origin which is a pole. In the neighbourhood of this pole the expansion of $\Phi(z)$ is as follows

$$A + \frac{B}{z} + \frac{C}{z^2}. \tag{6.13.4}$$

If $L-$region is the disc then the infinite point $z = \infty$ lies in $R-$region. Further $\bar{\Phi}(z), \bar{\Psi}(z)$ are series in terms of positive powers of z whereas

$\bar{\Phi}\left(a^2/z\right), \bar{\Psi}\left(a^2/z\right)$ are series in terms of negative powers of z and the constant terms. According to eq. (6.13.3) the constant term appears also in the analytic continuation of $\Phi(z)$ to R–region. Hence, in this region $\Phi(z)$ is holomorphic everywhere including the infinite point $z=\infty$ where the principal part of $\Phi(z)$ is constant.

Formulae (6.13.1) and (6.13.2) allow the following equality to be constructed

$$z \subset L: \quad \Phi(z) - \Phi\left(\frac{a^2}{\bar{z}^2}\right) + \bar{\Psi}(\bar{z})\frac{\bar{z}^2}{a^2}\left(1-\frac{a^2}{z\bar{z}}\right) = (\sigma_r + i\tau_{r\theta}). \quad (6.13.5)$$

Under the limiting process $|z| \to a$ both points $M(z)$ and $M^*\left(a^2/z\right)$ reach the same point $M^0\left(t = ae^{i\psi}\right)$ on Γ from regions L and R respectively. As $z\bar{z} - a^2 = 0$ we obtain

$$z = t \subset \Gamma: \quad \Phi^L(t) - \Phi^R(t) = (\sigma_r + i\tau_{r\theta})_\Gamma. \quad (6.13.6)$$

Referring now to eq. (5.11.9) we define $\Phi(z)$ by means of the integral of Cauchy's type

$$\Phi(z) = \frac{1}{2\pi i}\oint\limits_\Gamma \frac{(\sigma_r + i\tau_{r\theta})_\Gamma}{t-z}dt + \gamma(z). \quad (6.13.7)$$

The integral in this formula is holomorphic in the plane except Γ. The integration is performed in the direction such that L–region lies to the left. Function $\gamma(z)$ is introduced for accounting for possible singularities of $\Phi(z)$ at $z=0$ and $z=\infty$. According to the above-said it is taken as being equal to expressions (6.13.4) if L–region is a plane with a round opening. In the particular case in which L is a disc, $\gamma(z)$ is constant, say D. Therefore

$$\left.\begin{aligned}
\Phi(z) &= \frac{1}{2\pi i}\oint\limits_\Gamma \frac{(\sigma_r + i\tau_{r\theta})_\Gamma}{t-z}dt + D; \quad z \subset L, \quad |z| < a, \\
\Phi(z) &= \frac{1}{2\pi i}\oint\limits_\Gamma \frac{(\sigma_r + i\tau_{r\theta})_\Gamma}{t-z}dt + A + \frac{B}{z} + \frac{C}{z^2}; \quad z \subset L, \quad |z| > a.
\end{aligned}\right\} \quad (6.13.8)$$

Returning to formula (6.13.2) and using the complex conjugated expression we have

$$\Psi(z) = \frac{a^2}{z^2}\left[\Phi(z) + \bar{\Phi}\left(\frac{a^2}{z}\right) - z\Phi'(z)\right], \quad z \subset L. \quad (6.13.9)$$

This equality determines function $\Psi(z)$ holomorphic in L, function $\Phi(z)$ being given by one of formulae (6.13.8). The term $\bar{\Phi}\left(a^2/z\right)$ is calculated as follows: the integral in (6.13.8) is evaluated for $z \subset R$, the complex conjugate to $\Phi(z)$ is obtained and in the latter $\bar{z}$ is replaced by a^2/z in order to return to L–region. It is evident that the constructed function is holomorphic in L and is different from that obtained by the formal replacement of z by a^2/z in the expression for $\Phi(z)$ for $z \subset L$. Let us denote this function as $\Phi^{\times}\left(a^2/z\right)$, it is evident that this function is not holomorphic in L.

7.6.14 Solving the boundary-value problems of Subsections 7.6.2 and 7.6.10 by way of the continuation

Let us begin by considering the case of disc $|\zeta| < 1$ under the assumption that the surface forces are in equilibrium. The boundary conditions and the equilibrium conditions are written down in the form

$$\zeta = \sigma = e^{i\theta}: \quad \sigma_r + i\tau_{r\theta} = f(\theta) = A_0 + \sum_{k=1}^{\infty} A_k \sigma^k + \sum_{k=1}^{\infty} A_{-k}\sigma^{-k}, \tag{6.14.1}$$

$$\frac{1}{2\pi i}\oint_{\Gamma} f(\theta)\, d\sigma = A_{-1} = 0, \quad \frac{1}{2\pi i}\oint_{\Gamma}\left[f(\theta) - \bar{f}(\theta)\right]\frac{d\sigma}{\sigma} = A_0 - \bar{A}_0 = 0, \tag{6.14.2}$$

cf. eq. (6.2.4). Using eq. (6.13.8) and the formulae of Subsection 7.5.10 we have

$$\Phi(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\theta)\, d\sigma}{\sigma - \zeta} + D = \begin{cases} A_0 + \sum\limits_{k=1}^{\infty} A_k \zeta^k + D & (\zeta \subset L, \quad |\zeta| < 1), \\ -\sum\limits_{k=1}^{\infty} A_{-k}\zeta^{-k} + D & (\zeta \subset R, \quad |\zeta| > 1). \end{cases} \tag{6.14.3}$$

In agreement with Subsection 7.6.13 the latter equation determines the function

$$\bar{\Phi}\left(\frac{1}{\zeta}\right) = -\sum_{k=1}^{\infty} \bar{A}_{-k}\zeta^k + \bar{D}, \tag{6.14.4}$$

which is holomorphic in L. Taking into account that

$$\bar{f}(\theta) = \bar{A}_0 + \sum_{k=1}^{\infty} \bar{A}_k \sigma^{-k} + \sum_{k=1}^{\infty} \bar{A}_{-k}\sigma^k \tag{6.14.5}$$

we can rewrite eq. (6.14.4) in another form

$$\bar{\Phi}\left(\frac{1}{\zeta}\right) = -\frac{1}{2\pi i}\oint_\gamma \frac{\bar{f}(\theta)\,d\sigma}{\sigma-\zeta} + \bar{A}_0 + \bar{D}. \tag{6.14.6}$$

By virtue of eq. (6.13.9) we obtain

$$\Psi(\zeta) = \frac{1}{\zeta^2}\left[\frac{1}{2\pi i}\oint_\gamma \frac{f(\theta)-\bar{f}(\theta)}{\sigma-\zeta}d\sigma - \frac{\zeta}{2\pi i}\oint_\gamma \frac{f(\theta)\,d\sigma}{(\sigma-\zeta)^2} + \bar{A}_0 + D + \bar{D}\right].$$

Function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta| < 1$, thus the coefficient of ζ and the constant term in the expression in the square brackets must vanish. This yields the equalities

$$-\frac{1}{2\pi i}\oint_\gamma \frac{\bar{f}(\theta)}{\sigma^2}d\sigma = -A_{-1} = 0, \quad D + \bar{D} = -A_0 = -\frac{1}{2\pi i}\oint_\gamma \frac{f(\theta)}{\sigma}d\sigma.$$

The first one expresses the condition of zero principal vector of the surface forces. The second one can also be satisfied as it follows from the equation for the moments (A_0 is real-valued). Clearly, D is determined up to an imaginary term which can be set to zero. We arrive at the results obtained previously in Subsection 7.6.3

$$|\zeta| < 1: \quad \left.\begin{aligned} \Phi(\zeta) &= \frac{1}{2\pi i}\left(\oint_\gamma \frac{f(\theta)\,d\sigma}{\sigma-\zeta} - \frac{1}{2}\oint_\gamma f(\theta)\frac{d\sigma}{\sigma}\right), \\ \Psi(\zeta) &= \frac{1}{2\pi i}\frac{1}{\zeta^2}\left(\oint_\gamma \frac{f(\theta)-\bar{f}(\theta)}{\sigma-\zeta}d\sigma - \zeta\oint_\gamma \frac{f(\theta)}{(\sigma-\zeta)^2}d\sigma\right). \end{aligned}\right\} \tag{6.14.7}$$

In the case of the plane with an opening we set the boundary condition in the form

$$\zeta = \sigma = e^{\iota\theta}: \quad \sigma_r + i\tau_{r\theta} = -f(\theta) = -\left[A_0 + \sum_{k=1}^{\infty} A_k\sigma^k + \sum_{k=1}^{\infty} A_{-k}\sigma^{-k}\right], \tag{6.14.8}$$

and obtain by eq. (6.13.8) that

$$\begin{aligned} \Phi(\zeta) &= -\frac{1}{2\pi i}\oint_\gamma \frac{f(\theta)\,d\sigma}{\sigma-\zeta} + A + \frac{B}{\zeta} + \frac{C}{\zeta^2} \\ &= \begin{cases} -\sum\limits_{k=1}^{\infty}\dfrac{A_{-k}}{\zeta^k} + A + \dfrac{B}{\zeta} + \dfrac{C}{\zeta^2} & (\zeta \subset L, \quad |\zeta| > 1), \\ A_0 + \sum\limits_{k=1}^{\infty} A_k\zeta^k + A + \dfrac{B}{\zeta} + \dfrac{C}{\zeta^2} & (\zeta \subset R, \quad |\zeta| < 1). \end{cases} \end{aligned} \tag{6.14.9}$$

For this reason

$$\bar{\Phi}\left(\frac{1}{\zeta}\right) = \bar{A}_0 + \sum_{k=1}^{\infty} \frac{\bar{A}_k}{\zeta^k} + \bar{A} + \bar{B}\zeta + \bar{C}\zeta^2$$
$$= \frac{1}{2\pi i} \oint_{\gamma} \frac{\bar{f}(\theta)}{\sigma - \zeta} d\sigma + \bar{A}_0 + \bar{A} + \bar{B}\zeta + \bar{C}\zeta^2 \tag{6.14.10}$$

and by eq. (6.13.9)

$$\zeta \subset L, \quad |\zeta| > 1: \quad \Psi(\zeta) = \frac{1}{\zeta^2}\left[\frac{1}{2\pi i} \oint_{\gamma} \frac{\bar{f}(\theta) - f(\theta)}{\sigma - \zeta} d\sigma + \right.$$
$$\left. \frac{\zeta}{2\pi i} \oint_{\gamma} \frac{f(\theta)}{(\sigma - \zeta)^2} d\sigma + A + \bar{A} + \bar{A}_0 + \frac{2B}{\zeta} + \frac{3C}{\zeta^2}\right] + \frac{\bar{B}}{\zeta} + \bar{C}. \tag{6.14.11}$$

This function is holomorphic at the infinite point and its principal part is equal to $\bar{C}$ at infinity. The constants A, B, C can be expressed in terms of the principal vector of the surface forces and the stresses on the opening and the rotation at infinity. Referring to eqs. (6.10.1), (6.10.3) we have

$$A = \frac{1}{4}\left(\sigma_1^\infty + \sigma_2^\infty\right) + i\mu \frac{\varepsilon^\infty}{2(1-\nu)}, \quad \bar{B} = \frac{3 - 4\nu}{8\pi(1-\nu)}(X - iY),$$
$$\bar{C} = \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right) e^{-2i\alpha},$$

where

$$\frac{1}{2\pi r_0}(X + iY) = -\frac{1}{2\pi i} \oint_{\gamma} f(\theta)\, d\sigma = A_{-1}, \quad A_0 = -\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\theta)}{\sigma} d\sigma.$$

Returning to eqs. (6.14.9) and (6.14.11) we arrive at the following expressions for the sought functions

$$\zeta \subset L, \quad |\zeta| > 1: \qquad \Phi(\zeta) = \frac{1}{4}\left(\sigma_1^\infty + \sigma_2^\infty\right) + i\mu \frac{\varepsilon^\infty}{2(1-\nu)} +$$
$$\frac{3 - 4\nu}{8\pi(1-\nu)}(X + iY)\frac{1}{\zeta} + \frac{1}{2\zeta^2}\left(\sigma_2^\infty - \sigma_1^\infty\right) e^{2i\alpha} - \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\sigma)}{\sigma - \zeta} d\sigma,$$

$$\zeta \subset L, \quad |\zeta| > 1: \qquad \Psi(\zeta) = \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right) e^{-2i\alpha} +$$
$$\frac{3-4\nu}{8\pi(1-\nu)}(X - iY)\frac{1}{\zeta} + \frac{1}{\zeta^2}\left[\frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right) - \frac{1}{2\pi i}\oint_\gamma \bar{f}(\theta)\frac{d\sigma}{\sigma} + \right.$$
$$\frac{3-4\nu}{4\pi(1-\nu)}(X + iY)\frac{1}{\zeta} + \frac{3}{2\zeta^2}\left(\sigma_2^\infty - \sigma_1^\infty\right) e^{2i\alpha} +$$
$$\left.\frac{\zeta}{2\pi i}\oint_\gamma \frac{f(\theta)}{(\sigma-\zeta)^2} d\sigma + \frac{1}{2\pi i}\oint_\gamma \frac{\bar{f}(\theta) - f(\theta)}{\sigma - \zeta} d\sigma\right], \tag{6.14.12}$$

which are in full agreement with the results of Subsection 7.6.10.

7.7 Round ring

7.7.1 *The stresses due to distortion*

Let us begin by representing the stresses in Michell's form, eq. (5.6.7). When the external ($r = r_0$) and internal ($r = r_1$) circles bounding the ring are free of loads it is sufficient to keep the following terms in the above expression

$$U = \frac{\mu}{2\pi(1-\nu)}\left\{\left[\frac{1}{2}b_3 r^2 + (c_2 r\cos\theta - c_1 r\sin\theta)\right]\ln r + \right. \tag{7.1.1}$$
$$\left.\left[\alpha_1 r^2 + \alpha'_{-1}\ln r + r^3\left(C_1\cos\theta + S_1\sin\theta\right) + \frac{1}{r}\left(a'_1\cos\theta + b'_1\sin\theta\right)\right]\right\},$$

where the pairs of constants $\left(\alpha_1, \alpha'_{-1}\right), \left(C_1, a'_1\right), \left(S_1, b'_1\right)$ are determined from the independent systems of equations obtained from the boundary conditions

$$r = r_0, r = r_1: \quad \sigma_r = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right) = 0. \tag{7.1.2}$$

Obtaining these constants we arrive at the following states of stresses: an axially symmetric state due to the rotational component of distortion

$$\left.\begin{aligned} \sigma_r &= \frac{\mu b_3}{2\pi(1-\nu)}\left(\ln\frac{r}{r_0} - \frac{r_1^2}{r^2}\frac{r_0^2 - r^2}{r_0^2 - r_1^2}\ln\frac{r_1}{r_0}\right), \\ \sigma_\theta &= \frac{\mu b_3}{2\pi(1-\nu)}\left(1 + \ln\frac{r}{r_0} + \frac{r_1^2}{r^2}\frac{r_0^2 + r^2}{r_0^2 - r_1^2}\ln\frac{r_1}{r_0}\right), \end{aligned}\right\} \tag{7.1.3}$$

and the state of stress due to the translational distortions

$$\left.\begin{aligned}
\sigma_r &= \frac{\mu}{2\pi(1-\nu)r}(c_2\cos\theta - c_1\sin\theta)\left(1 - \frac{1}{r^2}\frac{r_0^2r_1^2+r^4}{r_0^2+r_1^2}\right),\\
\sigma_\theta &= \frac{\mu}{2\pi(1-\nu)r}(c_2\cos\theta - c_1\sin\theta)\left(1 + \frac{1}{r^2}\frac{r_0^2r_1^2-3r^4}{r_0^2+r_1^2}\right),\\
\tau_{r\theta} &= \frac{\mu}{2\pi(1-\nu)r}(c_2\sin\theta + c_1\cos\theta)\left(1 - \frac{1}{r^2}\frac{r_0^2r_1^2+r^4}{r_0^2+r_1^2}\right).
\end{aligned}\right\} \tag{7.1.4}$$

7.7.2 The second boundary-value problem for a ring

We consider the state of stress in a ring bounded by the concentric external (Γ_0) and internal (Γ_1) circles. For simplifying the notion, the radii of the external and internal circles are taken as being equal to 1 and $\alpha\,(0<\alpha<1)$ respectively. By eq. (1.14.9)

$$\sigma_r - i\tau_{r\theta} = \Phi(\zeta) + \bar{\Phi}(\zeta) - \zeta\Phi'(\zeta) - \frac{\zeta}{\bar{\zeta}}\Psi(\zeta), \tag{7.2.1}$$

and the boundary conditions can be set in the form

$$\text{on } \Gamma_0: \quad \zeta = \sigma, \quad \Phi(\sigma) + \bar{\Phi}\left(\frac{1}{\sigma}\right) - \sigma\Phi'(\sigma) - \sigma^2\Psi(\sigma) = \bar{f}(\theta), \tag{7.2.2}$$

$$\text{on } \Gamma_1: \quad \zeta = \alpha\sigma, \quad \Phi(\alpha\sigma) + \bar{\Phi}\left(\frac{\alpha}{\sigma}\right) - \alpha\sigma\Phi'(\alpha\sigma) - \sigma^2\Psi(\alpha\sigma) = -\bar{F}(\theta). \tag{7.2.3}$$

Here

$$f(\theta) = f_0 + \sum_{k=1}^{\infty} f_k\sigma^k + \sum_{k=1}^{\infty} f_{-k}\sigma^{-k}, \quad F(\theta) = F_0 + \sum_{k=1}^{\infty} F_k\sigma^k + \sum_{k=1}^{\infty} F_{-k}\sigma^{-k} \tag{7.2.4}$$

denote the vectors of the external surface forces distributed over Γ_0 and Γ_1 respectively.

The complex conjugated vectors are given by

$$\bar{f}(\theta) = \bar{f}_0 + \sum_{k=1}^{\infty} \bar{f}_k\sigma^{-k} + \sum_{k=1}^{\infty} \bar{f}_{-k}\sigma^{k}, \quad \bar{F}(\theta) = \bar{F}_0 + \sum_{k=1}^{\infty} \bar{F}_k\sigma^{-k} + \sum_{k=1}^{\infty} \bar{F}_{-k}\sigma^{k}. \tag{7.2.5}$$

The static equations

$$\oint f(\theta)\,d\sigma + \alpha\oint F(\theta)\,d\sigma = 0, \quad \oint (f-\bar{f})\frac{d\sigma}{\sigma} + \alpha^2\oint (F-\bar{F})\frac{d\sigma}{\sigma} = 0, \tag{7.2.6}$$

expressing the vanishing of the principal vector and the principal moment of the external forces applied to the ring yield the conditions imposed on the coefficients of the trigonometric series (7.2.5)

$$f_{-1} + \alpha F_{-1} = 0, \quad f_0 + \alpha^2 F_0 - \left(\bar{f}_0 + \alpha^2 \bar{F}_0\right) = 0. \tag{7.2.7}$$

Equation (7.2.2) can be satisfied by prescribing function $\Psi(\zeta)$ in the ring $\alpha \leq |\zeta| \leq 1$ by the following expression

$$\Psi(\zeta) = \frac{1}{\zeta^2}\left[\Phi(\zeta) + \bar{\Phi}\left(\frac{1}{\zeta}\right) - \zeta\Phi'(\zeta) - L(\zeta)\right], \tag{7.2.8}$$

where $L(\zeta)$ is given by the Laurent series

$$L(\zeta) = \bar{f}_0 + \sum_{k=1}^{\infty}\frac{\bar{f}_k}{\zeta^k} + \sum_{k=1}^{\infty}\bar{f}_{-k}\zeta^k. \tag{7.2.9}$$

Clearly, the boundary condition on Γ_0 conjugated to eq. (7.2.2) is satisfied.

Now we have

$$\sigma^2\Psi(\alpha\sigma) = \frac{1}{\alpha^2}\left[\Phi(\alpha\sigma) + \bar{\Phi}\left(\frac{1}{\alpha\sigma}\right) - \alpha\sigma\Phi'(\alpha\sigma) - L(\alpha\sigma)\right],$$

and the boundary condition (7.2.3) on Γ_1 is rewritten as follows

$$\begin{gathered}
\left(1-\alpha^2\right)\left[\Phi(\alpha\sigma) - \alpha\sigma\Phi'(\alpha\sigma)\right] - \alpha^2\bar{\Phi}\left(\frac{\alpha}{\sigma}\right) + \bar{\Phi}\left(\frac{1}{\alpha\sigma}\right) = \\
\alpha^2\bar{F}(\theta) + L(\alpha\sigma) = \alpha^2\bar{F}_0 + \bar{f}_0 + \\
\sum_{k=1}^{\infty}\left(\alpha^2\bar{F}_k + \frac{\bar{f}_k}{\alpha^k}\right)\sigma^{-k} + \sum_{k=1}^{\infty}\left(\alpha^2\bar{F}_{-k} + \alpha^k\bar{f}_{-k}\right)\sigma^k.
\end{gathered} \tag{7.2.10}$$

The conjugated boundary condition is given by

$$\begin{gathered}
\left(1-\alpha^2\right)\left[\bar{\Phi}\left(\frac{\alpha}{\sigma}\right) - \frac{\alpha}{\sigma}\bar{\Phi}'\left(\frac{\alpha}{\sigma}\right)\right] - \alpha^2\Phi(\alpha\sigma) + \Phi\left(\frac{\sigma}{\alpha}\right) = \\
= \alpha^2 F_0 + f_0 + \sum_{k=1}^{\infty}\left(\alpha^2 F_k + \frac{f_k}{\alpha^k}\right)\sigma^k + \sum_{k=1}^{\infty}\left(\alpha^2 F_{-k} + \alpha^k f_{-k}\right)\sigma^{-k}.
\end{gathered} \tag{7.2.11}$$

7.7.3 Determining functions $\Phi(\zeta), \Psi(\zeta)$

Let us present function $\Phi(\zeta)$ analytic in the ring $\alpha \leq |\zeta| \leq 1$ by the Laurent series

$$\Phi(\zeta) = C_0 + \sum_{k=1}^{\infty}C_k\zeta^k + \frac{C_{-1}}{\zeta} + \sum_{k=2}^{\infty}C_{-k}\zeta^{-k}. \tag{7.3.1}$$

Then by virtue of eq. (7.2.8) $\Psi(\zeta)$ is determined from the relationship

$$\Psi(\zeta) = \frac{1}{\zeta^2}\left[C_0 + \bar{C}_0 + \sum_{k=2}^{\infty}(1-k)\,C_k\zeta^k + \frac{2C_{-1}}{\zeta} + \sum_{k=2}^{\infty}(1+k)\,C_{-k}\zeta^{-k} + \right.$$
$$\left.\sum_{k=1}^{\infty}\frac{\bar{C}_k}{\zeta^k} + \bar{C}_{-1}\zeta + \sum_{k=2}^{\infty}\bar{C}_{-k}\zeta^k - L(\zeta)\right]. \tag{7.3.2}$$

It is important to mention that the coefficients of ζ^{-1} in the expressions for $\Phi(\zeta)$ and $\Psi(\zeta)$ are respectively equal to C_{-1} and $\left(\bar{C}_{-1} - \bar{f}_{-1}\right)$. Hence the expressions for $\varphi(\zeta)$ and $\psi(\zeta)$ contain the multivalued terms

$$\varphi(\zeta) = C_{-1}\ln\zeta + \ldots, \quad \psi(\zeta) = \left(\bar{C}_{-1} - \bar{f}_{-1}\right)\ln\zeta + \ldots,$$

and the condition for the single-valuedness of the displacement vector (1.14.5) allows us to determine C_{-1}

$$(3-4\nu)\,C_{-1} + (C_{-1} - f_{-1}) = 0, \quad C_{-1} = \frac{f_{-1}}{4(1-\nu)}. \tag{7.3.3}$$

Proceeding now to the boundary conditions (7.2.10), (7.2.11) and substituting expression (7.3.11) for $\Phi(\zeta)$ we arrive at the following system of equations

$$\left(1-\alpha^2\right)\left(C_0 + \bar{C}_0\right) = \alpha^2\bar{F}_0 + \bar{f}_0, \tag{7.3.4}$$

$$\left.\begin{aligned}
&\left(1-\alpha^2\right)(1-k)\,\alpha^k C_k + \left(\alpha^k - \alpha^{-k+2}\right)\bar{C}_{-k} = \alpha^2\bar{F}_{-k} + \alpha^2\bar{f}_{-k},\\
&\left(\alpha^{-k} - \alpha^{k+2}\right)C_k + \left(1-\alpha^2\right)(1+k)\,\alpha^{-k}\bar{C}_{-k} = \alpha^2 F_k + \frac{f_k}{\alpha^k}
\end{aligned}\right\} \tag{7.3.5}$$

and the complex conjugated system.

According to eq. (7.2.7), both sides of eq. (7.3.4) are real-valued. Coefficient C_0 is determined up to the imaginary part, thus, setting the imaginary part to zero, we obtain

$$C_0 = \frac{\alpha^2\bar{F}_0 + \bar{f}_0}{2(1-\alpha^2)} = \frac{\alpha^2 F_0 + f_0}{2(1-\alpha^2)}. \tag{7.3.6}$$

For $k=1$ the first equation in (7.3.5) holds identically whereas the second one yields

$$\left(1-\alpha^4\right)C_1 + 2\left(1-\alpha^2\right)\bar{C}_{-1} = \alpha^3 F_1 + f_1.$$

The case $k=-1$ leads to the latter equation for the complex conjugated values.

Equation (7.3.3) yields

$$C_1 = \frac{\alpha^3 F_1 + f_1}{1-\alpha^4} - \frac{\bar{f}_{-1}}{2(1-\nu)(1+\alpha^2)}. \tag{7.3.7}$$

For $k = \pm 2, \pm 3, \ldots$ the system of equations (7.3.5) has a single solution since its determinant

$$\Delta = \alpha^2 \left[\left(\alpha^{-k} + \alpha^k \right)^2 - k^2 \left(\alpha^{-1} - \alpha \right)^2 \right] = 4e^{-2t} \left[\sinh^2 kt - k^2 \sinh^2 t \right],$$

where $\alpha = e^{-t}$, does not vanish for real-valued t.

The sufficient condition for convergence of the series (the solution (7.3.1)) is that the coefficients of the trigonometric series (7.2.4) decrease as $n^{-(2+\mu)}$ $(0 < \mu < 1)$ for $n \to \infty$. To this aim, the first derivatives of functions $f(\theta), F(\theta)$ are required to satisfy Hölder's conditions with exponent μ.

7.7.4 Tube under uniform internal and external pressure (Lamé's problem)

In this simple case, the only non-vanishing coefficients are f_0 and F_0

$$(\sigma_r)_{r=r_0} = -p_0 = f_0, \quad (\sigma_r)_{r=r_1} = -p_1 = -F_0,$$

and by eqs. (7.3.6), (7.3.2) and (7.2.8)

$$C_0 = \frac{\alpha^2 p_1 - p_0}{2(1-\alpha^2)} = \Phi(\zeta), \quad \Psi(\zeta) = \frac{\alpha^2 (p_1 - p_0)}{1 - \alpha^2} \frac{1}{\zeta^2}.$$

Kolosov-Muskhelishvili's formulae yield

$$\begin{aligned} \sigma_r &= \frac{\alpha^2 p_1 - p_0}{1-\alpha^2} - \frac{\alpha^2}{\rho^2} \frac{p_1 - p_0}{1-\alpha^2}, \quad \tau_{r\theta} = 0, \\ \sigma_\theta &= \frac{\alpha^2 p_1 - p_0}{1-\alpha^2} + \frac{\alpha^2}{\rho^2} \frac{p_1 - p_0}{1-\alpha^2} \quad \left(\rho = |\zeta| = \frac{r}{r_0} \right). \end{aligned}$$

7.7.5 Thermal stresses in the ring

The temperature field in the ring is assumed to be stationary, then in the expression for the temperature it is sufficient to keep only the logarithmic term and the terms with ζ^{-1}. The external and internal radii of the ring is taken to be equal to 1 and α, respectively. The boundary condition (5.9.20) serves for determining functions $\varphi_{0_*}(\zeta), \psi_*(\zeta)$ holomorphic in the ring

$$\begin{aligned} &\varphi_{0_*}(\zeta) + \zeta \bar{\varphi}'_{0_*}(\bar{\zeta}) + \bar{\psi}_*(\bar{\zeta}) = \qquad (7.5.1) \\ &= \frac{1}{2} A \left[2 \left(\theta'_0 \zeta + \theta'_1 + i g'_1 \right) \ln \rho + \theta'_0 \zeta + \left(\theta'_1 - i g'_1 \right) \frac{\zeta}{\bar{\zeta}} \right] \quad \left(A = \mu \alpha \frac{1+\nu}{1-\nu} \right). \end{aligned}$$

Using the latter equation and eq. (5.9.919) we find functions $\varphi_0(\zeta), \psi(\zeta)$ which completes the solution of the problem.

The boundary conditions for functions Φ, Ψ can be obtained by differentiating relation (7.5.1) with respect to the arc on any circle $\zeta = \rho e^{i\theta}, \rho =$ const. Along this circle

$$d\zeta = \rho i e^{i\theta} d\theta = i\frac{\zeta}{\rho} ds, \quad \frac{d}{ds} = \frac{d\zeta}{ds}\frac{d}{d\zeta} = i\frac{\zeta}{\rho}\frac{d}{d\zeta} = -i\frac{\bar{\zeta}}{\rho}\frac{d}{d\bar{\zeta}}. \tag{7.5.2}$$

Assuming

$$\Phi_*(\zeta) = \varphi'_{0_*}(\zeta), \quad \Psi_*(\zeta) = \psi_*(\zeta),$$

differentiating the functions in eq. (7.5.1) (for $\rho =$ const) according to the rules (7.5.2) and calculating the complex conjugates we arrive at the equality

$$\Phi_*(\zeta) + \bar{\Phi}_*(\zeta) - \zeta\Phi'_*(\zeta) - \frac{\zeta}{\bar{\zeta}}\Psi_*(\zeta) = A\left[\theta'_0\left(\ln\rho + \frac{1}{2}\right) + \frac{\theta'_1 + ig'_1}{\zeta}\right]. \tag{7.5.3}$$

Here, by eqs. (7.2.1) and (7.2.4), the non-vanishing coefficients are

$$\bar{f}_0 = f_0 = \frac{1}{2}A\theta'_0, \qquad f_1 = A\left(\theta'_1 - ig'_1\right),$$
$$F_0 = \bar{F}_0 = -A\theta'_0\left(\frac{1}{2} + \ln\alpha\right), \quad F_1 = -A\alpha^{-1}\left(\theta'_1 - ig'_1\right),$$

and, by virtue of eqs. (7.3.36) and (7.3.37), we have

$$C_0 = \frac{1}{2}A\theta'_0\left(\frac{1}{2} - \frac{\alpha^2\ln\alpha}{1-\alpha^2}\right), \quad C_1 = \frac{A}{1+\alpha^2}\left(\theta'_1 - ig'_1\right),$$

and functions $\Phi_*(\zeta), \Psi_*(\zeta)$ are as follows

$$\Phi_*(\zeta) = A\left[\frac{1}{2}\theta'_0\left(\frac{1}{2} - \frac{\alpha^2\ln\alpha}{1-\alpha^2}\right) + \frac{\theta'_1 - ig'_1}{1+\alpha^2}\zeta\right],$$
$$\Psi_*(\zeta) = -\frac{A}{\zeta^2}\left[\theta'_0\frac{\alpha^2\ln\alpha}{1-\alpha^2} + \frac{\alpha^2}{1+\alpha^2}\frac{\theta'_1 + ig'_1}{\zeta}\right].$$

Referring now to relations (5.9.19) we obtain

$$\left.\begin{aligned} \Phi(\zeta) &= A\left[-\theta'_0\left(\frac{1}{4} + \frac{1}{2}\frac{\alpha^2\ln\alpha}{1-\alpha^2} + \frac{1}{2}\ln\zeta\right) + \right. \\ &\qquad\qquad \left. \frac{\theta'_1 - ig'_1}{1+\alpha^2}\zeta - \frac{1}{2}\frac{\theta'_1 + ig'_1}{\zeta}\right], \\ \Psi(\zeta) &= -\frac{A}{\zeta^2}\left[\frac{\theta'_0}{1-\alpha^2}\alpha^2\ln\alpha + \frac{\alpha^2}{1+\alpha^2}\frac{\theta'_1 + ig'_1}{\zeta} + \frac{1}{2}\left(\theta'_1 - ig'_1\right)\zeta\right], \end{aligned}\right\} \tag{7.5.4}$$

and the stresses obtained by Kolosov-Muskhelishvili's formulae are given by

$$\sigma_r+\sigma_\theta=-2A\theta_0'\left(\frac{1}{2}+\frac{\alpha^2\ln\alpha}{1-\alpha^2}+\ln\rho\right)+$$
$$A\left(\theta_1'-ig_1'\right)\left(\frac{2\zeta}{1+\alpha^2}-\frac{1}{\bar\zeta}\right)+A\left(\theta_1'+ig_1'\right)\left(\frac{2\bar\zeta}{1+\alpha^2}-\frac{1}{\zeta}\right), \quad (7.5.5)$$

$$\sigma_r-i\tau_{r\theta}=-A\theta_0'\left[\frac{\alpha^2\ln\alpha}{1-\alpha^2}\left(1-\frac{1}{\rho^2}\right)+\ln\rho\right]+$$
$$\frac{A\left(\theta_1'+ig_1'\right)}{\zeta}\left(\frac{\rho^2}{1+\alpha^2}+\frac{\alpha^2}{\alpha^2+1}\frac{1}{\rho^2}-1\right). \quad (7.5.6)$$

These formulae hold for any stationary distribution of the temperature in the hollow circular cylinder, the temperature field is needed for determining the displacement and stress σ_z.

Expression (7.5.1) for coefficient A assumes the case of plane strain. In the plane stress $A=\mu\alpha\left(1+\nu\right)$, where α denotes the coefficient of thermal expansion.

7.7.6 Tension of the ring by concentrated forces

Oppositely directed tensile forces of the same magnitude R are applied at the ends $(\rho=\pm1, r=\pm r_0)$ of the horizontal diameter of the external circle of the ring $(\alpha\le\rho\le1)$ whilst its internal boundary $(\rho=\alpha)$ is free.

The problem of loading the solid disc was considered in Subsection 7.6.1. Functions $\Phi_0\left(\zeta\right), \Psi_0\left(\zeta\right)$ determining the solution are obtained by eq. (6.1.10)

$$\Phi_0\left(\zeta\right)=q\frac{1+\zeta^2}{1-\zeta^2}, \quad \Psi_0\left(\zeta\right)=-\frac{4q}{\left(1-\zeta^2\right)^2} \quad \left(q=\frac{R}{2\pi r_0}\right). \quad (7.6.1)$$

The change in the sign is caused by those forces R that are assumed as being tensile. In accordance with eq. (1.14.9) the displacement vector on any circle in the disc is equal to

$$\frac{1}{q}\left(\sigma_r^0-\iota\tau_{r\theta}^0\right)=2\left[\frac{1}{1-\zeta^2}-\frac{\zeta^4-2\frac{\zeta}{\bar\zeta}+\zeta^2}{\left(1-\zeta^2\right)^2}\right], \quad (7.6.2)$$

and one can easily check that this expression turns to zero on the circle $|\zeta|=1$ except for the points $\zeta=\pm1$ where the concentrated forces are applied. On the circle $|\zeta| = \alpha\sigma$ we have

$$\left(\sigma_r^0 - i\tau_{r\theta}^0\right)_{\zeta=\alpha\sigma} = 2q\left[\frac{1}{1-\left(\frac{\alpha}{\sigma}\right)^2} - \sigma^2\frac{\sigma^2\alpha^4 - 2 + \alpha^2}{\left(1-\alpha^2\sigma^2\right)^2}\right] \tag{7.6.3}$$

$$= 2q\left[\sum_{k=0}^{\infty}\left(\frac{\alpha}{\sigma}\right)^{2k} - \frac{1}{\alpha^2}\left(\sigma^2\alpha^4 - 2 + \alpha^2\right)\sum_{k=1}^{\infty} k\left(\alpha\sigma\right)^{2k}\right].$$

The solution (7.6.1) for the disc should be corrected by terms $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$, that is,

$$\Phi(\zeta) = \Phi_0(\zeta) + \Phi_1(\zeta), \quad \Psi(\zeta) = \Psi_0(\zeta) + \Psi_1(\zeta). \tag{7.6.4}$$

The boundary conditions for the correcting functions must express the absence of the corresponding surface forces on the external surface of the ring

$$\zeta = \sigma: \quad \Phi_1(\sigma) + \bar{\Phi}_1\left(\frac{1}{\sigma}\right) - \sigma\Phi_1'(\sigma) - \sigma^2\Psi_1(\sigma) = 0, \tag{7.6.5}$$

and the absence of the surface forces corresponding to solution $\Phi_0(\zeta), \Psi_0(\zeta)$

$$\zeta = \alpha\sigma: \quad \Phi_1(\alpha\sigma) + \bar{\Phi}_1\left(\frac{\alpha}{\sigma}\right) - \alpha\sigma\Phi_1'(\alpha\sigma) - \sigma^2\Psi_1(\alpha\sigma) =$$

$$= \frac{1}{q}\left(\sigma_r - -i\tau_{r\theta}\right)_{\zeta=\alpha\sigma} = -\bar{F}(\theta), \tag{7.6.6}$$

where $\bar{F}(\theta)$ is given by eq. (7.6.3) and by virtue of eq. (7.2.5)

$$\bar{F}_0 = 2q, \quad \bar{F}_{2k} = 2q\alpha^{2k}, \quad \bar{F}_{-2k} = 2q\left[\frac{2-\alpha^2}{\alpha^2}k - (k-1)\right]\alpha^{2k}. \tag{7.6.7}$$

Due to eqs. (7.2.8) and (7.6.5), function $\Psi_1(\zeta)$ is given by

$$\Psi_1(\zeta) = \frac{1}{\zeta^2}\left[\Phi_1(\zeta) + \bar{\Phi}_1\left(\frac{1}{\zeta}\right) - \zeta\Phi_1'(\zeta)\right], \tag{7.6.8}$$

and the coefficients of Laurent series (7.3.1) for $\Phi_1(\zeta)$ are obtained from the system of equations (7.3.4), (7.3.5).

7.7.7 The way of continuation

The plane ζ is split into three regions: ring L $(\alpha \le |\zeta| \le 1)$ containing the elastic medium, region R_1 $(1 \le |\zeta| \le \infty)$ which is the plane outside the unit circle γ_0 and region R_2 $(0 \le |\zeta| \le \alpha)$ in the circle γ_1. The transformation by means of the inverse radii about γ_0 maps the point $\zeta \subset L$ to point $1/\zeta$

of region R_1 (the ring of the region $1 \le |\zeta| \le 1/\alpha$). The transformation by means of the inverse radii about γ_1 maps the point ζ to point $\alpha^2/\bar{\zeta}$ of region R_2 (the ring of the region $\alpha^2 \le |\zeta| \le \alpha$).

The continuation of $\Phi(\zeta)$ to regions R_1 and R_2 defined by (6.13.3) are given by

$$\left.\begin{aligned} \zeta \subset R_1: \quad \Phi(\zeta) &= -\bar{\Phi}\left(\frac{1}{\zeta}\right) + \frac{1}{\zeta}\bar{\Phi}'\left(\frac{1}{\zeta}\right) + \frac{1}{\zeta^2}\bar{\Psi}\left(\frac{1}{\zeta}\right), \\ \zeta \subset R_2: \quad \Phi(\zeta) &= -\bar{\Phi}\left(\frac{\alpha^2}{\zeta}\right) + \frac{\alpha^2}{\zeta}\bar{\Phi}\left(\frac{\alpha^2}{\zeta}\right) + \frac{\alpha^2}{\zeta^2}\bar{\Psi}\left(\frac{\alpha^2}{\zeta}\right). \end{aligned}\right\} \quad (7.7.1)$$

According to eq. (6.13.9), function $\Psi(z)$ in L is determined in two ways

$$\begin{aligned} \Psi(\zeta) &= \frac{1}{\zeta^2}\left[\Phi(\zeta) + \bar{\Phi}\left(\frac{1}{\zeta}\right) - \zeta\Phi'(\zeta)\right] \\ &= \frac{\alpha^2}{\zeta^2}\left[\Phi(\zeta) + \bar{\Phi}\left(\frac{\alpha^2}{\zeta}\right) - \zeta\Phi'(\zeta)\right] \quad (\zeta \subset L) \end{aligned} \quad (7.7.2)$$

These relationships lead to the "identity relation"

$$\zeta \subset L: \quad \left(1-\alpha^2\right)\left[\Phi(\zeta) - \zeta\Phi'(\zeta)\right] + \bar{\Phi}\left(\frac{1}{\zeta}\right) - \alpha^2\bar{\Phi}\left(\frac{\alpha^2}{\zeta}\right) = 0. \quad (7.7.3)$$

By virtue of eqs. (6.13.6) the boundary condition on γ_0 and γ_1 are set in the form

$$\left.\begin{aligned} \zeta = \sigma = e^{i\theta} \subset \gamma_0: &\quad \Phi^L(\sigma) - \Phi^{R_1}(\sigma) = \left(\sigma_r + i\tau_{r\theta}\right)_{\gamma_0} = f_0(\sigma), \\ \zeta = \alpha\sigma = \alpha e^{i\theta} \subset \gamma_1: &\quad \Phi^L(\sigma) - \Phi^{R_2}(\sigma) = \left(\sigma_r + i\tau_{r\theta}\right)_{\gamma_1} = f_1(\sigma), \end{aligned}\right\} \quad (7.7.4)$$

and by eq. (6.13.7) function $\Phi(\zeta)$ is determined by the equality

$$\Phi(\zeta) = \frac{1}{2\pi i}\oint_{\gamma_0}\frac{f_0(\sigma)}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\oint_{\gamma_1}\frac{f_1(\sigma)}{\sigma-\zeta}d\sigma + g(\zeta) \quad (7.7.5)$$

in the whole plane. Here function $g(\zeta)$ is holomorphic in the ring and can be represented by Laurent's series. The first integral in formula (7.7.5) is continuous on γ_1, while the second one is continuous on γ_0. For this reason, function $\Phi(\zeta)$ has discontinuities caused by the boundary conditions (7.7.4) on both γ_0 and γ_1.

Let us consider the trigonometric series for $f_0(\sigma)$ and $f_1(\sigma)$ and separate the terms $f^+(\sigma)$ with the positive powers of σ (the free term σ^0 included) from the terms $f^-(\sigma)$ with the negative degrees of σ

$$\left.\begin{aligned} f_0^+(\sigma) &= \alpha_0 + \sum_{k=1}^{\infty}\alpha_k\sigma^k, \quad f_0^-(\sigma) = \sum_{k=1}^{\infty}\alpha_{-k}\sigma^{-k}, \\ f_1^+(\sigma) &= \beta_0 + \sum_{k=1}^{\infty}\beta_k\sigma^k, \quad f_1^-(\sigma) = \sum_{k=1}^{\infty}\beta_{-k}\sigma^{-k}, \end{aligned}\right\} \quad (7.7.6)$$

so that

$$f_0(\sigma) = f_0^+(\sigma) + f_0^-(\sigma), \quad f_1(\sigma) = f_1^+(\sigma) + f_1^-(\sigma). \tag{7.7.7}$$

Let us notice that $f_0^+(\sigma)$ and $f_1^+(\sigma)$ are the values of functions $f_0^+(\zeta)$ and $f_1^+(\zeta/a)$ (holomorphic in γ_0, γ_1) on γ_0 and γ_1 respectively. In just the same way, $f_0^-(\sigma)$ and $f_1^-(\sigma)$ are the contour values of functions $f_0^-(\zeta)$ and $f_1^-(\zeta/a)$ holomorphic outside of γ_0 and γ_1.

Bearing this in mind and evaluating Cauchy's integrals, see Subsection 7.5.10, we obtain the following representations for function $\Phi(\zeta)$

$$\left.\begin{aligned} \zeta \subset R_1: &\quad \Phi(\zeta) = -f_0^-(\zeta) + f_1^-\left(\frac{\zeta}{\alpha}\right) + g(\zeta), \\ \zeta \subset L: &\quad \Phi(\zeta) = f_0^+(\zeta) + f_1^-\left(\frac{\zeta}{\alpha}\right) + g(\zeta), \\ \zeta \subset R_2: &\quad \Phi(\zeta) = f_0^+(\zeta) - f_1^+\left(\frac{\zeta}{\alpha}\right) + g(\zeta), \end{aligned}\right\} \tag{7.7.8}$$

which is in agreement with boundary conditions (7.7.4).

Let us proceed to the identity relation (7.7.3). Functions $\bar{\Phi}(1/\zeta), \bar{\Phi}(a^2/\zeta)$ are determined in terms of the expressions for $\Phi(\zeta)$ for $\zeta \subset R_1, \zeta \subset R_2$ provided that ζ is replaced by $1/\zeta$ and a^2/ζ respectively. Then we obtain

$$\zeta \subset L: \quad \left\{\begin{aligned} \bar{\Phi}\left(\frac{1}{\zeta}\right) &= -\bar{f}_0^-\left(\frac{1}{\zeta}\right) + \bar{f}_1^-\left(\frac{1}{\zeta}\right) + \bar{g}\left(\frac{1}{\zeta}\right), \\ \bar{\Phi}\left(\frac{\alpha^2}{\zeta}\right) &= \bar{f}_0^+\left(\frac{\alpha^2}{\zeta}\right) - \bar{f}_1^+\left(\frac{\alpha}{\zeta}\right) + \bar{g}\left(\frac{\alpha^2}{\zeta}\right), \end{aligned}\right. \tag{7.7.9}$$

and inserting into eq. (7.7.3) yields the equality

$$\begin{aligned} \left(1-\alpha^2\right)\left[\Phi(\zeta) - \zeta\Phi'(\zeta)\right] + \left[-\bar{f}_0^-\left(\frac{1}{\zeta}\right) + \bar{f}_1^-\left(\frac{1}{\zeta}\right) + \bar{g}\left(\frac{1}{\zeta}\right)\right] - \\ \alpha^2\left[\bar{f}_0^+\left(\frac{\alpha^2}{\zeta}\right) - \bar{f}_1^+\left(\frac{\alpha}{\zeta}\right) + \bar{g}\left(\frac{\alpha^2}{\zeta}\right)\right] = 0 \quad (\zeta \subset L). \end{aligned}$$

We can remove function g by using the expression for $\Phi(\zeta)$ for $\zeta \subset L$

$$\zeta \subset L: \quad \left\{\begin{aligned} \bar{g}\left(\frac{1}{\zeta}\right) &= \bar{\Phi}^\times\left(\frac{1}{\zeta}\right) - \bar{f}_0^+\left(\frac{1}{\zeta}\right) - \bar{f}_1^-\left(\frac{1}{\alpha\zeta}\right), \\ \bar{g}\left(\frac{\alpha^2}{\zeta}\right) &= \bar{\Phi}^\times\left(\frac{\alpha^2}{\zeta}\right) - \bar{f}_0^+\left(\frac{\alpha^2}{\zeta}\right) - \bar{f}_1^-\left(\frac{\alpha}{\zeta}\right), \end{aligned}\right. \tag{7.7.10}$$

where $\bar{\Phi}^\times(1/\zeta)$ and $\bar{\Phi}^\times(a^2/\zeta)$ are the results of the formal substitution of variable ζ by respectively $1/\zeta$ and a^2/ζ in $\bar{\Phi}(\zeta)$ for $\zeta \subset L$, see Subsection 7.6.13.

Using denotation (7.7.7) we arrive at the following functional equation for $\Phi(\zeta)$

$$\zeta \subset L: \quad \left(1-\alpha^2\right)\left[\Phi(\zeta) - \zeta\Phi'(\zeta)\right] + \\ \left[\bar{\Phi}^\times\left(\frac{1}{\zeta}\right) - \bar{f}_0\left(\frac{1}{\zeta}\right)\right] - \alpha^2\left[\bar{\Phi}^\times\left(\frac{\alpha^2}{\zeta}\right) - \bar{f}_1\left(\frac{\alpha}{\zeta}\right)\right] = 0. \quad (7.7.11)$$

From this equation we can obtain the equations determining the coefficients of Laurent's series of this function

$$\Phi(\zeta) = C_0 + \sum_{n=1}^{\infty}\left(C_n\zeta^n + C_{-n}\zeta^{-n}\right). \quad (7.7.12)$$

Inserting it into eq. (7.7.11) we obtain

$$\left(1-\alpha^2\right)\left\{C_0 + \sum_{n=1}^{\infty}\left[(1-n)C_n\zeta^n + (1+n)C_{-n}\zeta^{-n}\right]\right\} + \left(1-\alpha^2\right)\bar{C}_0 + \\ \sum_{n=1}^{\infty}\left(\bar{C}_n\zeta^{-n} + C_{-n}\zeta^n\right) - \alpha^2\sum_{n=1}^{\infty}\left(\bar{C}_n\alpha^{2n}\zeta^{-n} + \bar{C}_{-n}\alpha^{-2n}\zeta^n\right) = \quad (7.7.13) \\ = \bar{\alpha}_0 + \sum_{n=1}^{\infty}\left(\bar{\alpha}_n\zeta^{-n} + \bar{\alpha}_{-n}\zeta^n\right) - \alpha^2\left[\bar{\beta}_0 + \sum_{n=1}^{\infty}\left(\bar{\beta}_n\alpha^n\zeta^{-n} + \bar{\beta}_{-n}\alpha^{-n}\zeta^n\right)\right].$$

Equating the coefficients of $\zeta^0, \zeta^1, \zeta^{-1}$ in both sides of the latter equation yields

$$\left.\begin{array}{l} \left(1-\alpha^2\right)\left(C_0+\bar{C}_0\right) = \bar{\alpha}_0 - \alpha^2\bar{\beta}_0, \quad \bar{\alpha}_{-1} - \alpha\bar{\beta}_{-1} = 0, \\ \left(1-\alpha^2\right)2C_{-1} + \bar{C}_1\left(1-\alpha^4\right) = \bar{\alpha}_1 - \alpha^3\bar{\beta}_1. \end{array}\right\} \quad (7.7.14)$$

Comparing the terms with ζ^n, ζ^{-n} for $n = 2, 3, \ldots$ leads to the equations

$$\left.\begin{array}{l} \left(1-\alpha^2\right)(1-n)C_n + \bar{C}_{-n}\left(1-\alpha^{-2n+2}\right) = \bar{\alpha}_{-n} - \alpha^{-n+2}\bar{\beta}_{-n}, \\ \left(1-\alpha^{2n+2}\right)C_n + \left(1-\alpha^2\right)(1+n)\bar{C}_{-n} = \alpha_{-n} - \alpha^{n+2}\beta_n, \end{array}\right\} \quad (7.7.15)$$

where the equations for the coefficients of ζ^{-n} are written down for the complex conjugated values. These equations coincide with those in (7.3.5). This system of equations enables all coefficients C_n, C_{-n} to be determined since its determinant

$$\Delta_1 = \alpha^2\left(\alpha^n - \alpha^{-n}\right)^2 - n^2\left(1-\alpha^2\right)^2 \quad (7.7.16)$$

is not equal to zero.

The static equations of zero principal moment and zero principal vector of the surface forces distributed over γ_0, γ_1 have the form

$$\left.\begin{array}{c}\int\limits_0^{2\pi}\left[\tau_{r\theta}\big|_{\gamma_0}-\alpha^2\tau_{r\theta}\big|_{\gamma_1}\right]d\theta=\frac{\pi}{i}\left[(\alpha_0-\bar{\alpha}_0)-\alpha^2\left(\beta_0-\bar{\beta}_0\right)\right]=0,\\ \operatorname{Im}\left(\alpha_0-\alpha^2\beta_0\right)=0,\\ \int\limits_0^{2\pi}\left[(\sigma_r+i\tau_{r\theta})\big|_{\gamma_0}-\alpha\left(\sigma_r+i\tau_{r\theta}\right)\big|_{\gamma_1}\right]e^{i\theta}d\theta=2\pi\left(\alpha_{-1}-\alpha\beta_{-1}\right)=0.\end{array}\right\} \tag{7.7.17}$$

They corresponds to the first and second equations (7.7.14) and are conditions for existence of the solution. It follows from eq. (7.7.15) that

$$\left.\begin{array}{l}C_n=\dfrac{1}{\Delta_1}\left[\left(1-\alpha^2\right)(1+n)\left(\bar{\alpha}_{-n}-\alpha^{-n+2}\bar{\beta}_{-n}\right)-\right.\\ \qquad\qquad\left.\left(1-\alpha^{-2n+2}\right)\left(\alpha_n-\alpha^{n+2}\beta_n\right)\right],\\ C_{-n}=\dfrac{1}{\Delta_1}\left[-\left(1-\alpha^{2n+2}\right)\left(\alpha_{-n}-\alpha^{-n+2}\beta_{-n}\right)+\right.\\ \qquad\qquad\left.\left(1-\alpha^2\right)(1-n)\left(\bar{\alpha}_n-\alpha^{n+2}\bar{\beta}_n\right)\right].\end{array}\right\} \tag{7.7.18}$$

The third equation relating the unknowns C_{-1}, C_1 must be completed, due to eq. (5.4.15), by the relationship

$$C_{-1}=-\frac{X+iY}{8\pi(1-\nu)}, \tag{7.7.19}$$

obtained from the requirement of uniqueness of the displacement. We obtain

$$C_1=\frac{X-iY}{4\pi(1-\nu)}\frac{1}{1+\alpha^2}+\frac{\alpha_1-\alpha^3\beta_1}{1-\alpha^4}. \tag{7.7.20}$$

Function $\Psi(\zeta)$ determined, for example, by the first equality in eq. (7.7.2) is constructed by replacing $\bar{\Phi}(1/\zeta)$ by

$$\zeta^2\Psi(\zeta)=\Phi(\zeta)-\zeta\Phi'(\zeta)-\bar{f}_0^-\left(\frac{1}{\zeta}\right)+\bar{f}_1^-\left(\frac{1}{\alpha\zeta}\right)+\bar{g}\left(\frac{1}{\zeta}\right)$$

(as eq. (7.7.9) suggests) with the further substitution (7.7.10) for $\bar{g}(1/\zeta)$. The result is

$$\zeta\subset L:\quad \zeta^2\Psi(\zeta)=\Phi(\zeta)-\zeta\Phi'(\zeta)+\zeta\Phi'(\zeta)+\bar{\Phi}^\times\left(\frac{1}{\zeta}\right)-\bar{f}_0\left(\frac{1}{\zeta}\right). \tag{7.7.21}$$

With the help of the second equality (7.7.2) we obtain

$$\zeta\subset L:\quad \zeta^2\Psi(\zeta)=\alpha^2\left[\Phi(\zeta)-\zeta\Phi'(\zeta)+\bar{\Phi}^\times\left(\frac{\alpha^2}{\zeta}\right)-\bar{f}_1\left(\frac{\alpha}{\zeta}\right)\right],$$

which coincides with eq. (7.7.21) by virtue of eq. (7.7.11). Under such a definition of $\Psi(\zeta)$ the boundary conditions (1.14.9) are satisfied

$$\zeta = \sigma \subset \gamma_0 : \quad \Phi(\sigma) - \sigma\Phi'(\sigma) + \bar{\Phi}^{\times}\left(\frac{1}{\sigma}\right) - \sigma^2\Psi(\sigma) = \bar{f}_0\left(\frac{1}{\sigma}\right) =$$

$$= \bar{\alpha}_0 + \sum_{n=1}^{\infty}\left(\bar{\alpha}_n\sigma^{-n} + \bar{\alpha}_{-n}\sigma^n\right) = (\sigma_r - i\tau_{r\theta})_{\gamma_0},$$

$$\zeta = \alpha\sigma \subset \gamma_1 : \quad \Phi(\alpha\sigma) - \alpha\sigma\Phi'(\sigma) + \bar{\Phi}^{\times}\left(\frac{\alpha^2}{\sigma}\right) - \sigma^2\Psi(\sigma) = \bar{f}_1\left(\frac{1}{\sigma}\right) =$$

$$= \bar{\beta}_0 + \sum_{n=1}^{\infty}\left(\bar{\beta}_n\sigma^{-n} + \bar{\beta}_{-n}\sigma^n\right) = (\sigma_r - i\tau_{r\theta})_{\gamma_1},$$

which is required.

As expected, the results of the method of continuation are identical with those which can be obtained by directly constructing functions Φ, Ψ by means of the boundary conditions. However, the present method reduced to the single functional equation (7.7.11) is more efficient for some particular loads.

7.8 Applying the conformal transformation

7.8.1 *Infinite plane with an opening*

Assuming that the stresses at infinity are bounded we can set the boundary conditions on the contour Γ of the opening in the form of eqs. (5.4.15) and (5.4.17)

$$\Phi(z) + \bar{\Phi}(\bar{z}) + \left[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})\right]\frac{d\bar{z}}{dz} =$$

$$= -(F_n + iF_t) + \frac{X + iY}{8\pi(1-\nu)}\left[\frac{1}{z} - (3 - 4\nu)\frac{1}{\bar{z}}\frac{d\bar{z}}{dz}\right] +$$

$$\frac{X - iY}{8\pi(1-\nu)}\left(\frac{1}{z} - \frac{z}{\bar{z}^2}\right)\frac{d\bar{z}}{dz} - \frac{1}{2}\left[\sigma_1^{\infty} + \sigma_2^{\infty} + (\sigma_2^{\infty} - \sigma_1^{\infty})e^{2i\alpha}\frac{d\bar{z}}{dz}\right]. \quad (8.1.1)$$

Here $-(F_n + iF_t) = (F_x + iF_y)\bar{n}$ denotes the vector of the surface forces on the surface with the normal n to Γ directed into the medium, $X + iY$ is the principal vector of the surface forces $F_x + iF_y$, σ_1^{∞} and σ_2^{∞} are the principal stresses at infinity, α is the angle of the first principal stress to axis Ox, and z and $\bar{z}$ are related by the equation for contour Γ. Functions $\Phi(z), \Psi(z)$ (denoted as Φ_{**}, Ψ_{**} in Subsection 7.5.4) are holomorphic in L (the plane outside the opening) and their expansions in the power series in z^{-1} begin with the term z^{-2}.

The conformal transformation of the region $|\zeta| > 1$ outside the unit circle of plane ζ into the considered region L is assumed to be given by the relationship (5.4.1)

$$z = \omega(\zeta) = c_0\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots = c_0\zeta + m(\zeta) \quad (c_0 \neq 0), \qquad (8.1.2)$$

where $\omega'(\zeta) \neq 0$ for $|\zeta| \geq 1$ and c_0 is real-valued. Function $m(\zeta)$ is holomorphic for $|\zeta| > 1$. In what follows we will show that the problem is solvable by a finite number of steps if $m(\zeta)$ is a polynomial of ζ^{-1}. It is assumed that this condition is satisfied and the degree of the polynomial is denoted by n.

Making use of denotation (5.2.13) and talking into account that on Γ

$$z = \omega(\sigma), \; \bar{z} = \bar{\omega}\left(\frac{1}{\sigma}\right), \; dz = \omega'(\sigma)\, d\sigma, \; d\bar{z} = -\bar{\omega}'\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma^2}, \qquad (8.1.3)$$

we can transform the boundary condition (8.1.1) to the form

$$\text{on } \gamma : \zeta = \sigma = e^{i\theta}; \quad \omega'(\sigma)\Phi_1(\sigma) + \omega'(\sigma)\bar{\Phi}_1\left(\frac{1}{\sigma}\right) - \frac{\omega(\sigma)}{\sigma^2}\bar{\Phi}_1'\left(\frac{1}{\sigma}\right) -$$
$$\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2}\bar{\Psi}_1\left(\frac{1}{\sigma}\right) = -f(\theta)\,\omega'(\sigma) + \frac{X+iY}{8\pi(1-\nu)}\left[\frac{\omega'(\sigma)}{\omega(\sigma)} + \right. \qquad (8.1.4)$$
$$\left.(3-4\nu)\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2\bar{\omega}\left(\frac{1}{\sigma}\right)}\right] + \frac{X-iY}{8\pi(1-\nu)}\left[\frac{\omega'(\sigma)}{\bar{\omega}\left(\frac{1}{\sigma}\right)} + \frac{1}{\sigma^2}\frac{\omega(\sigma)}{\bar{\omega}^2\left(\frac{1}{\sigma}\right)}\bar{\omega}'\left(\frac{1}{\sigma}\right)\right]$$
$$\frac{1}{2}\left[(\sigma_1^\infty + \sigma_2^\infty)\,\omega'(\sigma) - (\sigma_2^\infty - \sigma_1^\infty)\,e^{2i\alpha}\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2}\right] \quad (f(\theta) = F_n + iF_t).$$

This relation can be slightly simplified by taking into account that

$$\frac{1}{\omega(\zeta)} = \frac{1}{c_0\zeta} - \frac{m(\zeta)}{c_0\zeta\omega(\zeta)},$$

and introducing into consideration the functions

$$\begin{aligned} \Phi_*(\zeta) &= \Phi_1(\zeta) + \frac{X+iY}{8\pi(1-\nu)}\frac{m(\zeta)}{c_0\omega(\zeta)\zeta}, \\ \Psi_*(\zeta) &= \Psi_1(\zeta) - \frac{(3-4\nu)}{8\pi(1-\nu)}(X-iY)\frac{m(\zeta)}{c_0\omega(\zeta)\zeta}, \end{aligned} \qquad (8.1.5)$$

which are also holomorphic and tending to zero not slower than ζ^{-2} at infinity. The boundary condition (8.1.4) is written down in the form

$$\omega'(\sigma)\Phi_*(\sigma)+\omega'(\sigma)\bar{\Phi}_*\left(\frac{1}{\sigma}\right)-\frac{\omega(\sigma)}{\sigma^2}\bar{\Phi}'_*\left(\frac{1}{\sigma}\right)-\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2}\bar{\Psi}_*\left(\frac{1}{\sigma}\right)=$$
$$=-f(\theta)\omega'(\sigma)+\frac{X+iY}{8\pi(1-\nu)c_0\sigma}\left[\omega'(\sigma)+(3-4\nu)\bar{\omega}'\left(\frac{1}{\sigma}\right)\right]+$$
$$\frac{X-iY}{8\pi(1-\nu)}\left(\frac{\omega'(\sigma)\sigma}{c_0}+\frac{\omega(\sigma)}{c_0}\right)-$$
$$\frac{1}{2}\left[\left(\sigma_1^\infty+\sigma_2^\infty\right)\omega'(\sigma)-\left(\sigma_2^\infty-\sigma_1^\infty\right)e^{2i\alpha}\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2}\right]=-F(\theta), \quad (8.1.6)$$

and the complex conjugated condition is given by

$$\bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\Phi}_*\left(\frac{1}{\sigma}\right)+\bar{\omega}'\left(\frac{1}{\sigma}\right)\Phi_*(\sigma)-\sigma^2\bar{\omega}\left(\frac{1}{\sigma}\right)\Phi'_*(\sigma)-\sigma^2\omega'(\sigma)\Psi_*(\sigma)=$$
$$=-\bar{f}(\theta)\bar{\omega}'\left(\frac{1}{\sigma}\right)+\frac{X-iY}{8\pi(1-\nu)}\frac{\sigma}{c_0}\left[\bar{\omega}'\left(\frac{1}{\sigma}\right)+(3-4\nu)\omega'(\sigma)\right]+$$
$$\frac{X+iY}{8\pi(1-\nu)}\left[\frac{1}{\sigma c_0}\bar{\omega}'\left(\frac{1}{\sigma}\right)+\frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{c_0}\right]-$$
$$\frac{1}{2}\left[\left(\sigma_1^\infty+\sigma_2^\infty\right)\bar{\omega}'\left(\frac{1}{\sigma}\right)-\left(\sigma_2^\infty-\sigma_1^\infty\right)e^{-2i\alpha}\sigma^2\omega'(\sigma)\right]=-\bar{F}(\theta). \quad (8.1.7)$$

7.8.2 The method of Cauchy's integrals

Recalling the character of the expansion of the sought functions

$$\begin{aligned}\Phi_*(\zeta)&=\frac{a^2}{\zeta^2}+\frac{a^3}{\zeta^3}+\ldots+\frac{a^n}{\zeta^n}+\ldots,\\ \Psi_*(\zeta)&=\frac{a_2'}{\zeta^2}+\frac{a_3'}{\zeta^3}+\ldots+\frac{a_n'}{\zeta^n}+\ldots,\end{aligned} \quad (8.2.1)$$

it is easy to obtain the behaviour of the functions

$$\omega'(\zeta)\bar{\Phi}_*\left(\frac{1}{\zeta}\right)-\frac{\omega(\zeta)}{\zeta^2}\bar{\Phi}'_*\left(\frac{1}{\zeta}\right)=\frac{d}{d\zeta}\omega(\zeta)\bar{\Phi}_*\left(\frac{1}{\zeta}\right); \quad \frac{1}{\zeta^2}\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Psi}_*\left(\frac{1}{\zeta}\right)$$

appearing in the boundary condition (8.1.6)) in the region $|\zeta| > 1$. We have

$$
\begin{aligned}
&\omega(\zeta)\,\bar{\Phi}_*\left(\frac{1}{\zeta}\right) = \qquad (8.2.2)\\
&= \left(c_0\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots + \frac{c_n}{\zeta^n}\right)\left(\bar{a}_2\zeta^2 + \bar{a}_3\zeta^3 + \ldots + \bar{a}_n\zeta^n + \ldots\right)\\
&= c_3\bar{a}_2\frac{1}{\zeta} + c_4\left(\frac{\bar{a}_2}{\zeta^2} + \frac{\bar{a}_3}{\zeta}\right) + \ldots + c_n\left(\frac{\bar{a}_2}{\zeta^{n-2}} + \frac{\bar{a}_3}{\zeta^{n-3}} + \ldots + \frac{\bar{a}_{n-1}}{\zeta}\right) + Z(\zeta)
\end{aligned}
$$

where $Z(\zeta)$ is holomorphic in the circle $|\zeta| < 1$ and the expansion of this function in the series contains only non-negative degrees of ζ. Hence,

$$
\frac{d}{d\zeta}\omega(\zeta)\,\bar{\Phi}_*\left(\frac{1}{\zeta}\right) = p_{n-1}\left(\frac{1}{\zeta}\right) + Z'(\zeta), \qquad (8.2.3)
$$

where $p_{n-1}(1/\zeta)$ is a polynomial of $(n-1)$-th degree in ζ^{-1} with the unknown coefficients $a_2, \ldots, a_{n-1}$

$$
\begin{aligned}
p_{n-1}\left(\frac{1}{\zeta}\right) = -\Bigg\{&c_3\bar{a}_2\frac{1}{\zeta^2} + c_4\left(\frac{2\bar{a}_2}{\zeta^3} + \frac{\bar{a}_3}{\zeta^2}\right) + \ldots\\
&\ldots + c_n\left[(n-2)\frac{\bar{a}_2}{\zeta^{n-1}} + \ldots + \frac{\bar{a}_{n-1}}{\zeta^2}\right]\Bigg\}. \qquad (8.2.4)
\end{aligned}
$$

The expansion of the functions $\frac{1}{\zeta^2}\bar{\omega}'\bar{\Psi}_*\left(\frac{1}{\zeta}\right)$ and $\bar{\omega}'(\zeta)\,\Phi_*(\zeta)$ in the power series contains respectively only non-negative and negative terms.

The function corresponding to the terms of the left hand side of the conjugated boundary condition is presented in the form

$$
\begin{aligned}
\frac{d}{d\left(\frac{1}{\zeta}\right)}\bar{\omega}\left(\frac{1}{\zeta}\right)\bar{\Phi}_*\left(\frac{1}{\zeta}\right) &= \bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Phi}_*(\zeta) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\bar{\Phi}'_*(\zeta)\\
&= \bar{p}_{n-1}(\zeta) + \bar{Z}'\left(\frac{1}{\zeta}\right), \qquad (8.2.5)
\end{aligned}
$$

whereas the functions

$$
\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Phi}_*\left(\frac{1}{\zeta}\right), \quad \zeta^2\omega'(\zeta)\,\Psi_*(\zeta)
$$

are respectively presented by the series with positive degrees and the series with negative degrees with the principal part $c_0a'_2$ at infinity.

Referring now to the rules of evaluating the integrals of Cauchy's type, Subsection 7.5.10, we have

$$
|\zeta|>1: \quad \left.\begin{aligned}
&\frac{1}{2\pi i}\oint_\gamma \omega'(\sigma)\,\Phi_*(\sigma)\,\frac{d\sigma}{\sigma-\zeta} = \omega'(\zeta)\,\Phi_*(\zeta),\\
&\frac{1}{2\pi i}\oint_\gamma \left[p_{n-1}\left(\frac{1}{\sigma}\right) + Z'(\sigma)\right]\frac{d\sigma}{\sigma-\zeta} = p_{n-1}\left(\frac{1}{\zeta}\right),\\
&\frac{1}{2\pi i}\oint_\gamma \bar\omega'\left(\frac{1}{\sigma}\right)\frac{\bar\Psi_*\left(\frac{1}{\sigma}\right)}{\sigma^2}\frac{d\sigma}{\sigma-\zeta} = 0,\\
&\frac{1}{2\pi i}\oint_\gamma \frac{\bar\omega'\left(\frac{1}{\sigma}\right)\bar\Phi_*\left(\frac{1}{\sigma}\right)}{\sigma-\zeta}d\sigma = 0,\\
&\frac{1}{2\pi i}\oint_\gamma \frac{\sigma^2\omega'(\sigma)\,\bar\Psi_*(\sigma)}{\sigma-\zeta}d\sigma = \zeta^2\omega'(\zeta)\,\bar\Psi_*(\zeta) - c_0 a_2',\\
&\frac{1}{2\pi i}\oint_\gamma \left[\bar p_{n-1}(\sigma) + \bar Z'\left(\frac{1}{\sigma}\right)\right]\frac{d\sigma}{\sigma-\zeta} = \bar Z'\left(\frac{1}{\zeta}\right) - \bar Z_0' =\\
&\quad = \bar\omega'\left(\frac{1}{\zeta}\right)\Phi_*(\zeta) - \zeta^2\bar\omega\left(\frac{1}{\zeta}\right)\Phi_*'(\zeta) - \bar p_{n-1}(\zeta) - \bar Z_0',
\end{aligned}\right\} \tag{8.2.6}
$$

where $\bar Z_0'$ is the principal part of function $\bar Z'(1/\zeta)$ as $\zeta \to \infty$ and can be expressed easily by means of eq. (8.2.2)

$$
Z_0' = c_1\bar a_2 + c_2\bar a_3 + \ldots + c_n\bar a_{n+1}, \quad \bar Z_0' = \bar c_1 a_2 + \bar c_2 a_3 + \ldots + \bar c_n a_{n+1}. \tag{8.2.7}
$$

Collecting these results we arrive at the relationships

$$
\left.\begin{aligned}
&\omega'(\zeta)\,\Phi_*(\zeta) + p_{n-1}\left(\frac{1}{\zeta}\right) = -\frac{1}{2\pi i}\oint_\gamma \frac{F(\theta)}{\sigma-\zeta}d\sigma,\\
&\bar\omega'\left(\frac{1}{\zeta}\right)\Phi_*(\zeta) - \zeta^2\bar\omega\left(\frac{1}{\zeta}\right)\Phi_*'(\zeta) - \bar p_{n-1}(\zeta) - \bar Z_0' -\\
&\qquad \zeta^2\omega'(\zeta)\,\Psi_*(\zeta) + c_0 a_2' = -\frac{1}{2\pi i}\oint_\gamma \frac{\bar F(\theta)}{\sigma-\zeta}d\sigma.
\end{aligned}\right\} \tag{8.2.8}
$$

Their right hand sides are calculated by the rules of Subsection 7.5.10. The system of equations determining the unknown coefficients $a_2, \bar a_2, \ldots,$ a_{n+1}, a_2' can be obtained by comparing the coefficients in the expansion in terms of degrees of $1/\zeta$ on the right and left hand sides of equalities (8.2.8). This will be explained through the following examples.

7.8.3 Elliptic opening

The conformal transformation of the region $|\zeta| > 1$ into the plane with an elliptic opening is given by the function

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) \quad (R > 0, \quad 0 \le m \le 1), \tag{8.3.1}$$

where the circle $|\zeta| = 1$ is mapped into the ellipse with semi- axes

$$a = R(1+m), \quad b = R(1-m),$$

the latter being a circle when $m = 0$. The value $m = 1$ corresponds to the transformation of region $|\zeta| > 1$ into the plane cut along the line $(-2R, 2R)$.

The derivative of the transforming function

$$\omega'(\zeta) = R\left(1 - \frac{m}{\zeta^2}\right) \tag{8.3.2}$$

does not vanish for $|\zeta| \ge 1$ if $m < 1$. For $m = 1$ it is zero at $\zeta = \pm 1$ which corresponds to the ends $-2R, 2R$ of the cut, i.e. the "corner points" on the contour of the region.

It is assumed in the following that the constant normal pressure p is applied to the edge of the opening and the stresses at infinity are given. Then $X + iY = 0$ and by eq. (8.1.6)

$$-F(\theta) = -\left[p + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right)\right]\omega'(\sigma) + \frac{1}{2}\left(\sigma_2^\infty - \sigma_1^\infty\right)\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma^2}e^{2i\alpha}, \tag{8.3.3}$$

so that

$$\left.\begin{aligned} -\frac{1}{2\pi i}\oint_\gamma \frac{F(\theta)\,d\sigma}{\sigma - \zeta} = \frac{Rm}{\zeta^2}\left[p + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right)\right] + \\ \frac{R}{2\zeta^2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{2i\alpha}, \qquad \oint_\gamma \frac{\bar{F}(\theta)}{\sigma - \zeta}d\sigma = 0. \end{aligned}\right\} \tag{8.3.4}$$

Repeating the calculation given by formula (8.2.2) we have

$$p_{n-1} = 0, \quad \bar{Z}_0' = Rma_2$$

and by eq. (8.2.8) we obtain

$$\begin{gathered}\left(1 - \frac{m}{\zeta^2}\right)\Phi_*(\zeta) = \frac{m}{\zeta^2}\left[p + \frac{1}{2}\left(\sigma_1^\infty + \sigma_2^\infty\right)\right] + \frac{1}{2\zeta^2}\left(\sigma_2^\infty - \sigma_1^\infty\right)e^{2i\alpha}, \\ \left(1 - m\zeta^2\right)\Phi_*(\zeta) - \left(\zeta + m\zeta^3\right)\Phi_*'(\zeta) - ma_2 - \left(\zeta^2 - m\right)\Psi_*(\zeta) + a_2' = 0. \end{gathered} \tag{8.3.5}$$

The second boundary condition (8.1.7) for the elliptic opening has the form

$$\left(1-m\sigma^2\right)\bar{\Phi}_*\left(\frac{1}{\sigma}\right)+\left(1-m\sigma^2\right)\Phi_*(\sigma)-$$
$$\sigma^2\left(\frac{1}{\sigma}+m\sigma\right)\Phi_*'(\sigma)-\left(\sigma^2-m\right)\Psi_*(\sigma)=$$
$$=-\left[p+\frac{1}{2}\left(\sigma_1^\infty+\sigma_2^\infty\right)\right]\left(1-m\sigma^2\right)+\frac{1}{2}\left(\sigma_2^\infty-\sigma_1^\infty\right)\left(\sigma^2-m\right)e^{-2i\alpha},$$

Comparing the terms which are independent of σ yields

$$ma_2-a_2'=-\left[p+\frac{1}{2}\left(\sigma_1^\infty+\sigma_2^\infty\right)\right]-\frac{1}{2}m\left(\sigma_2^\infty-\sigma_1^\infty\right)e^{-2i\alpha}. \tag{8.3.6}$$

Functions $\Phi_*(\sigma), \Psi_*(\sigma)$ are now determined. The formulae below are constructed for two special cases:

1. The stresses at infinity are absent and the constant normal pressure acts on the edge of the opening

$$\Phi_*(\zeta)=\frac{mp}{\zeta^2-m},\ \Psi_*(\zeta)=\frac{p}{\zeta^2-m}+\frac{mp}{\left(\zeta^2-m\right)^3}\left[m\zeta^4+\left(3+m^2\right)\zeta^2-m\right]. \tag{8.3.7}$$

Then we obtain

$$\sigma_\theta+\sigma_\rho=2\left[\Phi_*(\zeta)+\bar{\Phi}_*\left(\bar{\zeta}\right)\right]=2mp\left(\frac{1}{\zeta^2-m}+\frac{1}{\bar{\zeta}^2-m}\right)$$
$$=4mp\frac{\rho^2\cos 2\theta-m}{\rho^4-2m\rho^2\cos 2\theta+m^2}.$$

On the contour of the ellipse $\rho=1, \sigma_\rho=-p$, that is

$$\sigma_\theta|_{\rho=1}=4mp\frac{\cos 2\theta-m}{1-2m\cos 2\theta+m^2}+p.$$

The maximum is achieved at the ends of the major semi-axis ($\theta=0$) and is equal to

$$\frac{1}{p}\sigma_\theta\bigg|_{\substack{\rho=1\\ \theta=0}}=\frac{4m}{1-m}+1=2\frac{a}{b}-1. \tag{8.3.8}$$

2. The edge of the opening is free, $\sigma_1^\infty=q, \sigma_2^\infty=0$. Then

$$\frac{1}{q}\Phi_*(\zeta)=\frac{1}{2}\frac{m-e^{2i\alpha}}{\zeta^2-m},$$
$$\frac{1}{q}\Psi_*(\zeta)=\frac{1}{2}\frac{m+e^{-2i\alpha}}{\zeta^2-m}+\frac{1}{2}\frac{m-e^{2i\alpha}}{\left(\zeta^2-m\right)^3}\left[m\zeta^4+(3+m)\zeta^2-m\right] \tag{8.3.9}$$

and due to eq. (5.4.15)

$$\frac{1}{q}\Phi(\zeta) = \frac{1}{4}\frac{\zeta^2 + m - 2e^{2i\alpha}}{\zeta^2 - m}, \quad \frac{1}{q}\Psi(\zeta) = \frac{1}{q}\Psi_*(\zeta) - \frac{1}{2}e^{-2i\alpha}. \tag{8.3.10}$$

We obtain

$$\frac{1}{q}(\sigma_\rho + \sigma_\theta) = \frac{\rho^4 - 2\rho^2\cos(2\theta - 2\alpha) - m^2 + 2m\cos 2\alpha}{\rho^4 - 2m\rho^2\cos 2\theta + m^2} \tag{8.3.11}$$

and on the contour of the opening

$$\frac{1}{q}(\sigma_\theta)_{\rho=1} = \frac{1 - m^2 + 2m\cos 2\alpha - 2\cos(2\theta - 2\alpha)}{1 - 2m\cos 2\theta + m^2}. \tag{8.3.12}$$

7.8.4 Hypotrochoidal opening

By means of the conformal transformation

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{n\zeta^n}\right) \quad (0 < m < l)$$

the region $|\zeta| > 1$ outside the unit circle is mapped into region L of the plane z outside the opening with the hypotrochoidal contour

$$\begin{aligned} z &= R\left(\sigma + \frac{m}{n\sigma^n}\right), \\ x &= R\left(\cos\theta + \frac{m}{n}\cos n\theta\right), \quad y = R\left(\cos\theta - \frac{m}{n}\sin n\theta\right). \end{aligned} \tag{8.4.1}$$

Let us recall that the hypotrochoid is the locus of a point on the circle rolling inside a motionless circle. For $n = 1$ we return to the case of the ellipse and for integer $n > 1$ we obtain the regular curvilinear polygons with the rounded corners (a triangle for $n = 2$, a square for $n = 3$ etc.), Fig. 7.7.

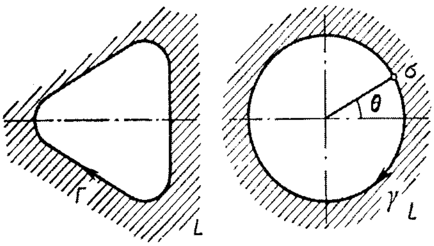

FIGURE 7.7.

By eqs. (8.2.2) and (8.2.7) we have

$$p_{n-1}\left(\frac{1}{\zeta}\right) = -\frac{Rm}{n}\left(\bar{a}_2\frac{n-2}{\zeta^{n-1}} + \bar{a}_3\frac{n-3}{\zeta^{n-2}} + \ldots + \frac{\bar{a}_{n-1}}{\zeta^2}\right), \quad Z_0' = \frac{Rm}{n}a_{n+1}, \tag{8.4.2}$$

and according to eq. (8.2.8) the expressions for functions $\Phi_*(\zeta), \Psi_*(\zeta)$ are written as follows

$$\left(1 - \frac{m}{\zeta^{n+1}}\right)\Phi_*(\zeta) - \frac{m}{n}\left(\bar{a}_2\frac{n-2}{\zeta^{n-1}} + \bar{a}_3\frac{n-3}{\zeta^{n-2}} + \ldots + \frac{\bar{a}_{n-1}}{\zeta^2}\right) =$$
$$= -\frac{1}{2\pi i R}\oint_\gamma \frac{F(\theta)}{\sigma - \zeta}d\sigma,$$

$$\left(1 - m\zeta^{n+1}\right)\Phi_*(\zeta) - \left(\zeta + \frac{m}{n}\zeta^{n+2}\right)\Phi_*'(\zeta) +$$
$$\frac{m}{n}\left[a_2(n-2)\zeta^{n-1} + a_3(n-3)\zeta^{n-2} + \ldots + a_{n-1}\zeta^2\right] -$$
$$\frac{m}{n}a_{n+1} - \zeta^2\left(1 - \frac{m}{\zeta^{n+1}}\right)\Psi_*(\zeta) + a_2' = -\frac{1}{2\pi i R}\oint_\gamma \frac{\bar{F}(\theta)}{\sigma - \zeta}d\sigma, \tag{8.4.3}$$

Let us next consider our consideration to the case of the free edge of the opening and the prescribed stress at infinity. Let for simplicity $\sigma_2^\infty = 0, \alpha = 0$. Then

$$F(\theta) = \frac{1}{2}\sigma_1^\infty R\left[1 - \frac{m}{\sigma^{n+1}} + \frac{1}{\sigma^2}\left(1 - m\sigma^{n+1}\right)\right],$$
$$-\frac{1}{2\pi i R}\oint_\gamma \frac{F(\theta)}{\sigma - \zeta}d\sigma = \frac{1}{2}\sigma_1^\infty\left(\frac{m}{\zeta^{n+1}} - \frac{1}{\zeta^2}\right),$$
$$-\frac{1}{2\pi i R}\oint_\gamma \frac{\bar{F}(\theta)\,d\sigma}{\sigma - \zeta} = \frac{1}{2}\sigma_1^\infty\frac{m}{\zeta^{n-1}} \quad (n > 1)$$

and by eq. (8.4.3)

$$\left(1 - \frac{m}{\zeta^{n+1}}\right)\left(\frac{a_2}{\zeta^2} + \frac{a_3}{\zeta^3} + \ldots + \frac{a_n}{\zeta^n} + \ldots\right) -$$
$$\frac{m}{n}\left(\bar{a}_2\frac{n-2}{\zeta^{n-1}} + \bar{a}_3\frac{n-3}{\zeta^{n-2}} + \ldots + \bar{a}_{n-1}\frac{1}{\zeta^2}\right) = \frac{1}{2}\sigma_1^\infty\left(\frac{m}{\zeta^{n+1}} - \frac{1}{\zeta^2}\right).$$

Comparing the coefficients at ζ^{-2}, ζ^{-n+1} results in the system of equations

$$a_2 - \frac{m}{n}\bar{a}_{n-1} = -\frac{1}{2}\sigma_1^\infty, \quad -\frac{m}{n}\bar{a}_2(n-2) + a_{n-1} = 0.$$

The remaining coefficients $a_3, a_4, \ldots, a_{n-2}$ are equal to zero. This determines the polynomial

$$p_{n-1}\left(\frac{1}{\zeta}\right) = \frac{1}{2}\sigma_1^\infty \frac{Rmn\,(n-2)}{n^2 - m^2\,(n-2)} \left(\frac{1}{\zeta^{n-1}} + \frac{m}{n}\frac{1}{\zeta^2}\right) \tag{8.4.4}$$

and then function $\Phi_*\,(\zeta)$

$$\Phi_*\,(\zeta) = \frac{1}{2}\sigma_1^\infty \left(\frac{m - \zeta^{n-1}}{\zeta^{n+1} - m} - \frac{m\,(n-2)}{n^2 - m^2\,(n-2)} \frac{n\zeta^2 + m\zeta^{n-1}}{\zeta^{n+1} - m}\right). \tag{8.4.5}$$

The constant a_2' appearing in the second equation (8.2.8) can be obtained by comparing the free terms in one of the boundary conditions, eqs. (8.1.6), (8.1.7). We have

$$a_2' = \frac{m}{n}a_{n+1} + \frac{1}{2}\sigma_1^\infty = \frac{1}{2}\sigma_1^\infty\left(\frac{m}{n} + 1\right), \quad \bar{Z}_0' = \frac{1}{2}\sigma_1^\infty\frac{m^2}{n}. \tag{8.4.6}$$

This determines function $\Psi_*\,(\zeta)$ in the following way

$$\begin{aligned}\zeta^2\left(1 - \frac{m}{\zeta^{n+1}}\right)\Psi_*\,(\zeta) &= \frac{1}{2}\sigma_1^\infty\left(1 - \frac{m}{\zeta^{n-1}}\right) + \left(1 - m\zeta^{n+1}\right)\Phi_*\,(\zeta) - \\ &\left(\zeta + \frac{m}{n}\zeta^{n+2}\right)\Phi_*'\,(\zeta) - \frac{1}{2}\sigma_1^\infty\frac{mn\,(n-2)}{n^2 - m^2\,(n-2)}\left(\zeta^{n-1} + \frac{m}{n}\zeta^2\right). \end{aligned} \tag{8.4.7}$$

As always, the stresses are obtained after some cumbersome algebraic manipulations. It is easy to find the sum of the normal stresses and then the value of σ_θ on the contour of the opening

$$\begin{aligned}\sigma_\theta = \frac{1}{1 + m^2 - 2m\cos\,(n+1)\,\theta}\Big\{\left(\sigma_1^\infty + \sigma_2^\infty\right)\left(1 - m^2\right) + \\ \frac{2n\,(\sigma_2^\infty - \sigma_1^\infty)}{n^2 - (n-2)\,m^2}\left[\left(n - m^2 n + 2m\right)\cos 2\,(\alpha - \theta) - 2m\cos\,(2\alpha + n\theta - \theta)\right]\Big\} \\ (n = 2, 3, \ldots).\end{aligned}$$

Here $\sigma_1^\infty, \sigma_2^\infty$ denote the principal stresses at infinity and α is the angle between the axes σ_1^∞ and x.

7.8.5 *Simply connected finite region*

The conformal transformation of the unit circle $|\zeta| \leq 1$ into the considered region bounded by the smooth closed contour Γ is given by the function $\omega\,(\zeta)$

$$z = \omega\,(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots,$$

which is holomorphic in the circle. Here we assume that c_1 is real-valued and $\omega'(\zeta) \neq 0$ for $|\zeta| \leq 1$. In what follows we can take that $\omega(\zeta)$ is a polynomial of order n. Then, similar to the case of the infinite region with an opening, a closed-form solution of the problem can be obtained.

By virtue of eq. (5.2.15) the boundary conditions are written (under slightly changes in denotations) in the form

$$\left.\begin{aligned} &\omega'(\sigma)\Phi(\sigma) + \frac{d}{d\sigma}\omega(\sigma)\bar{\Phi}\left(\frac{1}{\sigma}\right) - \frac{1}{\sigma^2}\bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\Psi}\left(\frac{1}{\sigma}\right) = \\ &\qquad = (F_n + iF_t)\,\omega'(\sigma) = F(\theta), \\ &\bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\Phi}\left(\frac{1}{\sigma}\right) + \frac{d}{d\left(\dfrac{1}{\sigma}\right)}\bar{\omega}\left(\frac{1}{\sigma}\right)\Phi(\sigma) - \sigma^2\omega'(\sigma)\Psi(\sigma) = \\ &\qquad = (F_n - iF_t)\,\bar{\omega}'\left(\frac{1}{\sigma}\right) = \bar{F}(\theta). \end{aligned}\right\} \tag{8.5.1}$$

Here, by eq. (5.2.9)

$$F_n + iF_t = \bar{n}\,(F_x + iF_y),$$

where $F_x + iF_y$ denotes the vector of the surface forces, whilst F_n and F_t denote its projections on the external normal and the tangent to Γ. The principal vector of the surface forces calculated with the help of formulae (5.2.6), (5.2.7) is presented as follows

$$\begin{aligned} \oint_\Gamma (F_x + iF_y)\,ds &= \oint_\Gamma (F_n + iF_t)\,n\,ds \\ &= \oint_\gamma (F_n + iF_t)\,\sigma\sqrt{\frac{\omega'(\sigma)}{\bar{\omega}'\left(\dfrac{1}{\sigma}\right)}}\sqrt{\omega'(\sigma)\,\bar{\omega}'\left(\frac{1}{\sigma}\right)}\,d\theta \\ &= \frac{1}{i}\oint_\gamma (F_n + iF_t)\,\omega'(\sigma)\,d\sigma = -i\oint_\gamma F(\theta)\,d\sigma. \end{aligned} \tag{8.5.2}$$

The principal moment of these forces about point $z = 0$ (into which the centre of the circle is mapped) is equal to

$$\begin{aligned} m^O &= \oint_\Gamma (xF_y - yF_x)\,ds = \frac{1}{2}i\oint_\Gamma \left[z\,(F_x - iF_y) - \bar{z}\,(F_x + iF_y)\right] ds \\ &= \frac{1}{2}\oint_\gamma \left[\frac{\omega(\sigma)}{\sigma^2}\bar{F}(\theta) - \bar{\omega}\left(\frac{1}{\sigma}\right)F(\theta)\right] d\sigma, \end{aligned} \tag{8.5.3}$$

and the conditions of the self-equilibrated system of surface forces are set in the form (compare eq. (6.2.4))

$$\oint_\gamma F(\theta)\, d\sigma = 0, \quad \oint_\gamma \left[\frac{\omega(\sigma)}{\sigma^2}\bar{F}(\theta) - \bar{\omega}\left(\frac{1}{\sigma}\right) F(\theta)\right] d\sigma = 0. \tag{8.5.4}$$

It is no problem to check that the left hand sides of the boundary conditions (8.5.1) satisfy the conditions (8.5.4) provided that functions $\Phi(\zeta), \Psi(\zeta)$ are holomorphic in the unit circle. Indeed,

$$\oint_\gamma F(\theta)\, d\sigma = \oint_\gamma \omega'(\sigma)\Phi(\sigma)\, d\sigma + \oint_\gamma d\left[\omega(\sigma)\bar{\Phi}\left(\frac{1}{\sigma}\right)\right] + \oint_\gamma \bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\Psi}\left(\frac{1}{\sigma}\right) d\left(\frac{1}{\sigma}\right) = 0,$$

The first and third integrals vanish as their integrands are the boundary values of the functions holomorphic for $|\zeta| < 1$ (and correspondingly for $|\zeta| > 1$), whilst the second integral vanishes inasmuch as $\omega(\sigma)\bar{\Phi}\left(\frac{1}{\sigma}\right)$ is a single-valued function. By analogy, we have for the equations of moments

$$-\oint_\gamma \left[\omega(\sigma)\bar{F}(\theta)\frac{1}{\sigma^2} - \bar{\omega}\left(\frac{1}{\sigma}\right) F(\theta)\right] d\sigma =$$
$$= \oint_\gamma d\left\{\bar{\omega}\left(\frac{1}{\sigma}\right)\omega(\sigma)\left[\Phi(\sigma) + \bar{\Phi}\left(\frac{1}{\sigma}\right)\right]\right\} + \frac{1}{2}\oint_\gamma \left[\bar{\Psi}\left(\frac{1}{\sigma}\right)\frac{d}{d\sigma}\bar{\omega}^2\left(\frac{1}{\sigma}\right) + \Psi(\sigma)\frac{d}{d\sigma}\omega^2(\sigma)\right] d\sigma = 0,$$

which is required.

Applying the method of Cauchy's integrals to the first boundary condition (8.5.1) leads to the relationship

$$\omega'(\zeta)\Phi(\zeta) + \frac{1}{2\pi i}\oint_\gamma \frac{d\omega(\sigma)\bar{\Phi}\left(\frac{1}{\sigma}\right)}{\sigma - \zeta} = \frac{1}{2\pi i}\oint_\gamma \frac{F(\theta)\, d\sigma}{\sigma - \zeta}. \tag{8.5.5}$$

On the other hand

$$\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{d\omega\left(\sigma\right)\bar{\Phi}\left(\frac{1}{\sigma}\right)}{\sigma-\zeta}=\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{\omega\left(\sigma\right)\bar{\Phi}\left(\frac{1}{\sigma}\right)}{\left(\sigma-\zeta\right)^{2}}d\sigma$$
$$=\frac{d}{d\zeta}\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{\omega\left(\sigma\right)\bar{\Phi}\left(\frac{1}{\sigma}\right)}{\sigma-\zeta}d\sigma. \tag{8.5.6}$$

Let $p_n\left(\zeta\right)$ denote the holomorphic part of the function

$$\omega\left(\zeta\right)\bar{\Phi}\left(\frac{1}{\zeta}\right)=\left(c_1\zeta+c_2\zeta^2+\ldots+c_n\zeta^n\right)\left(\bar{a}_0+\frac{\bar{a}_1}{\zeta}+\ldots+\frac{\bar{a}_n}{\zeta^n}+\ldots\right)$$
$$=p_n\left(\zeta\right)+Z\left(\frac{1}{\zeta}\right), \tag{8.5.7}$$

where

$$p_n\left(\zeta\right)=\left(c_1\bar{a}_1+c_2\bar{a}_2+\ldots+c_n\bar{a}_n\right)+\left(c_1\bar{a}_0+c_2\bar{a}_1+\ldots+c_n\bar{a}_{n-1}\right)\zeta+$$
$$\left(c_2\bar{a}_0+c_3\bar{a}_1+\ldots+c_n\bar{a}_{n-2}\right)\zeta^2+\ldots+c_n\bar{a}_0\zeta^n, \tag{8.5.8}$$

and $Z\left(\zeta\right)$ is a function holomorphic in the circle $|\zeta|<1$.

Equality (8.5.5) is now set in the form

$$\omega'\left(\zeta\right)\Phi\left(\zeta\right)+p_n'\left(\zeta\right)=\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{F\left(\theta\right)}{\sigma-\zeta}d\sigma. \tag{8.5.9}$$

Referring to eq. (8.5.7) we have

$$\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi\left(\zeta\right)=\bar{p}_n\left(\frac{1}{\zeta}\right)+\bar{Z}\left(\zeta\right),$$
$$\frac{d}{d\left(\frac{1}{\zeta}\right)}\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi\left(\zeta\right)=\bar{p}_n'\left(\frac{1}{\zeta}\right)-\zeta^2\bar{Z}'\left(\zeta\right). \tag{8.5.10}$$

Hence

$$\frac{1}{2\pi i}\oint\limits_{\gamma}\frac{d}{d\left(\frac{1}{\sigma}\right)}\bar{\omega}\left(\frac{1}{\sigma}\right)\Phi\left(\sigma\right)\frac{d\sigma}{\sigma-\zeta}=\bar{p}_n'\left(0\right)-\zeta^2\bar{Z}'\left(\zeta\right)=$$
$$=\bar{p}_n'\left(0\right)-\bar{p}_n'\left(\frac{1}{\zeta}\right)+\bar{\omega}'\left(\frac{1}{\zeta}\right)\Phi\left(\zeta\right)-\zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'\left(\zeta\right)$$

and the second boundary condition leads to the dependence

$$c_1\bar{a}_0 + \bar{\omega}'\left(\frac{1}{\zeta}\right)\Phi(\zeta) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) - \bar{p}_n'\left(\frac{1}{\zeta}\right) + \bar{p}_n'(0) - \frac{1}{2\pi i}\oint_\gamma \frac{\bar{F}(\theta)}{\sigma-\zeta}d\sigma = \zeta^2\omega'(\zeta)\Psi(\zeta), \quad (8.5.11)$$

where by eq. (8.5.8)

$$\bar{p}_n'(0) = \bar{c}_1 a_0 + \bar{c}_2 a_1 + \ldots + \bar{c}_n a_{n-1}. \quad (8.5.12)$$

Using eq. (8.5.9) and talking into account that $d\sigma = i\sigma d\theta$ we have

$$c_1 a_0 + p_n'(0) = \frac{1}{2\pi i}\oint_\gamma \frac{F(\theta)}{\sigma}d\sigma, \quad \bar{c}_1\bar{a}_0 + \bar{p}_n'(0) = \frac{1}{2\pi i}\oint_\gamma \frac{\bar{F}(\theta)}{\sigma}d\sigma.$$

Due to eq. (8.5.4)

$$\frac{1}{2\pi i}\oint_\gamma \frac{\bar{F}(\theta)}{\sigma^2}d\sigma = 0.$$

These relationships enable eq. (8.5.11) to be represented in the form

$$\bar{\omega}'\left(\frac{1}{\zeta}\right)\Phi(\zeta) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) - \bar{p}_n'\left(\frac{1}{\zeta}\right) - \frac{1}{2\pi i}\oint_\gamma F(\theta)\left(\frac{1}{\sigma-\zeta} - \frac{1}{\sigma} - \frac{\zeta}{\sigma^2}\right)d\sigma = \zeta^2\omega'(\zeta)\Psi(\zeta)$$

or

$$\omega'(\zeta)\Psi(\zeta) = \frac{1}{\zeta^2}\left[\bar{\omega}'\left(\frac{1}{\zeta}\right)\Phi(\zeta) - \zeta^2\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) - \bar{p}_n'\left(\frac{1}{\zeta}\right)\right] - \frac{1}{2\pi i}\oint_\gamma \frac{\bar{F}(\theta)\,d\sigma}{\sigma^2(\sigma-\zeta)}. \quad (8.5.13)$$

By eq. (8.5.10) the first group of terms is equal to $-Z'(\zeta)$. This confirms that function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta| < 1$ which was assumed earlier.

7.8.6 An example

Let us consider a finite region L bounded by the contour

$$x = R\left(\cos\theta + \frac{1}{3}m\cos 3\theta\right), \quad y = R\left(\sin\theta + \frac{1}{3}m\sin 3\theta\right),$$

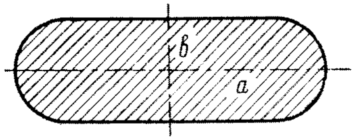

FIGURE 7.8.

denoting a epitrochoidal oval, Fig. 7.8. By means of the transformation

$$z = \omega(\zeta) = R\left(\zeta + \frac{1}{3}m\zeta^3\right), \tag{8.6.1}$$

the interior of the unit circle is mapped into region L. The condition $\omega'(\zeta) \neq 0$ for $|\zeta| \leq 1$ requires $0 \leq m \leq 1$. If $0 \leq m \leq 1/3$ then the points on Γ corresponding to $\theta = 0, \theta = \pi$ are furthest from the coordinate origin whilst the points $\theta = \pi/2, \theta = 3\pi/2$ are the closest. Curve (8.6.1) looks like an elongated ellipse with the semi-axes $a = R(1 + m/3), b = R(1 - m/3)$, region L lying within this oval.

By eq. (8.5.8) in the case considered we have

$$\bar{p}'_n(\zeta) = R\bar{a}_0 + \frac{1}{3}Rm\left(\bar{a}_2 + 2\bar{a}_1\zeta + 3\bar{a}_0\zeta^2\right), \tag{8.6.2}$$

and by eq. (8.5.9) we arrive at the following system of equations for the unknown coefficients $\bar{a}_0, \bar{a}_1, \bar{a}_2$

$$\left.\begin{aligned} a_0 + \bar{a}_0 + \frac{1}{3}m\bar{a}_2 &= \frac{1}{2\pi i R}\oint_\gamma \frac{F(\theta)}{\sigma}d\sigma, \\ a_1 + \frac{2}{3}m\bar{a}_1 &= \frac{1}{2\pi i R}\oint_\gamma \frac{F(\theta)}{\sigma^2}d\sigma, \\ a_2 + m(\bar{a}_0 + a_0) &= \frac{1}{2\pi i R}\oint_\gamma \frac{F(\theta)}{\sigma^3}d\sigma, \end{aligned}\right\} \tag{8.6.3}$$

and the complex conjugated system. The unknowns a_0, a_1, a_2 are easily expressed from these equations, then polynomial $p'_n(\zeta)$ is constructed and with the help of eqs. (8.5.9), (8.5.13) we obtain the sought functions $\Phi(\zeta)$ and $\Psi(\zeta)$.

7.8.7 The first boundary-value problem

We restrict ourselves to considering the case of the external problem and assume, as before, that mapping the exterior of the unit circle into the infinite region L bounded inside by the smooth contour Γ is possible by function (8.1.2) where $m(\zeta)$ is a polynomial of degree n in ζ^{-1}. Supposing zero stresses at infinity we have by eq. (5.4.18)

$$\begin{aligned}
&(3-4\nu)\,\varphi_1(z) - z\bar{\varphi}_1'(\bar{z}) - \bar{\psi}_1(\bar{z}) = \\
&= 2\mu\,(u+iv) + \frac{3-4\nu}{8\pi(1-\nu)}(X+iY)(\ln z + \ln \bar{z}) - \frac{X-iY}{8\pi(1-\nu)}\frac{z}{\bar{z}},
\end{aligned} \tag{8.7.1}$$

where $\varphi_1(z), \psi_1(z)$ are holomorphic in L and $X+iY$ denotes the prescribed principal vector of the surface forces on Γ.

Taking

$$\varphi_1(\omega(\zeta)) = \varphi_*(\zeta), \quad \psi_1(\omega(\zeta)) = \psi_*(\zeta)$$

we can write

$$\begin{aligned}
&(3-4\nu)\,\varphi_*(\zeta) - \frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})}\bar{\varphi}_*'(\bar{\zeta}) - \bar{\psi}_*(\bar{\zeta}) = \\
&= 2\mu\,(u+iv) + \frac{3-4\nu}{8\pi(1-\nu)}(X+iY)\left[\ln\omega(\zeta) + \ln\bar{\omega}(\bar{\zeta})\right] - \frac{X-iY}{8\pi(1-\nu)}\frac{\omega(\zeta)}{\bar{\omega}(\bar{\zeta})}.
\end{aligned}$$

The holomorphic function $\ln\dfrac{\omega(\zeta)}{\zeta}$ (for $|\zeta|>1$) can be included into the sought functions $\varphi_*(\zeta), \psi_*(\zeta)$ by setting

$$\left.\begin{aligned}
\varphi_*(\zeta) &= \varphi(\zeta) + \frac{X+iY}{8\pi(1-\nu)}\ln\frac{\omega(\zeta)}{\zeta}, \\
\psi_*(\zeta) &= \psi(\zeta) - \frac{3-4\nu}{8\pi(1-\nu)}(X-iY)\ln\frac{\omega(\zeta)}{\zeta}.
\end{aligned}\right\} \tag{8.7.2}$$

Then

$$\bar{\varphi}_*'(\bar{\zeta})\frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})} = \bar{\varphi}'(\bar{\zeta})\frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})} + \frac{X-iY}{8\pi(1-\nu)}\left(\frac{\omega(\zeta)}{\bar{\omega}(\bar{\zeta})} - \frac{1}{\zeta}\frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})}\right)$$

and the previous equality is written down as follows

$$\begin{aligned}
&(3-4\nu)\,\varphi(\zeta) - \frac{\omega(\zeta)}{\bar{\omega}'(\bar{\zeta})}\bar{\varphi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta}) = \\
&= 2\mu\,(u+iv) + \frac{3-4\nu}{4\pi(1-\nu)}(X+iY)\ln|\zeta| - \frac{X-iY}{8\pi(1-\nu)}\frac{\omega(\zeta)}{\zeta\bar{\omega}'(\bar{\zeta})}.
\end{aligned} \tag{8.7.3}$$

This leads to the following representation of the boundary conditions

$$
\left.
\begin{aligned}
&(3-4\nu)\,\bar{\omega}'\left(\frac{1}{\sigma}\right)\varphi(\sigma)-\omega(\sigma)\,\bar{\varphi}'\left(\frac{1}{\sigma}\right)-\bar{\omega}'\left(\frac{1}{\sigma}\right)\bar{\psi}\left(\frac{1}{\sigma}\right)=\\
&\qquad=2\mu\bar{\omega}'\left(\frac{1}{\sigma}\right)(u+iv)_{\Gamma}-\frac{X-iY}{8\pi(1-\nu)}\sigma\omega(\sigma),\\
&(3-4\nu)\,\omega'(\sigma)\,\bar{\varphi}\left(\frac{1}{\sigma}\right)-\bar{\omega}\left(\frac{1}{\sigma}\right)\varphi'(\sigma)-\omega'(\sigma)\,\psi(\sigma)=\\
&\qquad=2\mu\omega'(\sigma)\,(u-iv)_{\Gamma}-\frac{X+iY}{8\pi(1-\nu)}\frac{1}{\sigma}\bar{\omega}\left(\frac{1}{\sigma}\right),
\end{aligned}
\right\}
\tag{8.7.4}
$$

as $\ln|\zeta|=0$ on γ.

Following the approach of Subsection 7.8.2 we assume $\varphi(\infty)=0$ and consider the following function

$$
\begin{aligned}
\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta)&=\left(\bar{c}_0-\bar{c}_1\zeta^2-2\bar{c}_2\zeta^3-\ldots-n\bar{c}_n\zeta^{n+1}\right)\times\\
&\times\left(\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\ldots+\frac{a_{n+1}}{\zeta^{n+1}}+\ldots\right)=q_n(\zeta)+Z\left(\frac{1}{\zeta}\right).
\end{aligned}
\tag{8.7.5}
$$

Here $q_n(\zeta)$ denotes a polynomial of degree n in ζ and $Z(1/\zeta)$ is holomorphic in $|\zeta|>1$ and $Z(\infty)=0$. Then

$$
|\zeta|>1:\quad \oint_{\gamma}\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)\varphi(\sigma)}{\sigma-\zeta}d\sigma=Z\left(\frac{1}{\zeta}\right)=\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta)-q_n(\zeta).
$$

The product $\omega(\zeta)\,\bar{\varphi}'(1/\zeta)$ is a sum of the polynomial of ζ^{-1} and a function which is holomorphic in the circle $|\zeta|<1$

$$
\omega(\zeta)\,\bar{\varphi}'\left(\frac{1}{\zeta}\right)=r_{n-2}\left(\frac{1}{\zeta}\right)+Q(\zeta),
\tag{8.7.6}
$$

where

$$
r_{n-2}(\infty)=0,\quad Q(0)=-\left[c_2\bar{a}_1+2c_3\bar{a}_2+\ldots+(n-1)\,c_n\bar{a}_{n-1}\right].
\tag{8.7.7}
$$

Therefore

$$
\frac{1}{2\pi i}\oint_{\gamma}\frac{\omega(\sigma)\,\bar{\varphi}'\left(\frac{1}{\sigma}\right)d\sigma}{\sigma-\zeta}=r_{n-2}\left(\frac{1}{\zeta}\right).
$$

Applying the method of Cauchy's integral leads to the relationship

$$
\begin{aligned}
&(3-4\nu)\left[\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta)-q_n(\zeta)\right]-r_{n-2}\left(\frac{1}{\zeta}\right)= \qquad (8.7.8)\\
&=\frac{1}{2\pi i}\oint\limits_{\gamma} 2\mu(u+iv)_\Gamma \frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma-\zeta}d\sigma-\frac{X+iY}{8\pi(1-\nu)}\left[\zeta\omega(\zeta)-c_0\zeta^2-c_1\right].
\end{aligned}
$$

Analogous actions with respect to the second boundary condition yield

$$
\begin{aligned}
&(3-4\nu)\left[\bar{q}_n\left(\frac{1}{\zeta}\right)-q_n(0)\right]-\left[\bar{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta)-\bar{r}_{n-2}(\zeta)\right]+\\
&\qquad \bar{Q}(0)-\omega'(\zeta)\psi(\zeta)+c_0\psi(\infty)=\\
&\qquad =\frac{1}{2\pi i}\oint\limits_{\gamma} 2\mu(u-iv)_\Gamma \frac{\omega'(\sigma)}{\sigma-\zeta}d\sigma-\frac{X+iY}{8\pi(1-\nu)}\frac{c_0}{\zeta^2}. \qquad (8.7.9)
\end{aligned}
$$

The constant $\psi(\infty)$ is determined by comparing the terms which do not depend on σ (i.e. the free terms of the trigonometric series) on both sides of the second boundary condition (8.7.4). Multiplying them by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma}$ and integrating along the contour of the unit circle γ we obtain

$$
\begin{aligned}
&(3-4\nu)\frac{1}{2\pi i}\oint\limits_{\gamma}\omega'(\sigma)\bar{\varphi}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma}-\frac{1}{2\pi i}\oint\limits_{\gamma}\bar{\omega}\left(\frac{1}{\sigma}\right)\varphi'(\sigma)\frac{d\sigma}{\sigma}-\\
&\qquad \frac{1}{2\pi i}\oint\limits_{\gamma}\frac{\omega'(\sigma)}{\sigma}\psi(\sigma)\,d\sigma=\\
&\qquad =\frac{1}{2\pi i}\oint\limits_{\gamma} 2\mu(u-iv)_\Gamma\frac{\omega'(\sigma)}{\sigma}d\sigma-\frac{X+iY}{8\pi(1-\nu)}\oint\limits_{\gamma}\frac{\bar{\omega}\left(\frac{1}{\sigma}\right)}{\sigma^2}d\sigma,
\end{aligned}
$$

or

$$
\begin{aligned}
&(3-4\nu)\,\bar{q}_n(0)-\bar{Q}(0)-c_0\psi(\infty)=\\
&\qquad =\frac{1}{2\pi i}\oint\limits_{\gamma} 2\mu(u-iv)_\Gamma\frac{\omega'(\sigma)}{\sigma}d\sigma-\frac{X+iY}{8\pi(1-\nu)}\bar{c}_1. \qquad (8.7.10)
\end{aligned}
$$

7.8.8 Elliptic opening

In this case, we have by eqs. (8.3.1), (8.7.8) and (8.7.9)

$$(3-4\nu)\left[\left(1-m\zeta^2\right)\varphi(\zeta)+m(a_1\zeta+a_2)\right]=\frac{2\mu}{2\pi iR}\oint\limits_{\gamma}\bar{\omega}'\left(\frac{1}{\sigma}\right)\frac{(u+iv)_\Gamma}{\sigma-\zeta}d\sigma, \tag{8.8.1}$$

$$-(3-4\nu)\frac{m\bar{a}_1}{\zeta}-\left(\frac{1}{\zeta}+m\zeta\right)\varphi'(\zeta)-\left(1-\frac{m}{\zeta^2}\right)\psi(\zeta)+\zeta(\infty)=$$
$$=\frac{2\mu}{2\pi iR}\oint\limits_{\gamma}\bar{\omega}'(\sigma)\frac{(u-iv)_\Gamma}{\sigma-\zeta}d\sigma-\frac{X+iY}{8\pi(1-\nu)}\frac{1}{\zeta^2}. \tag{8.8.2}$$

In the particular case in which a rigid elliptic core placed in the elastic solid is subjected to the displacement

$$(u+iv)_\Gamma=u_0+iv_0+i\varepsilon z=u_0+iv_0+i\varepsilon z\omega(\zeta)$$

we have

$$\frac{1}{2\pi iR}\oint\limits_{\gamma}\bar{\omega}'\left(\frac{1}{\sigma}\right)\frac{(u+iv)_\Gamma}{\sigma-\zeta}d\sigma=$$
$$=\frac{1}{2\pi iR}\oint\limits_{\gamma}\left(1-m\sigma^2\right)\left[u_0+iv_0+i\varepsilon R\left(\sigma+\frac{m}{\sigma}\right)\right]\frac{d\sigma}{\sigma-\zeta}=\frac{i\varepsilon Rm}{\zeta}, \tag{8.8.3}$$

$$\frac{1}{2\pi iR}\oint\limits_{\gamma}\bar{\omega}'(\sigma)\frac{(u-iv)_\Gamma}{\sigma-\zeta}d\sigma=$$
$$=\frac{1}{2\pi i}\oint\limits_{\gamma}\left(1-\frac{m}{\sigma^2}\right)\left[u_0-iv_0-i\varepsilon R\left(\frac{1}{\sigma}+m\sigma\right)\right]\frac{d\sigma}{\sigma-\zeta}=$$
$$=-\frac{m}{\sigma^2}(u_0-iv_0)-i\varepsilon R\left[\left(1-m^2\right)\frac{1}{\zeta}-\frac{m}{\zeta^3}\right] \tag{8.8.4}$$

and furthermore by eqs. (8.8.1), (8.7.5) and (8.7.6)

$$\left.\begin{aligned}&(3-4\nu)\varphi(\zeta)=\frac{2\mu i\varepsilon mR}{\zeta}=(3-4\nu)a_1,\\&a_2=a_3=\ldots=0,\quad q_n(0)=0,\quad Q(0)=0.\end{aligned}\right\} \tag{8.8.5}$$

Turning to eq. (8.7.10) we obtain

$$\psi(\infty)=-2\mu(u_0-iv_0)+\frac{m(X+iY)}{8\pi(1-\nu)}$$

and by eq. (8.8.2)

$$\psi(\zeta) = -2\mu(u_0 - iv_0) + \frac{2\mu i\varepsilon R}{\zeta}\left(1 + m\frac{1+m\zeta^2}{\zeta^2 - m}\frac{1}{3-4\nu}\right) + \frac{X+iY}{8\pi(1-\nu)}\frac{1+m\zeta^2}{\zeta^2-m}. \tag{8.8.6}$$

All parts of the sought functions holomorphic in region $|\zeta| > 1$ are thus determined. The functions determining the solution of the problem denoted by $\varphi(\zeta), \psi(\zeta)$ are obtained by adding, according to eqs. (8.7.2) and (5.4.3), the logarithmic terms $-\frac{X+iY}{8\pi(1-\nu)}\ln\zeta$ and $\frac{X-iY}{8\pi(1-\nu)}(3-4\nu)\ln\zeta$ to eqs. (8.8.5), (8.8.6). Hence

$$\left.\begin{aligned}\varphi(\zeta) &= \frac{2\mu i\varepsilon mR}{3-4\nu}\frac{1}{\zeta} - \frac{X+iY}{8\pi(1-\nu)}\ln\zeta,\\ \psi(\zeta) &= -2\mu(u_0 - iv_0) + \frac{2\mu i\varepsilon R}{\zeta}\left(1 + m\frac{1+m\zeta^2}{\zeta^2-m}\frac{1}{3-4\nu}\right) + \\ &\qquad \frac{X+iY}{8\pi(1-\nu)}\frac{1+m\zeta^2}{\zeta^2-m} + \frac{X-iY}{8\pi(1-\nu)}(3-4\nu)\ln\zeta.\end{aligned}\right\} \tag{8.8.7}$$

The displacement vector is now given by

$$\begin{aligned}2\mu(u+iv) &= 2\mu(u_0+iv_0) + 2\mu i\varepsilon R\left(\frac{m}{\zeta} + \frac{1}{\zeta}\right) - \\ &\quad \frac{3-4\nu}{8\pi(1-\nu)}\left(\ln\zeta + \ln\bar{\zeta}\right)(X+iY) + \\ &\quad \frac{2\mu i\varepsilon R}{3-4\nu}\frac{1-\zeta\bar{\zeta}}{\bar{\zeta}^2-m}\left(\frac{1}{\bar{\zeta}} - \frac{m}{\zeta}\right) + \frac{X-iY}{8\pi(1-\nu)}\frac{1-\zeta\bar{\zeta}}{\bar{\zeta}^2-m}\left(\frac{m\bar{\zeta}}{\zeta} - 1\right).\end{aligned} \tag{8.8.8}$$

At a sufficient distance from the opening, the latter expression for the displacement vector contains logarithmically growing terms and the remaining bounded terms

$$2\mu(u+iv)^\infty = 2\mu\left(u^0+iv^0\right) - \frac{X-iY}{8\pi(1-\nu)}\left(m - e^{2i\theta}\right) - \frac{3-4\nu}{4\pi(1-\nu)}(X+iY)\ln\rho.$$

According to eq. (3.1.14) the moment of the external forces providing a turn ε to the core is determined in terms of the coefficient associated with z^{-1} in the expression for $\psi(z)$. By virtue of eq. (8.8.7) the term of degree ζ^{-1} is equal to

$$\frac{2\mu i\varepsilon R}{\zeta}\left(1 + \frac{m^2}{3-4\nu}\right) \approx \frac{2\mu i\varepsilon R^2}{z}\left(1 + \frac{m^2}{3-4\nu}\right)$$

and by eq. (3.1.14) we have

$$M^O = 4\pi\mu\varepsilon R^2 \left(1 + \frac{m^2}{3-4\nu}\right). \tag{8.8.9}$$

7.8.9 Double-connected region

The conformal transformation

$$z = \omega(\zeta) \tag{8.9.1}$$

maps the round ring in plane ζ bounded by the circles γ_0, γ_1 of the corresponding radii ρ_0, ρ_1 ($\rho_0 \leq |\zeta| \leq \rho_1$) into the double-connected region L in plane z. It is assumed that the external Γ_0 and the internal Γ_1 contours of this region correspond respectively to the internal γ_0 and external γ_1 circles of the ring.

We consider the case in which the system of surface forces is in equilibrium on each of the contours Γ_0, Γ_1. There exists the solution of the auxiliary problem of loading the single-connected region bounded by contour Γ_0. This solution, which determines the normal and shear stresses on contour Γ_1, i.e.$(\sigma_n + i\tau_{nt})^0_{\Gamma_0}$, is assumed to be given. Considering then the problem for region L with the boundary conditions

$$(\sigma_n + i\tau_{nt})_{\Gamma_0} = 0, \quad (\sigma_n + i\tau_{nt})_{\Gamma_1} = (\sigma_n + i\tau_{nt})^1_{\Gamma_1} - (\sigma_n + i\tau_{nt})^0_{\Gamma_1} \tag{8.9.2}$$

and imposing the solution of the auxiliary problem, we arrive at the solution of the problem in which the boundary conditions are satisfied on both contours. For this reason, in what follows we consider the case in which the external contour Γ_0 is not loaded. By eq. (5.2.10) on any contour Γ of region L which is transformed into the circle $\rho =$ const, the vector $\sigma_n - i\tau_{nt}$ is given by

$$\sigma_n - i\tau_{nt} = \left\{\Phi_1(z) + \bar{\Phi}_1(\bar{z}) - \left[\bar{z}\Phi_1'(z) + \Psi_1(z)\right] n^2\right\}_\Gamma, \tag{8.9.3}$$

where, by eq. (5.2.7) the square of the vector of the normal is as follows

$$n^2 = -\left(\frac{dz}{ds}\right)^2 = -\frac{\omega'(\zeta)}{\bar{\omega}'(\bar{\zeta})}\frac{d\zeta}{d\bar{\zeta}} = \frac{\omega'(\zeta)}{\bar{\omega}'(\bar{\zeta})}\frac{\zeta}{\bar{\zeta}}. \tag{8.9.4}$$

Transforming to variable ζ in eq. (8.9.3) and denoting

$$\Phi_1(z) = \Phi_1(\omega(\zeta)) = \Phi(\zeta), \quad \Psi_1(z) = \Psi_1(\omega(\zeta)) = \Psi(\zeta),$$

we arrive at the relation

$$\Phi(\zeta) + \bar{\Phi}(\bar{\zeta}) - \frac{\zeta}{\bar{\zeta}\bar{\omega}'(\bar{\zeta})}\left[\bar{\omega}(\bar{\zeta})\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right] = \sigma_n - i\tau_{nt}. \tag{8.9.5}$$

On the circle γ_0 we have $\zeta = \rho_0\sigma, \bar{\zeta} = \rho_0/\sigma$ and the first boundary condition (8.9.2) is reduced to the form

$$\Phi\left(\rho_0\sigma\right) + \bar{\Phi}\left(\frac{\rho_0}{\sigma}\right) - \frac{\sigma^2}{\bar{\omega}'\left(\frac{\rho_0}{\sigma}\right)}\left[\bar{\omega}\left(\frac{\rho_0}{\sigma}\right)\Phi'\left(\rho_0\sigma\right) + \omega'\left(\rho_0\sigma\right)\Psi\left(\rho_0\sigma\right)\right] = 0. \tag{8.9.6}$$

It is satisfied if $\Psi(\zeta)$ is defined in the ring $\rho_0 \le |\zeta| \le \rho_1$ in the following way

$$\begin{aligned}\omega'(\zeta)\Psi(\zeta) &= \frac{\rho_0^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_0^2}{\zeta}\right)\left[\Phi(\zeta) + \bar{\Phi}\left(\frac{\rho_0^2}{\zeta}\right)\right] - \bar{\omega}\left(\frac{\rho_0^2}{\zeta}\right)\Phi'(\zeta) \\ &= -\frac{d}{d\zeta}\bar{\omega}\left(\frac{\rho_0^2}{\zeta}\right)\Phi(\zeta) + \frac{\rho_0^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_0^2}{\zeta}\right)\bar{\Phi}\left(\frac{\rho_0^2}{\zeta}\right).\end{aligned} \tag{8.9.7}$$

The boundary condition on Γ_1 is set in the form

$$\begin{aligned}\Phi\left(\rho_1\sigma\right) + \bar{\Phi}\left(\frac{\rho_1}{\sigma}\right) - \frac{\sigma^2}{\bar{\omega}'\left(\frac{\rho_1}{\sigma}\right)}\left[\bar{\omega}\left(\frac{\rho_1}{\sigma}\right)\Phi'\left(\rho_1\sigma\right) + \omega'\left(\rho_1\sigma\right)\Psi\left(\rho_1\sigma\right)\right] &= \\ = \left(\sigma_n - i\tau_{nt}\right)_{\Gamma_1}.\end{aligned} \tag{8.9.8}$$

The surface force on Γ_1 given by the Fourier series

$$\left(\sigma_n - i\tau_{nt}\right)_{\Gamma_1} = \alpha_0 + \sum_{k=1}^{\infty}\left(a_k\sigma^k + \frac{a_{-k}}{\sigma^k}\right) \tag{8.9.9}$$

can be viewed as being the value of a function prescribed by the Laurent series

$$g(\zeta) = a_0 + \sum_{k=1}^{\infty}\left(\frac{a_k}{\rho_1^k}\zeta^k + \frac{a_{-k}}{\zeta^k}\rho_1^k\right) \tag{8.9.10}$$

in the ring. This allows us to give another representation of function $\Psi(\zeta)$

$$\begin{aligned}\omega'(\zeta)\Psi(\zeta) = \frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)\left[\Phi(\zeta) + \bar{\Phi}\left(\frac{\rho_1^2}{\zeta}\right)\right] - \bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)\Phi'(\zeta) - \\ \frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)g(\zeta) = -\frac{d}{d\zeta}\bar{\omega}\left(\frac{\rho_1^2}{\zeta}\right)\Phi(\zeta) + \frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)\bar{\Phi}\left(\frac{\rho_1^2}{\zeta}\right) - \\ \frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)g(\zeta)\end{aligned} \tag{8.9.11}$$

similar to that in eq. (8.9.7). Comparing these representations of the same function $\Psi(\zeta)$ leads to the following relation

$$\frac{d}{d\zeta}\left[\bar{\omega}\left(\frac{\rho_0^2}{\zeta}\right)-\bar{\omega}\left(\frac{\rho_1^2}{\zeta}\right)\right]\Phi(\zeta)+\frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)\bar{\Phi}\left(\frac{\rho_1^2}{\zeta}\right)-\frac{\rho_0^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_0^2}{\zeta}\right)\bar{\Phi}\left(\frac{\rho_0^2}{\zeta}\right)=\frac{\rho_1^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_1^2}{\zeta}\right)g(\zeta). \quad (8.9.12)$$

The solution $\omega'(\zeta)\Phi(\zeta)$ of this functional equation should be sought in the form of a Laurent series without term ζ^{-1}

$$\omega'(\zeta)\Phi(\zeta)=c_0+\sum_{k=1}^{\infty}c_k\zeta^k+\sum_{k=2}^{\infty}\frac{c_{-k}}{\zeta^k}, \quad (c_{-1}=0). \quad (8.9.13)$$

The same term will also be absent in Laurent's representation of the function

$$\frac{\rho_0^2}{\zeta^2}\bar{\omega}'\left(\frac{\rho_0^2}{\zeta}\right)\bar{\Phi}'\left(\frac{\rho_0^2}{\zeta}\right)=\bar{c}_0\frac{\rho_0^2}{\zeta^2}+\sum_{k=1}^{\infty}\frac{\bar{c}_k\rho_0^{2k+2}}{\zeta^{k+2}}+\sum_{k=2}^{\infty}\frac{\bar{c}_{-k}}{\rho_0^{2k-2}}\zeta^{k-2}.$$

It is seen from the second representation (8.9.7) that function $\omega'(\zeta)\Phi(\zeta)$ does not contain the term ζ^{-1} as well. Therefore the representations

$$\varphi(\zeta)=\int\omega'(\zeta)\Phi(\zeta)\,d\zeta, \quad \psi(\zeta)=\int\omega'(\zeta)\Psi(\zeta)\,d\zeta$$

have no logarithmic terms which guarantees the single-valuedness of the displacement vector or, in other words, it ensures no distortion.

7.8.10 The non-concentric ring

Let region L be bounded by the external circle Γ_0 of radius r_0 and the internal circle Γ_1 of radius r_1. The eccentricity, i.e. the distance between the centres of these circles is denoted as e. The conformal transformation of the round ring in this region was considered in Subsection 6.3.12. Here it is presented in another form

$$z=c\frac{\zeta+1}{\zeta-1}, \quad (8.10.1)$$

where c is a real-valued constant. By eq. (8.10.1)

$$\zeta=\frac{z+c}{z-c}=1+\frac{2c}{z-c}, \quad \zeta\bar{\zeta}-1=\frac{4cx}{(z-c)(\bar{z}-c)}, \quad (8.10.2)$$

which yields

$$\left(x-c\frac{\zeta\bar{\zeta}+1}{\zeta\bar{\zeta}-1}\right)^2+y^2=\left|\frac{2c\sqrt{\zeta\bar{\zeta}}}{\zeta\bar{\zeta}-1}\right|^2. \quad (8.10.3)$$

The concentric circles $\zeta\bar{\zeta} = \rho^2 = \mathrm{const}$ in the circle correspond to the circles of radius r in L with the centres on the abscissa axis at points d

$$d = c\frac{\rho^2+1}{\rho^2-1} = c\coth\alpha, \quad r = \frac{2c\rho}{\rho^2-1} = \frac{c}{\sinh\alpha} \quad (\rho = e^\alpha > 1, \quad \alpha > 0). \tag{8.10.4}$$

Thus, the concentric ring formed by the rings γ_0 and γ_1 of the corresponding radii ρ_0 and γ_1 is mapped into the round ring L, the external Γ_0 and internal Γ_1 circles correspond respectively to the internal γ_0 and external γ_1 circles of the ring. The values c, α_0, α_1 are related to each other as follows

$$r_0\sinh\alpha_0 = r_1\sinh\alpha_1 = c, \quad e = |r_0\cosh\alpha_0 - r_1\cosh\alpha_1|. \tag{8.10.5}$$

Eliminating α_0, α_1 we obtain the equation for c

$$e = \sqrt{r_0^2+c^2} - \sqrt{r_1^2+c^2}. \tag{8.10.6}$$

Let us limit our consideration to the simplest case of the uniform loading by pressure p_1 on circle Γ_1. By inserting eq. (8.10.1) for $\omega(\zeta)$ into relation (8.9.12) in which $g(\zeta) = -p$ we have

$$\frac{d}{d\zeta}\frac{\zeta\Phi(\zeta)}{Z_0Z_1} + \frac{1}{\rho_1^2-\rho_0^2}\left[\frac{\rho_0^2}{Z_0^2}\bar{\Phi}\left(\frac{\rho_0^2}{\zeta}\right) - \frac{\rho_1^2}{Z_1^2}\bar{\Phi}\left(\frac{\rho_1^2}{\zeta}\right)\right] = \frac{q}{Z_1^2}, \tag{8.10.7}$$

where it is denoted

$$Z_0 = \rho_0^2 - \zeta, \quad Z_1 = \rho_1^2 - \zeta, \quad q = \frac{\rho_1^2 p}{\rho_1^2 - \rho_0^2}. \tag{8.10.8}$$

This equation can be satisfied by taking

$$\begin{aligned} \omega'(\zeta)\Phi(\zeta) &= -2c\left(\frac{C_0}{(\zeta-1)^2} + C_1 + \frac{C_2}{\zeta^2}\right), \\ \Phi(\zeta) &= C_0 + (\zeta-1)^2\left(C_1 + \frac{C_2}{\zeta^2}\right), \end{aligned} \tag{8.10.9}$$

where C_0, C_1, C_2 are real-valued. Point $\zeta = 0$ is not a simple pole of function $\omega'(\zeta)\Phi(\zeta)$.

For this function $\Phi(\zeta)$ eq. (8.10.7) is lead to the form

$$Z_0Z_1[\Phi(\zeta) + \zeta\Phi'(\zeta)] + \zeta\Phi(\zeta)(Z_0+Z_1) + C_0\left(\rho_0^2\rho_1^2 - \zeta^2\right) - \frac{C_1}{\zeta^2}Z_0^2Z_1^2 + \frac{C_2}{\rho_0^2\rho_1^2}Z_0^2Z_1^2 = qZ_0^2. \tag{8.10.10}$$

It is easy to prove that under the condition

$$C_1\rho_0^2\rho_1^2 + C_2 = 0 \tag{8.10.11}$$

the expansion of the left hand side of eq. (8.10.10) has no negative degrees of ζ (i.e. ζ^{-2}, ζ^{-1}) and the coefficients associated with ζ^3 and ζ^4 vanish. We arrive at the relationship

$$C_1\left[\zeta^2\left(2\lambda+2\mu-2-\mu^2\right)+2\zeta\left(\lambda\mu-4\lambda+\mu\right)+\left(2\lambda-2\lambda^2+2\lambda\mu-\mu^2\right)\right] \\ +2C_0\left(\lambda-\zeta^2\right)=q\left(\rho_0^4-2\rho_0^2\zeta+\zeta^2\right) \qquad \left(\lambda=\rho_0^2\rho_1^2, \quad \mu=\rho_0^2+\rho_1^2\right) \tag{8.10.12}$$

and to the system of three equations for two unknowns C_1, C_0

$$\left.\begin{aligned} &C_1\left(2\lambda+2\mu-2-\mu^2\right)-2C_0-q=0, \\ &C_1\left(\lambda\mu-4\lambda+\mu\right)+q\rho_0^2=0, \\ &C_1\left(2\lambda-2\lambda^2+2\lambda\mu-\mu^2\right)+2C_0\lambda-q\rho_0^4=0. \end{aligned}\right\} \tag{8.10.13}$$

These equations have solutions since the determinant of this system is equal to zero. Using formulae (8.10.9) and (8.9.7) we can determine functions $\Phi(\zeta), \Psi(\zeta)$. The stresses are obtained by Kolosov-Muskhelishvili's formulae and the displacement vector is single-valued.

Part IV

Basic relationships in the nonlinear theory of elasticity

8

Constitutive laws for nonlinear elastic bodies

8.1 The strain energy

8.1.1 Ideally elastic body

Two groups of quantities were introduced for consideration in Chapters 1 and 2. The first group determines the stress tensor and serves to describe the state of stress due to the external mass and surface forces, whilst the quantities of the second group (the strain measures and the strain tensors) describe the change in the geometry of the objects (line, surface, volume) under deformation of the medium. No assumption about the relationships between the quantities of these groups, that is the constitutive laws, have been made. For this reason, the previous analysis is applicable to any medium. However it is not sufficient for describing the behaviour of a particular medium.

Establishing the constitutive laws, which are the dependences of the stress tensor on the strain tensors and the strain rate with account of the thermodynamic parameters and the influence of deformation history is the subject of rheology. As already mentioned in Subsections 3.1.1 and 3.1.3, the present book is concerned only with a single rheological model which is the ideally elastic solid. The fundamental property of this solid is that the processes in the solid are reversible. One can suggest two ways of defining this property. The first way is the ability to recuperate the shape of the solid and the second way is the loss-free return of the energy obtained by the solid under the deformation. It is assumed that the solid was in a certain natural state and it is subjected to such a "slow" loading that the solid is in equilibrium at any time instant, i.e. the dynamic effects are ignored. The accompanied deformation completely disappears and the solid returns to its initial shape under a slow unloading. The second way of constructing the model is that in terms of the energy, namely the work of the loads is accumulated in the solid in the form of the strain energy and (under the unloading) the solid returns the energy without loss under the unloading.

Mathematically, the first way consists in prescribing the law of the stress-strain relation. The second way deals with prescribing the potential energy in terms of the strain components which, in turn, determines the stress tensor in terms of the strain components.

The "elastic body" is characterised by a single-valued dependence of the stress tensor $\hat{T}$ on the strain measure $\hat{g}^{\times}$ (or $\hat{M}$). The requirement for the existence of the strain energy separates the "hyperelastic" solids (Truesdell) which are referred as to the ideally elastic solids in the present book. In what follows we consider only ideally elastic bodies and for this reason, the word "elastic" is often omitted.

The applicability of the model for the ideally elastic solid to real bodies must be confirmed experimentally. However it is feasible to prove only the results obtained theoretically from the constitutive law. The difficult problem of establishing the constitutive law for the material "should be transferred to the experimentalists as late as possible" (Signorini). It is also necessary to add that only strains can be directly measured whereas stresses can only be judged in terms of their integral characteristics, such as the tension force, the torque, the pressure of the specimens surface etc. This is why the tests are predominantly carried out on specimen of sufficiently simple geometric forms (a prismatic rod, a thin-walled cylindric tube) in problems in which the stress components are statically determinant. Experimental knowledge is concerned only with one-, two- and very seldom and fragmentary three-dimensional manifolds of the six-dimensional space of components of the stress tensor. More often than not, this information is not sufficient to choose the only constitutive law. One is satisfied with a particular constitutive law if this law is confirmed by the test data in the required range of the measured parameters.

8.1.2 The strain potentials

Let us consider the unit volume of the ideally elastic body in its initial state (v–volume). According to the first law of thermodynamics the specific elementary work of the external forces $\delta' A_{(e)}$ plus the supplied elementary amount of the heat $\delta' Q$ (expressed in terms of the mechanical units) is equal to the increment in the internal energy δE, see Subsection 3.2.1

$$\delta E = \delta' A_{(e)} + \delta' Q. \tag{1.2.1}$$

Adding the definition of the entropy under the reversible equilibrium process

$$\delta S = \frac{\delta' Q}{\Theta}, \tag{1.2.2}$$

where Θ denotes the absolute temperature and referring to the expression for the elementary work, eq. (3.5.8) of Chapter 1, we have

$$\delta E = \sqrt{\frac{G}{g}} \tilde{t}^{sq} \delta \mathcal{E}_{sq} + \Theta \delta S. \tag{1.2.3}$$

Considering now the internal energy as a function of seven independent variables, which are the six components of the strain tensor $\widehat{\mathcal{E}}$ and the entropy S, we have

$$\delta E = \frac{\partial E}{\partial \mathcal{E}_{st}} \delta \mathcal{E}_{st} + \frac{\partial E}{\partial S} \delta S; \quad \left(\frac{\partial E}{\partial \mathcal{E}_{st}} - \sqrt{\frac{G}{g}} \tilde{t}^{st} \right) \delta \mathcal{E}_{st} + \left(\frac{\partial E}{\partial S} - \Theta \right) \delta S = 0$$

and due to the independence of variations $\delta \mathcal{E}_{st}$ and δS

$$\tilde{t}^{st} = \sqrt{\frac{g}{G}} \left(\frac{\partial E}{\partial \mathcal{E}_{st}} \right)_S ; \quad \Theta = \left(\frac{\partial E}{\partial S} \right)_{\mathcal{E}_{st}} . \tag{1.2.4}$$

Let us recall that $\tilde{t}^{st}$ denotes the contravariant components of the stress tensor $\hat{T}$ in V–volume and $\mathcal{E}_{st}$ denotes the covariant components of $\widehat{\mathcal{E}}$ in the metric of v– volume. As adopted in thermodynamics the subscripts indicate the variables which are kept constant under differentiation. Thus, prescribing the internal energy $E(\mathcal{E}_{11}, \ldots, \mathcal{E}_{23}, S)$ determines the constitutive law for the medium which is the dependence of the components of the stress tensor and the temperature on the strains and the entropy. The constitutive law is given by the expression for the free energy F in terms of the temperature and the strain components. By virtue of eq. (1.2.3) the variation of this thermodynamic potential, eq. (2.2.3) of Chapter 3, is given by

$$\delta F = \delta E - \Theta \delta S - S \delta \Theta = \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta \mathcal{E}_{st} - S \delta \Theta = \frac{\partial F}{\partial \mathcal{E}_{st}} \delta \mathcal{E}_{st} + \frac{\partial F}{\partial \Theta} \delta \Theta, \tag{1.2.5}$$

so that

$$\tilde{t}^{st} = \sqrt{\frac{g}{G}} \left(\frac{\partial E}{\partial \mathcal{E}_{st}} \right)_\Theta , \quad S = - \left(\frac{\partial E}{\partial \Theta} \right)_{\mathcal{E}_{st}} . \tag{1.2.6}$$

The heat flux is absent under the adiabatic process of deformation, i.e. $\delta' Q = 0$, thus, by virtue of eq. (1.2.1)

$$\delta E = \delta' A_{(e)} = \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta \mathcal{E}_{st}. \tag{1.2.7}$$

Under the isothermal process $\delta\Theta = 0$ and it is the free energy whose complete differential is equal to the specific elementary work

$$\delta F = \delta' A_{(e)} = \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta \mathcal{E}_{st}. \tag{1.2.8}$$

Thus, under both processes one can introduce a quantity referred to as the specific strain energy. This quantity denoted by A depends upon the components of the strain tensor and its variation is as follows

$$\delta A = \delta' A_{(e)} = \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta \mathcal{E}_{st} = \frac{1}{2} \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta G_{st}. \tag{1.2.9}$$

Having introduced the same denotation for the specific strain energy in two processes we have to remember that we imply two different quantities. In the adiabatic and isothermal process A is identified with the internal and free energy respectively. The constitutive laws defined by both processes are formally coincident, however they contain the different moduli of elasticity ("adiabatic" and "isothermal" ones in the first and second process respectively), see Subsection 3.2.3.

The expression for the contravariant components of the strain tensor in terms of the specific strain energy is written down as follows

$$\tilde{t}^{st} = \sqrt{\frac{g}{G}} \frac{\partial A}{\partial \mathcal{E}_{st}} = 2 \sqrt{\frac{g}{G}} \frac{\partial A}{\partial G_{st}} \tag{1.2.10}$$

or in the equivalent form

$$\tilde{t}^{st} = \frac{1}{2} \sqrt{\frac{g}{G}} \left(\frac{\partial A}{\partial \mathcal{E}_{st}} + \frac{\partial A}{\partial \mathcal{E}_{ts}} \right) = \sqrt{\frac{g}{G}} \left(\frac{\partial A}{\partial G_{st}} + \frac{\partial A}{\partial G_{ts}} \right). \tag{1.2.11}$$

This notion prevents possible mistakes since $\mathcal{E}_{st}$ and $\mathcal{E}_{ts}$ (G_{st} and G_{ts}) are taken to be different.

The strain energy of the body is denoted by a and is equal to the integral of the specific strain energy over $v-$volume

$$a = \iiint\limits_{v} A d\tau_0. \tag{1.2.12}$$

Its variation is as follows

$$\delta a = \delta \iiint\limits_{v} A d\tau_0 = \iiint\limits_{v} \delta A d\tau_0, \tag{1.2.13}$$

and the sign of variation can be placed under the sign of the integral since $v-$volume is not varied. This provides one with the possibility of identifying δa with the elementary work of the external mass and surface forces. This explains the necessity of relating the specific strain energy to the unit of the initial volume rather than $V-$volume.

8.1.3 Homogeneous isotropic ideally elastic body

In the initial state the medium is assumed to be homogeneous, isotropic and have a constant density ρ_0. This ensures that the specific strain energy is independent of both the orientation of the basis and the coordinates of the particle.

The specific strain energy is thus a function of only three principal invariants of the strain tensor or Cauchy's strain measures (see Subsection 2.3.3)

$$A = A\left(I_1\left(\hat{\mathcal{E}}\right), I_2\left(\hat{\mathcal{E}}\right), I_3\left(\hat{\mathcal{E}}\right)\right) \tag{1.3.1}$$

or

$$A = A\left(I_1\left(\hat{G}^{\times}\right), I_2\left(\hat{G}^{\times}\right), I_3\left(\hat{G}^{\times}\right)\right). \tag{1.3.2}$$

The second form is often more preferable as using the strain measures simplifies the formulae. Equations (5.2.3)-(5.2.5) of Chapter 2 relating the invariants of the strain measures of Cauchy and Almansi as well as the inverse tensors allows us to consider A as a function of the invariants of Almansi's measure or Almansi's strain tensor

$$A = A\left(I_1\left(\hat{g}^{\times}\right), I_2\left(\hat{g}^{\times}\right), I_3\left(\hat{g}^{\times}\right)\right), \quad A = A\left(I_1\left(\hat{\tilde{\mathcal{E}}}\right), I_2\left(\hat{\tilde{\mathcal{E}}}\right), I_3\left(\hat{\tilde{\mathcal{E}}}\right)\right). \tag{1.3.3}$$

The representations for the specific strain energy in terms of the principal invariants of tensors $\hat{M}, \hat{m}$ do not differ from those in eqs. (1.3.2) and (1.3.3) because $I_k\left(\hat{G}^{\times}\right) = I_k\left(\hat{M}\right)$ and $I_k\left(\hat{g}^{\times}\right) = I_k\left(\hat{m}\right)$. Clearly, this does not mean that the constitutive laws in terms of, say, tensors $\hat{G}^{\times}$ and $\hat{M}$ are also coincident since these are different tensors.

In what follows we use the following notation

$$\left.\begin{array}{ll} I_k\left(\hat{G}^{\times}\right) = I_k\left(\hat{M}\right) = I_k, & I_k\left(\hat{\mathcal{E}}\right) = j_k, \\ I_k\left(\hat{g}^{\times}\right) = I_k\left(\hat{m}\right) = I'_k, & I_k\left(\hat{\tilde{\mathcal{E}}}\right) = j'_k. \end{array}\right\} \tag{1.3.4}$$

The specific strain energy can also be presented as a function of three independent invariants (not necessarily the principal invariants) of a strain measure, for instance, the first principal invariant, the second invariant of its deviator and a function of the third invariant. Clearly, the principal values of the strain measure, the principal extensions etc. are the invariants, too.

8.2 The constitutive law for the isotropic ideally-elastic body

8.2.1 General form for the constitutive law

The variation of the specific strain energy is equal to the specific elementary work of the external forces and is given by eq. (3.6.4) of Chapter 1

$$\delta A = \frac{1}{2}\sqrt{\frac{G}{g}}\hat{Q}\cdot\cdot\,\delta\hat{G}^{\times}. \tag{2.1.1}$$

By this equation we introduce the "energetic" stress tensor $\hat{Q}$ which is the tensor whose contravariant components in the basis $\mathbf{r}_s$ of the initial $v-$volume are equal to the contravariant components $\hat{t}^{st}$ of the stress tensor $\hat{T}$ in the basis of $V-$volume. Then by eq. (3.6.4) of Chapter 1 we have

$$\hat{T} = \nabla\mathbf{R}^{*}\cdot\hat{Q}\cdot\nabla\mathbf{R}. \tag{2.1.2}$$

The invariant definition of the gradient of the scalar invariant in terms of the tensor is given by relationship (A.12.7)

$$\delta A = \frac{\partial A}{\partial \hat{G}^{\times}}\cdot\cdot\,\delta\hat{G}^{\times} \quad \left(\delta\hat{G}^{\times} = \delta\hat{G}^{\times *}\right), \tag{2.1.3}$$

thus we obtain

$$\hat{Q} = 2\sqrt{\frac{g}{G}}\frac{\partial A}{\partial \hat{G}^{\times}}. \tag{2.1.4}$$

Considering A as being the prescribed function of the principal invariants $I_k\left(\hat{G}^{\times}\right)$ and referring to eqs. (A.12.12), (A.12.13) we arrive at the relationships

$$\hat{Q} = 2\sqrt{\frac{g}{G}}\left[\left(\frac{\partial A}{\partial I_1} + I_1\frac{\partial A}{\partial I_2}\right)\hat{g} - \frac{\partial A}{\partial I_2}\hat{G}^{\times} + I_3\frac{\partial A}{\partial I_3}\hat{G}^{\times^{-1}}\right], \tag{2.1.5}$$

$$\hat{Q} = 2\sqrt{\frac{g}{G}}\left[\left(\frac{\partial A}{\partial I_1} + I_1\frac{\partial A}{\partial I_2} + I_2\frac{\partial A}{\partial I_3}\right)\hat{g} - \left(\frac{\partial A}{\partial I_2} + I_1\frac{\partial A}{\partial I_3}\right)\hat{G}^{\times} + \frac{\partial A}{\partial I_3}\hat{G}^{\times^{2}}\right], \tag{2.1.6}$$

where $\hat{g}$ denotes the unit tensor in $v-$volume.

Introducing the notion

$$\overset{0}{c} = \frac{\partial A}{\partial I_1} + I_1 \frac{\partial A}{\partial I_2}, \quad \overset{1}{c} = \frac{\partial A}{\partial I_2}, \quad \overset{-1}{c} = I_3 \frac{\partial A}{\partial I_3}, \tag{2.1.7}$$

$$\overset{0}{d} = \frac{\partial A}{\partial I_1} + I_1 \frac{\partial A}{\partial I_2} + I_2 \frac{\partial A}{\partial I_3}, \quad \overset{1}{d} = \frac{\partial A}{\partial I_2} + I_1 \frac{\partial A}{\partial I_3}, \quad \overset{2}{d} = \frac{\partial A}{\partial I_3}. \tag{2.1.8}$$

and noting that

$$\hat{Q} = \tilde{t}^{st} \mathbf{r}_s \mathbf{r}_t, \; \hat{G}^\times = G_{mn} \mathbf{r}^m \mathbf{r}^n, \; \hat{G}^{\times^2} = G_{mn} G_{rq} g^{nr} \mathbf{r}^m \mathbf{r}^q, \; \hat{G}^{\times^{-1}} = G^{st} \mathbf{r}_s \mathbf{r}_t,$$

we arrive at the following notion for the constitutive law

$$\tilde{t}^{st} = 2\sqrt{\frac{g}{G}} \left(\overset{0}{c} g^{st} - \overset{1}{c} G_{mn} g^{sm} g^{tn} + \overset{-1}{c} G^{st} \right), \tag{2.1.9}$$

$$\tilde{t}^{st} = 2\sqrt{\frac{g}{G}} \left(\overset{0}{d} g^{st} - \overset{1}{d} G_{mn} g^{sm} g^{tn} + \overset{2}{d} G_{rq} g^{nr} g^{ms} g^{qt} \right). \tag{2.1.10}$$

Here the contravariant components of the stress tensor are expressed in terms of the components of the metric tensors of $v-$ and $V-$volumes. The values $\overset{k}{c}, \overset{k}{d}$ can be termed as the generalised moduli of elasticity.

Taking the Cartesian coordinates of the particle in the initial state as the material coordinates ($q^s = a_s$) we arrive at the constitutive law (2.1.9) in the form

$$\left.\begin{aligned} \tilde{t}^{11} &= \frac{2}{\sqrt{G}} \left[\overset{0}{c} - \overset{1}{c} G_{11} + \frac{\partial A}{\partial I_3} \left(G_{22} G_{33} - G_{32}^2 \right) \right], \\ \tilde{t}^{12} &= \frac{2}{\sqrt{G}} \left[- \overset{1}{c} G_{12} + \frac{\partial A}{\partial I_3} \left(G_{23} G_{31} - G_{33} G_{21} \right) \right] \end{aligned} \right\} \tag{2.1.11}$$

etc.

In the case of an incompressible medium, that is, a medium preserving volume under deformation, we have

$$I_3 \left(\hat{G}^\times \right) = \frac{G}{g} = 1, \quad A(I_1, I_2, 1) = A(I_1, I_2) \tag{2.1.12}$$

and by eqs. (A.12.7), (A.12.8)

$$\delta I_3 = \frac{\partial I_3}{\partial \hat{G}^\times} \cdot\cdot \delta \hat{G}^\times = I_3 \left(\hat{G}^\times \right) \hat{G}^{\times^{-1}} \cdot\cdot \delta \hat{G}^\times = 0. \tag{2.1.13}$$

We introduce into consideration a Lagrange multiplier denoted by $\overset{-1}{c} / I_3$ and rewrite the relationships (2.1.1), (2.1.2) in the form

$$\delta A = \left(\frac{\partial A}{\partial \hat{G}^\times} + \overset{-1}{c} \hat{G}^{\times^{-1}} \right) \cdot\cdot \delta \hat{G}^\times = \frac{1}{2} \hat{Q} \cdot\cdot \delta \hat{G}^\times. \tag{2.1.14}$$

This yields

$$\hat{Q} = 2\left(\frac{\partial A}{\partial \hat{G}^{\times}} + \overset{1}{c}\,\hat{G}^{\times^{-1}}\right). \tag{2.1.15}$$

Since A is independent of I_3 we arrive by means of eq. (A.12.12) at the familiar representation of the energetic stress tensor

$$\hat{Q} = 2\left(\overset{0}{c}\,\hat{g} - \overset{1}{c}\,\hat{G}^{\times} + \overset{-1}{c}\,\hat{G}^{\times^{-1}}\right). \tag{2.1.16}$$

Now there is no need to understand $\overset{-1}{c}$ as a "generalised modulus of elasticity". For the incompressible material this quantity (the Lagrange multiplier) is determined from the static equations completed by the incompressibility condition (2.1.12).

8.2.2 The initial and the natural states

Let us recall that the displacement of the particle is measured from its position in v−volume referred to as the initial volume. In this volume

$$\mathbf{u} = 0, \quad \mathbf{R} = \mathbf{r}, \quad \hat{G} = \hat{g}, \quad I_1 = I_2 = 3, \quad I_3 = 1, \tag{2.2.1}$$

so that

$$G_{mn} g^{mt} g^{ns} = g_n^t g^{ns} = g^{st}, \quad G^{st} = g^{st} \tag{2.2.2}$$

and the stresses in the initial state are given by the equality

$$\tilde{t}_0^{st} = 2\left(\overset{0}{c} - \overset{1}{c} + \overset{-1}{c}\right) G^{st} = 2\left(\frac{\partial A}{\partial I_1} + 2\frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3}\right)_0 G^{st}. \tag{2.2.3}$$

The subscript zero indicates that the quantity in the parentheses is calculated for the values I_k corresponding to the initial state. In this state we have a spherical stress tensor representing an all-round uniform compression or tension, see eq. (3.5.9) of Chapter 1. Only this state can be taken as being the initial one under a single assumption of the isotropy resulting in the constitutive law (2.1.9). Hence, the medium which is isotropic in the natural state remains isotropic while stressed only in the case of an all-round uniform compression or tension. For the initial states with the stresses different from the all-round uniform compression or tension the constitutive law (2.1.5) does not take place. Such states produce an anisotropy of the properties of the medium.

8.2.3 Relation between the generalised moduli under the different initial states

We consider two states of the elastic body. The first one (v_0−volume) is the natural state whereas the second one ($v^{\times}$−volume) is obtained from the first one by means of a similarity transformation with the factor K, that is

$$\mathbf{R} = \mathbf{r}^{\times} = K\mathbf{r}, \quad x_s = a_s^{\times} = K a_s. \tag{2.3.1}$$

The corresponding densities are related as follows

$$\rho_0 d\tau_0 = \rho^{\times} d\tau^{\times}, \quad \frac{\rho_0}{\rho^{\times}} = \frac{d\tau^{\times}}{d\tau_0} = K^3. \tag{2.3.2}$$

The strain measures for the first and second initial states and their invariants are related by eqs. (5.6.2), (5.6.3) of Chapter 2. For these states the strain energy in $V-$volume is denoted by a and $a^{\times}$ respectively. Clearly, $a = a^{\times}$ since the numerical value of the strain energy $a^{\times}$ is the sum of the strain energies accumulated under the passages $v_0 \to v^{\times}, v^{\times} \to V$ and can not differ from that under the passages $v_0 \to V$. Referring thus to eq. (1.2.12) we have

$$\begin{aligned} &\iiint\limits_{v_0} A(I_1, I_2, I_3)\, d\tau_0 = \iiint\limits_{v^{\times}} A^{\times}\left(I_1^{\times}, I_2^{\times}, I_3^{\times}\right) d\tau^{\times} = \\ &= \iiint\limits_{v_0} A^{\times}\left(I_1^{\times}, I_2^{\times}, I_3^{\times}\right) K^3 d\tau_0 = \iiint\limits_{v_0} A^{\times}\left(\frac{I_1}{K^2}, \frac{I_2}{K^4}, \frac{I_3}{K^6}\right) K^3 d\tau_0 \end{aligned} \tag{2.3.3}$$

and inasmuch as volume v_0 can be taken arbitrarily we have

$$A(I_1, I_2, I_3) = K^3 A^{\times}\left(\frac{I_1}{K^2}, \frac{I_2}{K^4}, \frac{I_3}{K^6}\right). \tag{2.3.4}$$

By virtue of eq. (2.3.4) we have

$$\frac{\partial A}{\partial I_1} = K\frac{\partial A^{\times}}{\partial I_1^{\times}}, \quad \frac{\partial A}{\partial I_2} = \frac{1}{K}\frac{\partial A^{\times}}{\partial I_2^{\times}}, \quad \frac{\partial A}{\partial I_2} = \frac{1}{K^3}\frac{\partial A^{\times}}{\partial I_3^{\times}}, \tag{2.3.5}$$

so that by eq. (2.1.4)

$$\left.\begin{aligned} &\overset{0}{c} = K\left(\frac{\partial A^{\times}}{\partial I_1^{\times}} + I_1^{\times}\frac{\partial A^{\times}}{\partial I_2^{\times}}\right) = K\, \overset{0}{c}{}^{\times}, \\ &\overset{1}{c} = \frac{1}{K}\frac{\partial A^{\times}}{\partial I_2^{\times}} = \frac{1}{K}\, \overset{1}{c}{}^{\times}, \quad \overset{-1}{c} = K^6 I_3^{\times}\frac{\partial A^{\times}}{\partial I_3^{\times}}\frac{1}{K^3} = K^3\, \overset{-1}{c}{}^{\times}, \end{aligned}\right\} \tag{2.3.6}$$

where $\overset{0}{c}, \overset{1}{c}, \overset{-1}{c}$ denote the generalised moduli of elasticity if the initial state is the natural one and $\overset{0}{c}{}^{\times}, \overset{1}{c}{}^{\times}, \overset{-1}{c}{}^{\times}$ denote the generalised moduli of elasticity if the initial state is $v^{\times}-$volume

$$\overset{0}{c}{}^{\times} = \frac{\partial A^{\times}}{\partial I_1^{\times}} + I_1^{\times}\frac{\partial A^{\times}}{\partial I_2^{\times}}, \quad \overset{1}{c}{}^{\times} = \frac{\partial A^{\times}}{\partial I_2^{\times}}, \quad \overset{-1}{c}{}^{\times} = \frac{\partial A^{\times}}{\partial I_3^{\times}} I_3^{\times}. \tag{2.3.7}$$

For this initial state the constitutive law is written down as follows

$$\tilde{t}^{\times^{st}} = \frac{2}{\sqrt{I_3^{\times}}}\left(\overset{0}{c}{}^{\times} g^{st} - \overset{1}{c}{}^{\times} g^{sk} g^{tq} G^{\times}_{kq} + \overset{-1}{c}{}^{\times} G^{\times^{st}}\right). \tag{2.3.8}$$

Replacing here $G^{\times}_{kq}, G^{\times^{st}}, I_3^{\times}$ respectively by $K^{-2}G_{kq}, K^2G^{st}, K^{-6}I_3$ and using eq. (2.3.6) we obtain

$$\begin{aligned}\tilde{t}^{\times^{st}} &= \frac{2K^3}{\sqrt{I_3}}\left(\frac{\overset{0}{c}}{K}g^{st} - K\,\overset{1}{c}\,g^{sk}g^{tq}\frac{1}{K^2}G_{kq} + \frac{\overset{-1}{c}}{K^3}K^2G^{st}\right)\\ &= \frac{2}{\sqrt{I_3}}K^2\left(\overset{0}{c}\,g^{st} - \overset{1}{c}\,g^{sk}g^{tq}G_{kq} + \overset{-1}{c}\,G^{st}\right),\end{aligned}$$

or

$$\tilde{t}^{\times^{st}} = K^2\tilde{t}^{st}. \tag{2.3.9}$$

For such a relation between the contravariant components, the stress tensor remains invariant, i.e. it is independent of the choice of the initial state

$$\hat{T}^{\times} = \tilde{t}^{\times^{st}}\mathbf{R}^{\times}_s\mathbf{R}^{\times}_t = K^2\tilde{t}^{st}\frac{1}{K^2}\mathbf{R}_s\mathbf{R}_t = \hat{T}, \tag{2.3.10}$$

which is required.

8.2.4 Representation of the stress tensor

The transformation from the energetic stress tensor to the stress tensor is carried out by means of the relationships (2.1.2). One uses the relationships (3.3.2), (3.3.7), (4.1.5), (4.1.6) of Chapter 2

$$\begin{gathered}\nabla\mathbf{R}^*\cdot\hat{g}\cdot\nabla\mathbf{R} = \mathbf{R}_s\mathbf{r}^s\cdot\mathbf{r}_t\mathbf{r}^t\cdot\mathbf{r}^q\mathbf{R}_q = g^{tq}\mathbf{R}_t\mathbf{R}_q = \hat{M},\\ \nabla\mathbf{R}^*\cdot\hat{G}^{\times}\cdot\nabla\mathbf{R} = \nabla\mathbf{R}^*\cdot\nabla\mathbf{R}\cdot\nabla\mathbf{R}^*\cdot\nabla\mathbf{R} = \hat{M}^2,\\ \nabla\mathbf{R}^*\cdot\hat{G}^{\times^{-1}}\cdot\nabla\mathbf{R} = \nabla\mathbf{R}^*\cdot(\nabla\mathbf{R}\cdot\nabla\mathbf{R}^*)^{-1}\cdot\nabla\mathbf{R} =\\ = \nabla\mathbf{R}^*\cdot(\nabla\mathbf{R}^*)^{-1}\cdot(\nabla\mathbf{R})^{-1}\cdot\nabla\mathbf{R} = \hat{G},\\ \nabla\mathbf{R}^*\cdot\hat{G}^{\times^2}\cdot\nabla\mathbf{R} = \hat{M}^3 = I_1\hat{M}^2 - I_2\hat{M} + I_3\hat{G}.\end{gathered}$$

Utilising eq. (2.1.5) or (2.1.6) we arrive at the Finger form for the constitutive law (1894)

$$\hat{T} = 2\sqrt{\frac{g}{G}}\left(\overset{0}{c}\,\hat{M} - \overset{1}{c}\,\hat{M}^2 + \overset{-1}{c}\,\hat{G}\right). \tag{2.4.1}$$

It contains the tensors determined with respect to the vector basis of V–volume where $\hat{G}$ denotes the unit tensor in this basis (not to be confused with $\hat{G}^{\times}$).

It is clear that instead of tensor $\hat{M}$ one can use the inverse tensor $\hat{M} = \hat{g}^{\times}$. Formulae (A.10.12), (A.10.13) yield the expressions for the integer negative degrees of the tensor and we arrive at the following notion for the constitutive law in the form

$$\hat{T} = 2\sqrt{\frac{g}{G}}\left(\overset{0}{e}\,\hat{G} - \overset{1}{e}\,\hat{g}^{\times} + \overset{2}{e}\,\hat{g}^{\times 2}\right). \tag{2.4.2}$$

Here the generalised moduli are given by

$$\overset{0}{e} = I_1\frac{\partial A}{\partial I_1} + I_2\frac{\partial A}{\partial I_2} + I_3\frac{\partial A}{\partial I_3}, \quad \overset{1}{e} = I_2\frac{\partial A}{\partial I_1} + I_3\frac{\partial A}{\partial I_2}, \quad \overset{2}{e} = I_3\frac{\partial A}{\partial I_1}. \tag{2.4.3}$$

They can also be expressed in terms of the invariants $I_k' = I_k\left(\hat{g}^{\times}\right)$. Turning to eq. (5.2.5) of Chapter 2 we obtain

$$\overset{0}{e} = -I_3'\frac{\partial A}{\partial I_3'}, \quad \overset{1}{e} = \frac{\partial A}{\partial I_1'} + I_1'\frac{\partial A}{\partial I_2'}, \quad \overset{2}{e} = \frac{\partial A}{\partial I_2'}. \tag{2.4.4}$$

Finger's constitutive law operates with tensors $\hat{M}$ and $\hat{g}$. Using eqs. (A.10.12), (A.10.14) and (A.10.15) we have

$$\hat{M}^2 = \hat{M}\cdot\hat{g}^{\times^{-1}} = \hat{M}\cdot\frac{1}{I_3'}\left(\hat{g}^{\times^2} - I_1'\hat{g}^{\times} + I_2'\hat{G}\right) = I_3\hat{g}^{\times} - I_2\hat{G} + I_1\hat{M}.$$

Inserting into eq. (2.4.1) and taking into account eq. (2.1.7) leads to the following relationship

$$\hat{T} = \frac{2}{\sqrt{I_3}}\left[\frac{\partial A}{\partial I_1}\hat{M} - I_3\frac{\partial A}{\partial I_2}\hat{g}^{\times} + \left(I_2\frac{\partial A}{\partial I_2} + I_3\frac{\partial A}{\partial I_3}\right)\hat{G}\right]. \tag{2.4.5}$$

Some authors introduce into consideration the tensor whose principal values (and thus the principal invariants) are equal to the principal values of the stress tensor $\hat{T}$, however its axes are coincident with the principal axes of the strain measure $\hat{G}^{\times}$. Noticing that the tensor is coaxial with tensor $\hat{g}^{\times}$ rather than $\hat{G}^{\times}$ and referring to eq. (5.3.3) of Chapter 2 we can define the "rotated" stress tensor $\hat{T}^{\backslash}$ by the relationship

$$\hat{T}^{\backslash} = \hat{A}\cdot\hat{T}\cdot\hat{A}^{*} \tag{2.4.6}$$

or due to eq. (2.4.1)

$$\hat{T}^{\backslash} = \frac{2}{\sqrt{I_3}}\left(\overset{0}{c}\,\hat{G}^{\times} - \overset{1}{c}\,\hat{G}^{\times 2} + \overset{-1}{c}\,\hat{g}\right), \tag{2.4.7}$$

as $\hat{A}\cdot\hat{G}\cdot\hat{A}^{*} = \hat{A}\cdot\hat{A}^{*} = \hat{g}$. By eq. (2.4.1) we also have

$$\hat{G}^{\times^{-1}}\cdot\hat{T}^{\backslash} = \hat{Q}, \quad \hat{T}^{\backslash\times} = \hat{G}^{\times}\cdot\hat{Q}. \tag{2.4.8}$$

8.2.5 *Expressing the constitutive law in terms of the strain tensors*

Transformation from the strain measures to the strain tensors are performed by means of relationships (3.6.3), (4.3.3) of Chapter 2

$$\widehat{\mathcal{E}} = \frac{1}{2}\left(\hat{G}^{\times} - \hat{g}\right), \quad \hat{\tilde{\mathcal{E}}} = \frac{1}{2}\left(\hat{G} - \hat{g}^{\times}\right), \tag{2.5.1}$$

and the formulae relating their principal invariants are listed in Subsection 2.5.4. From these formulae we obtain

$$\frac{\partial A}{\partial I_1} = \frac{1}{2}\frac{\partial A}{\partial j_1} - \frac{1}{2}\frac{\partial A}{\partial j_2} + \frac{1}{8}\frac{\partial A}{\partial j_3}, \quad \frac{\partial A}{\partial I_2} = \frac{1}{4}\frac{\partial A}{\partial j_2} - \frac{1}{8}\frac{\partial A}{\partial j_3}, \quad \frac{\partial A}{\partial I_3} = \frac{1}{8}\frac{\partial A}{\partial j_3}. \tag{2.5.2}$$

Using these relationships and substituting

$$\hat{G}^{\times 2} = 4\widehat{\mathcal{E}}^2 + 4\widehat{\mathcal{E}} + \hat{g}$$

one can represent eq. (2.1.5) for the energy stress tensor in the form

$$\sqrt{\frac{G}{g}}\hat{Q} = \overset{0}{d'}\,\hat{g} - \overset{1}{d'}\,\widehat{\mathcal{E}} + \overset{2}{d'}\,\widehat{\mathcal{E}}^2, \tag{2.5.3}$$

where the generalised moduli $\overset{k}{d'}$ expressed in terms of the principal invariants j_k of tensor $\widehat{\mathcal{E}}$ have the structure given by eq. (2.1.8)

$$\overset{0}{d'} = \frac{\partial A}{\partial j_1} + j_1\frac{\partial A}{\partial j_2} + j_2\frac{\partial A}{\partial j_3}, \quad \overset{1}{d'} = \frac{\partial A}{\partial j_2} + j_1\frac{\partial A}{\partial j_3}, \quad \overset{2}{d'} = \frac{\partial A}{\partial j_3}. \tag{2.5.4}$$

The representation for the "rotated" stress tensor $\hat{T}^{\backslash}$ is constructed by analogy. Instead of the third principal invariant it is convenient to introduce into consideration its square root which is the ratio of the volumes in V−state and v−state

$$\Theta = \sqrt{\frac{G}{g}} = \sqrt{I_3\left(\hat{G}^{\times}\right)} = \left(1 + 2j_1 + 4j_2 + 8j_3\right)^{1/2} = D + 1, \tag{2.5.5}$$

where, according to eq. (5.5.1) of Chapter 2, D denotes the relative change in the elementary volume. Instead of eq. (2.5.2) we have

$$\frac{\partial A}{\partial I_1} = \frac{1}{2}\left(\frac{\partial A}{\partial j_1} - \frac{\partial A}{\partial j_2}\right), \quad \frac{\partial A}{\partial I_2} = \frac{1}{4}\frac{\partial A}{\partial j_2}, \quad \frac{\partial A}{\partial I_3} = \frac{1}{2\Theta}\frac{\partial A}{\partial \Theta} \tag{2.5.6}$$

and the expression for the "rotated" stress tensor (2.4.7) is written down in the form

$$\hat{T}^{\backslash} = \frac{1}{\Theta}\left(\overset{0}{m}\,\hat{g} + \overset{1}{m}\,\tilde{\mathcal{E}} + \overset{2}{m}\,\tilde{\mathcal{E}}^2\right) \tag{2.5.7}$$

with the moduli

$$\left.\begin{aligned} &\overset{0}{m}=\frac{\partial A}{\partial j_1}+j_1\frac{\partial A}{\partial j_2}+\Theta\frac{\partial A}{\partial \Theta}, \\ &\overset{1}{m}=2\left(\frac{\partial A}{\partial j_1}+j_1\frac{\partial A}{\partial j_2}-\frac{1}{2}\frac{\partial A}{\partial j_2}\right), \quad \overset{2}{m}=-2\frac{\partial A}{\partial j_2}. \end{aligned}\right\} \tag{2.5.8}$$

By analogy, the stress tensor can be represented in terms of the Almansi-Hamel stress tensor

$$\hat{T}=\Theta'\left(\overset{0}{\tilde{m}}\,\hat{G}+\overset{1}{\tilde{m}}\,\hat{\tilde{\mathcal{E}}}+\overset{2}{\tilde{m}}\,\hat{\tilde{\mathcal{E}}}^{2}\right). \tag{2.5.9}$$

Here

$$\Theta'=\sqrt{\frac{g}{G}}=\sqrt{I_3\left(\hat{g}^{\times}\right)}=\left(1-2j_1'+4j_2'-8j_3'\right)^{1/2}=(D+1)^{-1}, \tag{2.5.10}$$

and the moduli $\overset{k}{\tilde{m}}$ are given by

$$\left.\begin{aligned} &\overset{0}{\tilde{m}}=\frac{\partial A}{\partial j_1'}+j_1'\frac{\partial A}{\partial j_2'}-\Theta'\frac{\partial A}{\partial \Theta'}, \\ &\overset{1}{\tilde{m}}=-2\left(\frac{\partial A}{\partial j_1'}+j_1'\frac{\partial A}{\partial j_2'}+\frac{1}{2}\frac{\partial A}{\partial j_2'}\right), \\ &\overset{2}{\tilde{m}}=2\frac{\partial A}{\partial j_2'}. \end{aligned}\right\} \tag{2.5.11}$$

In the initial state

$$\tilde{\mathcal{E}}=0,\quad \hat{\tilde{\mathcal{E}}}=0,\quad \Theta=\Theta'=1 \quad j_k=j_k'=0 \quad (k=1,2,3),$$

and the introduced tensors reduce to the following form

$$\hat{Q}_0=\left(\frac{\partial A}{\partial j_1}\right)_0\hat{g},\ \hat{T}_0^{\backslash}=\left(\frac{\partial A}{\partial j_1}+\frac{\partial A}{\partial \Theta}\right)_0\hat{g},\ \hat{T}_0=\left(\frac{\partial A}{\partial j_1'}-\frac{\partial A}{\partial \Theta'}\right)_0\hat{G}. \tag{2.5.12}$$

In the first, second and third formula the specific strain energy A is considered as the functions of j_1, j_2, j_3; j_1, j_2, Θ and j_1', j_2', Θ' respectively. This explains the difference in the form of the formulae for $\hat{Q}_0$ and $\hat{T}_0$. If the initial state is taken as being the natural one, then

$$\left[\frac{\partial}{\partial j_1}A\left(j_1,j_2,j_3\right)\right]_0=0,\quad \left[\left(\frac{\partial}{\partial j_1}+\frac{\partial}{\partial \Theta}\right)A\left(j_1,j_2,\Theta\right)\right]_0=0 \tag{2.5.13}$$

or

$$\left[\left(\frac{\partial}{\partial j_1'}-\frac{\partial}{\partial \Theta'}\right)A\left(j_1',j_2',\Theta'\right)\right]_0=0. \tag{2.5.14}$$

In the case of the incompressible material, the moduli $\overset{0}{m}$ and $\overset{0}{\tilde{m}}$ in eqs. (2.5.7) and (2.5.11) remain undetermined. They are determined from the static equations and the condition of incompressibility

$$\Theta = 1 \quad (\Theta' = 1).$$

8.2.6 *The principal stresses*

The result of Finger's constitutive law is that the stress tensor $\hat{T}$ is coaxial with the tensor of the strain measure $\hat{M}$ (or $\hat{g}^{\times}$). Remembering that the principal values of this measure are equal to the principal values G_s of tensor $\hat{G}^{\times}$ and denoting the principal stresses by t_s we have

$$\hat{M} = G_1 \overset{1}{\tilde{\mathbf{e}}}\overset{1}{\tilde{\mathbf{e}}} + G_2 \overset{2}{\tilde{\mathbf{e}}}\overset{2}{\tilde{\mathbf{e}}} + G_3 \overset{3}{\tilde{\mathbf{e}}}\overset{3}{\tilde{\mathbf{e}}}, \quad \hat{T} = t_1 \overset{1}{\tilde{\mathbf{e}}}\overset{1}{\tilde{\mathbf{e}}} + t_2 \overset{2}{\tilde{\mathbf{e}}}\overset{2}{\tilde{\mathbf{e}}} + t_3 \overset{3}{\tilde{\mathbf{e}}}\overset{3}{\tilde{\mathbf{e}}}, \tag{2.6.1}$$

where $\overset{s}{\tilde{\mathbf{e}}}$ denotes the unit vector of the principal directions of these tensors. Taking into account eq. (2.4.1) and $\hat{G} = \sum_{s=1}^{3} \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}$ for the unit tensor of V–volume we have

$$t_s = \frac{2}{\sqrt{I_3}} \left(\overset{0}{c}\, G_s - \overset{1}{c}\, G_s^2 + \overset{-1}{c} \right). \tag{2.6.2}$$

The principal relative extensions δ_s are introduced into consideration by means of eq. (3.4.4) of Chapter 2

$$G_s = (1 + \delta_s)^2, \quad s = 1, 2, 3. \tag{2.6.3}$$

The invariants $I_k\left(\hat{G}^{\times}\right)$ are expressed in terms of these extensions in the following way

$$\left.\begin{aligned} I_1\left(\hat{G}^{\times}\right) &= (1+\delta_1)^2 + (1+\delta_2)^2 + (1+\delta_3)^2, \\ I_2\left(\hat{G}^{\times}\right) &= (1+\delta_1)^2 (1+\delta_2)^2 + (1+\delta_2)^2 (1+\delta_3)^2 + \\ &\qquad (1+\delta_3)^2 (1+\delta_1)^2, \\ I_3\left(\hat{G}^{\times}\right) &= (1+\delta_1)^2 (1+\delta_2)^2 (1+\delta_3)^2. \end{aligned}\right\} \tag{2.6.4}$$

Therefore

$$\frac{\partial I_1}{\partial \delta_s} = 2(1+\delta_s), \quad \frac{\partial I_2}{\partial \delta_s} = 2(1+\delta_s)(I_1 - G_s), \quad \frac{\partial I_3}{\partial \delta_s} = 2(1+\delta_s)\frac{I_3}{(1+\delta_s)^2}. \tag{2.6.5}$$

Recalling now the definition (2.1.7) of the generalised moduli we can set eq. (2.6.2) in the form

$$t_s = \frac{2(1+\delta_s)^2}{\sqrt{I_3}} \left[\frac{\partial A}{\partial I_1} + (I_1 - G_s)\frac{\partial A}{\partial I_2} + \frac{I_3}{(1+\delta_s)^2}\frac{\partial A}{\partial I_3} \right], \tag{2.6.6}$$

where by means of eq. (2.6.5)

$$\frac{\partial A}{\partial \delta_s} = 2(1+\delta_s)\left[\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}(I_1 - G_s) + \frac{I_3}{(1+\delta_s)^2}\frac{\partial A}{\partial I_3}\right]. \qquad (2.6.7)$$

This allows one to put the expressions for the principal stresses in the simple form

$$t_s = \frac{1+\delta_s}{\sqrt{I_3}}\frac{\partial A}{\partial \delta_s} \quad (s = 1,2,3), \qquad (2.6.8)$$

so that

$$\left.\begin{array}{l} \dfrac{\partial A}{\partial \delta_1} = (1+\delta_2)(1+\delta_3)\,t_1, \quad \dfrac{\partial A}{\partial \delta_2} = (1+\delta_3)(1+\delta_1)\,t_2, \\ \dfrac{\partial A}{\partial \delta_3} = (1+\delta_1)(1+\delta_2)\,t_3. \end{array}\right\} \qquad (2.6.9)$$

In these formulae the specific strain energy is assumed to be prescribed in terms of the principal stresses.

The values on the right hand side of formulae (2.6.9) present the principal stresses ${}_0t_s$ related to the surfaces $d\overset{s}{\overset{\circ}{o}}$ in $v-$volume, the surface $d\overset{s}{\overset{\circ}{o}}$ being defined by the principal direction $\overset{s}{\mathbf{e}}$ of tensor $\hat{G}^{\times}$. Indeed

$${}_0t_s d\overset{s}{\overset{\circ}{o}} = t_s d\overset{s}{\overset{\circ}{O}}, \quad {}_0t_s = t_s \frac{d\overset{s}{\overset{\circ}{O}}}{d\overset{s}{\overset{\circ}{o}}} = t_s \frac{\sqrt{I_3}}{1+\delta_s},$$

and eq. (2.6.9) is set in the following simple form

$$\frac{\partial A}{\partial \delta_s} = {}_0t_s. \qquad (2.6.10)$$

Returning to eq. (2.6.8) let us construct the expression for the variation of the specific strain energy

$$\delta A = \sum_{s=1}^{3} \frac{\partial A}{\partial \delta_s}\delta\delta_s = t_1(1+\delta_2)(1+\delta_3)\,\delta(1+\delta_1) +$$
$$t_2(1+\delta_3)(1+\delta_1)\,\delta(1+\delta_2) + t_3(1+\delta_1)(1+\delta_2)\,\delta(1+\delta_3). \qquad (2.6.11)$$

The right hand side presents the increment in the strain energy of a unit cube in $v-$volume, the edges of the cube being directed in $V-$volume along the principal axes $\overset{s}{\tilde{\mathbf{e}}}$ of the stress tensor $\hat{T}$. The right hand side of eq. (2.6.11) is equal to the elementary work of the normal forces

$$t_1(1+\delta_2)(1+\delta_3), \; t_2(1+\delta_3)(1+\delta_1), \; t_3(1+\delta_1)(1+\delta_2)$$

applied to the cube faces. For the incompressible material

$$\sqrt{I_3} - 1 = (1+\delta_1)(1+\delta_2)(1+\delta_3) - 1 = 0, \tag{2.6.12}$$

and eq. (2.6.11) is completed by the relation between the variations of $1+\delta_s$

$$(1+\delta_2)(1+\delta_3)\,\delta(1+\delta_1) + (1+\delta_3)(1+\delta_1)\,\delta(1+\delta_2) + \\ (1+\delta_1)(1+\delta_2)\,\delta(1+\delta_3) = 0.$$

Introducing now a Lagrange multiplier p we arrive at the expressions for the principal stresses for the incompressible material in the form

$$t_s = p + (1+\delta_s)\frac{\partial A}{\partial \delta_s}. \tag{2.6.13}$$

Let us notice that $\overset{-1}{c}$ in the constitutive law (2.4.1) also plays the role of a Lagrange multiplier.

8.2.7 The stress tensor

Formulae (2.6.1), (2.6.7) allow the stress tensor to be represented in another form

$$\hat{T} = \frac{1}{\sqrt{I_3}} \sum_{s=1}^{3} (1+\delta_s)\frac{\partial A}{\partial \delta_s} \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}. \tag{2.7.1}$$

Replacing here $\overset{s}{\tilde{\mathbf{e}}}$ by the unit vectors $\overset{s}{\mathbf{e}}$ of the strain measure $\hat{G}^{\times}$

$$\overset{s}{\tilde{\mathbf{e}}} = \hat{A}^* \cdot \overset{s}{\mathbf{e}} = \overset{s}{\mathbf{e}} \cdot \hat{A} \quad \left(\hat{A} = \overset{t}{\mathbf{e}}\overset{t}{\tilde{\mathbf{e}}}\right),$$

where $\hat{A}$ denotes the rotation tensor (see eq. (5.3.3) of Chapter 2) we obtain

$$\hat{T} = \frac{1}{\sqrt{I_3}} \hat{A}^* \cdot \sum_{s=1}^{3} (1+\delta_s)\frac{\partial A}{\partial \delta_s} \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}} \cdot \hat{A}. \tag{2.7.2}$$

This expression can be set in the invariant form by considering the specific strain energy as a function of the following three arguments

$$s_1 = \delta_1 + \delta_2 + \delta_3, \quad s_2 = \delta_1^2 + \delta_2^2 + \delta_3^2, \quad s_3 = \delta_1^3 + \delta_2^3 + \delta_3^3. \tag{2.7.3}$$

Then

$$(1+\delta_s)\frac{\partial A}{\partial \delta_s} = (1+\delta_s)\left(\frac{\partial A}{\partial s_1} + 2\delta_s \frac{\partial A}{\partial s_2} + 3\delta_s^2 \frac{\partial A}{\partial s_3}\right),$$

and by means of the identical transformations

$$\delta_s (1+\delta_s) = (1+\delta_s)^2 - (1+\delta_s),$$
$$\delta_s^2 (1+\delta_s) = (1+\delta_s)^3 - 2(1+\delta_s)^2 + (1+\delta_s)$$

we can write the above expression in the form

$$(1+\delta_s)\frac{\partial A}{\partial \delta_s} = (1+\delta_s)\left(\frac{\partial A}{\partial s_1} - 2\frac{\partial A}{\partial s_2} + 3\frac{\partial A}{\partial s_3}\right) + \\ 2(1+\delta_s)^2\left(\frac{\partial A}{\partial s_2} - 3\frac{\partial A}{\partial s_3}\right) + 3(1+\delta_s)^3\frac{\partial A}{\partial s_3}.$$

Returning to eq. (2.7.2) and recalling the expression for the strain measure $\hat{G}^\times$

$$\hat{G}^\times = \sum_{s=1}^{3} G_s \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}} = \sum_{s=1}^{3}(1+\delta_s)^2 \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}, \quad \hat{G}^{\times^{1/2}} = \sum_{s=1}^{3}(1+\delta_s)\overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}},$$

we arrive at the expression

$$\hat{T} = \frac{1}{\sqrt{I_3}}\hat{A}^* \cdot \left[\left(\frac{\partial A}{\partial s_1} - 2\frac{\partial A}{\partial s_2} + 3\frac{\partial A}{\partial s_3}\right)\hat{G}^{\times^{1/2}} + \\ 2\left(\frac{\partial A}{\partial s_2} - 3\frac{\partial A}{\partial s_3}\right)\hat{G}^\times + 3\frac{\partial A}{\partial s_3}\hat{G}^{\times^{3/2}}\right] \cdot \hat{A}, \quad (2.7.4)$$

which, by virtue of eq. (5.3.5) of Chapter 2, can be reset as follows

$$\hat{T} = \frac{1}{\sqrt{I_3}}\nabla\mathbf{R}^* \cdot \left[\left(\frac{\partial A}{\partial s_1} - 2\frac{\partial A}{\partial s_2} + 3\frac{\partial A}{\partial s_3}\right)\hat{A} + \\ 2\left(\frac{\partial A}{\partial s_2} - 3\frac{\partial A}{\partial s_3}\right)\nabla\mathbf{R} + 3\frac{\partial A}{\partial s_3}\hat{G}^{\times^{1/2}} \cdot \nabla\mathbf{R}\right]. \quad (2.7.5)$$

Instead of eq. (2.7.2) we have for the incompressible material

$$\hat{T} = \hat{A}^* \cdot \sum_{s=1}^{3}(1+\delta_s)\frac{\partial A}{\partial \delta_s}\overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}} \cdot \hat{A} + p\hat{G}, \quad (2.7.6)$$

since $\hat{G} = \sum_{s=1}^{3} \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}$ denotes the unit tensor in $V-$volume. Using the incompressibility condition (2.6.12) and relationship (5.4.2) of Chapter 2 we can express the invariant s_3 in terms of s_1 and s_2

$$\delta_1\delta_2\delta_3 = -\left[s_1 + \frac{1}{2}\left(s_1^2 - s_2\right)\right] = \frac{1}{6}\left(2s_3 - 3s_1s_2 + s_1^3\right). \quad (2.7.7)$$

For this reason, s_3 is removed from the expression for the specific strain energy. Denoting the result of this as $\tilde{A}(s_1, s_2)$ we obtain

$$\hat{T} = \nabla\mathbf{R}^* \cdot \left[\left(\frac{\partial\tilde{A}}{\partial s_1} - 2\frac{\partial\tilde{A}}{\partial s_2}\right)\hat{A} + 2\frac{\partial\tilde{A}}{\partial s_2}\nabla\mathbf{R}\right] + p\hat{G} \tag{2.7.8}$$

instead of eq. (2.7.6).

8.2.8 The stress tensor of Piola (1836) and Kirchhoff (1850)

By definition, the product of the stress tensor and the vector of the oriented surface $\mathbf{N}dO$ in V−volume is equal to the force $\mathbf{F}dO$ acting on this surface, that is

$$\mathbf{F}dO = \mathbf{N} \cdot \hat{T}dO. \tag{2.8.1}$$

Using eq. (3.5.3) of Chapter 2 for transforming to the surface $\mathbf{n}do$ in v−volume we obtain

$$\mathbf{F}dO = \sqrt{I_3}\mathbf{n} \cdot \left(\tilde{\nabla}\mathbf{r}\right)^* \cdot \hat{T}do = \mathbf{n} \cdot \hat{D}do. \tag{2.8.2}$$

Thus we introduce the following non-symmetric tensor

$$\hat{D} = \sqrt{I_3}\left(\tilde{\nabla}\mathbf{r}\right)^* \cdot \hat{T}, \quad \hat{D}^* = \sqrt{I_3}\hat{T} \cdot \tilde{\nabla}\mathbf{r}, \tag{2.8.3}$$

referred to as the Piola-Kirchhoff stress tensor. The static equation in the volume which expresses the condition of zero principal vector of the forces acting on the arbitrary volume

$$\iint\limits_O \mathbf{F}dO + \iiint\limits_V \rho\mathbf{K}d\tau = \iint\limits_o \mathbf{n} \cdot \hat{D}do + \iiint\limits_v \rho_0\mathbf{K}d\tau_0$$
$$= \iiint\limits_v \left(\nabla \cdot \hat{D} + \rho_0\mathbf{K}\right)d\tau_0 = 0$$

reduces to the form

$$\nabla \cdot \hat{D} + \rho_0\mathbf{K} = 0. \tag{2.8.4}$$

The divergence is calculated in the vector basis of the initial volume v and this simplifies the solution of a number of problems. The static equation on the surface

$$\mathbf{n} \cdot \hat{D} = \mathbf{F}\frac{dO}{do} = \mathbf{F}\left(\frac{G}{g}\mathbf{n} \cdot \hat{G}^{\times^{-1}} \cdot \mathbf{n}\right)^{1/2}, \tag{2.8.5}$$

see eq. (3.5.4) of Chapter 2, assumes knowledge of the surface of V–volume, the latter being unknown in advance.

By eqs. (2.7.5), (2.8.3) and (3.2.6) of Chapter 2, the constitutive law for Piola's tensor is set in the form

$$\hat{D} = \left(\frac{\partial A}{\partial s_1} - 2\frac{\partial A}{\partial s_2} + 3\frac{\partial A}{\partial s_3}\right)\hat{A} + 2\left(\frac{\partial A}{\partial s_2} - 3\frac{\partial A}{\partial s_3}\right)\nabla\mathbf{R} + 3\frac{\partial A}{\partial s_3}\hat{G}^{\times^{1/2}} \cdot \nabla\mathbf{R}. \tag{2.8.6}$$

This law was applied in a series of papers (John, 1956) for materials of the "harmonic type". The designation "harmonic type" comes about because the plane strain of such a solid reduces to a nonlinear boundary-value problem of the theory of harmonic functions. This material can also be referred to as a semi-linear one. It is assumed that the strain energy of this material does not depend on the invariant s_3 whilst the dependence on s_1, s_2 is written down as follows

$$A = \frac{1}{2}\lambda s_1^2 + \mu s_2 = \frac{1}{2}\left[(\lambda+\mu)s_1^2 - 4\mu(\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1)\right], \tag{2.8.7}$$

where λ, μ are some constants. By means of identifying λ, μ with Lamé constants and the principal relative elongations δ_s with the diagonal components ε_{ss} of the linear strain tensor we arrive at the familiar expression for the specific strain energy in the linear theory of elasticity. For the material of the "harmonic type" we have

$$\hat{D} = (\lambda s_1 - 2\mu)\hat{A} + 2\mu\nabla\mathbf{R}. \tag{2.8.8}$$

8.2.9 Prescribing the specific strain energy

The choice of the dependence of the specific strain energy on the invariant characteristics of deformation presents a difficult problem which can not be solved uniquely. One can indicate a number of criteria which can be satisfied by a reasonable dependence.

The property of the elastic material to accumulate the energy under deformation leads to the requirement for the specific strain energy A to be positive for any non-rigid-body displacement from the natural state in which A is assumed to be equal to zero ($A = 0$).

In the natural state

$$\delta_s = 0, \quad t_s = 0, \quad \frac{\partial A}{\partial \delta_s} = 0 \qquad (s = 1, 2, 3),$$

see eq. (2.6.8), and thus the representation of $A(\delta_1, \delta_2, \delta_3)$ by a power series in the vicinity of the natural state begins with the terms which are quadratic in δ_s

$$A = \frac{1}{2}\left(\frac{\partial^2 A}{\partial\delta_s\partial\delta_k}\right)_0 \delta_s\delta_k + \ldots \tag{2.9.1}$$

The necessary criteria for this representation is the positiveness of the quadratic form

$$\left(\frac{\partial^2 A}{\partial \delta_s \partial \delta_k}\right)_0 \xi_s \xi_k ,$$

that is the matrix $\left\| \left(\frac{\partial^2 A}{\partial \delta_s \partial \delta_k}\right)_0 \right\|$ must satisfy Sylvester's inequalities. This guarantees the positiveness of A in a certain vicinity of the natural state, however not in the whole range of values of δ_s $(-1 < \delta_s < \infty)$.

By formulae (2.6.7) and (2.6.5) we have

$$\begin{aligned}
\frac{\partial^2 A}{\partial \delta_s \partial \delta_k} = {} & \frac{1}{1+\delta_s}\frac{\partial A}{\partial \delta_s}\frac{\partial \delta_s}{\partial \delta_k} + \\
& 4(1+\delta_s)(1+\delta_k)\left\{\frac{\partial^2 A}{\partial I_1^2} + \frac{\partial^2 A}{\partial I_1 \partial I_2}(2I_1 - G_s - G_k) + \right. \\
& \frac{\partial^2 A}{\partial I_1 \partial I_3} I_3 \left[\frac{1}{(1+\delta_k)^2} + \frac{1}{(1+\delta_s)^2}\right] + \frac{\partial^2 A}{\partial I_2^2}(I_1 - G_s)(I_1 - G_k) + \\
& \left. \frac{\partial^2 A}{\partial I_2 \partial I_3} I_3 \left[\frac{I_1 - G_s}{(1+\delta_k)^2} + \frac{I_1 - G_k}{(1+\delta_s)^2}\right] + \frac{\partial^2 A}{\partial I_3^2}\frac{I_3^2}{(1+\delta_s)^2(1+\delta_k)^2}\right\} + \\
& 4(1+\delta_s)\frac{\partial A}{\partial I_2}\left[1 + \delta_k - (1+\delta_s)\frac{\partial \delta_s}{\partial \delta_k}\right] + \\
& 4\frac{\partial A}{\partial I_3} I_3 \left[\frac{1}{(1+\delta_s)(1+\delta_k)} - \frac{1}{(1+\delta_s)^2}\frac{\partial \delta_s}{\partial \delta_k}\right]
\end{aligned} \tag{2.9.2}$$

and then

$$\left(\frac{\partial^2 A}{\partial \delta_s \partial \delta_k}\right)_0 = 4\left[\left(\frac{\partial}{\partial I_1} + 2\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right)^2 A\right]_0 + \begin{cases} 0, & s = k, \\ 4\left[\left(\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right) A\right]_0, & s \neq k. \end{cases} \tag{2.9.3}$$

Introducing the notation

$$4\left[\left(\frac{\partial}{\partial I_1} + 2\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right)^2 A\right]_0 = \lambda + 2\mu, \quad 4\left[\left(\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right) A\right]_0 = -2\mu, \tag{2.9.4}$$

we arrive at the following representation for the specific strain energy

$$A = \frac{1}{2}\left[(\lambda + 2\mu)(\delta_1 + \delta_2 + \delta_3)^2 - 4\mu(\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1)\right] + \ldots \tag{2.9.5}$$

or in terms of denotation (2.7.3)

$$A = \frac{1}{2}\lambda s_1^2 + \mu s_2 + \ldots \tag{2.9.6}$$

These terms provide one with the expression for the specific strain energy for the "harmonic" (or "semi-linear") material. The necessary (clearly, not sufficient) criteria of the positiveness of A are presented now in the form of the necessary and sufficient criteria of positiveness of these parameters in the linear theory of elasticity, see eq. (3.3.7) of Chapter 3,

$$\mu > 0, \quad 3\lambda + 2\mu > 0 \tag{2.9.7}$$

or by virtue of eq. (2.9.4)

$$\begin{gathered}
\left[\left(\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right) A\right]_0 < 0, \\
3\left[\left(\frac{\partial}{\partial I_1} + 2\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right)^2 A\right]_0 + 2\left[\left(\frac{\partial}{\partial I_2} + \frac{\partial}{\partial I_3}\right) A\right]_0 > 0.
\end{gathered} \tag{2.9.8}$$

One can also suggest some static criteria, namely the behaviour of the material must not be in conflict with the intuitively expected results. One of these criteria is formulated in terms of the following inequalities

$$\text{if} \quad \delta_1 > \delta_2 > \delta_3, \quad \text{then} \quad t_1 > t_2 > t_3. \tag{2.9.9}$$

Let $\delta_1 > \delta_2$, then $t_1 > t_2$ and by eqs. (2.6.6), (2.6.4) we obtain the inequality

$$\left[\frac{\partial A}{\partial I_1} + (1+\delta_3)^2 \frac{\partial A}{\partial I_2}\right]\left[(1+\delta_1)^2 - (1+\delta_2)^2\right] > 0. \tag{2.9.10}$$

As the second coefficient is positive we arrive at the inequalities (Truesdell)

$$\frac{\partial A}{\partial I_1} + (1+\delta_3)^2 \frac{\partial A}{\partial I_2} > 0 \quad (s = 1, 2, 3). \tag{2.9.11}$$

We proved only one of these inequalities, namely for $s = 3$, the remaining ones are proved by analogy. If $\delta_1 = \delta_2 > \delta_3$ then inequalities (2.9.11) hold true for $s = 1$ and $s = 2$. In this case $t_1 = t_2$ and the sign of the expression

$$\frac{\partial A}{\partial I_1} + (1+\delta_3)^2 \frac{\partial A}{\partial I_2}$$

is proved by a limiting passage: for $\delta_1 = \delta_2 + \varepsilon$ $(\varepsilon > 0)$ this value is positive for any small ε. As $\varepsilon \to 0$ it either retains the sign or vanishes, i.e.

$$\delta_1 = \delta_2, \quad \frac{\partial A}{\partial I_1} + (1+\delta_3)^2 \frac{\partial A}{\partial I_2} \geq 0. \tag{2.9.12}$$

8.3 Representing the constitutive law by a quadratic trinomial

8.3.1 Quadratic dependence between two coaxial tensors

The constitutive laws presented in Section 8.2 relate the pairs of the coaxial tensors $\left(\hat{Q}, \hat{G}^{\times}\right), \left(\hat{T}, \hat{G}^{\times}\right), \left(\hat{T}, \hat{M}\right)$ and $\left(\hat{T}, \hat{g}^{\times}\right)$. The structure of all of these relations is the same: the determined tensor is a quadratic trinomial equal to the sum of the second, first and zero degrees (the latter is the unit tensors $\hat{g}$ or $\hat{G}$ in $v-$ or $V-$volume respectively) of another tensor. The coefficients of this trinomial are functions of the invariants and are determined in terms of the specific strain energy.

As mentioned in Section A.12 the quadratic dependence between the coaxial tensors is the result of the Cayley-Hamilton theorem (A.10.11) stating that the degrees of the tensor higher than second are replaced in terms of the zero, first and second degrees of this tensor. This suggests another way for deriving the constitutive law. The relation between the stress tensor considered and the corresponding strain measure (or strain tensor) is given by a quadratic trinomial whose coefficients are determined from the condition of the integrability of variation of the specific strain energy. It is easy to explain in terms of the energetic stress tensor $\hat{Q}$ since this variation is explicitly expressed in tensor $\hat{Q}$, see eq. (2.1.1)

$$\delta A = \frac{1}{2}\sqrt{\frac{G}{g}}\hat{Q} \cdot\cdot\, \delta\hat{G}^{\times} = \sqrt{\frac{G}{g}}\hat{Q} \cdot\cdot\, \delta\widehat{\mathcal{E}}. \tag{3.1.1}$$

8.3.2 Representation of the energetic stress tensor

Cauchy's strain tensor $\widehat{\mathcal{E}}$ is coaxial with the energetic stress tensor and the quadratic trinomial relating these tensors has the form

$$\hat{Q} = \overset{0}{a}\, \hat{g} + \overset{1}{a}\, \widehat{\mathcal{E}} + \overset{2}{a}\, \widehat{\mathcal{E}}^2, \tag{3.2.1}$$

since $\hat{Q}$ and $\widehat{\mathcal{E}}$ are determined in the metric of the initial state ($v-$volume)

$$\hat{Q} = t^{st}\mathbf{r}_s\mathbf{r}_t, \quad \widehat{\mathcal{E}} = \mathcal{E}_{st}\mathbf{r}^s\mathbf{r}^t. \tag{3.2.2}$$

Furthermore we have

$$\hat{g} \cdot\cdot\, \delta\widehat{\mathcal{E}} = I_1\left(\hat{g} \cdot \delta\widehat{\mathcal{E}}\right) = I_1\left(\delta\widehat{\mathcal{E}}\right) = \delta I_1\left(\widehat{\mathcal{E}}\right),$$

$$\widehat{\mathcal{E}} \cdot\cdot\, \delta\widehat{\mathcal{E}} = I_1\left(\widehat{\mathcal{E}} \cdot \delta\widehat{\mathcal{E}}\right) = I_1\left(\delta\frac{\widehat{\mathcal{E}}^2}{2}\right) = \frac{1}{2}\delta I_1\left(\widehat{\mathcal{E}}^2\right),$$

$$\widehat{\mathcal{E}}^2 \cdot\cdot\, \delta\widehat{\mathcal{E}} = I_1\left(\widehat{\mathcal{E}}^2 \cdot \delta\widehat{\mathcal{E}}\right) = I_1\left(\delta\frac{\widehat{\mathcal{E}}^3}{2}\right) = \frac{1}{3}\delta I_1\left(\widehat{\mathcal{E}}^3\right),$$

as $\delta\widehat{P}^2 = \widehat{P}\cdot\delta\widehat{P} + \left(\delta\widehat{P}\right)\cdot\widehat{P} \neq 2\widehat{P}\cdot\delta\widehat{P}$. However $I_1\left(\widehat{P}\cdot\delta\widehat{P}\right) = I_1\left(\left(\delta\widehat{P}\right)\cdot\widehat{P}\right)$ and thus $I_1\left(\widehat{P}\cdot\delta\widehat{P}\right) = \frac{1}{2}\delta I_1\left(\widehat{P}^2\right)$. By analogy $\delta\widehat{P}^3 \neq 3\widehat{P}^2\cdot\delta\widehat{P}$, however $I_1\left(\widehat{P}^2\cdot\delta\widehat{P}\right) = \frac{1}{3}\delta I_1\left(\widehat{P}^3\right)$. By virtue of eqs. (A.10.10) and (A.10.11)

$$I_1\left(\widehat{\mathcal{E}}^2\right) = j_1^2 - 2j_2, \quad I_1\left(\widehat{\mathcal{E}}^3\right) = j_1^3 - 3j_1j_2 + 3j_3 \quad \left(j_k = I_k\left(\widehat{\mathcal{E}}\right)\right),$$

so that

$$\hat{g}\cdot\cdot\,\delta\widehat{\mathcal{E}} = \delta j_1, \quad \widehat{\mathcal{E}}\cdot\cdot\,\delta\widehat{\mathcal{E}} = j_1\delta j_1 - \delta j_2, \quad \widehat{\mathcal{E}}^2\cdot\cdot\,\delta\widehat{\mathcal{E}} = \left(j_1^2 - j_2\right)\delta j_1 - j_1\delta j_2 + \delta j_3. \tag{3.2.3}$$

The expression for variation of the specific strain energy is written in the form

$$\delta A = \sqrt{\frac{G}{g}}\left\{\left[\overset{0}{a} + \overset{1}{a}\,j_1 + \overset{2}{a}\left(j_1^2 - j_2\right)\right]\delta j_1 - \left(\overset{1}{a} + \overset{2}{a}\,j_1\right)\delta j_2 + \overset{2}{a}\,\delta j_3\right\}. \tag{3.2.4}$$

This expression must satisfy the integrability condition, that is the coefficients associated with δj_k should be equated to the derivatives of A with respect to j_k. We arrive at the three equations

$$\left.\begin{array}{l} \overset{0}{a} + \overset{1}{a}\,j_1 + \overset{2}{a}\left(j_1^2 - j_2\right) = \dfrac{\partial A}{\partial j_1}\sqrt{\dfrac{g}{G}}, \\ -\left(\overset{1}{a} + \overset{2}{a}\,j_1\right) = \dfrac{\partial A}{\partial j_2}\sqrt{\dfrac{g}{G}}, \qquad \overset{2}{a} = \dfrac{\partial A}{\partial j_3}\sqrt{\dfrac{g}{G}}. \end{array}\right\} \tag{3.2.5}$$

These equations yield representation (2.5.3) for the energetic stress tensor and formulae for its coefficients.

8.3.3 *Representation of the stress tensor*

The Almansi strain measure $\hat{g}^\times$ is coaxial with tensor $\hat{T}$. Both tensors are determined in the metric of V-volume, thus

$$\hat{T} = \overset{0}{b}\,\hat{G} + \overset{1}{b}\,\hat{g}^\times + \overset{2}{b}\,\hat{g}^{\times^2}. \tag{3.3.1}$$

The forthcoming calculation is slightly complicated by the fact that the expression for variation of the specific strain energy does not explicitly contain the stress tensor. The latter can be introduced by means of the energetic stress tensor $\hat{Q}$ with the help of the equality (2.1.2), i.e. $\hat{Q} = \left(\tilde{\nabla}\mathbf{r}\right)^*\cdot\hat{T}\cdot\tilde{\nabla}\mathbf{r}$.

We obtain

$$A\delta = \frac{1}{2}\sqrt{\frac{G}{g}}\left[\left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\left(\overset{0}{b}\,\hat{G}+\overset{1}{b}\,\hat{g}^{\times}+\overset{2}{b}\,\hat{g}^{\times^{2}}\right)\cdot\tilde{\nabla}\mathbf{r}\right]\cdot\cdot\,\delta\hat{G}^{\times}. \tag{3.3.2}$$

Referring to the definition of tensor $\tilde{\nabla}\mathbf{r}$, eq. (3.2.2) of Chapter 2, and using formulae (3.3.7) and (4.1.2) of Chapter 2 we have

$$\left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\hat{G}\cdot\tilde{\nabla}\mathbf{r} = \left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\tilde{\nabla}\mathbf{r} = \hat{G}^{\times^{-1}},$$

$$\left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\hat{g}^{\times}\cdot\tilde{\nabla}\mathbf{r} = \left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\tilde{\nabla}\mathbf{r}\cdot\left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\tilde{\nabla}\mathbf{r} = \hat{G}^{\times^{-2}},$$

$$\left(\tilde{\nabla}\mathbf{r}\right)^{*}\cdot\hat{g}^{\times^{2}}\cdot\tilde{\nabla}\mathbf{r} = \hat{G}^{\times^{-3}}.$$

Thus

$$\delta A = \frac{1}{2}\sqrt{\frac{G}{g}}\left(\overset{0}{b}\,\hat{G}^{\times^{-1}}+\overset{1}{b}\,\hat{G}^{\times^{-2}}+\overset{2}{b}\,\hat{G}^{\times^{-3}}\right)\cdot\cdot\,\delta\hat{G}^{\times}. \tag{3.3.3}$$

Further calculation is based on the tensor transformations (A.10.11), (A.10.11), (A.10.14)

$$\begin{aligned}\hat{G}^{\times^{-1}}\cdot\cdot\,\delta\hat{G}^{\times} &= \frac{\delta G_1}{G_1}+\frac{\delta G_2}{G_2}+\frac{\delta G_3}{G_3}\\ &= \frac{1}{G_1G_2G_3}\left(G_2G_3\delta G_1+G_3G_1\delta G_2+G_1G_2\delta G_3\right)\\ &= \frac{\delta\left(G_1G_2G_3\right)}{G_1G_2G_3} = \frac{\delta I_3}{I_3} = -\frac{\delta I_3'}{I_3'},\end{aligned}$$

$$\hat{G}^{\times^{-2}}\cdot\cdot\,\delta\hat{G}^{\times} = I_1\left(\hat{G}^{\times^{-2}}\cdot\delta\hat{G}^{\times}\right) = -\delta I_1\left(\hat{G}^{\times^{-1}}\right) = -\delta\frac{I_2}{I_3} = -\delta I_1',$$

$$\begin{aligned}\hat{G}^{\times^{-3}}\cdot\cdot\,\delta\hat{G}^{\times} = I_1\left(\hat{G}^{\times^{-3}}\cdot\delta\hat{G}^{\times}\right) = -\frac{1}{2}\delta I_1\left(\hat{G}^{\times^{-2}}\right) &= -\frac{1}{2}\delta\frac{I_2^2-2I_1I_3}{I_3^2}\\ &= -I_1'\delta I_1'+\delta I_2'.\end{aligned}$$

Inserting into eq. (3.3.3) yields

$$\delta A = -\frac{1}{2}\sqrt{I_3}\left[\left(\overset{1}{b}+\overset{2}{b}\,I_1'\right)\delta I_1'-\overset{2}{b}\,\delta I_2'+\overset{0}{b}\,\frac{\delta I_3'}{I_3'}\right], \tag{3.3.4}$$

so that

$$2\sqrt{I_3'}\frac{\partial A}{\partial I_1'} = -\left(\overset{1}{b}+\overset{2}{b}\,I_1'\right),\quad 2\sqrt{I_3'}\frac{\partial A}{\partial I_2'} = \overset{2}{b},\quad -2\sqrt{I_3'}\frac{\partial A}{\partial I_3'} = \frac{1}{I_3'}\,\overset{0}{b}\,. \tag{3.3.5}$$

Now we return to the constitutive law (2.4.2) and definition (2.4.3) for the generalised moduli $\overset{k}{e}$.

8.3.4 Splitting the stress tensor into the spherical tensor and the deviator

The general relation between two coaxial tensors studied in Section A.13 is applied to the energetic stress tensor $\hat{Q}$ and Cauchy's strain tensor and is set in the form, see eq. (A.13.15)

$$\hat{Q} = kj_1\hat{g} + \frac{2\mu}{\cos 3\psi}\Big\{\cos(\omega + 3\psi)\,\mathrm{Dev}\,\widehat{\mathcal{E}} - \frac{2\sqrt{3}}{\Gamma}\sin\omega\left[\left(\mathrm{Dev}\,\widehat{\mathcal{E}}\right)^2 - \frac{1}{6}\hat{g}\Gamma^2\right]\Big\}. \tag{3.4.1}$$

Tensor $\widehat{\mathcal{E}}$ is determined here by three characteristics, which are the first invariant $j_1 = I_1\left(\widehat{\mathcal{E}}\right)$, angle ψ appearing in the trigonometric characteristic of the principal values of $\mathrm{Dev}\,\widehat{\mathcal{E}}$, eq. (A.11.16)

$$\left.\begin{array}{l} \mathcal{E}_1 - \frac{1}{3}j_1 = \frac{\Gamma}{\sqrt{3}}\sin\psi, \quad \mathcal{E}_2 - \frac{1}{3}j_1 = \frac{\Gamma}{\sqrt{3}}\sin\left(\psi + \frac{2\pi}{3}\right), \\ \mathcal{E}_3 - \frac{1}{3}j_1 = \frac{\Gamma}{\sqrt{3}}\sin\left(\psi + \frac{4\pi}{3}\right), \quad \left(|\psi| < \frac{\pi}{6}\right) \end{array}\right\} \tag{3.4.2}$$

and the second invariant

$$I_2\left(\mathrm{Dev}\,\widehat{\mathcal{E}}\right) = -\frac{\Gamma^2}{4}. \tag{3.4.3}$$

Let us notice in passing that this invariant is denoted as Γ^2 in Section A.13 which explains the difference in the coefficients in equalities (3.4.1) and (A.13.15).

The constitutive law is given by three functions of these characteristics: (i) the ratio of two first invariants, see eq. (A.13.5)

$$3k = \frac{I_1\left(\hat{Q}\right)}{j_1}; \tag{3.4.4}$$

(ii) the ratio of the second invariants of the deviators, see eq. (A.13.9)

$$\mu = \frac{\tau_Q}{\Gamma} = \left[\frac{I_2\left(\mathrm{Dev}\,\hat{Q}\right)}{4I_2\left(\mathrm{Dev}\,\widehat{\mathcal{E}}\right)}\right]^{1/2}; \tag{3.4.5}$$

and (iii) the angle of similarity of the deviators

$$\omega = \chi - \psi. \tag{3.4.6}$$

The principal values of $\operatorname{Dev}\hat{Q}$ are expressed in terms of angle χ by formulae analogous to eq. (3.4.2), see eq. (A.13.12)

$$Q_1 - \frac{1}{3}I_1\left(\hat{Q}\right) = \frac{2\tau_Q}{\sqrt{3}}\sin\chi, \quad Q_2 - \frac{1}{3}I_1\left(\hat{Q}\right) = \frac{2\tau_Q}{\sqrt{3}}\sin\left(\chi + \frac{2\pi}{3}\right),$$
$$Q_3 - \frac{1}{3}I_1\left(\hat{Q}\right) = \frac{2\tau_Q}{\sqrt{3}}\sin\left(\chi + \frac{4\pi}{3}\right).$$

Let us recall that Γ denotes the intensity of the shear strain, eq. (3.7.6) of Chapter 2, and τ_Q denotes the intensity of the shear stress obtained with the help of tensor $\hat{Q}$, eq. (2.2.11) of Chapter 1.

The following three functions

$$k = k(j_1, \Gamma, \psi), \quad \mu = \mu(j_1, \Gamma, \psi), \quad \omega = \omega(j_1, \Gamma, \psi) \tag{3.4.7}$$

are related by three differential relationships determining the requirement for the existence of the specific strain energy. The latter can also be viewed as a function of three invariant characteristics j_1, Γ, ψ of Cauchy's strain tensor

$$A = A(j_1, \Gamma, \psi). \tag{3.4.8}$$

Referring to the basic equality (3.1.1) and making use of representation (3.4.1) of the energetic stress tensor we obtain

$$\delta A = \sqrt{\frac{G}{g}}\hat{Q}\cdot\cdot\,\delta\widehat{\mathcal{E}} = \sqrt{\frac{G}{g}}kj_1\delta j_1 + \sqrt{\frac{G}{g}}\frac{2\mu}{\cos 3\psi}\left\{\cos(\omega + 3\psi)\operatorname{Dev}\widehat{\mathcal{E}}\cdot\cdot\,\delta\widehat{\mathcal{E}} - \right.$$
$$\left.\frac{2\sqrt{3}}{\Gamma}\sin\omega\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\cdot\cdot\,\delta\widehat{\mathcal{E}} - \frac{1}{6}\Gamma^2\delta j_1\right]\right\} \tag{3.4.9}$$

where

$$\delta\widehat{\mathcal{E}} = \delta\operatorname{Dev}\widehat{\mathcal{E}} + \frac{1}{3}\hat{g}\delta j_1.$$

Then

$$\operatorname{Dev}\widehat{\mathcal{E}}\cdot\cdot\,\delta\widehat{\mathcal{E}} = I_1\left(\operatorname{Dev}\widehat{\mathcal{E}}\cdot\delta\widehat{\mathcal{E}}\right) = \frac{1}{2}\delta I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\right],$$
$$\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\cdot\cdot\,\delta\widehat{\mathcal{E}} = \frac{1}{3}\delta I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^3\right] + \frac{1}{3}I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\right]\delta j_1.$$

Referring to eq. (A.11.17) we have

$$I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\right] = \frac{1}{2}\Gamma^2,$$
$$I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^3\right] = -\frac{1}{4\sqrt{3}}\Gamma^3\sin 3\psi, \quad \delta I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^2\right] = \Gamma\delta\Gamma,$$
$$\delta I_1\left[\left(\operatorname{Dev}\widehat{\mathcal{E}}\right)^3\right] = \frac{\sqrt{3}}{4}\left(\Gamma^2\sin 3\psi\delta\Gamma + \Gamma^3\cos 3\psi\delta\psi\right),$$

and substitution into eq. (3.4.9) leads to a rather simple expression which is a generalisation of formula (2.4.1) of Chapter 3 of the linear theory of elasticity

$$\delta A = \sqrt{\frac{G}{g}}\left(kj_1\delta j_1 + \mu\cos\omega\Gamma\delta\Gamma + \mu\Gamma^2\sin\omega\delta\psi\right). \tag{3.4.10}$$

This equation yields the energy definitions of the "generalised moduli" k, μ, ω

$$\frac{\partial A}{\partial j_1} = \sqrt{\frac{G}{g}}kj_1, \quad \frac{\partial A}{\partial \Gamma} = \sqrt{\frac{G}{g}}\mu\Gamma\cos\omega, \quad \frac{\partial A}{\partial \psi} = \sqrt{\frac{G}{g}}\mu\Gamma^2\sin\omega \tag{3.4.11}$$

and the differential relationships between them

$$\left.\begin{aligned}
&j_1\frac{\partial}{\partial\Gamma}\sqrt{\frac{G}{g}}k = \Gamma\frac{\partial}{\partial j_1}\sqrt{\frac{G}{g}}\mu\cos\omega,\\
&\frac{\partial}{\partial\psi}\sqrt{\frac{G}{g}}\mu\cos\omega = 2\sqrt{\frac{G}{g}}\mu\sin\omega + \Gamma\frac{\partial}{\partial\Gamma}\sqrt{\frac{G}{g}}\mu\sin\omega,\\
&j_1\frac{\partial}{\partial\psi}\sqrt{\frac{G}{g}}k = \Gamma^2\frac{\partial}{\partial j_1}\left(\sqrt{\frac{G}{g}}\mu\sin\omega\right).
\end{aligned}\right\} \tag{3.4.12}$$

In the linear theory of elasticity, k and μ respectively are the bulk modulus and the shear modulus. However the linear theory indicates no analogy for the similarity angle ω.

The relation between the invariants Γ, ψ and the principal invariants j_1, j_2, j_3 of the strain tensor is obtained with the help of formulae (A.11.6), (A.11.7), (A.11.14), (A.11.15) by replacing Γ by $\Gamma/2$

$$\frac{\Gamma^2}{4} = \frac{1}{3}j_1^2 - j_2, \quad \frac{\Gamma^3}{12\sqrt{3}}\sin 3\psi = -j_3 + \frac{1}{3}j_1j_2 - \frac{2}{27}j_1^3, \tag{3.4.13}$$

so that

$$\left.\begin{aligned}
&\Gamma\delta\Gamma = \frac{4}{3}j_1\delta j_1 - 2\delta j_2,\\
&\frac{1}{4\sqrt{3}}\left(\Gamma^2\sin 3\psi\delta\Gamma + \Gamma^3\cos 3\psi\delta\psi\right) =\\
&= -\delta j_3 + \left(\frac{1}{3}j_2 - \frac{2}{9}j_1^2\right)\delta j_1 + \frac{1}{3}j_1\delta j_2.
\end{aligned}\right\} \tag{3.4.14}$$

Using these equations and relations (3.4.10) we obtain

$$\left.\begin{aligned}\frac{\partial A}{\partial j_3} &= -\sqrt{\frac{G}{g}}4\sqrt{3}\frac{\mu\sin\omega}{\Gamma\cos 3\psi},\\ \frac{\partial A}{\partial j_2} &= -2\sqrt{\frac{G}{g}}\mu\cos\omega - \frac{\partial A}{\partial j_3}\left(\frac{1}{3}j_1 + \frac{1}{2\sqrt{3}}\Gamma\sin 3\psi\right),\\ \frac{\partial A}{\partial j_1} &= \sqrt{\frac{G}{g}}\left(kj_1 + \frac{4}{3}\mu j_1\cos\omega\right) + \\ &\qquad \frac{1}{3}\frac{\partial A}{\partial j_3}\left(\frac{1}{3}j_1^2 + \frac{\Gamma^2}{4} + \frac{1}{\sqrt{3}}j_1\Gamma\sin 3\psi\right).\end{aligned}\right\} \tag{3.4.15}$$

The specific strain energy for materials with zero angle of similarity of the deviators ($\omega = 0$) do not depend upon the third principal invariant j_3. For such materials

$$A = A(j_1, j_2), \quad \frac{\partial A}{\partial j_1} = \sqrt{\frac{G}{g}}\left(k + \frac{4}{3}\mu\right)j_1, \quad \frac{\partial A}{\partial j_2} = -2\sqrt{\frac{G}{g}}\mu, \tag{3.4.16}$$

and the constitutive equation has a quasi-linear structure

$$\hat{Q} = kj_1\hat{g} + 2\mu\,\mathrm{Dev}\,\widehat{\mathcal{E}} = \left(k - \frac{2}{3}\mu\right)j_1\hat{g} + 2\mu\widehat{\mathcal{E}}. \tag{3.4.17}$$

The difference from the generalised Hooke law of the linear theory, see eqs. (1.3.9), (3.1.1) of Chapter 3, is not only the replacement of the linear strain tensor $\hat{\varepsilon}$ by tensor $\widehat{\mathcal{E}}$ but also the replacement of the constant moduli k, μ by "generalised" moduli, the latter depending upon all three invariants of tensor $\widehat{\mathcal{E}}$ (also on j_3 in terms of G). One should take

$$k\sqrt{\frac{G}{g}} = {}_0k(j_1, j_2), \quad \mu\sqrt{\frac{G}{g}} = {}_0\mu(j_1, j_2). \tag{3.4.18}$$

The constitutive law (3.4.17) takes the form

$$\sqrt{\frac{G}{g}}\hat{Q} = {}_0\hat{Q} = {}_0kj_1\hat{g} + 2{}_0\mu\,\mathrm{Dev}\,\widehat{\mathcal{E}} = \left({}_0k - \frac{2}{3}{}_0\mu\right)j_1\hat{g} + 2{}_0\mu\widehat{\mathcal{E}}. \tag{3.4.19}$$

According to Subsection 1.3.4 ${}_0\hat{Q}$ denotes the energetic stress tensor related to the unit area in the initial state.

According to eq. (A.13.15), Cauchy's stress tensor expressed in terms of the energetic stress tensor (i.e. the inversion of formula (3.4.1)) has the form

$$\begin{aligned}\widehat{\mathcal{E}} = \frac{1}{9k}I_1\left(\hat{Q}\right)\hat{g} + \frac{1}{2\mu\cos 3\chi}\Big\{&\cos(3\chi - \omega)\,\mathrm{Dev}\,\hat{Q} + \\ &\frac{\sqrt{3}}{\tau_Q}\sin\omega\left[\left(\mathrm{Dev}\,\hat{Q}\right)^2 - \frac{2}{3}\hat{g}\tau_Q^2\right]\Big\}.\end{aligned} \tag{3.4.20}$$

Evidently, the constitutive law relating the stress tensor $\hat{T}$ and the coaxial Almansi strain tensor can be put in a form analogous to eq. (3.4.1)

$$\hat{T} = \hat{k}j_1'\hat{G} + \frac{2\tilde{\mu}}{\cos 3\tilde{\psi}} \left\{ \cos\left(\tilde{\omega} + 3\tilde{\psi}\right) \operatorname{Dev} \hat{\tilde{\mathcal{E}}} - \frac{2\sqrt{3}}{\tilde{\Gamma}} \sin\tilde{\omega} \left[\left(\operatorname{Dev} \hat{\tilde{\mathcal{E}}}\right)^2 - \frac{1}{6}\hat{G}\tilde{\Gamma}^2 \right] \right\}. \quad (3.4.21)$$

Here the quantities with a tilde sign are constructed in terms of tensors $\hat{\tilde{\mathcal{E}}}, \hat{T}$ with the help of formulae analogous to eqs. (3.4.2)-(3.4.6). The calculation of the variation of the specific strain energy is more difficult

$$\delta A = \sqrt{\frac{G}{g}} \left(\tilde{\nabla}\mathbf{r}\right)^* \cdot \hat{T} \cdot \tilde{\nabla}\mathbf{r} \cdot\cdot \delta\hat{\mathcal{E}}$$

and the energetic definitions of moduli $\hat{k}, \hat{\Gamma}, \hat{\omega}$ are given by cumbersome formulae.

8.3.5 *Logarithmic strain measure*

This tensor is coaxial with tensor $\hat{M}$ and its principal values are equal to logarithms of the principal values of tensor $\hat{M}^{1/2}$, that is

$$\hat{N} = \sum_{s=1}^{3} \nu_s \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}, \quad \nu_s = \ln\left(1 + \delta_s\right), \quad \hat{M}^{1/2} = \sum_{s=1}^{3} \left(1 + \delta_s\right) \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}} = \sum_{s=1}^{3} e^{\nu_s} \overset{s}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}, \quad (3.5.1)$$

and additionally

$$\sqrt{I_3} = \left(1 + \delta_1\right)\left(1 + \delta_2\right)\left(1 + \delta_3\right) = e^{I_1(\hat{N})}, \quad I_1\left(\hat{N}\right) = \nu_1 + \nu_2 + \nu_3.$$

Making use of relationship (2.6.8) we obtain

$$t_s = \frac{(1 + \delta_s)}{\sqrt{I_3}} \frac{\partial A}{\partial \delta_s} = e^{-I_1(\hat{N})} \frac{\partial A}{\partial \nu_s}, \quad \delta A = \sum_{s=1}^{3} \frac{\partial A}{\partial \nu_s} \delta\nu_s = e^{I_1(\hat{N})} \sum_{s=1}^{3} t_s \delta\nu_s. \quad (3.5.2)$$

Tensors $\hat{T}$ and $\hat{N}$ are split into the spherical parts and the deviators

$$\hat{T} = \frac{1}{3}\hat{E}I_1\left(\hat{T}\right) + \operatorname{Dev}\hat{T}, \quad \hat{N} = \frac{1}{3}\hat{E}I_1\left(\hat{N}\right) + \operatorname{Dev}\hat{N},$$

where $\hat{E}$ denotes the unit tensor and we introduce into consideration the second and third invariants of the deviators, see Section A.11

$$\begin{aligned}
\tau^2 &= -I_2\left(\operatorname{Dev}\hat{T}\right) = \frac{1}{6}\left[(t_1-t_2)^2+(t_2-t_3)^2+(t_3-t_1)^2\right],\\
I_3\left(\operatorname{Dev}\hat{T}\right) &= -\frac{2}{3\sqrt{3}}\tau^3\sin 3\chi,\\
\frac{\Gamma^2}{4} &= -I_2\left(\operatorname{Dev}\hat{N}\right) = \frac{1}{6}\left[(\nu_1-\nu_2)^2+(\nu_2-\nu_3)^2+(\nu_3-\nu_1)^2\right],\\
I_3\left(\operatorname{Dev}\hat{N}\right) &= -\frac{1}{12\sqrt{3}}\Gamma^3\sin 3\psi.
\end{aligned}$$

The main components of the deviators are given by eq. (A.11.6)

$$\left.\begin{aligned}
t'_s &= \frac{2\tau}{\sqrt{3}}\sin\chi_s, \quad \chi_1=\chi, \quad \chi_2=\chi+\frac{2\pi}{3}, \quad \chi_3=\chi+\frac{4\pi}{3},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(|\chi|<\frac{\pi}{6}\right),\\
\nu'_s &= \frac{\Gamma}{\sqrt{3}}\sin\psi_s, \quad \psi_1=\psi, \quad \psi_2=\psi+\frac{2\pi}{3}, \quad \psi_3=\psi+\frac{4\pi}{3},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left(|\psi|<\frac{\pi}{6}\right),
\end{aligned}\right\} \tag{3.5.3}$$

where

$$t_s = \frac{1}{3}I_1\left(\hat{T}\right)+t'_s, \quad \sum_{s=1}^{3}t'_s=0; \quad \nu_s=\frac{1}{3}I_1\left(\hat{N}\right)+\nu'_s, \quad \sum_{s=1}^{3}\nu'_s=0.$$

Returning to formulae (3.5.2) we have

$$\begin{aligned}
\sum_{s=1}^{3}t_s\delta\nu_s &= \sum_{s=1}^{3}\left[\frac{1}{3}I_1\left(\hat{T}\right)+t'_s\right]\left[\frac{1}{3}\delta I_1\left(\hat{N}\right)+\delta\nu'_s\right]\\
&= \frac{1}{3}I_1\left(\hat{T}\right)\delta I_1\left(\hat{N}\right)+\sum_{s=1}^{3}t'_s\delta\nu'_s
\end{aligned}$$

and moreover

$$\begin{aligned}
\sum_{s=1}^{3}t'_s\delta\nu'_s &= \frac{2\tau}{3}\sum_{s=1}^{3}\sin\chi_s\left(\sin\psi_s\delta\Gamma+\Gamma\cos\psi_s\delta\psi_s\right)\\
&= \tau\left(\cos\omega\delta\Gamma+\Gamma\sin\omega\delta\psi\right),
\end{aligned}$$

where $\omega=\chi-\psi$ is the angle of similarity of the deviators. We arrive then at the relationship

$$\delta A = e^{I_1(\hat{N})}\left[\frac{1}{3}I_1\left(\hat{T}\right)\delta I_1\left(\hat{N}\right)+\tau\left(\cos\omega\delta\Gamma+\Gamma\sin\omega\delta\psi\right)\right]. \tag{3.5.4}$$

Considering A as a function of the invariant values $I_1\left(\hat{N}\right), \Gamma, \psi$ we obtain

$$\frac{\partial A}{\partial I_1\left(\hat{N}\right)} = \frac{1}{3} e^{I_1\left(\hat{N}\right)} I_1\left(\hat{T}\right), \quad \frac{\partial A}{\partial \Gamma} = e^{I_1\left(\hat{N}\right)} \tau \cos\omega, \quad \frac{\partial A}{\partial \psi} = e^{I_1\left(\hat{N}\right)} \tau \Gamma \sin\omega. \tag{3.5.5}$$

Denoting

$$\frac{1}{3} I_1\left(\hat{T}\right) = k I_1\left(\hat{N}\right), \quad \tau = \mu\Gamma \tag{3.5.6}$$

we return to formulae (3.4.11) with the difference that the invariants of the logarithmic strain measure have appeared.

The differential dependences between the invariants $I_1\left(\hat{T}\right), \tau, \omega$ of the stress tensor are now put in the form

$$\left.\begin{aligned}
\frac{1}{3}\frac{\partial I_1\left(\hat{T}\right)}{\partial \Gamma} &= \tau\cos\omega + \frac{\partial \tau\cos\omega}{\partial I_1\left(\hat{N}\right)}, \\
\frac{\partial \tau\cos\omega}{\partial\psi} &= \tau\sin\omega + \Gamma\frac{\partial\tau\sin\omega}{\partial\Gamma}, \\
\frac{1}{3}\frac{\partial I_1\left(\hat{T}\right)}{\partial \psi} &= \Gamma\left(\tau\sin\omega + \frac{\partial\tau\sin\omega}{\partial I_1\left(\hat{N}\right)}\right).
\end{aligned}\right\} \tag{3.5.7}$$

These are simplified for the materials with a zero angle of the deviator similarity, because A becomes independent of parameter ψ. However $I_1\left(\hat{T}\right), \tau$ are related to each other as follows

$$\frac{1}{3}\frac{\partial I_1\left(\hat{T}\right)}{\partial\Gamma} = \tau + \frac{\partial\tau}{\partial I_1\left(\hat{N}\right)} = e^{-I_1\left(\hat{N}\right)} \frac{\partial}{\partial I_1\left(\hat{N}\right)}\left(e^{I_1\left(\hat{N}\right)}\tau\right). \tag{3.5.8}$$

This relationship can be satisfied by assuming

$$\tau = e^{-I_1\left(\hat{N}\right)} f(\Gamma) = \frac{f(\Gamma)}{\sqrt{I_3}}. \tag{3.5.9}$$

In this case $I_1\left(\hat{T}\right)$ depends only on the first invariant of the logarithmic strain measure (the ratio of the volumes in the deformed and initial states). The second invariant of the stress deviator, and thus modulus μ, are dependent not only on Γ but also on the above ratio of the volumes.

According to eqs. (3.5.3), (3.5.6) the constitutive law for the materials with zero angle of similarity is written down in Hencky's form

$$t_s = k I_1\left(\hat{N}\right) + 2\mu\nu_s', \quad \hat{T} = k I_1\left(\hat{N}\right)\hat{E} + 2\mu \operatorname{Dev}\hat{N}, \tag{3.5.10}$$

and for experimentally determining the "moduli of compression k and shear μ" it is necessary to utilise the following relations

$$k = k\left(\sqrt{I_3}\right), \quad \mu = \frac{1}{\sqrt{I_3}} \frac{f(\Gamma)}{\Gamma}. \tag{3.5.11}$$

In the case of the incompressible material

$$I_1\left(\hat{N}\right) = \nu_1 + \nu_2 + \nu_3 = 0, \quad \hat{N} = \operatorname{Dev} \hat{N}, \tag{3.5.12}$$

the variation of the specific strain energy (3.5.4) takes the form

$$\delta A = \tau\left(\cos\omega\delta\Gamma + \Gamma\sin\omega\delta\psi\right),$$

where the value $I_1\left(\hat{T}\right) = 3p$ remains undetermined. For materials with zero angle of similarity of the deviators $(\omega = 0)$, we have

$$A = A(\Gamma), \tau = \frac{\partial A}{\partial \Gamma}, \mu = \frac{\tau}{\Gamma} = \mu(\Gamma), \quad \left(\frac{\Gamma^2}{4} = \nu_1^2 + \nu_2^2 + \nu_1\nu_2\right) \tag{3.5.13}$$

and the stress tensor is as follows

$$\hat{T} = p\hat{E} + 2\mu(\Gamma)\hat{N}. \tag{3.5.14}$$

8.4 Approximations of the constitutive law

8.4.1 Signorini's quadratic constitutive law

General constitutive laws for the nonlinear elastic media are made specific either by prescribing the explicit expression for the specific strain energy in terms of the strain tensors (or strain measures) or by assuming some explicit expressions for the laws. Considering certain simple states of stress allows using the experimental results for determining *a priori* introduced coefficients in these expressions.

Signorini studied the constitutive law with the quadratic dependence of components of the stress tensor on the components of Almansi's strain tensor $\hat{\hat{\mathcal{E}}}$, both tensors being coaxial. Instead of $\hat{\hat{\mathcal{E}}}$ the strain measure $\hat{g}^\times$ is introduced and the general expression for this dependence is as follows

$$\hat{T} = \left(m_1 I_2' + m_2 I_1'^2 + m_3 I_1' + m_4\right)\hat{G} - \left(m_5 I_1' + m_6\right)\hat{g}^\times + m_7 \hat{g}^{\times^2}, \tag{4.1.1}$$

where m_k are some constants and $I_k' = I_k\left(\hat{g}^\times\right)$. Comparing the latter equation with eq. (2.4.2) and using formulae (2.4.4) we arrive at the relationships

$$\left.\begin{array}{l} \dfrac{\partial A}{\partial I_1'} = \dfrac{1}{2\sqrt{I_3'}}\left[\left(m_5 - m_7\right) I_1' + m_6\right], \quad \dfrac{\partial A}{\partial I_2'} = \dfrac{1}{2\sqrt{I_3'}} m_7, \\ \dfrac{\partial A}{\partial I_3'} = -\dfrac{1}{2}\left(I_3'\right)^{-3/2}\left(m_1 I_2' + m_2 I_1'^2 + m_3 I_1' + m_4\right). \end{array}\right\} \tag{4.1.2}$$

Only two of the integrability conditions need to be considered as the third one holds identically

$$\frac{\partial^2 A}{\partial I_2' \partial I_3'} = -\frac{1}{4}\left(I_3'\right)^{-3/2} m_7 = -\frac{1}{2}\left(I_3'\right)^{-3/2} m_1,$$
$$\frac{\partial^2 A}{\partial I_3' \partial I_1'} = -\frac{1}{4}\left(I_3'\right)^{-3/2}\left[\left(m_5 - m_7\right) I_1' + m_6\right] = -\frac{1}{2}\left(I_3'\right)^{-3/2}\left(2m_2 I_1' + m_3\right).$$

We thus arrive at the equalities

$$m_7 = 2m_1, \quad m_5 = 2m_1 + 4m_2, \quad m_6 = 2m_3,$$

so that the constitutive law is proved to depend on four constants

$$\hat{T} = \left(m_1 I_2' + m_2 I_1'^2 + m_3 I_1' + m_4\right)\hat{G} - \left[\left(2m_1 + 4m_2\right) I_1' + 2m_3\right]\hat{g}^\times + 2m_1 \hat{g}^{\times^2}. \quad (4.1.3)$$

The specific strain energy is now given by

$$A = \frac{1}{\sqrt{I_3'}}\left(m_1 I_2' + m_2 I_1'^2 + m_3 I_1' + m_4\right) + \text{const}, \quad (4.1.4)$$

which is easy to prove by eq. (4.1.2).

The constant m_4 can be expressed in terms of the value of the uniform tension in the initial state when $I_1' = I_2' = 3, I_3' = 1, \hat{g}^\times = \hat{G}$. In this state

$$\hat{T}_0 = q\hat{G} = \left(-m_1 - 3m_2 + m_3 + m_4\right)\hat{G}, \quad (4.1.5)$$

and returning to eqs. (4.1.3), (4.1.4) we have

$$\hat{T} = \left[q + m_1\left(I_2' + 1\right) + m_2\left(I_1'^2 + 3\right) + m_3\left(I_1' - 1\right)\right]\hat{G} - \left[\left(2m_1 + 4m_2\right) I_1' + 2m_3\right]\hat{g}^\times + 2m_1 \hat{g}^{\times^2}, \quad (4.1.6)$$

$$A = \frac{1}{\sqrt{I_3'}}\left[m_1\left(I_2' + 1\right) + m_2\left(I_1'^2 + 3\right) + m_3\left(I_1' - 1\right) + q\right] + \text{const}. \quad (4.1.7)$$

If the initial state is the natural one, then $q = 0$ and the constitutive law is determined only by three coefficients. In order to compare this law with the generalised Hooke's law of the linear theory we replace the strain measure $\hat{g}^\times$ and its invariants $I_k' = I_k\left(\hat{g}^\times\right)$ by tensor $\hat{\hat{\mathcal{E}}}$ and its invariants $j_1' = I_k\left(\hat{\hat{\mathcal{E}}}\right)$ respectively. The calculation is carried out by formulae (4.3.3) and (5.4.6) of Chapter 2. Introducing

$$4m_1 = c, \; 4m_2 = \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right), \; 4m_1 + 12m_2 + 2m_3 = \mu + \frac{c}{2} \quad (4.1.8)$$

we can write eqs. (4.1.6) and (4.1.7) in the form

$$\hat{T} = \left[\lambda j_1' + c j_2' + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right) j_2'^2\right] \hat{G} + \\ 2\left[\mu - \left(\lambda + \mu - \frac{c}{2}\right) j_1'\right] \hat{\tilde{\mathcal{E}}} + 2c\hat{\tilde{\mathcal{E}}}^2, \quad (4.1.9)$$

$$A = \frac{1}{\sqrt{I_3'}}\left[c j_2' + \frac{1}{2}\left(\lambda + \mu - \frac{c}{2}\right) j_1'^2 + \left(\mu + \frac{c}{2}\right)(1 - j_1')\right] - \left(\mu + \frac{c}{2}\right), \quad (4.1.10)$$

where the additive constant is chosen such that $A = 0$ in the natural state.

Remark 1. The quadratic constitutive law for the ideally elastic body suggested by N.V. Zvolinsky and P.M. Riz (1939) contains five constants and is written down in the form

$$\hat{T} = \left[\lambda' j_1' + \left(B + \frac{\lambda'}{2}\right) j_1'^2 - (C + 3\lambda') j_2'\right] \hat{G} + \\ \left[2\mu' + (C + \lambda' - 2\mu') j_1'\right] \hat{\tilde{\mathcal{E}}} + (A + 5\mu') \hat{\tilde{\mathcal{E}}}^2 \quad (4.1.11)$$

and can be made consistent with Signorini's constitutive law (4.1.9) if we take

$$\lambda' = \lambda, \ \mu' = \mu, \ A = 2c - 5\mu, \ B = \frac{1}{2}\left(\mu - \frac{c}{2}\right), \ C = -c - 3\lambda, \quad (4.1.12)$$

where λ, μ, c are the constants appearing in eq. (4.1.9). This means that the suggestion $A = B = C = 0$ contradicts the assumption of the existence of the specific strain energy.

Remark 2. Seth considered a series of nonlinear problems by applying the following constitutive law

$$\hat{T} = \lambda j_1' \hat{G} + 2\mu \hat{\tilde{\mathcal{E}}}. \quad (4.1.13)$$

This law presents a seemingly natural generalisation of Hooke's law of the linear theory of elasticity which is obtained from eq. (4.1.13) by setting $\hat{\tilde{\mathcal{E}}} = \hat{\varepsilon}$ and $j_1' = \vartheta$. The law (4.1.3) is energetically unacceptable which is immediately seen by comparing to eq. (4.1.9). However this law allows certain peculiarities of the nonlinear theory to be taken into account, for example, the finiteness of the tension force causing the specimen break, the necessity of the normal forces for the simple shear etc., see Subsections 8.4.4 and 8.4.5. Under small relative elongations and shears the quantitative results due to the quasi-linear law (4.1.3) do not considerably differ from those due to Signorini's law. On the other hand, the quasilinear law imposes no restriction onto displacements and rotations, thus it is applicable to problems which can not be solved by Hooke's law.

8.4.2 Dependence of the coefficients of the quadratic law on the initial state

Similar to Subsection 8.2.3 we consider two initial states, namely the initial state v_0 and the state v_* which is obtained from v_0 by the similarity transformation. The invariants $I'_k = I_k(\hat{g}^\times)$ are related by the second group of formulae (5.6.3) of Chapter 2 and relation (2.3.4) is set in the form

$$A(I'_1, I'_2, I'_3) = K^3 A_*(I'_{1*}, I'_{2*}, I'_{3*}) = K^3 A_*\left(I'_1 K^2, I'_2 K^4, I'_3 K^6\right). \quad (4.2.1)$$

This equality can be satisfied by taking

$$A_* = \frac{1}{\sqrt{I'_{3*}}}\left[m_1^*\left(I'_{2*} + K^4\right) + m_2^*\left(I'^2_{1*} + 3K^4\right) + m_3^*\left(I'_{1*} - K^2\right)\right], \quad (4.2.2)$$

where

$$m_1^* = \frac{m_1}{K^4}, \quad m_2^* = \frac{m_2}{K^4}, \quad m_3^* = \frac{m_3}{K^2}. \quad (4.2.3)$$

Indeed, replacing m_k^*, I'_{k*} in terms of m_k, I'_k we arrive at relationship (4.2.1).

In the case in which the initial state is v_*- volume, the expression for the stress tensor is constructed with the help of formulae (2.4.3) and (2.4.4)

$$\hat{T} = \left[m_1^*\left(I'_{2*} + K^4\right) + m_2^*\left(I'^2_{1*} + 3K^4\right) + m_3^*\left(I'_{1*} - K^2\right)\right]\hat{G} - \\ \left[\left(2m_1^* + 4m_2^*\right)I'_{1*} + 2m_3^*\right]\hat{g}_*^\times + 2m_1^*\hat{g}_*^{\times 2}. \quad (4.2.4)$$

Proceeding now to strain tensor $\widehat{\widetilde{\mathcal{E}}}$ and introducing

$$4m_1^* = c^*, \quad 4m_2^* = \frac{1}{2}\left(\lambda^* - \mu^* - \frac{c^*}{2}\right), \quad 4m_1^* + 12m_2^* + 2m_3^* = \mu^* + \frac{c^*}{2}, \quad (4.2.5)$$

similar to (4.1.8), we arrive at the expression for the stress tensor which is analogous to eq. (4.1.9) but contains a nonvanishing term in v_*-volume (when $\widehat{\widetilde{\mathcal{E}}}_* = 0, {}^* j'_{k*} = 0$)

$$\hat{T} = q\hat{G} + \left[\lambda^* j'_{1*} + c^* j'_{2*} + \frac{1}{2}\left(\lambda^* + \mu^* - \frac{c^*}{2}\right)\right]\hat{G} + \\ 2\left[\mu^* - \left(\lambda^* + \mu^* + \frac{c^*}{2}\right)j'_{1*}\right]\widehat{\widetilde{\mathcal{E}}}_* + 2c^*\widehat{\widetilde{\mathcal{E}}}^2_*, \quad (4.2.6)$$

where

$$q = \frac{1}{8}\left(K^2 - 1\right)\left[\left(1 + K^2\right)\left(3\lambda^* + 3\mu^* + \frac{c^*}{2}\right) - c^* + 2\mu^* + 6\lambda^*\right]. \quad (4.2.7)$$

Using eqs. (4.2.3), (4.2.5) and (4.1.8) one can relate constants λ^*, μ^*, c^* to λ, μ, c in the following way

$$\left.\begin{aligned} \lambda &= K^4\lambda^* + \frac{1}{2}K^2\left(1-K^2\right)\left(3\lambda^* + \mu^* - \frac{c^*}{2}\right), \\ \mu &= K^4\lambda^* - \frac{1}{2}K^2\left(1-K^2\right)\left(3\lambda^* + \mu^* - \frac{c^*}{2}\right), \\ c &= K^4c^*. \end{aligned}\right\} \tag{4.2.8}$$

The deformation process under increasing temperature can be seen as consisting of two parts. The temperature of an imaginary cube with the initial zero temperature increases to θ which is accompanied by the deformation $\hat{\tilde{\mathcal{E}}}^0$ due to the similarity transformation with the coefficient

$$K^0 = 1 + \alpha\theta, \tag{4.2.9}$$

which causes no stresses. The stress is caused by the reactive interaction of the surrounding medium and is determined by the corresponding strain $\hat{\tilde{\mathcal{E}}}'$ such that $\hat{T} = \hat{T}\left(\hat{\tilde{\mathcal{E}}}'\right) = \hat{T}\left(\hat{\tilde{\mathcal{E}}} - \hat{\tilde{\mathcal{E}}}^0\right)$. Here $\tilde{\mathcal{E}}$ denotes the strain tensor vanishing in the medium with zero temperature and $\hat{T}^0 = \hat{T}\left(-\hat{\tilde{\mathcal{E}}}^0\right)$ denotes the spherical stress tensor describing the all-round compression needed for the cube in order that it has the sizes corresponding to zero temperature.

This means that the natural state (v_0—volume) of the cube is the state at temperature θ, i.e. when $\hat{T} = 0$. The transformed state (v_*—volume) with the similarity coefficient

$$K = \frac{1}{K^0} = \frac{1}{1+\alpha\theta} \tag{4.2.10}$$

is the state at temperature $\theta = 0$. By eq. (4.2.6) the stress tensor in this state is equal to

$$\hat{T}^0 = q\hat{G} = \frac{1}{8}\frac{1-K_0^2}{K_0^2}\left[\frac{1+K_0^2}{K_0^2}\left(3\lambda^* + 3\mu^* - \frac{c^*}{2}\right) - c^* + 2\mu^* + 6\lambda^*\right]\hat{G}, \tag{4.2.11}$$

where λ^*, μ^*, c^* are the values of the corresponding moduli at $\theta = 0$ whereas their values at temperature θ are given by eqs. (4.2.8) and (4.2.10).

Neglecting the degrees in $\alpha\theta$ which are higher than the first one we obtain by eq. (4.2.11)

$$\hat{T}^0 = -\left(3\lambda^* + 2\mu^*\right)\alpha\theta\hat{G}. \tag{4.2.12}$$

Such a temperature term is added into the expression of the stress tensor of the linear theory of elasticity, see eq. (3.4.8) of Chapter 3.

8.4.3 The sign of the strain energy

In what follows we consider a "simplified" constitutive law of Signorini. In this law $c = 0$ since a non-zero value of c results in a considerable complication of the results. By eqs. (4.1.9) and (4.1.10)

$$\hat{T} = \left[\lambda j_1' + \underline{\frac{1}{2}(\lambda+\mu)\, j_1'^2}\right]\hat{G} + 2\left[\mu - \underline{(\lambda+\mu)\, j_1'}\right]\hat{\hat{\mathcal{E}}}, \tag{4.3.1}$$

$$A = \frac{1}{\sqrt{I_3'}}\left[\frac{1}{2}(\lambda+\mu)\, j_1'^2 + \mu(1 - j_1') - \mu\sqrt{I_3'}\right], \tag{4.3.2}$$

where the additive constant is determined by the condition that $A = 0$ when the deformation is absent (i.e. at $I_3' = 1, I_1' = 3$).

Similar to Hooke's law of the linear theory, the simplified Signorini's law contains only two constants however the essential difference is not only in tensor $\hat{\hat{\mathcal{E}}}$ (instead of linear strain tensor) but also in the additional terms underlined in eq. (4.3.1).

It is required to establish the domain of parameters (λ, μ) in which the specific strain energy is positive for any state different from the natural state for any positive I_3' and I_1'. The necessary conditions (2.9.7) guarantee the positiveness of A only under small deformations, i.e. when I_3' and I_1' are close to 3 and 1 respectively. Consideration of a more difficult problem of constructing the criteria of positiveness of A for any deformations requires the estimates of I_3' for given I_1'. Denoting the principal values of the strain measure $\hat{g}^\times$ by g_s we have

$$I_1' = g_1 + g_2 + g_3, \quad I_3' = g_1 g_2 g_3.$$

It is known that the product of several positive numbers with a given sum reaches a maximum when all these numbers are equal to each other, thus

$$I_3' = g_1 g_2 g_3 \le \left[\frac{1}{3}(g_1 + g_2 + g_3)\right]^3 = \frac{I_1'^3}{27}.$$

Since

$$I_1'^2 (3 - I_1')^2 = I_1'^2 (3 + I_1')^2 - 12 I_1'^3 \ge 0,$$

we can improve the previous inequality

$$I_1'^2 (3 + I_1')^2 \ge 27 \cdot 12 I_3', \quad \sqrt{I_3'} \le \frac{1}{18}(3 + I_1')\, I_1'. \tag{4.3.3}$$

In particular, if $I_1' = 3$ we obtain $I_3' \le 1$ and the equality sign holds only in the natural state, that is the condition

$$A(I_1', I_3')_{I_1'=3} = \frac{\mu}{\sqrt{I_3'}}\left(1 - \sqrt{I_3'}\right) \ge 0$$

leads to the familiar criterion

$$\mu > 0. \tag{4.3.4}$$

The terms in eq. (4.3.2) are now set in the form

$$\begin{aligned}(\mu+\lambda) I_1'^2 &= \frac{4}{9}\mu I_1'^2 + \frac{1}{9}(9\lambda+5\mu) I_1'^2, \\ -2(3\lambda+\mu) I_1' &= \frac{4}{3}\mu I_1' - \frac{2}{3}(9\lambda+5\mu) I_1',\end{aligned}$$

hence

$$8A\sqrt{I_3'} = \frac{4}{9}\mu\left[(I_1'+3) I_1' - 18\sqrt{I_3'}\right] + (9\lambda+5\mu)\left(\frac{I_1'}{3}-1\right)^2. \tag{4.3.5}$$

Due to eq. (4.3.3) the expression in square brackets is non-negative and the sufficient condition for the positiveness of A is the following inequality

$$9\lambda + 5\mu > 0. \tag{4.3.6}$$

Criteria (4.3.6) and (4.3.4) ensuring the positiveness of the specific strain energy in the whole domain of the principal relative elongations $-1 \leq \delta_s < \infty$ (or the principal values $\infty < \tilde{\mathcal{E}}_s < 1/2$ of the strain tensor $\widetilde{\widetilde{\mathcal{E}}}$) increase the lower boundary (2.9.7) for parameter λ ($\lambda > -5/9$ instead of $\lambda > -2/3\mu$). This decrease in the domain of parameters (λ, μ) is expected since the necessary criterion (2.9.7) ensures the positiveness of A only for sufficiently small δ_s.

Introducing a new parameter

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad \lambda = \frac{2\mu\nu}{1-2\nu}, \tag{4.3.7}$$

which corresponds to Poisson's ratio in the linear theory we can put criterion (4.3.6) in the form

$$\frac{5+8\nu}{1-2\nu} > 0, \tag{4.3.8}$$

such that

$$-\frac{5}{8} \leq \nu \leq \frac{1}{2}. \tag{4.3.9}$$

8.4.4 Application to problems of uniaxial tension

In this case $\hat{T} = t_{11}\mathbf{i}_1\mathbf{i}_1$. By eqs. (4.3.1) and (4.3.7) we have

$$\left.\begin{aligned} \frac{t_{11}}{2\mu} &= \frac{\nu}{1-2\nu}j_1' + \frac{1}{1-2\nu}\frac{j_1'^2}{4} + \left(1 - \frac{1}{1-2\nu}j_1'\right)\tilde{\mathcal{E}}_{11}, \\ \frac{t_{kk}}{2\mu} &= \frac{\nu}{1-2\nu}j_1' + \frac{1}{1-2\nu}\frac{j_1'^2}{4} + \left(1 - \frac{1}{1-2\nu}j_1'\right)\tilde{\mathcal{E}}_{kk} = 0 \quad (k = 2,3), \\ \frac{t_{ks}}{2\mu} &= \left(1 - \frac{1}{1-2\nu}j_1'\right)\tilde{\mathcal{E}}_{ks} = 0 \quad (k \neq s), \end{aligned}\right\} \tag{4.4.1}$$

such that

$$\tilde{\mathcal{E}}_{ks} = 0 \quad (s \neq k), \quad \tilde{\mathcal{E}}_{22} = \tilde{\mathcal{E}}_{33}, \quad j_1' = \tilde{\mathcal{E}}_{11} + 2\tilde{\mathcal{E}}_{22}. \tag{4.4.2}$$

Inserting this value of j_1' in the second equation (4.4.1) for $k = 2$ we obtain the biquadratic equation

$$\tilde{\mathcal{E}}_{22}^2 - \tilde{\mathcal{E}}_{22} - \nu\tilde{\mathcal{E}}_{11} - \frac{1}{4}\tilde{\mathcal{E}}_{11}^2 = 0. \tag{4.4.3}$$

The discriminant of this equation

$$\frac{1}{4}\left(\tilde{\mathcal{E}}_{11}^2 + 4\nu\tilde{\mathcal{E}}_{11} + 1\right) = \frac{1}{4}\Delta$$

is positive for all values of ν in the interval (4.3.9) and for the values

$$-\infty \leq \tilde{\mathcal{E}}_{11} \leq \frac{1}{2}, \tag{4.4.4}$$

corresponding the relative elongations

$$-1 \leq \delta_1 \leq \infty, \tag{4.4.5}$$

see eqs. (4.2.2) and (4.3.3) of Chapter 2.

The root of eq. (4.4.3) which is smaller than 1/2 is equal to

$$\tilde{\mathcal{E}}_{22} = \frac{1}{2}\left[1 - \left(1 + 4\nu\tilde{\mathcal{E}}_{11} + \tilde{\mathcal{E}}_{11}^2\right)^{1/2}\right], \quad 1 - 2\tilde{\mathcal{E}}_{22} = \left(1 + 4\nu\tilde{\mathcal{E}}_{11} + \tilde{\mathcal{E}}_{11}^2\right)^{1/2}. \tag{4.4.6}$$

Referring to the equality

$$(1+\delta_k)^{-2} = 1 - 2\tilde{\mathcal{E}}_{kk}, \quad \delta_k = \left(1 - 2\tilde{\mathcal{E}}_{kk}\right)^{-1/2} - 1, \tag{4.4.7}$$

we obtain

$$-\tilde{\nu} = \frac{\delta_2}{\delta_1} = \frac{\Delta^{-1/4} - 1}{\left(1 - 2\tilde{\mathcal{E}}_{11}\right)^{-1/2} - 1} \approx -\nu - \frac{1}{4}\tilde{\mathcal{E}}_{11}\left(1 - 6\nu - 10\nu^2\right) + \ldots \tag{4.4.8}$$

Under this approximation the ratio $\tilde{\nu}$ of the transverse contraction to the elongation is proved to be a linear function of the elongation.

By virtue of eqs. (4.4.6), (4.4.2) and (4.4.1) we also have

$$\frac{t_{11}}{\mu}(1-2\nu) = 1 + 2\nu + \tilde{\mathcal{E}}_{11} - \tilde{\mathcal{E}}_{11}^2 - \Delta^{1/2}\left(1 + 2\nu - \tilde{\mathcal{E}}_{11}\right). \tag{4.4.9}$$

The cross-sectional area S of the elongated rod is related to the initial value by the equality

$$S = S_0 (1+\delta_2)^2 = S_0 \Delta^{-1/2},$$

enabling one to present the tensile force in the form

$$Q = t_{11} S = \frac{\mu S_0}{1-2\nu}\left[\tilde{\mathcal{E}}_{11} - 1 - 2\nu + \frac{1 + 2\nu + \tilde{\mathcal{E}}_{11} - \tilde{\mathcal{E}}_{11}^2}{\left(1 + 4\nu\tilde{\mathcal{E}}_{11} + \tilde{\mathcal{E}}_{11}^2\right)^{1/2}}\right]. \tag{4.4.10}$$

Expanding the right hand side in terms of $\tilde{\mathcal{E}}_{11}$ and retaining the terms up to the second degree we arrive at the equality

$$Q = ES_0\tilde{\mathcal{E}}_{11}\left[1 - \frac{3}{4}\frac{(1+2\nu)^2}{1+\nu}\tilde{\mathcal{E}}_{11}\right], \quad E = 2\mu(1+\nu), \tag{4.4.11}$$

where E denotes Young's modulus of the linear theory.

Under an infinite elongation $\left(\delta_1 \to \infty, \tilde{\mathcal{E}}_{11} \to 1/2\right)$ the tensile force is finite. In Signorini's theory, this force results in the specimen breaking and is equal to

$$Q_\infty = \frac{\mu S_0}{2(1-2\nu)}\left[\sqrt{5+8\nu} - (1+4\nu)\right]. \tag{4.4.12}$$

It remains finite for all admissible values of ν

$$-\frac{5}{8} \le \nu \le \frac{1}{2}, \quad \frac{1}{3}\mu S_0 \le Q_\infty \le \frac{2}{3}\mu S_0. \tag{4.4.13}$$

In contrast, the compression force corresponding to the zero length of the rod $\left(\delta_1 \to -1, \tilde{\mathcal{E}}_{11} \to -\infty\right)$ is infinitely great.

8.4.5 Simple shear

It is defined by the following tensor

$$\widehat{\tilde{\mathcal{E}}} = \frac{1}{2}(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1)s - \frac{1}{2}s^2\mathbf{i}_2\mathbf{i}_2, \quad j_1' = -\frac{1}{2}s^2, \tag{4.5.1}$$

see Subsection 2.6.3. In the framework of the simplified Signorini theory the components of the stress tensor are as follows, see eq. (4.3.1)

$$\left.\begin{aligned}
t_{11} &= t_{33} = -\frac{1}{2}\mu\frac{s^2\left(2\nu - \frac{1}{4}s^2\right)}{1-2\nu}, \\
t_{22} &= -\frac{\mu s^2}{1-2\nu}\left(1-\nu+\frac{3}{8}s^2\right), \\
t_{12} &= t_{21} = \mu s + \frac{1}{2}\mu\frac{s^3}{1-2\nu}, \quad t_{23} = t_{31} = 0.
\end{aligned}\right\} \tag{4.5.2}$$

Realisation of the shear requires applying normal loads to all faces of the tested cube. This is one of the essential differences between the nonlinear and linear theories because only shear stress $t_{12} = \mu s$ is needed in the latter theory.

8.4.6 Murnaghan's constitutive law

The specific strain energy is expanded as a power series in terms of the invariants of Cauchy's strain tensor with the constant coefficients

$$\begin{aligned}
A &= \alpha j_1 + \frac{1}{2}(\lambda+2\mu)j_1^2 - 2\mu j_2 + \frac{1}{3}(l+2m)j_1^3 - 2mj_1j_2 + nj_3 + \ldots \\
&= \left(\alpha - \frac{n}{4}\right)j_1 + \frac{1}{2}(\lambda+2\mu)j_1^2 - \left(2\mu+\frac{n}{2}\right)j_2 + \\
&\quad \frac{1}{3}(l+2m)j_1^3 - 2mj_1j_2 + \frac{n}{8}\left(\Theta^2-1\right) + \ldots
\end{aligned} \tag{4.6.1}$$

The denotations are coincident with those of Murnaghan. Keeping only the terms shown in the above equation and using eq. (2.5.7) we arrive at the following expressions for the moduli

$$\left.\begin{aligned}
\overset{0}{m} &= \alpha + \lambda j_1 + lj_1^2 - (2m-n)j_2 + 2nj_3, \\
\overset{1}{m} &= 2\left[\mu + \alpha + \left(\lambda+\mu+\frac{n}{2}\right)j_1 + lj_1^2 - 2mj_2\right], \\
\overset{2}{m} &= 2\left(2\mu + \frac{n}{2} + 2mj_1\right),
\end{aligned}\right\} \tag{4.6.2}$$

where $\alpha = 0$ if the initial state is the natural one. Provided that only the "classical" terms of the linear theory are kept, i.e.

$$A = \frac{1}{2}(\lambda+2\mu)j_1^2 - 2\mu j_2 \tag{4.6.3}$$

we obtain the constitutive law

$$\hat{T} = \frac{1}{\Theta}\left[\lambda j_1\hat{g} + 2(\mu+\lambda j_1)\hat{\mathcal{E}} + 4\mu\hat{\mathcal{E}}^2\right] \tag{4.6.4}$$

that differs from Hooke's law.

The data used to determine the coefficients l, m, n in Murnaghan's formulae are few in number and unreliable. A small table for several materials is shown below[1] $\left(\nu = \dfrac{\lambda}{2(\lambda+\mu)}\right)$

Material	ν	λ	l	m	n
polystyrene, [1]	0,338	0,138	-1,89	-1,33	-1,00
armco-iron, [1,2]	0,32	8,20	-3,48	-103	110
pyrex glass, [1,2]	0,165	2,75	1,40	92	42
nickel steel, [1,3]	0,28	7,8	-4,6	-59	-73
copper, [4]	0,34	4,9	-16	-62	-159
steel, [5]	0,3	8,1	-34	-63	-76

Table 8.1 Murnaghan's coefficients $\times\ 10^{-10}$, N/m^2

8.4.7 *Behaviour of the material under ultrahigh pressures*

As in Subsection 8.2.3 we consider the natural state (v_0−volume) and the state obtained from the natural state by a similarity transformation with coefficient K (v_*−volume). The coefficients of the constitutive law of V−volume for the first and second choices of the initial state are denoted respectively as

$$\alpha = 0, \lambda, \mu, l, m, n; \quad \alpha^*, \lambda^*, \mu^*, l^*, m^*, n^*.$$

Using eq. (2.3.4) we can write the equality

$$\begin{aligned} A &= \frac{1}{2}(\lambda+2\mu)j_1^2 - 2\mu j_2 + \frac{1}{3}(l+2m)j_1^3 - 2mj_1j_2 + nj_3 \qquad (4.7.1)\\ &= K^3\left[\alpha^* j_1^* + \frac{1}{2}(\lambda^*+2\mu^*){j_1^*}^2 - 2\mu^* j_2^* + \frac{1}{3}(l^*+2m^*){j_1^*}^3 - \right.\\ &\qquad \left. 2m^* j_1^* j_2^* + n^* j_3^*\right]. \end{aligned}$$

[1] The data for the table are taken from the following papers:

[1] Zarembo, A.K., Krasilnikov, B.A. Introduction into nonlinear acoustics (in Russian). Moscow, Nauka, 1966.

[2] Huges, D.S., Kelly, J.L. Second-order elastic deformation of solids. Physical Review, vol. 92, p. 1145, 1953.

[3] Creckkraft, D.J. Ultrasonic wave velocities on stressed nickel steel. Nature, vol. 92, No. 4847, p. 1193, 1962.

[4] Seeger, A., Buck, O. Die experimentelle Ermittlung der elastischen Konstanten höhere Ordnung. Z. Natur., vol. 15a, 12, 1960.

[5] Sekoyan, S.S., Eremeev, N.E. Measurement of the elasticity constants of the third order for the steel by means of the ultrasound (in Russian). Izmeritelnaya Tekhnika, vol. 7, 1966.

Replacing j_k^* in terms of j_k with the help of eq. (5.6.4) of Chapter 2 and equating the coefficients associated with the same degrees of j_k, we arrive at the following equalities (Brillouin)

$$
\begin{aligned}
n^* = nK^3, \quad m^* &= mK^3, \quad l^* = lK^3, \\
2\mu^* &= K\left[2\mu + \left(1-K^2\right)\left(\frac{n}{2} - 3m\right)\right], \\
\lambda^* + 2\mu^* &= K\left[\lambda + 2\mu + \left(1-K^2\right)(2m+3l)\right], \qquad (4.7.2) \\
K\alpha^* &= \left(1-K^2\right)\left[-\frac{1}{2}(3\lambda+2\mu) + \frac{1}{4}\left(1-K^2\right)(n+9l)\right]. \qquad (4.7.3)
\end{aligned}
$$

The latter formula describes the state of stress in v_*−volume under the all-round compression

$$
\hat{T}' = \hat{T}_0 = -p\hat{g} = \alpha^*\hat{g} = \frac{1}{K}\left(1-K^2\right)\left[-\frac{3}{2}k + \frac{1}{4}\left(1-K^2\right)(n+9l)\right]\hat{g}, \qquad (4.7.4)
$$

where the bulk modulus of the linear theory is denoted as

$$
k = \lambda + \frac{2}{3}\mu. \qquad (4.7.5)
$$

Taking into account that

$$
K^3 = \frac{V_*}{V_0} = 1 - \frac{\Delta V}{V_0} = 1 - \Theta,
$$

where V_* and V_0 denote the volume of the specimen in the compressed and natural states respectively and Θ is the dilatation, we arrive at the relationship

$$
p\left(\frac{V_*}{V_0}\right)^{1/3} = \frac{3}{2}k\left[1 - \left(\frac{V_*}{V_0}\right)^{2/3}\right] - \frac{1}{4}(n+9l)\left[1 - \left(\frac{V_*}{V_0}\right)^{2/3}\right]^2. \qquad (4.7.6)
$$

In Bridgman's tests the pressure reached a value of 100000 atm. The results of these test were presented by Murnaghan by formula (4.7.6) with the following numerical values for the parameters

$$
k = 0,628 \cdot 10^4 \text{ atm}, \quad n + 9l = -406,2 \cdot 10^4 \text{ atm}.
$$

These values were obtained by fitting the measured and calculated values of Θ for $p = 2,5 \cdot 10^4$ atm and $p = 10^5$ atm, the error being under 1% for the values of p between 10^4 atm and 10^5 atm. The dilatation Θ increases from 0,211 to 0,394 under the increase in pressure from $p = 2,5 \cdot 10^4$ atm to $p = 10^5$ atm. It is interesting to notice that $|n + 9l| \gg k$ which means that the terms of third order in the expression for the specific strain energy should be kept.

Retaining the terms of the second order in Θ in eq. (4.7.6) we obtain

$$p = k\Theta + \frac{1}{2}\left(k - l - \frac{1}{9}n\right)\Theta^2. \tag{4.7.7}$$

For $k = 11,85 \cdot 10^4$ atm (that corresponds to the measured value of $\Theta = 0,211$ for $p = 2,5 \cdot 10^4$ atm) the linear approximation deviates considerably from the results of the measurement for $p = 5 \cdot 10^5$ atm (linear approximation yields $\Theta = 0,422$ compared with the measured value of $\Theta = 0,292$). The second term in eq. (4.7.7) does not essentially improve the situation provided that the coefficients are determined in terms of those measured at pressures $p = 2,5 \cdot 10^4$ atm to $p = 10^5$ atm.

It is necessary to mention that we discussed "ultrahigh" pressures and compliant materials. For pressures less than 10^4 atm and standard engineering materials, formula (4.7.7) yields quite reliable results even with the linear approximation.

8.4.8 Uniaxial tension

In the problem of tension in a prismatic rod acted on by the forces having the direction of the rod's axis (X_3), the tensors $\hat{G}^\times$ and $\hat{g}^\times$ are coaxial, that is $\hat{T}^\backslash = \hat{T}$. Representing the strain tensor $\widehat{\mathcal{E}}$ in the form

$$\widehat{\mathcal{E}} = (\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2)\,\mathcal{E}_1 + \mathcal{E}_3\mathbf{i}_3\mathbf{i}_3 = \mathcal{E}_1\hat{E} + (\mathcal{E}_3 - \mathcal{E}_1)\,\mathbf{i}_3\mathbf{i}_3, \tag{4.8.1}$$

where $\mathcal{E}_k$ and $\hat{E}$ denote respectively the principal values and the unit tensor we have

$$\widehat{\mathcal{E}}^2 = \mathcal{E}_1^2\hat{E} + \left(\mathcal{E}_3^2 - \mathcal{E}_1^2\right)\mathbf{i}_3\mathbf{i}_3,$$
$$j_1 = 2\mathcal{E}_1 + \mathcal{E}_3, \quad j_2 = \mathcal{E}_1^2 + 2\mathcal{E}_1\mathcal{E}_3, \quad j_3 = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3,$$

see eq. (A.10.10). Using Murnaghan's constitutive law and requiring the components of the stress tensor to be zero, except for t_{33}, we arrive at the relationships

$$2(\lambda + \mu)\,\mathcal{E}_1 + \lambda\mathcal{E}_3 + l\left(4\mathcal{E}_1^2 + 4\mathcal{E}_1\mathcal{E}_3 + \mathcal{E}_3^2\right) + \\ 2m\mathcal{E}_1(\mathcal{E}_1 - \mathcal{E}_3) + n\mathcal{E}_1\mathcal{E}_3 = 0, \tag{4.8.2}$$

$$t_{33} = \frac{2}{\Theta}(\mathcal{E}_3 - \mathcal{E}_1)(1 + 2\mathcal{E}_3)\left[\mu + m(2\mathcal{E}_1 + \mathcal{E}_3) - \frac{1}{2}n\mathcal{E}_1\right], \tag{4.8.3}$$

where

$$\Theta = (1 + 2\mathcal{E}_1)\sqrt{1 + 2\mathcal{E}_3} = (1 + \delta_1)^2\sqrt{1 + 2\mathcal{E}_3}.$$

The stress corresponding to the unit of the cross-sectional area in the initial state is equal to

$$t_3 = t_{33}\left(1+\delta_1\right)^2 = 2\left(\mathcal{E}_3 - \mathcal{E}_1\right)\sqrt{1+2\mathcal{E}_3}\left[\mu + m\left(2\mathcal{E}_1 + \mathcal{E}_3\right) - \frac{1}{2}n\mathcal{E}_1\right]. \tag{4.8.4}$$

Following Murnaghan, we represent the result of the test of the specimen tension in the form

$$x_1 = \left(1 - \nu c + \beta c^2\right)a_1, \quad x_2 = \left(1 - \nu c + \beta c^2\right)a_2, \quad x_3 = \left(1 + c + \delta c^2\right)a_3, \tag{4.8.5}$$

where β, δ are the empirically fitted coefficients and c is a small parameter coinciding with the relative elongation of the rod in the linear theory. Truncating the expansion with terms of second order in c, we obtain by means of eq. (3.6.4) of Chapter 2

$$\mathcal{E}_1 = -\nu c + \left(\frac{1}{2}\nu^2 + \beta\right)c^2, \quad \mathcal{E}_3 = c + \left(\frac{1}{2} + \delta\right)c^2.$$

Inserting into eqs. (4.8.2), (4.8.4) and comparing the terms linear in c, we find

$$\nu = \frac{\lambda}{2\left(\lambda+\mu\right)}, \quad t_3^* = 2c\mu\left(1+\nu\right) = \frac{3\lambda+2\mu}{\lambda+\mu}\mu c = Ec. \tag{4.8.6}$$

Therefore, ν and E are related to λ, μ in the same fashion as Poisson's ratio and Young' modulus in the linear theory. Equating the remaining terms of second order in c to zero, we arrive at the following equations

$$2\left(\lambda+\mu\right)\beta + \lambda\delta = -\lambda\left(\frac{1}{2}+\nu^2\right) - \mu\nu^2 - l\left(1-2\nu\right)^2 - 2m\nu\left(1+\nu\right) + n\nu,$$
$$\mu\left(\delta-\beta\right) = -\mu\left(\frac{3}{2}+\nu-\frac{1}{2}\nu^2\right) - m\left(1+\nu\right)\left(1-2\nu\right) - \frac{1}{2}n,$$

relating the coefficients l, m, n of the constitutive law to the empirically determined coefficients δ, β.

8.4.9 Incompressible material

For incompressible materials

$$I_3\left(\hat{G}^{\times}\right) = G_1 G_2 G_3 = 1, \tag{4.9.1}$$

where G_s denotes the principal values of Cauchy's strain measure $\hat{G}^{\times}$. The expression for the specific strain energy suggested by Mooney (1940) has the form

$$A = C_1\left(I_1 - 3\right) + C_2\left(I_2 - 3\right), \tag{4.9.2}$$

where C_1 and C_2 are the non-negative constants

$$C_1 \geq 0, \quad C_2 \geq 0. \tag{4.9.3}$$

It is easy to prove that inequalities (4.9.3) are the necessary and sufficient conditions for the positiveness of A in all states different from the initial one when $A = 0$. Indeed, recalling that

$$I_1 = G_1 + G_2 + G_3, \quad I_2 = G_1G_2 + G_2G_3 + G_3G_1, \quad G_s \geq 0,$$

one can set eq. (4.9.2.) in the form

$$A = C_1 (G_1 + G_2 + G_3 - 3) + C_2 \left(\frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} - 3 \right). \tag{4.9.4}$$

The sum of three positive numbers, whose product is equal to one, has a minimum when these numbers are equal to each other, that is each of these numbers is equal to one. Hence, under condition (4.9.1)

$$G_1 + G_2 + G_3 - 3 \geq 0, \quad \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} - 3 \geq 0 \tag{4.9.5}$$

and equalities (4.9.3) are the sufficient conditions for the positiveness of A. They are also the necessary conditions. Indeed, at least one of the constants C_1, C_2, say C_1, must be positive. Assuming $C_2 < 0$ one can always find the positive numbers G_1, G_2, G_3, whose product is equal to one, such that the following inequalities

$$\frac{C_1}{|C_2|} < \frac{I_2 - 3}{I_1 - 3} = \frac{G_1^{-1} + G_2^{-1} + G_3^{-1} - 3}{G_1 + G_2 + G_3 - 3}, \quad A < 0 \tag{4.9.6}$$

hold.

The stress tensor for the Mooney material can be presented for example by Finger's formula (2.4.6) which takes the particular form

$$\hat{T} = 2 \left[(C_1 + I_1 C_2) \hat{M} - C_2 \hat{M}^2 + \overset{-1}{c} \hat{G} \right]. \tag{4.9.7}$$

Another representation for the specific strain energy in Mooney's form, in terms of Almansi's strain measure, is given by

$$A = C_1 (I_2' - 3) + C_2 (I_1' - 3) \quad \left[I_k' = I_k \left(\hat{g}^\times \right) \right], \tag{4.9.8}$$

see eq. (5.2.4) of Chapter 2. According to eq. (2.4.2) the stress tensor is as follows

$$\hat{T} = 2 \left[\overset{0}{e} \hat{G} - (C_2 + C_1 I_1') \hat{g}^\times + C_1 \hat{g}^{\times 2} \right]. \tag{4.9.9}$$

Clearly, the constitutive equations (4.9.7) and (4.9.9) are two forms of the same constitutive law. We can easily convince ourselves of this by replacing $\hat{g}^{\times}$ by tensor $\hat{M} = \hat{g}^{\times^{-1}}$.

By virtue of eq. (2.1.5) the energetic stress tensor $\hat{Q}$ is given by

$$\hat{Q} = 2\left[(C_1 + I_1 C_2)\,\hat{g} - C_2 \hat{G}^{\times} + \overset{-1}{c}\,\hat{G}^{\times^{-1}}\right]. \tag{4.9.10}$$

Based upon eq. (2.4.9), one could assume that $\partial A/\partial I_3 = 0$. However this is a mistake since the derivative $\partial A/\partial I_3$ is a function of the invariants I_1, I_2 which is not known in advance and is obtained from the static equation and the condition $I_3 = 1$. Formula (4.9.2) provides one with the value of A in the plane $I_3 = 1$ in the space of the parameters I_1, I_2, I_3 in this plane. Returning to definitions (2.1.7), (2.4.3) we have

$$\left.\frac{\partial A}{\partial I_3}\right|_{I_3=1} = \overset{-1}{c} = \overset{0}{c} - C_1 I_1 - C_2 I_2. \tag{4.9.11}$$

Representation of the specific strain energy in Mooney's form (4.9.2) was preceded by the simplified form

$$A = C_1 (I_1 - 3). \tag{4.9.12}$$

This form was suggested by Treloar as a result of representing rubber as a system of interacting long molecular chains. The same form referred to as the neo-Hookean body was applied in the first papers by Rivlin, 1948.

Mooney's formula reduces the void between the theory and experiment. The measurements in a large range of deformation discovered a dependence of the ratio $\dfrac{\partial A}{\partial I_1}\Big/\dfrac{\partial A}{\partial I_2}$ on I_2 (Rivlin and Saunders, 1951) and this gave rise to a more general Mooney's formula

$$A = C_1 (I_1 - 3) + f(I_2 - 3). \tag{4.9.13}$$

8.4.10 Materials with a zero angle of similarity of the deviators

Following the definition (3.4.1) of Chapter 1 we introduce into consideration the modified energetic stress tensor

$${}_0\hat{Q} = \sqrt{\frac{G}{g}}\,\hat{Q}. \tag{4.10.1}$$

The expression for the constitutive law in the form of eq. (3.4.1) can be kept for tensor ${}_0\hat{Q}$ if k, μ are replaced respectively by

$${}_0k = \frac{1}{3}\frac{I_1\left({}_0\hat{Q}\right)}{j_1} = \sqrt{\frac{G}{g}}\,k, \quad {}_0\mu = \left[\frac{I_2\left(\mathrm{Dev}_0\,\hat{Q}\right)}{4I_2\left(\mathrm{Dev}\,\hat{\mathcal{E}}\right)}\right]^{1/2} = \sqrt{\frac{G}{g}}\,\mu. \tag{4.10.2}$$

Formulae (3.4.11) are now reset in the form

$$\frac{\partial A}{\partial j_1} = {}_0k j_1, \quad \frac{\partial A}{\partial \Gamma} = {}_0\mu \Gamma \cos\omega, \quad \frac{\partial A}{\partial \psi} = {}_0\mu \Gamma^2 \sin\omega \tag{4.10.3}$$

and for materials with a zero similarity angle $\omega = 0$ we have

$$\frac{\partial A}{\partial j_1} = {}_0k j_1, \quad \frac{\partial A}{\partial \Gamma} = {}_0\mu \Gamma, \quad \frac{\partial A}{\partial \psi} = 0. \tag{4.10.4}$$

For such materials

$$A = A(j_1, j_2) = A(j_1, \Gamma), \quad \frac{\partial A}{\partial j_3} = 8\frac{\partial A}{\partial I_3} = 0, \tag{4.10.5}$$

and they are not incompressible. Considering A as being a function of j_1, j_2 we have, due to eq. (3.4.16)

$$\frac{\partial A}{\partial j_1} = \left({}_0k + \frac{4}{3}\,{}_0\mu\right) j_1, \quad \frac{\partial A}{\partial j_2} = -2\,{}_0\mu. \tag{4.10.6}$$

The constitutive equations (3.4.1) now take the classical structure of Hooke's law

$${}_0\hat{Q} = {}_0k j_1 \hat{g} + 2\,{}_0\mu \operatorname{Dev}\widehat{\mathcal{E}} = \left({}_0k - \frac{2}{3}\,{}_0\mu\right) j_1 \hat{g} + 2\,{}_0\mu \widehat{\mathcal{E}}, \tag{4.10.7}$$

where ${}_0k, {}_0\mu$ are functions of j_1, Γ (or j_1, j_2).

Hencky's body is a particular example of a material with a zero similarity angle for a constant ${}_0k$, then ${}_0\mu = {}_0\mu(\Gamma)$ which follows from the requirement of the compatibility of the relationships in eq. (4.10.4)

$${}_0k = \text{const}, \quad \frac{\partial^2 A}{\partial \Gamma \partial j_1} = 0 = \Gamma \frac{\partial\,{}_0\mu}{\partial j_1}; \quad {}_0\mu = {}_0\mu(\Gamma). \tag{4.10.8}$$

Another particular example is the constitutive law suggested by Neuber. Using the classical denotations (4.3.9), (4.7.5)

$${}_0k - \frac{2}{3}\,{}_0\mu = {}_0\lambda = {}_0\mu \frac{2\nu}{1 - 2\nu}, \tag{4.10.9}$$

we assume a constant "Poisson's ratio" ν in the constitutive law (4.10.7)

$${}_0\hat{Q} = 2\,{}_0\mu \left(\frac{\nu}{1 - 2\nu} j_1 \hat{g} + \widehat{\mathcal{E}}\right). \tag{4.10.10}$$

Then by eqs. (4.10.4) and (4.10.9) we have

$$\frac{\partial A}{\partial j_1^2} = \frac{1}{2}\,{}_0k = \frac{1}{3}\,{}_0\mu \frac{1 + \nu}{1 - 2\nu}, \quad \frac{\partial A}{\partial \Gamma^2} = \frac{1}{2}\,{}_0\mu,$$

so that

$$\frac{\partial A}{\partial j_1^2} - \frac{2}{3}\frac{1+\nu}{1-2\nu}\frac{\partial A}{\partial \Gamma^2} = 0, \tag{4.10.11}$$

and the general solution of this first order partial differential equation can be set in the form

$$A = 2\,{}_0\overset{0}{\mu}\, f\left(A_+\right), \quad A_+ = \frac{1}{4}\left(\frac{2}{3}\frac{1+\nu}{1-2\nu}j_1^2 + \Gamma^2\right) = \frac{1}{2}\frac{1-\nu}{1-2\nu}j_1^2 - j_2$$
$$= \frac{1}{2}\left[\frac{\nu}{1-2\nu}I_1\left(\widehat{\mathcal{E}}\right) + I_1\left(\widehat{\mathcal{E}}^2\right)\right]. \tag{4.10.12}$$

Here ${}_0\overset{0}{\mu}$ is a constant and A_+ denotes a classical (Hookean) expression for the specific strain energy of the elastic body with Poisson's ratio ν and shear modulus equal to $1/2$.

For Neuber's body, the "shear modulus" is a function of A_+ given, due to eqs. (4.10.4) and (4.10.12), by the equation

$${}_0\mu = 2\frac{\partial A}{\partial \Gamma^2} = 4\,{}_0\overset{0}{\mu}\,\frac{\partial f\left(A_+\right)}{\partial \Gamma^2} = {}_0\overset{0}{\mu}\,\frac{\partial f\left(A_+\right)}{\partial A_+}. \tag{4.10.13}$$

In order to construct the constitutive law in Neuber's form one needs the experimental dependence of the specific strain energy on A_+ calculated in terms of the measured strains.

8.5 Variational theorems of statics of the nonlinear-elastic body

8.5.1 Principle of virtual displacements

The statement of this principle for the solids was given in Subsection 1.3.5. This principle was used to derive eq. (3.5.6) of Chapter 1, which defines the elementary work of the external forces $\delta' a_{(e)}$ with the help of the static equation (3.3.1) of Chapter 1. Here we prove the inverse statement, namely the static equations in $V-$volume and on its surface O are contained in the principle of virtual displacements provided that the expression for the elementary work, eq. (3.5.6) of Chapter 1, is prescribed.

Indeed, by virtue of eqs. (3,5,3) and (3.5.5) of Chapter 1, we have

$$\delta' a_{(e)} = \frac{1}{2}\iiint\limits_V \tilde{t}^{st}\delta G_{st}d\tau = \iiint\limits_V \rho\mathbf{K}\cdot\delta\mathbf{R}d\tau + \iint\limits_O \mathbf{F}\cdot\delta\mathbf{R}dO. \tag{5.1.1}$$

Transforming the integration over v–volume by the integration over its surface o yields

$$\delta' a_{(e)} = \frac{1}{2} \iiint\limits_v \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta G_{st} d\tau_0$$
$$= \iiint\limits_v \rho_0 \mathbf{K} \cdot \delta \mathbf{R} d\tau_0 + \iint\limits_o \sqrt{\frac{G}{g}} \sqrt{G^{ik} n_i n_k} \mathbf{F} \cdot \delta \mathbf{R} do. \qquad (5.1.2)$$

Here we used eq. (3.5.4) of Chapter 2 relating the areas dO and do. Noticing that

$$\frac{1}{2}\sqrt{\frac{G}{g}} \tilde{t}^{st} \delta G_{st} = \frac{1}{2}\sqrt{\frac{G}{g}} \tilde{t}^{st} \left(\mathbf{R}_s \cdot \delta \mathbf{R}_t + \mathbf{R}_t \cdot \delta \mathbf{R}_s \right) = \sqrt{\frac{G}{g}} \tilde{t}^{st} \mathbf{R}_s \cdot \delta \mathbf{R}_t$$
$$= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial q^t} \sqrt{G} \tilde{t}^{st} \mathbf{R}_s \cdot \delta \mathbf{R} - \delta \mathbf{R} \cdot \frac{\partial}{\partial q^t} \sqrt{G} \tilde{t}^{st} \mathbf{R}_s \right] \qquad (5.1.3)$$

and using the transformation of a volume integral into a surface one, eq. (E.8.1), we have

$$\frac{1}{2} \iiint\limits_v \sqrt{\frac{G}{g}} \tilde{t}^{st} \delta G_{st} d\tau_0 =$$
$$= \iint\limits_o \sqrt{\frac{G}{g}} \tilde{t}^{st} n_t \mathbf{R}_s \cdot \delta \mathbf{R} do - \iiint\limits_v \frac{d\tau_0}{\sqrt{g}} \delta \mathbf{R} \cdot \frac{\partial}{\partial q^t} \sqrt{G} \tilde{t}^{st} \mathbf{R}_s.$$

Returning to eq. (5.1.2) we arrive at the relationship

$$\iiint\limits_v \frac{1}{\sqrt{g}} \left(\rho_0 \sqrt{g} \mathbf{K} + \frac{\partial}{\partial q^t} \sqrt{G} \tilde{t}^{st} \mathbf{R}_s \right) \cdot \delta \mathbf{R} d\tau_0 +$$
$$\iint\limits_o \sqrt{\frac{G}{g}} \left(\sqrt{G^{ik} n_i n_k} \mathbf{F} - \tilde{t}^{st} \mathbf{R}_s n_t \right) \cdot \delta \mathbf{R} do = 0. \qquad (5.1.4)$$

The requirement of that the volume integral vanishes due to the arbitrariness of the field of the virtual displacement $\delta \mathbf{u} = \delta \mathbf{R}$ in the volume of the body results in the static equations in the volume

$$\frac{\partial}{\partial q^t} \sqrt{G} \tilde{t}^{st} \mathbf{R}_s + \rho_0 \sqrt{g} \mathbf{K} = 0. \qquad (5.1.5)$$

In the surface integral we have $\delta \mathbf{u} = \delta \mathbf{R} = 0$ on the part o_1 of surface o where the displacement vector $\mathbf{u}$ is prescribed. On the part o_2, where the surface forces $\mathbf{F}$ are prescribed, the vector $\delta\mathbf{u} = \delta\mathbf{R}$ is arbitrary. Hence the requirement that the surface integral over o_2 vanishes yields the static equation on the surface

$$\tilde{t}^{st}\mathbf{R}_s n_t = \sqrt{G^{ik} n_i n_k}\mathbf{F}, \tag{5.1.6}$$

cf. eq. (3.3.6) of Chapter 1.

8.5.2 Stationarity of the potential energy of the system

In the ideally elastic medium the elementary work of the external forces $\delta' a_{(e)}$ is equal to the variation of the strain energy. Recalling its definition (1.2.13) and returning to eq. (5.1.1) we have

$$\delta \iiint\limits_{v} A d\tau_0 - \iiint\limits_{V} \rho\mathbf{K} \cdot \delta\mathbf{R} d\tau - \iint\limits_{O} \mathbf{F} \cdot \delta\mathbf{R} dO = 0, \tag{5.2.1}$$

where A denotes the specific strain energy.

In what follows the mass and the surface forces are assumed to be potential forces. The elementary work of the potential mass force can be determined by the relationship

$$\mathbf{K} \cdot \delta\mathbf{R} = -\delta\omega\,(x_1, x_2, x_3) = -\tilde{\nabla}\omega \cdot \delta\mathbf{R}, \tag{5.2.2}$$

where $\omega\,(x_1, x_2, x_3)$ denotes the potential energy of the external force, for example in the gravity field $\omega = g x_3$, $\tilde{\nabla}\omega = \mathbf{k} g$ where $\mathbf{k}$ denotes the vector of the ascending vertical. Therefore

$$\begin{aligned} \iiint\limits_{V} \rho\mathbf{K} \cdot \delta\mathbf{R} d\tau &= -\iiint\limits_{V} \rho\delta\omega d\tau = -\iiint\limits_{v} \rho_0 \delta\omega d\tau_0 \\ &= -\delta \iiint\limits_{v} \rho_0 \omega d\tau_0, \end{aligned} \tag{5.2.3}$$

where the variation sign is placed in front of the integral since neither volume v_0 nor density ρ_0 is varied.

The case of the potential surface force is that in which the force applied to the elementary surface retains the value and the direction ("dead load")

$$\mathbf{F}\,(x_1, x_2, x_3)\, dO = \mathbf{F}^0\,(a_1, a_2, a_3)\, do. \tag{5.2.4}$$

Here a_s denotes the coordinates of point M on o which moves to point M' on O under deformation. As vector $\mathbf{F}^0$ remains unchanged we have

$$\iint\limits_{O} \mathbf{F} \cdot \delta\mathbf{R} dO = \iint\limits_{o_2} \mathbf{F}^0 \cdot \delta\mathbf{R} do = \iint\limits_{o_2} \mathbf{F}^0 \cdot \delta\mathbf{u} do = \delta \iint\limits_{o_2} \mathbf{F}^0 \cdot \mathbf{u} do, \tag{5.2.5}$$

where, as above, o_2 denotes the part of surface o on which the forces are prescribed.

Another example of the surface forces having a potential is the normal pressure uniformly distributed over O. Then

$$\iint_O \mathbf{F} \cdot \delta \mathbf{R} dO = -p_0 \iint_O \mathbf{N} \cdot \delta \mathbf{R} dO = -p_0 \iiint_V \tilde{\nabla} \cdot \delta \mathbf{R} d\tau.$$

According to the definition of the nabla-operator in $V-$volume

$$\tilde{\nabla} \cdot \delta \mathbf{R} = \mathbf{R}^s \cdot \frac{\partial}{\partial q^s} \delta \mathbf{R} = \mathbf{R}^s \cdot \delta \mathbf{R}_s$$

and the previous equality is reset in the form

$$\iint_O \mathbf{F} \cdot \delta \mathbf{R} dO = -p_0 \iiint_v \sqrt{\frac{G}{g}} \mathbf{R}^s \cdot \delta \mathbf{R}_s d\tau_0. \tag{5.2.6}$$

The variation of value $\sqrt{G}$ can be presented in the form

$$\begin{aligned} \delta\sqrt{G} &= \frac{1}{2\sqrt{G}} \delta G = \frac{1}{2\sqrt{G}} \frac{\partial G}{\partial G_{st}} \delta G_{st} = \frac{1}{2} \sqrt{G} G^{st} \delta G_{st} \\ &= \frac{1}{2} \sqrt{G} \mathbf{R}^s \cdot \mathbf{R}^t \left(\mathbf{R}_t \cdot \delta \mathbf{R}_s + \mathbf{R}_s \cdot \delta \mathbf{R}_t \right) \\ &= \frac{1}{2} \sqrt{G} \left(\mathbf{R}^s \cdot \delta \mathbf{R}_s + \mathbf{R}^t \cdot \delta \mathbf{R}_t \right) = \sqrt{G} \mathbf{R}^s \cdot \delta \mathbf{R}_s, \end{aligned} \tag{5.2.7}$$

since $\mathbf{R}^t \mathbf{R}_t = \hat{G}$ is the unit tensor of $V-$volume and $\sqrt{g}$ is not varied. Equality (5.2.6) is then transformed to the expected form

$$\begin{aligned} \iint_O \mathbf{F} \cdot \delta \mathbf{R} dO &= -p_0 \iiint_v \delta \sqrt{\frac{G}{g}} d\tau_0 = -p_0 \delta \iiint_v \sqrt{\frac{G}{g}} d\tau_0 \\ &= -p_0 \delta \iiint_v d\tau = -p_0 \delta V, \end{aligned} \tag{5.2.8}$$

with V denoting the volume of the deformed body.

Expression (5.2.1) of the principle of virtual displacement takes the form

$$\delta \left[\iiint_v \left(A + \rho_0 \omega + p_0 \sqrt{\frac{G}{g}} \right) d\tau_0 - \iint_{o_2} \mathbf{F}^0 \cdot \mathbf{u} do \right] = \delta \Pi = 0, \tag{5.2.9}$$

where Π denotes the functional over the displacement vector $\mathbf{u}$

$$\Pi = \iiint_v \left(A + \rho_0 \omega + p_0 \sqrt{\frac{G}{g}} \right) d\tau_0 - \iint_{o_2} \mathbf{F}^0 \cdot \mathbf{u} do. \tag{5.2.10}$$

This functional is referred to as the potential energy of the system, i.e. the elastic body and the force field. Equality (5.2.9) verifies the stationarity of this functional, that is, among all possible displacement fields in the elastic body (i.e. those taking the prescribed values on o_1) that particular displacement field is realised for which the potential energy of the system takes a stationary value.

Let us recall that under the stationary value of the functional one understands such a value that variation $\delta \mathbf{u}$ results in the increment $\Delta\Pi$ of order higher than $|\delta \mathbf{u}|$. In the linear theory of elasticity it is proved that

$$\Delta\Pi > 0,$$

see Subsection 4.2.2, which implies that the stationary value of functional Π is a minimum. In the nonlinear theory of elasticity such a general statement does not occur.

Remark 1. The differential equations and the natural boundary conditions of the variational problem of stationarity of functional Π present the static equations in the volume and on the surface in which the stress tensor is expressed in terms of the strains by means of the constitutive law.

Indeed, repeating the transformation of Subsection 8.5.1 we have

$$\delta \iiint\limits_{v} A d\tau_0 = 2 \iint\limits_{o} \frac{\partial A}{\partial G_{st}} \mathbf{R}_t \cdot \delta \mathbf{R} n_s do - 2 \iiint\limits_{v} \frac{d\tau_0}{\sqrt{g}} \delta \mathbf{R} \cdot \frac{\partial}{\partial q^s} \sqrt{g} \frac{\partial A}{\partial G_{st}} \mathbf{R}_t.$$

The elementary work of the external (mass and surface) forces is represented in the form of eq. (5.1.2). We arrive then at the relationship

$$-\iiint\limits_{v} \frac{d\tau_0}{\sqrt{g}} \delta \mathbf{R} \cdot \left(2 \frac{\partial}{\partial q^s} \sqrt{g} \frac{\partial A}{\partial G_{st}} \mathbf{R}_t + \rho_0 \sqrt{g} \mathbf{K} \right) +$$

$$\iint\limits_{o} \delta \mathbf{R} \cdot \left(2 \sqrt{\frac{g}{G}} \frac{\partial A}{\partial G_{st}} n_s \mathbf{R}_t - \sqrt{G^{ik} n_i n_k} \mathbf{F} \right) \sqrt{\frac{G}{g}} do = 0 \quad (5.2.11)$$

and due to the arbitrariness of $\delta \mathbf{R}$ in the volume and on the part o_2 of the surface, where the forces are prescribed, we arrive at the differential equations

$$2 \frac{\partial}{\partial q^s} \sqrt{g} \frac{\partial A}{\partial G_{st}} \mathbf{R}_t + \rho_0 \sqrt{g} \mathbf{K} = 0 \quad (5.2.12)$$

with the boundary conditions

$$2 \sqrt{\frac{g}{G}} \frac{\partial A}{\partial G_{st}} n_s \mathbf{R}_t = \sqrt{G^{ik} n_i n_k} \mathbf{F}. \quad (5.2.13)$$

In order to return to the static equations (3.3.3) and (3.3.6) of Chapter 1 it is now sufficient to recall relationship (1.2.10).

The equilibrium equations for the isotropic elastic body in terms of the displacements are reduced to the form

$$2\frac{\partial}{\partial q^s}\sqrt{g}\mathbf{R}_t\left[\left(\frac{\partial A}{\partial I_1}+I_1\frac{\partial A}{\partial I_2}\right)g^{st}-\frac{\partial A}{\partial I_2}G_{mn}g^{mt}g^{ns}+\right.$$
$$\left.I_3\frac{\partial A}{\partial I_3}G^{st}\right]+\rho_0\sqrt{g}\mathbf{K}=0, \quad (5.2.14)$$

$$2\sqrt{\frac{g}{G}}\left[\left(\frac{\partial A}{\partial I_1}+I_1\frac{\partial A}{\partial I_2}\right)g^{st}-\frac{\partial A}{\partial I_2}G_{mn}g^{mt}g^{ns}+I_3\frac{\partial A}{\partial I_3}G^{st}\right]\mathbf{R}_t n_s=$$
$$=\sqrt{G^{ik}n_i n_k}\mathbf{F}. \quad (5.12.15)$$

They can be represented in terms of their contravariant components of the external forces as follows, see eq. (E.2.2)

$$\frac{2}{\sqrt{g}}\frac{\partial}{\partial q^s}\sqrt{g}\left[\left(\frac{\partial A}{\partial I_1}+I_1\frac{\partial A}{\partial I_2}\right)g^{sq}-\frac{\partial A}{\partial I_2}G_{mn}g^{mq}g^{ns}+I_3\frac{\partial A}{\partial I_3}G^{sq}\right]+$$
$$2\left[\left(\frac{\partial A}{\partial I_1}+I_1\frac{\partial A}{\partial I_2}\right)g^{st}-\frac{\partial A}{\partial I_2}G_{mn}g^{mt}g^{ns}+I_3\frac{\partial A}{\partial I_3}G^{st}\right]\widetilde{\left\{ {q \atop st} \right\}}+\rho_0 K^q=0,$$
$$(5.2.16)$$

$$\left[\left(\frac{\partial A}{\partial I_1}+I_1\frac{\partial A}{\partial I_2}\right)g^{sq}-\frac{\partial A}{\partial I_2}G_{mn}g^{mq}g^{ns}+I_3\frac{\partial A}{\partial I_3}G^{sq}\right]n_s=$$
$$=\frac{1}{2}\sqrt{\frac{G}{g}G^{ik}n_i n_k}F^q. \quad (5.2.17)$$

Remark 2. In the case of an incompressible body, the functional Π is varied under the side condition

$$I_3-1=0, \quad (5.2.18)$$

and the specific strain energy depends only on the invariants I_1, I_2. By introducing the Lagrange multiplier $p\left(q^1,q^2,q^3\right)$ one can write down the varied integral in the form

$$\delta\iiint\limits_v\left[A+p\left(I_3-1\right)\right]d\tau_0=\iiint\limits_v\left(\frac{\partial A}{\partial I_1}\delta I_1+\frac{\partial A}{\partial I_2}\delta I_2+p\delta I_3\right)d\tau_0.$$
$$(5.2.19)$$

In the equilibrium equations in terms of the displacements one should replace

$$I_3\frac{\partial A}{\partial I_3}G^{sq} \quad \text{by} \quad pG^{sq}.$$

For Mooney's body these equations simplify considerably and take the form

$$\left.\begin{array}{l}\dfrac{2}{\sqrt{g}}\dfrac{\partial}{\partial q^s}\left[(C_1+I_1C_2)\,g^{sq}-C_2G_{mn}g^{mq}g^{ns}+pG^{sq}\right]\sqrt{g}+\\ 2\left[(C_1+I_1C_2)\,g^{st}-C_2G_{mn}g^{mt}g^{ns}+pG^{st}\right]\widetilde{\left\{ {q \atop st} \right\}}+\rho_0K^q=0,\\ \left[(C_1+I_1C_2)\,g^{sq}-C_2G_{mn}g^{mq}g^{ns}+pG^{sq}\right]n_s=\dfrac{1}{2}\sqrt{G^{ik}n_in_k}F^q.\end{array}\right\} \tag{5.2.20}$$

Let us recall that according to Ricci's theorem, see Section E.3, the derivatives of the components of the metric tensors are expressed in terms of these components and Christoffel's symbols, for instance

$$\begin{aligned}\frac{\partial G_{mn}}{\partial q^r}&=\widetilde{\left\{ {q \atop rm} \right\}}G_{qn}+\widetilde{\left\{ {q \atop rn} \right\}}G_{mq},\\ \frac{\partial g^{sq}}{\partial q^r}&=-\left\{ {s \atop rt} \right\}g^{tq}-\left\{ {q \atop rt} \right\}g^{st}\end{aligned}$$

and so on.

8.5.3 Complementary work of deformation

Let us start with the static equations in the volume and on the surface, eqs. (2.8.4) and (2.8.5), expressed in terms of the Piola-Kirchhoff tensor

$$\nabla\cdot\hat{D}+\rho_0\mathbf{K}=0,\quad \mathbf{n}\cdot\hat{D}do=\mathbf{F}dO=\mathbf{N}\cdot\hat{T}dO. \tag{5.3.1}$$

The variation of the strain energy δa, which is equal to the work of the external forces under the admissible displacement $\delta\mathbf{R}=\delta\mathbf{u}$ from the equilibrium position, is given by

$$\begin{aligned}\delta a=\iiint\limits_V\rho\mathbf{K}\cdot\delta\mathbf{R}d\tau+\iint\limits_O\mathbf{F}\cdot\delta\mathbf{R}dO&=\iiint\limits_v\rho_0\mathbf{K}\cdot\delta\mathbf{R}d\tau_0+\iint\limits_o\mathbf{n}\cdot\hat{D}\cdot\delta\mathbf{R}do\\ =\iiint\limits_v\left(\rho_0\mathbf{K}+\nabla\cdot\hat{D}\right)\cdot\delta\mathbf{R}d\tau_0&+\iiint\limits_v\hat{D}\cdot\cdot\left(\nabla\delta\mathbf{R}\right)^*d\tau_0\\ &=\iiint\limits_v\hat{D}\cdot\cdot\nabla\delta\mathbf{R}^*d\tau_0.\end{aligned} \tag{5.3.2}$$

While deriving this formula we used the equilibrium equation (5.3.1) in the volume, the transformation of the surface integral into the volume one, the formula for the divergence of the product of a tensor and a vector (B.3.10) and the permutation of operations ∇ and δ.

The integrand in eq. (5.3.2) is the variation of the specific strain energy

$$\delta A = \hat{D} \cdot\cdot \nabla \delta \mathbf{R}^*. \tag{5.3.3}$$

Referring to the definition of the gradient of the scalar with respect to the tensor, eq. (A.12.7), we have

$$\hat{D} = \frac{\partial A}{\partial \nabla \mathbf{R}}. \tag{5.3.4}$$

Expression (5.3.3) is rewritten as follows

$$\delta A = \delta \left(\hat{D} \cdot\cdot \nabla \mathbf{R}^* \right) - \delta \hat{D} \cdot\cdot \nabla \mathbf{R}^*, \tag{5.3.5}$$

and we introduce into consideration the quantities referred to as the specific complementary work and the complementary work of the strains

$$B = \hat{D} \cdot\cdot \nabla \mathbf{R}^* - A, \quad b = \iiint\limits_{v} B d\tau_0. \tag{5.3.6}$$

It follows from eqs. (5.3.5) and (5.3.6) that

$$\delta B = \delta \hat{D} \cdot\cdot \nabla \mathbf{R}^* = \nabla \mathbf{R} \cdot\cdot \delta \hat{D}^*. \tag{5.3.7}$$

Assuming now that B is expressed in terms of tensor $\hat{D}$ and using eq. (A.12.7) we arrive at the relationship

$$\nabla \mathbf{R} = \frac{\partial B}{\partial \hat{D}} \tag{5.3.8}$$

which is the inverse relationship for eq. (5.3.4).

In the performed Legendre transformation the specific strain energy plays the part of the generating function for the mapping $\nabla \mathbf{R} \to \hat{D}$, whilst the generating function for the mapping $\hat{\mathbf{D}} \to \nabla \mathbf{R}$ is the specific complementary work of strains.

Relation (5.3.4) is the constitutive law for the nonlinear elastic body and expresses tensor $\hat{D}$ in terms of $\nabla \mathbf{R}$. This constitutive law is in fact a system of nine equations enabling tensor $\nabla \mathbf{R}$ to be determined. The solubility of the system requires a non-vanishing Hessian

$$H = \left| \frac{\partial^2 A}{\dfrac{\partial u_s}{\partial q^t} \dfrac{\partial u_s}{\partial q^r}} \right| \neq 0 \quad (s, t, r = 1, 2, 3),$$

where $\nabla \mathbf{u} = \nabla \mathbf{R} - \hat{E}$ and $\partial u_s / \partial q^t$ denotes the components of $\nabla \mathbf{u}$. Of course, this problem is challenging and the solution for the semi-linear material will be obtained in Subsection 8.5.5.

8.5.4 Stationarity of the complementary work

Let us consider a nonlinear elastic body subjected to the "dead" mass and surface forces. We also consider a statically admissible equilibrium of the same body subjected to the same body forces, this equilibrium being close to the true equilibrium. For this varied state the static equations in the volume and on the surfaces are given by

$$\nabla \cdot \left(\hat{D} + \delta\hat{D}\right) + \rho_0 \mathbf{K} = 0, \quad \mathbf{n} \cdot \left(\hat{D} + \delta\hat{D}\right) do = \begin{cases} \mathbf{F} dO & \text{on} \quad o_2, \\ (\mathbf{F} + \delta\mathbf{F})\, dO & \text{on} \quad o_1, \end{cases} \tag{5.4.1}$$

since $\mathbf{K}dm = \mathbf{K}_0 dm$ in the volume and $\mathbf{F}dO = \mathbf{P}do$ on the part of surface where the surface forces are prescribed ($\mathbf{K}_0$ and $\mathbf{P}$ denote the forces in the initial state). Vector $\mathbf{R} = \mathbf{r} + \mathbf{u}$ is given on o_1. Because vector $\mathbf{R}$ is prescribed on o_1 an unknown reaction force $\delta\mathbf{F}dO$ appears on o_1 in the varied state. By eqs. (5.3.1) and (5.4.1) we have

$$\nabla \cdot \delta\hat{D} = 0, \quad \mathbf{n} \cdot \delta\hat{D} do = \begin{cases} 0 & \text{on} \quad o_2, \\ \delta\mathbf{F} dO & \text{on} \quad o_1. \end{cases} \tag{5.4.2}$$

Taking into account these relations and eqs. (5.3.6) and (5.3.7) we have

$$\delta b = \iiint\limits_v \delta\hat{D} \cdot\cdot \nabla\mathbf{R}^* d\tau_0 = \iiint\limits_v \left[\nabla \cdot \left(\delta\hat{D} \cdot \mathbf{R}\right) - \left(\nabla \cdot \delta\hat{D}\right) \cdot \mathbf{R}\right] d\tau_0$$
$$= \iint\limits_o \mathbf{n} \cdot \delta\hat{D} \cdot \mathbf{R} do = \delta \iint\limits_{o_1} \mathbf{n} \cdot \hat{D} \cdot \mathbf{R} do. \tag{5.4.3}$$

The variation sign is placed beyond the integral as vector $\mathbf{R}$ is prescribed on o_1.

We thus arrive at the equality

$$\delta \left(b - \iint\limits_{o_1} \mathbf{n} \cdot \hat{D} \cdot \mathbf{R} do \right) = 0, \tag{5.4.4}$$

expressing the principle of stationarity of the complementary work, namely, the actual equilibrium state differs from any statically admissible state in that the value

$$b_* = b - \iint\limits_{o_1} \mathbf{n} \cdot \hat{D} \cdot \mathbf{R} do, \tag{5.4.5}$$

referred to as the complementary work, has a stationary value.

In the actual state of stress the solid is continuous, that is, tensor $\hat{C} = \nabla\mathbf{R}$ obtained in terms of tensor $\hat{D}$ by means of eq. (5.3.8) needs to be integrable. In other words, it should be the gradient of vector $\mathbf{R}$. By virtue of eq. (B.6.7) it means that the rotor of this tensor must vanish

$$\nabla \times \hat{C} = 0. \tag{5.4.6}$$

This condition is equivalent to the principle of stationarity of the complementary work. To put it differently, this condition presents Euler's equations for the variational problem of stationarity of functional b_* subjected to condition (5.4.2). This is verified by the method demonstrated in Subsection 4.2.5. Entering a Lagrange vector λ we have

$$\begin{aligned}
\delta b &= \iiint_v \left[\delta B + \left(\nabla \cdot \delta \hat{D}\right) \cdot \boldsymbol{\lambda}\right] d\tau_0 \\
&= \iiint_v \left[\delta \hat{D} \cdot\cdot \hat{C}^* + \nabla \cdot \left(\delta \hat{D} \cdot \boldsymbol{\lambda}\right) - \delta \hat{D} \cdot\cdot \nabla \boldsymbol{\lambda}^*\right] d\tau_0 \\
&= \iiint_v \delta \hat{D} \cdot\cdot \left(\hat{C}^* - \nabla \boldsymbol{\lambda}^*\right) d\tau_0 + \iint_{o_1} \mathbf{n} \cdot \delta \hat{D} \cdot \boldsymbol{\lambda} do
\end{aligned}$$

and by virtue of the principle of stationarity of the complementary work we obtain

$$\delta b_* = \iiint_v \delta \hat{D} \cdot\cdot \left(\hat{C}^* - \nabla \boldsymbol{\lambda}^*\right) d\tau_0 - \iint_{o_1} \mathbf{n} \cdot \delta \hat{D} \cdot (\mathbf{R} - \boldsymbol{\lambda})\, do = 0. \tag{5.4.7}$$

Under an appropriate choice of λ one can consider variation $\delta \hat{D}$ as being arbitrary in the volume and on o_1. Hence tensor $\hat{C}$ is the gradient of some vector λ in v whereas this vector is equal to $\mathbf{R}$ on o_1. We thus arrive at relationship (5.4.6) expressing the continuity condition in terms of Piola's tensor $\hat{D}$.

8.5.5 Specific complementary work of strains for the semi-linear material

The specific strain energy is given by eq. (2.8.7)

$$A = \frac{1}{2} \lambda s_1^2 + \mu s_2, \tag{5.5.1}$$

where s_1 and s_2 are expressed in terms of the invariants of tensor $\hat{G}^{\times^{1/2}}$ with the help of eqs. (2.7.3) and (2.6.4)

$$I_1\left(\hat{G}^{\times^{1/2}}\right) = s_1 + 3, \quad I_1\left(\hat{G}^{\times}\right) = s_2 + 2s_1 + 3. \tag{5.5.2}$$

The constitutive law is given by formula (2.8.8)

$$\hat{D} = \left[(\lambda s_1 - 2\mu)\,\hat{G}^{\times^{-1/2}} + 2\mu\hat{E}\right] \cdot \nabla \mathbf{R}. \tag{5.5.3}$$

This yields the following representations for the symmetric tensors

$$\hat{D} \cdot \hat{D}^* = \left[(\lambda s_1 - 2\mu)\,\hat{E} + 2\mu\hat{G}^{\times^{1/2}}\right]^2,$$
$$\left(\hat{D} \cdot \hat{D}^*\right)^{1/2} = (\lambda s_1 - 2\mu)\,\hat{E} + 2\mu\hat{G}^{\times^{1/2}}. \tag{5.5.4}$$

Taking into account eq. (5.5.2) one finds their first invariants denoted as f_1 and f_2

$$f_1 = I_1\left[\left(\hat{D} \cdot \hat{D}^*\right)^{1/2}\right] = (3\lambda + 2\mu)\, s_1,$$
$$f_2 = I_1\left(\hat{D} \cdot \hat{D}^*\right) = \lambda\,(3\lambda + 4\mu)\, s_1^2 + 4\mu^2 s_2. \tag{5.5.5}$$

The expressions for s_1 and s_2 in terms of the invariants f_1 and f_2 can be set as follows

$$s_1 = \frac{f_1}{3\lambda + 2\mu}, \quad s_2 = \frac{1}{4\mu^2}\left(f_2 - \frac{\lambda\,(3\lambda + 4\mu)}{(3\lambda + 2\mu)^2} f_1^2\right). \tag{5.5.6}$$

Inserting into eq. (5.5.1) allows the specific strain energy to be expressed in terms of the invariants of tensor (5.5.4)

$$A = \frac{1}{4\mu}\left(f_2 - \frac{\nu}{1+\nu} f_1^2\right) \quad \left(\nu = \frac{\lambda}{2\,(\lambda + \mu)}\right). \tag{5.5.7}$$

Using eqs. (5.5.2) and (5.5.6) one obtains

$$\hat{D} \cdot\cdot \nabla \mathbf{R}^* = I_1\left(\hat{D} \cdot \nabla \mathbf{R}^*\right) = I_1\left[(\lambda s_1 - 2\mu)\,\hat{G}^{\times^{1/2}} + 2\mu\hat{G}^{\times}\right] =$$
$$(\lambda s_1 - 2\mu)\,(s_1 + 3) + 2\mu\,(s_2 + 2s_1 + 3) = \lambda s_1^2 + 2\mu s_2 + (3\lambda + 2\mu)\, s_1 = 2A + f_1$$

and according to definition (5.3.6) the specific complementary work expressed in terms of tensor $\hat{D}$ is equal to

$$B = A + f_1 = \frac{1}{4\mu}\left(f_2 - \frac{\nu}{1+\nu} f_1^2\right) + f_1. \tag{5.5.8}$$

In order to determine tensor $\nabla \mathbf{R}$ by means of eq. (5.3.4) it is necessary to obtain the derivatives of the invariants f_1, f_2 with respect to $\hat{D}$. We have

$$\delta f_2 = \delta\left(\hat{D} \cdot\cdot \hat{D}^*\right) = \hat{D} \cdot\cdot \delta\hat{D}^* + \delta\hat{D} \cdot\cdot \hat{D}^* = 2\hat{D} \cdot\cdot \delta\hat{D}^*$$

and according to definition (A.12.7)

$$\frac{\partial f_2}{\partial \hat{D}} = 2\hat{D}. \tag{5.5.9}$$

Before we proceed to determining δf_1 we notice that the derivative of the first invariant of any symmetric tensor $\hat{Q}$ with respect to $\hat{Q}^2$ is equal to

$$\frac{\partial I_1\left(\hat{Q}\right)}{\partial \hat{Q}^2} = \frac{1}{2}\hat{Q}^{-1},$$

$$\delta I_1\left(\hat{Q}\right) = \frac{1}{2}\hat{Q}^{-1}\cdot\cdot\,\delta\hat{Q}^2 = \frac{1}{2}\hat{Q}^{-1}\cdot\cdot\,\hat{Q}\cdot\delta\hat{Q} + \frac{1}{2}\hat{Q}^{-1}\cdot\cdot\,\delta\hat{Q}\cdot\hat{Q},$$

as $\hat{Q}^{-1}\cdot\cdot\,\hat{Q}\cdot\delta\hat{Q} = \left(\hat{Q}^{-1}\right)^{*}\cdot\cdot\,\hat{Q}^{*}\cdot\delta\hat{Q}^{*} = \hat{E}\cdot\delta\hat{Q}^{*} = \delta I_1\left(\hat{Q}\right)$ for the symmetric tensor $\hat{Q}$.

Applying this result to the symmetric tensor $\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}$ we obtain

$$\begin{aligned}\delta f_1 &= \frac{1}{2}\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\cdot\cdot\,\delta\left(\hat{D}\cdot\hat{D}^{*}\right)\\ &= \frac{1}{2}\left[\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\cdot\hat{D}\cdot\cdot\,\delta\hat{D}^{*} + \left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\cdot\cdot\,\delta\hat{D}\cdot\hat{D}^{*}\right]\end{aligned}$$

and in the second term in the brackets one can replace tensor $\delta\hat{D}\cdot\hat{D}^{*}$ by $\left(\delta\hat{D}\cdot\hat{D}^{*}\right)^{*} = \hat{D}\cdot\delta\hat{D}^{*}$ due to the symmetry of tensor $\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}$. The results are

$$\delta f_1 = \left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\cdot\hat{D}\cdot\cdot\,\delta\hat{D}^{*}, \quad \frac{\partial f_1}{\partial \hat{D}} = \left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\cdot\hat{D}. \tag{5.5.10}$$

Turning now to eq. (5.5.8) we obtain

$$\nabla\mathbf{R} = \frac{\partial B}{\partial \hat{D}} = \frac{1}{4\mu}\left(\frac{\partial f_2}{\partial \hat{D}} - \frac{2\nu}{1+\nu}f_1\frac{\partial f_1}{\partial \hat{D}}\right) + \frac{\partial f_1}{\partial \hat{D}}$$

and by eqs. (5.5.9) and (5.5.10)

$$\nabla\mathbf{R} = \frac{1}{2\mu}\left[\hat{E} + \left(2\mu - \frac{\nu}{1+\nu}f_1\right)\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\right]\cdot\hat{D}. \tag{5.5.11}$$

In accordance with eq. (5.4.6) vector $\mathbf{R}$ is determined by the integral

$$\mathbf{R} = \frac{1}{2\mu}\int_{M^0}^{M}\left\{dr\cdot\left[\hat{E} + \left(2\mu - \frac{\nu}{1+\nu}f_1\right)\left(\hat{D}\cdot\hat{D}^{*}\right)^{-1/2}\right]\cdot\hat{D}\right\} + \mathbf{R}\left(M^0\right) \tag{5.5.12}$$

along any curve between the points M^0, M. The base vectors in the deformed volume are given by

$$\mathbf{R}_s = \frac{1}{2\mu}\mathbf{r}_s \cdot \left[\hat{E} + \left(2\mu - \frac{\nu}{1+\nu} f_1\right)\left(\hat{D}\cdot\hat{D}^*\right)^{-1/2}\right]\cdot\hat{D}. \tag{5.5.13}$$

Remark 1. When the mass force is absent the state of stress satisfying the static equation (5.3.1) in $v-$volume can be prescribed by assuming

$$\hat{D} = \nabla \times \hat{\Phi}, \tag{5.5.14}$$

where $\hat{\Phi}$ is any twice-differentiable tensor. It should be taken such that the boundary condition on o_2

$$\mathbf{n}\cdot\hat{D} = \mathbf{F}\frac{dO}{do} = \mathbf{P} \tag{5.5.15}$$

holds. The principle of complementary work can not be generalised to the case of an arbitrary (not "dead") loading inasmuch as prescribing $\mathbf{K}, \mathbf{F}$ requires knowledge of the geometry of the deformed body.

Remark 2. In the case of "dead" loading, Piola's tensor given in the vector basis of the initial state allows us to express the principle of stationarity of the complementary work in terms of the static quantities only and avoid the difficulty of removing the gradients of the displacement vector. The presentation of Subsections 8.5.3-8.5.5 is based on the paper by Zubov[2].

[2] Zubov, L.M. The stationarity principle in the nonlinear theory of elasticity (in Russian). Prikladnaya Matematika i Mechanika, vol. 34, No 2, 1970.

9

Problems and methods of the nonlinear theory of elasticity

9.1 The state of stress under affine transformation

9.1.1 The stress tensor under affine transformation

Sections 9.1-9.3 are devoted to those problems of the nonlinear theory of elasticity whose solutions can be obtained without prescribing a particular form of the specific strain energy as a function of the strain invariants. However the particular form is needed for numerical calculations.

The solution is constructed by "the inverse method" and includes the following steps: i) one chooses a particular form of transformation from $v-$volume into $V-$volume, ii) one constructs an expression for the strain measure (or strain tensor), iii) one chooses a constitutive law and proves whether the obtained stress tensor satisfies the static equations in $V-$volume, and iv) one determines the surface forces required for this state of stress. The obtained solutions are meaningful if the resulting surface forces can be realised with relative ease (the volume forces are assumed to be absent) and the obtained stress distribution can be replaced by another statically equivalent system of surface forces.

The affine transformation was defined in Subsection 2.6.1 where the strain measure $\hat{M} = \hat{g}^{\times -1}$ was given by expression (6.1.3)

$$\hat{M} = \hat{\Lambda} \cdot \hat{\Lambda}^*, \tag{1.1.1}$$

in which $\hat{\Lambda}$ denotes the transformation tensor.

The stress tensor given by the constitutive law in Finger's form, eq. (2.4.1) of Chapter 8, is as follows

$$\hat{T} = \frac{2}{\sqrt{I_3}} \left[\left(\frac{\partial A}{\partial I_1} + I_1 \frac{\partial A}{\partial I_2} \right) \hat{\Lambda} \cdot \hat{\Lambda}^* - \frac{\partial A}{\partial I_2} \left(\hat{\Lambda} \cdot \hat{\Lambda}^* \right)^2 + I_3 \frac{\partial A}{\partial I_3} \hat{E} \right]. \qquad (1.1.2)$$

Here $\hat{E} = \mathbf{i}_s \mathbf{i}_s$ denotes the unit tensor (in the Cartesian coordinate system) and $I_k \left(\hat{M} \right) = I_k \left(\hat{G}^\times \right)$.

Using the notation of Subsections 2.5.2 and 2.5.3 we have

$$\begin{aligned} & I_1 \hat{\Lambda} \cdot \hat{\Lambda}^* - \left(\hat{\Lambda} \cdot \hat{\Lambda}^* \right)^2 = I_1 \hat{M} - \hat{M}^2 \\ &= (G_1 + G_2 + G_3) \left(G_1 \overset{11}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + G_2 \overset{22}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + G_3 \overset{33}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right) - \left(G_1^2 \overset{11}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + G_2^2 \overset{22}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + G_3^2 \overset{33}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right) \\ &= G_1 G_2 G_3 \left[\left(\frac{1}{G_2} + \frac{1}{G_3} \right) \overset{11}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + \left(\frac{1}{G_3} + \frac{1}{G_1} \right) \overset{22}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + \left(\frac{1}{G_1} + \frac{1}{G_2} \right) \overset{33}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right] \\ &= I_3 \left[I_1 \left(\hat{g}^\times \right) \hat{E} - \hat{g}^\times \right], \end{aligned}$$

since $\hat{g}^\times = \hat{M}^{-1}$. We thus arrive at the relationship

$$I_1 \left(\hat{G}^\times \right) \hat{M} - \hat{M}^2 = I_2 \left(\hat{G}^\times \right) \hat{E} - I_3 \left(\hat{G}^\times \right) \hat{g}^\times,$$

the expressions for the invariants being given by formulae (6.1.8) of Chapter 2. In particular

$$\sqrt{I_3 \left(\hat{G}^\times \right)} = |\lambda_{st}| = \lambda.$$

Inserting this equation into eq. (1.1.2) leads to the following representation for the stress tensor

$$\frac{1}{2} \hat{T} = \frac{1}{\lambda} \frac{\partial A}{\partial I_1} \hat{M} - \lambda \frac{\partial A}{\partial I_2} \hat{M}^{-1} + \left(\frac{1}{\lambda} I_2 \frac{\partial A}{\partial I_2} + \lambda \frac{\partial A}{\partial I_3} \right) \hat{E}, \qquad (1.1.3)$$

as obtained by Truesdell.

In the case of no mass forces the equations of statics are satisfied since $\hat{T}$ is a constant tensor

$$\tilde{\nabla} \cdot \hat{T} = 0.$$

For the incompressible medium $\lambda = 1$, that is

$$\hat{T} = 2 \left(\frac{\partial A}{\partial I_1} \hat{M} - \frac{\partial A}{\partial I_2} \hat{M}^{-1} \right) - p\hat{E}, \quad A = A(I_1, I_2), \qquad (1.1.4)$$

where p denotes the unknown uniform pressure

$$p = -\left[I_2 \frac{\partial A}{\partial I_2} + \left. \frac{\partial A}{\partial I_3} \right|_{I_3=1} \right]. \tag{1.1.5}$$

The principal values of tensor $\hat{M}$ are easily expressed in terms of the principal elongations $\delta_q = \sqrt{G_q} - 1$. We have

$$\begin{aligned}
\hat{M} &= \overset{1}{\tilde{\mathbf{e}}}\overset{1}{\tilde{\mathbf{e}}} (1+\delta_1)^2 + \overset{2}{\tilde{\mathbf{e}}}\overset{2}{\tilde{\mathbf{e}}} (1+\delta_2)^2 + \overset{3}{\tilde{\mathbf{e}}}\overset{3}{\tilde{\mathbf{e}}} (1+\delta_3)^2, \\
\hat{M}^{-1} &= \overset{1}{\tilde{\mathbf{e}}}\overset{1}{\tilde{\mathbf{e}}} (1+\delta_1)^{-2} + \overset{2}{\tilde{\mathbf{e}}}\overset{2}{\tilde{\mathbf{e}}} (1+\delta_2)^{-2} + \overset{3}{\tilde{\mathbf{e}}}\overset{3}{\tilde{\mathbf{e}}} (1+\delta_3)^{-2}.
\end{aligned}$$

The expressions for the invariants are set in the form

$$\begin{aligned}
&I_1 = (1+\delta_1)^2 + (1+\delta_2)^2 + (1+\delta_3)^2, \\
&I_2 = (1+\delta_1)^2 (1+\delta_2)^2 + (1+\delta_2)^2 (1+\delta_3)^2 + (1+\delta_3)^2 (1+\delta_1)^2, \\
&I_3 = \lambda^2 = [(1+\delta_1)(1+\delta_2)(1+\delta_3)]^2.
\end{aligned}$$

The stress tensor has the same principal axes $\overset{q}{\tilde{\mathbf{e}}}$ as $\hat{M}$. Denoting the principal values of the stress tensor by t_q we have

$$\hat{T} = t_1 \overset{1}{\tilde{\mathbf{e}}}\overset{1}{\tilde{\mathbf{e}}} + t_2 \overset{2}{\tilde{\mathbf{e}}}\overset{2}{\tilde{\mathbf{e}}} + t_3 \overset{3}{\tilde{\mathbf{e}}}\overset{3}{\tilde{\mathbf{e}}},$$

and

$$\begin{aligned}
\frac{1}{2} t_1 = (1+\delta_1) \Bigg[&\frac{\partial A}{\partial I_1} \frac{1}{(1+\delta_2)(1+\delta_3)} + \frac{\partial A}{\partial I_2} \left(\frac{1+\delta_2}{1+\delta_3} + \frac{1+\delta_3}{1+\delta_2} \right) + \\
&\frac{\partial A}{\partial I_3} (1+\delta_2)(1+\delta_3) \Bigg].
\end{aligned} \tag{1.1.6}$$

The formulae for the remaining principal values are obtained by a circular permutation of the subscripts of δ_q. These formulae are just another form of the general relation (2.6.6) of Chapter 8.

Due to eq. (1.1.4) we have for the incompressible medium

$$(1+\delta_1)(1+\delta_2)(1+\delta_3) = 1, \; t_s = 2 \frac{\partial A}{\partial I_1} (1+\delta_s)^2 - 2 \frac{\partial A}{\partial I_2} (1+\delta_s)^{-2} - p. \tag{1.1.7}$$

9.1.2 Uniform compression

The affine transformation degenerates into the similarity transformation with the similarity coefficient

$$K = 1 + \delta = \left(\frac{\rho_0}{\rho} \right)^{1/3}, \quad (\delta_1 = \delta_2 = \delta_3 = \delta), \tag{1.2.1}$$

and due to eq. (1.1.6)

$$t_1 = t_2 = t_3 = 2\left(\frac{1}{K}\frac{\partial A}{\partial I_1} + 2K\frac{\partial A}{\partial I_2} + K^3\frac{\partial A}{\partial I_3}\right) = -p, \tag{1.2.2}$$

where p denotes the uniform pressure. At the same time

$$I_1 = 3K^2, \quad I_2 = 3K^4, \quad I_3 = K^6,$$

hence

$$\frac{dA}{dK} = 6K^2\left(\frac{1}{K}\frac{\partial A}{\partial I_1} + 2K\frac{\partial A}{\partial I_2} + K^3\frac{\partial A}{\partial I_3}\right).$$

We arrive at the following relationship

$$p = -\frac{1}{3K^2}\frac{dA}{dK} = f(\rho), \tag{1.2.3}$$

indicating that in the case of uniform compression any form of the relation between p and ρ is consistent with the relationships of nonlinear elasticity.

9.1.3 *Uniaxial tension*

Let the axis of the tensile rod be coincident with axis X_3, then $t_1 = t_2 = 0$, $\delta_1 = \delta_2$ and due to eq. (1.1.6) we have

$$\frac{\partial A}{\partial I_1}\frac{1}{(1+\delta_3)(1+\delta_1)} + \frac{\partial A}{\partial I_2}\left(\frac{1+\delta_1}{1+\delta_3} + \frac{1+\delta_3}{1+\delta_1}\right) + \frac{\partial A}{\partial I_3}(1+\delta_1)(1+\delta_3) = 0 \tag{1.3.1}$$

$$\frac{1}{2}t_3 = (1+\delta_3)\left[\frac{\partial A}{\partial I_1}(1+\delta_1)^{-2} + 2\frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3}(1+\delta_1)^2\right]. \tag{1.3.2}$$

We can express δ_1 in terms of δ_3 with the help of the first equation, then by means of the second equation we obtain the dependence $t_3 = t_3(\delta_3)$. However eq. (1.3.1) can have no real-valued roots and this indicates that the surface forces are required on the lateral surface $(t_1, t_2 \neq 0)$ for the simple uniaxial tension $\delta_1 = \delta_2$. Equation (1.3.1) can have a non-unique solution, i.e. the possibility of a non-unique dependence of the tensile force on the elongation δ_3 is not excluded.

Such complications do not take place for the incompressible medium. In this case we have three equations for determining the three unknowns t_3, p, δ_3. These equations are: the two equations (1.1.7)

$$\left.\begin{aligned} 2\frac{\partial A}{\partial I_1}(1+\delta_1)^2 - 2\frac{\partial A}{\partial I_2}(1+\delta_1)^{-2} - p = 0, \\ t_3 = 2\frac{\partial A}{\partial I_1}(1+\delta_3)^2 - 2\frac{\partial A}{\partial I_2}(1+\delta_3)^{-2} - p \end{aligned}\right\} \tag{1.3.3}$$

and the equation for volume conservation

$$1 + \delta_3 = (1 + \delta_1)^{-2}. \tag{1.3.4}$$

Then we obtain

$$t_3 (1 + \delta_1)^2 = t_3^* = 2 \left(\frac{\partial A}{\partial I_1} + \frac{1}{1 + \delta_3} \frac{\partial A}{\partial I_2} \right) \left[1 + \delta_3 - (1 + \delta_3)^{-2} \right], \tag{1.3.5}$$

where t_3^* denotes the force acting on a unit area of the initial cross-section of the tensile rod.

For Mooney's material, see Subsection 8.4.9, the first derivative of t_3^* with respect to δ_3 is positive for $-1 < \delta_3 < \infty$ since $C_1 \geq 0, C_2 \geq 0$, i.e. the tensile force increases monotonically as δ_3 increases, the growth decreasing with an increase in δ_3. However, in contrast to the material in the simplified Signorini theory (see Subsection 8.4.4) the tension diagram has no asymptote, that is, the tensile force breaking the rod ($\delta_3 \to \infty$) increases without bound.

9.1.4 Simple shear

Tensors $\hat{\Lambda}$ and $\hat{\Lambda}^*$ are given by formulae (6.3.2) of Chapter 2

$$\begin{aligned}
\hat{M} &= \hat{\Lambda} \cdot \hat{\Lambda}^* = \left(\hat{E} + s\mathbf{i}_1\mathbf{i}_2 \right) \cdot \left(\hat{E} + s\mathbf{i}_2\mathbf{i}_1 \right) = \hat{E} + s\left(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1 \right) + s^2 \mathbf{i}_1\mathbf{i}_1, \\
\hat{M}^{-1} &= \hat{\Lambda}^{*^{-1}} \cdot \hat{\Lambda}^{-1} = \left(\hat{E} - s\mathbf{i}_2\mathbf{i}_1 \right) \cdot \left(\hat{E} - s\mathbf{i}_1\mathbf{i}_2 \right) \\
&= \hat{E} - s\left(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1 \right) + s^2 \mathbf{i}_2\mathbf{i}_2, \quad I_1 = I_2 = 3 + s^2, \quad I_3 = \lambda^2 = 1.
\end{aligned}$$

By means of eq. (1.1.4) we obtain the expression for the stress tensor

$$\begin{aligned}
\frac{1}{2}\hat{T} = \left[\frac{\partial A}{\partial I_1} + \left(2 + s^2 \right) \frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3} \right] \hat{E} + \left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} \right) \left(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1 \right) s + \\
s^2 \left(\frac{\partial A}{\partial I_1} \mathbf{i}_1\mathbf{i}_1 + \frac{\partial A}{\partial I_2} \mathbf{i}_2\mathbf{i}_2 \right).
\end{aligned} \tag{1.4.1}$$

Its components along the axes of the Cartesian coordinates $OXYZ$ are equal to

$$\left.\begin{aligned}
\frac{1}{2} t_{11} &= \frac{\partial A}{\partial I_1} \left(1 + s^2 \right) + \frac{\partial A}{\partial I_2} \left(2 + s^2 \right) + \frac{\partial A}{\partial I_3}, \\
\frac{1}{2} t_{22} &= \frac{\partial A}{\partial I_1} + \left(1 + s^2 \right) \frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3}, \\
\frac{1}{2} t_{12} &= \left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} \right) s, \\
\frac{1}{2} t_{33} &= \frac{\partial A}{\partial I_1} + \left(2 + s^2 \right) \frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3}, \quad t_{23} = t_{31} = 0.
\end{aligned}\right\} \tag{1.4.2}$$

The unit normal vector $\mathbf{N}$ to the surface $a_1 = \text{const}$ is determined by eq. (3.3.5) of Chapter 2 in which

$$\mathbf{n} = \mathbf{i}_1, \quad \mathbf{n} \cdot \hat{G}^{\times^{-1}} \cdot \mathbf{n} = G^{11} = 1 + s^2,$$
$$\nabla\mathbf{R} = \hat{E} + \mathbf{i}_2\mathbf{i}_1 s, \quad (\nabla\mathbf{R})^{-1} = \hat{E} - \mathbf{i}_2\mathbf{i}_1 s,$$

therefore

$$\mathbf{N} = \frac{\mathbf{i}_1 - \mathbf{i}_2 s}{\sqrt{1+s^2}}$$

and the normal stress on this surface is

$$\sigma_N = \mathbf{N} \cdot \hat{T} \cdot \mathbf{N} = \frac{2}{1+s^2}\left[\frac{\partial A}{\partial I_1} + \left(2+s^2\right)\frac{\partial A}{\partial I_2} + \left(1+s^2\right)\frac{\partial A}{\partial I_3}\right]. \quad (1.4.3)$$

If the initial state is the natural one, then by eq. (2.2.3) of Chapter 8

$$(\sigma_N)_0 = 2\left(\frac{\partial A}{\partial I_1} + 2\frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3}\right)_0 = 0,$$

which means that the normal stress on surface $a_1 = \text{const}$, required for the simple shear, is proportional to s^2.

The shear stress on these surfaces is equal to

$$\tau_{Nt} = \mathbf{N} \cdot \hat{T} \cdot \mathbf{t} = \frac{2s}{1+s^2}\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\right), \quad (1.4.4)$$

where $\mathbf{t}$ denotes the unit vector of the tangent to surface $a_1 = \text{const}$

$$\mathbf{t} = \left|\frac{\partial\mathbf{R}}{\partial a_2}\right|^{-1}\frac{\partial\mathbf{R}}{\partial a_2} = \frac{\mathbf{i}_2 + \mathbf{i}_1 s}{\sqrt{1+s^2}}.$$

In the linear theory of elasticity the simple shear is caused only by shear stresses on surfaces $a_1 = \text{const}, a_2 = \text{const}$. In the nonlinear theory the realisation of the simple shear requires normal stresses on all faces of the parallelepiped. The normal stress on face $a_1 = \text{const}$ is needed to conserve the volume (the effect predicted by Kelvin) whereas the normal stress in the plane of shear is needed to ensure the dimension along the OY axis (the effect predicted by Poynting). The applied shear stresses are proportional to s^2 however their values on different faces are different. The shear stresses τ_{Nt} and t_{12} on faces $a_1 = \text{const}$ and $a_2 = \text{const}$ respectively differ in the terms of order s^2.

In the linear approximation for Mooney's material (Subsection 8.4.9) we have by eq. (1.4.2)

$$t_{12} = 2(C_1 + C_2)\,s \quad (1.4.5)$$

and by eq. (1.3.5)

$$t_3 = 6\left(C_1 + C_2\right)\delta_3. \tag{1.4.6}$$

This provides us with the right to refer to $2\left(C_1 + C_2\right)$ and $6\left(C_1 + C_2\right)$ as the shear modulus and Young modulus respectively

$$2\left(C_1 + C_2\right) = \mu, \quad 6\left(C_1 + C_2\right) = E. \tag{1.4.7}$$

"Poisson's ratio" turns out to be equal to 1/2

$$\frac{E}{\mu} = 2\left(1 + \nu\right) = 3, \quad \nu = \frac{1}{2},$$

which should be expected as the material is incompressible.

Rivlin took only the first term in Mooney's formula ($C_2 = 0$) and introduced the "neo-Hookean" body. According to eqs. (1.4.7) and (1.1.7) the constitutive equations for this body in terms of the principal axes are as follows

$$t_s = 2C_1\left(1 + \delta_3\right)^2 - p = \frac{1}{3}E\left(1 + \delta_s\right)^2 - p. \tag{1.4.8}$$

Varga considered a simplified variant of these relations, i.e.

$$t_s = \frac{2}{3}E\delta_s - p^*, \tag{1.4.9}$$

and mentioned an acceptable agreement between the calculations and some test results on rubber specimens.

9.2 Elastic layer

9.2.1 Cylindrical bending of the rectangular plate

The deformation of a plate from incompressible material was studied in Subsection 2.6.5. According to eqs. (6.5.5) and (6.5.6) of Chapter 2 the components of the strain measures $\hat{G}^{\times}, \hat{G}^{\times^{-1}}$ and the principal invariants $I_k\left(\hat{G}\right)$ are given by

$$\left.\begin{aligned}
G_{11} &= \frac{b^2}{\alpha^2 e^2}C^{-2}(a), \; G_{22} = \frac{\alpha^2}{b^2}C^2(a), \; G_{33} = e^2; \quad G_{sk} = 0 \;\; (s \neq k), \\
G^{11} &= \frac{\alpha^2 e^2}{b^2}C^2(a), \; G^{22} = \frac{b^2}{\alpha^2}C^{-2}(a), \; G^{33} = \frac{1}{e^2}; \quad G^{sk} = 0 \;\; (s \neq k), \\
I_1 &= \frac{b^2}{\alpha^2 e^2}C^{-2}(a) + \frac{\alpha^2}{b^2}C^2(a) + e^2, \quad I_2 = C^2\frac{\alpha^2 e^2}{b^2} + \frac{b^2}{\alpha^2}C^{-2} + \frac{1}{e^2},
\end{aligned}\right\} \tag{2.1.1}$$

the material coordinates being the Cartesian coordinates $a_1 = a, a_2, a_3$ of v–volume (the initial state).

Due to eqs. (2.1.5) and (2.1.16) of Chapter 8 the components of the energetic tensor expressed in terms of the orthogonal coordinate system are given by

$$\left.\begin{aligned}
\frac{1}{2}t^{11} &= \frac{\partial A}{\partial I_1} + e^2\frac{\partial A}{\partial I_2} + C^2(a)\frac{\alpha^2}{b^2}\left(\frac{\partial A}{\partial I_2} + e^2\overset{-1}{c}\right),\\
\frac{1}{2}t^{22} &= \frac{\partial A}{\partial I_1} + e^2\frac{\partial A}{\partial I_2} + \frac{b^2}{\alpha^2 e^2}C^{-2}(a)\left(\frac{\partial A}{\partial I_2} + e^2\overset{-1}{c}\right),\\
\frac{1}{2}t^{33} &= \frac{\partial A}{\partial I_1} + \left(\frac{b^2}{\alpha^2 e^2}C^{-2}(a) + \frac{\alpha^2}{b^2}C^2(a)\right)\frac{\partial A}{\partial I_2} + \frac{\overset{-1}{c}}{e^2},\\
t^{sk} &= 0 \quad (s \neq k).
\end{aligned}\right\} \tag{2.1.2}$$

The specific strain energy $A(I_1, I_2)$ can be viewed as being a function of $C(a)$

$$\frac{\partial A}{\partial C} = \frac{\partial A}{\partial I_1}\frac{\partial I_1}{\partial C} + \frac{\partial A}{\partial I_2}\frac{\partial I_2}{\partial C},$$

where

$$\frac{\partial I_1}{\partial C} = 2C\left(\frac{\alpha^2}{b^2} - \frac{b^2}{\alpha^2 e^2}C^{-4}\right) = \frac{1}{e^2}\frac{\partial I_2}{\partial C}. \tag{2.1.3}$$

For this reason

$$\frac{\partial A}{\partial I_1} + e^2\frac{\partial A}{\partial I_2} = \frac{\partial A}{\partial C}\frac{\partial C}{\partial I_1}$$

and excluding the term with the unknown $\overset{-1}{c}$ from the first two equations in (2.1.2) leads to the following equality

$$\frac{1}{2}t^{11}\frac{b^2}{\alpha^2 e^2}C^{-2} - \frac{1}{2}t^{22}C^2\frac{\alpha^2}{b^2} = \frac{\partial A}{\partial C}\frac{\partial C}{\partial I_1}\left(\frac{b^2}{\alpha^2 e^2}C^{-2} - C^2\frac{\alpha^2}{b^2}\right) = -\frac{1}{2}C\frac{\partial A}{\partial C}, \tag{2.1.4}$$

where we also used formula (2.1.3). Another relation between t^{11} and t^{22} follows from the equations of statics. Keeping in mind that due to eq. (6.5.1) of Chapter 2

$$\begin{aligned}
\mathbf{R}_1 &= \frac{\partial x_1}{\partial a_1}\mathbf{i}_1 + \frac{\partial x_2}{\partial a_1}\mathbf{i}_2 + \frac{\partial x_3}{\partial a_1}\mathbf{i}_3\\
&= C'(a)\left(\mathbf{i}_1\cos\frac{\alpha a_2}{b} + \mathbf{i}_2\sin\frac{\alpha a_2}{b}\right) = C'(a)\,\mathbf{e}_r,\\
\mathbf{R}_2 &= \frac{\partial x_1}{\partial a_2}\mathbf{i}_1 + \frac{\partial x_2}{\partial a_2}\mathbf{i}_2 + \frac{\partial x_3}{\partial a_2}\mathbf{i}_3\\
&= C(a)\frac{\alpha}{b}\left(-\mathbf{i}_1\sin\frac{\alpha a_2}{b} + \mathbf{i}_2\cos\frac{\alpha a_2}{b}\right) = C(a)\frac{\alpha}{b}\mathbf{e}_\theta,\\
\mathbf{R}_3 &= e\mathbf{i}_3, \quad \sqrt{G} = 1,
\end{aligned}$$

we can set the vectorial equation of statics in the form

$$\frac{\partial}{\partial a_1} t^{11} C'(a)\,\mathbf{e}_r + \frac{\partial}{\partial a_2} t^{22} C(a) \frac{\alpha}{b} \mathbf{e}_\theta + \frac{\partial}{\partial a_3} t^{33} e \mathbf{i}_3 = 0$$

or

$$\mathbf{e}_r \left[\frac{\partial}{\partial a} t^{11} C'(a) - \frac{\alpha^2}{b^2} t^{22} C(a) \right] + \frac{\alpha}{b} \mathbf{e}_\theta C(a) \frac{\partial t^{22}}{\partial a_2} + e \mathbf{i}_3 \frac{\partial t^{33}}{\partial a_3} = 0.$$

Only the first equation

$$\frac{\partial}{\partial a} t^{11} C'(a) - \frac{\alpha^2}{b^2} t^{22} C(a) = 0, \tag{2.1.5}$$

is of interest as the remaining two equations are satisfied identically because the stresses depend only on variable $a_1 = a$, see eq. (2.1.2).

It follows from eqs. (2.1.4) and (2.1.5) that after removing t^{22} we arrive at the following differential equation

$$\frac{\partial}{\partial a} t^{11} C'(a) - \frac{b^2}{\alpha^2 e^2} C^{-3} t^{11} = \frac{\partial A}{\partial C}. \tag{2.1.6}$$

Recalling now that by eq. (6.5.6) of Chapter 2

$$C'C = \frac{b}{\alpha e}, \quad C'^2 + CC'' = 0, \quad \frac{\partial}{\partial a} = C' \frac{\partial}{\partial C}, \tag{2.1.7}$$

we are led to the differential equation

$$C \frac{\partial t^{11}}{\partial C} - 2t^{11} = C^3 \frac{\partial A}{\partial C} \frac{\alpha^2 e^2}{b^2}. \tag{2.1.8}$$

Its general solution has the form

$$t^{11} = \frac{e^2 \alpha^2}{b^2} C^2 (A + D), \tag{2.1.9}$$

where D denotes the integration constant. Using eqs. (2.1.5) and (2.1.2) one easily finds the remaining contravariant components of the stress tensor $\hat{T}$ in the vector basis $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$

$$\left.\begin{aligned} t^{22} &= \frac{b^2}{\alpha^2 C^2} \left(A + D + C \frac{\partial A}{\partial C} \right) = \frac{b^2}{\alpha^2 C^2} \frac{d}{dC} C (A + D), \\ t^{33} &= \frac{1}{e^2} (A + D) + 2 \left(1 - \frac{b^2}{\alpha^2 e^4 C^2} \right) \left(\frac{\partial A}{\partial I_1} + \frac{\alpha^2}{b^2} C^2 \frac{\partial A}{\partial I_2} \right). \end{aligned}\right\} \tag{2.1.10}$$

Referring to the above expressions for the base vectors and relation (2.1.7) we obtain

$$\begin{aligned} \hat{T} &= t^{11} \mathbf{R}_1 \mathbf{R}_1 + t^{22} \mathbf{R}_2 \mathbf{R}_2 + t^{33} \mathbf{R}_3 \mathbf{R}_3 \\ &= \mathbf{e}_r \mathbf{e}_r (A + D) + \mathbf{e}_\theta \mathbf{e}_\theta \frac{d}{dC} C (A + D) + e^2 \mathbf{i}_3 \mathbf{i}_3 t^{33}. \end{aligned} \tag{2.1.11}$$

Here $\mathbf{e}_r, \mathbf{e}_\theta$ denote the unit base vectors of the normal to the cylindrical surfaces $a = \text{const}$ and the meridional cross-sections $a_2 = \text{const}$.

Let I_k^0 and I_k^1 denote respectively the values of the invariants on the cylindrical surfaces of radii r_0 and r_1 bounding the body in the deformable state. By eq. (6.5.3) of Chapter 2

$$r_0 = C\left(a^0\right), \quad r_1 = C\left(a^0 + h\right),$$

and by eq. (2.1.1) we have

$$I_1^0 - I_1^1 = \left(r_0^2 - r_1^2\right)\left(\frac{\alpha^2}{b^2} - \frac{b^2}{\alpha^2 e^2 r_0^2 r_1^2}\right),$$

$$I_2^0 - I_2^1 = \left(r_0^2 - r_1^2\right) e^2 \left(\frac{\alpha^2}{b^2} - \frac{b^2}{\alpha^2 e^2 r_0^2 r_1^2}\right),$$

that is, at $r_0 r_1 = b^2/\alpha^2 e^2$

$$I_1^0 = I_1^1, \quad I_2^0 = I_2^1, \quad A\left(I_1^0, I_2^0\right) = A\left(I_1^1, I_2^1\right). \tag{2.1.12}$$

The conditions of no loading on both surfaces $r = r_0$ and $r = r_1$ reduce to the conditions imposed on the only constant D

$$\left.\begin{aligned} \left(\mathbf{e}_r \cdot \hat{T} \cdot \mathbf{e}_r\right)_{r=r_0} &= A\left(I_1^0, I_2^0\right) + D = 0, \\ \left(\mathbf{e}_r \cdot \hat{T} \cdot \mathbf{e}_r\right)_{r=r_1} &= A\left(I_1^1, I_2^1\right) + D = 0. \end{aligned}\right\} \tag{2.1.13}$$

We arrive at the following expressions for the physical components of the stress tensor in the system of cylindrical coordinate r, θ, z

$$\left.\begin{aligned} \sigma_r &= A\left(I_1, I_2\right) - A\left(I_1^0, I_2^0\right), \\ \sigma_\theta &= \frac{d}{dr} r A\left(I_1, I_2\right) - A\left(I_1^0, I_2^0\right), \\ \sigma_z &= A\left(I_1, I_2\right) - A\left(I_1^0, I_2^0\right) + 2\left(e^2 - \frac{r_0 r_1}{e r^2}\right)\left(\frac{\partial A}{\partial I_1} + \frac{\alpha^2 r^2}{b^2}\frac{\partial A}{\partial I_2}\right), \end{aligned}\right\} \tag{2.1.14}$$

where $C(a)$ is replaced by r. Due to eq. (6.5.8) of Chapter 2 we obtain

$$r = C(a) = \sqrt{\left(r_1^2 - r_0^2\right)\frac{a - a^0}{h} + r_0^2}. \tag{2.1.15}$$

According to eq. (2.1.3) the resulting vector of the forces in any meridional section $\theta = \alpha a_2 = \text{const}$ of the cylinder in the axial direction acting on the unit of length vanishes since

$$\int_{r_1}^{r_0} \left[\frac{d}{dr} r A\left(I_1, I_2\right) - A\left(I_1^0, I_2^0\right)\right] dr =$$

$$= r_0 A\left(I_1^0, I_2^0\right) - r_1 A\left(I_1^1, I_2^1\right) - \left(r_0 - r_1\right) A\left(I_1^0, I_2^0\right) = 0,$$

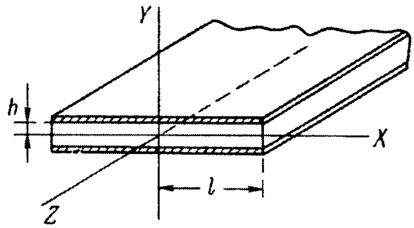

FIGURE 9.1.

which can be predicted from symmetry. The principal moment of the forces about the cylinder axis is determined by the equality

$$M = \int_{r_1}^{r_0} \left[\frac{d}{dr} rA(I_1, I_2) - A\left(I_1^0, I_2^0\right) \right] rdr =$$

$$= \frac{1}{2}\left(r_0^2 - r_1^2\right) A\left(I_1^0, I_2^0\right) - \int_{r_0}^{r_1} rA(I_1, I_2)\, dr. \quad (2.1.16)$$

Further calculations require the specific strain energy to be prescribed as an explicit function of the invariants.

9.2.2 *Compression and tension of the elastic strip*

We consider an elastic layer of the incompressible material which in its initial state occupies the region $|a_1| < l, |a_2| < h$ of plane XOY and extends without bound along axis Z. At planes $a_2 = \pm h$ the layer is fixed to rigid plates whilst the end surfaces $a_1 = \pm l$ are free[1]. The plates are subjected to displacements which are parallel to axis OY, equal and opposite in direction. The thickness $2h$ of the layer becomes $2H$, such that $H > h$ under tension of the layer and $H < h$ under compression, see Fig 9.1.

The Cartesian coordinates a_1, a_2, a_3 of the particle in the initial state are deemed as the material coordinates. The sought quantities are the Cartesian coordinates x_1, x_2 of the particle in the deformed state ($V-$volume). It is additionally assumed that x_2 is independent of a_1, i.e. the planes a_2 remain parallel planes in $V-$volume

$$x_1 = x_1(a_1, a_2), \quad x_2 = x_2(a_2), \quad x_3 = a_3. \quad (2.2.1)$$

These functions are subjected to the geometric boundary conditions

$$x_1(a_1, \pm h) = a_1, \quad x_2(\pm h) = \pm H \quad (2.2.2)$$

[1] Klingbeil, W.W., Schield, R.T. Large deformation analysis of bonded elastic mounts. Zeitschrift für angewandte Mathematik und Physik, vol. 17, No.2, 1966, pp. 281-305.

and the requirement for the system of stresses on surfaces $a_1 = \pm l$ to be statically equivalent to zero.

Due to the incompressibility of the material

$$\sqrt{G} = \begin{vmatrix} \dfrac{\partial x_1}{\partial a_1} & \dfrac{\partial x_1}{\partial a_2} & 0 \\ 0 & \dfrac{\partial x_2}{\partial a_2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} = 1, \tag{2.2.3}$$

that is, x_1 is a linear function of a_1. Hence, instead of eq. (2.2.1) we can take

$$x_1 = a_1 f(a_2) + k(a_2), \quad x_2 = g(a_2), \quad x_3 = a_3,$$

where

$$g'(a_2) f(a_2) = 1. \tag{2.2.4}$$

The statement of the problem indicates the symmetric properties of the sought functions, namely x_1 is an odd function of a_1 and an even function of a_2, while x_2 is an odd function of a_2. Hence $k(a_2) = 0$ and the deformation considered is determined by two functions $f(a_2)$ and $g(a_2)$

$$x_1 = a_1 f(a_2), \quad x_2 = g(a_2), \quad x_3 = a_3, \tag{2.2.5}$$

subjected to condition (2.2.4).

The coordinate basis of V-volume is given by the vectors

$$\mathbf{R}_1 = \mathbf{i}_1 f(a_2), \quad \mathbf{R}_2 = \mathbf{i}_1 a_1 f'(a_2) + \mathbf{i}_2 g'(a_2), \quad \mathbf{R}_3 = \mathbf{i}_3. \tag{2.2.6}$$

The covariant and contravariant components of the metric tensor $\hat{G}$ are equal to

$$\begin{Vmatrix} G_{11} = f^2 & G_{12} = a_1 f f' & G_{13} = 0 \\ & G_{22} = a_1^2 f'^2 + g'^2 & G_{23} = 0 \\ & & G_{33} = 1 \end{Vmatrix}, \tag{2.2.7}$$

$$\begin{Vmatrix} G^{11} = a_1^2 f'^2 + g'^2 & G^{12} = -a_1 f f' & G^{13} = 0 \\ & G^{22} = f^2 & G^{23} = 0 \\ & & G^{33} = 1 \end{Vmatrix}, \tag{2.2.8}$$

and the principal invariants are equal to each other

$$I_1 = I_2 = 1 + f^2 + a_1^2 f'^2 + g'^2 = 1 + f^2 + f^{-2} + a_1^2 f'^2. \tag{2.2.9}$$

The specific strain energy is assumed to be given in Mooney's form (4.9.2) of Chapter 8 and due to eqs. (2.1.5), (2.1.4) of Chapter 8 the contravariant components $\tilde{t}^{st}$ of the stress tensor are as follows

$$\left.\begin{aligned}
\frac{1}{2}\tilde{t}^{11} &= C_1 + C_2 + \left(C_2 + \overset{-1}{c}\right)\left(a_1^2 f'^2 + f^{-2}\right),\\
\frac{1}{2}\tilde{t}^{22} &= C_1 + C_2 + \left(C_2 + \overset{-1}{c}\right) f^2,\\
\frac{1}{2}\tilde{t}^{33} &= \left(C_1 + I_1 C_2\right) - C_2 + \overset{-1}{c},\\
\frac{1}{2}\tilde{t}^{12} &= \frac{1}{2}\tilde{t}^{21} = -\left(C_2 + \overset{-1}{c}\right) a_1 f f', \quad \tilde{t}^{23} = \tilde{t}^{31} = 0.
\end{aligned}\right\} \tag{2.2.10}$$

The physical components of the stress tensor are given by the equalities

$$\left.\begin{aligned}
\sigma_x &= \mathbf{i}_1 \cdot \hat{T} \cdot \mathbf{i}_1 = \tilde{t}^{11} f^2 + \tilde{t}^{22} a_1 f'^2 + 2\tilde{t}^{12} a_1 f f',\\
\sigma_y &= \mathbf{i}_2 \cdot \hat{T} \cdot \mathbf{i}_2 = \tilde{t}^{22} g'^2,\\
\tau_{xy} &= \mathbf{i}_1 \cdot \hat{T} \cdot \mathbf{i}_2 = \tilde{t}^{22} a_1 f' g' + \tilde{t}^{12} f g',\\
\sigma_z &= \tilde{t}^{33}, \quad \tau_{xz} = \tau_{yz} = 0.
\end{aligned}\right\} \tag{2.2.11}$$

9.2.3 Equations of statics

The vector equation of statics is set in the form

$$\frac{\partial}{\partial a_1}\left(\tilde{t}^{11}\mathbf{R}_1 + \tilde{t}^{12}\mathbf{R}_2\right) + \frac{\partial}{\partial a_2}\left(\tilde{t}^{12}\mathbf{R}_1 + \tilde{t}^{22}\mathbf{R}_2\right) = 0. \tag{2.3.1}$$

Referring to eq. (2.2.6) we arrive at two equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial a_1}\left(\tilde{t}^{11} f + \tilde{t}^{12} a_1 f'\right) + \frac{\partial}{\partial a_2}\left(\tilde{t}^{21} f + \tilde{t}^{22} a_1 f'\right) = 0,\\
&\frac{\partial}{\partial a_1}\tilde{t}^{12} g' + \frac{\partial}{\partial a_2}\tilde{t}^{22} g' = 0.
\end{aligned}\right\} \tag{2.3.2}$$

Using the incompressibility relation (2.24) we can reduce these equations to the form

$$\left.\begin{aligned}
&\frac{\partial \tilde{t}^{11}}{\partial a_1} + \frac{\partial \tilde{t}^{12}}{\partial a_2} + 2\frac{f'}{f}\tilde{t}^{12} + a_1 \frac{f'^2 + f f''}{f^2}\tilde{t}^{22} = 0,\\
&\frac{\partial \tilde{t}^{12}}{\partial a_1} + \frac{\partial \tilde{t}^{22}}{\partial a_2} - \frac{f'}{f}\tilde{t}^{22} = 0.
\end{aligned}\right\} \tag{2.3.3}$$

Substituting the stress components (2.2.10) in the latter equations and using formulae (2.2.7)-(2.2.9) results in the following system of equations

$$\left.\begin{aligned}
&\frac{\partial \overset{-1}{c}}{\partial a_1}\left(a_1^2 f'^2 + \frac{1}{f^2}\right) - \frac{\partial \overset{-1}{c}}{\partial a_2} a_1 f f' = -\frac{f'^2 + f'' f}{f^2} a_1 \left(C_1 + C_2\right),\\
&-\frac{\partial \overset{-1}{c}}{\partial a_1} a_1 f' + f\frac{\partial \overset{-1}{c}}{\partial a_2} = \frac{f'}{f^2}\left(C_1 + C_2\right).
\end{aligned}\right\} \tag{2.3.4}$$

From this system we obtain

$$\frac{\partial \overset{-1}{c}}{\partial a_1} = -(C_1 + C_2)\, a_1 f f'', \quad \frac{\partial \overset{-1}{c}}{\partial a_2} = -(C_1 + C_2)\left(a_1^2 f' f'' - \frac{f'}{f^3}\right), \tag{2.3.5}$$

and the incompressibility condition

$$\frac{\partial^2 \overset{-1}{c}}{\partial a_2 \partial a_1} = \frac{\partial^2 \overset{-1}{c}}{\partial a_1 \partial a_2}, \quad a_1 (f' f'' + f f''') = 2 a_1 f' f''$$

leads to the differential equation for the unknown function f

$$f f''' = f' f'', \quad \frac{f'''}{f''} = \frac{f'}{f}, \quad (\ln f'')' = (\ln f)', \quad \frac{f''}{f} = c, \tag{2.3.6}$$

where c is a constant.

Function $\overset{-1}{c}$ is determined from eq. (2.3.5) by means of the incompressibility condition

$$\begin{aligned}\overset{-1}{c} &= -\frac{1}{2}(C_1 + C_2)\left(a_1^2 f f'' + \frac{1}{f^2}\right) + \overset{-1}{c}_0 - C_2 \\ &= -\frac{1}{2}(C_1 + C_2)\left(a_1^2 c f^2 + \frac{1}{f^2}\right) + \overset{-1}{c}_0 - C_2,\end{aligned} \tag{2.3.7}$$

where $\overset{-1}{c} - C_2$ denotes the integration constant.

The stress tensor is now given by formulae (2.2.10) up to an additive constant. The latter is obtained from the condition that the principal vector $\mathbf{P}$ of the forces on surface $a_1 = l$ of the deformed body vanishes. The principal moment of these forces vanishes due to the symmetry of the surface about plane $a_2 = 0$.

Referring to formulae (3.2.3) of Chapter 1 and taking into account that normal $\mathbf{n}$ has the direction of $\mathbf{i}_1$ we have

$$\mathbf{P} = \int_{-h}^{h} \left(\tilde{t}^{11}\mathbf{R}_1 + \tilde{t}^{12}\mathbf{R}_2\right)\Big|_{a_1 = l} da_2 = \mathbf{i}_1 \int_{-h}^{h} \left(\tilde{t}^{11} f + \tilde{t}^{12} l f'\right) da_2 = 0, \tag{2.3.8}$$

the $\mathbf{i}_2$ term disappears due to the symmetry of the problem

$$\int_{-h}^{h} \tilde{t}^{12} \frac{da_2}{f} = 2 \int_{-h}^{h} \left[-\overset{-1}{c}_0 + \frac{1}{2}(C_1 + C_2)\left(l^2 c f^2 + \frac{1}{f^2}\right)\right] l f' da_2 = 0,$$

as function $f(a_2)$ is even.

Inserting the stress components into eq. (2.3.8) and using relations (2.2.4), (2.2.2) yields

$$\overset{-1}{c}_0 = \frac{C_1 + C_2}{2H}\left[\left(l^2 c - 2\right)\int_0^h f da_2 + \int_0^h \frac{da_2}{f^3}\right]. \tag{2.3.9}$$

Force $\mathbf{Q}$ providing the displacement of the rigid plates (related to the unit length of axis Z) is determined by eq. (3.2.3) of Chapter 1

$$\mathbf{Q} = \int_{-l}^{l}\left(\tilde{t}^{21}\mathbf{R}_1 + \tilde{t}^{22}\mathbf{R}_2\right) da_1 = \mathbf{i}_2\int_{-l}^{l}\tilde{t}^{22} g'(h)\, da_1 = \mathbf{i}_2\int_{-l}^{l}\tilde{t}^{22} da_1,$$

since the terms associated with $\mathbf{i}_1$ vanish due to their evenness with respect to a_1 and $g'(h) = [f(h)]^{-1} = 1$. Substituting $\tilde{t}^{22}$ into the latter equation yields

$$\mathbf{Q} = 2\mathbf{i}_2 l\left\{2\overset{-1}{c}_0\left(1 + \frac{1}{3}l^2 f'^2(h)\right) + \right.$$
$$\left.\left(C_1 + C_2\right)\left[1 - \frac{1}{3}l^2\left(c + f'^2(h)\right) - \frac{1}{5}l^4 c f'^2(h)\right]\right\}. \tag{2.3.10}$$

Taking into account eq. (2.2.5) and the evenness of function $f(a_2)$ we have at point $a_1 = l, a_2 = 0$

$$x_1(l,0) = lf(0), \quad \left.\frac{\partial x_1(l,a_2)}{\partial a_2}\right|_{a_2=0} = lf'(0) = 0,$$
$$\left.\frac{\partial x_1(l,a_2)}{\partial a_2^2}\right|_{a_2=0} = lf''(0) = clf(0)$$

and at $a_2 = 0$ quantity $x_1(l,a_2)$ has a minimum for $c > 0$ and a maximum for $c < 0$. In the first case the material "becomes drawn" into the layer in the neigbourhood of this point, whilst in the second case the material "bulges" outward. For this reason, $c > 0$ and $c < 0$ correspond to tension and compression of the layer respectively.

9.2.4 Compression of the layer

Assuming $c = -\alpha^2$ in eq. (2.3.6) and taking into account eq. (2.2.2) we have

$$f(a_2) = \frac{\cos\alpha a_2}{\cos\alpha h}, \quad x_1 = a_1\frac{\cos\alpha a_2}{\cos\alpha h}. \tag{2.4.1}$$

Due to eq. (2.2.4) and the evenness of function $g(\alpha_2)$ we obtain

$$g(a_2) = \frac{\cos\alpha h}{\alpha}\Lambda(\alpha a_2). \tag{2.4.2}$$

Here $\Lambda(x)$ denotes the lambda-function

$$\Lambda(x) = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} = \int_0^x \frac{dx}{\cos x}. \tag{2.4.3}$$

The constant α is determined from the boundary condition (2.2.2)

$$\frac{H}{h} = \frac{\cos\alpha h}{\alpha h}\Lambda(\alpha h). \tag{2.4.4}$$

9.2.5 *Tension of the layer*

The solutions obtained in the previous subsection are now expressed in terms of the hyperbolic functions. Assuming $c = \alpha^2$ we have

$$f(a_2) = \frac{\cosh\alpha a_2}{\cosh\alpha h}, \quad x_1 = a_1\frac{\cosh\alpha a_2}{\cosh\alpha h}. \tag{2.5.1}$$

Function $g(a_2)$ is represented in the form

$$g(a_2) = \frac{\cosh\alpha a_2}{\alpha}\Lambda^*(\alpha a_2), \quad \Lambda^*(x) = \arctan\sinh x, \tag{2.5.2}$$

where α is obtained from the following equation

$$\frac{H}{h} = \frac{\cosh\alpha h}{\alpha h}\Lambda^*(\alpha h). \tag{2.5.3}$$

The formulae for stresses are obtained from the relations of Subsection 9.2.3 and are not shown here as they are very cumbersome, see the above cited paper by Klingbeil and Schield.

Figures 9.2a and 9.2b display the schematics of the deformed lines for the cases of tension and compression, respectively, for the ratio of the loaded and unloaded areas

$$S = \frac{l}{2h}$$

and the elongations

$$e = \frac{H-h}{h} = 0,25\ (-0,25).$$

One of the loading curves calculated by means of the above formulae for the same value of S is shown in Fig. 9.2c. One can see that for the elongation $e > 40\%$, the value of the force decreases with the growth of e. This can be explained by decreasing the cross-sectional area caused by retraction of the material into the layer. Figure 9.2 is taken from the above cited paper by Klingbeil and Schield which contains a large number of comparisons with the experimental results of other authors.

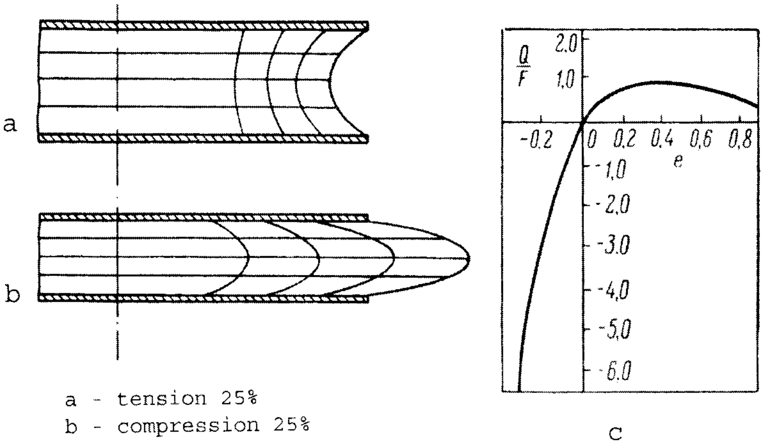

FIGURE 9.2.

9.3 Elastic cylinder and elastic sphere

9.3.1 Cylindrical tube under pressure (Lame's problem for the nonlinear elastic incompressible material)

The cylindric coordinates $q^1 = r, q^2 = 0, q^3 = z$ of the initial volume ($v-$volume) are considered as being the material coordinates. It is assumed that the deformation preserves axial symmetry, that is, the position of point $M(r,\theta,z)$ in $V-$volume in the same meridional plane can be prescribed in the same cylindric coordinate system by means of the values

$$R = R(r), \quad \theta = \theta, \quad z' = \alpha z, \tag{3.1.1}$$

where $\alpha = \text{const}$.

The position radii of the initial and final states of point M and the base vectors in $v-$ and $V-$volumes are as follows

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{i}_3, \quad \mathbf{r}_1 = \mathbf{e}_r, \quad \mathbf{r}_2 = r\mathbf{e}_\theta, \quad \mathbf{r}_3 = \mathbf{i}_3, \tag{3.1.2}$$

$$\mathbf{R} = R(r)\,\mathbf{e}_r + \alpha z\mathbf{i}_3, \quad \mathbf{R}_1 = R'(r)\,\mathbf{e}_r, \quad \mathbf{R}_2 = R\mathbf{e}_\theta, \quad \mathbf{R}_3 = \alpha\mathbf{i}_3. \tag{3.1.3}$$

Only the diagonal components of the metric tensors $\hat{g}$ and $\hat{G}$ differ from zero, namely in $v-$volume

$$\left.\begin{array}{llll} g_{11} = 1, & g_{22} = r^2, & g_{33} = 1 & \left(g = r^2\right), \\ g^{11} = 1, & g^{22} = \dfrac{1}{r^2}, & g^{33} = 1, & \end{array}\right\} \tag{3.1.4}$$

and in V-volume

$$\left.\begin{aligned} &G_{11} = R'^2, \quad G_{22} = R^2, \quad G_{33} = \alpha^2 \quad \left(G = R'^2 R^2 \alpha^2\right), \\ &G^{11} = \frac{1}{R'^2}, \quad G^{22} = \frac{1}{R^2}, \quad G^{33} = \frac{1}{\alpha^2}. \end{aligned}\right\} \tag{3.1.5}$$

Due to the condition of the material incompressibility

$$I_3 = \frac{G}{g} = 1, \quad R'R\alpha = r. \tag{3.1.6}$$

Let r_0 and r_1 denote respectively the internal and external radii of the cylinder in the initial state, and R_0 and R_1 denote respectively the values of r_0 and r_1 in the deformed cylinder, that is

$$\alpha\left(R^2 - R_0^2\right) = r^2 - r_0^2, \quad \alpha\left(R_1^2 - R^2\right) = r_1^2 - r^2. \tag{3.1.7}$$

Since $r_0 \leq r \leq r_1$ then $R_0 \leq R \leq R_1$ for $\alpha > 0$ where R_0 and R_1 denote the internal and external radii of the deformed cylinder. For $\alpha < 0$ we have $R_1 \leq R \leq R_0$, i.e. in the deformed cylinder R_1 and R_0 become respectively the internal and external radii. In other words, the cylinder is "turned inside out".

The expressions for the covariant and the contravariant components of the metric tensor $\hat{G}$ are presented in the form

$$\left.\begin{aligned} &G_{11} = \frac{r^2}{\alpha^2 R^2}, \quad G_{22} = R^2, \quad G_{33} = \alpha^2, \\ &G^{11} = \frac{\alpha^2 R^2}{r^2}, \quad G^{22} = \frac{1}{R^2}, \quad G^{33} = \frac{1}{\alpha^2}, \end{aligned}\right\} \tag{3.1.8}$$

and the principal invariants of Cauchy's strain measure are

$$I_1 = \frac{r^2}{\alpha^2 R^2} + \frac{R^2}{r^2} + \alpha^2, \quad I_2 = \frac{\alpha^2 R^2}{r^2} + \frac{r^2}{R^2} + \frac{1}{\alpha^2}, \quad I_3 = 1. \tag{3.1.9}$$

It follows from these equalities and eq. (3.1.6) that

$$\frac{dI_1}{dr} = \alpha^{-2}\frac{\partial I_2}{\partial r} = \frac{2}{r}\left(1 - \frac{r^2}{\alpha R^2}\right)\left(\frac{r^2}{\alpha^2 R^2} - \frac{R^2}{r^2}\right). \tag{3.1.10}$$

9.3.2 Stresses

Using eq. (2.1.5) of Chapter 8 we arrive at the following expressions for the nontrivial contravariant components of the stress tensor

$$\left.\begin{aligned} &\frac{1}{2}\tilde{t}^{11} = \overset{0}{c} - \overset{1}{c}\,\frac{r^2}{\alpha^2 R^2} + \overset{-1}{c}\,\frac{\alpha^2 R^2}{r^2}, \\ &\frac{1}{2}\tilde{t}^{22} = \frac{\overset{0}{c}}{r^2} - \overset{1}{c}\,\frac{R^2}{r^4} + \overset{-1}{c}\,\frac{1}{R^2}, \\ &\frac{1}{2}\tilde{t}^{33} = \overset{0}{c} - \overset{1}{c}\,\alpha^2 + \frac{\overset{-1}{c}}{\alpha^2}. \end{aligned}\right\} \tag{3.2.1}$$

Eliminating the unknown (for the incompressible material) value $\overset{-1}{c}$ from the first two equations we obtain the equation relating $\tilde{t}^{11}$ and $\tilde{t}^{22}$

$$\frac{1}{2}\left(\tilde{t}^{11}\frac{r^2}{\alpha^2 R^2} - \tilde{t}^{22}R^2\right) = \left(\frac{r^2}{\alpha^2 R^2} - \frac{R^2}{r^2}\right)\left[\overset{0}{c} - \overset{1}{c}\left(\frac{r^2}{\alpha^2 R^2} + \frac{R^2}{r^2}\right)\right]. \tag{3.2.2}$$

By means of eqs. (2.1.7) of Chapter 8 and eq. (3.1.9) the right hand side is transformed to the form

$$\left(\frac{r^2}{\alpha^2 R^2} - \frac{R^2}{r^2}\right)\left(\frac{\partial A}{\partial I_1} + \alpha^2\frac{\partial A}{\partial I_2}\right).$$

On the other hand, by eq. (3.1.10)

$$\begin{aligned}\frac{dA}{dr} &= \frac{\partial A}{\partial I_1}\frac{dI_1}{dr} + \frac{\partial A}{\partial I_2}\frac{dI_2}{dr} = \frac{dI_1}{dr}\left(\frac{\partial A}{\partial I_1} + \alpha^2\frac{\partial A}{\partial I_2}\right)\\ &= \frac{2}{r}\left(1 - \frac{r^2}{\alpha R^2}\right)\left(\frac{r^2}{\alpha^2 R^2} - \frac{R^2}{r^2}\right)\left(\frac{\partial A}{\partial I_1} + \alpha^2\frac{\partial A}{\partial I_2}\right),\end{aligned} \tag{3.2.3}$$

and relationship (3.2.2) takes the form

$$\tilde{t}^{11}\frac{r^2}{\alpha^2 R^2} - \tilde{t}^{22}R^2 = \frac{\alpha r R^2}{\alpha R_0^2 - r_0^2}\frac{dA}{dr}. \tag{3.2.4}$$

Removing $\overset{-1}{c}$ from the first and third equations (3.2.1) we obtain

$$\tilde{t}^{33} = \tilde{t}^{11}\frac{r^2}{\alpha^4 R^2} + 2\left(1 - \frac{r^2}{\alpha^4 R^2}\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{r^2}\frac{\partial A}{\partial I_2}\right). \tag{3.2.5}$$

Let us turn to the equations of statics. Taking into account eqs. (3.1.6) and (3.1.3) and the absence of the mass forces we can write these equations in the vectorial form

$$\frac{\partial}{\partial q^s} r\tilde{t}^{st}\mathbf{R}_t = \frac{\partial}{\partial r}\tilde{t}^{11}\frac{r^2}{\alpha R}\mathbf{e}_r + \frac{\partial}{\partial \theta} rR\tilde{t}^{22}\mathbf{e}_\theta + \frac{\partial}{\partial z} r\alpha\tilde{t}^{33}\mathbf{i}_3 = 0. \tag{3.2.6}$$

In this equation only $\mathbf{e}_\theta$ depends on θ, namely

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r,$$

whereas the other quantities depend only on r. We arrive at two equations

$$\frac{d}{dr}\frac{r^2}{\alpha R}\tilde{t}^{11} - rR\tilde{t}^{22} = 0, \quad \frac{\partial \tilde{t}^{33}}{\partial z} = 0, \tag{3.2.7}$$

the second one being satisfied identically.

In order to obtain the expressions for the physical components of the stress tensor we write down the stress tensor in terms of both contravariant and physical components

$$\hat{T} = \tilde{t}^{11}\mathbf{R}_1\mathbf{R}_1 + \tilde{t}^{22}\mathbf{R}_2\mathbf{R}_2 + \tilde{t}^{33}\mathbf{R}_3\mathbf{R}_3 = \tilde{t}^{11}R'^2\mathbf{e}_r\mathbf{e}_r + \tilde{t}^{22}R^2\mathbf{e}_\theta\mathbf{e}_\theta + \tilde{t}^{33}\alpha^2\mathbf{i}_3\mathbf{i}_3,$$

thus

$$\sigma_r = R'^2 t^{11} = \frac{r^2}{\alpha^2 R^2}\tilde{t}^{11}, \quad R^2\tilde{t}^{22}, \quad \sigma_z = \alpha^2\tilde{t}^{33}. \tag{3.2.8}$$

This allows one to set eqs. (3.2.4), (3.2.5) and (3.2.7) in the following form

$$\sigma_\theta = \sigma_r - \frac{\alpha r R^2}{\alpha R_0^2 - r_0^2}\frac{dA}{dr}, \tag{3.2.9}$$

$$\sigma_z = \sigma_r + 2\left(\alpha^2 - \frac{r^2}{\alpha^2 R^2}\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{r^2}\frac{\partial A}{\partial I_2}\right), \tag{3.2.10}$$

$$\frac{d}{dr}\alpha R\sigma_r - \frac{r}{R}\sigma_\theta = 0. \tag{3.2.11}$$

Removing σ_θ from eqs. (3.2.9) and (3.2.11) we arrive at the differential equation

$$\frac{d}{dr}\alpha R\sigma_r - \frac{r}{R}\sigma_r = \frac{\alpha r^2 R}{\alpha R_0^2 - r_0^2}\frac{dA}{dr}. \tag{3.2.12}$$

Introducing a new independent variable R

$$\alpha R dR = r dr$$

we transform eq. (3.2.12) to the differential equation

$$\frac{d\sigma_r}{dR} = -\frac{r^2}{\alpha R_0^2 - r_0^2}\frac{dA}{dR},$$

whose solution is as follows

$$\sigma_r = -\frac{1}{\alpha R_0^2 - r_0^2}\left(\int_{R_0}^{R} r^2\frac{dA}{dR}dR + C\right) = -\frac{1}{\alpha R_0^2 - r_0^2}\left(\int_{r_0}^{r} r^2\frac{dA}{dr}dr + C\right). \tag{3.2.13}$$

9.3.3 Determination of the constants

Along with the integration constant C, the values determining the geometric sizes of the deformed cylinder are unknown. These values are α, R_0, R_1 and are related to each other by eq. (3.1.7)

$$\alpha\left(R_1^2 - R_0^2\right) = r_1^2 - r_0^2. \tag{3.3.1}$$

The constants C, α, R_0 are determined with the help of the prescribed pressure q_0 and q_1 on the corresponding surfaces $r = r_0$ and $r = r_1$ in $V-$volume and the axial force on the end surfaces. The first two conditions are written as follows

$$(\sigma_r)_{r=r_0} = -q_0, \quad (\sigma_r)_{r=r_1} = -q_1, \tag{3.3.2}$$

such that by eq. (3.2.13)

$$C = q_0 \left(\alpha R_0^2 - r_0^2\right), \tag{3.3.3}$$

$$(q_0 - q_1)\left(\alpha R_0^2 - r_0^2\right) = -\int_{r_0}^{r_1} r^2 \frac{dA}{dr} dr. \tag{3.3.4}$$

By eqs. (3.2.13), (3.2.3), (3.2.9) and (3.2.10) the physical stress components are set in the form

$$\sigma_r = -q_0 + \frac{q_0 - q_1}{\int_{r_0}^{r_1} r^2 \frac{dA}{dr} dr} \int_{r_0}^{r} r^2 \frac{dA}{dr} dr, \tag{3.3.5}$$

$$\sigma_\theta = \sigma_r - 2\left(\frac{\partial A}{\partial I_1} + \alpha^2 \frac{\partial A}{\partial I_2}\right)\left(\frac{r^2}{\alpha^2 R^2} - \frac{R^2}{r^2}\right), \tag{3.3.6}$$

$$\sigma_z = \sigma_r + 2\left(\alpha^2 - \frac{r^2}{\alpha^2 R^2}\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{r^2}\frac{\partial A}{\partial I_2}\right). \tag{3.3.7}$$

The force acting on surface $d\overset{3}{O}$ of the cross-section of the cylinder is given by formula (3.2.3) of Chapter 1 for $\mathbf{n} = \mathbf{i}_3$

$$\overset{N}{\mathbf{t}}\, d\overset{3}{O} = \tilde{t}^{33}\mathbf{R}_3 r dr d\theta = \frac{1}{\alpha}\sigma_z r dr d\theta \mathbf{i}_3,$$

and the axial force Z is as follows

$$Z = \frac{2\pi}{\alpha}\left[\int_{r_0}^{r} \sigma_r r dr + 2\int_{r_0}^{r}\left(\alpha^2 - \frac{r^2}{\alpha^2 R^2}\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{r^2}\frac{\partial A}{\partial I_2}\right) r dr\right]. \tag{3.3.8}$$

Together with eqs. (3.3.4) and (3.3.1) this equation determines the constants α, R_0, R_1. If the end faces are free, then $Z = 0$. If the cylinder is placed between two rigid plates then its length does not change and $\alpha = 1$. In this case eq. (3.3.8) serves to determine the axial force and the unknowns R_0, R_1 are obtained from eqs. (3.3.4) and (3.3.1).

The above formulae contain the derivative of the specific potential energy with respect to r which should be replaced by the expression in terms of the derivatives with respect to the invariants, i.e. "the generalised moduli of elasticity". To this end, formula (3.2.3) is transformed as follows

$$\frac{dA}{dr}=-\frac{2c^2\left(2r^2+c\right)}{\alpha\left(r^2+c\right)^2 r^3}\left(\frac{\partial A}{\partial I_1}+\alpha^2\frac{\partial A}{\partial I_2}\right). \tag{3.3.9}$$

Here and in what follows we introduce the denotation

$$c=\alpha R_0^2-r_0^2=\alpha R_1^2-r_1^2=\alpha R^2-r^2, \tag{3.3.10}$$

where by eq. (3.3.4)

$$\frac{q_0-q_1}{\int\limits_{r_0}^{r_1} r^2\frac{dA}{dr}dr}=-\frac{1}{c}. \tag{3.3.11}$$

9.3.4 Mooney's material

In the above calculation the specific strain energy is assumed to be prescribed in Mooney's form, see eq. (4.9.2) of Chapter 8. Equation (3.3.4) is written in the form

$$\frac{q_0-q_1}{2c}\alpha=\left(C_1+\alpha^2 C_2\right)\int\limits_{r_0}^{r_1}\frac{2r^2+c}{r\left(r^2+c\right)^2}dr,$$

and the integral in the latter equation is equal to

$$\int\limits_{r_0}^{r_1}\frac{2r^2+c}{r\left(r^2+c\right)^2}dr=\frac{1}{2c}\left[\ln\frac{r_1^2}{r_0^2}\frac{R_0^2}{R_1^2}+\left(\frac{r_1^2}{\alpha R_1^2}-\frac{r_0^2}{\alpha R_0^2}\right)\right]. \tag{3.4.1}$$

Introducing the denotation

$$f_*(x)=x+\ln x \tag{3.4.2}$$

one can represent eq. (3.3.4) in the form

$$\alpha\left(q_0-q_1\right)=\left[f_*\left(\frac{r_1^2}{\alpha R_1^2}\right)-f_*\left(\frac{r_0^2}{\alpha R_0^2}\right)\right]\left(C_1+\alpha^2 C_2\right). \tag{3.4.3}$$

The similar representation of expression (3.3.8) for the axial force requires estimation of the double integral

$$\int\limits_{r_0}^{r_1} rdr\int\limits_{r_0}^{r}\frac{\partial A}{\partial\rho}\rho^2 d\rho=\frac{1}{2}r_1^2\int\limits_{r_0}^{r_1}\frac{\partial A}{\partial r}r^2 dr-\frac{1}{2}\int\limits_{r_0}^{r_1}\frac{\partial A}{\partial r}r^4 dr.$$

By means of eq. (3.3.5) we obtain

$$\int_{r_0}^{r_1} \sigma_r r dr = \frac{1}{2}\left(q_0 r_0^2 - q_1 r_1^2\right) + \frac{1}{2c}\int_{r_0}^{r_1} \frac{\partial A}{\partial r} r^4 dr$$
$$= \frac{1}{2}\left(q_0 r_0^2 - q_1 r_1^2\right) - \frac{c}{2\alpha}\left(C_1 + \alpha^2 C_2\right)\left(2\ln\frac{R_1^2}{R_0^2} + \frac{r_0^2}{\alpha R_0^2} - \frac{r_1^2}{\alpha R_1^2}\right),$$

and the expression for the axial force reduces to the following form

$$\frac{\alpha Z}{2\pi} = \frac{1}{2}\left(q_0 r_0^2 - q_1 r_1^2\right) - \frac{c}{2\alpha}\left(C_1 + \alpha^2 C_2\right)\left(2\ln\frac{R_1^2}{R_0^2} + \frac{r_0^2}{\alpha R_0^2} - \frac{r_1^2}{\alpha R_1^2}\right) +$$
$$\left(r_1^2 - r_0^2\right)\left[C_1\left(\alpha^2 - \frac{1}{\alpha}\right) + C_2\left(\alpha - \frac{1}{\alpha^2}\right)\right] + \frac{c}{\alpha}\left(C_1 \ln\frac{R_1^2}{R_0^2} + C_2\alpha^2 \ln\frac{r_1^2}{r_0^2}\right). \tag{3.4.4}$$

9.3.5 Cylinder "turned inside out"

The internal and external radii of the cylinder in the initial state are denoted respectively as r_0 and r_1. In the final state, i.e. for the cylinder "turned inside out", R_0 and R_1 are respectively the external and internal radii. The external forces in the final state are absent

$$q_0 = q_1 = 0, \quad Z = 0. \tag{3.5.1}$$

Further $\alpha < 0$ and $c < 0$. Introducing the denotation

$$-\alpha = \beta, \quad -c = K r_0^2, \quad \frac{R_0^2}{r_0^2} = \mu_0^2, \quad \frac{R_1^2}{r_1^2} = \mu_1^2, \quad \frac{r_1^2}{r_0^2} = n > 1, \tag{3.5.2}$$

one can set relationship (3.3.10) in the following form

$$K = \beta\mu_0^2 + 1 = n\left(\beta\mu_1^2 + 1\right). \tag{3.5.3}$$

It follows from the latter that

$$\beta\left(\mu_0^2 - n\mu_1^2\right) = n - 1 > 0, \quad \mu_0^2 > n\mu_1^2, \quad \mu_0^2 > \mu_1^2.$$

Stress σ_r determined in terms of (3.3.3), (3.3.4) and (3.3.8) is equal to

$$\sigma_r = \frac{1}{K r_0^2}\int_{r_0}^{r} \frac{\partial A}{\partial r} r^2 dr = \frac{1}{\beta}\left(C_1 + \beta^2 C_2\right)\left(\ln\frac{R^2}{\mu_0^2 r^2} + \frac{r^2}{\beta R^2} - \frac{1}{\beta\mu_0^2}\right). \tag{3.5.4}$$

It is zero at $r = r_0$ and $r = r_1$

$$\ln\frac{\mu_1^2}{\mu_0^2} + \frac{1}{\beta\mu_1^2} - \frac{1}{\beta\mu_0^2} = \left(\frac{1}{\beta\mu_1^2} - \ln\frac{1}{\beta\mu_1^2}\right) - \left(\frac{1}{\beta\mu_0^2} - \ln\frac{1}{\beta\mu_0^2}\right) = 0. \tag{3.5.5}$$

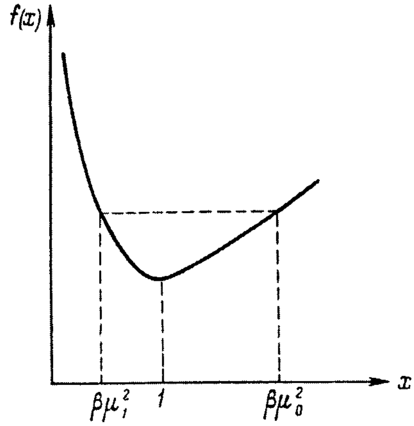

FIGURE 9.3.

The latter relationship can be written down in the form

$$f\left(\frac{1}{\beta\mu_0^2}\right)=f\left(\frac{1}{\beta\mu_1^2}\right), \quad f(x)=x-\ln x \tag{3.5.6}$$

and presents another form of condition (3.4.3) if one utilises denotation (3.5.2).

One can see from Fig. 9.3 that function $f(x)$ has a minimum at $x=1$. It is also clear that eq. (3.5.6) determines a unique value $\beta\mu_1^2$ for any $\beta\mu_0^2>0$.

The meridional stress σ_θ on the external $(R=R_0, r=r_0)$ and internal $(R=R_1, r=r_1)$ sides of the cylinder turned inside out is determined by eq. (3.3.6)

$$\begin{aligned}
\sigma_\theta\big|_{r=r_0} &= \frac{2}{\beta}\left(C_1+\beta^2C_2\right)\left(\beta\mu_0^2-\frac{1}{\beta\mu_0^2}\right),\\
\sigma_\theta\big|_{r=r_1} &= \frac{2}{\beta}\left(C_1+\beta^2C_2\right)\left(\beta\mu_1^2-\frac{1}{\beta\mu_1^2}\right).
\end{aligned} \tag{3.5.7}$$

As one would expect this stress is tensile and compressive on the external and internal sides respectively.

Using eq. (3.4.4) the condition whereby the axial force vanishes is set in the form

$$\begin{aligned}
&2(n-1)\frac{1+\beta^3}{\beta^2}\left(C_1-\frac{C_2}{\beta}\right)+\\
&\left(1+\beta\mu_0^2\right)\left[2C_2\ln\frac{\mu_0^2}{\mu_1^2}-\frac{1}{\beta^2}\left(\frac{1}{\beta\mu_1^2}-\frac{1}{\beta\mu_0^2}\right)\left(C_1+\beta^2C_2\right)\right]=0.
\end{aligned} \tag{3.5.8}$$

Referring to eq. (3.5.5) this condition can be transformed to the form

$$2(n-1)\frac{C_1\beta - C_2}{C_1 - \beta^2 C_2}\frac{1+\beta^3}{\beta} = \left(1+\beta\mu_0^2\right)\left(\frac{1}{\beta\mu_1^2} - \frac{1}{\beta\mu_0^2}\right). \qquad (3.5.9)$$

For a given n, the unknowns $\beta\mu_0^2, \beta\mu_1^2$ are determined by equations (3.5.3) and (3.5.6), then for the measured value of β one can find the ratio C_2/C_1 for the considered Mooney material. The limits of possible values of β are determined by the positiveness of this ratio, see Subsection 8.4.9.

9.3.6 Torsion of a circular cylinder

Similar to Subsection 9.3.1 the material coordinates are identified as the cylindric coordinates of the point in the initial position. The torsional deformation is described by the rotation of the cross-sections of the cylinder accompanied by the axial displacement. The latter is introduced by means of the constant parameter α

$$z = \alpha a_3.$$

The position radius $\mathbf{R}$ of the deformed cylinder is given by formula (6.4.2) of Chapter 2

$$\mathbf{R} = \mathbf{r} \cdot \hat{A},$$

where expression (6.4.2) of Chapter 2 for tensor $\hat{A}$ needs to be corrected in order to take into account the axial displacement and conservation of the volume

$$\hat{A} = \frac{1}{\sqrt{\alpha}}\left[(\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta)\cos\chi - (\mathbf{e}_\theta\mathbf{e}_r - \mathbf{e}_r\mathbf{e}_\theta)\sin\chi\right] + \alpha\mathbf{i}_3\mathbf{i}_3,$$

$$\chi = \chi_0 + \psi a_3.$$

The constant ψ denotes the angle of torsion per unit length in the axial direction.

Presenting $\mathbf{R}$ in the form

$$\mathbf{R} = \mathbf{r} \cdot \hat{A} = (r\mathbf{e}_r + \mathbf{i}_3 a_3) \cdot \hat{A} = \frac{r}{\sqrt{\alpha}}(\mathbf{e}_r\cos\chi + \mathbf{e}_\theta\sin\chi) + \alpha a_3\mathbf{i}_3$$

and recalling the formula for differentiation

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta, \qquad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r,$$

we obtain the following representations for the base vectors in the deformed cylinder

$$\begin{aligned}
\mathbf{R}_1 &= \frac{1}{\sqrt{\alpha}}\left(\mathbf{e}_r \cos\chi + \mathbf{e}_\theta \sin\chi\right), \\
\mathbf{R}_2 &= \frac{r}{\sqrt{\alpha}}\left(\mathbf{e}_\theta \cos\chi - \mathbf{e}_r \sin\chi\right), \\
\mathbf{R}_3 &= \psi\frac{r}{\sqrt{\alpha}}\left(-\mathbf{e}_r \sin\chi + \mathbf{e}_\theta \cos\chi\right) + \alpha\mathbf{i}_3
\end{aligned}$$

or

$$\mathbf{R}_1 = \frac{1}{\sqrt{\alpha}}\mathbf{e}_r^*, \quad \mathbf{R}_2 = \frac{r}{\sqrt{\alpha}}\mathbf{e}_\theta^*, \quad \mathbf{R}_3 = \psi\frac{r}{\sqrt{\alpha}}\mathbf{e}_\theta^* + \alpha\mathbf{i}_3, \tag{3.6.1}$$

where

$$\mathbf{e}_r^* = \mathbf{e}_r \cos\chi + \mathbf{e}_\theta \sin\chi, \quad \mathbf{e}_\theta^* = -\mathbf{e}_r \sin\chi + \mathbf{e}_\theta \cos\chi. \tag{3.6.2}$$

This determines the covariant components of the metric tensor

$$\left.\begin{array}{lll}
G_{11} = \dfrac{1}{\alpha}, & G_{12} = 0, & G_{13} = 0, \\
 & G_{22} = \dfrac{r^2}{\alpha}, & G_{23} = \dfrac{r^2}{\alpha}\psi, \\
 & & G_{33} = \alpha^2 + \dfrac{r^2\psi^2}{\alpha},
\end{array}\right\} \tag{3.6.3}$$

such that

$$G = r^2. \tag{3.6.4}$$

The contravariant components and the principal invariants of tensor $\hat{G}^\times$ are

$$\left.\begin{array}{lll}
G^{11} = \alpha, & G^{12} = 0, & G^{13} = 0, \\
 & G^{22} = \dfrac{\alpha}{r^2} + \dfrac{\psi^2}{\alpha^2}, & G^{23} = -\dfrac{\psi}{\alpha^2}, \\
 & & G^{33} = \dfrac{1}{\alpha^2},
\end{array}\right\} \tag{3.6.5}$$

$$I_1 = \frac{2}{\alpha} + \alpha^2 + \frac{r^2\psi^2}{\alpha}, \quad I_2 = 2\alpha + \frac{r^2\psi^2}{\alpha} + \frac{1}{\alpha^2}, \quad I_3 = 1. \tag{3.6.6}$$

The expressions for the contravariant components of the stress tensor are set in the form

$$\left.\begin{aligned}
\frac{1}{2}\tilde{t}^{11} &= \overset{0}{c} - \frac{\overset{1}{c}}{\alpha} + \overset{-1}{c}\,\alpha,\\
\frac{1}{2}\tilde{t}^{22} &= \frac{\overset{0}{c}}{r^2} - \frac{\overset{1}{c}}{\alpha r^2} + \overset{-1}{c}\left(\frac{\alpha}{r^2} + \frac{\psi^2}{\alpha^2}\right),\\
\frac{1}{2}\tilde{t}^{33} &= \overset{0}{c} - \overset{1}{c}\left(\alpha^2 + \frac{r^2\psi^2}{\alpha}\right) + \frac{\overset{-1}{c}}{\alpha^2},\\
\frac{1}{2}\tilde{t}^{23} &= -\overset{1}{c}\,\frac{\psi}{\alpha} - \overset{-1}{c}\,\frac{\psi}{\alpha^2},\\
\tilde{t}^{12} &= \tilde{t}^{13} = 0.
\end{aligned}\right\} \tag{3.6.7}$$

From the representation of the stress tensor in the vector basis of the deformed volume

$$\begin{aligned}
\hat{T} &= \tilde{t}^{sk}\mathbf{R}_s\mathbf{R}_k = \tilde{t}^{11}\mathbf{R}_1\mathbf{R}_1 + \tilde{t}^{22}\mathbf{R}_2\mathbf{R}_2 + \tilde{t}^{33}\mathbf{R}_3\mathbf{R}_3 + \tilde{t}^{23}\left(\mathbf{R}_2\mathbf{R}_3 + \mathbf{R}_3\mathbf{R}_2\right)\\
&= \frac{1}{\alpha}\tilde{t}^{11}\mathbf{e}_r^*\mathbf{e}_r^* + \frac{r^2}{\alpha}\tilde{t}^{22}\mathbf{e}_\theta^*\mathbf{e}_\theta^* + \tilde{t}^{33}\left[\frac{\psi^2 r^2}{\alpha}\mathbf{e}_\theta^*\mathbf{e}_\theta^* + \alpha^2\mathbf{i}_3\mathbf{i}_3 + \psi r\sqrt{\alpha}\left(\mathbf{e}_\theta^*\mathbf{i}_3 + \mathbf{i}_3\mathbf{e}_\theta^*\right)\right]\\
&\qquad + \tilde{t}^{23}\left[2\frac{\psi r^2}{\alpha}\mathbf{e}_\theta^*\mathbf{e}_\theta^* + r\sqrt{\alpha}\left(\mathbf{e}_\theta^*\mathbf{i}_3 + \mathbf{i}_3\mathbf{e}_\theta^*\right)\right],
\end{aligned}$$

we obtain the expressions for the physical stress components

$$\left.\begin{aligned}
\sigma_r &= \frac{1}{\alpha}\tilde{t}^{11}, \quad \sigma_\theta = \frac{r^2}{\alpha}\tilde{t}^{22} + \frac{\psi^2 r^2}{\alpha}\tilde{t}^{33} + 2\frac{\psi r^2}{\alpha}\tilde{t}^{23},\\
\sigma_z &= \alpha^2\tilde{t}^{33}, \quad \tau_{\theta z} = r\sqrt{\alpha}\left(\tilde{t}^{23} + \psi\tilde{t}^{33}\right).
\end{aligned}\right\} \tag{3.6.8}$$

The constitutive law (3.6.7) for these components is written down in the form

$$\left.\begin{aligned}
\frac{1}{2}\sigma_r &= \frac{1}{\alpha}\frac{\partial A}{\partial I_1} + \left(\frac{1}{\alpha^2} + \alpha + \frac{r^2\psi^2}{\alpha^2}\right)\frac{\partial A}{\partial I_2} + \overset{-1}{c},\\
\frac{1}{2}\sigma_\theta &= \frac{1}{\alpha}\left(1 + r^2\psi^2\right)\frac{\partial A}{\partial I_1} + \left(\frac{1}{\alpha^2} + \alpha + \frac{r^2\psi^2}{\alpha^2}\right)\frac{\partial A}{\partial I_2} + \overset{-1}{c},\\
\frac{1}{2}\sigma_z &= \alpha^2\frac{\partial A}{\partial I_1} + 2\alpha\frac{\partial A}{\partial I_2} + \overset{-1}{c},\\
\frac{1}{2}\tau_{\theta z} &= \frac{r\psi}{\sqrt{\alpha}}\left(\alpha\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\right).
\end{aligned}\right\} \tag{3.6.9}$$

Here we used formula (2.1.4) of Chapter 8 in order to replace "moduli" $\overset{0}{c}$ and $\overset{1}{c}$ in terms of the derivatives of the strain energy with respect to the invariants.

Having removed the unknown $\overset{-1}{c}$ we arrive at the relationships

$$\left.\begin{aligned}
\sigma_r - \sigma_\theta &= -2\frac{r^2\psi^2}{\alpha}\frac{\partial A}{\partial I_1}, \\
\sigma_r - \sigma_z &= 2\left(\frac{1}{\alpha} - \alpha^2\right)\left(\frac{\partial A}{\partial I_1} + \frac{1}{\alpha}\frac{\partial A}{\partial I_2}\right) + 2\frac{r^2\psi^2}{\alpha^2}\frac{\partial A}{\partial I_2}, \\
\tau_{\theta z} &= 2r\psi\sqrt{\alpha}\left(\frac{\partial A}{\partial I_1} + \frac{1}{\alpha}\frac{\partial A}{\partial I_2}\right),
\end{aligned}\right\} \tag{3.6.10}$$

where

$$2\left(\frac{\partial A}{\partial I_1} + \frac{1}{\alpha}\frac{\partial A}{\partial I_2}\right) = \frac{\alpha}{r\psi^2}\frac{dA}{dr}. \tag{3.6.11}$$

9.3.7 Stresses, torque and axial force

The equation of statics has the form

$$\frac{\partial}{\partial q^s} = \sqrt{G}\tilde{t}^{st}\mathbf{R}_t = \frac{\partial}{\partial r} r\tilde{t}^{11}\frac{1}{\sqrt{\alpha}}\mathbf{e}_r^* + \frac{\partial}{\partial \theta} r\left[\tilde{t}^{22}\frac{r}{\sqrt{\alpha}}\mathbf{e}_\theta^* + \tilde{t}^{23}\left(\frac{\psi r}{\sqrt{\alpha}}\mathbf{e}_\theta^* + \alpha\mathbf{i}_3\right)\right]$$
$$+ \frac{\partial}{\partial z} r\left[\tilde{t}^{23}\frac{r}{\sqrt{\alpha}}\mathbf{e}_\theta^* + \tilde{t}^{33}\left(\frac{\psi r}{\sqrt{\alpha}}\mathbf{e}_\theta^* + \alpha\mathbf{i}_3\right)\right] = 0. \tag{3.7.1}$$

Applying the differentiation formulae, see eq. (3.6.2),

$$\frac{\partial \mathbf{e}_\theta^*}{\partial \theta} = -\mathbf{e}_r^*, \quad \frac{\partial \mathbf{e}_\theta^*}{\partial z} = -\psi\mathbf{e}_r^*$$

and taking into account that the remaining quantities depend only on r we arrive at the single equation

$$\frac{d}{dr}\frac{1}{\sqrt{\alpha}} r\tilde{t}^{11} - \frac{1}{\sqrt{\alpha}} r^2\left(\tilde{t}^{22} + 2\tilde{t}^{23}\psi + \tilde{t}^{33}\psi^2\right) = 0. \tag{3.7.2}$$

Transforming it to physical components we obtain the well-know equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{3.7.3}$$

Using the first relationship in eq. (3.6.10) we obtain

$$\sigma_r = -2\frac{\psi^2}{\alpha}\int_r^{r_0}\frac{\partial A}{\partial I_1} r\,dr, \quad \sigma_\theta = \sigma_r + 2\frac{r^2\psi^2}{\alpha}\frac{\partial A}{\partial I_1}. \tag{3.7.4}$$

The integration constant is determined from the condition of vanishing external forces on the outer surface $r = r_0$ of the cylinder. The adopted deformation does not provide us with the solution to the problem of torsion of the hollow cylinder since only one constant is at our disposal and we can not satisfy the requirement of vanishing stress σ_r on the inner surface. It is a serious error to think that the problem of torsion of the hollow cylinder can be solved by superposition of the solution of "Lame's problem" of Subsections 9.3.1-9.3.4 on the obtained solution. Solving the problem of torsion of the hollow cylinder requires the rejection of the assumption that the radial displacement is absent and the axial displacement is independent of r.

The force acting on surface $d\overset{3}{O}$ in the cross-section of the cylinder is given by eq. (3.2.3) of Chapter 1

$$\begin{aligned}\overset{N}{\mathbf{t}}\, d\overset{3}{O} &= \tilde{t}^{s3}\mathbf{R}_s r dr d\theta = \left(\tilde{t}^{23}\mathbf{R}_2 + \tilde{t}^{33}\mathbf{R}_3\right) r dr d\theta \\ &= \left[\frac{r}{\sqrt{\alpha}}\left(\tilde{t}^{23} + \psi\tilde{t}^{33}\right)\mathbf{e}_\theta^* + \alpha\tilde{t}^{33}\mathbf{i}_3\right] r dr d\theta = \frac{1}{\alpha}\left(\tau_{\theta z}\mathbf{e}_\theta^* + \sigma_z\mathbf{i}_3\right) r dr d\theta.\end{aligned}$$

The principal force and the principal moment of the system of forces are as follows

$$\begin{aligned}\mathbf{V} &= \frac{1}{\alpha}\int_0^{2\pi} d\theta \int_0^{r_0} r dr\left(\tau_{\theta z}\mathbf{e}_\theta^* + \sigma_z\mathbf{i}_3\right), \\ \mathbf{m}^O &= \frac{1}{\alpha}\int_0^{2\pi} d\theta \int_0^{r_0} \mathbf{R}\times\left(\tau_{\theta z}\mathbf{e}_\theta^* + \sigma_z\mathbf{i}_3\right) r dr \\ &= \frac{1}{\alpha}\int_0^{2\pi} d\theta \int_0^{r_0}\left(\frac{r}{\sqrt{\alpha}}\mathbf{e}_r^* + \alpha\mathbf{i}_3 z\right)\times\left(\tau_{\theta z}\mathbf{e}_\theta^* + \sigma_z\mathbf{i}_3\right) r dr.\end{aligned}$$

It follows from the relationships

$$\int_0^{2\pi}\mathbf{e}_\theta^* d\theta = \mathbf{e}_r^*\Big|_0^{2\pi} = 0, \quad \int_0^{2\pi}\mathbf{e}_r^* d\theta = -\mathbf{e}_\theta^*\Big|_0^{2\pi} = 0$$

that these vectors are parallel to axis $\mathbf{i}_3$, as expected. We arrive at the expressions for the axial force and the torque

$$Z = \frac{2\pi}{\alpha}\int_0^{r_0}\sigma_z r dr, \quad m_z = \frac{2\pi}{\alpha\sqrt{\alpha}}\int_0^{r_0}\tau_{\theta z} r^2 dr. \tag{3.7.5}$$

Noticing that

$$\int_0^{r_0} r dr \int_r^{r_0}\frac{\partial A}{\partial I_1}\rho d\rho = \frac{1}{2}\int_0^{r_0}\frac{\partial A}{\partial I_1} r^3 dr,$$

and referring to eqs. (3.6.10) and (3.6.11) we arrive at the formulae

$$Z = -2\pi \left\{ \frac{\psi^2}{\alpha^2} \int_0^{r_0} \left(\frac{\partial A}{\partial I_1} + \frac{2}{\alpha} \frac{\partial A}{\partial I_2} \right) r^3 dr + \frac{1-\alpha^3}{\alpha \psi^2} \left[A(r_0) - A(0) \right] \right\}, \tag{3.7.6}$$

$$m_z = \frac{2\pi}{\psi} \int_0^{r_0} \frac{\partial A}{\partial r} r^2 dr = \frac{2}{\psi} \left[\pi r_0^2 A(r_0) - a \right]. \tag{3.7.7}$$

Here $A(r_0)$ and $A(0)$ denote the values of the specific strain energy on the surface and on the axis of the rod. Further

$$a = 2\pi \int_0^{r_0} A r dr$$

denotes the strain energy per unit length of the rod under torsion.

Here one clearly sees the Poynting effect, namely the necessity for a compressive axial force in order to preserve the unchanged length of the twisted rod ($\alpha = 1$)

$$Z|_{\alpha=1} = -2\pi\psi^2 \int_0^{r_0} \left(\frac{\partial A}{\partial I_1} + 2 \frac{\partial A}{\partial I_2} \right) r^3 dr. \tag{3.7.8}$$

In the case of no axial force, the change in the rod length (parameter α) is determined by eq. (3.7.6) for $Z = 0$.

In the nonlinear theory, the torsion is accompanied not only by shear stress $\tau_{\theta z}$ but also by all normal stresses.

For Mooney's material the compressive force needed for preserving the length of the rod is equal to

$$Z|_{\alpha=1} = -(C_1 + 2C_2) I_p \psi^2, \tag{3.7.9}$$

where I_p denotes the polar moment of inertia of the cross-section. In this case

$$I_1 - 3 = I_2 - 3 = r^2\psi^2, \quad A = (C_1 + C_2) r^2 \psi^2, \quad a = (C_1 + C_2) I_p \psi^2.$$

Using eq. (3.7.7) and introducing denotation (1.4.7) for the shear modulus, we arrive at the expression for the torque

$$m_z = 2(C_1 + C_2) I_p \psi = \mu I_p \psi, \tag{3.7.10}$$

which coincides with that of the linear theory.

In the case of vanishing axial force the expression for the torque is set in the form

$$m_z = \frac{2}{\alpha}\left(C_1 + \frac{C_2}{\alpha}\right) I_p \psi, \tag{3.7.11}$$

where α is a function of ψ determined by means of eq. (3.7.6) from the following equation

$$4\left(1-\alpha^3\right)\left(\alpha C_1 + C_2\right) + r_0^2\psi^2\left(\alpha C_1 + 2C_2\right) = 0. \tag{3.7.12}$$

For example, for the neo-Hookean material ($C_2 = 0$)

$$\alpha = \sqrt[3]{1+\frac{r_0^2\psi^2}{4}}, \quad m_z = \mu I_p \frac{\psi}{\sqrt[3]{1+\frac{1}{4}r_0^2\psi^2}}, \tag{3.7.13}$$

that is, the rod extends and the growth of the torque decreases as the twist angle ψ increases.

9.3.8 Symmetric deformation of the hollow sphere (Lame's problem for a sphere)

The material coordinates of a point are the spherical coordinates (see Section C.8) of the point in the initial state of the sphere (v—volume)

$$q^1 = \rho, \quad q^2 = \theta, \quad q^3 = \lambda.$$

Only the diagonal components of the metric tensor $\hat{g}$ of this state differ from zero and are equal to

$$\left.\begin{array}{l} g_{11} = 1, \quad g_{22} = \rho^2, \quad g_{33} = \rho^2\sin^2\vartheta \quad \left(g = \rho^4\sin^2\vartheta\right), \\ g^{11} = 1, \quad g^{22} = \dfrac{1}{\rho^2}, \quad g^{33} = \dfrac{1}{\rho^2\sin^2\vartheta}. \end{array}\right\} \tag{3.8.1}$$

The deformation is assumed to be radially symmetric, that is, the displacements of the sphere from the initial state into the final state are directed radially and depend only on coordinate ρ. The hollow sphere remains a hollow sphere. Its external and internal radii in the initial and final states are denoted respectively as ρ_0, ρ_1 and R_0, R_1.

The position vector of the point and the base vectors $\mathbf{R}_s$ (determined in terms of the position vector) in the deformed sphere are given by expressions (C.8.4)

$$\mathbf{R} = R(\rho)\,\mathbf{e}_\rho, \; \mathbf{R}_1 = R'(\rho)\,\mathbf{e}_\rho, \; \mathbf{R}_2 = R(\rho)\,\mathbf{e}_\vartheta, \; \mathbf{R}_3 = R(\rho)\,\mathbf{e}_\lambda \sin\vartheta. \tag{3.8.2}$$

It is assumed that the sphere is subjected to a constant pressure on both the inside and outside surfaces

$$(\sigma_\rho)_{\rho=\rho_0} = -q_0, \quad (\sigma_\rho)_{\rho=\rho_1} = -q_1. \tag{3.8.3}$$

Only normal stresses σ_ρ,σ_ϑ,σ_λ appear under this loading, i.e.

$$\hat{T} = \sigma_\rho \mathbf{e}_\rho \mathbf{e}_\rho + \sigma_\vartheta \mathbf{e}_\vartheta \mathbf{e}_\vartheta + \sigma_\lambda \mathbf{e}_\lambda \mathbf{e}_\lambda = \frac{\sigma_\rho}{R'^2}\mathbf{R}_1\mathbf{R}_1 + \frac{\sigma_\vartheta}{R^2}\mathbf{R}_2\mathbf{R}_2 + \frac{\sigma_\lambda}{R^2\sin^2\vartheta}\mathbf{R}_3\mathbf{R}_3$$

and the formulae relating the contravariant and physical components of the stress tensor have the form

$$\sigma_\rho = \tilde{t}^{11}R'^2, \quad \sigma_\vartheta = \tilde{t}^{22}R^2, \quad \sigma_\lambda = \tilde{t}^{33}R^2\sin^2\vartheta. \tag{3.8.4}$$

The components of the metric tensor $\hat{G}$ in the deformed sphere and the invariants of the Cauchy strain measure are equal to

$$\left.\begin{aligned}
&G_{11} = R'^2, \quad G_{22} = R^2, \quad G_{33} = R^2\sin^2\vartheta \quad \left(G = R'^2R^4\sin^2\vartheta\right),\\
&G^{11} = \frac{1}{R'^2}, \quad G^{22} = \frac{1}{R^2}, \quad G^{33} = \frac{1}{R^2\sin^2\vartheta},\\
&I_1 = R'^2 + 2\frac{R^2}{\rho^2}, \quad I_2 = \frac{R^4}{\rho^4} + 2R'^2\frac{R^2}{\rho^2}, \quad I_3 = R'^2\frac{R^4}{\rho^4}.
\end{aligned}\right\} \tag{3.8.5}$$

The constitutive law is given by the following system of formulae

$$\left.\begin{aligned}
\frac{1}{2}\tilde{t}^{11} &= \frac{\rho^2}{R'R^2}\left(\overset{0}{c} - \overset{1}{c}R'^2 + \frac{\overset{-1}{c}}{R'^2}\right),\\
\frac{1}{2}\tilde{t}^{22} &= \frac{\rho^2}{R'R^2}\left(\frac{\overset{0}{c}}{\rho^2} - \overset{1}{c}\frac{R^2}{\rho^4} + \frac{\overset{-1}{c}}{R^2}\right),\\
\frac{1}{2}\tilde{t}^{33} &= \frac{1}{2}\frac{\tilde{t}^{22}}{\sin^2\vartheta}.
\end{aligned}\right\} \tag{3.8.6}$$

The static equation

$$\frac{\partial}{\partial q^s}\sqrt{G}\tilde{t}^{st}\mathbf{R}_t = \frac{\partial}{\partial\rho}R'^2R^2\sin\vartheta\tilde{t}^{11}\mathbf{e}_\rho + \frac{\partial}{\partial\vartheta}R'R^3\tilde{t}^{22}\mathbf{e}_\vartheta\sin\vartheta + \frac{\partial}{\partial\lambda}R'R^3\tilde{t}^{33}\sin^2\vartheta\mathbf{e}_\lambda = 0$$

is rewritten by means of formulae (C.8.4)

$$\sin\vartheta\left[\frac{d}{d\rho}R'^2R^2\tilde{t}^{11} - R'R^3\left(\tilde{t}^{22} + \tilde{t}^{33}\sin^2\vartheta\right)\right]\mathbf{e}_\rho + R'R^3\cos\vartheta\left(\tilde{t}^{22} - \tilde{t}^{33}\sin^2\vartheta\right)\mathbf{e}_\vartheta = 0.$$

It yields the above mentioned relationship

$$\tilde{t}^{22} = \tilde{t}^{33} \sin^2 \vartheta \tag{3.8.7}$$

and the differential equation

$$\frac{d}{d\rho} R'^2 R^2 \tilde{t}^{11} - 2R' R^3 \tilde{t}^{22} = 0. \tag{3.8.8}$$

The latter can be reset in a simpler form in terms of the physical components of tensor $\hat{T}$

$$\frac{d}{d\rho} R^2 \sigma_\rho - 2R' R \sigma_\vartheta = 0, \quad \frac{d\sigma_\rho}{d\rho} + 2\frac{R'}{R}(\sigma_\rho - \sigma_\vartheta) = 0. \tag{3.8.9}$$

Noticing that $R' d\rho = dR$ we can reduce eqs. (3.8.6) and (3.8.7) to the form

$$\frac{d\sigma_\rho}{dR} + 2\frac{\sigma_\rho - \sigma_\vartheta}{R} = 0, \quad \sigma_\vartheta = \sigma_\lambda. \tag{3.8.10}$$

This form can be obtained directly from the equations of statics in spherical coordinates of the deformed volume.

It remains to substitute the expressions for the difference

$$\sigma_\rho - \sigma_\vartheta = \frac{2}{R'} \left(\frac{\rho^2 R'^2}{R^2} - 1 \right) \left(\frac{\partial A}{\partial I_1} + \frac{R^2}{\rho^2} \frac{\partial A}{\partial I_2} \right) \tag{3.8.11}$$

and stress σ_ρ

$$\sigma_\rho = 2\frac{\rho^2 R'}{R^2} \left(\frac{\partial A}{\partial I_1} + 2\frac{R^2}{\rho^2} \frac{\partial A}{\partial I_2} + \frac{R^4}{\rho^4} \frac{\partial A}{\partial I_3} \right) \tag{3.8.12}$$

into the equation of statics.

We arrive at the nonlinear differential equation of second order for function $R(\rho)$. The boundary conditions given by relationships (3.8.3) and (3.8.12) are also nonlinear. The problem is cumbersome even for the simple constitutive laws for compressible elastic material.

9.3.9 Incompressible material

The above mentioned complications do not exist for incompressible material. The incompressibility condition determines the derivative R' of the sought function R

$$I_1 = 1, \quad R' = \frac{\rho^2}{R^2}, \tag{3.9.1}$$

and by eq. (3.8.11) we have

$$\sigma_\rho - \sigma_\vartheta = 2\frac{R^2}{\rho^2}\left(\frac{\rho^6}{R^6} - 1\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{\rho^2}\frac{\partial A}{\partial I_2}\right). \tag{3.9.2}$$

The equation of statics (3.8.10) reduces to the form

$$\frac{d\sigma_\rho}{d\rho} + \frac{4}{R}\left(\frac{\rho^6}{R^6} - 1\right)\left(\frac{\partial A}{\partial I_1} + \frac{R^2}{\rho^2}\frac{\partial A}{\partial I_2}\right) = 0, \tag{3.9.3}$$

and the problem reduces to considering the system of equations (3.9.1) and (3.9.3). Its solution is simplified by introducing the new independent variable

$$\chi = \frac{\rho}{R}, \quad d\chi = \frac{d\rho}{R} - \frac{\rho R'}{R^2}d\rho = \frac{d\rho}{R}\left(1 - \chi^3\right).$$

The expression for the invariants, eq. (3.8.5), takes the form

$$I_1 = \chi^4 + \frac{2}{\chi^2}, \quad I_2 = \frac{1}{\chi^4} + 2\chi^2. \tag{3.9.4}$$

For this reason

$$\frac{dA}{d\chi} = 4\left(\frac{\partial A}{\partial I_1} + \frac{1}{\chi^2}\frac{\partial A}{\partial I_2}\right)\frac{\chi^6 - 1}{\chi^3}, \tag{3.9.5}$$

and the equation of statics is transformed to the form

$$\frac{d\sigma_\rho}{d\chi} = \frac{\chi^3}{\chi^3 - 1}\frac{dA}{d\chi}. \tag{3.9.6}$$

Taking into account the boundary conditions (3.8.3) we arrive at the expression for the normal stress

$$\sigma_\rho = -q_0 + \int\limits_{\chi_0}^{\chi} \frac{\chi^3}{\chi^3 - 1}\frac{dA}{d\chi}d\chi \tag{3.9.7}$$

and the formula relating the unknowns χ_1 and χ_0

$$q_0 - q_1 = \int\limits_{\chi_0}^{\chi_1} \frac{\chi^3}{\chi^3 - 1}\frac{dA}{d\chi}d\chi. \tag{3.9.8}$$

The second relationship between them is given by the incompressibility condition

$$R^2 R' = \rho^2, \quad R^3 - \rho^3 = R_0^3 - \rho_0^3 = R_1^3 - \rho_1^3, \tag{3.9.9}$$

which is written in the following form

$$\frac{1 - \chi_1^3}{\chi_1^3} = \left(\frac{\rho_0}{\rho_1}\right)^3 \frac{1 - \chi_0^3}{\chi_0^3}. \tag{3.9.10}$$

9.3.10 Applying the principle of stationarity of strain energy

In the above problems for the cylinder and sphere, the invariants are expressed in terms of the functions and constant parameters of deformations in a rather simple way. This fact allows us to suggest a simple derivation of the equilibrium equations with the help of the principle of stationarity of the specific strain energy.

For instance, let us consider the sphere. The invariants depend on the sought function $R(\rho)$ and its derivative, hence

$$\delta A = \sum_{k=1}^{3} \frac{\partial A}{\partial I_k} \delta I_k = \sum_{k=1}^{3} \frac{\partial A}{\partial I_k} \left(\frac{\partial I_k}{\partial R} \delta R + \frac{\partial I_k}{\partial R'} \delta R' \right). \tag{3.10.1}$$

The element of the volume is $d\tau_0 = 4\pi\rho^2 d\rho$ and the varied quantity is transformed to the form

$$\begin{aligned} 4\pi \int_{\rho_0}^{\rho_1} \rho^2 \sum_{k=1}^{3} \frac{\partial A}{\partial I_k} \delta I_k d\rho = 4\pi \sum_{k=1}^{3} \int_{\rho_0}^{\rho_1} \Bigg[\rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R} \delta R + \\ \frac{d}{d\rho} \rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R'} \delta R - \delta R \frac{d}{d\rho} \rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R'} \Bigg] d\rho \qquad (3.10.2) \\ = 4\pi \left[\sum_{k=1}^{3} \int_{\rho_0}^{\rho_1} \left(-\frac{d}{d\rho} \rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R'} + \rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R} \right) \delta R d\rho + \rho^2 \frac{\partial A}{\partial I_k} \frac{\partial I_k}{\partial R'} \delta R \Bigg|_{\rho_0}^{\rho_1} \right]. \end{aligned}$$

The elementary work of the surface forces of pressure $-q_0 \overset{0}{\mathbf{N}}$ and $q_1 \overset{1}{\mathbf{N}}$ on the concentric spheres O_0 and O_1 respectively is equal to

$$\begin{aligned} -q_0 \iint_{O_0} \overset{0}{\mathbf{N}} \cdot \delta \mathbf{R}_0 d\overset{0}{O} - q_1 \iint_{O_1} \overset{1}{\mathbf{N}} \cdot \delta \mathbf{R}_1 d\overset{1}{O} = \\ = q_0 \iint_{O_0} \delta R_0 d\overset{0}{O} - q_1 \iint_{O_1} \delta R_1 d\overset{1}{O} = 4\pi \left(q_0 R_0^2 \delta R_0 - q_1 R_1^2 \delta R_1 \right), \end{aligned}$$

where we took into account the evident relationships

$$\overset{0}{\mathbf{N}} \cdot \delta \mathbf{R}_0 = -\frac{\mathbf{R}_0}{R_0} \cdot \delta \mathbf{R}_0 = -\delta R_0, \quad \overset{1}{\mathbf{N}} \cdot \delta \mathbf{R}_1 = -\frac{\mathbf{R}_1}{R_1} \cdot \delta \mathbf{R}_1 = \delta R_1.$$

On any sphere

$$dO = \frac{R^2}{\rho^2} do, \quad \iint \delta R dO = \frac{R^2}{\rho^2} \delta R \iint do = 4\pi R^2 \delta R.$$

We arrive at the equality

$$-\sum_{k=1}^{3}\left\{\int_{\rho_0}^{\rho_1}\left(\frac{d}{d\rho}\rho^2\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}-\rho^2\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R}\right)\delta R d\rho+\right.$$
$$\left.\left[\frac{\rho_1^2}{R_1^2}\left(\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}\right)_1+q_1\right]R_1^2\delta R_1-\left[\frac{\rho_0^2}{R_0^2}\left(\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}\right)_0+q_0\right]R_0^2\delta R_0\right\}=0.$$

From the latter we obtain the differential equation

$$\sum_{k=1}^{3}\left(\frac{d}{d\rho}\rho^2\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}-\rho^2\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R}\right)=0 \tag{3.10.3}$$

and the boundary conditions

$$\left.\begin{aligned}\rho=\rho_1:&\quad \frac{\rho_1^2}{R_1^2}\left(\sum_{k=1}^{3}\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}\right)=-q_1;\\ \rho=\rho_0:&\quad \frac{\rho_0^2}{R_0^2}\left(\sum_{k=1}^{3}\frac{\partial A}{\partial I_k}\frac{\partial I_k}{\partial R'}\right)=-q_0.\end{aligned}\right\} \tag{3.10.4}$$

According to eq. (3.8.5) the extended form of the boundary conditions repeats eqs. (3.8.12) and (3.8.3)

$$2\frac{\rho_s^2}{R_s^2}R_s'\left(\frac{\partial A}{\partial I_1}+2\frac{\partial A}{\partial I_2}\frac{R^2}{\rho^2}+\frac{\partial A}{\partial I_3}\frac{R^4}{\rho^4}\right)_s=-q_s\quad(s=0,1), \tag{3.10.5}$$

while differential equation (3.10.3) is equivalent to the equation of statics (3.8.8) (or (3.8.9))

$$\frac{d}{d\rho}\rho^2R'\left(\frac{\partial A}{\partial I_1}+2\frac{\partial A}{\partial I_2}\frac{R^2}{\rho^2}+\frac{\partial A}{\partial I_3}\frac{R^4}{\rho^4}\right)-$$
$$2R\left[\frac{\partial A}{\partial I_1}+\frac{\partial A}{\partial I_2}\left(\frac{R^2}{\rho^2}+R'^2\right)+\frac{\partial A}{\partial I_3}\frac{R^2R'^2}{\rho^2}\right]=0. \tag{3.10.6}$$

This derivation requires only the expressions for the invariants and performing the standard operations for constructing equations of the variational problem. There is no need to take account of the distinction between the contravariant and physical components, to write down the expressions of the constitutive equations etc.

The extended form of equations is cumbersome since it contains derivatives of the type

$$\frac{d}{d\rho}\frac{\partial A}{\partial I_s}=\sum_{k=1}^{3}\frac{\partial^2 A}{\partial I_k\partial I_s}\frac{\partial I_k}{\partial\rho}=\sum_{k=1}^{3}\frac{\partial^2 A}{\partial I_k\partial I_s}\left(\frac{\partial I_k}{\partial\rho}+\frac{\partial I_k}{\partial R}R'+\frac{\partial I_k}{\partial R'}R''\right).$$

Utilising Ritz's method, admitted by the variational statement of the problem, results in a nonlinear system of finite equations, the number of equations coinciding with the number of introduced parameters.

9.4 Small deformation in the case of the initial loading

9.4.1 Small deformation of the deformed volume

In what follows three states of the elastic body are considered: the initial state in $v-$volume bounded by surface o, the first state of deformation ($V-$volume and surface O) and the second state obtained from the first one by means of a small displacement $\eta\mathbf{w}$. The volume and the surface of the body in this state are denoted respectively as V^* and O^*. Further, η is a small parameter and only terms first order in this parameter are retained in what follows.

A Cartesian coordinate system $OXYZ$ is introduced and the fixed directions of their axes are given by the unit vectors $\mathbf{i}_s$. The coordinates of a point in the medium in $v-, V-, V^*-$volumes are denoted respectively as a_s, x_s, x_s^* and the corresponding position vectors are as follows

$$\mathbf{r} = a_s\mathbf{i}_s, \quad \mathbf{R} = \mathbf{i}_s x_s, \quad \mathbf{R}^* = \mathbf{i}_s x_s^* = \mathbf{R} + \eta\mathbf{w}. \tag{4.1.1}$$

For the material coordinates of the point, we preserve the denotation q^1, q^2, q^3. The vector basis and cobasis in $v-$ and $V-$volumes are given by the vectorial triples $\mathbf{r}_s, \mathbf{R}_s$ and $\mathbf{r}^s, \mathbf{R}^s$ respectively. As usual, the metric tensors are denoted as $\hat{g}$ and $\hat{G}$.

The coordinate basis in V^*-volume is determined by the vectorial triple

$$\mathbf{R}_s^* = \frac{\partial \mathbf{R}^*}{\partial q^s} = \mathbf{R}_s + \eta\frac{\partial \mathbf{w}}{\partial q^s}. \tag{4.1.2}$$

Up to now we have denoted the operations in $V-$volume (in contrast to operations in $v-$volume) by a tilde ($\sim$). There is no need for such a complication of notation here, as all operations will be carried out in $V-$ and V^*-volumes, the operations in V^*-volume being marked by an asterisk.

By definition of the nabla-operator ∇ and the gradient of the vector we have

$$\nabla\mathbf{w} = \mathbf{R}^s\frac{\partial \mathbf{w}}{\partial q^s}, \quad \frac{\partial \mathbf{w}}{\partial q^s} = \mathbf{R}^s \cdot \nabla\mathbf{w},$$

see eqs. (E.4.2) and (E.4.5). Introducing into consideration the unit (metric) tensor $\hat{G}$ we have

$$\mathbf{R}_s^* = \mathbf{R}_s + \eta\mathbf{R}_s \cdot \nabla\mathbf{w} = \mathbf{R}_s \cdot \left(\hat{G} + \eta\nabla\mathbf{w}\right) = \left(\hat{G} + \eta\nabla\mathbf{w}^T\right) \cdot \mathbf{R}_s, \tag{4.1.3}$$

where superscript T denotes the transposition of tensor, i.e.

$$\nabla\mathbf{w} = \mathbf{R}^s\mathbf{R}^q\nabla_s w_q, \quad \nabla\mathbf{w}^T = \mathbf{R}^q\mathbf{R}^s\nabla_s w_q. \tag{4.1.4}$$

The covariant components of the metric tensor $\hat{G}^*$ in V^*–volume are given by

$$G^*_{sk} = \mathbf{R}^*_s \cdot \mathbf{R}^*_k = (\mathbf{R}_s + \eta \mathbf{R}_s \cdot \nabla \mathbf{w}) \cdot \left(\mathbf{R}_k + \eta \nabla \mathbf{w}^T \cdot \mathbf{R}_k\right)$$
$$= G_{sk} + \eta \mathbf{R}_s \cdot \left(\nabla \mathbf{w} + \nabla \mathbf{w}^T\right) \cdot \mathbf{R}_k.$$

Recalling the definition of the linear strain tensor for vector $\mathbf{w}$

$$\hat{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{w} + \nabla \mathbf{w}^T\right), \qquad (4.1.5)$$

we obtain

$$G^*_{sk} = G_{sk} + 2\eta \mathbf{R}_s \cdot \hat{\varepsilon} \cdot \mathbf{R}_k = G_{sk} + 2\eta \varepsilon_{sk}. \qquad (4.1.6)$$

Determining the cobasis $\mathbf{R}^{*s}$ relies on the identity of the metric tensors $\hat{G}$ and $\hat{G}^*$

$$\hat{G} = \hat{G}^*. \qquad (4.1.7)$$

Representing them in the form of dyadic products

$$\mathbf{R}^{*s}\mathbf{R}^*_s = \mathbf{R}^s \mathbf{R}_s, \quad \mathbf{R}^{*s}\mathbf{R}^*_s \cdot \mathbf{R}^k = \mathbf{R}^k,$$

substituting eq. (4.1.3) for $\mathbf{R}^*_s$ and replacing $\eta \mathbf{R}^{*s}$ by $\eta \mathbf{R}^s$ we have

$$\mathbf{R}^{*s}\mathbf{R}_s \cdot \mathbf{R}^k + \eta \mathbf{R}^s \mathbf{R}_s \cdot \nabla \mathbf{w} \cdot \mathbf{R}^k = \mathbf{R}^{*k} + \eta \nabla \mathbf{w} \cdot \mathbf{R}^k = \mathbf{R}^k,$$

or

$$\mathbf{R}^{*k} = \mathbf{R}^k - \eta \mathbf{R}^k \cdot \nabla \mathbf{w}^T = \mathbf{R}^k \cdot \left(\hat{G} - \eta \nabla \mathbf{w}^T\right) = \left(\hat{G} - \eta \nabla \mathbf{w}\right) \cdot \mathbf{R}^k. \qquad (4.1.8)$$

Expressions for the contravariant components G^{*sk} of the metric tensor are now set in the form

$$G^{*sk} = \mathbf{R}^{*s} \cdot \mathbf{R}^{*k} = \left(\mathbf{R}^s - \eta \mathbf{R}^s \cdot \nabla \mathbf{w}^T\right) \cdot \left(\mathbf{R}^k - \eta \nabla \mathbf{w} \cdot \mathbf{R}^k\right)$$
$$= G^{sk} - 2\eta \mathbf{R}^s \cdot \hat{\varepsilon} \cdot \mathbf{R}^k. \qquad (4.1.9)$$

The easiest way to find determinant $G^* = |G^*_{st}|$ is to use the equality

$$G^* = G + \eta \left(\frac{\partial G^*}{\partial \eta}\right)_{\eta=0} = G + \eta \left(\frac{\partial G^*}{\partial G^*_{st}}\right)_{\eta=0} \frac{\partial G^*_{st}}{\partial \eta} = G + 2\eta G G^{st} \varepsilon_{st}.$$

Introducing the first invariant of tensor $\hat{\varepsilon}$ in V–metric

$$\vartheta = G^{st} \varepsilon_{st}, \qquad (4.1.10)$$

we obtain

$$G^* = G(1 + 2\eta\vartheta), \quad \sqrt{G^*} = \sqrt{G}(1 + \eta\vartheta). \tag{4.1.11}$$

The strain measure of V^*–volume is determined by tensor $\hat{G}^{\times^*}$ which has the covariant components in the metric of the initial v–volume, these components being equal to the covariant components G^*_{st} of the metric tensor of V^*–volume

$$\hat{G}^{\times^*} = G^*_{st}\mathbf{r}^s\mathbf{r}^t = \hat{G}^\times + 2\eta\varepsilon_{st}\mathbf{r}^s\mathbf{r}^t. \tag{4.1.12}$$

The principal invariants of this tensor are given by formulae (5.2.6)-(5.2.8) of Chapter 2

$$\begin{aligned}
I_1\left(\hat{G}^{\times^*}\right) &= g^{sk}G^*_{sk} = I_1\left(\hat{G}^\times\right) + 2\eta g^{st}\varepsilon_{st}, \\
I_2\left(\hat{G}^{\times^*}\right) &= \frac{G^*}{g}g_{sk}G^{*sk} \\
&= I_2\left(\hat{G}^\times\right) + 2\eta\left[I_2\left(\hat{G}^\times\right)\vartheta - I_3\left(\hat{G}^\times\right)g_{sk}G^{sq}G^{kt}\varepsilon_{qt}\right], \\
I_3\left(\hat{G}^{\times^*}\right) &= \frac{G^*}{g} = I_3\left(\hat{G}^\times\right)(1 + 2\eta\vartheta).
\end{aligned}$$

In the following ηl_s denotes the difference in the principal invariants

$$\eta l_s = I_s\left(\hat{G}^{\times^*}\right) - I_s\left(\hat{G}^\times\right), \tag{4.1.13}$$

such that

$$l_1 = 2g^{st}\varepsilon_{st}, l_2 = 2\left[I_2\left(\hat{G}^\times\right)\vartheta - I_3\left(\hat{G}^\times\right)g_{sk}G^{sq}G^{kt}\varepsilon_{qt}\right], l_3 = 2\vartheta I_3\left(\hat{G}^\times\right). \tag{4.1.14}$$

The vector of the oriented surface is determined by eq. (3.5.3) of Chapter 2

$$\begin{aligned}
\mathbf{N}^*dO^* &= \sqrt{\frac{G^*}{G}}\mathbf{R}^{*s}N_s dO = (1 + \eta\vartheta)(\mathbf{R}^s - \eta\nabla\mathbf{w}\cdot\mathbf{R}^s)N_s dO \\
&= [\mathbf{N} + \eta(\vartheta\mathbf{N} - \nabla\mathbf{w}\cdot\mathbf{N})]\,dO.
\end{aligned} \tag{4.1.15}$$

Therefore

$$\left(\frac{dO^*}{dO}\right)^2 = 1 + 2\eta(\vartheta - \mathbf{N}\cdot\nabla\mathbf{w}\cdot\mathbf{N}), \quad \frac{dO^*}{dO} = 1 + \eta(\vartheta - \mathbf{N}\cdot\nabla\mathbf{w}\cdot\mathbf{N})$$

and

$$\mathbf{N}^* = \mathbf{N} + \eta(\mathbf{NN}\cdot\nabla\mathbf{w}\cdot\mathbf{N} - \nabla\mathbf{w}\cdot\mathbf{N}) = \mathbf{N} + \eta\mathbf{N}\times[\mathbf{N}\times(\nabla\mathbf{w}\cdot\mathbf{N})]. \tag{4.1.16}$$

The difference in vectors $\mathbf{N}^*$ and $\mathbf{N}$ is a vector orthogonal to $\mathbf{N}$.

Using eq. (1.2.13) of Chapter 2 for decomposing tensors $\nabla \mathbf{w}$ and $\nabla \mathbf{w}^T$ into symmetric and skew-symmetric parts

$$\nabla \mathbf{w} = \hat{\varepsilon} - \hat{\Omega}, \quad \nabla \mathbf{w}^T = \hat{\varepsilon} + \hat{\Omega}$$

and referring to formulae (A.5.8) and (A.5.11) we can present the base vectors and the normal vector in the form

$$\left.\begin{array}{l} \mathbf{R}_s^* - \mathbf{R}_s = \eta \left(\mathbf{R}_s \cdot \hat{\varepsilon} + \boldsymbol{\omega} \times \mathbf{R}_s \right), \\ \mathbf{R}^{*s} - \mathbf{R}^s = -\eta \left(\mathbf{R}^s \cdot \hat{\varepsilon} - \boldsymbol{\omega} \times \mathbf{R}^s \right), \\ \mathbf{N}^* - \mathbf{N} = \eta \left\{ \mathbf{N} \times \left[\mathbf{N} \times (\mathbf{N} \cdot \hat{\varepsilon}) \right] + \boldsymbol{\omega} \times \mathbf{N} \right\}. \end{array} \right\} \tag{4.1.17}$$

9.4.2 Stress tensor

Let ηp^{st} denote the difference in the contravariant components t^{*st} and t^{st} of tensors $\hat{T}^*$ and $\hat{T}$ respectively

$$\hat{T}^* = t^{*st} \mathbf{R}_s^* \mathbf{R}_t^* = \left(t^{st} + \eta p^{st} \right) \mathbf{R}_s^* \mathbf{R}_t^*. \tag{4.2.1}$$

Referring to eq. (4.1.3) we have

$$\begin{aligned} \hat{T}^* &= t^{st} \left(\mathbf{R}_s + \eta \nabla \mathbf{w}^T \cdot \mathbf{R}_s \right) \left(\mathbf{R}_t + \eta \mathbf{R}_t \cdot \nabla \mathbf{w} \right) + \eta p^{st} \mathbf{R}_s \mathbf{R}_t \\ &= \hat{T} + \eta \left(\hat{P} + t^{st} \mathbf{R}_s \mathbf{R}_t \cdot \nabla \mathbf{w} + t^{st} \nabla \mathbf{w}^T \cdot \mathbf{R}_s \mathbf{R}_t \right), \end{aligned}$$

or

$$\hat{T}^* = \hat{T} + \eta \hat{S}, \quad \hat{S} = \hat{P} + \hat{T} \cdot \nabla \mathbf{w} + \nabla \mathbf{w}^T \cdot \hat{T}, \quad \hat{P} = p^{st} \mathbf{R}_s \mathbf{R}_t. \tag{4.2.2}$$

The constitutive law is now presented in Finger's form, eq. (2.4.1) of Chapter 8

$$\hat{T} = 2\sqrt{\frac{g}{G}} \left(\overset{0}{c}\, \hat{M} - \overset{1}{c}\, \hat{M}^2 + \overset{-1}{c}\, \hat{G} \right). \tag{4.2.3}$$

Taking into account eq. (4.1.7) we have in V^*–volume

$$\hat{T}^* = 2\sqrt{\frac{g}{G^*}} \left(\overset{0}{c^*}\, \hat{M}^* - \overset{1}{c^*}\, \hat{M}^{*2} + \overset{-1}{c^*}\, \hat{G} \right). \tag{4.2.4}$$

By introducing the denotation

$$\overset{s}{c^*} - \overset{s}{c} = \eta\, \overset{s}{b} \quad (s = 0, 1, -1) \tag{4.2.5}$$

we can rewrite eq. (4.2.2) as follows

$$\begin{aligned} \hat{T}^* = \hat{T} - \eta \vartheta \hat{T} + 2\eta \sqrt{\frac{g}{G}} \left(\overset{0}{b}\, \hat{M} - \overset{1}{b}\, \hat{M}^2 + \overset{-1}{b}\, \hat{G} \right) + \\ 2\sqrt{\frac{g}{G}} \left[\overset{0}{c} \left(\hat{M}^* - \hat{M} \right) - \overset{1}{c} \left(\hat{M}^{*2} - \hat{M}^2 \right) \right]. \end{aligned} \tag{4.2.6}$$

Here by definition of tensor $\hat{M}$

$$\left.\begin{aligned}
&\hat{M}^* - \hat{M} = g^{sk}\left(\mathbf{R}_s^*\mathbf{R}_k^* - \mathbf{R}_s\mathbf{R}_k\right) = \eta\left(\hat{M}\cdot\nabla\mathbf{w} + \nabla\mathbf{w}^T\cdot\hat{M}\right),\\
&\hat{M}^{*2} - \hat{M}^2 = \left(\hat{M}^* - \hat{M}\right)\cdot\hat{M}^* + \hat{M}\cdot\left(\hat{M}^* - \hat{M}\right)\\
&\qquad\qquad = \left(\hat{M}^* - \hat{M}\right)\cdot\hat{M} + \hat{M}\cdot\left(\hat{M}^* - \hat{M}\right)\\
&= \eta\left(\hat{M}\cdot\nabla\mathbf{w}\cdot\hat{M} + \nabla\mathbf{w}^T\cdot\hat{M}^2 + \hat{M}^2\cdot\nabla\mathbf{w} + \hat{M}\cdot\nabla\mathbf{w}^T\cdot\hat{M}\right)
\end{aligned}\right\} \tag{4.2.7}$$

or

$$\hat{M}^{*2} - \hat{M}^2 = \eta\left(2\hat{M}\cdot\hat{\varepsilon}\cdot\hat{M} + \hat{M}^2\cdot\nabla\mathbf{w} + \nabla\mathbf{w}^T\cdot\hat{M}^2\right). \tag{4.2.8}$$

This allows us to set the last term in eq. (4.2.6) in the following form

$$\begin{aligned}
&2\sqrt{\frac{g}{G}}\left[\overset{0}{c}\left(\hat{M}^* - \hat{M}\right) - \overset{1}{c}\left(\hat{M}^{*2} - \hat{M}^2\right)\right] =\\
&\qquad = 2\eta\sqrt{\frac{g}{G}}\left(\overset{0}{c}\,\hat{M} - \overset{1}{c}\,\hat{M}^2 + \overset{-1}{c}\,\hat{G}\right)\cdot\nabla\mathbf{w} +\\
&2\eta\sqrt{\frac{g}{G}}\nabla\mathbf{w}^T\cdot\left(\overset{0}{c}\,\hat{M} - \overset{1}{c}\,\hat{M}^2 + \overset{-1}{c}\,\hat{G}\right) - 4\eta\sqrt{\frac{g}{G}}\left(\overset{1}{c}\,\hat{M}\cdot\hat{\varepsilon}\cdot\hat{M} + \overset{-1}{c}\,\hat{\varepsilon}\right)\\
&\qquad = \eta\left(\hat{T}\cdot\nabla\mathbf{w} + \nabla\mathbf{w}^T\cdot\hat{T}\right) - 4\eta\sqrt{\frac{g}{G}}\left(\overset{1}{c}\,\hat{M}\cdot\hat{\varepsilon}\cdot\hat{M} + \overset{-1}{c}\,\hat{\varepsilon}\right).
\end{aligned}$$

Returning to eq. (4.2.6) we obtain

$$\begin{aligned}
\hat{S} = -\vartheta T + 2\sqrt{\frac{g}{G}}\left(\overset{0}{b}\,\hat{M} - \overset{1}{b}\,\hat{M}^2 + \overset{-1}{b}\,\hat{G}\right) + \hat{T}\cdot\nabla\mathbf{w} +&\\
\nabla\mathbf{w}^T\cdot\hat{T} - 4\sqrt{\frac{g}{G}}\left(\overset{1}{c}\,\hat{M}\cdot\hat{\varepsilon}\cdot\hat{M} + \overset{-1}{c}\,\hat{\varepsilon}\right),&
\end{aligned} \tag{4.2.9}$$

and tensor $\hat{P}$ can be represented as follows

$$\begin{aligned}
\hat{P} = 2\sqrt{\frac{g}{G}}\Bigg[\left(\overset{0}{b} - \vartheta\,\overset{0}{c}\right)\hat{M} - \left(\overset{1}{b} - \vartheta\,\overset{1}{c}\right)\hat{M}^2 + \left(\overset{-1}{b} - \vartheta\,\overset{-1}{c}\right)\hat{G} -&\\
2\left(\overset{1}{c}\,\hat{M}\cdot\hat{\varepsilon}\cdot\hat{M} + \overset{-1}{c}\,\hat{\varepsilon}\right)\Bigg].&
\end{aligned} \tag{4.2.10}$$

Its components in the metric of $V-$volume are determined by the formulae

$$\begin{aligned}
p^{st} = 2\sqrt{\frac{g}{G}}\Bigg[g^{st}\left(\overset{0}{b} - \vartheta\,\overset{0}{c}\right) - \left(\overset{1}{b} - \vartheta\,\overset{1}{c}\right)g^{sr}g^{tq}G_{rq} +&\\
\left(\overset{-1}{b} - \vartheta\,\overset{-1}{c}\right)G^{st} - 2\left(\overset{1}{c}\,g^{sr}g^{tq} + \overset{-1}{c}\,G^{sr}G^{tq}\right)\varepsilon_{rq}\Bigg].&
\end{aligned} \tag{4.2.11}$$

9.4.3 Necessary conditions of equilibrium

The vectors of mass and volume force in V^*–volume are denoted respectively as $\mathbf{K}^*$ and $\rho^*\mathbf{K}^*$

$$\mathbf{K}^* = \mathbf{K} + \eta\mathbf{k}, \quad \rho^*\mathbf{K}^* = \sqrt{\frac{G}{G^*}}\rho\mathbf{K} + \eta\rho\mathbf{k}. \tag{4.3.1}$$

The equilibrium equations in V^*– and V–volume are written down in the form of eq. (3.3.2) of Chapter 1

$$\frac{1}{\sqrt{G^*}}\frac{\partial}{\partial q^s}\sqrt{G^*}t^{*st}\mathbf{R}_t^* + \rho^*\mathbf{K}^* = 0, \quad \frac{1}{\sqrt{G}}\frac{\partial}{\partial q^s}\sqrt{G}t^{st}\mathbf{R}_t + \rho\mathbf{K} = 0. \tag{4.3.2}$$

It follows from these equations that

$$\frac{1}{\sqrt{G}}\frac{\partial}{\partial q^s}\left(\sqrt{G^*}t^{*st}\mathbf{R}_t^* - \sqrt{G}t^{st}\mathbf{R}_t\right) + \eta\rho\mathbf{k} = 0. \tag{4.3.3}$$

The value in the parentheses is as follows

$$\sqrt{G}\left[(1+\eta\vartheta)\left(t^{st} + \eta p^{st}\right)\left(\mathbf{R}_t + \eta\mathbf{R}_t\cdot\nabla\mathbf{w}\right) - t^{st}\mathbf{R}_t\right] = \\ = \eta\sqrt{G}\left(\vartheta t^{st}\mathbf{R}_t + p^{st}\mathbf{R}_t + t^{st}\mathbf{R}_t\cdot\nabla\mathbf{w}\right).$$

On the other hand, estimating the divergence of tensor $\hat{T}\cdot\nabla\mathbf{w}$ we have (see eq. (D.4.7))

$$\nabla\cdot\hat{T}\cdot\nabla\mathbf{w} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial q^s}\sqrt{G}t^{st}\mathbf{R}_t\cdot\nabla\mathbf{w}$$

and eq. (4.3.3) takes the form

$$\nabla\cdot\left(\vartheta\hat{T} + \hat{P} + \hat{T}\cdot\nabla\mathbf{w}\right) + \rho\mathbf{k} = 0. \tag{4.3.4}$$

Let us proceed to the equation of equilibrium on the surface. Force $\mathbf{F}^*dO^*$ acting on the surface element dO^* is equal to

$$\mathbf{F}^*dO^* = \mathbf{F}dO + \eta\mathbf{f}dO, \tag{4.3.5}$$

and according to eqs. (4.1.15) and (4.2.2) the equilibrium equation on dO^* is presented in the form

$$\mathbf{F}^*dO^* = \mathbf{N}\cdot\hat{T}^*dO^* = \left[\mathbf{N} + \eta\left(\vartheta\mathbf{N} - \nabla\mathbf{w}\cdot\mathbf{N}\right)\right]\cdot\left(\hat{T} + \eta\hat{S}\right)dO \\ = \mathbf{F}dO + \eta\mathbf{f}dO.$$

Substituting

$$\nabla\mathbf{w}\cdot\mathbf{N} = \mathbf{N}\cdot\nabla\mathbf{w}^T, \quad \mathbf{N}\cdot\hat{T} = \mathbf{F}$$

we obtain

$$\mathbf{N} \cdot \left(\hat{S} + \vartheta \hat{T} - \nabla \mathbf{w}^T \cdot \hat{T} \right) = \mathbf{f}$$

or referring to eq. (4.2.2) we have

$$\mathbf{N} \cdot \left(\vartheta \hat{T} + \hat{P} + \hat{T} \cdot \nabla \mathbf{w} \right) = \mathbf{f}. \tag{4.3.6}$$

The equilibrium equations in the volume and on the surface are proved to be expressed in terms of the same non-symmetric tensor of second rank

$$\hat{\Theta} = \vartheta \hat{T} + \hat{P} + \hat{T} \cdot \nabla \mathbf{w} = \vartheta \hat{T} + \hat{P} + \hat{T} \cdot \hat{\varepsilon} + \left(\boldsymbol{\omega} \times \hat{T} \right)^T. \tag{4.3.7}$$

This could be foreseen as the equilibrium equation in the volume expresses the condition of zero principal vector of external forces acting on any volume $V^{*'}$ contained in V^*–volume

$$\iint\limits_{O^{*'}} \mathbf{N}^* \cdot \hat{T}^* dO + \iiint\limits_{V^{*'}} \rho^* \mathbf{K}^* d\tau^* = 0.$$

Replacing the integrands in the latter equation by the following expressions

$$\mathbf{N}^* \cdot \hat{T}^* dO = \mathbf{N} \cdot \hat{T} dO + \eta \mathbf{N} \cdot \hat{\Theta} dO, \quad \rho^* \mathbf{K}^* d\tau^* = \rho \mathbf{K} d\tau + \eta \rho \mathbf{k} d\tau,$$

we obtain

$$\iint\limits_{O'} \mathbf{N} \cdot \hat{T} dO + \iiint\limits_{V'} \rho \mathbf{K} d\tau + \eta \left(\iint\limits_{O'} \mathbf{N} \cdot \hat{\Theta} dO + \iiint\limits_{V'} \rho \mathbf{k} d\tau \right) = 0.$$

Utilising denotation (4.3.7) we again arrive at the equilibrium equation (4.3.4). The obtained equilibrium equation

$$\nabla \cdot \hat{\Theta} + \rho \mathbf{k} = 0, \quad \mathbf{N} \cdot \hat{\Theta} = \mathbf{f} \tag{4.3.8}$$

looks like the equilibrium equation

$$\nabla \cdot \hat{T} + \rho \mathbf{K} = 0, \quad \mathbf{N} \cdot \hat{T} = \mathbf{F},$$

however one should remember that tensor $\hat{\Theta}$ is not equal to the difference $\hat{T}^* - \hat{T}$ which is determined by tensor $\hat{S}$

$$\hat{\Theta} = \hat{S} + \vartheta \hat{T} - \nabla \mathbf{w}^T \cdot \hat{T} = \hat{S} + \vartheta \hat{T} - \hat{\varepsilon} \cdot \hat{T} - \boldsymbol{\omega} \times \hat{T}. \tag{4.3.9}$$

Tensor $\hat{\Theta}$ is not symmetric due to the term $\boldsymbol{\omega} \times \hat{T}$ and this is explained by rotation of the volume element under deformation of V–volume.

Tensor $\hat{\Theta}$ is a linear differential operator over vector $\mathbf{w}$. When the mass and surface forces are absent ($\mathbf{k} = 0, \mathbf{f} = 0$) or forces $\mathbf{k}$ and $\mathbf{f}$ are linear differential operators over $\mathbf{w}$ then the problem of determining $\mathbf{w}$ reduces to a homogeneous system of linear (in $\mathbf{w}$) differential equations of second order with homogeneous boundary conditions. These are the so-called equations of neutral equilibrium. Clearly, they admit the trivial solution $\mathbf{w} = 0$. However there can be non-trivial solutions with the equilibrium states which are close to the considered equilibrium of V–volume loaded by forces $\mathbf{K}$ and $\mathbf{F}$. The values of the loading parameters for which the equations of the neutral equilibrium have a nontrivial solution are referred to as the bifurcation parameters. The formulated boundary value allows one to find the bifurcation parameters. However, it does not determine the equilibrium forms of V–volume which differ from the original form. Obtaining these forms requires consideration of the complete equations of equilibrium of V^*–volume.

9.4.4 Representation of tensor $\hat{\Theta}$

According to eqs. (4.3.7), (4.2.3) and (4.2.10) tensor $\hat{\Theta}$ is the sum of two tensors

$$\hat{\Theta} = \hat{R} + \hat{T} \cdot \nabla \mathbf{w}, \tag{4.4.1}$$

$$\hat{R} = \frac{2}{\sqrt{I_3}} \left[\left(\overset{0}{b}\, \hat{M} - \overset{1}{b}\, \hat{M}^2 + \overset{-1}{b}\, \hat{G} \right) - 2 \left(\overset{1}{c}\, \hat{M} \cdot \hat{\varepsilon} \cdot \hat{M} + \overset{-1}{c}\, \hat{\varepsilon} \right) \right]$$

where, by eqs. (4.1.13) and (4.2.5),

$$\overset{s}{b} = \frac{\partial \overset{s}{c}}{\partial I_k} l_k \quad (s = 0, 1, -1). \tag{4.4.2}$$

The values of I_k are given by formulae (4.1.14) reduced to the form

$$\left.\begin{aligned} l_1 &= 2 g^{mn} \varepsilon_{mn} = 2 \hat{M} \cdot\cdot\, \hat{\varepsilon}, \\ l_2 &= 2 \left(\vartheta I_2 - I_3 g_{sk} G^{sm} G^{kn} \varepsilon_{mn} \right) = 2 \left(I_2 \hat{G} - I_3 \hat{M}^{-1} \right) \cdot\cdot\, \hat{\varepsilon}, \\ l_3 &= 2 I_3 \vartheta = 2 I_3 \hat{G} \cdot\cdot\, \hat{\varepsilon}, \end{aligned}\right\} \tag{4.4.3}$$

where $I_k = I_k \left(\hat{G}^\times \right) = I_k \left(\hat{M} \right)$. This transformation is used in what follows and is based upon the relationships

$$\vartheta = G^{mn} \varepsilon_{mn}, \quad \hat{M} = g^{sk} \mathbf{R}_s \mathbf{R}_k, \quad \hat{M}^{-1} = g_{sk} \mathbf{R}^s \mathbf{R}^k = g_{sk} G^{sq} G^{kt} \mathbf{R}_q \mathbf{R}_t.$$

The symmetric tensor $\hat{R}$ is set in the form

$$\frac{1}{4}\sqrt{I_3}\hat{R} = \left[\left(\frac{\partial \overset{0}{c}}{\partial I_1} g^{st} - \frac{\partial \overset{1}{c}}{\partial I_1} g^{sk} g^{rt} G_{kr} + \frac{\partial \overset{-1}{c}}{\partial I_1} G^{st}\right) g^{mn} + \right.$$
$$\left(\frac{\partial \overset{0}{c}}{\partial I_2} g^{st} - \frac{\partial \overset{1}{c}}{\partial I_2} g^{sk} g^{rt} G_{kr} + \frac{\partial \overset{-1}{c}}{\partial I_2} G^{st}\right) \left(I_2 G^{mn} - I_3 g_{qk} G^{qm} G^{kn}\right) +$$
$$\left(\frac{\partial \overset{0}{c}}{\partial I_3} g^{st} - \frac{\partial \overset{1}{c}}{\partial I_3} g^{sk} g^{rt} G_{kr} + \frac{\partial \overset{-1}{c}}{\partial I_3} G^{st}\right) I_3 G^{mn} -$$
$$\left.\left(\overset{1}{c}\, g^{sm} g^{nt} + \overset{-1}{c}\, G^{sm} G^{nt}\right)\right] \mathbf{R}_s \mathbf{R}_t \varepsilon_{mn}. \quad (4.4.4)$$

Its contravariant components are determined by the formulae

$$R^{st} = c^{stmn}\varepsilon_{mn}, \quad c^{stmn} = c^{tsmn} = c^{stnm}, \quad (4.4.5)$$

where c^{stmn} are the components of the tensor of fourth rank ${}^{(4)}\hat{C}$. This tensor is defined in terms of the components in the square brackets in eq. (4.4.4).

In terms of the principal axes $\overset{r}{\tilde{\mathbf{e}}}$ of tensor $\hat{M}$ the tensors $\hat{M}, \hat{M}^{-1}, \hat{G}$ and $\hat{\varepsilon}$ take the form

$$\hat{M} = \sum_{r=1}^{3} G_r \overset{r}{\tilde{\mathbf{e}}}\overset{r}{\tilde{\mathbf{e}}}, \quad \hat{M}^{-1} = \sum_{r=1}^{3} \frac{\overset{r}{\tilde{\mathbf{e}}}\overset{r}{\tilde{\mathbf{e}}}}{G_r}, \quad \hat{G} = \sum_{r=1}^{3} \overset{r}{\tilde{\mathbf{e}}}\overset{r}{\tilde{\mathbf{e}}}, \quad \hat{\varepsilon} = \sum_{r=1}^{3}\sum_{s=1}^{3} \varepsilon_{(rs)} \overset{r}{\tilde{\mathbf{e}}}\overset{s}{\tilde{\mathbf{e}}}$$
$$\left(\varepsilon_{(rs)} = \overset{r}{\tilde{\mathbf{e}}} \cdot \hat{\varepsilon} \cdot \overset{s}{\tilde{\mathbf{e}}}\right).$$

recalling the expressions for the invariants

$$I_1 = G_1 + G_2 + G_3, \quad I_2 = G_1 G_2 + G_2 G_3 + G_3 G_1, \quad I_3 = G_1 G_2 G_3$$

and the relations which can be easily proved

$$I_2 - \frac{I_3}{G_r} = G_r (I_1 - G_r) = G_r \frac{\partial I_2}{\partial G_r}, \quad \frac{\partial I_3}{\partial G_r} = \frac{I_3}{G_r}, \quad (4.4.6)$$

we can write expressions for l_k in another form

$$l_1 = 2G_r \varepsilon_{(rr)}, \quad l_2 = 2G_r \frac{\partial I_2}{\partial G_r} \varepsilon_{(rr)}, \quad l_3 = 2I_3 \varepsilon_{(rr)},$$

where here and in what follows the summation over the repeated indices is adopted.

It allows one to simplify the expression for $\hat{R}$

$$\frac{1}{4}\sqrt{I_3}\hat{R} = G_r\left(\frac{\partial \overset{0}{c}}{\partial G_r}\hat{M} - \frac{\partial \overset{1}{c}}{\partial G_r}\hat{M}^2 + \frac{\partial \overset{-1}{c}}{\partial G_r}\hat{G}\right)\varepsilon_{(rr)} - \left(\overset{1}{c}\, G_r G_s + \overset{-1}{c}\right)\overset{r\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\,\varepsilon_{(rs)}. \qquad (4.4.7)$$

Replacing $\hat{M}$ by its expression in terms of the principal axes and using the following formulae

$$\overset{0}{c} = \frac{\partial A}{\partial I_1} + I_1\frac{\partial A}{\partial I_2}, \quad \overset{1}{c} = \frac{\partial A}{\partial I_2}, \quad \overset{-1}{c} = I_3\frac{\partial A}{\partial I_3}$$

we obtain

$$\frac{1}{4}\sqrt{I_3}\hat{R} = G_r G_s\left\{\left[\frac{\partial}{\partial G_r}\frac{\partial A}{\partial I_1} + (I_1 - G_s)\frac{\partial}{\partial G_r}\frac{\partial A}{\partial I_2} + \frac{I_3}{G_s}\frac{\partial}{\partial G_r}\frac{\partial A}{\partial I_3}\right]\overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\,\varepsilon_{(rr)} + \left(\frac{\partial A}{\partial I_2} + \frac{I_3}{G_r G_s}\frac{\partial A}{\partial I_3}\right)\left(\overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\,\varepsilon_{(rr)} - \overset{r\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\,\varepsilon_{(rs)}\right)\right\}.$$

The value in the square brackets denoted by $C(G_r, G_s)$ is symmetric about its arguments and is given by

$$C(G_r, G_s) = \frac{\partial^2 A}{\partial I_1^2} + (I_1 - G_s)(I_1 - G_r)\frac{\partial^2 A}{\partial I_2^2} + \frac{I_3^2}{G_r G_s}\frac{\partial^2 A}{\partial I_3^2} + (2I_1 - G_r - G_s)\frac{\partial^2 A}{\partial I_1 \partial I_2} + I_3\left(\frac{I_1 - G_r}{G_s} + \frac{I_1 - G_s}{G_r}\right)\frac{\partial^2 A}{\partial I_2 \partial I_3} + I_3\frac{G_r + G_s}{G_r G_s}\frac{\partial^2 A}{\partial I_3 \partial I_1}. \qquad (4.4.8)$$

At the same time

$$\left.\begin{aligned} \frac{\partial A}{\partial G_s} &= \frac{\partial A}{\partial I_1} + (I_1 - G_s)\frac{\partial A}{\partial I_2} + \frac{I_3}{G_s}\frac{\partial A}{\partial I_3}, \\ \frac{\partial^2 A}{\partial G_r \partial G_s} &= C(G_r, G_s) + (1 - \delta_{sr})\frac{\partial A}{\partial I_2} + \frac{I_3}{G_s G_r}\left(1 - \frac{G_r}{G_s}\delta_{sr}\right)\frac{\partial A}{\partial I_3} \end{aligned}\right\} \qquad (4.4.9)$$

and the expression for $\hat{R}$ takes the form

$$\frac{1}{4}\sqrt{I_3}\hat{R} = G_r G_s \left[C(G_r, G_s) \overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\, \varepsilon_{(rr)} + \left(\frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3} \frac{I_3}{G_r G_s} \right) \left(\overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\, \varepsilon_{(rr)} - \overset{r\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\, \varepsilon_{(rs)} \right) \right] =$$
$$= \left[G_r G_s \frac{\partial^2 A}{\partial G_r \partial G_s} \varepsilon_{(rr)} + \left(G_s^2 \frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3} \right) \varepsilon_{(ss)} \right] \overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} - \left(G_r G_s \frac{\partial A}{\partial I_2} + I_3 \frac{\partial A}{\partial I_3} \right) \overset{r\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}}\, \varepsilon_{(rs)}. \quad (4.4.10)$$

An extended form is as follows

$$\frac{1}{4}\sqrt{I_3}\hat{R} = G_r G_s \frac{\partial^2 A}{\partial G_r \partial G_s} \varepsilon_{(rr)} \overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} - \left[\left(G_1 G_2 \frac{\partial A}{\partial I_2} + \frac{\partial A}{\partial I_3} \right) \varepsilon_{(12)} \left(\overset{1\,2}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + \overset{2\,1}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right) + \left(G_2 G_3 \frac{\partial A}{\partial I_2} + I_3 \frac{\partial A}{\partial I_3} \right) \varepsilon_{(23)} \left(\overset{2\,3}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + \overset{3\,2}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right) + \left(G_3 G_1 \frac{\partial A}{\partial I_2} + I_3 \frac{\partial A}{\partial I_3} \right) \varepsilon_{(31)} \left(\overset{3\,1}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} + \overset{1\,3}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \right) \right]. \quad (4.4.11)$$

The values of $\dfrac{\partial^2 A}{\partial G_r \partial G_s}$ can be expressed in terms of the principal stresses t_s and the derivatives of t_s with respect to G_r. Indeed, referring to eq. (2.6.8) of Chapter 8 we have

$$\frac{2G_s}{\sqrt{I_3}} \frac{\partial A}{\partial G_s} = t_s, \quad \frac{2G_s}{\sqrt{I_3}} \frac{\partial^2 A}{\partial G_r \partial G_s} = \frac{\partial t_s}{\partial G_r} + t_s \left(\frac{1}{2G_r} - \frac{\delta_{rs}}{G_s} \right). \quad (4.4.12)$$

Let us also notice that $\hat{T} \cdot \nabla \mathbf{w}$ can be represented in the following form

$$\hat{T} \cdot \nabla \mathbf{w} = t_s \overset{s\,s}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \cdot \left(\hat{\varepsilon} - \hat{\Omega} \right) = t_s \overset{s\,r}{\tilde{\mathbf{e}}\tilde{\mathbf{e}}} \left[\varepsilon_{(sr)} + \left(\overset{s}{\tilde{\mathbf{e}}} \times \overset{r}{\tilde{\mathbf{e}}} \right) \cdot \boldsymbol{\omega} \right]. \quad (4.4.13)$$

9.4.5 Triaxial state of stress

Under the uniform tension along the directions of the axes of Cartesian system $OXYZ$ the principal directions $\overset{s}{\tilde{\mathbf{e}}}$ are coincident with the direction $\mathbf{i}_s$ of the above axes. The coordinates of point x_s are taken as the material coordinates in V–volume. Then

$$\overset{r}{\tilde{\mathbf{e}}} = \mathbf{i}_r, \quad \varepsilon_{(rs)} = \varepsilon_{rs} = \frac{1}{2} \left(\frac{\partial w_r}{\partial x_s} + \frac{\partial w_s}{\partial x_r} \right),$$

where w_r denotes the projections of vector $\mathbf{w}$ on the coordinate axes. Let us denote

$$G_s = \varkappa_s^2 = (1 + \delta_s)^2, \quad (4.5.1)$$

where δ_s denotes the principal elongations. Tensors $\hat{R}$ and $\hat{T} \cdot \nabla \mathbf{w}$ are presented by linear differential operators over vector $\mathbf{w}$ with constant coefficients.

The mass force is assumed to be absent. Let us also assume that the surface load does not change under displacement $\mathbf{w}$ of the points of V–volume ("dead load") and presents a pressure p which is uniformly distributed over the whole surface normal to the deformed surface O^* ("follower pressure"). By eqs. (4.1.15) and (4.3.5) we have

$$\mathbf{f} = -p\left(\vartheta \mathbf{N} - \mathbf{N} \cdot \nabla \mathbf{w}^T\right). \tag{4.5.2}$$

The equilibrium equations in the volume and on the surface, eq. (4.3.8) are written in the form

$$\begin{aligned}
&\nabla \cdot \hat{\Theta} = \nabla \cdot \hat{R} + \nabla \cdot \hat{T} \cdot \nabla \mathbf{w} = 0, \\
&\mathbf{N} \cdot \hat{\Theta} = \mathbf{N} \cdot \hat{R} + \mathbf{N} \cdot \hat{T} \cdot \nabla \mathbf{w} = -p\left(\vartheta \mathbf{N} - \mathbf{N} \cdot \nabla \mathbf{w}^T\right).
\end{aligned}$$

Repeating the proof of Kirchhoff's theorem on uniqueness of solution of the equations of the linear theory of elasticity, Subsection 4.4.1, let us consider the integral

$$\begin{aligned}
\iiint\limits_V \mathbf{w} \cdot \nabla \cdot \hat{\Theta} d\tau &= \iiint\limits_V \left[\nabla \cdot \left(\hat{\Theta} \cdot \mathbf{w}\right) - \hat{\Theta} \cdot\cdot \nabla \mathbf{w}^T\right] d\tau \\
&= \iint\limits_O \mathbf{N} \cdot \hat{\Theta} \cdot \mathbf{w} dO - \iiint\limits_V \hat{\Theta} \cdot\cdot \nabla \mathbf{w}^T d\tau = 0
\end{aligned}$$

or

$$p \iint\limits_O \left(\mathbf{N} \cdot \vartheta \mathbf{w} - \mathbf{N} \cdot \nabla \mathbf{w}^T \cdot \mathbf{w}\right) dO + \iiint\limits_V \left(\hat{R} + \hat{T} \cdot \nabla \mathbf{w}\right) \cdot\cdot \nabla \mathbf{w}^T d\tau = 0.$$

The surface integral is transformed into a volume one

$$\begin{aligned}
\iint\limits_O \left(\mathbf{N} \cdot \vartheta \mathbf{w} - \mathbf{N} \cdot \nabla \mathbf{w}^T \cdot \mathbf{w}\right) dO &= \iiint\limits_V \left[\nabla \cdot \vartheta \mathbf{w} - \nabla \cdot \left(\nabla \mathbf{w}^T \cdot \mathbf{w}\right)\right] d\tau \\
&= \iiint\limits_V \left(\vartheta^2 + \mathbf{w} \cdot \nabla \vartheta - \mathbf{w} \cdot \nabla \cdot \nabla \mathbf{w}^T - \nabla \mathbf{w}^T \cdot\cdot \nabla \mathbf{w}^T\right) d\tau \\
&= \iiint\limits_V \left(\vartheta^2 - \nabla \mathbf{w}^T \cdot\cdot \nabla \mathbf{w}^T\right) d\tau,
\end{aligned}$$

since $\nabla \cdot \nabla \mathbf{w}^T = \nabla \vartheta$. We arrive at the relation

$$\iiint\limits_V \left\{\left[\hat{R} + \left(\hat{T} \cdot \nabla \mathbf{w} - p \nabla \mathbf{w}^T\right)\right] \cdot\cdot \nabla \mathbf{w}^T + p\vartheta^2\right\} d\tau = \iiint\limits_V 2\Phi d\tau = 0. \tag{4.5.3}$$

The quadratic form of the components of tensor $\nabla \mathbf{w}$ in the integrand is as follows

$$2\Phi = \frac{4}{\varkappa_1\varkappa_2\varkappa_3}\left\{\varkappa_r^2\varkappa_s^2\frac{\partial^2 A}{\partial\varkappa_r^2\partial\varkappa_s^2}\varepsilon_{rr}\varepsilon_{ss} - 2\left[\varkappa_1^2\varkappa_2^2\left(\frac{\partial A}{\partial I_2}+\varkappa_3^2\frac{\partial A}{\partial I_3}\right)\varepsilon_{12}^2 + \right.\right.$$
$$\left.\left.\varkappa_2^2\varkappa_3^2\left(\frac{\partial A}{\partial I_2}+\varkappa_1^2\frac{\partial A}{\partial I_3}\right)\varepsilon_{12}^2 + \varkappa_3^2\varkappa_1^2\left(\frac{\partial A}{\partial I_2}+\varkappa_2^2\frac{\partial A}{\partial I_3}\right)\varepsilon_{31}^2\right]\right\} +$$
$$p\vartheta^2 + \left[\left(\hat{T} - p\hat{E}\right)\cdot\hat{\varepsilon} - \left(\hat{T}+p\hat{E}\right)\cdot\hat{\Omega}\right]\cdot\cdot\nabla\mathbf{w}^T. \quad (4.5.4)$$

The sign of this form depends on the prescribed strain energy and the load. Of course, there may exist nontrivial values of $\partial w_q/\partial x_k$ for which integral (4.5.3) differs from zero.

9.4.6 Hydrostatic state of stress

This is the simplest case in which

$$\varkappa_1 = \varkappa_2 = \varkappa_3 = \varkappa,\ \hat{T} = -p\hat{E},\ I_1 = 3\varkappa^2,\ I_2 = 3\varkappa^4,\ I_3 = \varkappa^6. \quad (4.6.1)$$

Since the specific strain energy is a function of the invariants it can be viewed as a function of $\varkappa^2$

$$A(I_1, I_2, I_3) = A\left(3\varkappa^2, 3\varkappa^4, \varkappa^6\right) = \bar{A}\left(\varkappa^2\right). \quad (4.6.2)$$

Hence

$$\frac{d\bar{A}}{d\varkappa^2} = 3\left(\frac{\partial A}{\partial I_1} + 2\varkappa^2\frac{\partial A}{\partial I_2} + \varkappa^4\frac{\partial A}{\partial I_3}\right) = 3\left(\frac{\partial A}{\partial\varkappa_r^2}\right)_{\varkappa_r=\varkappa}$$

and due to eqs. (4.4.8) and (4.4.9)

$$\frac{d^2\bar{A}}{(d\varkappa^2)^2} = 6\left(\frac{\partial A}{\partial I_2}+\varkappa^2\frac{\partial A}{\partial I_3}\right) + 9C\left(\varkappa^2,\varkappa^2\right) = -3\left(\frac{\partial A}{\partial I_2}+\varkappa^2\frac{\partial A}{\partial I_3}\right) +$$
$$9\left[\frac{\partial^2 A}{\partial G_r\partial G_s} + \delta_{sr}\left(\frac{\partial A}{\partial I_2}+\varkappa^2\frac{\partial A}{\partial I_3}\right)\right]_{\varkappa_r=\varkappa_s=\varkappa}. \quad (4.6.3)$$

Expression (4.4.11) for tensor $\hat{R}$ and the differential equation of equilibrium are set in the form

$$\hat{R} = 4\varkappa\left\{\left[\frac{1}{9}\frac{d^2\bar{A}}{(d\varkappa^2)^2} + \frac{1}{3}\left(\frac{\partial A}{\partial I_2}+\varkappa^2\frac{\partial A}{\partial I_3}\right)\right]\vartheta\hat{E} - \left(\frac{\partial A}{\partial I_2}+\varkappa^2\frac{\partial A}{\partial I_3}\right)\hat{\varepsilon}\right\}, \quad (4.6.4)$$

$$\nabla \cdot \hat{\Theta} = \nabla \cdot \hat{R} - p\nabla \cdot \nabla \mathbf{w} = 4\varkappa \left\{ \left[\frac{1}{9} \frac{d^2 \bar{A}}{(d\varkappa^2)^2} + \frac{1}{3} \left(\frac{\partial A}{\partial I_2} + \varkappa^2 \frac{\partial A}{\partial I_3} \right) \right] \nabla \vartheta - \frac{1}{2} \left(\frac{\partial A}{\partial I_2} + \varkappa^2 \frac{\partial A}{\partial I_3} \right) \left(\nabla^2 \mathbf{w} + \nabla \vartheta \right) \right\} - p\nabla^2 \mathbf{w} = 0. \quad (4.6.5)$$

This form of the equilibrium equation suggests the following denotations for Lame's coefficients

$$\left.\begin{aligned} \tilde{\lambda} + \tilde{\mu} &= \frac{4}{9}\varkappa \left[\frac{d^2 \bar{A}}{(d\varkappa^2)^2} - \frac{2}{3} \left(\frac{\partial A}{\partial I_2} + \varkappa^2 \frac{\partial A}{\partial I_3} \right) \right], \\ \tilde{\mu} &= -\left[p + 2\varkappa \left(\frac{\partial A}{\partial I_2} + \varkappa^2 \frac{\partial A}{\partial I_3} \right) \right]. \end{aligned}\right\} \quad (4.6.6)$$

The equilibrium equation takes the standard form of the equation for a linear elastic isotropic body in terms of displacement

$$\left(\tilde{\lambda} + \tilde{\mu} \right) \nabla \vartheta + \tilde{\mu} \nabla^2 \mathbf{w} = 0, \quad (4.6.7)$$

and tensor $\hat{\Theta}$ is as follows

$$\hat{\Theta} = \tilde{\lambda} \vartheta \hat{E} + 2\tilde{\mu} \hat{\varepsilon} - p \left(\vartheta \hat{E} - \nabla \mathbf{w}^T \right). \quad (4.6.8)$$

In follows from formulae (4.6.6) that

$$\tilde{\lambda} + \frac{2}{3}\tilde{\mu} = \frac{4}{9} \left(\varkappa \frac{d^2 \bar{A}}{(d\varkappa^2)^2} + \frac{3}{4} p \right). \quad (4.6.9)$$

Introducing a new variable $\varkappa^3 = \tau$ which is equal to the ratio of the body volume under hydrostatic compression to the initial volume $\left(\tau = \sqrt{I_3} \right)$ we have

$$\frac{d\bar{A}}{d\tau} = \frac{2}{3\varkappa} \frac{d\bar{A}}{d\varkappa^2}, \quad \frac{d^2 \bar{A}}{d\tau^2} = \frac{4}{9\varkappa} \frac{d}{d\varkappa^2} \left(\frac{1}{\varkappa} \frac{d\bar{A}}{d\varkappa^2} \right) = \frac{4\varkappa}{9\tau} \frac{d^2 \bar{A}}{(d\varkappa^2)^2} - \frac{1}{3\tau} \frac{d\bar{A}}{d\tau}.$$

Referring to eq. (1.2.3) we arrive at the relation

$$\tilde{\lambda} + \frac{2}{3}\tilde{\mu} = \tau \frac{d^2 \bar{A}}{d\tau^2}, \quad (4.6.10)$$

which is an expression for the "reduced" bulk modulus in terms of the second derivative of the specific strain energy with respect to the parameter determining the volume ratio.

Taking into account relationships (4.6.3) and (4.6.6) we can transform the quadratic form (4.5.4) in the following way

$$\begin{aligned} 2\Phi &= \left(\tilde{\lambda} - p \right) \vartheta^2 + 2 \left(\tilde{\mu} + p \right) I_1 \left(\hat{\varepsilon}^2 \right) + p\vartheta^2 - 2p\hat{\varepsilon} \cdot\cdot \nabla \mathbf{w}^T \\ &= \tilde{\lambda} \vartheta^2 + 2\tilde{\mu} I_1 \left(\hat{\varepsilon}^2 \right) = 2\tilde{A}. \end{aligned}$$

Here

$$\tilde{A} = \frac{1}{2}\tilde{\lambda}\vartheta^2 + \tilde{\mu}I_1\left(\hat{\varepsilon}^2\right) \tag{4.6.11}$$

denotes the specific strain energy of the linear elastic isotropic solid with the "reduced" Lame's coefficients.

It follows from eq. (4.5.3) that in the case of the "follower pressure" and positive definite function $\tilde{A}$, displacement $\mathbf{w}$ is nothing else than the rigid body displacement. In agreement with eqs. (4.6.6), (4.6.10) and (3.3.5) of Chapter 4 these conditions lead to the inequalities

$$\tilde{k} = \tilde{\lambda} + \frac{2}{3}\tilde{\mu} = \tau\frac{d^2\bar{A}}{(d\tau^2)^2} > 0,\ \tilde{\mu} = -p - 2\varkappa\left(\frac{\partial A}{\partial I_2} + \varkappa^2\frac{\partial A}{\partial I_3}\right) > 0. \tag{4.6.12}$$

The first one is the intuitively clear requirement for the behaviour of the elastic material which is that the rate of growth of the strain energy increases as the strain increases.

It was shown in Subsection 8.2.9 that inequality (2.9.8) of Chapter 8 is one of the conditions of positiveness of A under sufficiently small deformations ($\varkappa \approx 1$). It can be assumed that it is satisfied under not small deformations

$$-2\varkappa\left(\frac{\partial A}{\partial I_2} + \varkappa^2\frac{\partial A}{\partial I_3}\right) > 0.$$

The second inequality in (4.6.12) means that in the case of hydrostatic compression this value is a function of pressure and exceeds the applied pressure p.

The hydrostatically compressed body remains stable even under superhigh pressures. Prescribing the expression for the specific strain energy must satisfy these requirements.

For example, according to eqs. (4.1.7) and (4.1.8) of Chapter 8 the specific strain energy in the simplified constitutive law of Signorini ($c = 0$) is given by

$$\begin{aligned} A &= \frac{\sqrt{I_3}}{8}\left[(\lambda+\mu)\left(\frac{I_2^2}{I_3^2}+3\right) - 2(3\lambda+\mu)\left(\frac{I_2}{I_3}-1\right)\right] \\ &= \frac{1}{8}\left[\frac{9(\lambda+\mu)}{\varkappa} - 6(3\lambda+\mu)\varkappa + (9\lambda+5\mu)\varkappa^3\right], \end{aligned}$$

and using formulae (4.6.12) leads to the following relations between the "reduced Lame's moduli" and the "Lame moduli" of the Signorini constitutive law

$$\tilde{\lambda} = \frac{1}{2\varkappa^2}\left[(3\lambda+\mu) - \frac{\lambda+\mu}{\varkappa^2}\right],\ \tilde{\mu} = \frac{1}{2\varkappa^2}\left[\frac{3(\lambda+\mu)}{\varkappa^2} - (3\lambda+\mu)\right]. \tag{4.6.13}$$

Of course, $\tilde{\lambda} = \lambda$ and $\tilde{\mu} = \mu$ for $\kappa = 1$. Here $\tilde{\mu} > 0$ and $\tilde{\lambda} + \frac{2}{3}\tilde{\mu} > 0$ since $\varkappa < 1$.

9.4.7 Uniaxial tension

Under tension of the rod along axis x_3 we have

$$\varkappa_1 = \varkappa_2 = \chi, \ \varkappa_3 = \varkappa; \ I_1 = 2\chi^2 + \varkappa^2, \ I_2 = \chi^2\left(\chi^2 + 2\varkappa^2\right), \ I_3 = \chi^4\varkappa^2. \tag{4.7.1}$$

The condition of vanishing the principal stresses t_1 and t_2

$$\frac{\partial A}{\partial I_1} + \left(\chi^2 + \varkappa^2\right)\frac{\partial A}{\partial I_2} + \chi^2\varkappa^2\frac{\partial A}{\partial I_3} = 0 \tag{4.7.2}$$

allows the principal stress t_3 to be represented in one of two form

$$t_3 = \frac{2\varkappa}{\chi^2}\left(\frac{\partial A}{\partial I_1} + 2\chi^2\frac{\partial A}{\partial I_2} + \chi^4\frac{\partial A}{\partial I_3}\right) \tag{4.7.3}$$

$$t_3 = \frac{2}{\chi^2\varkappa}\left(\varkappa^2 - \chi^2\right)\left(\frac{\partial A}{\partial I_1} + \chi^2\frac{\partial A}{\partial I_2}\right) = \frac{2\varkappa}{\chi^2}\left(\chi^2 - \varkappa^2\right)\left(\frac{\partial A}{\partial I_2} + \chi^2\frac{\partial A}{\partial I_3}\right). \tag{4.7.4}$$

By eq. (4.4.10) tensor $\hat{R}$ is set as follows

$$\begin{aligned}
\frac{1}{4}\chi^2\varkappa\hat{R} = {} & \chi^4\left[C\left(\chi^2,\chi^2\right)\left(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2\right)\left(\varepsilon_{11} + \varepsilon_{22}\right) + \right.\\
& \left.\left(\frac{\partial A}{\partial I_2} + \varkappa^2\frac{\partial A}{\partial I_3}\right)\left(\mathbf{i}_1\mathbf{i}_1\varepsilon_{22} + \mathbf{i}_2\mathbf{i}_2\varepsilon_{11}\right)\right] + \\
& \chi^2\varkappa^2\left[C\left(\varkappa^2,\chi^2\right) + \left(\frac{\partial A}{\partial I_2} + \chi^2\frac{\partial A}{\partial I_3}\right)\right]\left[\left(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2\right)\varepsilon_{33} + \mathbf{i}_3\mathbf{i}_3\left(\varepsilon_{11} + \varepsilon_{22}\right)\right] \\
& + \varkappa^4 C\left(\varkappa^2,\varkappa^2\right)\mathbf{i}_3\mathbf{i}_3\varepsilon_{33} - \chi^4\left(\frac{\partial A}{\partial I_2} + \varkappa^2\frac{\partial A}{\partial I_3}\right)\varepsilon_{12}\left(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1\right) - \\
& \chi^2\varkappa^2\left(\frac{\partial A}{\partial I_2} + \chi^2\frac{\partial A}{\partial I_3}\right)\left[\varepsilon_{23}\left(\mathbf{i}_2\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_2\right) + \varepsilon_{31}\left(\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_1\mathbf{i}_3\right)\right].
\end{aligned} \tag{4.7.5}$$

Here due to eq. (4.4.2)

$$\left.\begin{array}{l}
C\left(\chi^2,\chi^2\right) = \dfrac{\partial^2 A}{\partial I_1^2} + \left(\chi^2+\varkappa^2\right)^2 \dfrac{\partial^2 A}{\partial I_2^2} + \chi^4\varkappa^4 \dfrac{\partial^2 A}{\partial I_3^2} + \\
\quad 2\left(\chi^2+\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_1 \partial I_2} + 2\varkappa^2\chi^2\left(\chi^2+\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_2 \partial I_3} + 2\varkappa^2\chi^2 \dfrac{\partial^2 A}{\partial I_3 \partial I_1}, \\
C\left(\varkappa^2,\varkappa^2\right) = \dfrac{\partial^2 A}{\partial I_1^2} + 4\chi^4 \dfrac{\partial^2 A}{\partial I_2^2} + \chi^8 \dfrac{\partial^2 A}{\partial I_3^2} + 4\chi^2 \dfrac{\partial^2 A}{\partial I_1 \partial I_2} + \\
\qquad\qquad 4\chi^6 \dfrac{\partial^2 A}{\partial I_2 \partial I_3} + 2\chi^4 \dfrac{\partial^2 A}{\partial I_3 \partial I_1}, \\
C\left(\varkappa^2,\chi^2\right) = \dfrac{\partial^2 A}{\partial I_1^2} + 2\chi^2\left(\chi^2+\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_2^2} + \chi^6\varkappa^2 \dfrac{\partial^2 A}{\partial I_3^2} + \\
\left(3\chi^2+\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_1 \partial I_2} + \chi^4\left(\chi^2+2\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_2 \partial I_3} + \chi^2\left(\chi^2+\varkappa^2\right) \dfrac{\partial^2 A}{\partial I_3 \partial I_1}.
\end{array}\right\} \tag{4.7.6}$$

In outward appearance tensor $\hat{R}$ looks like the stress tensor of the transversely isotropic solid. However it is necessary to bear in mind that the quadratic form

$$\begin{gathered}
\frac{1}{2}\hat{R}\cdot\cdot\,\hat{\varepsilon} = \frac{2}{\varkappa\chi^2}\left\{\chi^4 C\left(\chi^2,\chi^2\right)\left(\varepsilon_{11}+\varepsilon_{22}\right)^2 + 2\chi^4\left(\frac{\partial A}{\partial I_2} + \varkappa^2\frac{\partial A}{\partial I_3}\right)\times\right. \\
\left(\varepsilon_{11}\varepsilon_{22}-\varepsilon_{12}^2\right) + 2\chi^2\varkappa^2 C\left(\varkappa^2,\chi^2\right)\left(\varepsilon_{11}+\varepsilon_{22}\right)\varepsilon_{33} + \varkappa^4 C\left(\varkappa^2,\varkappa^2\right)\varepsilon_{33}^2 + \\
\left. 2\chi^2\varkappa^2\left(\frac{\partial A}{\partial I_2} + \chi^2\frac{\partial A}{\partial I_3}\right)\left[\left(\varepsilon_{11}+\varepsilon_{22}\right)\varepsilon_{33} - \varepsilon_{23}^2 - \varepsilon_{31}^2\right]\right\}
\end{gathered} \tag{4.7.7}$$

in the transversely isotropic solid presents the specific strain energy and thus is positive definite whereas in the problem of tension (or compression at $t_3 < 0$) of a nonlinear elastic solid, the sign of this form relies on the properties of the material (prescribed $A(I_1, I_2, I_3)$) and the loading intensity (the value of t_3).

Tensor $\hat{\Theta}$ is given by the formula

$$\hat{\Theta} = \hat{R} + \hat{T}\cdot\nabla\mathbf{w} = \hat{R} + t_3\mathbf{i}_3\left[\mathbf{i}_1\left(\varepsilon_{31}+\omega_2\right) + \mathbf{i}_2\left(\varepsilon_{23}-\omega_1\right) + \mathbf{i}_3\varepsilon_{33}\right]. \tag{4.7.8}$$

9.4.8 Torsional deformation of the compressed rod

Under the deformation of torsion we have

$$w_1 = -\alpha x_2 x_3, \quad w_2 = \alpha x_3 x_1, \quad w_3 = \alpha\varphi\left(x_1,x_2\right), \tag{4.8.1}$$

such that

$$\begin{gathered}
\varepsilon_{ss} = \varepsilon_{12} = 0 \ \ (s=1,2,3), \quad \varepsilon_{31} = \frac{1}{2}\alpha\left(\frac{\partial\varphi}{\partial x_1} - x_2\right), \\
\varepsilon_{23} = \frac{1}{2}\alpha\left(\frac{\partial\varphi}{\partial x_2} + x_1\right), \qquad \varepsilon_{31}+\omega_2 = -\alpha x_2, \qquad \varepsilon_{23}-\omega_1 = \alpha x_1.
\end{gathered}$$

Turning to eqs. (4.7.5), (4.7.8) and (4.7.4) we obtain

$$\hat{\Theta} = \alpha t_3 \frac{\chi^2}{\varkappa^2 - \chi^2} \left[(\mathbf{i}_2\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_2) \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) + (\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_1\mathbf{i}_3) \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) \right] + \alpha t_3 \mathbf{i}_3 \mathbf{i}_3 \left(-\mathbf{i}_1 x_2 + \mathbf{i}_2 x_1 \right), \quad (4.8.2)$$

and the equilibrium equation in the volume reduces to the single equation

$$\nabla \cdot \hat{\Theta} = 0 : \quad \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = 0. \quad (4.8.3)$$

On the lateral surface of the rod ($\mathbf{N} = N_1\mathbf{i}_1 + N_2\mathbf{i}_2$)

$$\mathbf{N} \cdot \hat{\Theta} = \alpha t_3 \frac{\chi^2}{\varkappa^2 - \chi^2} \left[N_1 \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) + N_2 \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) \right],$$

and when the surface force is absent, then

$$\frac{\partial \varphi}{\partial N} = N_1 x_2 - N_2 x_1. \quad (4.8.4)$$

The stress vector in the cross-section is determined by the relationship

$$\mathbf{f} = \mathbf{i}_3 \cdot \hat{\Theta} = \alpha t_3 \left\{ \frac{\chi^2}{\varkappa^2 - \chi^2} \left[\mathbf{i}_1 \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) + \mathbf{i}_2 \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) \right] + (\mathbf{i}_2 x_1 - \mathbf{i}_1 x_2) \right\}. \quad (4.8.5)$$

Axes OX and OY are directed along the principal central axes of inertia of the cross-section of the rod in the initial state. However the cross-section of the compressed rod is subjected to the similarity transformation

$$x_1 = \chi a_1, \quad x_2 = \chi a_2. \quad (4.8.6)$$

For this reason axes OX and OY remain the principal central axes of the rod

$$\iint\limits_S x_1 dO = 0, \quad \iint\limits_S x_2 dO = 0, \quad \iint\limits_S x_1 x_2 dO = 0, \quad (4.8.7)$$

whilst the cross-sectional area, the polar moment of inertia and the torsional rigidity are expressed in terms of these values in the initial state with the help of the following equalities

$$S = \chi^2 S_0, \quad I_p = \chi^4 I_p^0, \quad C = \chi^4 C_0. \quad (4.8.8)$$

Proceeding now to calculation of the principal vector of forces in the cross-section of the compressed rod, we have by eqs. (4.8.7) and (4.8.5)

$$\iint\limits_S \mathbf{f} dO = \alpha t_3 \frac{\chi^2}{\varkappa^2 - \chi^2} \left[\mathbf{i}_1 \iint\limits_S \left(\frac{\partial \varphi}{\partial x_1} - x_2 \right) dO + \mathbf{i}_2 \iint\limits_S \left(\frac{\partial \varphi}{\partial x_2} + x_1 \right) dO \right].$$

It is easy to prove that this value is equal to zero. Indeed, by eqs. (4.8.3) and (4.8.4) we have for instance

$$\iint_S \left(\frac{\partial\varphi}{\partial x_1} - x_2\right) dO = \iint_S \left[\frac{\partial}{\partial x_1} x_1 \left(\frac{\partial\varphi}{\partial x_1} - x_2\right) + \right.$$
$$\left. \frac{\partial}{\partial x_2} x_1 \left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\right] dO = \oint_\Gamma x_1 \left[\frac{\partial\varphi}{\partial N} - (N_1 x_2 - N_2 x_1)\right] dl = 0.$$

Let us proceed to calculation of the principal moment of stresses in the cross-section. We have

$$\iint_S (\mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 x_3) \times \mathbf{f} dO = \mathbf{i}_3 m_3 + x_3 \mathbf{i}_3 \times \iint_S \mathbf{f} dO = \mathbf{i}_3 m_3,$$

since the principal vector of these stresses vanishes. According to eq. (4.8.5)

$$m_3 = \alpha t_3 \left\{\frac{\chi^2}{\varkappa^2 - \chi^2} \iint_S \left[x_1 \left(\frac{\partial\varphi}{\partial x_2} + x_1\right) - x_2 \left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\right] dO + I_p\right\}$$

and by eqs. (4.8.8) and (3.13.13) of Chapter 6

$$m_3 = \alpha t_3 \chi^4 \left(\frac{\chi^2}{\varkappa^2 - \chi^2} C_0 + I_p^0\right). \tag{4.8.9}$$

The torque is equal to zero if

$$\frac{\varkappa^2}{\chi^2} = \frac{I_p^0 - C_0}{I_p^0}. \tag{4.8.10}$$

It is known that $I_p^0 \geq C_0$, the equality sign being possible only for circular cross-sections and concentric rings. For the majority of materials the rod length reduces under compression whereas its cross-sectional dimensions increase ($\chi > 1$). Therefore, both left and right hand sides of equality (4.8.10) lie in interval $(0, 1)$

$$0 < \frac{\varkappa^2}{\chi^2} = 1 - \frac{C_0}{I_p^0} < 1.$$

If the cross-section of the rod differs from a circle or ring then for given geometric characteristics of the cross-section one can find such a (bifurcation) value of parameter $\varkappa^2/\chi^2$ that the compression is accompanied by torsion with zero torque on the rod ends.

The value of the compressive force Q for which this phenomenon can occur referred to as the "critical value" is determined by formula (4.7.4)

$$Q = |t_3|\,\chi^2 S_0 = \frac{2S_0}{\varkappa}\left(\chi^2 - \varkappa^2\right)\left(\frac{\partial A}{\partial I_1} + \chi^2 \frac{\partial A}{\partial I_2}\right). \tag{4.8.11}$$

Three unknowns $Q, \chi, \varkappa$ are determined from three equations: (4.8.10), (4.8.11) and (4.7.2)

$$\frac{\partial A}{\partial I_1} + \left(\chi^2 + \varkappa^2\right)\frac{\partial A}{\partial I_2} + \chi^2\varkappa^2 \frac{\partial A}{\partial I_3} = 0. \tag{4.8.12}$$

For the incompressible material this equation makes no sense because it serves for determining the unknown $\partial A/\partial I_3$ and must be replaced by the incompressibility condition $\chi^2\varkappa = 1$. We obtain

$$Q = \frac{2S_0 C_0}{I_0 - C_0}\left[\left(1 - \frac{C_0}{I_0}\right)^{1/3}\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\right], \tag{4.8.13}$$

where A is a function of invariants I_1, I_2

$$I_1 = \frac{\varkappa^3 + 2}{\varkappa}, \quad I_2 = \frac{1 + 2\varkappa^3}{\varkappa^2}, \quad \varkappa = \left(1 - \frac{C_0}{I_0}\right)^{1/3}. \tag{4.8.14}$$

For instance, for a rod which is made of "neo-Hookean" material and has the elliptic cross-section, see Subsection 9.1.4, we have

$$\left.\begin{aligned} A = \frac{1}{6}EI_1, \quad 1 - \frac{C_0}{I_0} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2, \\ Q = \frac{1}{3}ES_0\frac{C_0}{I_0}\left(1 - \frac{C_0}{I_0}\right)^{-2/3} = \frac{2}{3}ES_0\frac{a^2b^2}{(a^2+b^2)^2}\left(\frac{a^2+b^2}{a^2-b^2}\right)^{4/3}. \end{aligned}\right\} \tag{4.8.15}$$

9.5 Second order effects

9.5.1 *Extracting linear terms in the constitutive law*

In the solution of further nonlinear problems the smallness of the derivatives of the displacement with respect to the coordinates of the initial state is assumed

$$\left|\frac{\partial u_s}{\partial a_k}\right| \ll 1. \tag{5.1.1}$$

Additionally, neglecting terms of third and higher powers of these values is admitted, however their products and squares are retained.

We start from the constitutive law in Finger's form, eq. (2.4.1) of Chapter 8. Taking the coordinates of a point in the initial state ($q^s = a_s$) we obtain

$$\hat{T} = \frac{2}{\sqrt{I_3}}\left(\overset{0}{c}\hat{M} - \overset{1}{c}\hat{M}^2 + \overset{-1}{c}\hat{E}\right), \tag{5.1.2}$$

where $\hat{E} = \mathbf{i}_s\mathbf{i}_s$ denotes the unit tensor. The base vectors of $v-$ and $V-$volumes are respectively

$$\mathbf{r}_s = \mathbf{r}^s = \mathbf{i}_s, \quad \mathbf{R}_s = \frac{\partial}{\partial a_s}(\mathbf{r}+\mathbf{u}) = \mathbf{i}_s + \frac{\partial \mathbf{u}}{\partial a_s}. \tag{5.1.3}$$

Recalling definition (5.1.1) of tensor $\hat{M}$ we have

$$\begin{aligned}\hat{M} &= \nabla\mathbf{R}^T \cdot \nabla\mathbf{R} = \mathbf{R}_s\mathbf{r}^s \cdot \mathbf{r}^k\mathbf{R}_k \\ &= \left(\mathbf{i}_s + \frac{\partial \mathbf{u}}{\partial a_s}\right)\left(\mathbf{i}_s + \frac{\partial \mathbf{u}}{\partial a_s}\right) = \hat{E} + \mathbf{i}_s\frac{\partial \mathbf{u}}{\partial a_s} + \frac{\partial \mathbf{u}}{\partial a_s}\mathbf{i}_s + \frac{\partial \mathbf{u}}{\partial a_s}\frac{\partial \mathbf{u}}{\partial a_s} \\ &= \hat{E} + (\mathbf{i}_s\mathbf{i}_k + \mathbf{i}_k\mathbf{i}_s)\frac{\partial u_k}{\partial a_s} + \mathbf{i}_k\mathbf{i}_r\frac{\partial u_k}{\partial a_s}\frac{\partial u_r}{\partial a_s}\end{aligned}$$

and then

$$\begin{aligned}(\mathbf{i}_s\mathbf{i}_k + \mathbf{i}_k\mathbf{i}_s)\frac{\partial u_k}{\partial a_s} &= \mathbf{i}_s\mathbf{i}_k\left(\frac{\partial u_k}{\partial a_s} + \frac{\partial u_s}{\partial a_k}\right) = 2\hat{\varepsilon}, \\ \nabla\mathbf{u}^T \cdot \nabla\mathbf{u} = \mathbf{i}_k\mathbf{i}_m\frac{\partial u_k}{\partial a_m} &\cdot \mathbf{i}_s\mathbf{i}_r\frac{\partial u_r}{\partial a_s} = \mathbf{i}_k\mathbf{i}_r\frac{\partial u_k}{\partial a_s}\frac{\partial u_r}{\partial a_s},\end{aligned}$$

where $\hat{\varepsilon}$ is the linear strain tensor, $\nabla\mathbf{u}$ is the gradient of vector $\mathbf{u}$ and T denotes the transposition. Recalling eq. (1.2.13) of Chapter 2 and eq. (A.6.12) we have

$$\nabla\mathbf{u}^T \cdot \nabla\mathbf{u} = \left(\hat{\varepsilon} + \hat{\Omega}\right)\cdot\left(\hat{\varepsilon} - \hat{\Omega}\right) = \hat{\varepsilon}^2 + \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T.$$

Therefore

$$\begin{aligned}\hat{M} &= \hat{E} + 2\hat{\varepsilon} + \nabla\mathbf{u}^T \cdot \nabla\mathbf{u} \\ &= \hat{E} + 2\hat{\varepsilon} + \hat{\varepsilon}^2 + \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T\end{aligned} \tag{5.1.4}$$

and under the above assumptions

$$\hat{M}^2 = \hat{E} + 4\hat{\varepsilon} + 6\hat{\varepsilon}^2 + 2\left[\hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T\right]. \tag{5.1.5}$$

Referring to the formulae

$$\begin{aligned}I_2\left(\hat{M}\right) &= \frac{1}{2}\left[I_1^2\left(\hat{M}\right) - I_1\left(\hat{M}^2\right)\right], \\ I_3\left(\hat{M}\right) &= \frac{1}{3}\left\{I_1\left(\hat{M}^3\right) - I_1\left(\hat{M}\right)\left[I_1\left(\hat{M}^2\right) - I_2\left(\hat{M}\right)\right]\right\},\end{aligned}$$

taking into account that the first invariant of tensor $\boldsymbol{\omega} \times \hat{\varepsilon}$ is equal to zero (as $\hat{\varepsilon}$ is the symmetric tensor, see eq. (A.5.9)) and recalling that $I_k\left(\hat{M}\right) = I_k\left(\hat{G}^\times\right) = I_k$, we obtain

$$\left.\begin{array}{l}
I_1 = 3 + 2I_1\left(\hat{\varepsilon}\right) + I_1\left(\hat{\varepsilon}^2\right) + 2\boldsymbol{\omega}\cdot\boldsymbol{\omega} = 3 + 2\vartheta + I_1\left(\hat{\varepsilon}^2\right) + 2\boldsymbol{\omega}\cdot\boldsymbol{\omega}, \\
I_2 = 3 + 4\vartheta + 2\vartheta^2 + 4\boldsymbol{\omega}\cdot\boldsymbol{\omega}, \\
I_3 = 1 + 2\vartheta + 2\vartheta^2 + 2\boldsymbol{\omega}\cdot\boldsymbol{\omega} - I_1\left(\hat{\varepsilon}^2\right).
\end{array}\right\} \tag{5.1.6}$$

Substitution into eq. (5.1.2) allows the stress tensor to be represented in the form

$$\begin{aligned}
\hat{T} = \frac{2}{\sqrt{I_3}}\Bigg\{&\left[\frac{\partial A}{\partial I_1} + (I_1 - 1)\frac{\partial A}{\partial I_2} + I_3\frac{\partial A}{\partial I_3}\right]\hat{E} + \\
&\left[\frac{\partial A}{\partial I_1} + (I_1 - 2)\frac{\partial A}{\partial I_2}\right]2\hat{\varepsilon} + \left[\frac{\partial A}{\partial I_1} + (I_1 - 6)\frac{\partial A}{\partial I_2}\right]\hat{\varepsilon}^2 + \\
&\left[\frac{\partial A}{\partial I_1} + (I_1 - 2)\frac{\partial A}{\partial I_2}\right]\left[\hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T\right]\Bigg\}.
\end{aligned} \tag{5.1.7}$$

The invariants $I_k\left(\hat{\mathcal{E}}\right) = j_k$ of the strain tensor $\hat{\mathcal{E}}$ are determined by formulae (5.4.3) of Chapter 2

$$j_1 = \vartheta + \boldsymbol{\omega}\cdot\boldsymbol{\omega} + \frac{1}{2}I_1\left(\hat{\varepsilon}^2\right), \quad j_2 = \frac{1}{2}\vartheta^2 - \frac{1}{2}I_1\left(\hat{\varepsilon}^2\right), \quad j_3 = 0. \tag{5.1.8}$$

As one expects, invariants j_1, j_2, j_3 contain respectively terms of the first, second and third power in quantities (5.1.1).

Referring to eq. (2.5.2) of Chapter 8 we have

$$\begin{aligned}
a_0 &= \frac{\partial A}{\partial I_1} + (I_1 - 1)\frac{\partial A}{\partial I_2} + I_3\frac{\partial A}{\partial I_3} \\
&= \frac{1}{2}\frac{\partial A}{\partial j_1} + \frac{1}{2}\frac{\partial A}{\partial j_2}\left[\vartheta + \boldsymbol{\omega}\cdot\boldsymbol{\omega} + \frac{1}{2}I_1\left(\hat{\varepsilon}^2\right)\right] + \frac{1}{4}\frac{\partial A}{\partial j_3}\left[\vartheta^2 - I_1\left(\hat{\varepsilon}^2\right)\right] \\
&= \frac{1}{2}\frac{\partial A}{\partial j_1} + \frac{1}{2}j_1\frac{\partial A}{\partial j_2} + \frac{1}{2}j_2\frac{\partial A}{\partial j_3}, \\
a_1 &= \frac{\partial A}{\partial I_1} + (I_1 - 2)\frac{\partial A}{\partial I_2} = \frac{1}{2}\frac{\partial A}{\partial j_1} - \frac{1}{4}\frac{\partial A}{\partial j_2}(1 - 2\vartheta) - \frac{1}{4}\frac{\partial A}{\partial j_3}\vartheta, \\
a_2 &= \frac{\partial A}{\partial I_1} + (I_1 - 6)\frac{\partial A}{\partial I_2} = \frac{1}{2}\frac{\partial A}{\partial j_1} - \frac{5}{4}\frac{\partial A}{\partial j_2} + \frac{1}{2}\frac{\partial A}{\partial j_3},
\end{aligned}$$

where in the second and third formulae it is sufficient to keep the linear terms and those terms containing no displacement derivatives.

The specific strain energy is approximated by Murnaghan's formula, eq. (4.6.1) of Chapter 8, under the assumption that the initial state is the natural one

$$A = \frac{1}{2}(\lambda + 2\mu) j_1^2 - 2\mu j_2^2 + \frac{1}{3}(l + 2m) j_1^3 - 2m j_1 j_2 + n j_3. \tag{5.1.9}$$

Then

$$\left.\begin{aligned} a_0 &= \frac{1}{2}\lambda j_1 + \frac{1}{2} l j_1^2 - \left(m - \frac{1}{2} n\right) j_2, \\ a_1 &= \frac{1}{2}\left[\mu + \left(\lambda + m - \frac{n}{2}\right)\vartheta\right], \quad a_2 = \frac{1}{2}(5\mu + n) \end{aligned}\right\} \tag{5.1.10}$$

and the expression for the stress tensor calculated up to terms of second order reduces to the form

$$\begin{aligned} \hat{T} = \frac{1}{\sqrt{I_3}} \hat{T}^0 + \left[\lambda \boldsymbol{\omega} \cdot \boldsymbol{\omega} + \left(l - m + \frac{n}{2}\right)\vartheta^2 + \frac{1}{2}(\lambda + 2m - n) I_1\left(\hat{\varepsilon}^2\right)\right]\hat{E} + \\ 2\left(\lambda\vartheta\hat{\varepsilon} + 2\mu\hat{\varepsilon}^2\right) + 2\left(m - \frac{n}{2}\right)\vartheta\hat{\varepsilon} + n\hat{\varepsilon}^2 + \\ \mu\left[\hat{\varepsilon}^2 + \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T\right], \end{aligned} \tag{5.1.11}$$

where $\hat{T}^0$ denotes the stress tensor of the linear theory of elasticity

$$\hat{T}^0 = \lambda\vartheta\hat{E} + 2\mu\hat{\varepsilon}. \tag{5.1.12}$$

Noticing that

$$\frac{1}{\sqrt{I_3}} = 1 - \vartheta + \dots, \quad \lambda\vartheta\hat{\varepsilon} + 2\mu\hat{\varepsilon}^2 = \hat{T}^0 \cdot \hat{\varepsilon} = \hat{\varepsilon} \cdot \hat{T}^0,$$

one can present eq. (5.1.11) in another form

$$\hat{T} = \hat{T}^0 - \vartheta\hat{T}^0 + 2\hat{\varepsilon}\cdot\hat{T}^0 + \hat{T}' \tag{5.1.13}$$

in terms of tensor $\hat{T}'$

$$\begin{aligned} \hat{T}' = \left[\lambda\boldsymbol{\omega}\cdot\boldsymbol{\omega} + \left(l - m + \frac{n}{2}\right)\vartheta^2 + \frac{1}{2}(\lambda + 2m - n) I_1\left(\hat{\varepsilon}^2\right)\right]\hat{E} + \\ (2m - n)\vartheta\hat{\varepsilon} + n\hat{\varepsilon}^2 + \mu\left[\hat{\varepsilon}^2 + \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega}\times\hat{\varepsilon} + (\boldsymbol{\omega}\times\hat{\varepsilon})^T\right], \end{aligned} \tag{5.1.14}$$

where the latter term can be also set in the following form

$$\mu\nabla\mathbf{u}^T \cdot \nabla\mathbf{u}. \tag{5.1.15}$$

9.5.2 Equilibrium equations

The equations of statics in the volume are written in the metric of the deformed body. In this metric the vector basis is prescribed by the triple of vectors $\mathbf{R}_s$ given by eq. (5.1.3)

$$\mathbf{R}_s = \mathbf{i}_s + \frac{\partial \mathbf{u}}{\partial a_s} = \mathbf{i}_s + \mathbf{i}_t \frac{\partial u_t}{\partial a_s}.$$

With the required accuracy the vectors of the cobasis (i.e. up to the first power in the derivatives of displacement) are as follows

$$\mathbf{R}^s = \mathbf{i}_s - \mathbf{i}_t \frac{\partial u_s}{\partial a_t}. \tag{5.2.1}$$

Indeed, under this definition

$$\mathbf{R}_s \mathbf{R}^s = \left(\mathbf{i}_s + \mathbf{i}_t \frac{\partial u_t}{\partial a_s}\right)\left(\mathbf{i}_s - \mathbf{i}_k \frac{\partial u_s}{\partial a_k}\right) = \hat{E} + \mathbf{i}_t \mathbf{i}_s \frac{\partial u_t}{\partial a_s} - \mathbf{i}_s \mathbf{i}_t \frac{\partial u_s}{\partial a_t} = \hat{E},$$

which is required.

Let $\mathbf{F}$ denote the surface force acting on the unit area of surface O of the deformed volume V, then referring to eq. (3.5.3) of Chapter 2 we have

$$\mathbf{F} dO = \mathbf{N} \cdot \hat{T} dO = \sqrt{I_3} n_s \mathbf{R}^s \cdot \hat{T} do.$$

By eqs. (5.1.13) and (5.2.1)

$$\sqrt{I_3}\hat{T} = \hat{T}^0 + 2\hat{\varepsilon} \cdot \hat{T}^0 + \hat{T}' = \hat{T}^0 + \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \cdot \hat{T}^0 + \hat{T}',$$
$$n_s \mathbf{R}^s = \mathbf{n} - \mathbf{n} \cdot \nabla \mathbf{u}^T,$$

and thus

$$\mathbf{F} dO = \left(\mathbf{n} \cdot \hat{T}^0 + \mathbf{n} \cdot \nabla \mathbf{u} \cdot \hat{T}^0 + \mathbf{n} \cdot \hat{T}'\right) do = \mathbf{n} \cdot \hat{\Theta} do, \tag{5.2.2}$$

where we introduced the tensor

$$\hat{\Theta} = \hat{T}^0 + \nabla \mathbf{u} \cdot \hat{T}^0 + \hat{T}'. \tag{5.2.3}$$

It is evident that relationship (5.2.2) is applicable to surface O^* of any V^*–volume in the deformed body. For this reason the equilibrium equation of this volume can be put in the form

$$\iiint\limits_{V^*} \rho \mathbf{K} d\tau + \iint\limits_{O^*} \mathbf{F} dO = \iiint\limits_{v^*} \rho_0 \mathbf{K} d\tau_0 + \iint\limits_{o^*} \mathbf{n} \cdot \hat{\Theta} do$$
$$= \iiint\limits_{v^*} \left(\rho_0 \mathbf{K} + \nabla \cdot \hat{\Theta}\right) d\tau_0 = 0.$$

The equilibrium equations in the volume and on the surface are now expressed in terms of tensor $\hat{\Theta}$ in the metric of the initial v—volume

$$\text{in} \quad v: \quad \nabla \cdot \hat{\Theta} + \rho_0 \mathbf{K} = 0, \tag{5.2.4}$$

$$\text{on} \quad o: \quad \mathbf{F}\frac{dO}{do} = \mathbf{F}^0 = \mathbf{n} \cdot \hat{\Theta}. \tag{5.2.5}$$

Here $\mathbf{F}^0$ denotes the surface force acting on the unit area of surface o.

Tensor $\hat{\Theta}$ is a differential operator over vector $\mathbf{u}$. Its explicit expression is as follows

$$\hat{\Theta} = \hat{T}^0 + \hat{\varepsilon} \cdot \hat{T}^0 + \hat{T}' - \boldsymbol{\omega} \times \hat{T}^0, \tag{5.2.6}$$

the last term representing the non-symmetric part of this tensor.

9.5.3 *Effects of second order*

The approximate solution of the boundary-value problem (5.2.4), (5.2.5) is presented by the displacement vector which is the geometric sum

$$\mathbf{u} = \mathbf{v} + \mathbf{w}. \tag{5.3.1}$$

Here $\mathbf{v}$ is the vector describing the solution of the equations of linear theory under the prescribed surface forces

$$\nabla \cdot \hat{T}^0 (\mathbf{v}) + \rho_0 \mathbf{K} = 0, \quad \mathbf{n} \cdot \hat{T}^0 (\mathbf{v}) = \mathbf{F}^0, \tag{5.3.2}$$

$$\hat{T}^0 (\mathbf{v}) = \lambda \vartheta (\mathbf{v}) \hat{E} + 2\mu \hat{\varepsilon} (\mathbf{v}). \tag{5.3.3}$$

This solution is assumed to be known. Vector $\mathbf{w}$ is the correcting term and is added to satisfy the equations taking into accounts the terms of second power in the derivatives of the displacement vector $\mathbf{u}$ (5.1.1). Since $\mathbf{w}$ has the same order the differential operators in $\hat{\Theta}$ (except for $\hat{T}^0$) over $\mathbf{u}$ can be replaced by the operators over $\mathbf{v}$

$$\left.\begin{aligned} &\nabla \mathbf{u} \cdot \hat{T}^0 (\mathbf{u}) = \nabla (\mathbf{v} + \mathbf{w}) \cdot \hat{T}^0 (\mathbf{v} + \mathbf{w}) \approx \nabla \mathbf{v} \cdot \hat{T}^0 (\mathbf{v}), \\ &\hat{T}' (\mathbf{u}) = \hat{T}' (\mathbf{v} + \mathbf{w}) \approx \hat{T}' (\mathbf{v}), \end{aligned}\right\} \tag{5.3.4}$$

the suggested accuracy being retained. The terms of third and higher order are omitted. As $\hat{T}^0$ is a linear operator we have

$$\hat{T}^0 (\mathbf{u}) = \hat{T}^0 (\mathbf{v} + \mathbf{w}) = \hat{T}^0 (\mathbf{v}) + \hat{T}^0 (\mathbf{w}) \tag{5.3.5}$$

and by eq. (5.2.3)

$$\hat{\Theta} (\mathbf{u}) = \hat{\Theta} (\mathbf{v} + \mathbf{w}) = \hat{T}^0 (\mathbf{v}) + \hat{T}^0 (\mathbf{w}) + \nabla \mathbf{v} \cdot \hat{T}^0 (\mathbf{v}) + \hat{T}' (\mathbf{v}). \tag{5.3.6}$$

The equilibrium equations (5.2.4), (5.2.5) in the volume and the surface are now presented in the form

$$\left.\begin{aligned} \nabla\cdot\hat{T}^0(\mathbf{v})+\rho_0\mathbf{K}+\nabla\cdot\hat{T}^0(\mathbf{w})+\nabla\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]&=0,\\ \mathbf{n}\cdot\hat{T}^0(\mathbf{v})-\mathbf{F}^0+\mathbf{n}\cdot\hat{T}^0(\mathbf{w})+\mathbf{n}\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]&=0. \end{aligned}\right\} \tag{5.3.7}$$

Recalling now that vector $\mathbf{v}$ is determined by eq. (5.3.2) we also arrive at the equations of the linear theory

$$\left.\begin{aligned} \nabla\cdot\hat{T}^0(\mathbf{w})+\nabla\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]&=0,\\ \mathbf{n}\cdot\hat{T}^0(\mathbf{w})+\mathbf{n}\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]&=0, \end{aligned}\right\} \tag{5.3.8}$$

determining vector $\mathbf{w}$ in terms of the given "volume" and "surface" forces

$$\mathbf{k}=\nabla\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right],\quad \mathbf{f}=-\mathbf{n}\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]. \tag{5.3.9}$$

Taking account of the terms of second order in the expressions for the mass and surface forces $\mathbf{K}$ and $\mathbf{F}$ presents no difficulty.

The problem has a solution if the "external" forces $\mathbf{k}$ and $\mathbf{f}$ satisfy the equations of statics

$$\iiint\limits_v \mathbf{k}d\tau_0+\iint\limits_o \mathbf{f}do=0,\quad \iiint\limits_v r\times\mathbf{k}d\tau_0+\iint\limits_o \mathbf{r}\times\mathbf{f}do=0. \tag{5.3.10}$$

The first equation is satisfied which can be easily proved by transforming the surface integral into a volume integral

$$\begin{aligned} \iint\limits_o \mathbf{f}do&=-\iint\limits_o \mathbf{n}\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]do\\ &=-\iiint\limits_v \nabla\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]d\tau_0=-\iiint\limits_v \mathbf{k}d\tau_0, \end{aligned}$$

which is required. The second equation presents a challenge because tensor $\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})$ in $\hat{\Theta}(\mathbf{v})$ is not symmetric. Referring to eq. (B.5.6) we have

$$\begin{aligned} \iint\limits_o \mathbf{r}\times\mathbf{f}do&=-\iint\limits_o \mathbf{r}\times\mathbf{n}\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]do\\ &=-\iiint\limits_v \mathbf{r}\times\nabla\cdot\left[\nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v})+\hat{T}'(\mathbf{v})\right]d\tau_0+2\iiint\limits_v \mathbf{a}d\tau_0\\ &=-\iiint\limits_v \mathbf{r}\times\mathbf{k}d\tau_0+2\iiint\limits_v \mathbf{a}d\tau_0. \end{aligned}$$

Here $\mathbf{a}$ denotes the vector accompanying the nonsymmetric part

$$-\hat{\Omega}(\mathbf{v}) \cdot \hat{T}^0(\mathbf{v}) = -\boldsymbol{\omega} \times \hat{T}^0(\mathbf{v}) \tag{5.3.11}$$

of tensor $\nabla \mathbf{v} \cdot \hat{T}^0(\mathbf{v})$. This vector can de defined by the equality

$$2\mathbf{a} = \mathbf{i}_s \times \left\{ \mathbf{i}_s \cdot \left[\boldsymbol{\omega} \times \hat{T}^0(\mathbf{v}) \right] \right\}, \tag{5.3.12}$$

and the second equation of statics is satisfied under the condition

$$2 \iiint\limits_v \mathbf{a} d\tau_0 = \iiint\limits_v \mathbf{i}_s \times \left\{ \mathbf{i}_s \cdot \left[\boldsymbol{\omega} \times \hat{T}^0(\mathbf{v}) \right] \right\} d\tau_0 = 0.$$

An extended form of vector $2\mathbf{a}$ is as follows

$$\begin{aligned} 2\mathbf{a} &= \mathbf{i}_s \times \left\{ \mathbf{i}_s \cdot \left[\boldsymbol{\omega} \times \hat{T}^0(\mathbf{v}) \right] \right\} = \mathbf{i}_s \times \left[\mathbf{i}_s \cdot (\mathbf{i}_r \times \mathbf{i}_q \mathbf{i}_t) \, \omega_r t^0_{qt} \right] \\ &= e_{srq} e_{stm} \mathbf{i}_m \omega_r t^0_{qt} = (\delta_{rt}\delta_{qm} - \delta_{rm}\delta_{qt}) \, \mathbf{i}_m \omega_r t^0_{qt} = \omega_t t^0_{qt} \mathbf{i}_q - \omega_r t^0_{qq} \mathbf{i}_r \end{aligned}$$

or

$$2a = \boldsymbol{\omega} \cdot \hat{T}^0(\mathbf{v}) - \boldsymbol{\omega}\sigma, \quad \sigma = t_{qq}(\mathbf{v}) = I_1\left(\hat{T}^0(\mathbf{v})\right). \tag{5.3.13}$$

Hence, the necessary condition for existence of a solution of the boundary-value problem (5.3.8) is the equality

$$\iiint\limits_v \boldsymbol{\omega} \cdot \left(\hat{T}^0 - \hat{E}\sigma \right) d\tau_0 = 0. \tag{5.3.14}$$

In the second boundary-value problem (surface forces $\mathbf{F}^0$ are prescribed on o) vector $\boldsymbol{\omega}$ is determined up to an additive constant term $\boldsymbol{\omega}_0$. Therefore assuming $\boldsymbol{\omega} = \boldsymbol{\omega}' + \boldsymbol{\omega}_0$ where for example $\boldsymbol{\omega}'(0,0,0) = 0$, the choice of vector $\boldsymbol{\omega}_0$ should be subjected to the condition

$$\boldsymbol{\omega}_0 \cdot \iiint\limits_v \left(\hat{T}^0 - \hat{E}\sigma \right) d\tau_0 = -\iiint\limits_v \boldsymbol{\omega}' \cdot \left(\hat{T}^0 - \hat{E}\sigma \right) d\tau_0 = v\mathbf{b},$$

where $\mathbf{b}$ is given. We thus arrive at the system of equations for the unknowns ω_{0r}

$$\omega_{0r} \left(c_{rq} - c\delta_{rq} \right) = b_q \quad q = (1,2,3). \tag{5.3.15}$$

The coefficients

$$c_{rq} = \frac{1}{v} \iiint\limits_v t^0_{rq} d\tau_0, \quad c = c_{11} + c_{22} + c_{33} \tag{5.3.16}$$

are the stresses averaged over the volume and can be expressed in terms of the external forces $\rho_0\mathbf{K}$ and $\mathbf{F}^0$ by means of formulae (4.3.2) of Chapter 1. The determinant of this system must not be zero

$$\Delta = |c_{rq} - c\delta_{rq}| \neq 0, \tag{5.3.17}$$

and if this condition is not satisfied (i.e. $\Delta = 0$) then boundary-value problem (5.3.8) may have no solution, that is, the account of the effect of nonlinearity is not achieved by introducing a correcting term into the solution of the linear problem. The proof of criterion (5.3.17) is carried out only by prescribing external forces $\rho_0\mathbf{K}$ and $\mathbf{F}^0$ and does not require solution of the linear boundary-value problem (5.3.2) and (5.3.3). The additive constant vector ω_0 appearing in the solution is determined in the process of accounting for the second order nonlinear effect (i.e. introducing vector $\mathbf{w}$).

Solving boundary-value problem (5.3.8) is made difficult by the complexity of the expressions for the "volume and surface" forces $\mathbf{k}$ and $\mathbf{f}$. Applying the reciprocity theorem allows the mean values of the strains and stresses to be determined in terms of these forces. This turns out to be sufficient in many problems when the details of the strain distribution are not needed.

The calculations required by the reciprocity theorem are slightly simplified because of the special structure of vectors $\mathbf{k}$ and $\mathbf{f}$. According to eq. (3.3.5) of Chapter 4 we have

$$\frac{\nu}{1-2\nu}\vartheta'\vartheta_m(\mathbf{w}) + \hat{\varepsilon}' \cdot\cdot\, \hat{\varepsilon}_m(\mathbf{w}) =$$
$$= \frac{1}{2\mu\nu}\left(\iiint\limits_v \mathbf{k}\cdot\hat{\varepsilon}'\cdot\mathbf{r}\,d\tau_0 + \iint\limits_o \mathbf{f}\cdot\hat{\varepsilon}'\cdot\mathbf{r}\,do\right). \tag{5.3.18}$$

Here $\hat{\varepsilon}'$ is an auxiliary constant symmetric tensor of second rank and ϑ' is its first invariant, $\hat{\varepsilon}_m(\mathbf{w})$ and $\vartheta_m(\mathbf{w})$ denote respectively the values of the strain tensor $\varepsilon_m(\mathbf{w})$ and the dilatation $\vartheta_m(\mathbf{w})$ averaged over the volume, $\mathbf{r} = \mathbf{i}_s a_s$ is the position vector, $\hat{\varepsilon}' \cdot\cdot\, \hat{\varepsilon}_m(\mathbf{w}) = I_1(\hat{\varepsilon}'\cdot\hat{\varepsilon}_m)$ is the first invariant of the product of tensors $\hat{\varepsilon}'$ and $\hat{\varepsilon}_m$. Assuming for brevity

$$\hat{Q} = \nabla\mathbf{v}\cdot\hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v}),$$

we have

$$\iint\limits_o \mathbf{n}\cdot\hat{Q}\cdot\hat{\varepsilon}'\cdot\mathbf{r}\,do = \iint\limits_o n_s q_{st}\hat{\varepsilon}'_{tm} a_m\,do$$
$$= \iiint\limits_v \left(\frac{\partial q_{st}}{\partial a_s}\varepsilon'_{tm}a_m + q_{st}\varepsilon'_{ts}\right)d\tau_0$$
$$= \iiint\limits_v \left(\nabla\cdot\hat{Q}\right)\cdot\hat{\varepsilon}'\cdot\mathbf{r}\,d\tau_0 + \iiint\limits_v \hat{Q}\,d\tau_0 \cdot\cdot\, \hat{\varepsilon}' \tag{5.3.19}$$

and recalling formula (5.3.9) we have

$$\iiint\limits_{v} \mathbf{k} \cdot \hat{\varepsilon}' \cdot \mathbf{r} d\tau_0 + \iint\limits_{o} \mathbf{f} \cdot \hat{\varepsilon}' \cdot \mathbf{r} do = -\iiint\limits_{v} \hat{Q} d\tau_0 \cdot\cdot\, \hat{\varepsilon}'.$$

Relationship (5.3.18) is written in the form

$$\frac{\nu}{1-2\nu}\vartheta' \vartheta_m(\mathbf{w}) + \tag{5.3.20}$$

$$\hat{\varepsilon}' \cdot\cdot \left\{ \hat{\varepsilon}_m(\mathbf{w}) + \frac{1}{2\mu v} \iiint\limits_{v} \left[\nabla \mathbf{v} \cdot \hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v}) \right] d\tau_0 \right\} = 0.$$

Assuming for instance $\hat{\varepsilon}' = \hat{E}$, $\vartheta' = 3$ we obtain the mean value $\vartheta_m(\mathbf{w})$

$$\vartheta_m(\mathbf{w}) = -\frac{1-2\nu}{2\mu(1+\nu)v} \iiint\limits_{v} I_1\left(\nabla \mathbf{v} \cdot \hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v}) \right) d\tau_0. \tag{5.3.21}$$

Taking $\hat{\varepsilon}' = \mathbf{i}_1\mathbf{i}_1$ and $\hat{\varepsilon}' = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2$ we obtain respectively the mean values

$$\frac{\nu}{1-2\nu}\vartheta_m(\mathbf{w}) + [\hat{\varepsilon}_{11}(\mathbf{w})]_m = -\frac{1}{2\mu v} \iiint\limits_{v} \left[\frac{\partial v_k}{\partial a_1} t^0_{k1}(\mathbf{v}) + t'_{11}(\mathbf{v}) \right] d\tau_0, \tag{5.3.22}$$

$$2[\varepsilon_{12}(\mathbf{w})]_m = -\frac{1}{2\mu v} \iiint\limits_{v} \left[\frac{\partial v_k}{\partial a_2} t^0_{k1}(\mathbf{v}) + \frac{\partial v_k}{\partial a_1} t^0_{k2}(\mathbf{v}) + 2t'_{12}(\mathbf{v}) \right] d\tau_0, \tag{5.3.23}$$

such that

$$[\hat{\varepsilon}_{11}(\mathbf{w})]_m = \frac{1}{vE} \left\{ \nu \iiint\limits_{v} \left[\frac{\partial v_k}{\partial a_s} t^0_{ks}(\mathbf{v}) + I_1\left(\hat{T}'(\mathbf{v})\right) \right] d\tau_0 - \right.$$

$$\left. (1+\nu) \iiint\limits_{v} \left[\frac{\partial v_k}{\partial a_1} t^0_{k1}(\mathbf{v}) + t'_{11}(\mathbf{v}) \right] d\tau_0 \right\}. \tag{5.3.24}$$

Change in volume of the body subjected to distortion

To begin with, we write down the expression for the mean value of stress tensor in the body volume, see eq. (4.3.2) of Chapter 1. Introducing the dyadics $\rho\mathbf{KR}$ and $\mathbf{FR}$ we can write

$$\iiint\limits_{V} \rho\mathbf{KR} d\tau + \iint\limits_{O} \mathbf{FR} dO = \iiint\limits_{V} \rho\mathbf{KR} d\tau + \iint\limits_{O} \mathbf{N} \cdot \hat{T}\mathbf{R} dO$$

$$= \iiint\limits_{V} \left(\rho\mathbf{KR} + \tilde{\nabla} \cdot \hat{T}\mathbf{R} \right) d\tau. \tag{5.3.25}$$

It is easy to prove that the divergence of tensor $\hat{T}\mathbf{R}$ of third rank is transformed to the form

$$\tilde{\nabla}\cdot\hat{T}\mathbf{R} = \mathbf{i}_s\frac{\partial}{\partial x_s}\cdot\hat{T}\mathbf{R} = \left(\tilde{\nabla}\cdot\hat{T}\right)\mathbf{R} + \mathbf{i}_s\cdot\hat{T}\mathbf{i}_t\delta_{ts} = \left(\tilde{\nabla}\cdot\hat{T}\right)\mathbf{R} + \hat{T}^T\cdot\mathbf{i}_s\mathbf{i}_s$$
$$= \left(\tilde{\nabla}\cdot\hat{T}\right)\mathbf{R} + \hat{T}\cdot\hat{E} = \left(\tilde{\nabla}\cdot\hat{T}\right)\mathbf{R} + \hat{T},$$

as $\hat{T}^T = \hat{T}$. Returning now to eq. (5.3.25) we have

$$\iiint\limits_V \rho\mathbf{K}\mathbf{R}d\tau + \iint\limits_O \mathbf{F}\mathbf{R}dO =$$
$$= \iiint\limits_V \left(\rho\mathbf{K} + \tilde{\nabla}\cdot\hat{T}\right)\mathbf{R}d\tau + \iiint\limits_V \hat{T}d\tau = \iiint\limits_V \hat{T}d\tau. \quad (5.3.26)$$

Though the forthcoming analysis is valid for the dislocation of a more general nature, we consider Volterra's distortion. In the body subjected to this distortion the stress tensor $\hat{T}$ is not zero even when the volume and surface forces vanish. However its mean value in the volume is zero

$$\iiint\limits_V \hat{T}d\tau = 0. \quad (5.3.27)$$

It follows that in the linear elastic body the mean value of the linear strain tensor $\hat{\varepsilon}$ is zero and the change in the volume of the body subjected to distortion can be explained only in the framework of the nonlinear theory of elasticity.

Replacing tensor $\hat{T}$ in eq. (5.3.27) by its approximate expression (5.1.13) we have with the adopted accuracy

$$\iiint\limits_V \hat{T}d\tau = \iiint\limits_v \sqrt{I_3}\hat{T}d\tau_0 = \iiint\limits_v \left(\hat{T}^0 + 2\hat{\varepsilon}\cdot\hat{T}^0 + \hat{T}'\right)d\tau_0 = 0. \quad (5.3.28)$$

At the same time by eq. (5.1.6)

$$\frac{d\tau}{d\tau_0} = \sqrt{I_3} = 1 + \vartheta + \frac{1}{2}\vartheta^2 + \boldsymbol{\omega}\cdot\boldsymbol{\omega} - \frac{1}{2}I_1\left(\hat{\varepsilon}^2\right), \quad (5.3.29)$$

and the relative change in the volume of the body is as follows

$$\frac{V-v}{v} = D_{av} = \iiint\limits_v \left[\vartheta + \frac{1}{2}\vartheta^2 + \boldsymbol{\omega}\cdot\boldsymbol{\omega} - \frac{1}{2}I_1\left(\hat{\varepsilon}^2\right)\right]d\tau_0. \quad (5.3.30)$$

Returning to eq. (5.3.28) we construct the expression for the first invariant of the integrand. Referring to eqs. (5.1.12) and (5.1.14) we obtain

$$I_1\left(\hat{T}^0 + 2\hat{\varepsilon}\cdot\hat{T}^0 + \hat{T}'\right) = (3\lambda + 2\mu)\,\vartheta + (3\lambda + 2\mu)\,\omega\cdot\omega +$$
$$\vartheta^2\left(2\lambda + 3l - m + \frac{1}{2}n\right) + I_1\left(\hat{\varepsilon}^2\right)\left(5\mu + \frac{3}{2}\lambda + 3m - \frac{1}{2}n\right) =$$
$$= (3\lambda + 2\mu)\left(\frac{d\tau}{d\tau_0} - 1\right) + \vartheta^2\left(\frac{1}{2}\lambda - \mu + 3l - m + \frac{1}{2}n\right) +$$
$$I_1\left(\hat{\varepsilon}^2\right)\left(3\lambda + 6\mu + 3m - \frac{1}{2}n\right)$$

and referring to eqs. (5.3.28) and (5.3.30) we arrive at the following relationship (Toupin and Rivlin, 1960)

$$D_{av} = -\frac{1}{3\lambda + 2\mu}\left[\left(\frac{1}{2}\lambda - \mu + 3l - m + \frac{1}{2}n\right)\iiint\limits_v \vartheta^2 d\tau_0 + \right.$$
$$\left.\left(3\lambda + 6\mu + 3m - \frac{1}{2}n\right)\iiint\limits_v I_1\left(\hat{\varepsilon}^2\right) d\tau_0\right].$$

For the adopted accuracy, the invariants ϑ and $I_1\left(\hat{\varepsilon}^2\right)$ are determined by solution of the problem on the state of stress of the linear elastic body in v–volume.

9.5.4 Choice of the first approximation

It is not necessary to take, as the first approximation, the solution of the linear boundary-value problem (5.3.2), (5.3.3) which exactly corresponds to forces $\mathbf{K}$ and $\mathbf{F}^0$. As often happens, it is preferable to include the terms of second order of assumed small parameters in vector $\mathbf{v}$. We take

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \quad \mathbf{u} = \mathbf{v} + \mathbf{w} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{w}, \tag{5.4.1}$$

where $\mathbf{v}_0$ denotes the solution of the linear problem satisfying the mentioned system. Instead of eq. (5.3.6) we have

$$\begin{aligned}\Theta(\mathbf{u}) &= \hat{T}^0(\mathbf{u}) + \nabla\mathbf{u}\cdot\hat{T}^0(\mathbf{u}) + \hat{T}'(\mathbf{u}) \\ &= \hat{T}^0(\mathbf{v}_0) + \hat{T}^0(\mathbf{v}_1) + \hat{T}^0(\mathbf{w}) + \nabla\mathbf{v}_0\cdot\hat{T}^0(\mathbf{v}_0) + \hat{T}'(\mathbf{v}_0),\end{aligned} \tag{5.4.2}$$

where the terms of third order have been neglected. Insertion into the equilibrium equations (5.2.7) and (5.2.8) under the condition

$$\nabla\cdot\hat{T}^0(\mathbf{v}_0) + \rho_0\mathbf{K} = 0, \quad \mathbf{n}\cdot\hat{T}^0(\mathbf{v}_0) = \mathbf{F}^0 \tag{5.4.3}$$

requires only adding the correcting "forces"

$$\left.\begin{aligned}\mathbf{k} &= \nabla \cdot \left[\nabla \mathbf{v}_0 \cdot \hat{T}^0(\mathbf{v}_0) + \hat{T}'(\mathbf{v}_0) + \hat{T}^0(\mathbf{v}_1)\right], \\ \mathbf{f} &= -\mathbf{n} \cdot \left[\nabla \mathbf{v}_0 \cdot \hat{T}^0(\mathbf{v}_0) + \hat{T}'(\mathbf{v}_0) + \hat{T}^0(\mathbf{v}_1)\right]\end{aligned}\right\} \tag{5.4.4}$$

in expression (5.3.9). Vector $\mathbf{w}$ should be obtained, as above, from the system of equations

$$\nabla \cdot \hat{T}^0(\mathbf{w}) + \mathbf{k} = 0, \quad \mathbf{n} \cdot \hat{T}^0(\mathbf{w}) = \mathbf{f}, \tag{5.4.5}$$

vectors $\mathbf{k}$ and $\mathbf{f}$ being determined by eq. (5.4.4). The additional terms $\nabla \cdot \hat{T}^0(\mathbf{w})$ and $\mathbf{n} \cdot \hat{T}^0(\mathbf{w})$ present a statically self-equilibrated system

$$\left.\begin{aligned}\iiint\limits_v \nabla \cdot \hat{T}^0(\mathbf{v}_1)\, d\tau_0 - \iint\limits_o \mathbf{n} \cdot \hat{T}^0(\mathbf{v}_1)\, do = 0, \\ \iiint\limits_v \mathbf{r} \times \nabla \cdot \hat{T}^0(\mathbf{v}_1)\, d\tau_0 - \iint\limits_o \mathbf{r} \times \mathbf{n} \cdot \hat{T}^0(\mathbf{v}_1)\, do = 0.\end{aligned}\right\} \tag{5.4.6}$$

The first relation is an immediate consequence of transformation of the surface integral into a volume integral whereas the second follows from the first and the symmetry of tensor $\hat{T}^0(\mathbf{v}_1)$. Hence the necessary criteria (5.3.14) and (5.3.17) for the correcting vector $\mathbf{w}$ existing are kept. In these criteria it is necessary to replace $\mathbf{v}$ by $\mathbf{v}_0$.

An additional term needs to be added in eq. (5.3.20) which now takes the form

$$\frac{\nu}{1-2\nu}\vartheta'\vartheta_m(\mathbf{w}) + \hat{\varepsilon}' \cdot\cdot \left\{\hat{\varepsilon}_m(\mathbf{w}) + \frac{1}{2\mu v}\iiint\limits_v \left[\nabla \mathbf{v}_0 \cdot \hat{T}^0(\mathbf{v}_0) + \hat{T}'(\mathbf{v}_0) + \hat{T}^0(\mathbf{v}_1)\right] d\tau_0\right\}, \tag{5.4.7}$$

and formulae (5.3.21), (5.3.23) and (5.3.24) are as follows

$$\vartheta_m(\mathbf{w}) = -\frac{1-2\nu}{2\mu(1+\nu)v}\iiint\limits_v I_1\left[\nabla \mathbf{v}_0 \cdot \hat{T}^0(\mathbf{v}_0) + \hat{T}'(\mathbf{v}_0) + \hat{T}^0(\mathbf{v}_1)\right] d\tau_0, \tag{5.4.8}$$

$$[\varepsilon_{12}(\mathbf{w})]_m = -\frac{1}{2\mu v}\iiint\limits_v \left[\frac{\partial v_{0k}}{\partial a_2} t^0_{k1}(\mathbf{v}_0) + \frac{\partial v_{0k}}{\partial a_1} t^0_{k2}(\mathbf{v}_0) + 2t'_{12}(\mathbf{v}_0) + 2t^0_{12}(\mathbf{v}_1)\right] d\tau_0, \tag{5.4.9}$$

$$[\varepsilon_{11}(\mathbf{w})]_m = \frac{1}{vE}\left\{\nu \iiint\limits_v \left[\frac{\partial v_{0k}}{\partial a_s} t^0_{ks}(\mathbf{v}_0) + I_1\left(\hat{T}'(\mathbf{v}_0)\right) + I_1\left(\hat{T}^0(\mathbf{v}_1)\right)\right] d\tau_0 \right.$$
$$\left. - (1+\nu)\iiint\limits_v \left[\frac{\partial v_{0k}}{\partial a_1} t^0_{k1}(\mathbf{v}_0) + t'_{11}(\mathbf{v}_0) + t^0_{11}(\mathbf{v}_1)\right] d\tau_0 \right\}. \tag{5.4.10}$$

9.5.5 Effects of second order in the problem of rod torsion

In the classical solution, assuming the smallness of the angle of torsion per unit length, vector v_0 is given by the formulae

$$v_{01} = -\alpha a_2 a_3, \quad v_{02} = \alpha a_3 a_1, \quad v_{03} = \alpha\varphi(a_1, a_2). \tag{5.5.1}$$

The nonvanishing stresses are

$$t^0_{31}(\mathbf{v}_0) = \mu\alpha\left(\frac{\partial\varphi}{\partial a_1} - a_2\right), \quad t^0_{23}(\mathbf{v}) = \mu\alpha\left(\frac{\partial\varphi}{\partial a_2} + a_1\right).$$

By eq. (2.2.10) of Chapter 6 the components of the linear vector of rotation are given by

$$\omega_1(\mathbf{v}_0) = \frac{1}{2\mu} t^0_{23}(\mathbf{v}_0) - \alpha a_1, \quad \omega_2(\mathbf{v}_0) = -\frac{1}{2\mu} t^0_{31}(\mathbf{v}_0) - \alpha a_2, \quad \omega_3 = \alpha a_3. \tag{5.5.2}$$

For this reason

$$\omega(\mathbf{v}_0)\cdot\hat{T}^0(\mathbf{v}_0) - \sigma\omega(\mathbf{v}_0) = -\mu\alpha^2\left(a_1\frac{\partial\varphi}{\partial a_1} + a_2\frac{\partial\varphi}{\partial a_2}\right)\mathbf{i}_3 + \alpha a_3\left(t^0_{31}\mathbf{i}_1 + t^0_{23}\mathbf{i}_2\right).$$

In the problem of torsion

$$\iint\limits_S t^0_{31}(\mathbf{v}_0)\, do = P = 0, \quad \iint\limits_S t^0_{23}(\mathbf{v}_0)\, do = Q = 0,$$

and referring to eq. (2.5.5) of Chapter 6 we have

$$\iint\limits_S \left(a_1\frac{\partial\varphi}{\partial a_1} + a_2\frac{\partial\varphi}{\partial a_2}\right) do = -2\iint\limits_S \varphi\, do + \oint\limits_\Gamma \varphi(n_1a_1 + n_2a_2)\, ds$$
$$= -2\iint\limits_S \varphi\, do + 2\oint\limits_\Gamma \varphi\, d\omega = \oint\limits_\Gamma \varphi\frac{\partial\varphi}{\partial s} ds = \frac{1}{2}\oint\limits_\Gamma \frac{\partial\varphi^2}{\partial s} ds = 0,$$

since function φ^2 is single-valued. This proves the necessary condition for existence of the correcting vector $\mathbf{w}$

$$\iiint\limits_v \left[\boldsymbol{\omega} \cdot \hat{T}^0(\mathbf{v}_0) - \boldsymbol{\omega}\sigma(\mathbf{v}_0)\right] d\tau_0 = 0. \tag{5.5.3}$$

In the expression for vector $\mathbf{v}$ we take into account the terms of second power in parameter α. Representing the Cartesian coordinates x_1, x_2, x_3 of point (a_1, a_2, a_3) after deformation in the form

$$\begin{aligned}
x_1 &= \sqrt{a_1^2 + a_2^2}\cos(\theta + \alpha a_3) = a_1 \cos\alpha a_3 - a_2 \sin\alpha a_3 \\
&= a_1 - \alpha a_2 a_3 - \frac{1}{2}\alpha^2 a_3^2 a_1, \\
x_2 &= \sqrt{a_1^2 + a_2^2}\sin(\theta + \alpha a_3) = a_1 \sin\alpha a_3 - a_2 \cos\alpha a_3 \\
&= a_2 + \alpha a_1 a_3 - \frac{1}{2}\alpha^2 a_3^2 a_2, \\
x_3 &= \alpha\varphi(a_1, a_2) + a_3 \quad \left(a_1 = \sqrt{a_1^2 + a_2^2}\cos\theta, \quad a_2 = \sqrt{a_1^2 + a_2^2}\sin\theta\right),
\end{aligned}$$

we have

$$\begin{aligned}
v_1 &= x_1 - a_1 = -\alpha a_2 a_3 - \frac{1}{2}\alpha^2 a_3^2 a_1, \\
v_2 &= x_2 - a_2 = \alpha a_3 a_1 - \frac{1}{2}\alpha^2 a_3^2 a_2, \\
v_3 &= \alpha\varphi(a_1, a_2)
\end{aligned}$$

and by eq. (5.5.1)

$$v_{11} = -\frac{1}{2}\alpha^2 a_3^2 a_1, \quad v_{12} = -\frac{1}{2}\alpha^2 a_3^2 a_2, \quad v_{13} = 0.$$

The linear stress tensor obtained in terms of these displacements is equal to

$$\hat{T}^0(\mathbf{v}_1) = -\alpha^2\left\{\lambda a_3^2 \hat{E} + \mu\left[a_3^2(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2) + a_3\mathbf{i}_3(a_1\mathbf{i}_1 + a_2\mathbf{i}_2) + a_3(a_1\mathbf{i}_1 + a_2\mathbf{i}_2)\mathbf{i}_3\right]\right\}. \tag{5.5.4}$$

The tensors which are parts of the expressions for vectors $\mathbf{k}$ and $\mathbf{f}$ are as follows

$$\nabla\mathbf{v}_0 \cdot \hat{T}^0(\mathbf{v}_0) = \mathbf{i}_s\mathbf{i}_3\left(\frac{\partial v_{01}}{\partial a_s}t_{13}^0 + \frac{\partial v_{02}}{\partial a_s}t_{23}^0\right) + \mathbf{i}_s\frac{\partial v_{03}}{\partial a_s}\left(\mathbf{i}_1 t_{13}^0 + \mathbf{i}_2 t_{23}^0\right), \tag{5.5.5}$$

$$\hat{T}'(\mathbf{v}_0) = \left\{ \lambda \left[\frac{1}{2\mu^2} \left(t_{23}^{0^2} + t_{31}^{0^2} \right) - \frac{\alpha}{\mu} \left(a_1 t_{23}^0 - a_2 t_{31}^0 \right) + \alpha^2 \left(a_1^2 + a_2^2 + a_3^2 \right) \right] \right.$$
$$\left. + \frac{1}{4\mu^2} (2m - n) \left(t_{23}^{0^2} + t_{31}^{0^2} \right) \right\} \hat{E} + \frac{n}{4\mu^2} \left[\left(t_{31}^{0^2} + t_{23}^{0^2} \right) \mathbf{i}_3 \mathbf{i}_3 + t_{31}^{0^2} \mathbf{i}_1 \mathbf{i}_1 + \right.$$
$$\left. + t_{32}^{0^2} \mathbf{i}_2 \mathbf{i}_2 + t_{31}^0 t_{23}^0 \left(\mathbf{i}_1 \mathbf{i}_2 + \mathbf{i}_2 \mathbf{i}_1 \right) \right] + \mu \mathbf{i}_s \mathbf{i}_q \frac{\partial v_{0s}}{\partial a_k} \frac{\partial v_{0q}}{\partial a_k}. \quad (5.5.6)$$

As one can see, the system of "volume and surface" forces $\mathbf{k}$ and $\mathbf{f}$ are rather complex. Let us restrict our consideration to the effect of change in the rod length which is equal to $l\,[\varepsilon_{33}(\mathbf{w})]$. It requires only the first invariants of the considered tensors and their $[3, 3]$ components

$$I_1 \left[\hat{T}^0 (\mathbf{v}_1) \right] = -\alpha^2 (3\lambda + 2\mu) a_3^2; \quad \mathbf{i}_3 \cdot \hat{T}^0 (\mathbf{v}_1) \cdot \mathbf{i}_3 = -\lambda \alpha^2 a_3^2;$$
$$I_1 \left[\nabla \mathbf{v}_0 \cdot \hat{T}^0 (\mathbf{v}_0) \right] = \frac{1}{\mu} \left(t_{31}^{0^2} + t_{23}^{0^2} \right); \ \mathbf{i}_3 \cdot \nabla \mathbf{v}_0 \cdot \hat{T}^0 (\mathbf{v}_0) \cdot \mathbf{i}_3 = \alpha \left(a_1 t_{23}^0 - a_2 t_{31}^0 \right);$$
$$I_1 \left[\hat{T}' (\mathbf{v}_0) \right] = (3\lambda + 2\mu) \left[\frac{1}{2\mu^2} \left(t_{31}^{0^2} + t_{23}^{0^2} \right) - \frac{\alpha}{\mu} \left(a_1 t_{23}^0 - a_2 t_{31}^0 \right) + \right.$$
$$\left. \alpha^2 \left(a_1^2 + a_2^2 + a_3^2 \right) \right] + \frac{1}{4\mu^2} (6m - n) \left(t_{31}^{0^2} + t_{23}^{0^2} \right);$$
$$\mathbf{i}_3 \cdot \hat{T}' (\mathbf{v}_0) \cdot \mathbf{i}_3 = \lambda \left[\frac{1}{2\mu^2} \left(t_{31}^{0^2} + t_{23}^{0^2} \right) - \frac{\alpha}{\mu} \left(a_1 t_{23}^0 - a_2 t_{31}^0 \right) + \alpha^2 \left(a_1^2 + a_2^2 + a_3^2 \right) \right]$$
$$+ \mu \alpha^2 \left[\left(\frac{\partial \varphi}{\partial a_1} \right)^2 + \left(\frac{\partial \varphi}{\partial a_2} \right)^2 \right] + \frac{m}{2\mu^2} \left(t_{31}^{0^2} + t_{23}^{0^2} \right).$$

It remains only to refer to Subsection 6.3.13 and write down the expressions for the following integrals

$$\frac{1}{\mu^2} \iint\limits_S \left(t_{31}^{0^2} + t_{23}^{0^2} \right) do = \alpha^2 C, \quad \frac{\alpha}{\mu} \iint \left(a_1 t_{23}^0 - a_2 t_{31}^0 \right) do = \alpha^2 C,$$
$$\iint\limits_S \left[\left(\frac{\partial \varphi}{\partial a_1} \right)^2 + \left(\frac{\partial \varphi}{\partial a_2} \right)^2 \right] do = I_p - C.$$

After calculation by formula (5.4.10) we arrive at the following expression for the mean value of the elongation of the rod which accompanies torsion

$$[\varepsilon_{33}(\mathbf{w})]_m = -\frac{\alpha^2}{2S(1+\nu)} \left\{ \left[\frac{\nu n}{4\mu} + \frac{1}{2}(1 - 2\nu) - 2\nu \right] C + I_p (1 + \nu) \right\}, \quad (5.5.7)$$

the latter differing from the result of Rivlin, 1953 only in notation.

It is evident that $I_p (1 + \nu) > 2\nu C$ as $I_p > C, \nu < 1/2$, however one can not conclude that torsion of the rod is accompanied by a decrease in its length since for the majority of materials $n < 0$ and $m < 0$.

9.5.6 Incompressible media

The specific strain energy is given by Murnaghan's formula

$$A = C_1 (I_1 - 3) + C_2 (I_2 - 3),$$

and expression (5.1.7) for the stress tensor takes the form

$$\frac{1}{2}\hat{T}(\mathbf{u}) = \left[C_1 + 2C_2 + \left(\frac{\partial A}{\partial I_3}\right)_{I_3=1} \right] \hat{E} + 2(C_1 + C_2)\hat{\varepsilon}(\mathbf{u}) + C_2 (I_1 - 3)\hat{E}$$
$$- 4C_2\hat{\varepsilon}^2(\mathbf{u}) + (C_1 + C_2)\nabla\mathbf{u}^T \cdot \nabla\mathbf{u} + C_2 (I_1 - 3)\left(3\hat{\varepsilon} + \nabla\mathbf{u}^T \cdot \nabla\mathbf{u}\right), \tag{5.6.1}$$

where replacement (5.1.15) was used. We denote

$$2(C_1 + C_2) = \mu$$

and introduce the scalar (pressure)

$$C_1 + 2C_2 + \left.\frac{\partial A}{\partial I_3}\right|_{I_3=1} = -\frac{1}{2}q,$$

defined by the condition

$$I_3 = 1: \quad 2\vartheta(\mathbf{u}) + 2\vartheta^2(\mathbf{u}) - I_1\left(\hat{\varepsilon}^2(\mathbf{u})\right) + 2\boldsymbol{\omega}(\mathbf{u}) \cdot \boldsymbol{\omega}(\mathbf{u}) = 0. \tag{5.6.2}$$

As in Subsection 9.5.3 we take

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \tag{5.6.3}$$

where $\mathbf{v}$ denotes the displacement vector in the linear approximation and $\mathbf{w}$ is the correcting vector describing the second order effects. Then by eq. (5.6.2)

$$2\vartheta(\mathbf{v}) + 2\vartheta(\mathbf{w}) + 2\vartheta^2(\mathbf{v}) - I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) + 2\boldsymbol{\omega}(\mathbf{v}) \cdot \boldsymbol{\omega}(\mathbf{v}) = 0,$$

where the terms of third and higher order are omitted. The incompressibility condition must be satisfied both in the linear approximation and with account of the second order terms, thus

$$\vartheta(\mathbf{v}) = \nabla \cdot \mathbf{v} = 0, \tag{5.6.4}$$

$$\vartheta(\mathbf{w}) = \frac{1}{2}I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) - \boldsymbol{\omega}(\mathbf{v}) \cdot \boldsymbol{\omega}(\mathbf{v}). \tag{5.6.5}$$

Due to eq. (5.1.6) we have

$$\left.\begin{aligned} I_1 - 3 &= 2\vartheta(\mathbf{w}) + I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) + 2\boldsymbol{\omega}(\mathbf{v}) \cdot \boldsymbol{\omega}(\mathbf{v}) = 2I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right), \\ I_2 - 3 &= 4\vartheta(\mathbf{w}) + 4\boldsymbol{\omega}(\mathbf{v}) \cdot \boldsymbol{\omega}(\mathbf{v}) = 2I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right). \end{aligned}\right\} \tag{5.6.6}$$

Scalar q is also presented by the sum of the value in the linear approximation q^0 and the correcting term q^1

$$q = q^0 + q^1. \tag{5.6.7}$$

With the adopted accuracy, expression (5.6.1) for the stress tensor can be taken in the following form

$$\hat{T}(\mathbf{u}) = \hat{T}^0(\mathbf{v}) + \hat{T}^0(\mathbf{w}) + \hat{T}'(\mathbf{v}), \tag{5.6.8}$$

where $\hat{T}^0(\mathbf{v})$ is the stress tensor in the linear approximation

$$\hat{T}^0(\mathbf{v}) = -q^0\hat{E} + 2\mu\hat{\varepsilon}(\mathbf{v}), \tag{5.6.9}$$

$\hat{T}^0(\mathbf{w})$ is the stress tensor calculated in terms of the correcting vector

$$\hat{T}^0(\mathbf{w}) = -q^1\hat{E} + 2\mu\hat{\varepsilon}(\mathbf{w}),$$

and $\hat{T}'(\mathbf{v})$ is the term corresponding to the linear approximation

$$\hat{T}'(\mathbf{v}) = 8C\left[\frac{1}{2}\hat{E}I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) - \hat{\varepsilon}^2(\mathbf{v})\right] + \mu\nabla\mathbf{v}^T\cdot\nabla\mathbf{v}. \tag{5.6.10}$$

9.5.7 Equilibrium equations

Similar to Subsection 9.5.2 we determine force $\mathbf{F}dO$ by the equality

$$\mathbf{F}\frac{dO}{do} = \mathbf{N}\cdot\hat{T}\frac{dO}{do} = \sqrt{I_3}n_s\mathbf{R}^s\cdot\hat{T} = \sqrt{I_3}\left(\mathbf{n} - \mathbf{n}\cdot\nabla\mathbf{u}^T\right)\cdot\hat{T}. \tag{5.7.1}$$

Taking into account representation (5.6.8) of the stress tensor and the incompressibility condition (5.6.2) we arrive at the equilibrium equation on the surface

$$\mathbf{F}^0 = \mathbf{F}\frac{dO}{do} = \mathbf{n}\cdot\hat{T}^0(\mathbf{v}) + \mathbf{n}\cdot\left[\hat{T}^0(\mathbf{w}) - \nabla\mathbf{v}^T\cdot\hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v})\right]. \tag{5.7.2}$$

Making use of this equation and repeating the derivation of Subsection 9.5.2 we obtain the equilibrium equation in the volume

$$\nabla\cdot\hat{T}^0(\mathbf{v}) + \rho_0\mathbf{K} + \nabla\cdot\left[\hat{T}^0(\mathbf{w}) - \nabla\mathbf{v}^T\cdot\hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v})\right] = 0. \tag{5.7.3}$$

In the linear approximation

$$\nabla\cdot\hat{T}^0(\mathbf{v}) + \rho_0\mathbf{K} = 0, \quad \mathbf{n}\cdot\hat{T}^0(\mathbf{v}) - \mathbf{F}^0 = 0. \tag{5.7.4}$$

Taking into account relationships (5.6.9) and (5.6.4)

$$\nabla\cdot q^0\hat{E} = \nabla q^0, \quad \nabla\cdot 2\hat{\varepsilon}(\mathbf{v}) = \nabla^2\mathbf{v} + \nabla\nabla\cdot\mathbf{v} = \nabla^2\mathbf{v} \tag{5.7.5}$$

the obtained relations take the form of equations of motion of viscous incompressible fluids (the Navier-Stokes equations)

$$\mu\nabla^2\mathbf{v} + \rho_0\mathbf{K} = \nabla q^0, \quad \nabla\cdot\mathbf{v} = 0; \quad 2\mu\mathbf{n}\cdot\hat{\varepsilon}(\mathbf{v}) = \mathbf{n}q^0 + \mathbf{F}^0, \qquad (5.7.6)$$

where $\mathbf{v}$ denotes the velocity vector.

Returning to eqs. (5.7.2), (5.7.3) and (5.6.5) we arrive at the following system of equations determining the correcting vector $\mathbf{w}$

$$\left.\begin{aligned} &\nabla\cdot\hat{T}^0(\mathbf{w}) + \nabla\cdot\left[-\nabla\mathbf{v}^T\cdot\hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v})\right] = 0,\\ &\nabla\cdot\mathbf{w} = \frac{1}{2}I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) - \boldsymbol{\omega}(\mathbf{v})\cdot\boldsymbol{\omega}(\mathbf{v}),\\ &\mathbf{n}\cdot\hat{T}^0(\mathbf{w}) + \mathbf{n}\cdot\left[-\nabla\mathbf{v}^T\cdot\hat{T}^0(\mathbf{v}) + \hat{T}'(\mathbf{v})\right] = 0. \end{aligned}\right\} \qquad (5.7.7)$$

The notion in the form of Navier-Stokes equations has the form

$$\left.\begin{aligned} &\mu\nabla^2\mathbf{w} + \mu\nabla\left[\frac{1}{2}I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) - \boldsymbol{\omega}(\mathbf{v})\cdot\boldsymbol{\omega}(\mathbf{v})\right] + \mathbf{k} = 0,\\ &\nabla\cdot\mathbf{w} = \frac{1}{2}I_1\left(\hat{\varepsilon}^2(\mathbf{v})\right) - \boldsymbol{\omega}(\mathbf{v})\cdot\boldsymbol{\omega}(\mathbf{v}),\\ &2\mu\mathbf{n}\cdot\hat{\varepsilon}(\mathbf{w}) = \mathbf{n}q^1 + \mathbf{f}. \end{aligned}\right\} \qquad (5.7.8)$$

Here vectors $\mathbf{k}$ and $\mathbf{f}$ play respectively the role of volume and surface forces and are defined by the equalities

$$\left.\begin{aligned} \mathbf{k} &= \nabla\cdot\left\{q^0\nabla\mathbf{v}^T - 2\mu\nabla\mathbf{v}^T\cdot\hat{\varepsilon}(\mathbf{v}) + \hat{T}'(\mathbf{v})\right\},\\ \mathbf{f} &= -\mathbf{n}\cdot\left\{q^0\nabla\mathbf{v}^T - 2\mu\nabla\mathbf{v}^T\cdot\hat{\varepsilon}(\mathbf{v}) + \hat{T}'(\mathbf{v})\right\}. \end{aligned}\right\} \qquad (5.7.9)$$

The nonsymmetric part of the tensor in the braces is equal to

$$q^0\hat{\Omega} - 2\mu\hat{\Omega}\cdot\hat{\varepsilon}(\mathbf{v}) = -\boldsymbol{\omega}(\mathbf{v})\times\hat{T}^0(\mathbf{v}),$$

and according to eq. (5.3.14) the necessary condition for the existence of the solution of boundary-value problem (5.7.8) is set in the form

$$\iiint\limits_v \left[\boldsymbol{\omega}(\mathbf{v})\cdot\hat{T}^0(\mathbf{v}) + 3\boldsymbol{\omega}(\mathbf{v})\,q^0\right]d\tau_0 = 0$$

which, after eliminating $\hat{T}^0(\mathbf{v})$ takes another form

$$\iiint\limits_v \left[\mu\boldsymbol{\omega}(\mathbf{v})\cdot\hat{\varepsilon}(\mathbf{v}) + \boldsymbol{\omega}(\mathbf{v})\,q^0\right]d\tau_0 = 0. \qquad (5.7.10)$$

9.6 Plane problem

9.6.1 Geometric relationships

In plane strain, the coordinates x_s of the medium particles are related to the coordinates a_s in the initial state as follows

$$x_1 = x_1(a_1, a_2), \quad x_2 = x_2(a_1, a_2), \quad x_3 = \lambda a_3 \quad (\lambda = \mathrm{const}). \tag{6.1.1}$$

Introducing the material coordinates $q^1, q^2, q^3 = x_3$ we have

$$a_\alpha = a_\alpha\left(q^1, q^2\right), \quad a_3 = \frac{q^3}{\lambda}; \quad x_\alpha = x_\alpha\left(q^1, q^2\right), \quad x_3 = q^3, \tag{6.1.2}$$

where here and in Subsections 9.6.1-9.6.9 the Greek indices take values 1,2 whereas the Latin indices, as above, take values 1,2,3. The position vectors in the initial ($v-$volume) and the deformed ($V-$volume) states are equal to

$$\mathbf{r} = \mathbf{i}_\alpha a_\alpha + \mathbf{i}_3 \frac{q^3}{\lambda} = \mathbf{b} + \mathbf{i}_3 \frac{q^3}{\lambda}, \quad \mathbf{R} = \mathbf{i}_\alpha x_\alpha + \mathbf{i}_3 x_3 = \mathbf{B} + \mathbf{i}_3 q^3, \tag{6.1.3}$$

and the base vectors in these states are respectively equal to

$$\mathbf{r}_\alpha = \frac{\partial \mathbf{b}}{\partial q^\alpha} = \mathbf{b}_\alpha, \quad \mathbf{r}_3 = \frac{\mathbf{i}_3}{\lambda}, \quad \mathbf{R}_\alpha = \frac{\partial \mathbf{B}}{\partial q^\alpha} = \mathbf{B}_\alpha, \quad \mathbf{R}_3 = \mathbf{i}_3. \tag{6.1.4}$$

The covariant components of the metric tensors $\hat{g}$ and $\hat{G}$ are determined by the formulae

$$\left.\begin{aligned} g_{\alpha\beta} &= \mathbf{b}_\alpha \cdot \mathbf{b}_\beta = b_{\alpha\beta}, \quad & g_{\alpha 3} &= 0, \quad & g_{33} &= \frac{1}{\lambda^2}, \\ G_{\alpha\beta} &= \mathbf{B}_\alpha \cdot \mathbf{B}_\beta = B_{\alpha\beta}, \quad & G_{\alpha 3} &= 0, \quad & G_{33} &= 1, \end{aligned}\right\} \tag{6.1.5}$$

such that

$$g = |g_{sk}| = \frac{b}{\lambda^2}, \quad b = b_{11}b_{22} - b_{12}^2; \quad G = |G_{sk}| = B = B_{11}B_{22} - B_{12}^2. \tag{6.1.6}$$

The nonvanishing contravariant components are as follows

$$\left.\begin{aligned} & g^{11} = b^{11} = \frac{g_{22}}{g\lambda^2} = \frac{b_{22}}{b}, \quad g^{22} = b^{22} = \frac{b_{11}}{b}, \\ & g^{12} = b^{12} = -\frac{b_{12}}{b}, \quad g^{33} = \lambda^2, \quad G^{11} = B^{11} = \frac{G_{22}}{G} = \frac{B_{22}}{B}, \\ & G^{22} = B^{22} = \frac{B_{11}}{B}, \quad G^{12} = B^{12} = -\frac{B_{12}}{B}, \quad G^{33} = 1. \end{aligned}\right\} \tag{6.1.7}$$

This introduces the plane metric tensors $\hat{b}$ and $\hat{B}$ in $v-$ and $V-$volumes respectively

$$\hat{b} = b^{\alpha\beta}\mathbf{b}_\alpha\mathbf{b}_\beta = b_{\alpha\beta}\mathbf{b}^\alpha\mathbf{b}^\beta, \quad \hat{B} = B^{\alpha\beta}\mathbf{B}_\alpha\mathbf{B}_\beta = B_{\alpha\beta}\mathbf{B}^\alpha\mathbf{B}^\beta, \tag{6.1.8}$$

the cobasis vectors being determined by the standard formulae

$$\mathbf{b}^\alpha = b^{\alpha\beta}\mathbf{b}_\beta, \quad \mathbf{B}^\alpha = B^{\alpha\beta}\mathbf{B}_\beta. \tag{6.1.9}$$

The third index 3 is omitted in the notion of the Levi-Civita tensors $\tilde{\in}, \hat{\epsilon}$

$$\sqrt{b}\epsilon^{\alpha\beta} = \sqrt{B}\tilde{\in}^{\alpha\beta} = e^{\alpha\beta}, \quad \frac{1}{\sqrt{b}}\epsilon_{\alpha\beta} = \frac{1}{\sqrt{B}}\tilde{\in}_{\alpha\beta} = e_{\alpha\beta}, \tag{6.1.10}$$

where

$$e^{\alpha\beta} = e_{\alpha\beta} = \begin{cases} 0, & \alpha = \beta, \\ 1, & \alpha = 1, \quad \beta = 2, \\ 1, & \alpha = 2, \quad \beta = 1. \end{cases} \tag{6.1.11}$$

Using these tensors one can rewrite formulae (6.1.7) and the inverse ones in the following form

$$\left.\begin{array}{ll} b^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}b_{\gamma\delta}, & B^{\alpha\beta} = \tilde{\in}^{\alpha\gamma}\tilde{\in}^{\beta\delta}B_{\gamma\delta}, \\ b_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}b^{\gamma\delta}, & B_{\alpha\beta} = \tilde{\in}_{\alpha\gamma}\tilde{\in}_{\beta\delta}B^{\gamma\delta}. \end{array}\right\} \tag{6.1.12}$$

The invariants of strain measure $\hat{G}^\times$ determined by formulae (5.2.6)-(5.2.8) of Chapter 2 are equal to

$$\left.\begin{array}{l} I_1\left(\hat{G}^\times\right) = g^{sk}G_{sk} = b^{\alpha\beta}B_{\alpha\beta} + \lambda^2, \\ I_2\left(\hat{G}^\times\right) = \dfrac{G}{g}g_{sk}G^{sk} = \dfrac{B}{b}\left(\lambda^2 b_{\alpha\beta}B^{\alpha\beta} + 1\right), \\ I_3\left(\hat{G}^\times\right) = \dfrac{G}{g} = \lambda^2\dfrac{B}{b}. \end{array}\right\} \tag{6.1.13}$$

Referring to eq. (6.1.12) we have

$$b^{\alpha\beta}B_{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}b_{\gamma\delta}\tilde{\in}_{\alpha\lambda}\tilde{\in}_{\beta\mu}B^{\lambda\mu} = \frac{B}{b}e^{\alpha\gamma}e_{\alpha\lambda}e^{\beta\delta}e_{\beta\mu}b_{\gamma\delta}B^{\lambda\mu}.$$

By using definition (6.1.11) one can easily prove the relationships

$$e^{\alpha\gamma}e_{\alpha\lambda} = \delta^\gamma_\lambda = \begin{cases} 0, & \gamma \neq \lambda, \\ 1, & \gamma = \lambda. \end{cases} \tag{6.1.14}$$

Hence

$$b^{\alpha\beta}B_{\alpha\beta} = \frac{B}{b}\delta^\gamma_\lambda\delta^\delta_\mu B^{\lambda\mu} = \frac{B}{b}b_{\gamma\delta}B^{\gamma\delta}, \tag{6.1.15}$$

which results in the formula relating the invariants

$$I_1 - \lambda^2 = \frac{I_2}{\lambda^2} - \frac{I_3}{\lambda^4} \quad \text{or} \quad I_3 - I_2\lambda^2 + I_1\lambda^4 - \lambda^6 = 0. \tag{6.1.16}$$

9.6.2 Constitutive equation

The specific strain energy is now considered as being a function of invariants I_1 and I_3

$$A\left(I_1, I_2, I_3\right) = A\left(I_1, \frac{I_3}{\lambda^2} + I_1\lambda^2 - \lambda^4, I_3\right) = A\left(I_1, I_3\right), \tag{6.2.1}$$

such that

$$\frac{\partial A_1}{\partial I_1} = \frac{\partial A}{\partial I_1} + \lambda^2 \frac{\partial A}{\partial I_2}, \quad \frac{\partial A_1}{\partial I_3} = \frac{\partial A}{\partial I_3} + \frac{1}{\lambda^2}\frac{\partial A}{\partial I_2}. \tag{6.2.2}$$

The constitutive equation (2.1.9) of Chapter 8 can be written in the form

$$\tilde{t}^{st} = \frac{2}{\sqrt{I_3}}\left[\frac{\partial A}{\partial I_1} g^{st} + \frac{\partial A}{\partial I_2}\left(I_1 g^{st} - g^{sm} g^{tn} G_{mn}\right) + I_3 \frac{\partial A}{\partial I_3} G^{st}\right]$$

and by eqs. (6.1.5) and (6.1.7)

$$\tilde{t}^{\alpha 3} = 0, \quad \tilde{t}^{33} = \frac{2}{\sqrt{I_3}}\left[\frac{\partial A}{\partial I_1}\lambda^2 + \frac{\partial A}{\partial I_2}\lambda^2\left(I_1 - \lambda^2\right) + I_3 \frac{\partial A}{\partial I_3}\right]. \tag{6.2.3}$$

Replacing invariant I_1 according to eq. (6.1.13) and using eq. (6.2.2) we have

$$\tilde{t}^{\alpha\beta} = \frac{2}{\sqrt{I_3}}\left[\frac{\partial A_1}{\partial I_1} b^{\alpha\beta} + \frac{\partial A}{\partial I_2}\left(b^{\alpha\beta} b^{\gamma\delta} - b^{\alpha\gamma} b^{\beta\delta}\right) B_{\gamma\delta} + I_3 \frac{\partial A}{\partial I_3} B^{\alpha\beta}\right]. \tag{6.2.4}$$

For fixed values of α, β there are four possible combinations of values of γ, δ

$$\gamma = \alpha,\ \delta = \alpha; \quad \gamma = \alpha,\ \delta = \beta; \quad \gamma = \beta,\ \delta = \alpha; \quad \gamma = \beta,\ \delta = \beta.$$

For this reason (do not sum over α, β!)

$$b^{\alpha\beta} b^{\gamma\delta} B_{\gamma\delta} = b^{\alpha\beta}\left(b^{\alpha\alpha} B_{\alpha\alpha} + b^{\alpha\beta} B_{\alpha\beta} + b^{\beta\alpha} B_{\beta\alpha} + b^{\beta\beta} B_{\beta\beta}\right),$$
$$b^{\alpha\gamma} b^{\beta\delta} B_{\gamma\delta} = b^{\alpha\alpha} b^{\beta\alpha} B_{\alpha\alpha} + b^{\alpha\alpha} b^{\beta\beta} B_{\alpha\beta} + b^{\alpha\beta} b^{\beta\alpha} B_{\beta\alpha} + b^{\alpha\beta} b^{\beta\beta} B_{\beta\beta},$$

such that by eqs. (6.1.6) and (6.1.7)

$$\left(b^{\alpha\beta} b^{\gamma\delta} - b^{\alpha\gamma} b^{\beta\delta}\right) B_{\gamma\delta} = \left[\left(b^{\alpha\beta}\right)^2 - b^{\alpha\alpha} b^{\beta\beta}\right] B_{\alpha\beta} = \frac{B}{b} B^{\alpha\beta} = \frac{I_3}{\lambda^2} B^{\alpha\beta},$$

and returning to eqs. (6.2.4) and (6.2.2) we have

$$\tilde{t}^{\alpha\beta} = \frac{2}{\sqrt{I_3}}\left(\frac{\partial A_1}{\partial I_1} b^{\alpha\beta} + \frac{\partial A_1}{\partial I_3} I_3 B^{\alpha\beta}\right). \tag{6.2.5}$$

For the incompressible medium

$$I_3 = 1, \quad A_1 = A_1(I_1), \quad I_2\lambda^2 - I_1\lambda^4 = 1 - \lambda^6, \tag{6.2.6}$$

such that

$$\tilde{t}^{\alpha\beta} = 2\frac{\partial A_1}{\partial I_1} b^{\alpha\beta} + pB^{\alpha\beta}, \tag{6.2.7}$$

$$\tilde{t}^{\alpha 3} = 0, \quad \tilde{t}^{33} = 2\left[\frac{\partial A}{\partial I_1}\lambda^2 + \lambda^2\left(I_1 - \lambda^2\right)\frac{\partial A}{\partial I_2}\right] + p, \tag{6.2.8}$$

where p denotes a function of the coordinates which is not known in advance.

9.6.3 Equations of statics

When the mass forces are absent the equilibrium equations in the volume are written in the form

$$\frac{\partial}{\partial q^\alpha}\left(\sqrt{G}\tilde{t}^{\alpha s}\mathbf{R}_s\right) = 0, \quad \frac{\partial}{\partial q^3}\left(\sqrt{G}\tilde{t}^{3s}\mathbf{R}_s\right) = 0.$$

However $\tilde{t}^{\alpha 3} = 0$ and $\tilde{t}^{33}, G$ do not depend on q^3, hence the latter equation is identically satisfied and the first set of equations is set as follows

$$\frac{\partial}{\partial q^\alpha}\sqrt{B}\tilde{t}^{\alpha\beta}\mathbf{B}_\beta = 0 \tag{6.3.1}$$

or

$$\frac{\partial}{\partial q^a}\sqrt{B}\tilde{t}^{\alpha\beta} + \sqrt{B}\tilde{t}^{\alpha\gamma}\widetilde{\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\}} = 0, \tag{6.3.2}$$

where Christoffel's symbols are calculated in terms of the metric tensor $\hat{B}$.

The equilibrium equations on the surface can be written down in one of the forms (3.3.7) and (3.3.8) of Chapter 1

$$\tilde{t}^{\alpha\beta} n_\alpha = \tilde{F}^\beta\sqrt{B^{\gamma\delta}n_\gamma n_\delta}, \quad \tilde{t}^{\alpha\beta}\tilde{N}_\alpha = \tilde{F}^\beta, \tag{6.3.3}$$

where n_α and $\tilde{N}_\alpha$ denote the covariant components of the unit vectors of the outward normals $\mathbf{n}$ and $\mathbf{N}$ to contours γ and Γ of the cross-sections of the body in the initial and deformed states respectively

$$n_\alpha = \mathbf{n}\cdot\mathbf{b}_\alpha, \quad \tilde{N}_\alpha = \mathbf{N}\cdot\mathbf{B}_\alpha. \tag{6.3.4}$$

Here $\tilde{F}^\beta$ denotes the contravariant components of the surface force

$$\tilde{F}^\beta = \mathbf{F}\cdot\mathbf{B}^\beta. \tag{6.3.5}$$

9.6.4 Stress function

The tensor of stress functions, Subsection 1.1.6, is given in the form

$$\hat{\Phi} = U\left(q^1, q^2\right)\mathbf{i}_3\mathbf{i}_3,$$

whilst the stress tensor is determined in terms of the tensor of stress functions by eq. (1.6.6) of Chapter 1

$$\hat{T} = \tilde{t}^{\alpha\beta}\mathbf{B}_\alpha\mathbf{B}_\beta = \operatorname{inc}\hat{\Phi} = \tilde{\nabla}\times\left[\tilde{\nabla}\times U\left(q^1,q^2\right)\mathbf{i}_3\mathbf{i}_3\right]^*, \tag{6.4.1}$$

The component $\tilde{t}^{33}$ is not considered here since it is determined independently by eq. (6.2.3).

The calculation is carried out in the following way

$$\tilde{\nabla}\times U\mathbf{i}_3\mathbf{i}_3 = \mathbf{R}^s\frac{\partial}{\partial q^s}\times U\mathbf{R}^3\mathbf{i}_3 = \mathbf{R}^\beta\times\mathbf{R}^3\frac{\partial U}{\partial q^\beta}\mathbf{i}_3 = \tilde{\in}^{\lambda\beta}\mathbf{R}_\lambda\mathbf{i}_3\frac{\partial U}{\partial q^\beta},$$

$$\left(\tilde{\nabla}\times U\mathbf{i}_3\mathbf{i}_3\right)^* = \tilde{\in}^{\lambda\beta}\mathbf{i}_3\mathbf{R}_\lambda\frac{\partial U}{\partial q^\beta},$$

$$\operatorname{inc}U\mathbf{i}_3\mathbf{i}_3 = \mathbf{R}^\alpha\times\mathbf{i}_3\frac{\partial}{\partial q^\alpha}\tilde{\in}^{\lambda\beta}\mathbf{R}_\lambda\frac{\partial U}{\partial q^\beta} =$$

$$= \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\left(\mathbf{R}_\mu\mathbf{R}_\lambda\frac{\partial^2 U}{\partial q^\alpha\partial q^\beta} + \mathbf{R}_\mu\frac{\partial U}{\partial q^\beta}\widetilde{\left\{\begin{matrix}\rho\\ \lambda\alpha\end{matrix}\right\}}\mathbf{R}_\rho\right) + \tilde{\in}^{\mu\alpha}\mathbf{R}_\mu\mathbf{R}_\lambda\frac{\partial U}{\partial q^\beta}\frac{\partial\tilde{\in}^{\lambda\beta}}{\partial q^\alpha}. \tag{6.4.2}$$

According to the condition whereby the covariant derivative of the Levi-Civita tensor vanishes

$$\tilde{\nabla}_\alpha\tilde{\in}^{\lambda\beta} = \frac{\partial\tilde{\in}^{\lambda\beta}}{\partial q^\alpha} + \widetilde{\left\{\begin{matrix}\lambda\\ \alpha\rho\end{matrix}\right\}}\tilde{\in}^{\rho\beta} + \widetilde{\left\{\begin{matrix}\beta\\ \alpha\rho\end{matrix}\right\}}\tilde{\in}^{\lambda\rho} = 0$$

we obtain by renaming the dummy indices

$$\mathbf{R}_\lambda\frac{\partial U}{\partial q^\beta}\frac{\partial\tilde{\in}^{\lambda\beta}}{\partial q^\alpha} = -\mathbf{R}_\rho\widetilde{\left\{\begin{matrix}\rho\\ \lambda\alpha\end{matrix}\right\}}\tilde{\in}^{\lambda\rho}\frac{\partial U}{\partial q^\beta} - \mathbf{R}_\lambda\tilde{\in}^{\lambda\beta}\widetilde{\left\{\begin{matrix}\rho\\ \alpha\beta\end{matrix}\right\}}\frac{\partial U}{\partial q^\rho},$$

and substitution into eq. (6.4.2) leads to the following representation of the stress tensor

$$\hat{T} = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\rho}\mathbf{R}_\mu\mathbf{R}_\lambda\left(\frac{\partial^2 U}{\partial q^\alpha\partial q^\beta} - \widetilde{\left\{\begin{matrix}\rho\\ \alpha\beta\end{matrix}\right\}}\frac{\partial U}{\partial q^\rho}\right). \tag{6.4.3}$$

The value in the parentheses represents the covariant derivative of the covariant components of the gradient of U

$$\tilde{\nabla}U = \mathbf{R}^\beta\frac{\partial U}{\partial q^\beta}, \quad \tilde{\nabla}_\alpha\frac{\partial U}{\partial q^\beta} = \frac{\partial^2 U}{\partial q^\alpha\partial q^\beta} - \widetilde{\left\{\begin{matrix}\rho\\ \alpha\beta\end{matrix}\right\}}\frac{\partial U}{\partial q^\rho} = \tilde{\nabla}_\beta\frac{\partial U}{\partial q^\alpha} = \tilde{\nabla}_\alpha\tilde{\nabla}_\beta U.$$

Hence

$$\hat{T} = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\mathbf{R}_\mu\mathbf{R}_\lambda\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U; \quad \tilde{t}^{\mu\lambda} = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = \frac{b}{B}\epsilon^{\mu\alpha}\epsilon^{\lambda\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U. \tag{6.4.4}$$

The inverse relationships have the form

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = \tilde{\in}_{\mu\alpha}\tilde{\in}_{\lambda\beta}\tilde{t}^{\mu\lambda} = \frac{B}{b}\epsilon_{\mu\alpha}\epsilon_{\lambda\beta}\tilde{t}^{\mu\lambda}, \tag{6.4.5}$$

and by eqs. (6.2.5), (6.1.12) and (6.1.13)

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = 2\sqrt{I_3}\left(\frac{\partial A_1}{\partial I_1}\frac{1}{\lambda^2}b_{\alpha\beta} + \frac{\partial A_1}{\partial I_3}B_{\alpha\beta}\right). \tag{6.4.6}$$

The expression for the first invariant of the stress tensor is set in the form

$$I_1\left(\hat{T}\right) = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}B_{\mu\lambda}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = B^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = \tilde{\nabla}^2 U. \tag{6.4.7}$$

As expected it turns out to be equal to the Laplace operator over U (in the metric of the deformed body)

$$\tilde{\nabla}^2 U = \tilde{\nabla}\cdot\tilde{\nabla}U = \mathbf{R}^\alpha\frac{\partial}{\partial q^\alpha}\cdot\mathbf{R}^\beta\frac{\partial U}{\partial q^\beta} = B^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U. \tag{6.4.8}$$

The unit vectors of the tangent $\mathbf{t}$ and normal $\mathbf{N}$ to plane curve Γ in the deformed body

$$\mathbf{R} = \mathbf{B}(S) + \mathbf{i}_3 x_3 \quad (x_3 = \mathrm{const})$$

are determined under the assumption that $\mathbf{t}, \mathbf{N}, \mathbf{i}_3$ are oriented along axes of system $OXYZ$ in the following way

$$\mathbf{t} = \frac{\partial\mathbf{B}}{\partial S} = \mathbf{B}_\gamma\frac{dq^\gamma}{dS}, \quad \mathbf{N} = \mathbf{t}\times\mathbf{i}_3 = \tilde{\in}_{\rho\gamma}\mathbf{B}^\rho\frac{dq^\gamma}{dS}. \tag{6.4.9}$$

The stress vector describing action of the part of medium "over G" on the medium "under G" is

$$\begin{aligned}\mathbf{F} = \mathbf{N}\cdot\hat{T} &= \tilde{\in}_{\rho\gamma}\mathbf{B}^\rho\cdot\tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\mathbf{B}_\mu\mathbf{B}_\lambda\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U\frac{dq^\gamma}{dS}\\ &= \tilde{\in}^{\lambda\beta}\mathbf{B}_\lambda\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U\frac{dq^\alpha}{dS} = \tilde{\nabla}_\alpha\tilde{\nabla}_\beta U\frac{dq^\alpha}{dS}\mathbf{B}^\beta\times\mathbf{i}_3.\end{aligned} \tag{6.4.10}$$

As indicated above

$$\begin{aligned}d\tilde{\nabla}U &= dq^\alpha\frac{\partial}{\partial q^\alpha}\mathbf{B}^\beta\tilde{\nabla}_\beta U = dq^\alpha\mathbf{B}^\beta\left(\frac{\partial^2 U}{\partial q^\alpha\partial q^\beta} - \widetilde{\left\{\begin{matrix}\rho\\ \alpha\beta\end{matrix}\right\}}\frac{\partial U}{\partial q^\rho}\right)\\ &= dq^\alpha\mathbf{B}^\beta\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U,\end{aligned}$$

and hence

$$\mathbf{F}dS = d\tilde{\nabla}U \times \mathbf{i}_3, \quad \int_A^M \mathbf{F}dS = \mathbf{P} = \tilde{\nabla}U \times \mathbf{i}_3. \tag{6.4.11}$$

This formula determines the principal vector $\mathbf{P}$ of stresses on curve AM of curve Γ. This vector can be presented in the form

$$\mathbf{P} = \tilde{\in}^{\lambda\beta}\mathbf{R}_\lambda \frac{\partial U}{\partial q^\beta}, \quad \tilde{P}^\lambda = \tilde{\in}^{\lambda\beta} \frac{\partial U}{\partial q^\beta}, \tag{6.4.12}$$

such that

$$\frac{\partial U}{\partial q^\beta} = \tilde{\in}_{\lambda\beta}\tilde{P}^\lambda; \quad \frac{\partial U}{\partial q^1} = -\sqrt{B}\tilde{P}^2, \quad \frac{\partial U}{\partial q^2} = \sqrt{B}\tilde{P}^1. \tag{6.4.13}$$

The latter is a generalisation of formulae (1.8.4) of Chapter 7. In these formulae $\tilde{P}^\alpha$ denotes contravariant components of the principal vector of stresses in the metric of the deformed body.

The principal moment of stresses on curve Γ about the coordinate origin in plane $x_3 = \text{const}$ is given by the following integral

$$\mathbf{m}^O = \int_A^M \mathbf{B} \times \mathbf{F}dS = \int_A^M \mathbf{B} \times \left(d\tilde{\nabla}U \times \mathbf{i}_3\right) = -\mathbf{i}_3 \int_A^M \mathbf{B} \cdot d\tilde{\nabla}U.$$

Returning to the relationship

$$\begin{aligned} \mathbf{B} \cdot d\tilde{\nabla}U &= d\left(\mathbf{B} \cdot \tilde{\nabla}U\right) - dq^\beta \mathbf{B}_\beta \cdot \tilde{\nabla}U \\ &= d\left(\mathbf{B} \cdot \tilde{\nabla}U\right) - \frac{\partial U}{\partial q^\beta} dq^\beta = d\left(\mathbf{B} \cdot \tilde{\nabla}U\right) - dU, \end{aligned}$$

we arrive at the following expression for the principal moment about axis Oz

$$m_z = U - \mathbf{B} \cdot \tilde{\nabla}U = U - \mathbf{B} \cdot \mathbf{B}^\beta \frac{\partial U}{\partial q^\beta}. \tag{6.4.14}$$

This is a generalisation of formula (1.8.5) of Chapter 7.

9.6.5 Plane stress

We consider a body having, in the initial state (v—volume), the form of a plate of constant thickness h_0 which is small relative to the plate sizes, i.e. $a_3 \le h_0$. The faces of the plate are not loaded and the surface forces on the lateral surface are parallel to the plate midplane $a_3 = 0$ and distributed symmetrically about this plane. These properties are assumed to be kept in the deformed plate ($V-$volume), such that the state of stress is symmetric about the midplane $x_3 = 0$.

The material coordinates q^1, q^2, q^3 are introduced by the relations

$$x_\alpha = x_\alpha\left(q^1, q^2\right), \quad x_3 = q^3; \quad \mathbf{R} = x_\alpha \mathbf{i}_\alpha + \mathbf{i}_3 q^3. \tag{6.5.1}$$

Hence the vectors of the initial basis and cobasis in $V-$volume are equal to each other

$$\mathbf{R}_\alpha = \mathbf{B}_\alpha = \frac{\partial x_\gamma}{\partial q^\alpha}\mathbf{i}_\gamma, \ \mathbf{R}_3 = \mathbf{i}_3; \quad \mathbf{R}^\alpha = B^{\alpha\beta}\mathbf{B}_\beta = \mathbf{B}^\alpha, \ \mathbf{R}^3 = \mathbf{i}_3 \tag{6.5.2}$$

and the covariant and contravariant components of metric tensor $\hat{G}$ of $V-$volume are determined by formulae (6.1.5) and (6.1.7).

The unit vector of the normal $\overset{\alpha}{\mathbf{N}}$ to surface $q^\alpha = \text{const}$ has the direction of vector $\mathbf{R}^\alpha$ of the cobasis and is given by

$$\overset{\alpha}{\mathbf{N}} = \frac{\mathbf{R}^\alpha}{|\mathbf{R}^\alpha|} = \frac{1}{\sqrt{B^{\alpha\alpha}}}\mathbf{R}^\alpha. \tag{6.5.3}$$

The stress vector $\overset{\alpha}{\mathbf{t}}$ on this surface is thus equal to

$$\overset{\alpha}{\mathbf{t}} = \overset{\alpha}{\mathbf{N}} \cdot \hat{T} = \frac{1}{\sqrt{B^{\alpha\alpha}}}\mathbf{R}^\alpha \cdot \tilde{t}^{mk}\mathbf{R}_m\mathbf{R}_k = \frac{1}{\sqrt{B^{\alpha\alpha}}}\tilde{t}^{\alpha k}\mathbf{R}_k, \tag{6.5.4}$$

and the principal vector integrated over the thickness of the plate is denoted by $\overset{\alpha}{\mathbf{T}}$ and is as follows

$$\overset{\alpha}{\mathbf{T}} = -\int_{-h}^{h} \overset{\alpha}{\mathbf{t}}\, dq^3 = \frac{1}{\sqrt{B^{\alpha\alpha}}}\mathbf{R}_k \int_{-h}^{h} \tilde{t}^{\alpha k} dq^3. \tag{6.5.5}$$

Introducing the denotation

$$\int_{-h}^{h} \tilde{t}^{\alpha\beta} dq^3 = \tilde{p}^{\alpha\beta} = \tilde{p}^{\beta\alpha} \tag{6.5.6}$$

and noticing that due to the symmetry of the state of stress $\tilde{t}^{\alpha 3}$ is even with respect to q^3

$$\int_{-h}^{h} \tilde{t}^{\alpha 3} dq^3 = 0, \tag{6.5.7}$$

we arrive at the formula

$$\sqrt{B^{\alpha\alpha}}\, \overset{\alpha}{\mathbf{T}} = \tilde{p}^{\alpha\beta}\mathbf{B}_\beta. \tag{6.5.8}$$

Functions $\tilde{p}^{\alpha\beta}$ can be deemed as the contravariant components of the surface symmetric tensor

$$\hat{P} = \tilde{p}^{\alpha\beta}\mathbf{B}_\alpha\mathbf{B}_\beta. \tag{6.5.9}$$

On surface $q^3 = h\left(q^1.q^2\right)$ bounding the deformed plate

$$\mathbf{R} = \mathbf{B}\left(q^1, q^2\right) + \mathbf{i}_3 h\left(q^1, q^2\right), \tag{6.5.10}$$

and vectors $\mathbf{R}'_\alpha$ (which differ from the base vectors $\mathbf{R}_\alpha = \mathbf{B}_\alpha$)

$$\mathbf{R}'_1 = \mathbf{B}_1 + \mathbf{i}_3\frac{\partial h}{\partial q^1}, \quad \mathbf{R}'_2 = \mathbf{B}_2 + \mathbf{i}_3\frac{\partial h}{\partial q^2}$$

lie in the tangent plane to this surface, while the vector

$$\begin{aligned}\mathbf{R}'_1 \times \mathbf{R}'_2 &= \mathbf{B}_1 \times \mathbf{B}_2 + \mathbf{B}_1 \times \mathbf{i}_3\frac{\partial h}{\partial q^2} + \mathbf{i}_3 \times \mathbf{B}_2\frac{\partial h}{\partial q^1} \\ &= \mathbf{B}_1 \times \mathbf{B}_2 + \tilde{\in}_{21}\mathbf{B}^2\frac{\partial h}{\partial q^2} - \mathbf{B}^1\tilde{\in}_{12}\frac{\partial h}{\partial q^1} = \mathbf{B}_1 \times \mathbf{B}_2 - \sqrt{B}\mathbf{B}^\alpha\frac{\partial h}{\partial q^a}\end{aligned}$$

has the direction of the unit vector of the normal $\overset{3}{\mathbf{N}}$ to the plane. Noticing that

$$\mathbf{B}_1 \times \mathbf{B}_2 = \mathbf{i}_3\sqrt{B}, \quad \mathbf{R}'_1 \times \mathbf{R}'_2 = \sqrt{B}\left(\mathbf{i}_3 - \mathbf{B}^\alpha\frac{\partial h}{\partial q^\alpha}\right),$$

$$\left|\mathbf{R}'_1 \times \mathbf{R}'_2\right| = \sqrt{B}\left(1 + B^{\alpha\beta}\frac{\partial h}{\partial q^\alpha}\frac{\partial h}{\partial q^\beta}\right)^{1/2},$$

we obtain

$$\overset{3}{\mathbf{N}} = \left(1 + B^{\alpha\beta}\frac{\partial h}{\partial q^\alpha}\frac{\partial h}{\partial q^\beta}\right)^{-1/2}\left(\mathbf{i}_3 - \mathbf{B}^\alpha\frac{\partial h}{\partial q^\alpha}\right). \tag{6.5.11}$$

The condition of absence of loading in this plane takes the form

$$\overset{3}{\mathbf{N}} \cdot \hat{T} = 0, \quad \left(\mathbf{i}_3 - \mathbf{B}^\lambda\frac{\partial h}{\partial q^\lambda}\right) \cdot \left[\tilde{t}^{\alpha\beta}\mathbf{B}_\alpha\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\left(\mathbf{B}_\alpha\mathbf{i}_3 + \mathbf{i}_3\mathbf{B}_\alpha\right) + \tilde{t}^{33}\mathbf{i}_3\mathbf{i}_3\right] = 0$$

or

$$\left(\tilde{t}^{\alpha 3} - \tilde{t}^{\alpha\beta}\frac{\partial h}{\partial q^\beta}\right)\mathbf{B}_\alpha + \mathbf{i}_3\left(\tilde{t}^{33} - \tilde{t}^{\alpha 3}\frac{\partial h}{\partial q^\alpha}\right) = 0. \tag{6.5.12}$$

From the obtained three relations

$$\tilde{t}^{\alpha 3} - \tilde{t}^{\alpha\beta}\frac{\partial h}{\partial q^\beta} = 0, \quad \tilde{t}^{33} - \tilde{t}^{\alpha 3}\frac{\partial h}{\partial q^\alpha} = 0$$

one can eliminate components $\tilde{t}^{\alpha 3}$ of the stress tensor. We thus arrive at the equation

$$\text{for } \ q^3 = \pm h\left(q^\alpha, q^\beta\right) \quad \tilde{t}^{33} = \tilde{t}^{\alpha\beta}\frac{\partial h}{\partial q^\alpha}\frac{\partial h}{\partial q^\beta}. \tag{6.5.13}$$

9.6.6 Equilibrium equations

In the case of absent volume forces, the equations of statics in the volume

$$
\begin{aligned}
\frac{\partial}{\partial q^s}\sqrt{G}\tilde{t}^{sk}\mathbf{R}_k &= \frac{\partial}{\partial q^\alpha}\sqrt{G}\tilde{t}^{\alpha k}\mathbf{R}_k + \frac{\partial}{\partial q^3}\sqrt{G}\tilde{t}^{3k}\mathbf{R}_k \\
&= \frac{\partial}{\partial q^\alpha}\sqrt{G}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right) + \frac{\partial}{\partial q^3}\sqrt{G}\left(\tilde{t}^{3\alpha}\mathbf{B}_\alpha + \tilde{t}^{33}\mathbf{i}_3\right) = 0
\end{aligned}
$$

are integrated over the thickness of the plate. We arrive at the equality

$$
\int_{-h}^{h}\frac{\partial}{\partial q^\alpha}\sqrt{B}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right)dq^3 + \sqrt{B}\left(\tilde{t}^{3\alpha}\mathbf{B}_\alpha + \tilde{t}^{33}\mathbf{i}_3\right)\Big|_{-h}^{h} = 0. \qquad (6.6.1)
$$

The integration limits depend on q^1 and q^2, hence

$$
\begin{aligned}
&\frac{\partial}{\partial q^\alpha}\int_{-h}^{h}\sqrt{B}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right)dq^3 = \\
&= \int_{-h}^{h}\frac{\partial}{\partial q^\alpha}\sqrt{B}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right)dq^3 + \sqrt{B}\left[\frac{\partial h}{\partial q^\alpha}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right)\right]\Big|_{-h}^{h}
\end{aligned}
$$

and equality (6.6.1) is set in the form

$$
\begin{aligned}
&\frac{\partial}{\partial q^\alpha}\int_{-h}^{h}\sqrt{B}\left(\tilde{t}^{\alpha\beta}\mathbf{B}_\beta + \tilde{t}^{\alpha 3}\mathbf{i}_3\right)dq^3 + \\
&\qquad \sqrt{B}\left[\left(\tilde{t}^{33} - \tilde{t}^{\alpha 3}\frac{\partial h}{\partial q^\alpha}\right)\mathbf{i}_3 + \left(\tilde{t}^{3\alpha} - \tilde{t}^{\alpha\beta}\frac{\partial h}{\partial q^\beta}\right)\mathbf{B}_\alpha\right]\Big|_{-h}^{h} = 0.
\end{aligned}
$$

The value in the square brackets vanishes by means of eq. (6.5.12). Proceeding to formulae (6.5.6) and (6.5.7) we arrive at the equilibrium equation which contains only the values determined on the mid-surface

$$
\frac{\partial}{\partial q^\alpha}\sqrt{B}\tilde{p}^{\alpha\beta}\mathbf{B}_\beta = 0. \qquad (6.6.2)
$$

This equation is completely coincident with the equation of statics (6.3.1) of the problem of plane strain. Thus it can be satisfied by introducing the stress function U, such that by eq. (6.4.4)

$$
\tilde{p}^{\mu\lambda} = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = \frac{b}{B}\epsilon^{\mu\alpha}\epsilon^{\lambda\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U. \qquad (6.6.3)
$$

On the lateral surface of the plate

$$\mathbf{R} = \mathbf{B}\left(q^1(S), q^2(S)\right) + \mathbf{i}_3 q^3, \tag{6.6.4}$$

where S is the arc of curve Γ bounding the cross-section in the mid-plane $q^3 = 0$. The vectors

$$\frac{\partial \mathbf{R}}{\partial S} = \mathbf{B}_\alpha \frac{\partial q^\alpha}{\partial S}, \quad \frac{\partial \mathbf{R}}{\partial q^3} = \mathbf{i}_3$$

lie in the tangent plane to this surface while the vector

$$\mathbf{B}_\alpha \frac{\partial q^\alpha}{\partial S} \times \mathbf{i}_3 = \tilde{\in}_{\lambda\alpha} \mathbf{B}^\lambda \frac{\partial q^\alpha}{\partial S} = \mathbf{N} \tag{6.6.5}$$

has the direction of the normal to this surface. This vector is the unit vector of the normal because

$$\tilde{\in}_{\lambda\alpha} \mathbf{B}^\lambda \cdot \tilde{\in}_{\mu\beta} \mathbf{B}^\mu \frac{\partial q^\alpha}{\partial S} \frac{\partial q^\beta}{\partial S} = \tilde{\in}_{\lambda\alpha} \tilde{\in}_{\mu\beta} \mathbf{B}^{\lambda\mu} \frac{\partial q^\alpha}{\partial S} \frac{\partial q^\beta}{\partial S} = B_{\alpha\beta} \frac{\partial q^\alpha}{\partial S} \frac{\partial q^\beta}{\partial S} = 1.$$

Here formula (6.1.12) was used and it was taken into account that

$$B_{\alpha\beta} dq^\alpha dq^\beta = dS^2.$$

The surface force $\mathbf{F}$ on surface (6.6.4) is determined by the equality

$$\mathbf{F} = \mathbf{N} \cdot \hat{T} = \tilde{\in}_{\lambda\gamma} \frac{dq^\gamma}{dS} \mathbf{B}^\lambda \cdot \left[\tilde{t}^{\alpha\beta} \mathbf{B}_\alpha \mathbf{B}_\beta + \tilde{t}^{\alpha 3} \mathbf{B}_\alpha \mathbf{i}_3 + \mathbf{i}_3 \left(\mathbf{B}_\alpha \tilde{t}^{\alpha 3} + \mathbf{i}_3 \tilde{t}^{33}\right)\right]$$

or

$$\mathbf{F} = \tilde{\in}_{\alpha\gamma} \frac{dq^\gamma}{dS} \left(\tilde{t}^{\alpha\beta} \mathbf{B}_\beta + \tilde{t}^{\alpha 3} \mathbf{i}_3\right). \tag{6.6.6}$$

By eqs. (6.5.6) and (6.5.7) the principal vector of these forces integrated over the plate thickness is

$$\mathbf{f} = \int_{-h}^{h} \mathbf{F} dq^3 = \tilde{\in}_{\alpha\gamma} \tilde{p}^{\alpha\beta} \mathbf{B}_\beta \frac{dq^\gamma}{dS} = \tilde{\in}^{\beta\mu} \mathbf{B}_\beta \tilde{\nabla}_\lambda \tilde{\nabla}_\mu U \frac{dq^\gamma}{dS}. \tag{6.6.7}$$

We arrive at equality (6.4.10) from which, by repeating derivation of Subsection 9.6.4, one can obtain the boundary conditions (6.4.13) and (6.4.14) for the stress function U.

9.6.7 Constitutive equation

As a result of the assumed symmetry of the deformation about plane $q^3 = x_3 = 0$ the Cartesian coordinates of point $a_\alpha\left(q^1, q^2, q^3\right)$ and $a_3\left(q^1, q^2, q^3\right)$

in the initial state are respectively even and odd in q^3. Hence

$$a_3\left(q^1, q^2, 0\right) = 0, \quad \left(\frac{\partial a_\alpha}{\partial q^3}\right)_{q^3=0} = 0, \quad \left(\frac{\partial a_3}{\partial q^\alpha}\right)_{q^3=0} = 0, \quad \left(\frac{\partial a_3}{\partial q^3}\right)_{q^3=0} = \frac{1}{\lambda}, \tag{6.7.1}$$

where λ is an unknown function of q^1, q^2.

These relations admit the following expressions for the components of the metric tensor $\hat{g}$ and the determinant g in the form (6.1.5) and (6.1.7)

$$q^3 = 0: \left\{ \begin{array}{l} g_{\alpha\beta} = b_{\alpha\beta}, \quad g_{33} = \dfrac{1}{\lambda^2}, \quad g_{\alpha 3} = 0, \\ g^{\alpha\beta} = b^{\alpha\beta}, \quad g^{33} = \lambda^2, \quad g^{\alpha 3} = 0, \\ g = |g_{sk}| = \dfrac{b}{\lambda^2}, \quad b = b_{11}b_{22} - b_{12}^2. \end{array} \right. \tag{6.7.2}$$

Repeating the calculation of Subsection 9.6.1 we arrive at expressions (6.1.13) and (6.1.15) for the invariants of the strain measure $\hat{G}^\times$

$$q^3 = 0: \quad I_1 = b^{\alpha\beta}B_{\alpha\beta} + \lambda^2, \; I_2 = \frac{B}{b}\left(\lambda^2 b^{\alpha\beta}B_{\alpha\beta} + 1\right), \; I_3 = \frac{B}{b}\lambda^2 \tag{6.7.3}$$

and relationship (6.1.16) between them. This allows us to write the expressions for the components of the stress tensor in the mid-plane in the form of eqs. (6.2.3) and (6.2.5)

$$\left. \begin{array}{l} q^3 = 0: \quad \tilde{t}^{\alpha 3} = 0, \quad \tilde{t}^{33} = \dfrac{2}{\sqrt{I_3}}\left[\dfrac{\partial A}{\partial I_1}\lambda^2 + \dfrac{\partial A}{\partial I_2}\lambda^2\left(I_1 - \lambda^2\right) + I_3\dfrac{\partial A}{\partial I_3}\right], \\ \qquad\qquad \tilde{t}^{\alpha\beta} = \dfrac{2}{\sqrt{I_3}}\left(\dfrac{\partial A_1}{\partial I_1}b^{\alpha\beta} + I_3\dfrac{\partial A_1}{\partial I_3}B^{\alpha\beta}\right). \end{array} \right\} \tag{6.7.4}$$

In the case of incompressible medium we arrive at formulae (6.2.7) and (6.2.8). For a very thin plate we have, due to eq. (6.7.1)

$$a_3\left(q^1, q^2, q^3\right) = a_3\left(q^1, q^2, 0\right) + \left(\frac{\partial a_3}{\partial q^3}\right)_{q^3=0} q^3 = \frac{1}{\lambda}q^3$$

and for $q^3 = x_3 = h, a_3 = h_0$

$$h = h_0\lambda. \tag{6.7.5}$$

With the same degree of accuracy, the components $\tilde{t}^{\alpha\beta}$ of the stress tensor in the mid-plane are related to the components of the averaged internal forces in the following way

$$\frac{\tilde{p}^{\alpha\beta}}{2h_0} = \frac{1}{2h_0}\int\limits_{-h}^{h}\tilde{t}^{\alpha\beta}dx_3 \approx \left(\tilde{t}^{\alpha\beta}\right)_{x_3=0}\frac{2h}{2h_0} = \lambda\left(\tilde{t}^{\alpha\beta}\right)_{x_3=0},$$

such that

$$\tilde{p}^{\alpha\beta} = 2h_0\lambda \left(\tilde{t}^{\alpha\beta}\right)_{q^3=0}. \tag{6.7.6}$$

By eqs. (6.5.13) and (6.7.5) we also have

$$x^3 = q^3 = \pm h: \quad \tilde{t}^{33} = \tilde{t}^{\alpha\beta}\frac{\partial h}{\partial q^\alpha}\frac{\partial h}{\partial q^\beta} \approx h_0^2 \tilde{t}^{\alpha\beta}\frac{\partial \lambda}{\partial q^\alpha}\frac{\partial \lambda}{\partial q^\beta},$$

and for $h_0 \to 0$ one can take

$$\left(\tilde{t}^{33}\right)_{q^3=0} = 0. \tag{6.7.7}$$

9.6.8 *System of equations in the problem of plane stress*

Under the assumption of small thickness of the plate and symmetric loading on its lateral surface, the problem reduces to considering values in the mid-plane. We seek the averaged values $\tilde{p}^{\alpha\beta}$ of the main stresses $\tilde{t}^{\alpha\beta}$ rather than the stress tensor. The remaining components $\tilde{t}^{\alpha 3}$, $\tilde{t}^{33}$ of tensor $\tilde{t}^{\alpha\beta}$ are excluded from consideration because of their smallness compared to the main ones.

Two sets of the relationships determining the symmetric tensor of the averaged stresses reduce to the equation of statics

$$\frac{\partial}{\partial q^\alpha}\sqrt{B}\tilde{p}^{\alpha\beta}\mathbf{B}_\beta = 0 \tag{6.8.1}$$

and the constitutive equation

$$\tilde{p}^{\alpha\beta} = \frac{2}{\sqrt{I_3}}2h_0\lambda\left(\frac{\partial A_1}{\partial I_1}b^{\alpha\beta} + I_3\frac{\partial A_1}{\partial I_3}B^{\alpha\beta}\right). \tag{6.8.2}$$

Here A_1 is the function of invariants I_1 and I_3

$$I_1 = b^{\alpha\beta}B_{\alpha\beta} + \lambda^2, \quad I_3 = \frac{B}{b}\lambda^2 \tag{6.8.3}$$

and presents the specific strain energy $A(I_1, I_2, I_3)$ in which invariant I_2 is eliminated by means of the equation

$$I_3 - I_2\lambda^2 + I_1\lambda^4 - \lambda^6 = 0. \tag{6.8.4}$$

The additional condition serving to determine unknown function $\lambda\left(q^1, q^2\right)$ expresses the requirement of vanishing stress $\tilde{t}^{33}$ in the mid-plane

$$\frac{\partial A}{\partial I_1}\lambda^2 + \frac{\partial A}{\partial I_2}\lambda^2\left(I_1 - \lambda^2\right) + I_3\frac{\partial A}{\partial I_3} = 0. \tag{6.8.5}$$

The boundary condition on contour Γ (the section of the plate by the mid-plane) is given by the relationship

$$\text{on} \quad \Gamma: \quad \tilde{\in}_{\alpha\gamma}\tilde{p}^{\alpha\beta}\mathbf{B}_\beta \frac{dq^\gamma}{dS} = \mathbf{f}, \tag{6.8.6}$$

where $\mathbf{f}$ denotes the principal vector of the surface forces on the lateral surface

$$\mathbf{f} = \int_{-h}^{h} \mathbf{F} dq^3, \quad h = h_0 \lambda. \tag{6.8.7}$$

The equation of statics can be satisfied by expressing the stresses in terms of Airy's stress function

$$\tilde{p}^{\mu\lambda} = \tilde{\in}^{\mu\alpha}\tilde{\in}^{\lambda\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U. \tag{6.8.8}$$

The inverse relationships have the form

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\beta U = \tilde{\in}_{\mu\alpha}\tilde{\in}_{\lambda\beta}\tilde{p}^{\mu\lambda} = 4h_0\sqrt{\frac{B}{b}}\left(\frac{\partial A_1}{\partial I_1} b_{\alpha\beta} + \lambda^2 \frac{\partial A_1}{\partial I_3} B_{\alpha\beta}\right). \tag{6.8.9}$$

The boundary conditions for the stress function are given by formulae (6.4.11)-(6.4.14).

In the case of the incompressible body

$$I_3 = \frac{B}{b}\lambda^2 = 1 \tag{6.8.10}$$

and the unknown scalar function p introduced instead of $\partial A/\partial I_3$ is determined from the equations of statics augmented by this condition.

9.6.9 Using the logarithmic measure in the problem of plane strain

In the plane field of displacements (6.2.1) of Chapter 2 the principal values of tensor $\hat{M}^{1/2}$ or $\hat{G}^{\times^{1/2}}$ are equal to $1 + \delta_\alpha = e^{\nu_\alpha}$ $(\alpha = 1, 2)$, $\delta_3 = 0, e^{\nu_3} = 1$, see Subsection 8.3.5. Let χ denote the angle of rotation between axes $OXYZ$ and the principal axes $\overset{s}{\mathbf{e}}$ of tensor $\hat{G}^{\times^{1/2}}$

$$\overset{1}{\mathbf{e}} = \mathbf{i}_1 \cos\chi + \mathbf{i}_2 \sin\chi, \quad \overset{2}{\mathbf{e}} = -\mathbf{i}_1 \sin\chi + \mathbf{i}_2 \cos\chi.$$

These axes become coincident with axes $\overset{1}{\tilde{\mathbf{e}}}, \overset{2}{\tilde{\mathbf{e}}}$ of tensor $\hat{M}^{1/2}$ by rotation through angle α such that

$$\begin{aligned}
&\overset{1}{\tilde{\mathbf{e}}} = \overset{1}{\mathbf{e}} \cos\alpha + \overset{2}{\mathbf{e}} \sin\alpha, && \overset{2}{\tilde{\mathbf{e}}} = -\overset{1}{\mathbf{e}} \sin\alpha + \overset{2}{\mathbf{e}} \cos\alpha, \\
&\overset{1}{\tilde{\mathbf{e}}} = \mathbf{i}_1 \cos(\chi + \alpha) + \mathbf{i}_2 \sin(\chi + \alpha), && \overset{1}{\tilde{\mathbf{e}}} = -\mathbf{i}_1 \sin(\chi + \alpha) + \mathbf{i}_2 \cos(\chi + \alpha).
\end{aligned}$$

At the same time, by eq. (3.4.5) of Chapter 2 we have

$$e^{\nu_k} \overset{k}{\tilde{\mathbf{e}}} = \overset{k}{\mathbf{e}} \cdot \nabla \mathbf{R} = \overset{k}{\mathbf{e}} \cdot \mathbf{i}_s \mathbf{i}_t \lambda_{ts} \qquad \left(\lambda_{ts} = \frac{\partial x_t}{\partial a_s} \right)$$

or

$$e^{\nu_1} \overset{1}{\tilde{\mathbf{e}}} \cdot \mathbf{i}_t = \overset{1}{\mathbf{e}} \cdot \mathbf{i}_s \lambda_{ts}, \quad e^{\nu_2} \overset{2}{\tilde{\mathbf{e}}} \cdot \mathbf{i}_t = \overset{2}{\mathbf{e}} \cdot \mathbf{i}_s \lambda_{ts}.$$

We arrive at the relationships

$$\begin{aligned} e^{\nu_1} \cos(\chi + \alpha) &= \lambda_{11} \cos \chi + \lambda_{12} \sin \chi, \\ e^{\nu_2} \sin(\chi + \alpha) &= \lambda_{11} \sin \chi - \lambda_{12} \cos \chi, \\ e^{\nu_2} \sin(\chi + \alpha) &= \lambda_{21} \cos \chi + \lambda_{22} \sin \chi, \\ e^{\nu_2} \cos(\chi + \alpha) &= -\lambda_{21} \sin \chi + \lambda_{22} \cos \chi, \end{aligned}$$

enabling expressions for λ_{ts} to be written in terms of four invariant parameters $e^{\nu_1}, e^{\nu_2}, \chi, \alpha$

$$\left. \begin{aligned} \lambda_{11} &= \frac{1}{2} \left[\left(e^{\nu_1} + e^{\nu_2} \right) \cos \alpha + \left(e^{\nu_1} - e^{\nu_2} \right) \cos (2\chi + \alpha) \right], \\ \lambda_{22} &= \frac{1}{2} \left[\left(e^{\nu_1} + e^{\nu_2} \right) \cos \alpha - \left(e^{\nu_1} - e^{\nu_2} \right) \cos (2\chi + \alpha) \right], \\ \lambda_{21} &= \frac{1}{2} \left[\left(e^{\nu_1} + e^{\nu_2} \right) \sin \alpha + \left(e^{\nu_1} - e^{\nu_2} \right) \sin (2\chi + \alpha) \right], \\ \lambda_{12} &= \frac{1}{2} \left[- \left(e^{\nu_1} + e^{\nu_2} \right) \sin \alpha + \left(e^{\nu_1} - e^{\nu_2} \right) \sin (2\chi + \alpha) \right]. \end{aligned} \right\} \tag{6.9.1}$$

These parameters are related by the integrability conditions

$$\frac{\partial \lambda_{11}}{\partial a_2} = \frac{\partial \lambda_{12}}{\partial a_1}, \quad \frac{\partial \lambda_{21}}{\partial a_2} = \frac{\partial \lambda_{22}}{\partial a_1}. \tag{6.9.2}$$

Let us also notice that the representations of the components of tensor $\hat{M}^{1/2}$ in axes $OXYZ$ are given by the formulae, see for example (A.3.14)

$$\begin{aligned} M_{11}^{1/2} &= \frac{1}{2} \left[\left(e^{\nu_1} + e^{\nu_2} \right) + \left(e^{\nu_1} - e^{\nu_2} \right) \cos 2 (\chi + \alpha) \right], \\ M_{22}^{1/2} &= \frac{1}{2} \left[\left(e^{\nu_1} + e^{\nu_2} \right) - \left(e^{\nu_1} - e^{\nu_2} \right) \cos 2 (\chi + \alpha) \right], \\ M_{12}^{1/2} &= \frac{1}{2} \left(e^{\nu_1} - e^{\nu_2} \right) \sin 2 (\chi + \alpha), \quad M_{33}^{1/2} = 1. \end{aligned}$$

The logarithmic strain measure $\hat{N}$ having the same principal axes is determined by the components

$$\left. \begin{aligned} N_{11} &= \frac{1}{2} \left[(\nu_1 + \nu_2) + (\nu_1 - \nu_2) \cos 2 (\chi + \alpha) \right], \\ N_{22} &= \frac{1}{2} \left[(\nu_1 + \nu_2) - (\nu_1 - \nu_2) \cos 2 (\chi + \alpha) \right], \\ N_{12} &= \frac{1}{2} (\nu_1 - \nu_2) \sin 2 (\chi + \alpha), \quad N_{33} = 0. \end{aligned} \right\} \tag{6.9.3}$$

In the case of plane strain of the incompressible material $\nu_1 + \nu_2 = 0$, $\nu_1 = -\nu_2 = \nu > 0$ and the above expressions reduce to the form

$$\left.\begin{array}{c} N_{11} = \nu\cos 2(\chi+\alpha), \quad N_{22} = -\nu\cos 2(\chi+\alpha), \\ N_{12} = \nu\sin 2(\chi+\alpha), \end{array}\right\} \tag{6.9.4}$$

whereas values $\lambda_{\alpha\beta}$ are set in the following way

$$\left.\begin{aligned} \lambda_{11} &= \cosh\nu\cos\alpha + \sinh\nu\cos(2\chi+\alpha), \\ \lambda_{22} &= \cosh\nu\cos\alpha - \sinh\nu\cos(2\chi+\alpha), \\ \lambda_{21} &= \cosh\nu\sin\alpha + \sinh\nu\sin(2\chi+\alpha), \\ \lambda_{12} &= -\cosh\nu\sin\alpha + \sinh\nu\sin(2\chi+\alpha). \end{aligned}\right\} \tag{6.9.5}$$

The components of the stress tensor $\hat{T}$ for the incompressible material with zero angle of similarity ω are, in accordance with eq. (3.5.14) of Chapter 8, given by

$$\left.\begin{aligned} t_{11} &= p + 2\mu(\Gamma)\,\nu\cos 2(\chi+\alpha), \\ t_{22} &= p - 2\mu(\Gamma)\,\nu\cos 2(\chi+\alpha), \\ t_{12} &= 2\mu(\Gamma)\,\nu\sin 2(\chi+\alpha), \\ t_{33} &= p \quad (\Gamma = 2\nu). \end{aligned}\right\} \tag{6.9.6}$$

In order to determine four unknown parameters we have the same number of equations, which are two equations of statics and two integrability conditions (6.9.2).

9.6.10 Plane strain of incompressible material with zero angle of similarity of deviators

This problem is the subject of the paper by L.A. Tolokonnikov[2].

In the case of no mass forces, the equations of statics can be expressed in the form

$$\frac{\partial}{\partial z}(t_{11} - t_{22} + 2it_{12}) + \frac{\partial}{\partial \bar{z}}(t_{11} + t_{22}) = 0 \quad (z = x_1 + ix_2),$$

where the coordinates of the point are taken as the independent variables. Turning to formulae (6.9.6) we obtain

$$\frac{\partial}{\partial z} f(\nu)\, e^{2i(\chi+\alpha)} + \frac{\partial p}{\partial \bar{z}} = 0 \quad (f(\nu) = 2\nu\mu(\Gamma)). \tag{6.10.1}$$

This equation can be satisfied by assuming

$$f(\nu) = -2\frac{\partial \Phi}{\partial \bar{z}} e^{-2i(\chi+\alpha)}, \quad p = 2\frac{\partial \Phi}{\partial z}, \tag{6.10.2}$$

[2]Tolokonnikov, L.A. "Finite plane strains of incompressible material" (in Russian), Prikladnaya Matematika i Mekhanika, vol. 23, No.1, 1959, pp.146-158.

and introducing a real-valued function Φ

$$\Phi = \frac{\partial U}{\partial \bar{z}}, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial^2 U}{\partial z \partial \bar{z}},$$

since p is real-valued. We arrive at the relations which could be foreseen

$$\left.\begin{aligned} 2f(\nu)\, e^{2i(\chi+\alpha)} &= t_{11} - t_{22} + 2it_{12} \\ &= -4\frac{\partial^2 U}{\partial \bar{z}^2} = -\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - 2i\frac{\partial^2 U}{\partial x \partial y}, \\ 2p = t_1 + t_2 &= 4\frac{\partial^2 U}{\partial z \partial \bar{z}} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}, \end{aligned}\right\} \tag{6.10.3}$$

and function Φ is proved to be Airy's function of the plane problem

$$\left.\begin{aligned} t_{11} &= p + f(\nu)\cos 2(\chi+\alpha) = \frac{\partial^2 U}{\partial y^2}, \\ t_{22} &= p - f(\nu)\cos 2(\chi+\alpha) = \frac{\partial^2 U}{\partial x^2}, \\ t_{12} &= f(\nu)\sin 2(\chi+\alpha) = -\frac{\partial^2 U}{\partial x \partial y}. \end{aligned}\right\} \tag{6.10.4}$$

A consequence of the first relationship in eq. (6.10.3) is the formula relating ν to the stress function

$$f^2(\nu) = 4\frac{\partial^2 U}{\partial z^2}\frac{\partial^2 U}{\partial \bar{z}^2}. \tag{6.10.5}$$

It allows us to find the derivatives of ν with respect to z and $\bar{z}$

$$f(\nu) f'(\nu)\frac{\partial \nu}{\partial z} = 2\frac{\partial}{\partial z}\left(\frac{\partial^2 U}{\partial z^2}\frac{\partial^2 U}{\partial \bar{z}^2}\right), \quad f(\nu) f'(\nu)\frac{\partial \nu}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\left(\frac{\partial^2 U}{\partial z^2}\frac{\partial^2 U}{\partial \bar{z}^2}\right). \tag{6.10.6}$$

From the same relationship in eq. (6.10.3) we have

$$e^{2i(\chi+\alpha)} = -\frac{2}{f(\nu)}\frac{\partial^2 U}{\partial \bar{z}^2} \tag{6.10.7}$$

and taking into account eqs. (6.10.5) and (6.10.6)

$$\frac{\partial e^{2i(\chi+\alpha)}}{\partial z} = \frac{1}{2i}\frac{\partial}{\partial z}\ln\frac{\dfrac{\partial^2 U}{\partial \bar{z}^2}}{\dfrac{\partial^2 U}{\partial z^2}} \tag{6.10.8}$$

and a similar equation for the derivative with respect to $\bar{z}$.

Returning to relationships (6.9.5) we present them as follows

$$
\left.
\begin{aligned}
&\frac{\partial z}{\partial \zeta} = \frac{1}{2}\left[(\lambda_{11}+\lambda_{22}) + i(\lambda_{21}-\lambda_{12})\right] = \cosh\nu e^{i\alpha}, \\
&\frac{\partial z}{\partial \bar\zeta} = \frac{1}{2}\left[(\lambda_{11}-\lambda_{22}) + i(\lambda_{21}+\lambda_{12})\right] = \sinh\nu e^{i(2\chi+\alpha)}, \\
&\frac{\partial \bar z}{\partial \bar\zeta} = \cosh\nu e^{-i\alpha}, \quad \frac{\partial \bar z}{\partial \zeta} = \sinh\nu e^{-i(2\chi+\alpha)},
\end{aligned}
\right\} \tag{6.10.9}
$$

where $\zeta = a_1 + ia_2$. From these equations we obtain the inverse relationships

$$
\left.
\begin{aligned}
&\frac{\partial \zeta}{\partial z} = \cosh\nu e^{-i\alpha}, && \frac{\partial \bar\zeta}{\partial \bar z} = \cosh\nu e^{i\alpha}, \\
&\frac{\partial \zeta}{\partial \bar z} = -\sinh\nu e^{i(2\chi+\alpha)}, && \frac{\partial \bar\zeta}{\partial z} = -\sinh\nu e^{-i(2\chi+\alpha)}.
\end{aligned}
\right\} \tag{6.10.10}
$$

Writing down the integrability conditions

$$
\frac{\partial^2 \zeta}{\partial z \partial \bar z} = \frac{\partial^2 \zeta}{\partial \bar z \partial z}, \quad \frac{\partial^2 \bar\zeta}{\partial z \partial \bar z} = \frac{\partial^2 \bar\zeta}{\partial \bar z \partial z},
$$

we arrive at the following two equations

$$
\left.
\begin{aligned}
&\sinh\nu \frac{\partial \nu}{\partial \bar z} - i\cosh\nu \frac{\partial \alpha}{\partial \bar z} = \\
&\qquad = -\left[\cosh\nu \frac{\partial \nu}{\partial z} + i\sinh\nu\left(\frac{\partial 2(\chi+\alpha)}{\partial z} - \frac{\partial \alpha}{\partial z}\right)\right] e^{2i(\chi+\alpha)}, \\
&\sinh\nu \frac{\partial \nu}{\partial z} + i\cosh\nu \frac{\partial \alpha}{\partial z} = \\
&\qquad = -\left[\cosh\nu \frac{\partial \nu}{\partial \bar z} - i\sinh\nu\left(\frac{\partial 2(\chi+\alpha)}{\partial \bar z} - \frac{\partial \alpha}{\partial \bar z}\right)\right] e^{-2i(\chi+\alpha)}.
\end{aligned}
\right\} \tag{6.10.11}
$$

The values in these equations

$$
e^{\pm 2i(\chi+\alpha)}, \quad \frac{\partial \nu}{\partial z}, \quad \frac{\partial \nu}{\partial \bar z}, \quad \frac{\partial 2(\chi+\alpha)}{\partial z}, \quad \frac{\partial 2(\chi+\alpha)}{\partial \bar z} \tag{6.10.12}
$$

should be replaced according to eq. (6.10.6)-(6.10.8). They allow the derivatives of α with respect to z and $\bar z$ to be expressed in terms of the derivatives of the stress function and ν. The integrability condition

$$
\frac{\partial^2 \alpha}{\partial z \partial \bar z} = \frac{\partial^2 \alpha}{\partial \bar z \partial z} \tag{6.10.13}
$$

leads to the differential equation of fourth order for the stress function U. This equation also contains quantity ν however the latter relates to U by relationship (6.10.5).

Transformation to the original independent variables ζ and $\bar{\zeta}$ is carried out with the help of the relationships

$$\frac{\partial U}{\partial \zeta} = \left[\frac{\partial U}{\partial z} \cosh \nu + \frac{\partial U}{\partial \bar{z}} \sinh \nu e^{-2i(\chi+\alpha)} \right] e^{i\alpha} \quad \text{etc.}$$

by further replacing the derivatives (6.10.12) by means of eqs. (6.10.6)-(6.10.8).

Transformation of the boundary conditions presents no difficulty and, for this reason, is not presented here.

9.6.11 Example of radially symmetric deformation

Under this deformation

$$z = R\left(\rho^2\right)\zeta = R\left(\zeta\bar{\zeta}\right)\zeta \quad \left(\zeta = \rho e^{i\theta} = a_1 + ia_2\right), \tag{6.11.1}$$

where $R\left(\rho^2\right)$ is a real-valued function. By eq. (6.10.9) we have

$$\begin{aligned}
\frac{\partial z}{\partial \zeta} &= R'\left(\rho^2\right)\rho^2 + R\left(\rho^2\right) = \cosh \nu e^{i\alpha}, \\
\frac{\partial z}{\partial \bar{\zeta}} &= R'\left(\rho^2\right)\zeta^2 = R'\left(\rho^2\right)\rho^2 \frac{\zeta}{\bar{\zeta}} = \sinh \nu e^{i(2\chi+\alpha)},
\end{aligned}$$

such that

$$\alpha = 0,\ R'\left(\rho^2\right)\rho^2 + R\left(\rho^2\right) = \cosh \nu,\ R'\left(\rho^2\right)\rho^2 = \sinh \nu,\ e^{2i\chi} = \frac{\zeta}{\bar{\zeta}} = e^{2i\theta}. \tag{6.11.2}$$

Then we obtain

$$2R\left(\rho^2\right)R'\left(\rho^2\right)\rho^2 + R^2\left(\rho^2\right) = 1, \quad R^2 = 1 - \frac{C}{\rho^2} \tag{6.11.3}$$

and further

$$R\left(\rho^2\right) = e^{-\nu} = \sqrt{1 - \frac{C}{\rho^2}}, \tag{6.11.4}$$

such that $0 < C < \rho_0^2$, where ρ_0 denotes the radius of the opening in the deformed cylindrical body.

Referring, for example, to formula (1.13.8) of Chapter 8 we have by eqs. (6.10.3) and (6.11.2)

$$t_r - t_\theta + 2it_{r\theta} = 2f(\nu) = -4\frac{\bar{\zeta}}{\zeta}\frac{\partial^2 U}{\partial \bar{z}^2},\ t_r - t_\theta = 2f(\nu) =,\ t_{r\theta} = 0 \tag{6.11.5}$$

and further

$$\frac{\partial U}{\partial \bar{z}} = \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial \bar{z}} + \frac{\partial U}{\partial \bar{\zeta}}\frac{\partial \bar{\zeta}}{\partial \bar{z}} = U'\left(-\bar{\zeta}e^{2i\theta}\sinh\nu + \zeta\cosh\nu\right) = \zeta e^{\nu}U'. \quad (6.11.6)$$

Here, as above, a prime denotes the derivative with respect to $\rho^2 = \zeta\bar{\zeta}$. Now we find

$$\begin{aligned}\frac{\partial^2 U}{\partial \bar{z}^2} &= \zeta^2 e^{2\nu}U'' + U'\left[\frac{\partial \zeta}{\partial \bar{z}}e^{\nu} + \zeta e^{\nu}\left(\frac{\partial \nu}{\partial \zeta}\frac{\partial \zeta}{\partial \bar{z}} + \frac{\partial \nu}{\partial \bar{\zeta}}\frac{\partial \bar{\zeta}}{\partial \bar{z}}\right)\right] \\ &= \zeta^2 e^{2\nu}U'' - e^{\nu}U'\left[e^{2i\theta}\sinh\nu + \frac{\zeta^2 C e^{\nu}}{2\rho^4\left(1-\dfrac{C}{\rho^2}\right)}\right],\end{aligned}$$

and the differential equation of equilibrium (6.11.5) takes the form

$$\rho^2 e^{2\nu}U'' - e^{\nu}\left[\sinh\nu + \frac{e^{\nu}C}{2\rho^2\left(1-\dfrac{C}{\rho^2}\right)}\right]U' = -\frac{1}{2}f(\nu).$$

After replacing ρ^2 according to eq. (6.11.4)

$$U'' - \frac{1}{C}\left(1-e^{-2\nu}\right)\sinh 2\nu U' = -\frac{1}{2C}f(\nu)\,e^{-2\nu}\left(1-e^{-2\nu}\right) \quad (6.11.7)$$

and introducing the new independent variable

$$q = e^{-2\nu} = 1 - \frac{C}{\rho^2}$$

instead of q, we arrive at the differential equation for U'

$$\frac{dU'}{dq} - \frac{1+q}{q}U' = -\frac{1}{2}f(\nu)\frac{q}{1-q},$$

which can be easily integrated. The result is

$$\frac{dU}{d\rho^2} = U' = -\frac{1}{2}qe^{q}\left[C_1 + \int^{q} f(\nu)\frac{e^{-q}}{1-q}dq\right], \quad (6.11.8)$$

which allows one to determine $\dfrac{\partial U}{\partial \bar{z}}$ by eq. (6.11.6) and then $\dfrac{\partial^2 U}{\partial z \partial \bar{z}}$. By eq. (6.10.4) the latter is a sum of the normal stresses, i.e. one determines stresses t_r and t_θ. We omit this calculation as well as the calculation of the constants in terms of the boundary conditions[3].

[3] see the paper by V.G. Gromov in the book by Savin, G.N. "Distribution of stresses near openings" (in Russian), Chapter 9, pp. 676-691, Naukova Dumka, Kiev, 1968 for detail.

9.7 Semi-linear material

9.7.1 *Equilibrium equations for the semi-linear material*

The specific strain energy of the "semi-linear" or "harmonic" material introduced in Subsection 8.2.8 is given by eq. (2.8.7) of Chapter 8. The constitutive law, eq. (2.8.8) of Chapter 8, relates the Piola-Kirchhoff stress tensor $\hat{D}$ to the quantities characterising deformation, namely the tensor of rotation $\hat{A}$ of the principal axes of the strain measure $\hat{G}^{\times}$ and the tensor-gradient $\nabla\mathbf{R}$

$$\hat{D} = (\lambda s_1 - 2\mu)\,\hat{A} + 2\mu\nabla\mathbf{R}. \tag{7.1.1}$$

The equilibrium equation (2.8.4) of Chapter 8 for Piola's tensor written in the vector basis of the initial volume is set in the form

$$(\lambda s_1 - 2\mu)\,\nabla\cdot\hat{A} + \lambda\hat{A}^*\cdot\nabla s_1 + 2\mu\nabla^2\mathbf{R} + \rho_0\mathbf{K} = 0, \tag{7.1.2}$$

where an asterisk denotes transposition of the tensor. This equation is an analogue of the equilibrium equation in terms of displacements of the linear theory of elasticity. When the surface forces $\mathbf{F}$ are prescribed, the boundary condition takes the form

$$(\lambda s_1 - 2\mu)\,\mathbf{n}\cdot\hat{A} + 2\mu\mathbf{n}\cdot\nabla\mathbf{R} = \mathbf{F}\frac{dO}{do} = \mathbf{F}\left(\frac{G}{g}\mathbf{n}\cdot\hat{G}^{\times^{-1}}\cdot\mathbf{n}\right)^{1/2}. \tag{7.1.3}$$

Hence, the equations of statics in the volume and on the surface are presented in the basis of the initial state and this explains the simplification due to applying the Piola-Kirchhoff stress tensor in problems of the nonlinear theory of elasticity. However the complication is that this tensor contains the rotation tensor $\hat{A}$ and invariant s_1. Their representation requires tensors $\hat{G}^{\times^{1/2}}$ and $\hat{G}^{\times^{-1/2}}$

$$\hat{A} = \hat{G}^{\times^{-1/2}}\cdot\nabla\mathbf{R}, \quad s_1 = \delta_1 + \delta_2 + \delta_3 = I_1\left(\hat{G}^{\times^{1/2}}\right) - 3, \tag{7.1.4}$$

and obtaining these tensors assumes the principal values G_s and the principal directions $\overset{s}{\mathbf{e}}$ of the strain measure $\hat{G}^{\times}$

$$\hat{G}^{\times} = G_1\,\overset{11}{\mathbf{ee}} + G_2\,\overset{22}{\mathbf{ee}} + G_3\,\overset{33}{\mathbf{ee}}\,. \tag{7.1.5}$$

9.7.2 *Conserving the principal directions*

The mentioned complications are no longer relevant if the principal directions of tensors $\hat{G}^{\times}$ and $\hat{T}$ (or $\hat{G}^{\times}$ and $\hat{M}$) are coincident. Then $\overset{s}{\mathbf{e}} = \overset{s}{\tilde{\mathbf{e}}}$ and

$$\hat{A} = \overset{s}{\mathbf{e}}\overset{s}{\tilde{\mathbf{e}}} = \hat{g}, \quad \nabla\mathbf{R} = \hat{G}^{\times^{1/2}} = \nabla\mathbf{R}^*, \quad s_1 = \nabla\cdot\mathbf{R} - 3, \tag{7.2.1}$$

where $\hat{g}$ denotes the unit tensor in $v-$volume. The equilibrium equation (7.1.2) which is linear in vector $\mathbf{R}$ takes the form

$$\lambda\nabla\nabla\cdot\mathbf{R} + 2\mu\nabla^2\mathbf{R} + \rho_0\mathbf{K} = 0, \tag{7.2.2}$$

and nonlinearity of the problem is only because of the right hand side of the boundary condition

$$[\lambda\nabla\cdot\mathbf{R} - (3\lambda + 2\mu)]\,\mathbf{n} + 2\mu\mathbf{n}\cdot\nabla\mathbf{R} = \mathbf{F}\frac{dO}{do}. \tag{7.2.3}$$

Replacing in these equations vector $\mathbf{R}$ by the displacement vector $\mathbf{u} = \mathbf{R} - \mathbf{r}$ and taking into account $\nabla\mathbf{u} = \nabla\mathbf{u}^*$, we arrive at the equations

$$(\lambda + 2\mu)\,\nabla^2\mathbf{u} + \rho_0\mathbf{K} = 0, \quad \lambda\mathbf{n}\nabla\cdot\mathbf{u} + 2\mu\mathbf{n}\cdot\nabla\mathbf{u} = \mathbf{F}\frac{dO}{do}. \tag{7.2.4}$$

The left hand sides of these equations present a particular case of the equilibrium equations in terms of displacements of the linear theory of elasticity (here $\nabla\nabla\cdot\mathbf{u} = \nabla^2\mathbf{u}$).

9.7.3 Examples: cylinder and sphere

Two cases of conserving the principal directions take place for the case of axially symmetric deformation of a round cylinder and radially symmetric deformation of a sphere.

1. *Cylinder.* Introducing the cylindric coordinates r, θ, z and assuming that the axial displacement does not depend on r we have

$$\left.\begin{array}{c} \mathbf{R} = f(r)\,\mathbf{e}_r + \mathbf{k}\alpha z,\ \mathbf{R}_1 = f'(r)\,\mathbf{e}_r,\ \mathbf{R}_2 = f(r)\,\mathbf{e}_\theta,\ \mathbf{R}_3 = \alpha\mathbf{k}, \\ \mathbf{r}' = \mathbf{e}_r, \quad \mathbf{r}^2 = \dfrac{1}{r}\mathbf{e}_\theta, \quad \mathbf{r}^3 = \mathbf{k}. \end{array}\right\} \tag{7.3.1}$$

Hence

$$\left.\begin{array}{c} \nabla\mathbf{R} = \mathbf{e}_r\mathbf{e}_r f'(r) + \mathbf{e}_\theta\mathbf{e}_\theta\dfrac{f(r)}{r} + \mathbf{k}\mathbf{k}\alpha = \nabla\mathbf{R}^* = \hat{\mathbf{G}}^{\times^{1/2}}, \\ \hat{G}^\times = \mathbf{e}_r\mathbf{e}_r f'^2(r) + \mathbf{e}_\theta\mathbf{e}_\theta\dfrac{f^2(r)}{r^2} + \mathbf{k}\mathbf{k}\alpha = \hat{g}^{\times^{-1}}, \end{array}\right\} \tag{7.3.2}$$

such that the principal axes of tensors $\hat{\mathbf{G}}^\times$ and $\hat{g}^\times$ coincide with the directions of vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$. Then

$$\nabla\cdot\mathbf{R} = f' + \frac{f}{r} + \alpha, \quad \nabla\nabla\cdot\mathbf{R} = \mathbf{e}_r\left(f' + \frac{f}{r}\right)' = \nabla\cdot\nabla\mathbf{R} = \nabla^2\mathbf{R}$$

and when the mass forces are absent the equilibrium equation (7.2.2) reduces to the relationship

$$\left(f' + \frac{f}{r}\right)' = 0, \quad f' + \frac{f}{r} = 2C_1, \quad f = C_1 r + \frac{C_2}{r}. \tag{7.3.3}$$

The constants C_1, C_2, α are determined in terms of the pressures prescribed on the external $r = r_0$ and internal $r = r_1$ surfaces of the hollow cylinder and the axial force Q on the end face S. We arrive at three equations which can be written down in the form

$$\left.\begin{aligned} &2(\lambda+\mu)(C_1-1)+\lambda(\alpha-1)-2\mu\frac{C_2}{r_0^2} = -p_0\left(C_1+\frac{C_2}{r_0^2}\right)\alpha, \\ &2(\lambda+\mu)(C_1-1)+\lambda(\alpha-1)-2\mu\frac{C_2}{r_1^2} = p_1\left(C_1+\frac{C_2}{r_1^2}\right)\alpha, \\ &2\lambda(C_1-1)+(\lambda+2\mu)(\alpha-1) = \frac{Q}{\pi\left(r_0^2-r_1^2\right)} \qquad \left(\mathbf{k}Q = \iint \mathbf{F}\,dO\right). \end{aligned}\right\} \tag{7.3.4}$$

For example in the case of the cylinder placed between two motionless smooth plates ($\alpha = 1$) and loaded by the uniform pressure on the external surface ($p_0 = p, p_1 = 0$) we have

$$f(r_1) = C_1 r_1 + \frac{C_2}{r_1} = \frac{(\lambda+\mu)(2\mu-p)(1-k)}{[\mu+(\lambda+\mu)k]\,p+2\mu(\lambda+\mu)(1-k)}r_1 \quad \left(k = \frac{r_1^2}{r_0^2}\right)$$

and the internal radius of the deformed cylinder $f(r_1) \to 0$ when $p \to 2\mu$.

2. *Sphere.* In the case of the centrally symmetric deformation of a sphere we introduce the spherical coordinates R, ϑ, λ, to get

$$\left.\begin{aligned} &\mathbf{R} = f(R)\,\mathbf{e}_R,\ \mathbf{R}_1 = f'(R)\,\mathbf{e}_R,\ \mathbf{R}_2 = f(R)\,\mathbf{e}_\vartheta,\ \mathbf{R}_3 = f(R)\,\mathbf{e}_\lambda \sin\vartheta, \\ &\qquad \mathbf{r}^1 = \mathbf{e}_R, \quad \mathbf{r}^2 = \frac{\mathbf{e}_\vartheta}{R}, \quad \mathbf{r}^3 = \frac{\mathbf{e}_\lambda}{R\sin\vartheta}, \end{aligned}\right\} \tag{7.3.5}$$

such that

$$\begin{aligned} \nabla\mathbf{R} &= f'(R)\,\mathbf{e}_R\mathbf{e}_R + \frac{f(R)}{R}\left(\mathbf{e}_\vartheta\mathbf{e}_\vartheta + \mathbf{e}_\lambda\mathbf{e}_\lambda\right) \\ &= \left[f'(R) - \frac{f(R)}{R}\right]\mathbf{e}_R\mathbf{e}_R + \hat{g}\frac{f(R)}{R}, \end{aligned}$$

where $\hat{g}$ denotes the unit tensor. The principal axes of the tensor

$$\hat{G}^\times = f'^2(R)\,\mathbf{e}_R\mathbf{e}_R + \frac{f^2(R)}{R^2}\left(\mathbf{e}_\vartheta\mathbf{e}_\vartheta + \mathbf{e}_\lambda\mathbf{e}_\lambda\right) = \left(f'^2 - \frac{f^2}{R^2}\right)\mathbf{e}_R\mathbf{e}_R + \frac{f^2}{R^2}\hat{g} \tag{7.3.6}$$

also have the directions $\mathbf{e}_R, \mathbf{e}_\vartheta, \mathbf{e}_\lambda$ conserved under the deformation. We obtain

$$\left.\begin{aligned} &\nabla\cdot\mathbf{R} = f'(R) + 2\frac{f(R)}{R}, \quad \nabla\nabla\cdot\mathbf{R} = \left(f' + 2\frac{f}{R}\right)'\mathbf{e}_R, \\ &\nabla\cdot\nabla\mathbf{R} = \nabla^2\mathbf{R} = \mathbf{e}_R\left(f' + 2\frac{f}{R}\right)' = \nabla\nabla\cdot\mathbf{R} \end{aligned}\right\} \tag{7.3.7}$$

and by eq. (7.2.2)

$$f'(R) + 2\frac{f(R)}{R} = 3c_1, \quad f(R) = c_1 R + \frac{c_2}{R^2}. \tag{7.3.8}$$

Next, we have by eq. (7.2.3)

$$\frac{dO}{do} = \frac{f^2}{R^2}, \quad \left[(3\lambda + 2\mu)(c_1 - 1) - \frac{4\mu c_2}{R^3} \right] \frac{1}{\left(c_1 + \frac{c_2}{R^3} \right)^2} = \sigma_R, \tag{7.3.9}$$

and the constants c_1 and c_2 are obtained in terms of the prescribed pressure of the external ($R = R_0$) and internal ($R = R_1$) surfaces of the hollow sphere. For instance, in the case of only external pressure ($p_0 = p, p_1 = 0$) we arrive at the equation

$$p = \frac{(3\lambda + 2\mu)(1 - c_1)(1 - k)}{\left[c_1 + \frac{c_1 - 1}{4\mu}(3\lambda + 2\mu) k \right]^2} \quad \left(k = \frac{R_1^3}{R_0^3} \right). \tag{7.3.10}$$

The internal radius of the deformed sphere is given by the equality

$$f(R_1) = \frac{R_1}{4\mu} \left[4\mu c_1 + (3\lambda + 2\mu)(c_1 - 1) \right],$$

and since $p > 0, f(R_1) \geq 0$ we have

$$0 \leq 1 - c_1 \leq \frac{4\mu}{3(\lambda + 2\mu)}.$$

The obtained solution is realised for the following values of external pressure

$$0 \leq p \leq \frac{12\mu(\lambda + 2\mu)}{(1 - k)(3\lambda + 2\mu)}.$$

9.7.4 Plane strain

We consider the field of displacements in which the Cartesian coordinates x_s of the point in the deformed prismatic body are related to its Cartesian coordinates a_s in the initial state by the following relationships

$$x_\alpha = x_\alpha(a_1, a_2), \quad \alpha = 1, 2; \quad x_3 = c a_3. \tag{7.4.1}$$

The unit vectors $\mathbf{i}_s$ of the coordinate axes present the vector basis $\mathbf{r}_s$ and cobasis $\mathbf{r}^s$ of the initial state. In the final state the vector basis and the tensor-gradient $\nabla\mathbf{R}$ are given by the equalities

$$\mathbf{R}_\alpha = \mathbf{i}_\beta \frac{\partial x_\beta}{\partial a_\alpha}, \quad \mathbf{R}_3 = c\mathbf{i}_3; \quad \nabla\mathbf{R} = \mathbf{i}_\alpha \mathbf{i}_\beta \frac{\partial x_\beta}{\partial a_\alpha} + c\mathbf{i}_3\mathbf{i}_3,$$

where the Greek indices take values of 1 and 2. Now we have

$$\hat{G}^{\times} = \nabla\mathbf{R}\cdot\nabla\mathbf{R}^{*} = \mathbf{i}_\alpha\mathbf{i}_\beta\frac{\partial x_\gamma}{\partial a_\alpha}\frac{\partial x_\gamma}{\partial a_\beta} + c^2\mathbf{i}_3\mathbf{i}_3,$$

$$G_{\alpha\beta} = \frac{\partial x_\gamma}{\partial a_\alpha}\frac{\partial x_\gamma}{\partial a_\beta}, \quad G_{\alpha 3} = 0, \quad G_{33} = c^2,$$

$$G = c^2\left(G_{11}G_{22} - G_{12}^2\right) = c^2\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} - \frac{\partial x_1}{\partial a_2}\frac{\partial x_2}{\partial a_1}\right)^2.$$

Tensors $\hat{G}^{\times^{1/2}}$ and $\hat{A}$ are calculated as shown in Subsection 2.6.2

$$\left.\begin{aligned} \hat{G}^{\times^{1/2}} &= \left(\mathbf{i}_\alpha\mathbf{i}_\beta\cos\chi - \mathbf{i}_3\times\mathbf{i}_\alpha\mathbf{i}_\beta\sin\chi\right)\frac{\partial x_\alpha}{\partial a_\beta} + c\mathbf{i}_3\mathbf{i}_3, \\ \hat{A} &= \left(\mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2\right)\cos\chi + \left(\mathbf{i}_1\mathbf{i}_2 - \mathbf{i}_2\mathbf{i}_1\right)\sin\chi + \mathbf{i}_3\mathbf{i}_3. \end{aligned}\right\} \tag{7.4.2}$$

Here

$$\left.\begin{aligned} \cos\chi &= \frac{1}{q}\left(\frac{\partial x_1}{\partial a_1} + \frac{\partial x_2}{\partial a_2}\right), \quad \sin\chi = \frac{1}{q}\left(\frac{\partial x_2}{\partial a_1} - \frac{\partial x_1}{\partial a_2}\right), \\ q &= \sqrt{\left(\frac{\partial x_1}{\partial a_1} + \frac{\partial x_2}{\partial a_2}\right)^2 + \left(\frac{\partial x_2}{\partial a_1} - \frac{\partial x_1}{\partial a_2}\right)^2}, \end{aligned}\right\} \tag{7.4.3}$$

and

$$I_1\left(\hat{G}^{\times^{1/2}}\right) = q + c, \quad s_1 = q + c - 3. \tag{7.4.4}$$

Next we introduce complex notation for the coordinates of the point in the cross-section of the body

$$\zeta = a_1 + ia_2, \quad z = x_1 + ix_2.$$

This allows us to write formula (7.4.3) in the form

$$qe^{i\chi} = 2\frac{\partial z}{\partial\zeta}, \quad e^{2i\chi} = \frac{\partial z}{\partial\zeta}\left(\frac{\partial\bar{z}}{\partial\bar{\zeta}}\right)^{-1}, \quad q = 2\left|\frac{\partial z}{\partial\zeta}\right| = 2\left(\frac{\partial z}{\partial\zeta}\frac{\partial\bar{z}}{\partial\bar{\zeta}}\right)^{1/2}. \tag{7.4.5}$$

Tensor $\hat{D}$, by eq. (7.1.1), is led to the form

$$\hat{D} = \left[\lambda q - 2(\lambda+\mu) + \lambda(c+1)\right]\hat{A} + 2\mu\nabla\mathbf{R},$$

and its components are

$$\left.\begin{aligned} \partial^{11} &= \psi(q)\cos\chi - 2\mu\frac{\partial x_2}{\partial a_2}, \quad \partial^{22} = \psi(q)\cos\chi - 2\mu\frac{\partial x_1}{\partial a_1}, \\ \partial^{12} &= \psi(q)\sin\chi + 2\mu\frac{\partial x_1}{\partial a_2}, \quad \partial^{21} = -\psi(q)\sin\chi + 2\mu\frac{\partial x_2}{\partial a_1}, \\ \partial^{33} &= \lambda(q+c-3) + 2\mu(c-1), \quad \partial^{\alpha 3} = 0, \end{aligned}\right\} \tag{7.4.6}$$

where

$$\psi(q) = (\lambda+2\mu)(q-2) + 2\mu - \lambda(c-1)$$
$$= (\lambda+2\mu)\left[q - \frac{1-\nu(c-1)}{1-\nu}\right] \quad \left(\nu = \frac{\lambda}{2(\lambda+\mu)}\right). \tag{7.4.7}$$

It is convenient to enter the complex expressions

$$\partial^{11} + i\partial^{12} = \psi(q)e^{i\chi} + 2\mu i\frac{\partial z}{\partial a_2}, \quad \partial^{22} - i\partial^{21} = \psi(q)e^{i\chi} - 2\mu\frac{\partial z}{\partial a_1}. \tag{7.4.8}$$

Then in the case of no mass forces, the equations of statics

$$\frac{\partial\partial^{11}}{\partial a_1} + \frac{\partial\partial^{21}}{\partial a_2} = 0, \quad \frac{\partial\partial^{12}}{\partial a_1} + \frac{\partial\partial^{22}}{\partial a_2} = 0, \quad \frac{\partial\partial^{33}}{\partial a_3} = 0$$

are written in the form

$$\frac{\partial}{\partial a_1}\left(\partial^{11} + i\partial^{12}\right) + i\frac{\partial}{\partial a_2}\left(\partial^{22} - i\partial^{21}\right) = 0, \quad \frac{\partial\partial^{33}}{\partial a_3} = 0$$

and, with the help of eq. (7.4.8), they reduce to the relationships

$$2\frac{\partial}{\partial\zeta}\psi(q)e^{i\chi} = 0, \quad \partial^{33} = \partial^{33}(a_1, a_2).$$

The quantity $\psi(q)e^{i\chi}$ is the function of the complex variable ζ

$$\psi(q)e^{i\chi} = \Phi'(\zeta), \quad e^{i\chi} = \frac{\Phi'(\zeta)}{|\Phi'(\zeta)|}. \tag{7.4.9}$$

Apparently, this explains the notion "harmonic" for the material with the specific strain energy given by expression (2.8.7) of Chapter 8.

Equalities (7.4.5) and (7.4.7) allow the following equalities to be established

$$2\frac{\partial z}{\partial\zeta} = \frac{\Phi'(\zeta)}{\lambda+2\mu} + \frac{1-\nu(c-1)}{1-\nu}\frac{\Phi'(\zeta)}{|\Phi'(\zeta)|}. \tag{7.4.10}$$

These relate the sought functions $z(\zeta,\bar{\zeta})$ and $\Phi'(\zeta)$.

Let us consider an arc L in the cross-section of the deformed body (which is arc l in the initial state). Denoting an element on arc L by dS (ds on l) and using the equilibrium equations on the surface (7.1.3) we have

$$\text{on } l: cF_1\frac{dS}{ds} = n_1\partial^{11} + n_2\partial^{21}, \; cF_2\frac{dS}{ds} = n_1\partial^{12} + n_2\partial^{22} \quad \left(\frac{dO}{do} = c\frac{dS}{ds}\right),$$

where F_1 and F_2 denote the projections of the surface force on L on axes OX and OY respectively. Denoting

$$F = F_1 + iF_2, \quad n = n_1 + in_2 = \frac{da_2}{ds} - i\frac{da_1}{ds} = -i\frac{d\zeta}{ds},$$

and using equalities (7.4.8) and (7.4.9) we obtain

$$\text{on } l: \quad icF\frac{dS}{ds} = \psi(q)\, e^{i\chi}\frac{\partial\zeta}{ds} - 2\mu\frac{dz}{ds} = \Phi'(\zeta)\frac{d\zeta}{ds} - 2\mu\frac{dz}{ds}. \tag{7.4.11}$$

This yields the value of the principal vector of the surface forces on the arc M_0M

$$\text{on } l: \quad icV = ic\int_{s_0}^{s} FdS = \Phi(\zeta) - \Phi(\zeta_0) - 2\mu\left[z\left(\zeta,\bar{\zeta}\right) - z\left(\zeta_0,\bar{\zeta}_0\right)\right]. \tag{7.4.12}$$

For example, under the uniformly distributed normal load we have

$$\text{on } l: \quad F = -pN = pi\frac{dz}{dS}, \quad \Phi(\zeta) = (2\mu - cp)\, z\left(\zeta,\bar{\zeta}\right) + \text{const}. \tag{7.4.13}$$

On the straight lines which are parallel to the coordinate axes OX and OY we have

$$\begin{aligned} dS &= dx_2, \quad F = \overset{1}{F}_1 + i\overset{1}{F}_2 = \sigma_1 + i\tau_{12}, \\ d\zeta &= \frac{\partial\zeta}{\partial x_2}dx_2, \quad dz = idx_2 \quad (dx_1 = 0), \\ dS &= dx_1, \quad F = \overset{2}{F}_1 + i\overset{2}{F}_2 = -(\tau_{21} + i\sigma_2), \\ d\zeta &= \frac{\partial\zeta}{\partial x_1}dx_1, \quad dz = dx_1 \quad (dx_2 = 0), \end{aligned}$$

so that, proceeding to eq. (7.4.11), we obtain

$$\left.\begin{aligned} i(c\sigma_1 + 2\mu) - c\tau_{12} &= \psi(q)\, e^{i\chi}\frac{\partial\zeta}{\partial x_2}, \\ -ic\tau_{21} + (c\sigma_2 + 2\mu) &= \psi(q)\, e^{i\chi}\frac{\partial\zeta}{\partial x_1}. \end{aligned}\right\} \tag{7.4.14}$$

Utilising the relationships

$$\frac{\partial\zeta}{\partial\zeta} = 1 = \frac{\partial\zeta}{\partial z}\frac{\partial z}{\partial\zeta} + \frac{\partial\zeta}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\zeta}, \quad \frac{\partial\zeta}{\partial\bar{\zeta}} = 0 = \frac{\partial\zeta}{\partial z}\frac{\partial z}{\partial\bar{\zeta}} + \frac{\partial\zeta}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\bar{\zeta}},$$

and taking into account eq. (7.4.5) and the identity (which is straightforward to prove)

$$\frac{\partial z}{\partial\zeta}\frac{\partial\bar{z}}{\partial\bar{\zeta}} - \frac{\partial\bar{z}}{\partial\zeta}\frac{\partial z}{\partial\bar{\zeta}} = \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} - \frac{\partial x_1}{\partial a_2}\frac{\partial x_2}{\partial a_1} = \frac{1}{c}\sqrt{G},$$

we obtain

$$\frac{\partial\zeta}{\partial z} = \frac{c}{\sqrt{G}}\frac{\partial\bar{z}}{\partial\bar{\zeta}} = \frac{1}{2}\frac{c}{\sqrt{G}}qe^{-i\chi}, \quad \frac{\partial\zeta}{\partial\bar{z}} = -\frac{c}{\sqrt{G}}\frac{\partial z}{\partial\bar{\zeta}}$$

and by eq. (7.4.14) we arrive at the formulae

$$\left.\begin{aligned} \tau_{12} = \tau_{21}, \quad c(\sigma_1 + \sigma_2) + 4\mu &= \frac{c}{\sqrt{G}} q\psi(q), \\ c(\sigma_2 - \sigma_1) - 2ic\tau_{12} &= -\frac{4c}{\sqrt{G}} \frac{\psi(q)}{q} \frac{\partial z}{\partial \zeta} \frac{\partial z}{\partial \bar{\zeta}}. \end{aligned}\right\} \tag{7.4.15}$$

The normal stresses in the cross-section of the body are determined from the relationship

$$\sigma_3 dO = \partial^{33} do, \quad \sigma_3 = \frac{c}{\sqrt{G}} \partial^{33} = \frac{c}{\sqrt{G}} \left[\lambda(q-2) + (\lambda + 2\mu)(c-1)\right], \tag{7.4.16}$$

and the axial force which is the principal vector of these normal stresses is equal to

$$Q = \iint_{\Omega} \sigma_3 dO = \lambda \iint_{\Omega_0} (q-2)\, do + (\lambda + 2\mu)\Omega_0 (c-1). \tag{7.4.17}$$

Here Ω_0 denotes the cross-sectional area in the initial state.

9.7.5 *State of stress under a plane affine transformation*

The transformation is given by the linear relationships

$$x_1 = \lambda_{11}a_1 + \lambda_{12}a_2, \quad x_2 = \lambda_{21}a_1 + \lambda_{22}a_2, \quad x_3 = ca_3$$

or in another form

$$2z = [\lambda_{11} + \lambda_{22} + i(\lambda_{21} - \lambda_{12})]\zeta + [\lambda_{11} - \lambda_{22} + i(\lambda_{21} + \lambda_{12})]\bar{\zeta}, \; x_3 = ca_3. \tag{7.5.1}$$

The stresses calculated by means of formulae (7.4.15) and the axial force are equal to

$$\left.\begin{aligned} c\sigma_1 + 2\mu &= \frac{\psi(q)}{q}\left(\frac{\lambda_{11}^2 + \lambda_{12}^2}{\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}} + 1\right), \\ c\sigma_2 + 2\mu &= \frac{\psi(q)}{q}\left(\frac{\lambda_{22}^2 + \lambda_{21}^2}{\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}} + 1\right), \\ c\tau_{12} &= \frac{\psi(q)}{q} \frac{\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22}}{\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}}, \end{aligned}\right\} \tag{7.5.2}$$

$$Q = \Omega_0 \left[\lambda(q-2) + (\lambda + 2\mu)(c-1)\right]. \tag{7.5.3}$$

1. *Uniaxial tension.* In this state $\sigma_1 = 0, \sigma_2 = 0, \tau_{12} = 0$, the latter condition yielding $\lambda_{12} = -\lambda_{21}\dfrac{\lambda_{11}}{\lambda_{22}}$. From the first and second conditions we have

$$2\mu = \frac{\psi(q)}{q} \frac{\lambda_{11} + \lambda_{22}}{\lambda_{22}} = \frac{\psi(q)}{q} \frac{\lambda_{11} + \lambda_{22}}{\lambda_{11}}, \quad \lambda_{11} = \lambda_{22}, \quad \lambda_{12} = -\lambda_{21}.$$

Assuming

$$c-1=\varepsilon_3,\quad \lambda_{11}=\lambda_{22}=1-\varepsilon_1,\quad \lambda_{12}=-\lambda_{21}=-\omega_0,$$

we have

$$\mu=\frac{\psi(q)}{q},\quad q=\frac{\lambda+2\mu}{\lambda+\mu}\frac{1-\nu\varepsilon_3}{1-\nu}=2(1-\nu)\varepsilon_3,\quad \varepsilon_3=\nu\varepsilon_1$$

and by eqs. (7.5.3) and (7.5.4)

$$\frac{Q}{\Omega_0}=[\lambda(1-2\nu)+2\mu]\varepsilon_3=E\varepsilon_3\quad [E=2\mu(1+\nu)]. \tag{7.5.4}$$

In the considered case of no stresses $\sigma_1,\sigma_2,\tau_{12}$, the obtained solution of eq. (7.5.2.) leads to the equations of linear theory

$$x_1=a_1(1-\nu\varepsilon_3)-\omega_0 a_2,\quad x_2=a_2(1-\nu\varepsilon_3)+\omega_0 a_1,\quad x_3=(1+\varepsilon_3)a_3, \tag{7.5.5}$$

determining the displacements up to an arbitrary small rotation.

2. *Simple shear.* This deformation is given by

$$x_1=a_1+sa_2,\ x_2=a_2,\ x_3=a_3;\ \lambda_{11}=\lambda_{22}=1,\ \lambda_{12}=s,\ \lambda_{21}=0,\ c=1.$$

The stresses are calculated by the formulae

$$\sigma_1=\frac{\psi(q)}{q}\left(2+s^2\right)-2\mu,\quad \sigma_2=2\frac{\psi(q)}{q}-2\mu,$$
$$\tau_{12}=\frac{\psi(q)}{q}s,\quad \frac{Q}{\Omega_0}=\lambda(q-2),$$

where

$$q=\sqrt{4+s^2},\quad \frac{\psi(q)}{q}=(\lambda+2\mu)\left(1-\frac{1}{q(1-\nu)}\right)\approx\mu\left(1+\frac{s^2}{8(1-2\nu)}\right).$$

Only terms of order s^2 are kept here. With this degree of accuracy we have

$$\tau_{12}=\mu s,\quad \sigma_1=\mu\frac{5-8\nu}{4(1-2\nu)}s^2,\quad \sigma_2=\frac{\mu s^2}{4(1-2\nu)},\quad \frac{Q}{\Omega_0}=\mu\frac{\nu s^2}{4(1-2\nu)}. \tag{7.5.6}$$

9.7.6 Bending a strip into a cylindrical panel

This problem was considered in Subsection 9.2.1 for the case of incompressible material.

Deforming the rectangular parallelepiped

$$-h\le a_1\le h,\quad -b\le a_2\le b,\quad -l\le a_3\le l$$

into the cylindrical panel

$$r_0 \leq r \leq r_1, \quad -\alpha \leq \theta \leq \alpha, \quad -L \leq x_3 \leq L$$

is performed by means of the transformation, see the example in Subsection 2.6.5,

$$z = x_1 + ix_2 = C(a_1) \exp\left(i\tfrac{\alpha}{b} a_2\right), \quad x_3 = ca_3, \tag{7.6.1}$$

where

$$C(-h) = r_0, \quad C(h) = r_1, \quad cl = L. \tag{7.6.2}$$

Here

$$2\frac{dz}{d\zeta} = \frac{\partial z}{\partial a_1} - i\frac{\partial z}{\partial a_3} = \left[C'(a_1) + \frac{\alpha}{b} C(a_1)\right] \exp\left(i\tfrac{\alpha}{b} a_2\right) = qe^{i\chi} \tag{7.6.3}$$

and one can take

$$q = C'(a_1) + \frac{\alpha}{b} C(a_1), \quad \chi = \frac{\alpha}{b} a_2.$$

Now by eqs. (7.4.7) and (7.4.9) we have

$$\psi(q) = (\lambda + \mu)\left[C'(a_1) + \frac{\alpha}{b} C(a_1) - \frac{1 - \nu(c-1)}{1-\nu}\right], \quad \Phi(\zeta) = \psi(q)\, e^{i\chi}, \tag{7.6.4}$$

and the differential equation determining the unknown function $C(a_1)$ is as follows

$$\left.\begin{aligned} \frac{\partial \Phi'(\zeta)}{\partial \zeta} &= \frac{1}{2}\left(\frac{\partial}{\partial a_1} + i\frac{\partial}{\partial a_2}\right) \psi(q)\, e^{i\chi} = 0, \\ C''(a_1) - \frac{\alpha^2}{b^2} C(a_1) &= -\frac{\alpha}{b} \frac{1 - \nu(c-1)}{1-\nu}. \end{aligned}\right\} \tag{7.6.5}$$

Its solution subjected to boundary conditions (7.6.2) is presented in the form

$$C(a_1) = \frac{1}{2}\left[(r_1 + r_0)\frac{\cosh \frac{\alpha a_1}{b}}{\cosh \gamma} + (r_1 - r_0)\frac{\sinh \frac{\alpha a_1}{b}}{\sinh \gamma}\right] +$$
$$\frac{b}{\alpha} \frac{1 - \nu(c-1)}{1-\nu}\left(1 - \frac{\cosh \frac{\alpha a_1}{b}}{\cosh \gamma}\right) \quad \left(\gamma = \frac{\alpha h}{b}\right). \tag{7.6.6}$$

The conditions for no loads on the panel surfaces $a_1 = \pm h$ serve for determining constants r_1 and r_0. Along these surfaces $ds = da_2$ and by eq.

(7.4.11) for $F = 0$ and eqs. (7.6.1) and (7.6.4) we obtain

$$2\mu \frac{dz}{ds} = \Phi'(\zeta) \frac{d\zeta}{ds},$$
$$2\mu \frac{\alpha}{b} C(\pm h) = (\lambda + 2\mu) \left[C'(\pm h) + \frac{\alpha}{b} C(\pm h) - \frac{1 - \nu(c-1)}{1-\nu} \right]$$

or

$$C'(\pm h) + \frac{\alpha}{b} \frac{\nu}{1-\nu} C(\pm h) = \frac{1 - \nu(c-1)}{1-\nu} \qquad [C(h) = r_1, \ C(-h) = r_0].$$

From this equation we obtain

$$r_1 + r_0 = 2\frac{b}{\alpha}[1 - \nu(c-1)], \ r_1 - r_0 = 2\frac{b}{\alpha}[1 - \nu(c-1)]\tanh\gamma, \quad (7.6.7)$$

which enables the following representations

$$\left.\begin{aligned} C(a_1) &= \frac{b}{\alpha}[1 - \nu(c-1)] \left[\frac{\sinh \frac{\alpha a_1}{b}}{\cosh\gamma} + \frac{1}{1-\nu} \left(1 - \nu \frac{\cosh \frac{\alpha a_1}{b}}{\cosh\gamma} \right) \right], \\ C'(a_1) &= [1 - \nu(c-1)] \left(\frac{\cosh \frac{\alpha a_1}{b}}{\cosh\gamma} - \frac{\nu}{1-\nu} \frac{\sinh \frac{\alpha a_1}{b}}{\cosh\gamma} \right). \end{aligned}\right\} \quad (7.6.8)$$

The distribution of surface forces on planes $a_2 = \pm b$ is obtained from eq. (7.4.11) at $ds = \mp da_1$. The result is

$$\mp ice^{\mp i\alpha} \frac{dS}{da_1} = (\lambda + 2\mu) \left[C'(a_1) + \frac{\alpha}{b} C(a_1) - \frac{1 - \nu(c-1)}{1-\nu} \right] - 2\mu C'(a_1), \quad (7.6.9)$$

which yields the distribution of normal stresses on these boundaries

$$\begin{aligned} c\sigma_N \frac{dS}{da_1} &= c(-F_1 \sin\alpha \pm F_2 \cos\alpha) \\ &= (\lambda + 2\mu) \left[\frac{\nu}{1-\nu} C'(a_1) + \frac{\alpha}{b} C(a_1) - \frac{1 - \nu(c-1)}{1-\nu} \right] \\ &= \frac{E}{1-\nu^2}[1 - \nu(c-1)] \frac{\sinh \frac{\alpha a_1}{b}}{\cosh\gamma} \quad \left(\lambda + 2\mu = E \frac{1-\nu}{(1+\nu)(1-2\nu)} \right) \end{aligned} \quad (7.6.10)$$

and demonstrates the absence of shear stresses on them (as the right hand side of eq. (7.6.9) is real-valued).

The principal vector of the normal stresses is evidently equal to zero whereas the principal moment of these stresses, i.e. in any cross-section $\theta = \text{const}$ the bending moment per unit length of axis OX_3 is given by

$$m^O = \int_{r_0}^{r_1} C(a_1)\,\sigma_N dS = \frac{Eh^2}{c(1-\nu^2)}\left[1-\nu(c-1)\right]^2 \frac{1}{\gamma^2}\left(\tanh\gamma - \frac{\gamma}{\cosh^2\gamma}\right). \tag{7.6.11}$$

The axial force in the direction OX_3 per unit length of axis OX_3 is determined by eq. (7.4.17)

$$\begin{aligned} \frac{Q}{2b} &= \lambda \int_{-h}^{h} (q-2)\,da_1 + (\lambda + 2\mu)\,2h\,(c-1) \\ &= \frac{2Eh}{1-\nu^2}\left[(c-1)\left(1-\nu^2\frac{\tanh\gamma}{\gamma}\right) - \nu\left(1-\frac{\tanh\gamma}{\gamma}\right)\right]. \end{aligned} \tag{7.6.12}$$

For the strip with free end surfaces we have

$$Q = 0, \quad c - 1 = \nu\frac{\gamma - \tanh\gamma}{\gamma - \nu^2\tanh\gamma}. \tag{7.6.13}$$

Formulae (7.6.11) and (7.6.13) serve for determining the unknowns c and γ (or α) in terms of the prescribed bending moment.

9.7.7 Superimposing a small deformation

As is the case of Subsection 9.4.1 it is assumed that the points of the elastic body are subjected to a small displacement $\eta\mathbf{w}\left(q^1, q^2, q^3\right)$ from the equilibrium state, the latter being suggested to be given. In other words, we consider three states of the body: the initial one ($v-$volume), the given state of stress ($\overset{0}{V}$ $-$volume) and the second state of stress ($V-$volume) which is close to the given one. The position vector of a point in these states is denoted respectively as $\mathbf{r}$, $\overset{0}{\mathbf{R}}$, $\mathbf{R}$ where

$$\mathbf{R} = \overset{0}{\mathbf{R}} + \eta\mathbf{w}.$$

For the values in $V-$volume we keep the earlier taken denotations (for example $\mathbf{R}$,$\hat{D}$, $\hat{A}$ etc.) whereas their values in $\overset{0}{V}$ $-$volume have a "zero" at the top ($\overset{0}{\mathbf{R}}$, $\overset{0}{\hat{D}}$, $\overset{0}{\hat{A}}$ etc.). The differences, i.e. "perturbations", are calculated by keeping only the first degree of the small parameter η and are written as the products of this parameter and the quantities with a dot at the top

$$\mathbf{R} = \overset{0}{\mathbf{R}} + \eta\dot{\mathbf{R}}, \quad \hat{D} = \overset{0}{\hat{D}} + \eta\,\overset{\bullet}{\hat{D}}, \quad \hat{A} = \overset{0}{\hat{A}} + \eta\,\overset{\bullet}{\hat{A}} \quad \text{etc.}$$

It is evident that $\dot{\mathbf{R}} = \mathbf{w}$ and all quantities with a dot at the top are linear operators over vector $\mathbf{w}$. They can be treated as being the derivatives of the corresponding quantities with respect to η at $\eta = 0$

$$\overset{\bullet}{\hat{D}}= \left(\frac{\partial}{\partial\eta}\hat{D}\right)_{\eta=0}, \quad \overset{\bullet}{\hat{A}}= \left(\frac{\partial}{\partial\eta}\hat{A}\right)_{\eta=0} \quad \text{etc.}$$

According to the definition of the strain measure $\hat{G}^{\times}$, eq. (3.3.2) of Chapter 2, we can set the tensor in the form

$$\overset{\bullet}{\hat{G}^{\times}}= \nabla\dot{\mathbf{R}} \cdot \nabla \overset{0}{\mathbf{R}}{}^{*} + \nabla \overset{0}{\mathbf{R}} \cdot \nabla\dot{\mathbf{R}}^{*} = \nabla\mathbf{w} \cdot \nabla \overset{0}{\mathbf{R}}{}^{*} + \nabla \overset{0}{\mathbf{R}} \cdot \nabla\mathbf{w}^{*}, \tag{7.7.1}$$

where an asterisk denotes, as always, the operation of transposing a tensor of second rank.

Further, the principal values $\overset{0}{G}_s, M_s = \overset{0}{G}_s$ and the principal directions $\overset{0}{\mathbf{e}}_s, \overset{0}{\tilde{\mathbf{e}}}_s$ of the strain measures $\overset{0}{\hat{G}}, \overset{0}{\hat{M}}$, eq. (5.3.1) of Chapter 2, are assumed to be given[4]. For constructing the tensor

$$\overset{\bullet}{\hat{A}}= \dot{\mathbf{e}}_s\, \overset{0}{\tilde{\mathbf{e}}}_s + \overset{0}{\tilde{\mathbf{e}}}_s\, \dot{\mathbf{e}}_s \tag{7.7.2}$$

it is necessary to determine vectors $\dot{\mathbf{e}}_s, \overset{\bullet}{\tilde{\mathbf{e}}}_s$ which also allows us to obtain the values of $\overset{\bullet}{\hat{G}}_s$. It is easy to note that these vectors are orthogonal to $\overset{0}{\mathbf{e}}_s, \overset{0}{\tilde{\mathbf{e}}}_s$, indeed

$$\mathbf{e}_s \cdot \mathbf{e}_s = 1, \quad \dot{\mathbf{e}}_s \cdot \overset{0}{\mathbf{e}}_s + \overset{0}{\mathbf{e}}_s\, \dot{\mathbf{e}}_s = 0, \quad \dot{\mathbf{e}}_s \cdot \overset{0}{\mathbf{e}}_s = 0 \quad (s = 1, 2, 3)\,. \tag{7.7.3}$$

Referring to definitions (A.9.1) and (A.9.4) of the principal values and the principal directions of the tensor, we have

$$\hat{G}^{\times} \cdot \mathbf{e}_s - G_s\mathbf{e}_s = 0, \quad \overset{0}{\hat{G}^{\times}} \cdot\dot{\mathbf{e}}_s - \overset{0}{G}_s\, \dot{\mathbf{e}}_s = \dot{G}_s\, \overset{0}{\mathbf{e}}_s - \overset{\bullet}{\hat{G}^{\times}} \cdot \overset{0}{\mathbf{e}}_s, \tag{7.7.4}$$

where in $V-$volume

$$\overset{0}{\hat{G}^{\times}} \cdot \overset{0}{\mathbf{e}}_k - \overset{0}{G}_k\overset{0}{\mathbf{e}}_k= 0 \quad \text{or} \quad \overset{0}{\mathbf{e}}_k \cdot \overset{0}{\hat{G}^{\times}}=\overset{0}{G}_k\overset{0}{\mathbf{e}}_k\,. \tag{7.7.5}$$

Premultiplying both sides of eq. (7.7.4) by $\overset{0}{\mathbf{e}}_k$ we have

$$\overset{0}{\mathbf{e}}_k \cdot \overset{0}{\hat{G}^{\times}_k} \cdot\dot{\mathbf{e}}_s - \overset{0}{G}_s\overset{0}{\mathbf{e}}_k \cdot\dot{\mathbf{e}}_s = \dot{G}_s\, \overset{0}{\mathbf{e}}_s \cdot \overset{0}{\mathbf{e}}_k - \overset{0}{\mathbf{e}}_k \cdot \overset{\bullet}{\hat{G}^{\times}} \cdot \overset{0}{\mathbf{e}}_s,$$

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[4] In order to avoid the denotation $\overset{0}{\mathbf{e}}{}^{s}, \overset{0}{\tilde{\mathbf{e}}}{}^{s}$ the index s is placed below.

or by means of eq. (7.7.5)

$$\left(\overset{0}{G}_k - \overset{0}{G}_s\right) \overset{0}{\mathbf{e}}_k \cdot \dot{\mathbf{e}}_s = \dot{G}_s \delta_{sk} - \overset{0}{\mathbf{e}}_k \cdot \overset{\bullet}{\hat{G}}{}^{\times} \cdot \overset{0}{\mathbf{e}}_s \,.$$

For $k = s$ we obtain

$$\dot{G}_s = \overset{0}{\mathbf{e}}_s \cdot \overset{\bullet}{\hat{G}}{}^{\times} \cdot \overset{0}{\mathbf{e}}_s, \qquad (7.7.6)$$

while for $k \neq s$ we have

$$\dot{\mathbf{e}}_s \cdot \overset{0}{\mathbf{e}}_k = \frac{\overset{0}{\mathbf{e}}_k \cdot \overset{\bullet}{\hat{G}}{}^{\times} \cdot \overset{0}{\mathbf{e}}_s}{\overset{0}{G}_s - \overset{0}{G}_k}.$$

Together with eq. (7.7.3) this equation determines the projections of vector $\dot{\mathbf{e}}_s$ on the axes of trihedron $\overset{0}{\mathbf{e}}_k$, hence

$$\dot{\mathbf{e}}_s = \sum_k{}' \frac{\overset{0}{\mathbf{e}}_k \cdot \overset{\bullet}{\hat{G}}{}^{\times} \cdot \overset{0}{\mathbf{e}}_s}{\overset{0}{G}_s - \overset{0}{G}_k} \overset{0}{\mathbf{e}}_k \quad (s = 1, 2, 3) \qquad (7.7.7)$$

where a prime implies that the term $k = s$ is omitted. Similarly for tensor $\overset{\bullet}{\hat{M}}$ we have

$$\overset{0}{\tilde{\mathbf{e}}}_k \cdot \overset{\bullet}{\hat{M}} \cdot \overset{0}{\tilde{\mathbf{e}}}_s = \dot{G}_s, \quad \dot{\tilde{\mathbf{e}}}_k = \sum_s{}' \frac{\overset{0}{\tilde{\mathbf{e}}}_s \cdot \overset{\bullet}{\hat{M}} \cdot \overset{0}{\tilde{\mathbf{e}}}_k}{\overset{0}{G}_k - \overset{0}{G}_s} \overset{0}{\tilde{\mathbf{e}}}_s, \qquad (7.7.8)$$

where, by analogy to eq. (7.7.1)

$$\overset{\bullet}{\hat{M}} = \nabla \overset{0}{\mathbf{R}}{}^* \cdot \nabla \mathbf{w} + \nabla \mathbf{w}^* \cdot \nabla \overset{0}{\mathbf{R}} \,.$$

Expression (7.7.2) for tensor $\overset{\bullet}{\hat{A}}$ is now written in the following form (the summation sign is omitted)

$$\overset{\bullet}{\hat{A}} = \frac{\overset{0}{\tilde{\mathbf{e}}}_k \overset{0}{\tilde{\mathbf{e}}}_s}{\overset{0}{G}_s - \overset{0}{G}_k} \left[\overset{0}{\mathbf{e}}_k \cdot \left(\nabla \mathbf{w} \cdot \nabla \overset{0}{\mathbf{R}}{}^* + \nabla \overset{0}{\mathbf{R}} \cdot \nabla \mathbf{w}^* \right) \cdot \overset{0}{\mathbf{e}}_s - \right.$$
$$\left. \overset{0}{\tilde{\mathbf{e}}}_s \cdot \left(\nabla \overset{0}{\mathbf{R}}{}^* \cdot \nabla \mathbf{w} + \nabla \mathbf{w}^* \cdot \nabla \overset{0}{\mathbf{R}} \right) \cdot \overset{0}{\tilde{\mathbf{e}}}_k \right]. \qquad (7.7.9)$$

Use is made of the relationships

$$\nabla \overset{0}{\mathbf{R}} = \overset{0}{\hat{G}}{}^{\times 1/2} \cdot \overset{0}{\hat{A}} = \sqrt{\overset{0}{G}_m}\, \overset{0}{\mathbf{e}}_m \overset{0}{\tilde{\mathbf{e}}}_m, \quad \nabla \overset{0}{\mathbf{R}}{}^* = \sqrt{\overset{0}{G}_m}\, \overset{0}{\tilde{\mathbf{e}}}_m \overset{0}{\mathbf{e}}_m,$$

which yield

$$\overset{0}{\mathbf{e}}_k \cdot\nabla \overset{0}{\mathbf{R}} = \sqrt{\overset{0}{G}_k}\, \overset{0}{\tilde{\mathbf{e}}}_k = \nabla \overset{0}{\mathbf{R}}{}^* \cdot \overset{0}{\mathbf{e}}_k, \quad \nabla \overset{0}{\mathbf{R}}\cdot\overset{0}{\tilde{\mathbf{e}}}_k = \sqrt{\overset{0}{G}_k}\,\overset{0}{\mathbf{e}}_k = \overset{0}{\tilde{\mathbf{e}}}_k \cdot\nabla \overset{0}{\mathbf{R}}{}^* .$$

This admits the following representation of eq. (7.7.9)

$$\begin{aligned}\overset{\bullet}{\hat{A}} &= \frac{\overset{0}{\tilde{\mathbf{e}}}_k \overset{0}{\tilde{\mathbf{e}}}_s}{\overset{0}{G}_s - \overset{0}{G}_k} \left[\left(\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_s - \overset{0}{\mathbf{e}}_s \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k \right) \sqrt{\overset{0}{G}_s} + \right. \\ &\qquad \left. \left(\overset{0}{\tilde{\mathbf{e}}}_k \cdot\nabla\mathbf{w}^*\cdot \overset{0}{\mathbf{e}}_s - \overset{0}{\tilde{\mathbf{e}}}_s \cdot\nabla\mathbf{w}^*\cdot \overset{0}{\mathbf{e}}_k \right) \sqrt{\overset{0}{G}_k} \right] \\ &= \frac{\overset{0}{\mathbf{e}}_k \overset{0}{\tilde{\mathbf{e}}}_s}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \left(\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_s - \overset{0}{\mathbf{e}}_s \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k \right),\end{aligned}$$

where the following relationships were used

$$\overset{0}{\tilde{\mathbf{e}}}_k \cdot\nabla\mathbf{w}^*\cdot \overset{0}{\mathbf{e}}_s = \overset{0}{\mathbf{e}}_s \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k .$$

We arrive at the expression

$$\overset{\bullet}{\hat{A}} = \frac{\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_s}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \left(\overset{0}{\mathbf{e}}_k \overset{0}{\tilde{\mathbf{e}}}_s - \overset{0}{\mathbf{e}}_s \overset{0}{\tilde{\mathbf{e}}}_k \right). \tag{7.7.10}$$

For calculation of the tensor

$$\overset{\bullet}{\hat{D}} = \left(\lambda \overset{0}{s}_1 - 2\mu \right) \overset{\bullet}{\hat{A}} + \lambda \dot{s} \overset{0}{\hat{A}} + 2\mu \nabla \dot{\mathbf{R}} \tag{7.7.11}$$

we need the following expressions

$$\dot{s}_1 = \dot{I}_1 \left(\hat{G}^{\times 1/2} \right) = \left[\frac{\partial}{\partial\eta} I_1 \left(\hat{G}^{\times 1/2} \right) \right]_{\eta=0} = \frac{\partial}{\partial\eta} \left(\sum_k \sqrt{G_k} \right)_{\eta=0} = \frac{1}{2} \frac{\dot{G}_k}{\sqrt{\overset{0}{G}_k}} .$$

Referring to eqs. (7.7.6) and (7.7.1) we have

$$\dot{s}_1 = \frac{1}{2} \sum_k \frac{\overset{0}{\mathbf{e}}_k \cdot \overset{\bullet}{\hat{G}}{}^{\times} \cdot \overset{0}{\mathbf{e}}_k}{\sqrt{\overset{0}{G}_k}} = \frac{1}{2} \sum_k \left(\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k + \overset{0}{\tilde{\mathbf{e}}}_k \cdot\nabla\mathbf{w}^*\cdot \overset{0}{\mathbf{e}}_k \right)$$

or, omitting the summation sign,

$$\dot{s}_1 = \overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k . \tag{7.7.12}$$

Thus we are led to the following representation of formula (7.7.11)

$$\overset{\bullet}{\hat{D}}= \frac{\lambda \overset{0}{s}_1 -2\mu}{\sqrt{\overset{0}{G}_s}+\sqrt{\overset{0}{G}_k}} \overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_s \left(\overset{0}{\mathbf{e}}_k\overset{0}{\tilde{\mathbf{e}}}_s - \overset{0}{\mathbf{e}}_s\overset{0}{\tilde{\mathbf{e}}}_k\right) + \lambda \overset{0}{\hat{A}}\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k +2\mu\nabla\mathbf{w}. \tag{7.7.13}$$

The mass force is assumed to be unchanged under transformation of $\overset{0}{V}$ −volume into V−volume. The equilibrium equation in V−volume takes the form

$$\nabla\cdot \overset{\bullet}{\hat{D}} +\dot{\rho}\mathbf{K} = 0, \tag{7.7.14}$$

where according to the mass conservation law

$$\rho\sqrt{G} = \rho^0\sqrt{\overset{0}{G}}, \qquad \dot{\rho}\sqrt{\overset{0}{G}} + \rho^0 \overset{\bullet}{\sqrt{G}}= 0,$$

$$\dot{\rho} = -\rho^0 \left(\overset{0}{G}\right)^{-1/2} \left(\frac{\partial}{\partial\eta}\sqrt{G}\right)_{\eta=0} = -\frac{1}{2}\rho^0\frac{\dot{G}}{\overset{0}{G}} = -\frac{1}{2}\rho^0\sum_k \frac{\dot{G}_k}{\overset{0}{G}_k}$$

or by eq. (7.7.1)

$$\dot{\rho} = -\rho^0\sum_k \frac{\overset{0}{\mathbf{e}}_k \cdot\nabla\mathbf{w}\cdot \overset{0}{\tilde{\mathbf{e}}}_k}{\sqrt{\overset{0}{G}_k}}. \tag{7.7.15}$$

The equilibrium equation on the surface

$$\mathbf{F}dO = \mathbf{n}\cdot\hat{D}do$$

reduces to the form

$$\mathbf{n}\cdot\hat{D} = \dot{\mathbf{F}}\frac{d\overset{0}{O}}{do} + \overset{0}{\mathbf{F}}\frac{d\dot{O}}{do} = \left(\dot{\mathbf{F}} + \overset{0}{\mathbf{F}}\frac{d\dot{O}}{d\overset{0}{O}}\right)\frac{d\overset{0}{O}}{do}.$$

Here

$$dO = \sqrt{\frac{G}{g}}\left(\mathbf{n}\cdot\hat{G}^{\times^{-1}}\cdot\mathbf{n}\right)^{1/2} do, \quad d\dot{O} = \frac{1}{2}\left(\frac{\dot{G}}{\overset{0}{G}} + \frac{\mathbf{n}\cdot\left(\hat{G}^{\times^{-1}}\right)^{\bullet}\cdot\mathbf{n}}{\mathbf{n}\cdot\overset{0}{\hat{G}^{\times}}\cdot\mathbf{n}}\right) d\overset{0}{O},$$

and the equilibrium equation on surface O of volume V takes the form

$$\mathbf{n}\cdot\overset{\bullet}{\hat{D}}= \sqrt{\frac{\overset{0}{G}}{g}}\left[\left(\dot{\mathbf{F}} + \frac{1}{2}\overset{0}{\mathbf{F}}\frac{\dot{G}}{\overset{0}{G}}\right)\left(\mathbf{n}\cdot\overset{0}{\hat{G}^{\times^{-1}}}\cdot\mathbf{n}\right)^{1/2} + \frac{1}{2}\overset{0}{\mathbf{F}}\frac{\mathbf{n}\cdot\left(\hat{G}^{\times^{-1}}\right)^{\bullet}\cdot\mathbf{n}}{\left(\mathbf{n}\cdot\overset{0}{\hat{G}^{\times^{-1}}}\cdot\mathbf{n}\right)^{1/2}}\right]. \tag{7.7.16}$$

Representation of value $\left(\hat{G}^{\times^{-1}}\right)^{\bullet}$ in terms of vector $\mathbf{w}$ is based upon the relationships

$$\mathbf{R}_s\mathbf{R}^s = \hat{G} = \overset{0}{\mathbf{R}}_s\overset{0}{\mathbf{R}}{}^s, \quad \dot{\mathbf{R}}_s\,\overset{0}{\mathbf{R}}{}^s + \overset{0}{\mathbf{R}}_s\,\dot{\mathbf{R}}^s = 0.$$

It follows from these that

$$\dot{\mathbf{R}}^t = -\overset{0}{\mathbf{R}}{}^t\cdot\dot{\mathbf{R}}_s\,\overset{0}{\mathbf{R}}{}^s, \quad \dot{G}^{qt} = \overset{0}{\mathbf{R}}{}^q\cdot\dot{\mathbf{R}}^t + \overset{0}{\mathbf{R}}{}^t\cdot\dot{\mathbf{R}}^q = -\left(\overset{0}{G}{}^{sq}\overset{0}{\mathbf{R}}{}^t + \overset{0}{G}{}^{st}\overset{0}{\mathbf{R}}{}^q\right)\cdot\dot{\mathbf{R}}_s, \tag{7.7.17}$$

so that

$$\left(\hat{G}^{\times^{-1}}\right)^{\bullet} = -\left(\overset{0}{G}{}^{sq}\overset{0}{\mathbf{R}}{}^t + \overset{0}{G}{}^{st}\overset{0}{\mathbf{R}}{}^q\right)\cdot\frac{\partial\mathbf{w}}{\partial q^s}\mathbf{r}_q\mathbf{r}_t.$$

In the particular case of a constant pressure p remaining normal to surface O bounding V−volume we have by eq. (3.5.7) of Chapter 2

$$\mathbf{n}\cdot\overset{\bullet}{\hat{D}}\,do = -p\,(\mathbf{N}dO)^{\bullet} = -p\left(\sqrt{\frac{\overset{0}{G}}{g}}\dot{\mathbf{R}}^s + \overset{\bullet}{\sqrt{\frac{G}{g}}}\overset{0}{\mathbf{R}}{}^s\right)n_s do$$

and since

$$\frac{\overset{\bullet}{\sqrt{G}}}{\sqrt{\overset{0}{G}}} = \sum_k\frac{\overset{0}{\mathbf{e}}_k\cdot\nabla\mathbf{w}\cdot\overset{0}{\tilde{\mathbf{e}}}_k}{\sqrt{\overset{0}{G}_k}},$$

we obtain

$$\mathbf{n}\cdot\overset{\bullet}{\hat{D}} = -p\sqrt{\frac{\overset{0}{G}}{g}}\left(\dot{\mathbf{R}}^s + \sum_k\frac{\overset{0}{\mathbf{e}}_k\cdot\nabla\mathbf{w}\cdot\overset{0}{\tilde{\mathbf{e}}}_k}{\sqrt{\overset{0}{G}_k}}\overset{0}{\mathbf{R}}{}^s\right)n_s. \tag{7.7.18}$$

9.7.8 The case of conserved principal directions

Similar to Subsection 9.7.2 it is assumed that the principal directions $\overset{0}{\mathbf{e}}_s, \overset{0}{\tilde{\mathbf{e}}}_s$ of tensors $\overset{0}{\hat{G}}{}^{\times}$ and $\overset{0}{\hat{M}}$ are coincident. This occurs in the case of the diagonal matrices of tensors $\hat{g}$ and $\hat{G}$, then the unit vectors $\mathbf{t}_s$ of the tangents to the coordinate lines $q^s = \text{const}$ forming the orthogonal trihedron have the directions $\overset{0}{\mathbf{e}}_s, \overset{0}{\tilde{\mathbf{e}}}_s$.

Referring to eq. (7.2.1) and replacing tensor $\nabla \mathbf{w}$ by representation (1.2.13) of Chapter 2 one can transform expression (7.7.10) for tensor $\overset{\bullet}{\hat{A}}$ with the help of the symmetry of strain tensor $\hat{\varepsilon}$

$$\overset{\bullet}{\hat{A}} = \frac{1}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \left(\overset{0}{\mathbf{e}}_k \cdot\hat{\varepsilon}\cdot \overset{0}{\mathbf{e}}_s - \overset{0}{\mathbf{e}}_k \cdot\hat{\Omega}\cdot \overset{0}{\mathbf{e}}_s \right) \left(\overset{0}{\mathbf{e}}_k \overset{0}{\mathbf{e}}_s - \overset{0}{\mathbf{e}}_s \overset{0}{\mathbf{e}}_k \right) \tag{7.8.1}$$

$$= -\frac{2}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \overset{0}{\mathbf{e}}_k \cdot\hat{\Omega}\cdot \overset{0}{\mathbf{e}}_s \overset{0}{\mathbf{e}}_k \overset{0}{\mathbf{e}}_s = -\frac{2 \overset{0}{\mathbf{e}}_k \overset{0}{\mathbf{e}}_s}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \boldsymbol{\omega} \cdot \left(\overset{0}{\mathbf{e}}_s \times \overset{0}{\mathbf{e}}_k \right).$$

Here $\boldsymbol{\omega}$ denotes the linear vector of rotation determined in terms of vector $\mathbf{w}$.

Next, by eq. (7.7.12) we have

$$\dot{s}_1 = \overset{0}{\mathbf{e}}_k \cdot\hat{\varepsilon}\cdot \overset{0}{\mathbf{e}}_k - \overset{0}{\mathbf{e}}_k \cdot\hat{\Omega}\cdot \overset{0}{\mathbf{e}}_k = I_1(\hat{\varepsilon}) = \nabla \cdot \mathbf{w}, \tag{7.8.2}$$

and substitution into eq. (7.7.13) leads to the equality

$$\overset{\bullet}{\hat{D}} = -2 \frac{\lambda s_1^0 - 2\mu}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} \boldsymbol{\omega} \cdot \left(\overset{0}{\mathbf{e}}_s \times \overset{0}{\mathbf{e}}_k \right) \overset{0}{\mathbf{e}}_k \overset{0}{\mathbf{e}}_s - 2\mu\hat{\Omega} + \lambda \hat{g} \nabla \cdot \mathbf{w} + 2\mu\hat{\varepsilon}$$

or, if eq. (A.4.6) is used, we obtain

$$\overset{\bullet}{\hat{D}} = \hat{T}(\mathbf{w}) + 2 \left(\frac{\lambda s_1^0 - 2\mu}{\sqrt{\overset{0}{G}_s} + \sqrt{\overset{0}{G}_k}} + \mu \right) \boldsymbol{\omega} \cdot \left(\overset{0}{\mathbf{e}}_k \times \overset{0}{\mathbf{e}}_s \right) \overset{0}{\mathbf{e}}_k \overset{0}{\mathbf{e}}_s, \tag{7.8.3}$$

where $\hat{T}(\mathbf{w})$ denotes the stress tensor determined by means of the Hooke generalised law in terms of vector $\mathbf{w}$

$$\hat{T}(\mathbf{w}) = 2\mu \left[\frac{\nu}{1-2\nu} \hat{g} \nabla \cdot \mathbf{w} + \frac{1}{2} \left(\nabla \mathbf{w} + \nabla \mathbf{w}^* \right) \right]. \tag{7.8.4}$$

9.7.9 Southwell's equations of neutral equilibrium (1913)

We consider the deformation of v–volume into $\overset{0}{V}$ –volume described by the transformation

$$x_s = (1+\varepsilon_s) a_s, \quad \overset{0}{\mathbf{R}} = \sum_s \mathbf{i}_s a_s (1+\varepsilon_s). \tag{7.9.1}$$

Then

$$\overset{0}{\mathbf{R}}_s = (1+\varepsilon_s) \mathbf{i}_s, \qquad \nabla \overset{0}{\mathbf{R}} = \sum_s (1+\varepsilon_s) \mathbf{i}_s \mathbf{i}_s = \nabla \overset{0}{\mathbf{R}}{}^* = \hat{G}^{\times^{1/2}},$$

$$\overset{0}{\hat{A}} = \hat{E}, \qquad \overset{0}{s}_1 = I_1 \left(\hat{G}^{\times^{1/2}} \right) - 3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \overset{0}{\vartheta}$$

and tensor $\overset{0}{\hat{D}}$, eq. (7.1.1), is set in the form coinciding with the generalised Hooke law of the linear theory of elasticity

$$\overset{0}{\hat{D}} = \lambda \overset{0}{\vartheta} \hat{E} + 2\mu \left(\nabla \overset{0}{\mathbf{R}} - \hat{E} \right) = \lambda \overset{0}{\vartheta} \hat{E} + 2\mu \overset{0}{\hat{\varepsilon}}, \qquad \overset{0}{\hat{\varepsilon}} = \sum_s \varepsilon_s \mathbf{i}_s \mathbf{i}_s. \tag{7.9.2}$$

For this reason

$$\varepsilon_s = \frac{1}{2\mu} \left(\overset{0}{\partial^{ss}} - \frac{\lambda}{3\lambda + 2\mu} \sum_k \overset{0}{\partial^{kk}} \right), \qquad \overset{0}{\vartheta} = \frac{1}{3\lambda + 2\mu} \sum_k \overset{0}{\partial^{kk}},$$

so that

$$2 \left(\frac{\lambda \overset{0}{s_1} - 2\mu}{\sqrt{\overset{0}{G_s}} + \sqrt{\overset{0}{G_k}}} + \mu \right) = \frac{\overset{0}{\partial^{ss}} + \overset{0}{\partial^{kk}}}{2 + \varepsilon_s + \varepsilon_k},$$

and tensor $\overset{\bullet}{\hat{D}}$ takes the form

$$\overset{\bullet}{\hat{D}} = \hat{T}(\mathbf{w}) + 2\mu \left[A_1 \omega_1 (\mathbf{i}_2 \mathbf{i}_3 - \mathbf{i}_3 \mathbf{i}_2) + A_2 \omega_2 (\mathbf{i}_3 \mathbf{i}_1 - \mathbf{i}_1 \mathbf{i}_3) + A_3 \omega_3 (\mathbf{i}_1 \mathbf{i}_2 - \mathbf{i}_2 \mathbf{i}_1) \right], \tag{7.9.3}$$

where for brevity

$$A_1 = \frac{1}{2\mu} \frac{\overset{0}{\partial^{22}} + \overset{0}{\partial^{33}}}{2 + \varepsilon_2 + \varepsilon_3}, \quad A_2 = \frac{1}{2\mu} \frac{\overset{0}{\partial^{33}} + \overset{0}{\partial^{11}}}{2 + \varepsilon_3 + \varepsilon_1}, \quad A_3 = \frac{1}{2\mu} \frac{\overset{0}{\partial^{11}} + \overset{0}{\partial^{22}}}{2 + \varepsilon_1 + \varepsilon_2}. \tag{7.9.4}$$

The values $\overset{0}{\partial^{ss}}$ relate to the principal stresses $\overset{0}{\sigma_s}$ in $\overset{0}{V}$ –volume in the following way

$$\overset{0}{\partial^{11}} = \overset{0}{\sigma_1} (1 + \varepsilon_2)(1 + \varepsilon_3), \quad \overset{0}{\partial^{22}} = \overset{0}{\sigma_2} (1 + \varepsilon_3)(1 + \varepsilon_1),$$
$$\overset{0}{\partial^{33}} = \overset{0}{\sigma_3} (1 + \varepsilon_1)(1 + \varepsilon_2).$$

In the case of no mass forces the equilibrium equations in V–volume (7.7.14) are written in the form

$$\left.\begin{aligned}
\frac{1}{1 - 2\nu} \frac{\partial \vartheta}{\partial a_1} + \nabla^2 u + 2A_2 \frac{\partial \omega_2}{\partial a_3} - 2A_3 \frac{\partial \omega_3}{\partial a_2} &= 0, \\
\frac{1}{1 - 2\nu} \frac{\partial \vartheta}{\partial a_2} + \nabla^2 v + 2A_3 \frac{\partial \omega_3}{\partial a_1} - 2A_1 \frac{\partial \omega_1}{\partial a_3} &= 0, \\
\frac{1}{1 - 2\nu} \frac{\partial \vartheta}{\partial a_3} + \nabla^2 w + 2A_1 \frac{\partial \omega_1}{\partial a_2} - 2A_2 \frac{\partial \omega_2}{\partial a_1} &= 0.
\end{aligned}\right\} \tag{7.9.5}$$

Here we introduced the denotation

$$\mathbf{w} = \mathbf{i}_1 u + \mathbf{i}_2 v + \mathbf{i}_3 w, \quad \vartheta = \nabla \cdot \mathbf{w} = \frac{\partial u}{\partial a_1} + \frac{\partial v}{\partial a_2} + \frac{\partial w}{\partial a_3},$$

$$2\omega_1 = \frac{\partial w}{\partial a_2} - \frac{\partial v}{\partial a_3} \quad \text{etc.}$$

Taking into account the well-known relationship

$$\nabla^2 \mathbf{w} = \nabla\nabla \cdot \mathbf{w} - \nabla \times (\nabla \times \mathbf{w}) = \nabla\vartheta - 2\nabla \times \boldsymbol{\omega},$$

one can present these equations in another form

$$\left.\begin{aligned} \frac{\partial \vartheta}{\partial a_1} + 2B_2 \frac{\partial \omega_2}{\partial a_3} - 2B_3 \frac{\partial \omega_3}{\partial a_2} &= 0, \\ \frac{\partial \vartheta}{\partial a_2} + 2B_3 \frac{\partial \omega_3}{\partial a_1} - 2B_1 \frac{\partial \omega_1}{\partial a_3} &= 0, \\ \frac{\partial \vartheta}{\partial a_3} + 2B_1 \frac{\partial \omega_1}{\partial a_2} - 2B_2 \frac{\partial \omega_2}{\partial a_1} &= 0, \end{aligned}\right\} \tag{7.9.6}$$

where

$$B_s = \frac{1-2\nu}{2(1-\nu)} (1 + A_s). \tag{7.9.7}$$

The equations of the "neutral equilibrium" in the form of eq. (7.9.5) were obtained by Southwell from different considerations.

9.7.10 *Solution of Southwell's equations*

Equations (7.9.6) are rewritten in the form

$$\left.\begin{aligned} \left(\partial_1^2 + B_2\partial_3^2 + B_3\partial_2^2\right) u + (1-B_3)\,\partial_1\partial_2 v + (1-B_2)\,\partial_1\partial_3 w &= 0, \\ (1-B_3)\,\partial_1\partial_2 u + \left(\partial_2^2 + B_3\partial_1^2 + B_1\partial_3^2\right) v + (1-B_1)\,\partial_2\partial_3 w &= 0, \\ (1-B_2)\,\partial_1\partial_3 u + (1-B_1)\,\partial_2\partial_3 v + \left(\partial_3^2 + B_1\partial_2^2 + B_2\partial_1^2\right) w &= 0, \end{aligned}\right\} \tag{7.10.1}$$

$(\partial_s = \partial/\partial a_s)$ or in a compact form

$$e_{j1} u + e_{j2} v + e_{j3} w = 0 \quad (j = 1, 2, 3), \tag{7.10.2}$$

where $e_{js} = e_{sj}$ denote the above operators.

Presenting one of the particular solutions of this system in the form

$$u_1 = \Delta_{11}\chi_1, \quad v_1 = \Delta_{12}\chi_1, \quad w_1 = \Delta_{13}\chi_1, \tag{7.10.3}$$

we determine the operators of differentiation Δ_{1s} as solutions of the homogeneous system of two equations (7.10.2) for $j = 2, 3$. Insertion in these equations yields

$$e_{21}\Delta_{11} + e_{22}\Delta_{12} + e_{23}\Delta_{13} = 0,$$
$$e_{31}\Delta_{11} + e_{32}\Delta_{12} + e_{33}\Delta_{13} = 0,$$

where up to a common multiplier we obtain

$$\Delta_{11} = e_{22}e_{33} - e_{23}e_{32}, \quad \Delta_{12} = e_{23}e_{31} - e_{21}e_{33}, \quad \Delta_{13} = e_{21}e_{32} - e_{22}e_{31}.$$

As expected, Δ_{sk} is the algebraic adjunct of element e_{sk} of the determinant

$$\Delta = |e_{sk}| = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix}. \tag{7.10.4}$$

Substituting expressions (7.10.3) for the sought functions into the first equation of system (7.10.2) we arrive at the differential equation for χ_1

$$(e_{11}\Delta_{11} + e_{12}\Delta_{12} + e_{13}\Delta_{13})\,\chi_1 = \Delta\chi_1 = 0. \tag{7.10.5}$$

The solutions of systems of equations (7.10.2) for $j = 3, 1$ and $j = 1, 2$ are written down by analogy

$$\begin{aligned} u_2 &= \Delta_{21}\chi_2, & v_2 &= \Delta_{22}\chi_2, & w_2 &= \Delta_{23}\chi_2, \\ u_3 &= \Delta_{31}\chi_3, & v_3 &= \Delta_{32}\chi_3, & w_3 &= \Delta_{33}\chi_3, \end{aligned}$$

and substituting in the remaining equation ($j = 2$ and $j = 3$ respectively) leads to differential equations of the type of eq. (7.10.5). Hence

$$\Delta\chi_s = 0 \quad (s = 1, 2, 3). \tag{7.10.6}$$

It remains to write the general solution

$$\left.\begin{aligned} u &= \Delta_{11}\chi_1 + \Delta_{21}\chi_2 + \Delta_{31}\chi_3 = |\chi_s, e_{s2}, e_{s3}|, \\ v &= \Delta_{12}\chi_1 + \Delta_{22}\chi_2 + \Delta_{32}\chi_3 = |e_{s1}, \chi_s, e_{s3}|, \\ w &= \Delta_{13}\chi_1 + \Delta_{23}\chi_2 + \Delta_{33}\chi_3 = |e_{s1}, e_{s2}, \chi_s|, \end{aligned}\right\} \tag{7.10.7}$$

where the expression on the right hand side of any equation is the determinant (7.10.4) in which the corresponding column is replaced by the column χ_1, χ_2, χ_3, for instance

$$u = \begin{vmatrix} \chi_1 & e_{12} & e_{13} \\ \chi_2 & e_{22} & e_{23} \\ \chi_3 & e_{32} & e_{33} \end{vmatrix}.$$

Being applied to system (7.10.1) this calculation leads to the following solution

$$\left.\begin{aligned} u = {} & \partial_1 \left(D_1^2 + D_2^2 - B_1\nabla^2\right)(\partial_1\chi_1 + \partial_2\chi_2 + \partial_3\chi_3) - \\ & \nabla^2 \left[\partial_2 (\partial_1 B_2\chi_2 - \partial_2 B_1\chi_1) - \partial_3 (\partial_3 B_1\chi_1 - \partial_1 B_3\chi_3)\right], \\ v = {} & \partial_2 \left(D_1^2 + D_2^2 - B_2\nabla^2\right)(\partial_1\chi_1 + \partial_2\chi_2 + \partial_3\chi_3) - \\ & \nabla^2 \left[\partial_3 (\partial_2 B_3\chi_3 - \partial_3 B_2\chi_2) - \partial_1 (\partial_1 B_2\chi_2 - \partial_2 B_1\chi_1)\right], \\ w = {} & \partial_1 \left(D_1^2 + D_2^2 - B_3\nabla^2\right)(\partial_1\chi_1 + \partial_2\chi_2 + \partial_3\chi_3) - \\ & \nabla^2 \left[\partial_1 (\partial_3 B_1\chi_1 - \partial_1 B_3\chi_3) - \partial_2 (\partial_2 B_3\chi_3 - \partial_3 B_2\chi_2)\right], \end{aligned}\right\} \tag{7.10.8}$$

where the denotation for the differential operators is as follows

$$D_1^2 = B_1\partial_1^2 + B_2\partial_2^2 + B_3\partial_3^2, \quad D_2^2 = B_2B_3\partial_1^2 + B_3B_1\partial_2^2 + B_1B_2\partial_3^2, \tag{7.10.9}$$

and the differential equations (7.10.6) for functions χ_s take the form

$$\Delta\chi_s = \nabla^4 D_2^2 \chi_s = 0 \quad (s = 1, 2, 3). \tag{7.10.10}$$

The obtained representation of the solution can be simplified by taking

$$\chi_s = \chi_s' + \chi_s'' \quad (s = 1, 2, 3)$$

and defining χ_s' and χ_s'' in the following way

$$B_s\chi_s' = \partial_s\varphi \ (s = 1, 2, 3); \quad \partial_1\chi_1' + \partial_2\chi_2' + \partial_3\chi_3' = \frac{D_2^2\varphi}{B_1B_2B_3} = \Phi, \tag{7.10.11}$$

$$\nabla^2\chi_s'' = \Psi_s \ (s = 1, 2, 3); \quad \partial_1\chi_1'' + \partial_2\chi_2'' + \partial_3\chi_3'' = 0. \tag{7.10.12}$$

Then by eq. (7.10.10)

$$\nabla^4\Phi = 0, \quad D_2^2\nabla^2\Psi_s = 0. \tag{7.10.13}$$

Solution (7.10.8) is now presented by a sum of the vector $(B_1\chi_1', B_2\chi_2', B_3\chi_3')$ with zero rotor and a solenoidal vector $(\chi_1'', \chi_2'', \chi_3'')$. The first and second ones are expressed in terms of the biharmonic scalar Φ and solenoidal vector (Ψ_1, Ψ_2, Ψ_3) respectively. The particular solutions corresponding to these components are set in the form

$$\left.\begin{aligned}
u' &= \left(D_1^2 + D_2^2 - B_1\nabla^2\right)\partial_1\Phi, \\
u'' &= B_1\nabla^2\Psi_1 - \partial_1\left(B_1\partial_1\Psi_1 + B_2\partial_2\Psi_2 + B_3\partial_3\Psi_3\right), \\
v' &= \left(D_1^2 + D_2^2 - B_2\nabla^2\right)\partial_2\Phi, \\
u'' &= B_2\nabla^2\Psi_2 - \partial_2\left(B_1\partial_1\Psi_1 + B_2\partial_2\Psi_2 + B_3\partial_3\Psi_3\right), \\
w' &= \left(D_1^2 + D_2^2 - B_3\nabla^2\right)\partial_3\Phi, \\
w'' &= B_3\nabla^2\Psi_3 - \partial_3\left(B_1\partial_1\Psi_1 + B_2\partial_2\Psi_2 + B_3\partial_3\Psi_3\right).
\end{aligned}\right\} \tag{7.10.14}$$

The corresponding dilatations and the linear vectors of rotation are given by

$$\vartheta' = D_2^2\nabla^2\Phi, \quad \vartheta'' = 0, \tag{7.10.15}$$

$$\left.\begin{aligned}
\omega_1' &= (B_2 - B_3)\,\partial_2\partial_3\nabla^2\Phi, & \omega_1'' &= \nabla^2\left(B_3\partial_2\Psi_3 - B_2\partial_3\Psi_2\right), \\
\omega_2' &= (B_3 - B_1)\,\partial_3\partial_1\nabla^2\Phi, & \omega_2'' &= \nabla^2\left(B_1\partial_3\Psi_1 - B_3\partial_1\Psi_3\right), \\
\omega_1' &= (B_1 - B_2)\,\partial_1\partial_2\nabla^2\Phi, & \omega_3'' &= \nabla^2\left(B_2\partial_1\Psi_2 - B_1\partial_2\Psi_1\right).
\end{aligned}\right\} \tag{7.10.16}$$

Referring to eqs. (7.10.11)-(7.10.13) it is easy to prove that the starting equations (7.9.6) are satisfied.

Let us also notice that for $\overset{0}{\partial}{}^{ss}= 0$, i.e. in the case of the natural state of $\overset{0}{V}$ −volume (coinciding with v−volume), we have by eqs. (7.9.4) and (7.9.7)

$$A_s = 0, \quad B_s = \frac{1-2\nu}{2(1-\nu)} = \alpha \quad (s = 1, 2, 3).$$

Introducing into consideration vector $\mathbf{G}$

$$\mathbf{G} = -\alpha\nabla^2 (\chi_1 \mathbf{i}_1 + \chi_2 \mathbf{i}_2 + \chi_3 \mathbf{i}_3),$$

one can write solution (7.10.8) in the form

$$\begin{aligned}\mathbf{w} &= -\alpha\nabla\nabla\cdot\mathbf{G} + \nabla\times(\nabla\times\mathbf{G}) = (1-\alpha)\nabla\nabla\cdot\mathbf{G} - \nabla^2\mathbf{G} \\ &= \frac{1}{2(1-\nu)}\nabla\nabla\cdot\mathbf{G} - \nabla^2\mathbf{G},\end{aligned} \tag{7.10.17}$$

where, according to eq. (7.10.10) $\mathbf{G}$ is the biharmonic vector. We arrive at the solution of the equation of the theory of elasticity, eq. (1.7.4) of Chapter 4, in the Boussinesq-Galerkin form.

9.7.11 Bifurcation of equilibrium of a compressed rod

It is assumed that the rod is placed between two horizontal, rigid and smooth plates and its lateral surface is not loaded. The uniaxial state of stress in $\overset{0}{V}$ −volume is caused by the vertical displacement of the upper plate downward on value $L\varepsilon_3$, i.e. $a_3 = L$ whilst the lower plate is motionless. In this state $\overset{0}{\partial}{}^{11}=\overset{0}{\partial}{}^{22}= 0$ and by eqs. (7.4.16), (7.4.17), (7.5.4) and (7.9.4) we have

$$\overset{0}{\partial}{}^{33}= \frac{Q}{\Omega_0} = E\varepsilon_3,\ \varepsilon_1 = \varepsilon_2 = -\nu\varepsilon_3,\ A_1 = A_2 = A = \frac{(1+\nu)\varepsilon_3}{2+(1-\nu)\varepsilon_3},\ A_3 = 0$$

and by eq. (7.9.7)

$$\begin{aligned}B_3 &= \alpha = \frac{1-2\nu}{2(1-\nu)}, \quad B_1 = B_2 = \alpha\sigma, \quad \sigma = 2\frac{(1+\varepsilon_3)}{2+(1-\nu)\varepsilon_3}, \\ \varepsilon_3 &= -\frac{2(1-\sigma)}{2-(1-\nu)\sigma},\end{aligned} \tag{7.11.1}$$

where $-1 < \varepsilon_3 < 0$, such that $0 < \sigma < 1$.

According to eq. (7.9.3) we have on the unloaded lateral surface of V−volume

$$\mathbf{n}\cdot \overset{\bullet}{\hat{D}}= \mathbf{n}\cdot\hat{T}(\mathbf{w}) + 2\mu A(\omega_1 n_2 - \omega_2 n_1)\mathbf{i}_3 = 0,$$

and three boundary conditions are written in the form

$$\left.\begin{aligned}
n_1\left(\frac{\nu}{1-2\nu}\vartheta+\partial_1 u\right)+\frac{1}{2}n_2\left(\partial_2 u+\partial_1 v\right)=0,\\
\frac{1}{2}n_1\left(\partial_2 u+\partial_1 v\right)+n_2\left(\frac{\nu}{1-2\nu}\vartheta+\partial_2 v\right)=0,\\
(2-\sigma)\left(n_1\partial_3 u+n_2\partial_3 v\right)+\sigma\frac{\partial w}{\partial n}=0.
\end{aligned}\right\} \tag{7.11.2}$$

On the end faces of the rod

$$\mathbf{i}_3\cdot \overset{\bullet}{\hat{D}}=\mathbf{i}_3\cdot\hat{T}(\mathbf{w})-2\mu A\left(\omega_1\mathbf{i}_2-\omega_2\mathbf{i}_1\right),$$

and the requirement of no horizontal surface forces and vertical displacement on the end surfaces yields

$$\text{for } a_3=L,\quad a_3=0:\quad \mathbf{i}_3\cdot\overset{\bullet}{\hat{D}}\cdot\mathbf{i}_1=0,\quad \mathbf{i}_3\cdot\overset{\bullet}{\hat{D}}\cdot\mathbf{i}_2=0,\quad w=0,$$

They reduce to the conditions

$$w=0,\quad \partial_3 u=0,\quad \partial_3 v=0, \tag{7.11.3}$$

which are automatically satisfied if the particular solutions (7.10.14) are chosen such that u, v and w are respectively proportional to $\cos\frac{n\pi a_3}{L}$ and $\sin\frac{n\pi a_3}{L}$ $(n=1,2,\ldots)$. Then we assume

$$\Phi=\varphi_n\left(a_1,a_2\right)\cos\frac{n\pi a_3}{L}, \tag{7.11.4}$$

$$\Psi_1=\partial_2\Psi,\ \Psi_2=-\partial_1\Psi,\ \nabla^2\Psi=\frac{n^2\pi^2}{L^2}\psi_n\left(a_1,a_2\right)\cos\frac{n\pi a_3}{L},\ \Psi_3=0, \tag{7.11.5}$$

with vector (Ψ_1,Ψ_2,Ψ_3) being solenoidal. Omitting the nonessential constant multiplier we obtain by means of eq. (7.10.4)

$$\left.\begin{aligned}
u'&=\left[\alpha\sigma\nabla_1^2-\frac{n^2\pi^2}{L^2}\left(1-\sigma+\alpha\sigma^2\right)\right]\partial_1\varphi_n\cos\frac{n\pi a_3}{L},\\
v'&=\left[\alpha\sigma\nabla_1^2-\frac{n^2\pi^2}{L^2}\left(1-\sigma+\alpha\sigma^2\right)\right]\partial_2\varphi_n\cos\frac{n\pi a_3}{L},\\
w'&=\left[\left(1-\sigma+\alpha\sigma\right)\nabla_1^2+\frac{n^2\pi^2}{L^2}\alpha\sigma^2\right]\frac{n\pi}{L}\varphi_n\sin\frac{n\pi a_3}{L}
\end{aligned}\right\} \tag{7.11.6}$$

and further

$$u''=\frac{n^2\pi^2}{L^2}\partial_2\psi_n\cos\frac{n\pi a_3}{L},\ v''=-\frac{n^2\pi^2}{L^2}\partial_1\psi_n\cos\frac{n\pi a_3}{L},\ w''=0. \tag{7.11.7}$$

Functions φ_n and ψ_n are determined from the differential equations

$$\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)^2 \varphi_n(a_1, a_2) = 0, \quad \left(\nabla_1^2 - \sigma\frac{n^2\pi^2}{L^2}\right)\psi_n(a_1, a_2) = 0, \quad (7.11.8)$$

where $\nabla_1^2 = \partial_1^2 + \partial_2^2$. On the lateral surface these functions are subjected to boundary conditions (7.11.2) taking the following form

$$\left.\begin{array}{l}
n_1 \dfrac{\nu}{2(1-\nu)}\sigma(1-\sigma)\dfrac{n^2\pi^2}{L^2}\left(\nabla_1^2 - \dfrac{n^2\pi^2}{L^2}\right)\varphi_n + \dfrac{\partial}{\partial n}\left[\alpha\sigma\left(\nabla_1^2 - \dfrac{n^2\pi^2}{L^2}\right)\right. \\
\quad \left. - \dfrac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\partial_1\varphi_n + \dfrac{n^2\pi^2}{2L^2}\left(\dfrac{\partial}{\partial n}\partial_2\psi_n + \dfrac{\partial}{\partial s}\partial_1\psi_n\right) = 0, \\
n_2 \dfrac{\nu}{2(1-\nu)}\sigma(1-\sigma)\dfrac{n^2\pi^2}{L^2}\left(\nabla_1^2 - \dfrac{n^2\pi^2}{L^2}\right)\varphi_n + \dfrac{\partial}{\partial n}\left[\alpha\sigma\left(\nabla_1^2 - \dfrac{n^2\pi^2}{L^2}\right)\right. \\
\quad \left. - \dfrac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\partial_2\varphi_n + \dfrac{n^2\pi^2}{2L^2}\left(\dfrac{\partial}{\partial s}\partial_2\psi_n - \dfrac{\partial}{\partial n}\partial_1\psi_n\right) = 0, \\
\dfrac{\partial}{\partial n}\left[\left(2\alpha\sigma - \sigma + \sigma^2\right)\nabla_1^2 - \left(2 - 3\sigma + 2\alpha\sigma^2 + \sigma^2\right)\dfrac{n^2\pi^2}{L^2}\right]\varphi_n + \\
\qquad\qquad\qquad\qquad \dfrac{n^2\pi^2}{L^2}(2-\sigma)\dfrac{\partial\psi_n}{\partial s} = 0.
\end{array}\right\} \quad (7.11.9)$$

9.7.12 *Rod of circular cross-section*

Let us limit our consideration to solutions of the form

$$\varphi_n = R_n(r)\cos\theta, \quad \psi_n = Q_n(r)\sin\theta.$$

Functions $R_n(r), Q_n(r)$ are determined by the differential equations

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\left(1 + \frac{n^2\pi^2 r^2}{L^2}\right)\right]^2 R_n = 0,$$
$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\left(1 + \sigma\frac{n^2\pi^2 r^2}{L^2}\right)\right] Q_n = 0,$$

whose solutions, in the case of a solid cylinder, are expressed in terms of the modified Bessel functions

$$R_n(r) = C_1 I_1(x) + C_2 x I_0(x), \; Q_n = C_3 I_1\left(x\sqrt{\sigma}\right); \quad x = \frac{\pi n r}{L}. \quad (7.12.1)$$

The boundary conditions are reduced to the form: at $r = r_0$

$$\left.\begin{aligned}
&\frac{\nu}{2(1-\nu)}\sigma(1-\sigma)\frac{n^2\pi^2}{L^2}\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)R_n + \frac{d}{dr}\left[\alpha\sigma\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right) - \right.\\
&\qquad\left.\frac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\frac{dR_n}{dr} + \frac{n^2\pi^2}{L^2}\left(\frac{1}{r}\frac{dQ_n}{dr} - \frac{Q_n}{r^2}\right) = 0,\\
&\frac{d}{dr}\left[\alpha\sigma\left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right) - \frac{n^2\pi^2}{L^2}(1-\sigma)(1-\alpha\sigma)\right]\frac{R_n}{r} + \\
&\qquad\qquad\frac{n^2\pi^2}{2L^2}\left(\frac{d^2Q_n}{dr^2} - \frac{1}{r}\frac{dQ_n}{dr} + \frac{Q_n}{r^2}\right) = 0,\\
&\frac{d}{dr}\left[\left(2\alpha\sigma - \sigma + \sigma^2\right)\nabla_1^2 - \left(2 - 3\sigma + 2\alpha\sigma^2 + \sigma^2\right)\right]R_n + \\
&\qquad\qquad\frac{n^2\pi^2}{L^2}(2-\sigma)\frac{Q_n}{r} = 0.
\end{aligned}\right\} \tag{7.12.2}$$

Here

$$\nabla_1^2 = \frac{n^2\pi^2}{L^2}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{x^2}\right),$$

and the following properties of Bessel functions

$$\nabla_1^2 I_1(x) = \frac{n^2\pi^2}{L^2}I_1(x), \quad \left(\nabla_1^2 - \frac{n^2\pi^2}{L^2}\right)xI_0(x) = \frac{n^2\pi^2}{L^2}2I_1(x) \quad \text{etc.}$$

are useful while substituting expressions (7.12.1) for $R_n(r), Q_n(r)$ into eq. (7.12.2). By cancelling out the common multiplier $n\pi/L$ we arrive at the system of three homogeneous equations which are linear in the constants C_1, C_2, C_3. The bifurcation values σ, and in turn ε_3, are roots of the determinant of this system $(0 < \sigma < 1)$. They are functions of parameter $x_0 = n\pi r_0/L$ and depend on Poisson's ratio.

9.7.13 Bifurcation of equilibrium of the hollow sphere compressed by uniformly distributed pressure

The radially symmetrical state of equilibrium was considered in Subsection 9.7.3. The close axially symmetrical states of equilibrium can be obtained by superimposing the displacement which does not depend on coordinate λ (longitude)

$$\eta\mathbf{w}(R,\vartheta) = \eta\left[w_R(R,\vartheta)\mathbf{e}_r + w_0(R,\vartheta)\mathbf{e}_\vartheta\right]. \tag{7.13.1}$$

The only nonvanishing component of vector $\boldsymbol{\omega}$ is ω_λ

$$2\omega_\lambda = (\nabla\times\mathbf{w})\cdot\mathbf{e}_\lambda = \frac{\partial w_\vartheta}{\partial R} - \frac{1}{R}\left(\frac{\partial\omega_R}{\partial\vartheta} - w_\vartheta\right). \tag{7.13.2}$$

Referring to eqs. (7.2.1), (7.3.7) and (7.8.3) we obtain

$$\overset{\bullet}{\hat{D}} = \hat{T}(\mathbf{w}) + 2\psi(R)\,\omega_\lambda(\mathbf{e}_R\mathbf{e}_\vartheta - \mathbf{e}_\vartheta\mathbf{e}_R). \tag{7.13.3}$$

Here

$$\psi(R) = \frac{\lambda \overset{0}{s}_1 - 2\mu}{\frac{f(R)}{R} + f'(R)} + \mu = \frac{(3\lambda + 2\mu)(c_1 - 1)\left(1 - \frac{R_1^3}{4R^3}\right)}{2c_1 - \frac{R_1^3}{4R^3\mu}(3\lambda + 2\mu)(c_1 - 1)}, \tag{7.13.4}$$

where constant c_2 is determined from formula (7.3.9) by the condition of free internal surface $R = R_1$ of the hollow sphere. Further we have

$$\left.\begin{aligned}
\nabla\cdot\hat{T}(\mathbf{w}) &= (\lambda+2\mu)\nabla\nabla\cdot\mathbf{w} - 2\mu\nabla\times\boldsymbol{\omega}\\
&= \mathbf{e}_R\left[(\lambda+2\mu)(\nabla\cdot\mathbf{w})' - \frac{2\mu}{R\sin\vartheta}\frac{\partial}{\partial\vartheta}(\omega_\lambda\sin\vartheta)\right] +\\
&\qquad \mathbf{e}_\vartheta\left[(\lambda+2\mu)\frac{\partial\nabla\cdot\mathbf{w}}{R\partial\vartheta} + 2\mu\left(\omega_\lambda' + \frac{\omega_\lambda}{R}\right)\right],\\
\nabla\cdot 2\psi(R)\,\omega_\lambda(\mathbf{e}_R\mathbf{e}_\vartheta - \mathbf{e}_\vartheta\mathbf{e}_R) &=\\
= -\frac{2\psi(R)}{R\sin\vartheta}\mathbf{e}_R\frac{\partial}{\partial\vartheta}(\omega_\lambda\sin\vartheta) &+ 2\mathbf{e}_\vartheta\left[\frac{\psi(R)}{R}\omega_\lambda + (\psi(R)\,\omega_\lambda)'\right],
\end{aligned}\right\} \tag{7.13.5}$$

where a prime indicates a differentiation with respect to R. In the case of no mass forces the equilibrium equations (7.7.14) take the form

$$\left.\begin{aligned}
(\lambda+2\mu)(\nabla\cdot\mathbf{w})' - \frac{2g(R)}{R\sin\vartheta}\frac{\partial}{\partial\vartheta}\omega_\lambda\sin\vartheta = 0,\\
(\lambda+2\mu)\frac{\partial\nabla\cdot\mathbf{w}}{R\partial\vartheta} + \frac{2}{R}[g(R)\,R\omega_\lambda]' = 0,
\end{aligned}\right\} \tag{7.13.6}$$

where

$$g(R) = \psi(R) + \mu. \tag{7.13.7}$$

Assuming that the pressure remains normal to the deformed surface of the sphere, we turn to the boundary condition (7.7.18). By eq. (7.7.17) we have

$$\begin{aligned}
\dot{\mathbf{R}}^s n_s &= \dot{\mathbf{R}}^1 = -\frac{1}{f'^2}\frac{\partial w_R}{\partial R}\mathbf{e}_R - \frac{1}{f'f}\left(\frac{\partial w_R}{\partial\vartheta} - w_\vartheta\right)\mathbf{e}_\vartheta,\\
\overset{0}{\mathbf{R}}{}^s\, n_s &= \overset{0}{\mathbf{R}}{}^1 = \frac{\mathbf{e}_R}{f'}
\end{aligned}$$

and further

$$\sum_k \frac{\overset{0}{\mathbf{e}}_k \cdot \nabla \mathbf{w} \cdot \overset{0}{\mathbf{e}}_k}{\sqrt{\overset{0}{G}_k}} = \frac{1}{f'} \mathbf{e}_R \cdot \nabla \mathbf{w} \cdot \mathbf{e}_R + \frac{R}{f} \left(\mathbf{e}_\vartheta \cdot \nabla \mathbf{w} \cdot \mathbf{e}_\vartheta + \mathbf{e}_\lambda \cdot \nabla \mathbf{w} \cdot \mathbf{e}_\lambda \right)$$
$$= \frac{1}{f'} \frac{\partial w_R}{\partial R} + \frac{1}{f} \left(\frac{\partial \mathbf{w}}{\partial \vartheta} \cdot \mathbf{e}_\vartheta + \frac{1}{\sin \vartheta} \frac{\partial \mathbf{w}}{\partial \lambda} \cdot \mathbf{e}_\lambda \right)$$
$$= \frac{1}{f'} \frac{\partial w_R}{\partial R} + \frac{R}{f} \left(\nabla \cdot \mathbf{w} - \frac{\partial w_R}{\partial R} \right),$$

because

$$\nabla \cdot \mathbf{w} = \frac{\partial w_R}{\partial R} + 2 \frac{w_R}{R} + \frac{1}{R \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(w_\vartheta \sin \vartheta \right). \tag{7.13.8}$$

For this reason

$$\sqrt{\frac{\overset{0}{G}}{g}} \left(\dot{\mathbf{R}}^s + \overset{0}{\mathbf{R}^s} \sum_k \frac{\overset{0}{\mathbf{e}}_k \cdot \nabla \mathbf{w} \cdot \overset{0}{\mathbf{e}}_k}{\sqrt{\overset{0}{G}_k}} \right) =$$
$$= \frac{f}{R} \left[\mathbf{e}_R \left(\nabla \cdot \mathbf{w} - \frac{\partial w_R}{\partial R} \right) - \frac{\mathbf{e}_\vartheta}{R} \left(\frac{\partial w_R}{\partial \vartheta} - w_\vartheta \right) \right].$$

On the surface of the sphere ($\mathbf{e}_R = \mathbf{n}$)

$$\mathbf{n} \cdot \overset{\bullet}{\hat{D}} = \mathbf{e}_R \cdot \hat{T}(\mathbf{w}) + 2\psi(R)\,\omega_\lambda \mathbf{e}_\vartheta$$
$$= \lambda \left(\nabla \cdot \mathbf{w} + 2\mu \frac{\partial w_R}{\partial R} \right) \mathbf{e}_R + 2 \left[g(R)\,\omega_\lambda + \frac{\mu}{R} \left(\frac{\partial w_R}{\partial \vartheta} - w_\vartheta \right) \right] \mathbf{e}_\vartheta .$$

The boundary conditions are now led to the form: on the external surface of the sphere

$$R = R_0 : \quad \left. \begin{aligned} \left[\lambda + \frac{f(R)}{R} p \right] \nabla \cdot \mathbf{w} + \left(2\mu - p \frac{f}{R} \right) \frac{\partial w_R}{\partial R} &= 0, \\ 2g(R)\,\omega_\lambda + \left(2\mu - p \frac{f}{R} \right) \left(\frac{\partial w_R}{\partial \vartheta} - w_\vartheta \right) &= 0, \end{aligned} \right\} \tag{7.13.9}$$

and on the internal surface

$$R = R^1 : \ \lambda \nabla \cdot \mathbf{w} + 2\mu \frac{\partial w_R}{\partial R} = 0, \quad g(R)\,\omega_\lambda + \mu \left(\frac{\partial w_R}{\partial \vartheta} - w_\vartheta \right) = 0. \tag{7.13.10}$$

Differential equations (7.13.6) together with the boundary conditions (7.13.9), (7.13.10) provide us with the statement of the homogeneous boundary-value problem. The conditions for the existence of nontrivial solutions determine the bifurcation values of parameter p, the minimum value being the critical pressure. Similar calculations lead to the statement of the boundary-value problem yielding the critical external pressure in the case of a hollow circular cylinder.

The solution of the system of differential equations of equilibrium (7.13.6) is sought in the form

$$w_R = a_n(R) P_n(\cos\vartheta), \quad w_\vartheta = -b_n(R) P_n^1(\cos\vartheta) = b_n(R) P_n'(\cos\vartheta)\sin\vartheta. \tag{7.13.11}$$

This solution remains bounded at both poles of the sphere ($\vartheta = 0, \vartheta = \pi$) only for integer n. Using eqs. (7.13.8), (7.13.2) and the recurrent relations for Legendre's polynomials we obtain

$$\left.\begin{aligned}
\nabla\cdot\mathbf{w} = \left(a_n' + 2\frac{a_n}{R}\right) P_n + \frac{b_n}{R}\left(-P_n''\sin^2\vartheta + 2\cos\vartheta P_n'\right) = \\
= \varphi_n(R) P_n(\cos\vartheta), \\
2\omega_\lambda = \chi_n(R) P_n'(\cos\vartheta)\sin\vartheta, \qquad\qquad\qquad\qquad \\
\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta} 2\omega_\lambda \sin\vartheta = \chi_n(R)\, n(n+1) P_n(\cos\vartheta), \qquad
\end{aligned}\right\} \tag{7.13.12}$$

where the following denotations are introduced

$$\varphi_n(R) = a_n' + 2\frac{a_n}{R} + n(n+1)\frac{b_n}{R}, \quad \chi_n(R) = b_n' + \frac{a_n + b_n}{R}. \tag{7.13.13}$$

The variables are separated in eq. (7.13.6) and we arrive at the system of ordinary differential equations written in the form

$$\left.\begin{aligned}
(\lambda + 2\mu) R^2 \varphi_n'(R) - Rg(R)\, n(n+1)\chi_n(R) = 0, \\
(\lambda + 2\mu)\varphi_n(R) - [Rg(R)\chi_n(R)]' = 0.
\end{aligned}\right\} \tag{7.13.14}$$

The solution is as follows

$$\varphi_n(R) = (n+1) A_n R^n - \frac{nB_n}{R^{n+1}}, \quad \chi_n(R) = \frac{\lambda + 2\mu}{g(R)}\left(A_n R^n + \frac{B_n}{R^{n+1}}\right), \tag{7.13.15}$$

where A_n, B_n are arbitrary constants. Inserting now $\varphi_n(R)$ and $\chi_n(R)$ in system (7.13.13) we arrive at the following system of linear inhomogeneous equations for the unknown functions $a_n(R)$ and $b_n(R)$

$$a_n = -n\tilde{C}_n R^{n-1} + (n+1)\frac{\tilde{D}_n}{R^{n+2}}, \quad b_n = \tilde{C}_n R^{n-1} + \frac{\tilde{D}_n}{R^{n+2}}. \tag{7.13.16}$$

Here

$$\left.\begin{aligned}\tilde{C}_n &= \frac{1}{2n+1}\int\limits_{R_1}^{R}\left[(n+1)\,\chi_n - \varphi_n\right]\frac{dR}{R^{n-1}} + C_n,\\ \tilde{D}_n &= \frac{1}{2n+1}\int\limits_{R_1}^{R}\left(\varphi_n + n\chi_n\right)R^{n+2}dR + D_n,\end{aligned}\right\} \tag{7.13.17}$$

where C_n and D_n are constants.

Variables R and ϑ are separated also in the linear boundary conditions (7.13.9), (7.13.10). Inserting the obtained functions $a_n\,(R)$ and $b_n\,(R)$ in these equations yields a system of four homogeneous linear equations for constants A_n, B_n, C_n, D_n. The vanishing determinant of the system results in the equation for the bifurcation values of parameter p. The latter appears also in this equation in terms of function $g\,(R)$ from eqs. (7.13.7), (7.13.4), the constant c_1 being related nonlinearly to p by means of eq. (7.3.10). The critical pressure p_{cr} is the minimum bifurcation value of p determined by a proper choice of the number n of nodes of the sought form of equilibrium for the taken ratio R_1/R_0.

Part V

Appendices

Appendix A
Basics tensor algebra

A.1 Scalars and vectors

In natural philosophy one considers the quantities determining the properties of physical objects and the processes in them. Prescribing numerical values in a chosen coordinate system contains an arbitrariness due to the choice of the coordinate system. However the relationships between the quantities are independent of the introduced methods of description. Tensor calculus presents a mathematical means for formulating the invariant (i.e. independent of coordinate systems) relationships between the studied objects.

The simplest physical quantity is a scalar which is given by a numerical value, this value being unchanged in all coordinate systems. Examples of scalars are density, temperature, work, kinetic energy etc. A scalar is an invariant by definition.

The vector is a more complex physical quantity which has a prescribed direction in addition to a numerical value. Examples of vectors are velocity, acceleration, force etc. For denotation of vectors the bold font and low-case Latin alphabet is used. The operations of linear algebra are assumed to be known, namely the scalar and vector products of vectors $\mathbf{a}$ and $\mathbf{b}$ are denoted as $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ respectively. For the forthcoming analysis it is necessary to recall the rules of transforming the vector projections under the rotation of the orthogonal Cartesian coordinate system since only these systems are used in what follows, unless stated otherwise. Let us notice in passing that the quantities are considered in the Euclidean three-dimensional space (E_3).

Let $Ox_1x_2x_3$ be the initial system of axes ("old axes") and $Ox_1'x_2'x_3'$ ("new axes") be obtained from the initial system by a rotation. Let also $\mathbf{i}_s'$ and $\mathbf{i}_k$ denote the unit vectors prescribing the directions of new axis Ox_s' and old axis Ox_k , the cosine function of the angle between these axes being designated as $\alpha_{ks} = \mathbf{i}_s' \cdot \mathbf{i}_k$. Vector $\mathbf{a}$ can be prescribed by the projections a_s' a_k onto axes of the new and old systems respectively and thus can be presented by the expansions in terms of the above unit vectors

$$\left.\begin{aligned}\mathbf{a} = a_1'\mathbf{i}_1' + a_2'\mathbf{i}_2' + a_3'\mathbf{i}_3' = \sum_{s=1}^{3} a_s'\mathbf{i}_s' = a_s'\mathbf{i}_s',\\ \mathbf{a} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3 = \sum_{k=1}^{3} a_k\mathbf{i}_k = a_k\mathbf{i}_k.\end{aligned}\right\} \tag{A.1.1}$$

Here and throughout the book the sign of summation over the repeating index (dummy index) is omitted. The case in which the summation is not required, i.e. a single component of the sum is considered, is specified by the symbol $\not\Sigma_s$, i.e. do not sum over s!. Clearly, the dummy index can be changed arbitrarily, for instance $a_s b_s = a_k b_k, \alpha_{st} b_s b_t = \alpha_{rq} b_r b_q$ are the sums having respectively three and nine terms. The non-repeating indices are called free and they are ascribed values of 1,2,3. Free indices on both sides of the equality must coincide, for example, denotation $q_r = b_{rk} a_k$ presents three equalities, whilst $q_{rt} = C_{mnrt} b_{mn}$ means nine equalities and nine terms on the right hand side of each equality.

Being applied to the unit vectors $\mathbf{i}_s'$ and $\mathbf{i}_k$ formulae (A.1.2) are written in the form

$$\mathbf{i}_s' = \alpha_{sk}\mathbf{i}_k, \quad \mathbf{i}_k = \alpha_{sk}\mathbf{i}_s', \tag{A.1.2}$$

since α_{sk} denotes the projection of $\mathbf{i}_s'$ onto $\mathbf{i}_k$ (or $\mathbf{i}_k$ onto $\mathbf{i}_s'$). Introducing Kronecker's symbol

$$\delta_{st} = \begin{cases} 0, & s \neq t, \\ 1, & s = t, \end{cases} \tag{A.1.3}$$

and writing the conditions of the orthonormality of these vectors

$$\mathbf{i}_s' \cdot \mathbf{i}_t' = \delta_{st}, \quad \mathbf{i}_k \cdot \mathbf{i}_m = \delta_{km}, \tag{A.1.4}$$

we arrive at the formulae relating the cosine functions of the angles between the axes of the new and old systems

$$\left.\begin{aligned}\mathbf{i}_s' \cdot \mathbf{i}_t' = \delta_{st} = \alpha_{sk}\mathbf{i}_k \cdot \alpha_{tm}\mathbf{i}_m = \alpha_{sk}\alpha_{tm}\delta_{km} = \alpha_{sk}\alpha_{tk},\\ \mathbf{i}_k' \cdot \mathbf{i}_m' = \delta_{km} = \alpha_{sk}\mathbf{i}_s' \cdot \alpha_{tm}\mathbf{i}_t' = \alpha_{sk}\alpha_{tm}\delta_{st} = \alpha_{sk}\alpha_{sm}.\end{aligned}\right\} \tag{A.1.5}$$

Inserting eq. (A.1.2) into formulae (A.1.1) leads to the law of transformation of projections of a vector

$$\left.\begin{array}{ll} \mathbf{a} = a_k \mathbf{i}_k = a_k \alpha_{sk} \mathbf{i}'_s = a'_s \mathbf{i}'_s, & a'_s = \alpha_{sk} a_k, \\ \mathbf{a} = a'_s \mathbf{i}'_s = a'_s \alpha_{sk} \mathbf{i}_k = a_k \mathbf{i}_k, & a_k = \alpha_{sk} a'_s. \end{array}\right\} \tag{A.1.6}$$

Clearly, the numerical value of the vector projection depends on the direction of the axis that the vector is projected onto. For this reason the wording "projection of a vector onto an axis is a scalar" is confusing as scalar is an invariant physical quantity. The invariant of vector **a** is its absolute value denoted by a. Of course, it follows from the transformation law (A.1.6) that

$$a^2 = \mathbf{a} \cdot \mathbf{a} = a_k a_k = \alpha_{sk} \alpha_{tk} a'_s a'_t = \delta_{st} a'_s a'_t = a'_s a'_s. \tag{A.1.7}$$

A scalar invariant of two vectors is its scalar product

$$\mathbf{a} \cdot \mathbf{b} = a_s b_s = a'_k b'_k. \tag{A.1.8}$$

If it is known that $a'_k b'_k = a_s b_s$ where b_s and b'_k denote respectively the projections of vector **b** onto old and new axes, then a_s and a'_k are also the projections of vector a onto these axes. Indeed,

$$a'_k b'_k = a_s b_s = a_s \alpha_{ks} b'_k, \quad a'_k = \alpha_{ks} a_s,$$

since directions $\mathbf{i}'_k$ can be taken arbitrarily. Values a'_k obey the law of transformation of vector projections, which is required.

Provided that both the old and new systems are right-handed or left-handed under the transformation of rotation, then the determinant of the cosine matrix is equal to unity

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = |\alpha_{sk}| = 1. \tag{A.1.9}$$

If however the rotation is associated with a mirror transformation (for example $\mathbf{i}'_1 = \mathbf{i}_1, \mathbf{i}'_2 = \mathbf{i}_2, \mathbf{i}'_3 = -\mathbf{i}_3$) then $|\alpha_{sk}| = -1$. In what follows we consider only transformations of rotation. This allows us to avoid some complications, for example the distinction between true vectors and pseudovectors.

A.2 The Levi-Civita symbols

These are as follows

$$e_{rst} = \mathbf{i}_r \cdot (\mathbf{i}_s \times \mathbf{i}_t), \quad e'_{rst} = \mathbf{i}'_r \cdot (\mathbf{i}'_s \times \mathbf{i}'_t). \tag{A.2.1}$$

They are equal to zero if any of indices r, s, t are coincident. If indices r, s, t are different and follow in the order 123 or 231 or 312 then the symbols are equal to $+1$. Finally if this order is violated then the symbols are equal to -1. This definition is valid in any orthogonal Cartesian system obtained from the initial one by the transformation of rotation ($e_{rst} = e'_{rst}$). Using the Levi-Civita symbols we can write the vector products of the unit vectors in the form

$$\mathbf{i}_s \times \mathbf{i}_t = e_{rst}\mathbf{i}_r, \quad \mathbf{i}'_s \times \mathbf{i}'_t = e'_{rst}\mathbf{i}'_r, \tag{A.2.2}$$

hence

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = a_s\mathbf{i}_s \times b_t\mathbf{i}_t = e_{rst}\mathbf{i}_r a_s b_t, \quad c_r = e_{rst}a_s b_t. \tag{A.2.3}$$

For example

$$c_1 = e_{1st}a_s b_t = e_{123}a_2b_3 + e_{132}a_3b_2 = a_2b_3 - a_3b_2.$$

Using eq. (A.2.2) and expanding the double vector product, we have

$$\mathbf{i}_r \times (\mathbf{i}_s \times \mathbf{i}_t) = \mathbf{i}_s\delta_{rt} - \mathbf{i}_t\delta_{rs} = \mathbf{i}_r \times e_{qst}\mathbf{i}_q = e_{mrq}e_{stq}\mathbf{i}_m.$$

Multiplying both sides of this equation by $\mathbf{i}_k$ we arrive at the relationship

$$e_{krq}e_{stq} = \delta_{ks}\delta_{rt} - \delta_{kt}\delta_{rs}. \tag{A.2.4}$$

For $r = t$, i.e. after summation over t and q we obtain

$$e_{ktq}e_{stq} = \delta_{ks}\delta_{tt} - \delta_{kt}\delta_{ts} = 3\delta_{ks} - \delta_{ks} = 2\delta_{ks}. \tag{A.2.5}$$

Finally after summation over all three indices we find

$$e_{ktq}e_{ktq} = 2\delta_{kk} = 6. \tag{A.2.6}$$

Using eq. (A.2.5) we have also

$$e_{qst}\mathbf{i}_s \times \mathbf{i}_t = e_{qst}e_{rst}\mathbf{i}_r = 2\delta_{qr}\mathbf{i}_r = 2\mathbf{i}_q, \quad \mathbf{i}_q = \frac{1}{2}e_{qst}\mathbf{i}_s \times \mathbf{i}_t. \tag{A.2.7}$$

For instance

$$\mathbf{i}_1 = \frac{1}{2}\left(\mathbf{i}_2 \times \mathbf{i}_3 - \mathbf{i}_3 \times \mathbf{i}_2\right).$$

Introducing the Levi-Civita symbols and utilising the above rules considerably reduces the number of calculations in tensor algebra.

A.3 Tensor of second rank

The definition of this physical object which is more complicated than the vector can be expressed in different ways. The first one is as follows.

A square matrix

$$\begin{Vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{Vmatrix} = \|q_{st}\|, \tag{A.3.1}$$

is considered in the system of axes $Ox_1x_2x_3$. With the help of the elements of this matrix the projections a_k of vector $\mathbf{a}$ are associated with numbers b_s by means of the following rule

$$\left.\begin{aligned} b_1 &= q_{11}a_1 + q_{12}a_2 + q_{13}a_3, \\ b_2 &= q_{21}a_1 + q_{22}a_2 + q_{23}a_3, \\ b_3 &= q_{31}a_1 + q_{32}a_2 + q_{33}a_3 \end{aligned}\right\} \tag{A.3.2}$$

or in a shorthand notation

$$b_s = q_{sk}a_k. \tag{A.3.3}$$

The same rule defines multiplication of square matrix $\|q_{st}\|$ by a column a resulting in a column b.

Definition. Matrix $\|q_{st}\|$ defines a tensor of second rank $\hat{Q}$ if for any vector $\mathbf{a}$, b_s are the projections of vector $\mathbf{b}$. The elements q_{st} of the matrix are referred to as the components of tensor $\hat{Q}$ in the assumed system of axes. The operation of association of vector $\mathbf{b}$ to vector $\mathbf{a}$ by means of the tensor is called postmultiplication of this tensor by vector $\mathbf{a}$, i.e.

$$\mathbf{b} = \hat{Q} \cdot \mathbf{a}. \tag{A.3.4}$$

One physical quantity (vector $\mathbf{a}$) is transformed into another quantity (vector $\mathbf{b}$) with the help of matrix (A.3.1). This means that this matrix in the system of axes $Ox_1x_2x_3$ describes a quantity having an independent physical meaning. It remains to require that this ability of associating vector $\mathbf{b}$ to vector $\mathbf{a}$ holds in any coordinate system. This implies that, under transition to new axes $Ox'_1x'_2x'_3$, elements q_{sk} of matrix $\|q_{st}\|$ must obey the transformation law ensuring transformation of b_s as being the vector projections (due to rule (A.1.6)) when a_k is transformed using this rule. Hence

$$b'_s = \alpha_{sk}b_k = \alpha_{sk}q_{kt}a_t, \quad a_t = a_{pt}a'_p,$$

which yields

$$b'_s = \alpha_{sk}\alpha_{pt}q_{kt}a'_p = q'_{sp}a'_p, \tag{A.3.5}$$

where

$$q'_{sp} = \alpha_{sk}\alpha_{pt}q_{kt}. \tag{A.3.6}$$

Comparison of eqs. (A.3.5) and (A.3.3) shows that the rule of postmultiplication of tensor $\hat{Q}$ by vector $\mathbf{a}$ is valid in the new system of axes provided that the components of this tensor obey the transformation law (A.3.6). The inverse transformation has the form

$$q_{sk} = \alpha_{ps}\alpha_{tk}q'_{pt}. \tag{A.3.7}$$

The matrix should not be confused with the tensor. The latter is an independent physical quantity whose description requires a matrix. Based on the transformation laws (A.3.6) and (A.3.7) we can give the second definition of the tensor of second rank as a physical quantity whose components obey these laws under rotation of the coordinate system.

The determinant of matrix $\|q_{st}\|$ is denoted as $|q_{st}|$ and is one of the invariants of tensor $\hat{Q}$. Indeed

$$q = |q_{st}| = \left|q'_{pq}\boldsymbol{\alpha}_{ps}\boldsymbol{\alpha}_{qt}\right| = \left|q'_{pq}\right| \left|\boldsymbol{\alpha}_{ps}\right| \left|\boldsymbol{\alpha}_{qt}\right| = \left|q'_{pq}\right| = q'. \tag{A.3.8}$$

Here we have used the rule of multiplication of the determinants as well as eq. (A.1.9).

Example 1. The vector of the angular momentum $\mathbf{k}^0$ of a rigid body rotating about a fixed point O is determined in terms of the vector of angular velocity as follows

$$\left.\begin{aligned} k_1^0 &= \Theta_{11}\omega_1 + \Theta_{12}\omega_2 + \Theta_{13}\omega_3, \\ k_2^0 &= \Theta_{21}\omega_1 + \Theta_{22}\omega_2 + \Theta_{23}\omega_3, \\ k_3^0 &= \Theta_{31}\omega_1 + \Theta_{32}\omega_2 + \Theta_{33}\omega_3, \end{aligned}\right\} \tag{A.3.9}$$

where $\Theta_{ik} = -\Theta_{ki}\,(i \neq k)$ denote the products of inertia and Θ_{ii} denote the moments of inertia about the axes fixed in the body. The table of these values determines the tensor of inertia $\hat{\Theta}$ of the body about point O and formulae (A.3.9) can be set in the form

$$\mathbf{k}^0 = \hat{\Theta}^0 \cdot \boldsymbol{\omega}. \tag{A.3.10}$$

The tensor of inertia describes the behaviour of the rotating rigid body in terms of the matrix with elements Θ_{ik} transformed due to the rule (A.3.6) under rotation of the system of axes.

Example 2. Let us consider a rod with a straight axis (axis x_3) whose left end is fixed. Let the origin of the system of axes $Cx_1x_2x_3$ be placed at the centre of inertia of the right cross-section of the rod, axes Cx_2 and Cx_3 being directed along the principal central axes of inertia of this cross-section. Applying a transverse force F_1 (F_2) of direction Cx_1 (Cx_2) at point C results in the displacement of this point $f_1 = \sigma_1 F_1$ ($f_2 = \sigma_2 F_2$) in the direction of the force. Let $f_3 = \sigma_3 F_3$ denote the displacement along axis Cx_3 due to the axial force F_3. The formulae

$$f_1 = \sigma_1 F_1, \quad f_2 = \sigma_2 F_2, \quad f_3 = \sigma_3 F_3 \tag{A.3.11}$$

show the particular case of eq. (A.3.2) when matrix (A.3.1) has a diagonal form

$$\begin{Vmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{Vmatrix}. \tag{A.3.12}$$

This determines the compliance tensor at point C denoted as $\hat{\Sigma}$ and formulae (A.3.11) can be written in the form

$$\mathbf{f} = \hat{\Sigma} \cdot \mathbf{F}.$$

The absence of the non-diagonal components in matrix (A.3.12) is caused by a special choice of the coordinate system and of course is not an invariant property of tensor $\hat{\Sigma}$. For example, under rotation of the system of axes through angle φ about axis Cx_3 the table of cosine functions takes the form

	x_1	x_2	x_3
x_1'	$\cos\varphi$	$\sin\varphi$	0
x_2'	$-\sin\varphi$	$\cos\varphi$	0
x_3'	0	0	1

(A.3.13)

and by eq. (A.3.6) the matrix of components of tensor $\hat{\Sigma}$ is as follows

$$\begin{Vmatrix} \frac{1}{2}\left[(\sigma_1+\sigma_2) - (\sigma_2-\sigma_1)\cos 2\varphi\right] & \frac{1}{2}(\sigma_2-\sigma_1)\sin 2\varphi & 0 \\ \frac{1}{2}(\sigma_2-\sigma_1)\sin 2\varphi & \frac{1}{2}\left[(\sigma_1+\sigma_2) + (\sigma_2-\sigma_1)\cos 2\varphi\right] & 0 \\ 0 & 0 & \sigma_3 \end{Vmatrix} \tag{A.3.14}$$

so that formulae (A.3.2) yield

$$\begin{aligned} f_1' &= \frac{1}{2}\left[(\sigma_1+\sigma_2)F_1' + (\sigma_2-\sigma_1)\left(-F_1'\cos 2\varphi + F_2'\sin 2\varphi\right)\right], \\ f_2' &= \frac{1}{2}\left[(\sigma_1+\sigma_2)F_2' + (\sigma_2-\sigma_1)\left(F_1'\sin 2\varphi + F_2'\cos 2\varphi\right)\right], \\ f_3' &= \sigma_3 F_3'. \end{aligned}$$

Example 3. The affine transformation is described by the formulae

$$\left.\begin{aligned} x_1 &= \lambda_{11}a_1 + \lambda_{12}a_2 + \lambda_{13}a_3, \\ x_2 &= \lambda_{21}a_1 + \lambda_{22}a_2 + \lambda_{23}a_3, \\ x_3 &= \lambda_{31}a_1 + \lambda_{32}a_2 + \lambda_{33}a_3, \end{aligned}\right\} \tag{A.3.15}$$

where a_s and x_s denote respectively the coordinates of points M and M' with the position vectors $\mathbf{r}$ and $\mathbf{R}$ in the system of axes $OXYZ$. Under this transformation a straight line remains a straight line, a straight-line segment rotates and changes length, a rectangular transforms into a parallelogram and a circle is transformed into an ellipse. Matrix $\|\lambda_{st}\|$ determines a tensor of second rank $\hat{\Lambda}$ and formulae (A.3.15) are written in the form

$$\mathbf{R} = \hat{\Lambda} \cdot \mathbf{r}.$$

A.4 Basic tensor operations

The sum of tensors $\hat{P}$ and $\hat{Q}$ is termed tensor $\hat{T}$ which being postmultiplied by a vector $\mathbf{a}$ defines the vector which is equal to the geometric sum of $\hat{P}\cdot$ $\mathbf{a}$ and $\dot{Q}\cdot$ $\mathbf{a}$, i.e.

$$\hat{T} = \hat{P} + \hat{Q} = \hat{Q} + \hat{P}.$$

It follows from the definition that the components t_{st} of tensor $\hat{T}$ are equal to the sums of the corresponding components of tensors $\hat{P}$ and $\hat{Q}$, i.e. $t_{st} = p_{st} + q_{st}$. By analogy the product $\lambda\hat{Q}$ of tensor $\hat{Q}$ and scalar λ is defined. The components of this tensor are equal to λq_{st}.

Let us construct the scalar product of vectors $\mathbf{c}$ and $\mathbf{b} = \hat{Q} \cdot \mathbf{a}$

$$\mathbf{c} \cdot \mathbf{b} = \mathbf{c} \cdot \hat{Q} \cdot \mathbf{a} = q_{st} c_s a_t = q_{ts} c_t a_s.$$

Dummy indices s and t are interchanged which does not change the sum. Denoting $q_{ts} c_t = e_s$ we have

$$\mathbf{c} \cdot \mathbf{b} = a_s e_s.$$

Since on the left hand side we have an invariant and $\mathbf{a}$ is a vector, we can refer to the remark at the end of Section A.1 and conclude that e_s are the vector projections. They are obtained by means of the matrix

$$\|q_{ts}\| = \begin{Vmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{Vmatrix}, \tag{A.4.1}$$

transposing matrix (A.3.1). The above-said suggests that this matrix defines the components of the tensor of second rank $\hat{Q}^*$ referred to as the transpose of $\hat{Q}$, such that

$$\mathbf{e} = \hat{Q}^* \cdot \mathbf{a}.$$

Premultiplication of tensor $\hat{Q}$ by vector $\mathbf{a}$ is defined as postmultiplying $\hat{Q}^*$ by the same vector $\mathbf{a}$

$$\mathbf{a} \cdot \hat{Q} = \hat{Q}^* \cdot \mathbf{a}. \tag{A.4.2}$$

In contrast to eq. (A.3.3) the projections of this vector are equal to $q_{ts}a_t$. Based on this observation we can write the identities

$$\mathbf{c} \cdot \left(\hat{Q} \cdot \mathbf{a}\right) = \left(\mathbf{c} \cdot \hat{Q}\right) \cdot \mathbf{a} = \mathbf{a} \cdot \left(\hat{Q}^* \cdot \mathbf{c}\right) = q_{st}a_t c_s$$

and omit the parentheses when writing the bilinear form of values a_t and c_s

$$\mathbf{c} \cdot \hat{Q} \cdot \mathbf{a} = \mathbf{a} \cdot \hat{Q}^* \cdot \mathbf{c}. \tag{A.4.3}$$

The tensor of second rank is referred to as being symmetric if it is equal to its transpose

$$\hat{S} = \hat{S}^*, \quad s_{kt} = s_{tk}. \tag{A.4.4}$$

This tensor is given by six components. The tensor

$$\hat{\Omega} = -\hat{\Omega}^*, \quad \omega_{kt} = -\omega_{tk} \tag{A.4.5}$$

having zero diagonal components is called skew-symmetric. It is given by three components denoted by ω_r

$$\omega_{st} = -e_{rst}\omega_r, \quad \omega_q = -\frac{1}{2}e_{qst}\omega_{st}, \tag{A.4.6}$$

so that the matrix of components Ω is written as follows

$$\left\| \begin{matrix} 0 & \omega_{12} = -\omega_3 & \omega_{13} = \omega_2 \\ \omega_{21} = \omega_3 & 0 & \omega_{23} = -\omega_1 \\ \omega_{31} = -\omega_2 & \omega_{32} = \omega_1 & 0 \end{matrix} \right\|. \tag{1.4.7}$$

The identity

$$\hat{Q} = \frac{1}{2}\left(\hat{Q} + \hat{Q}^*\right) + \frac{1}{2}\left(\hat{Q} - \hat{Q}^*\right) = \hat{S} + \hat{\Omega} \tag{A.4.8}$$

describes splitting the tensor into symmetric and skew-symmetric parts. Their components are given by

$$s_{ik} = \frac{1}{2}\left(q_{ik} + q_{ki}\right), \quad \omega_r = \frac{1}{2}e_{rst}q_{ts}. \tag{A.4.9}$$

For example

$$\omega_1 = \frac{1}{2}e_{1st}q_{ts} = \frac{1}{2}\left(e_{123}q_{32} + e_{132}q_{23}\right) = \frac{1}{2}\left(q_{32} - q_{23}\right).$$

The property of the tensor to be symmetric is invariant with respect to rotation of the system of axes. Indeed, by eq. (A.3.6)

$$q'_{pq} = q_{st}\alpha_{ps}\alpha_{qt} = q_{ts}\alpha_{pt}\alpha_{qs} = q_{st}\alpha_{qs}\alpha_{pt} = q'_{qp}.$$

The components of the skew-symmetric tensor are transformed under rotation of the coordinate system as the projections of vector $\boldsymbol{\omega}$, indeed

$$
\begin{aligned}
\omega_r' = \frac{1}{2} e_{rst}' \omega_{ts}' = \frac{1}{2} \mathbf{i}_r' \cdot \left(\mathbf{i}_s' \times \mathbf{i}_t'\right) \alpha_{tp} \alpha_{sq} \omega_{pq} = \frac{1}{2} \mathbf{i}_r' \cdot \left(\mathbf{i}_s' \alpha_{sq} \times \mathbf{i}_t' \alpha_{tp}\right) \omega_{pq} \\
= \frac{1}{2} \mathbf{i}_r' \cdot \left(\mathbf{i}_q \times \mathbf{i}_p\right) \omega_{pq} = \frac{1}{2} \mathbf{i}_r' \cdot \mathbf{i}_m e_{mqp} \omega_{pq} = \mathbf{i}_r' \cdot \mathbf{i}_m \omega_m,
\end{aligned}
$$

so that

$$
\omega_r' = \alpha_{rm} \omega_m,
$$

which is in agreement with law (A.1.6) of transformation of projections of the vector. This vector, defined by eq. (A.4.9), is called the vector accompanying tensor $\hat{Q}$. If it vanishes, then the tensor is symmetric. Vector $\mathbf{b} = \hat{\Omega} \cdot \mathbf{a}$, by eq. (A.3.3), has the projections

$$
b_s = \omega_{st} a_t = -e_{rst} \omega_r a_t = e_{srt} \omega_r a_t.
$$

Thus referring to eq. (A.2.3) we arrive at the frequently used relationships

$$
\hat{\Omega} \cdot \mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}, \quad \mathbf{a} \cdot \hat{\Omega} = \hat{\Omega}^* \cdot \mathbf{a} = -\hat{\Omega} \cdot \mathbf{a} = \mathbf{a} \times \boldsymbol{\omega}. \tag{A.4.10}
$$

Turning to eq. (A.4.8) we have

$$
\hat{Q} \cdot \mathbf{a} = \hat{S} \cdot \mathbf{a} + \boldsymbol{\omega} \times \mathbf{a}, \quad \mathbf{a} \cdot \hat{Q} = \hat{S} \cdot \mathbf{a} - \boldsymbol{\omega} \times \mathbf{a}. \tag{A.4.11}
$$

A consequence of this formula is the following relationship

$$
\mathbf{a} \cdot \hat{Q} \cdot \mathbf{a} = \mathbf{a} \cdot \hat{S} \cdot \mathbf{a}, \tag{A.4.12}
$$

expressing the easily foreseen result that only the symmetric part of tensor $\hat{Q}$ contributes to the quadratic form produced by this tensor.

Let us consider a bilinear form $q_{sk} a_s b_k$ where a_s and b_k denote the projections of vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The assumption that it is invariant, that is, its numerical value is independent of the choice of the coordinate system is expressed by the equality

$$
q_{qt}' a_p' b_t' = q_{sk} a_s b_k.
$$

It holds if the coefficients of the form and the variables are transformed according to laws (A.3.6) and (A.1.6) respectively. This allows us to give the third definition of the tensor of second rank as the quantity prescribed by matrix $\|q_{st}\|$ of the coefficients of the invariant bilinear form. The coefficients of the invariant quadratic form prescribe a symmetric tensor of second rank.

Example 1. It is known that twice the kinetic energy of a rigid body rotating about a fixed point O with the angular velocity given by vector $\boldsymbol{\omega}$ is as follows

$$\begin{aligned} 2T &= \Theta_{ik}\omega_i\omega_k \\ &= \Theta_{11}\omega_1^2 + \Theta_{22}\omega_2^2 + \Theta_{33}\omega_3^2 + 2\Theta_{12}\omega_1\omega_2 + 2\Theta_{23}\omega_2\omega_3 + 2\Theta_{31}\omega_3\omega_1. \end{aligned}$$

This form is invariant because the numerical value of the kinetic energy does not depends upon the choice of the coordinate system. Hence Θ_{ik} are the components of the symmetric tensor of second rank $\hat{\Theta}^0$ which is the tensor of inertia about point O and the expression for $2T$ can be set in another form

$$2T = \boldsymbol{\omega} \cdot \hat{\Theta}^0 \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathbf{k}^0,$$

see eq. (A.3.10).

Example 2. In the second example in Section A.3 the quadratic form

$$2\Pi = \mathbf{F} \cdot \mathbf{f} = \mathbf{F} \cdot \hat{\Sigma} \cdot \mathbf{F} = \sigma_1 F_1^2 + \sigma_2 F_2^2 + \sigma_3 F_3^2$$

represents twice the potential energy of an elastic rod. This confirms that $\hat{\Sigma}$ is a symmetric tensor of second rank.

A.5 Vector dyadic and dyadic representation of tensors of second rank

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, we introduce the following matrix

$$\left\| \begin{array}{ccc} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{array} \right\| = \|a_s b_k\| . \qquad \text{(A.5.1)}$$

It determines a tensor of second rank referred to as the dyadic products of vectors $\mathbf{a}$ and $\mathbf{b}$ (or simply a dyadic) and is denoted as $\mathbf{ab}$. This is in agreement with the definition of a tensor, Section A.3, since for any vector $\mathbf{c}$ the values due to rule (A.3.2)

$$a_s\left(b_1c_1 + b_2c_2 + b_3c_3\right) = a_s\mathbf{b} \cdot \mathbf{c},$$

differ from the projections of vector $\mathbf{a}$ only in the scalar multiplier $\mathbf{b} \cdot \mathbf{c}$. These formulae define the postmultiplication of the dyadic by a vector

$$(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{ab} \cdot \mathbf{c}. \qquad \text{(A.5.2)}$$

Transposing matrix (A.5.1) leads to interchanging $\mathbf{a}$ and $\mathbf{b}$

$$(\mathbf{ab})^* = \mathbf{ba}, \qquad \text{(A.5.3)}$$

such that by eq. (A.4.2)

$$\mathbf{ab}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{ba}=\mathbf{c}\cdot(\mathbf{ab})^{*}. \tag{A.5.4}$$

In the matrix defining dyadic $\mathbf{i}_s\mathbf{i}_k$ only the element of the $s-th$ row and $k-th$ column differs from zero and equal to 1. The sum of three dyadics

$$\hat{E}=\mathbf{i}_1\mathbf{i}_1+\mathbf{i}_2\mathbf{i}_2+\mathbf{i}_3\mathbf{i}_3=\delta_{st}\mathbf{i}_s\mathbf{i}_t \tag{A.5.5}$$

defines the unit tensor in the orthogonal Cartesian system of coordinates. This is the tensor whose nontrivial components lie on the diagonal and are equal to unity. This property of the unit tensor is invariant with respect to rotation since

$$\delta'_{pq}=\alpha_{ps}\alpha_{qt}\delta_{st}=\alpha_{pt}\alpha_{qt}=\delta_{pq}.$$

It is also evident that

$$\hat{E}\cdot\mathbf{a}=\mathbf{a}\cdot\hat{E}=\mathbf{a}, \tag{A.5.6}$$

and this equality can be taken as a definition of the unit tensor: post- or premultiplying it by any vector $\mathbf{a}$ yields the same vector $\mathbf{a}$.

Tensor $\hat{Q}$ can be presented by a sum of nine dyadics

$$\hat{Q}=q_{st}\mathbf{i}_s\mathbf{i}_t. \tag{A.5.7}$$

This follows from the fact that the tensor on the right hand side is prescribed by the same matrix of components as $\hat{Q}$. Such a dyadic representation of the tensor essentially simplifies the operations of tensor algebra. The above operations of postmultiplication and premultiplication

$$\left.\begin{aligned}\hat{Q}\cdot\mathbf{a}&=q_{st}\mathbf{i}_s\mathbf{i}_t\cdot a_k\mathbf{i}_k=q_{st}a_k\mathbf{i}_s\delta_{tk}=q_{st}a_t\mathbf{i}_s,\\ \mathbf{a}\cdot\hat{Q}&=a_k\mathbf{i}_k\cdot q_{st}\mathbf{i}_s\mathbf{i}_t=q_{st}a_k\delta_{ks}\mathbf{i}_t=q_{st}a_s\mathbf{i}_t,\end{aligned}\right\}$$

can serve as an example.

The vector product of the tensor of second rank and a vector leads to new tensors of the same rank

$$\left.\begin{aligned}\hat{Q}\times\mathbf{a}&=q_{st}a_k\mathbf{i}_s\mathbf{i}_t\times\mathbf{i}_k=e_{rtk}q_{st}a_k\mathbf{i}_s\mathbf{i}_r,\\ \mathbf{a}\times\hat{Q}&=q_{st}a_k\mathbf{i}_k\times\mathbf{i}_s\mathbf{i}_t=e_{rks}q_{st}a_k\mathbf{i}_r\mathbf{i}_t.\end{aligned}\right\} \tag{A.5.8}$$

For example, the matrix of components of the latter tensor is as follows

$$\left\|\begin{matrix} q_{31}a_2-q_{21}a_3 & q_{32}a_2-q_{22}a_3 & q_{33}a_2-q_{23}a_3\\ q_{11}a_3-q_{31}a_1 & q_{12}a_3-q_{32}a_1 & q_{13}a_3-q_{33}a_1\\ q_{21}a_1-q_{11}a_2 & q_{22}a_1-q_{12}a_2 & q_{23}a_1-q_{13}a_2\end{matrix}\right\|. \tag{A.5.9}$$

In particular, dyadic $\mathbf{ab}$ and vector $\mathbf{c}$ produce the following dyadics

$$(\mathbf{ab})\times\mathbf{c}=\mathbf{a}(\mathbf{b}\times\mathbf{c}),\quad \mathbf{c}\times(\mathbf{ab})=(\mathbf{c}\times\mathbf{a})\mathbf{b}. \tag{A.5.10}$$

It it straightforward to prove the identities

$$\left(\mathbf{a} \times \hat{Q}\right)^* = -\hat{Q}^* \times \mathbf{a}, \quad \mathbf{a} \times \left(\hat{P} \cdot \mathbf{b}\right) = \left(\mathbf{a} \times \hat{P}\right) \cdot \mathbf{b}. \qquad \text{(A.5.11)}$$

Let us introduce into consideration the following values

$$q_{st}\mathbf{i}_s = \mathbf{q}_t^*, \quad q_{st}\mathbf{i}_t = \mathbf{q}_s. \qquad \text{(A.5.12)}$$

These are not vectors since q_{st} are not transformed in the way that the vector projection are transformed. Nonetheless introducing these "quasi-vectors" in a fixed coordinate system is admitted since this simplifies the formulae. Using these, tensor $\hat{Q}$ is written as a sum of three dyadics

$$\hat{Q} = \mathbf{q}_t^* \mathbf{i}_t = \mathbf{i}_s \mathbf{q}_s. \qquad \text{(A.5.13)}$$

The accompanying vector $\boldsymbol{\omega}$ is presented in the form

$$\boldsymbol{\omega} = -\frac{1}{2} q_t^* \times \mathbf{i}_t = -\frac{1}{2}\mathbf{i}_s \times \mathbf{q}_s = \frac{1}{2} e_{rts} q_{st} \mathbf{i}_r. \qquad \text{(A.5.14)}$$

In particular, the vector accompanying the dyadic $\mathbf{ab}$ is equal to

$$\boldsymbol{\omega} = \frac{1}{2} e_{rts} a_s b_t \mathbf{i}_r = -\frac{1}{2}\mathbf{a} \times \mathbf{b}. \qquad \text{(A.5.15)}$$

Let us also notice that the $st-$component of tensor $\hat{Q}$ can be presented as follows

$$q_{st} = \mathbf{i}_s \cdot \hat{Q} \cdot \mathbf{i}_t. \qquad \text{(A.5.16)}$$

A.6 Tensors of higher ranks, contraction of indices

Let us agree to refer to the scalar and the vector as the tensor of zero rank and first rank respectively. From three operations on two vectors $\mathbf{a}$ and $\mathbf{b}$, which are dyadic, scalar and vector multiplication, the most general is the first one. From two tensors of first rank it produces the tensor $\mathbf{ab}$ of second rank prescribed by the matrix of components $\|a_s b_t\|$. The rank of this tensor decreases by one when the accompanying vector $\frac{1}{2} e_{rst} a_t b_s \mathbf{i}_r$, which is the tensor of first rank, is calculated. The rank decreases by two (i.e. tensor produces a scalar) when the value $\delta_{st} a_s b_t = a_s b_s = \mathbf{a} \cdot \mathbf{b}$ is introduced. In other words, a contraction with respect to a pair of indices s, t takes place.

The value

$$^{(n)}\hat{Q} = q_{s_1 s_2 \ldots s_n} \mathbf{i}_{s_1} \mathbf{i}_{s_2} \ldots \mathbf{i}_{s_n} \qquad \text{(A.6.1)}$$

determines a tensor of $n-th$ rank provided that its components (altogether 3^n) obey the transformation law

$$q'_{t_1 t_2 \ldots t_n} = q_{s_1 s_2 \ldots s_n} \alpha_{t_1 s_1} \alpha_{t_2 s_2} \ldots \alpha_{t_n s_n} \tag{A.6.2}$$

under rotation of the coordinate system. Equality (A.6.2) is a direct generalisation of rule (A.3.6). The rank of the tensor decreases by two under any contraction. For example, under contraction with respect to the last indices we arrive at the tensor

$$^{(n-2)}\hat{Q} = q_{s_1 s_2 \ldots s_{n-1} s_{n-1}} \mathbf{i}_{s_1} \mathbf{i}_{s_2} \ldots \mathbf{i}_{s_{n-2}}$$

(to sum over index s_{n-1}). The rank of the tensor can be decreased by one by introducing an "accompanying" tensor. One of these tensors is the following one

$$\begin{aligned} ^{(n-1)}\hat{Q} &= -\frac{1}{2} q_{s_1 s_2 \ldots s_n} \left(\mathbf{i}_{s_1} \times \mathbf{i}_{s_2}\right) \mathbf{i}_{s_3} \ldots \mathbf{i}_{s_n} \\ &= -\frac{1}{2} e_{s_1 s_2 r} \mathbf{i}_r \mathbf{i}_{s_3} \ldots \mathbf{i}_{s_n} q_{s_1 s_2 \ldots s_n}. \end{aligned}$$

Contraction of the tensor leads to the scalar referred to as the trace of the tensor or its first invariant

$$\operatorname{tr} \hat{Q} = I_1\left(\hat{Q}\right) = q_{st} \mathbf{i}_s \cdot \mathbf{i}_t = q_{st} \delta_{st} = q_{ss}. \tag{A.6.3}$$

In particular, for the unit tensor

$$I_1\left(\hat{E}\right) = 3, \tag{A.6.4}$$

and for the dyadic

$$I_1\left(\mathbf{ab}\right) = \operatorname{tr} \mathbf{ab} = \mathbf{a} \cdot \mathbf{b}. \tag{A.6.5}$$

By means of two tensors $\hat{Q}$ and $\hat{P}$ of second rank one can obtain tensors of fourth rank, for example

$$\hat{Q}\hat{P} = q_{st} p_{rq} \mathbf{i}_s \mathbf{i}_t \mathbf{i}_r \mathbf{i}_q, \quad \hat{P}\hat{Q} = p_{rq} q_{st} \mathbf{i}_r \mathbf{i}_q \mathbf{i}_s \mathbf{i}_t.$$

Tensors of second rank can be obtained from the above tensors by means of a single contraction. In particular, postmultiplying tensor $\hat{Q}$ by $\hat{P}$ results in the following tensor

$$\hat{Q} \cdot \hat{P} = q_{st} p_{rq} \mathbf{i}_s \mathbf{i}_t \cdot \mathbf{i}_r \mathbf{i}_q = q_{st} p_{tq} \mathbf{i}_s \mathbf{i}_q. \tag{A.6.6}$$

The second contraction yields the invariant which is the trace of tensor (A.6.6)

$$\hat{Q} \cdot \cdot \hat{P} = \operatorname{tr}\left(\hat{Q} \cdot \hat{P}\right) = q_{st} p_{tq} \mathbf{i}_s \cdot \mathbf{i}_q = q_{st} p_{ts}. \tag{A.6.7}$$

It is easy to prove the identity

$$\left(\hat{Q}\cdot\hat{P}\right)^* = \hat{P}^*\cdot\hat{Q}^* = q_{st}p_{tq}\mathbf{i}_q\mathbf{i}_s, \tag{A.6.8}$$

which suggests that tensor $\hat{Q}\cdot\hat{Q}^*$ is symmetric, however $\hat{Q}^*\cdot\hat{Q}\neq\hat{Q}\cdot\hat{Q}^*$, see eq. (A.6.14).

For $\hat{Q}=\hat{P}$ we arrive at the tensor of second rank, termed the square of tensor,

$$\hat{Q}\cdot\hat{Q} = \hat{Q}^2 = q_{st}q_{tr}\mathbf{i}_s\mathbf{i}_r. \tag{A.6.9}$$

By analogy one constructs higher powers of tensor, for example

$$\hat{Q}^3 = \hat{Q}^2\cdot\hat{Q} = q_{st}q_{tr}q_{rq}\mathbf{i}_s\mathbf{i}_q.$$

The traces of these tensors are

$$\operatorname{tr}\hat{Q}^2 = q_{st}q_{ts}, \quad \operatorname{tr}\hat{Q}^3 = q_{st}q_{tr}q_{rs}. \tag{A.6.10}$$

The higher powers than second are expressed in terms of $\hat{Q}^2, \hat{Q}$ and $\hat{Q}^0=\hat{E}$. This is the Cayley-Hamilton theorem proved in Sections A.10 and A.12.

Post- or premultiplication of a tensor of second rank by the unit tensor $\hat{E}$ results in the same tensor

$$\hat{Q}\cdot\hat{E} = \hat{E}\cdot\hat{Q} = \hat{Q}. \tag{A.6.11}$$

Further analysis requires an expression for tensor $\hat{Q}^*\cdot\hat{Q}$ in terms of the symmetric part $\hat{S}$ of tensor $\hat{Q}$ and the accompanying vector $\boldsymbol{\omega}$. We have

$$\hat{Q}^*\cdot\hat{Q} = \left(\hat{S}-\hat{\Omega}\right)\cdot\left(\hat{S}+\hat{\Omega}\right) = \hat{S}^2 - \hat{\Omega}\cdot\hat{S} + \hat{S}\cdot\hat{\Omega} - \hat{\Omega}^2$$

and referring to eqs. (A.4.6) and (A.5.8) we find

$$\hat{\Omega}\cdot\hat{S} = \omega_{tr}s_{rq}\mathbf{i}_t\mathbf{i}_q = e_{mrt}\omega_m s_{rq}\mathbf{i}_t\mathbf{i}_q = \boldsymbol{\omega}\times\hat{S}.$$

Taking into account that $\hat{\Omega}^* = -\hat{\Omega}, \hat{S}^* = \hat{S}$ we find by eq. (1.6.8)

$$\left(\hat{\Omega}\cdot\hat{S}\right)^* = \hat{S}^*\cdot\hat{\Omega}^* = -\hat{S}\cdot\hat{\Omega} = e_{mrt}\omega_m s_{rq}\mathbf{i}_q\mathbf{i}_t = \left(\boldsymbol{\omega}\times\hat{S}\right)^*$$

and

$$\begin{aligned}\hat{\Omega}\cdot\hat{\Omega} &= \omega_{st}\omega_{tr}\mathbf{i}_s\mathbf{i}_r = e_{tsm}e_{rtq}\omega_m\omega_q\mathbf{i}_s\mathbf{i}_r = \left(\delta_{sq}\delta_{mr} - \delta_{sr}\delta_{mq}\right)\omega_m\omega_q\mathbf{i}_s\mathbf{i}_r \\ &= \omega_r\omega_s\mathbf{i}_s\mathbf{i}_r - \omega_m\omega_m\mathbf{i}_s\mathbf{i}_s = \boldsymbol{\omega}\boldsymbol{\omega} - \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega}.\end{aligned}$$

Hence

$$\begin{aligned}\hat{Q}^*\cdot\hat{Q} &= \hat{S}^2 + \hat{E}\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} - \boldsymbol{\omega}\times\hat{S} - \left(\boldsymbol{\omega}\times\hat{S}\right)^* \\ &= \left[s_{ts}s_{rq} - \delta_{tq}\omega_m\omega_m - \omega_t\omega_q - \omega_m\left(e_{mrt}s_{rq} + e_{mrq}s_{rt}\right)\right]\mathbf{i}_t\mathbf{i}_q \end{aligned} \tag{A.6.12}$$

and by analogy

$$\hat{Q} \cdot \hat{Q}^* = \hat{S}^2 + \hat{E}\boldsymbol{\omega} \cdot \boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\omega} + \boldsymbol{\omega} \times \hat{S} + \left(\boldsymbol{\omega} \times \hat{S}\right)^*, \qquad \text{(A.6.13)}$$

so that

$$\hat{Q} \cdot \hat{Q}^* - \hat{Q}^* \cdot \hat{Q} = 2\left[\boldsymbol{\omega} \times \hat{S} + \left(\boldsymbol{\omega} \times \hat{S}\right)^*\right]. \qquad \text{(A.6.14)}$$

An example of a tensor of third rank is the Levi-Civita tensor. From 27 Levi-Civita's symbols only six are nontrivial. Their tensorial character can be detected easily by using definition (A.2.1), formulae (A.1.2) and eq. (A.6.2)

$$e'_{str} = \mathbf{i}'_s \cdot (\mathbf{i}'_t \times \mathbf{i}'_r) = \alpha_{sk}\alpha_{tl}\alpha_{rm}\mathbf{i}_k \cdot (\mathbf{i}_l \times \mathbf{i}_m) = \alpha_{sk}\alpha_{tl}\alpha_{rm}e_{klm}.$$

The values $e_{str}e_{klm}$ determine a tensor of sixth rank. The contraction of it with respect to three pairs of indices leads to the invariant (A.2.6) whereas the contraction with respect to two pairs of indices results in the double unit vector and a contraction with respect to a single pair of indices yields a tensor of fourth rank (A.2.4).

A tensor is said to be isotropic if its components are unchanged in all coordinate systems obtained from each other by rotation. An example of the isotropic tensor of second rank is the product of a scalar and the unit tensor $\left(\lambda\hat{E}\right)$ and the product of a scalar and the Levi-Civita tensor is the isotropic tensor of third rank. It can be proved that there exist no other isotropic tensor of second and third rank. A more general form for the components of isotropic tensor of fourth rank is presented by the formula

$$c_{ikmp} = \lambda\delta_{ik}\delta_{mp} + \mu\left(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}\right) + \nu\left(\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}\right) \qquad \text{(A.6.15)}$$

containing three scalar multipliers λ, μ, ν. Under the symmetry requirement

$$c_{ikmp} = c_{mpik}, \quad c_{ikmp} = c_{ikpm} \qquad \text{(A.6.16)}$$

the third term in eq. (A.6.5) vanishes, i.e. $\nu = 0$.

A.7 Inverse tensor

Relationships (A.3.2) can be considered as a system of linear equations for unknowns a_r. It has a solution if matrix $\|q_{st}\|$ is nonsingular, that is, its determinant is not zero

$$q = |q_{st}| \neq 0. \qquad \text{(A.7.1)}$$

The components a_r are given by the standard equalities

$$a_r = \frac{1}{q}\left(b_1 A_{r1} + b_2 A_{r2} + b_3 A_{r3}\right) = \frac{1}{q} b_t A_{rt}, \tag{A.7.2}$$

in which A_{rt} denotes the algebraic adjunct of element q_{tr} of determinant q. The tensor with the following matrix of components

$$q^{rt} = \frac{1}{q} A_{rt} \tag{A.7.3}$$

is called the tensor inverse of $\hat{Q}$ and is denoted as follows

$$\hat{Q}^{-1} = \mathbf{i}_r \mathbf{i}_t q^{rt} = \mathbf{i}_r \mathbf{i}_t \frac{A_{rt}}{q}. \tag{A.7.4}$$

Relations (A.7.2) are now written in the form

$$\mathbf{a} = \hat{Q}^{-1} \cdot \mathbf{b}, \tag{A.7.5}$$

and substitution into eq. (A.3.4) leads to the equality

$$\mathbf{b} = \hat{Q} \cdot \hat{Q}^{-1} \cdot \mathbf{b}, \quad \hat{Q} \cdot \hat{Q}^{-1} = \hat{E}. \tag{A.7.6}$$

It follows from this equation that

$$\hat{Q} \cdot \hat{Q}^{-1} = q_{sq} \mathbf{i}_s \mathbf{i}_q \cdot q^{rt} \mathbf{i}_r \mathbf{i}_t = q_{sr} q^{rt} \mathbf{i}_s \mathbf{i}_t = \hat{E} = \delta_{st} \mathbf{i}_s \mathbf{i}_t, \tag{A.7.7}$$

and we arrive at the well-known property of the determinants

$$q_{sr} q^{rt} = \delta_{st}. \tag{A.7.8}$$

Let us notice that

$$A_{tr} = \frac{\partial q}{\partial q_{rt}}, \quad q^{tr} = \frac{1}{q} \frac{\partial q}{\partial q_{rt}}. \tag{A.7.9}$$

It is known that the determinant q is a sum of the products of the type $q_{st} q_{qr} q_{mn}$ taken with the corresponding signs. These products must have no repeating indices in the triples sqm and trn otherwise the sum vanishes. It is straightforward to prove that

$$q = \frac{1}{6} e_{sqm} e_{trn} q_{st} q_{qr} q_{mn}. \tag{A.7.10}$$

Let us introduce the denotation

$$q^{ts} = \frac{1}{6q} e_{sqm} e_{trn} q_{qr} q_{mn}. \tag{A.7.11}$$

Then

$$q_{lt}q^{ts} = \frac{1}{6q} e_{sqm} e_{trn} q_{lt} q_{qr} q_{mn} = \begin{cases} 0, & s \neq l \\ 1, & s = l \end{cases} = \delta_{sl},$$

since for $s \neq l$ the indices in the triples sqm and trn are repeating. Thus we arrive at equality (A.7.8) which implies that formula (A.7.11) yields another representation of the tensor components

$$\hat{Q}^{-1} = q^{ts} \mathbf{i}_t \mathbf{i}_s.$$

Other relationships are

$$\left(\hat{Q}^{-1}\right)^{-1} = \hat{Q}, \quad \left(\hat{Q}^{-1}\right)^{*} = \left(\hat{Q}^{*}\right)^{-1}, \tag{A.7.12}$$

that is, the inverse of the inverse of the tensor is the tensor itself. The inverse of the transpose is equal to the transpose of the inverse. We notice also the permutation property

$$\hat{Q} \cdot \hat{Q}^{-1} = \hat{Q}^{-1} \cdot \hat{Q} = \hat{E} \tag{A.7.13}$$

and the relation defining the tensor inverse of the product of tensors

$$\left(\hat{P} \cdot \hat{Q}\right)^{-1} = \hat{Q}^{-1} \cdot \hat{P}^{-1}, \tag{A.7.14}$$

which is proved by multiplying both sides by $\hat{P} \cdot \hat{Q}$

$$\hat{P} \cdot \hat{Q} \cdot \left(\hat{P} \cdot \hat{Q}\right)^{-1} = \hat{E} = \hat{P} \cdot \hat{Q} \cdot \hat{Q}^{-1} \cdot \hat{P}^{-1} = \hat{P} \cdot \hat{E} \cdot \hat{P}^{-1} = \hat{P} \cdot \hat{P}^{-1} = \hat{E},$$

which is required.

A.8 Rotation tensor

Let $\mathbf{i}_s$ denote the unit vectors of the axes of trihedron $Ox_1x_2x_3$ and $\mathbf{i}'_s$ denote those of trihedron $Ox'_1x'_2x'_3$, the latter being obtained from the first by a rotation. We introduce tensor $\hat{A}$ which is a sum of three dyadics $\mathbf{i}_s\mathbf{i}'_s$ and the transpose $\hat{A}^*$

$$\hat{A} = \mathbf{i}_s \mathbf{i}'_s, \quad \hat{A}^* = \mathbf{i}'_s \mathbf{i}_s. \tag{A.8.1}$$

Premultiplying $\hat{A}$ by vector $\mathbf{a}$ or postmultiplying $\hat{A}^*$ by $\mathbf{a}$ leads to the vector $\grave{\mathbf{a}}$

$$\grave{\mathbf{a}} = \mathbf{a} \cdot \hat{A} = \mathbf{a} \cdot \mathbf{i}_s \mathbf{i}'_s = a_s \mathbf{i}'_s, \quad \hat{A}^* \cdot \mathbf{a} = \mathbf{i}'_s a_s = \grave{\mathbf{a}}. \tag{A.8.2}$$

The projections of this vector on new axes $\mathbf{i}'_s$ are equal to the projections a_s of vector $\mathbf{a}$ in the old axes $\mathbf{i}_s$. This means that vector $\grave{\mathbf{a}}$ is obtained from $\mathbf{a}$ by a rotation together with trihedron $Ox_1x_2x_3$. Tensors $\hat{A}$ and $\hat{A}^*$ performing this operation are referred to as the rotation tensors.

The projections of vectors $\mathbf{i}_1$ and $\mathbf{i}'_1$ on the old axes are equal $(1, 0, 0)$ and $(\alpha_{11}, \alpha_{12}, \alpha_{13})$ respectively and the matrix of the components of dyadic $\mathbf{i}\mathbf{i}'_1$ is as follows

$$\begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Clearly, the components of tensor $\left\|\hat{A}\right\|$ are presented by the matrix of cosines $\|\alpha_{st}\|$, hence

$$\hat{A} = \alpha_{st}\mathbf{i}_s\mathbf{i}_t, \quad \grave{\mathbf{a}} = \mathbf{a} \cdot \hat{A} = \alpha_{st}\mathbf{i}_t a_s = \mathbf{i}'_s a_s, \quad \grave{a}_s = a_s, \tag{A.8.3}$$

which is required.

Let us notice the relationship

$$\hat{A} \cdot \hat{A}^* = \mathbf{i}_s\mathbf{i}'_s \cdot \mathbf{i}'_k\mathbf{i}_k = \mathbf{i}_s\mathbf{i}_k\delta_{sk} = \hat{E}, \quad \hat{A}^* = \hat{A}^{-1}, \tag{A.8.4}$$

indicating that transposing the rotation tensor leads to the inverse tensor. This is a characteristic property of the rotation tensor and any tensor possessing this property is a rotation tensor. Indeed, let

$$\hat{Q} \cdot \hat{Q}^* = \hat{E}, \quad \hat{Q}^* = \hat{Q}^{-1}.$$

As the determinants q and q^* are equal to each other and determinant $|\delta_{st}| = 1$ we have $q^2 = 1$. Let us take $q = 1$, in eq. (A.7.8) we have $q^{rt} = q_{tr}$ and we arrive at six equalities

$$q_{sr}q_{tr} = \delta_{st},$$

describing the matrix of cosines along with the condition $q = 1$, see eq. (A.1.5).

It is known that a rigid body rotation (the rotation of the fixed axes from the old position $Ox_1x_2x_3$ into a new one $Ox'_1x'_2x'_3$) can be described by a vector of finite rotation $\boldsymbol{\theta}$. This vector has the direction of the axis of rotation and the value $2\tan\frac{\chi}{2}$

$$\boldsymbol{\theta} = 2\mathbf{k}\tan\frac{\chi}{2},$$

where χ denotes the angle of rotation. Vector $\mathbf{a}$ given in axes $Ox_1x_2x_3$ becomes vector $\grave{\mathbf{a}}$ determined by Rodrigues's formula

$$\grave{\mathbf{a}} = \mathbf{a} + \frac{\boldsymbol{\theta}}{1 + \frac{1}{4}\theta^2} \times \left(\mathbf{a} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{a}\right), \tag{A.8.5}$$

see Subsection A.10.1. Another form of the latter equation is

$$\mathbf{\grave{a}} = \mathbf{a} + \sin\chi \left[\mathbf{k}\times\mathbf{a} + (\mathbf{kk}\cdot\mathbf{a} - \mathbf{a})\tan\frac{\chi}{2}\right]. \tag{A.8.6}$$

Let us introduce a skew-symmetric tensor

$$\hat{\Omega} = \mathbf{k}\times\hat{E} = k_1(\mathbf{i}_3\mathbf{i}_2 - \mathbf{i}_2\mathbf{i}_3) + k_2(\mathbf{i}_1\mathbf{i}_3 - \mathbf{i}_3\mathbf{i}_1) + k_3(\mathbf{i}_2\mathbf{i}_1 - \mathbf{i}_1\mathbf{i}_2), \tag{A.8.7}$$

so that

$$\mathbf{k}\times\mathbf{a} = \hat{\Omega}\cdot\mathbf{a}.$$

Then eq. (A.8.6) can be transformed to the form

$$\mathbf{\grave{a}} = \left[\left(\hat{E} - \mathbf{kk}\right)\cos\chi + \mathbf{kk} + \hat{\Omega}\sin\chi\right]\cdot\mathbf{a} = \hat{A}^*\cdot\mathbf{a},$$

admitting an invariant (i.e. independent of the choice of axes) representation for the rotation tensor

$$\hat{A} = \left(\hat{E} - \mathbf{kk}\right)\cos\chi + \mathbf{kk} - \hat{\Omega}\sin\chi,\ \hat{A}^* = \left(\hat{E} - \mathbf{kk}\right)\cos\chi + \mathbf{kk} + \hat{\Omega}\sin\chi. \tag{A.8.8}$$

A.9 Principal axes and principal values of symmetric tensors

One seeks such a direction described by the unit vector $\mathbf{e}$ that vector $\hat{Q}\cdot\mathbf{e}$ is parallel to this direction, i.e.

$$\hat{Q}\cdot\mathbf{e} = \lambda\mathbf{e},\quad \left(\hat{Q} - \lambda\hat{E}\right)\cdot\mathbf{e} = 0, \tag{A.9.1}$$

where $\hat{Q}$ is a prescribed symmetric tensor of second rank and λ is an unknown scalar. Assuming $\mathbf{e} = e_t\mathbf{i}_t$ one can rewrite this equality in the form

$$(q_{st} - \lambda\delta_{st})\,e_t = 0\quad (s = 1,2,3). \tag{A.9.2}$$

We arrive at the three equations

$$\left.\begin{array}{l} (q_{11} - \lambda)\,e_1 + q_{12}e_2 + q_{13}e_3 = 0, \\ q_{21}e_1 + (q_{22} - \lambda)\,e_2 + q_{23}e_3 = 0, \\ q_{31}e_1 + q_{32}e_2 + (q_{33} - \lambda)\,e_3 = 0, \end{array}\right\} \tag{A.9.3}$$

augmented by the equation expressing that $\mathbf{e}$ is a unit vector

$$e_1^2 + e_2^2 + e_3^2 = 1. \tag{A.9.4}$$

The latter condition excludes the trivial solution ($e_s = 0$) of the system of linear equations (A.9.3). The determinant of the system must vanish

$$P_3(\lambda) = |q_{st} - \lambda\delta_{st}| = \begin{vmatrix} q_{11}-\lambda & q_{12} & q_{13} \\ q_{21} & q_{22}-\lambda & q_{23} \\ q_{31} & q_{32} & q_{33}-\lambda \end{vmatrix} = 0, \qquad \text{(A.9.5)}$$

so that λ are the roots of this cubic equation which is called the characteristic equation of tensor $\hat{Q}$. These values are invariant with rotation of the coordinate system which immediately follows from relationship (A.3.8) applied to tensor $\hat{Q} - \lambda\hat{E}$.

Let λ_1 and λ_2 be two different (not equal to each other) roots of equation (A.9.5) and let the corresponding vectors $\mathbf{e}$ be denoted as $\overset{1}{\mathbf{e}}=\overset{1}{e_t}\,\mathbf{i}_t$ and $\overset{2}{\mathbf{e}}=\overset{2}{e_t}\,\mathbf{i}_t$. Then

$$\left(\hat{Q} - \lambda_1\hat{E}\right)\cdot\overset{1}{\mathbf{e}}= 0, \quad \left(\hat{Q} - \lambda_2\hat{E}\right)\cdot\overset{2}{\mathbf{e}}= 0, \qquad \text{(A.9.6)}$$

and therefore

$$\overset{2}{\mathbf{e}}\cdot\hat{Q}\cdot\overset{1}{\mathbf{e}}= \lambda_1\,\overset{2}{\mathbf{e}}\cdot\overset{1}{\mathbf{e}}, \quad \overset{1}{\mathbf{e}}\cdot\hat{Q}\cdot\overset{2}{\mathbf{e}}= \lambda_2\,\overset{1}{\mathbf{e}}\cdot\overset{2}{\mathbf{e}}.$$

For the symmetric tensor

$$\overset{2}{\mathbf{e}}\cdot\hat{Q}\cdot\overset{1}{\mathbf{e}}= q_{st}\,\overset{2}{e_s}\overset{1}{e_t}= q_{ts}\,\overset{2}{e_t}\overset{1}{e_s}= q_{st}\,\overset{1}{e_s}\overset{2}{e_t}=\overset{1}{\mathbf{e}}\cdot\hat{Q}\cdot\overset{2}{\mathbf{e}},$$

and thus

$$(\lambda_1 - \lambda_2)\,\overset{1}{\mathbf{e}}\cdot\overset{2}{\mathbf{e}}= 0, \quad \overset{1}{\mathbf{e}}\cdot\overset{2}{\mathbf{e}}= 0. \qquad \text{(A.9.7)}$$

If the roots λ_1 and λ_2 were complex conjugate numbers, then the corresponding solutions $\overset{1}{e_s}$ and $\overset{2}{e_s}$ of the system of equations (A.9.3) would also be complex conjugate. However the sum $\overset{1}{e_s}\overset{2}{e_s}= \left|\overset{1}{e_s}\right|^2$ can not vanish since it is equal to the sum of the squares of the absolute values of $\overset{1}{e_s}$. This proves that the roots of polynomial $P_3(\lambda)$ are real-valued and the vectors corresponding to two different roots are mutually orthogonal.

1. *Simple roots.* Let the roots of polynomial $P_3(\lambda)$ be simple, i.e. $\lambda_1 \neq \lambda_2 \neq \lambda_3$. then we have

$$-P'(\lambda_s) = \Delta_{11}(\lambda_s) + \Delta_{22}(\lambda_s) + \Delta_{33}(\lambda_s), \qquad \text{(A.9.8)}$$

where $\Delta_{sk}(\lambda_s)$ denotes the algebraic adjunct of the element of the $i-th$ row and $k-th$ column of determinant (A.9.5) for $\lambda = \lambda_s$. At least one term of this sum (for instance the third one) is not equal to zero otherwise

$P'(\lambda_s) = 0$ and the root λ_s is no longer simple. Then we obtain from the first and second equations in (A.9.3)

$$\overset{s}{e}_1 = \overset{s}{e}_3 \frac{\Delta_{31}(\lambda_s)}{\Delta_{33}(\lambda_s)}, \quad \overset{s}{e}_2 = \overset{s}{e}_3 \frac{\Delta_{32}(\lambda_s)}{\Delta_{33}(\lambda_s)},$$

see also eq. (A.10.32) and by eq. (A.9.4) we find

$$\overset{s}{e}_k = \pm \frac{1}{D} \Delta_{sk}(\lambda_s), \quad D^2 = \Delta_{31}^2(\lambda_s) + \Delta_{32}^2(\lambda_s) + \Delta_{33}^2(\lambda_s), \tag{A.9.9}$$

the third equation (A.9.3) being satisfied since

$$q_{31}\Delta_{31}(\lambda_s) + q_{32}\Delta_{32}(\lambda_s) + (q_{33} - \lambda_s)\Delta_{33}(\lambda_s) = P_3(\lambda_s) = 0.$$

Thus, for any root λ_s of equation (A.9.5) there exists a direction $\overset{s}{\mathbf{e}}$ given by the directional cosines that

$$\hat{Q} \cdot \overset{s}{\mathbf{e}} = \lambda_s \overset{s}{\mathbf{e}} \quad (\not{\Sigma}_s).$$

These three directions are mutually orthogonal and are referred to as the principal directions of tensor $\hat{Q}$ whereas λ_s are called the principal values. It follows from the relationships

$$\overset{k}{\mathbf{e}} \cdot \hat{Q} \cdot \overset{s}{\mathbf{e}} = \lambda_s \overset{k}{\mathbf{e}} \cdot \overset{s}{\mathbf{e}} = \lambda_s \delta_{ks} \tag{A.9.10}$$

that in the orthogonal system of directions $\overset{1}{\mathbf{e}}, \overset{2}{\mathbf{e}}, \overset{3}{\mathbf{e}}$ the matrix of the components of tensor $\hat{Q}$ becomes diagonal

$$\begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{Vmatrix}, \tag{A.9.11}$$

and the dyadic representation of the tensor has the trinomial form[1]

$$\hat{Q} = \lambda_1 \overset{1}{\mathbf{e}}\overset{1}{\mathbf{e}} + \lambda_2 \overset{2}{\mathbf{e}}\overset{2}{\mathbf{e}} + \lambda_3 \overset{3}{\mathbf{e}}\overset{3}{\mathbf{e}} = \sum_{s=1}^{3} \lambda_s \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}. \tag{A.9.12}$$

2. *Double root.* The above-said remains valid for direction $\overset{3}{\mathbf{e}}$ which is orthogonal to the other directions

$$\overset{3}{\mathbf{e}} \cdot \overset{s}{\mathbf{e}} = \overset{3}{e}_1 \overset{s}{e}_1 + \overset{3}{e}_2 \overset{s}{e}_2 + \overset{3}{e}_3 \overset{s}{e}_3 = 0 \quad (s = 1, 2). \tag{A.9.13}$$

[1] It is preferable to keep the summation sign in those cases when the dummy index repeats three times (rather than two times)

For the double root

$$P'(\lambda_s) = 0, \quad \frac{1}{2}P''(\lambda_s) = (q_{11} - \lambda_s) + (q_{22} - \lambda_s) + (q_{33} - \lambda_s),$$

and at least one term in this sum, say the first, does not vanish. For determining three unknowns $\overset{1}{e}_k$ we have two equations: first equation in (A.9.3) and eq. (A.9.4). The second and third equations in (A.9.3) are satisfied identically which is the result from $P'(\lambda_1) = 0, P(\lambda_1) = 0$. The determination of direction $\overset{2}{\mathbf{e}}$ is completed by the requirement $\overset{1}{\mathbf{e}} \cdot \overset{2}{\mathbf{e}} = 0$ of orthogonality to direction $\overset{1}{\mathbf{e}}$. Hence in the case of double root only one of three principal directions is determined uniquely while the remaining two directions $\overset{1}{\mathbf{e}}$ and $\overset{2}{\mathbf{e}}$ are oriented arbitrarily in the plane orthogonal to $\overset{3}{\mathbf{e}}$. Thus, the system of directions $\overset{1}{\mathbf{e}}, \overset{2}{\mathbf{e}}, \overset{3}{\mathbf{e}}$ is determined up to a rotation about $\overset{3}{\mathbf{e}}$. The dyadic representation of tensor $\hat{Q}$ is written in the form

$$\hat{Q} = \lambda_1 \left(\overset{11}{\mathbf{ee}} + \overset{22}{\mathbf{ee}} \right) + \lambda_3 \overset{33}{\mathbf{ee}} = \lambda_1 \hat{E} + (\lambda_3 - \lambda_1) \overset{33}{\mathbf{ee}}. \tag{A.9.14}$$

It follows from this representation that only direction $\overset{3}{\mathbf{e}}$ is the characteristic for tensor $\hat{Q}$.

3. *Equal roots.* In this case $\lambda_1 = \lambda_2 = \lambda_3$ and

$$\hat{Q} = \lambda_1 \hat{E}, \tag{A.9.15}$$

that is, the principal directions are arbitrary. Tensor $\hat{Q}$ is isotropic and is also referred to as spherical.

To conclude, we notice that the determinant of the product of two tensors is equal to the product of their determinants

$$\det \hat{A} \cdot \hat{B} = \det \hat{A} \det \hat{B} = \det \hat{B} \cdot \hat{A}, \tag{A.9.16}$$

and it follows from eq. (A.9.5) that the principal values of tensors $\hat{A} \cdot \hat{B}$ and $\hat{B} \cdot \hat{A}$ are coincident.

Let us also consider the tensor

$$\hat{Q}' = \hat{A} \cdot \hat{Q} \cdot \hat{A} = \mathbf{i}'_s \mathbf{i}_s \cdot q_{rt} \mathbf{i}_r \mathbf{i}_t \cdot \mathbf{i}_m \mathbf{i}'_m = q_{sm} \mathbf{i}'_s \mathbf{i}'_m. \tag{A.9.17}$$

This is a "turned tensor $\hat{Q}$" since the principal values of tensors $\hat{Q}$ and $\hat{Q}'$ coincide, whilst the trihedrons of their principal axes are related by the transformation of rotation.

A.10 Tensor invariants, the Cayley-Hamilton theorem

Let us make the old axes coincident with the principal directions $\overset{s}{\mathbf{e}}$ and denote, for simplicity, the unit vectors of the new axes as $\mathbf{i}_s$. Then by eq.

(A.3.6)

$$q_{st} = \lambda_1 \alpha_{s1}\alpha_{t1} + \lambda_2 \alpha_{s2}\alpha_{t2} + \lambda_3 \alpha_{s3}\alpha_{t3}, \tag{A.10.1}$$

where α_{sm} denotes the cosine function of the angle between axis $\mathbf{i}_s$ and the principal direction $\overset{m}{\mathbf{e}}$. These formulae express the tensor components in the arbitrary directed axes in terms of its principal values.

The principal values of the symmetric tensor of second rank are its invariants. It follows from the remark of Section A.9 stating that the roots of polynomial $P_3(\lambda)$ are independent of the choice of the coordinate system in which the matrix of components is prescribed. It is evident that any function of the principal values of the tensor $\Phi(\lambda_1, \lambda_2, \lambda_3)$ is an invariant. The invariants which are the symmetric functions of the principal values, i.e. the roots of polynomial $P_3(\lambda)$, are the most convenient because they are expressed in terms of the coefficients of this polynomial. They are referred to as the principal invariants. Of course, the invariants of the tensor do not depend on the orientation of the trihedron of its principal axes, that is, tensors $\hat{Q}$ and $\hat{P}$ have the same invariants.

An extended form of polynomial $P_3(\lambda)$ has the form

$$\begin{aligned} P_3(\lambda) = {} & -\lambda^3 + \lambda^2 (q_{11} + q_{22} + q_{33}) - \\ & \lambda \left(q_{11}q_{22} + q_{22}q_{33} + q_{33}q_{11} - q_{12}^2 - q_{23}^2 - q_{31}^2\right) + |q_{st}| . \end{aligned} \tag{A.10.2}$$

On the other hand,

$$\begin{aligned} P_3(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \\ &= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \lambda (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \lambda_1\lambda_2\lambda_3, \end{aligned} \tag{A.10.3}$$

and comparison of these forms allows the following expressions for the principal invariants to be written

$$I_1\left(\hat{Q}\right) = \lambda_1 + \lambda_2 + \lambda_3 = q_{11} + q_{22} + q_{33}, \tag{A.10.4}$$

$$\begin{aligned} I_2\left(\hat{Q}\right) &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ &= q_{11}q_{22} - q_{12}^2 + q_{22}q_{33} - q_{23}^2 + q_{33}q_{11} - q_{31}^2 = q\left(q^{11} + q^{22} + q^{33}\right), \end{aligned} \tag{A.10.5}$$

$$I_3\left(\hat{Q}\right) = \lambda_1\lambda_2\lambda_3 = |q_{st}| = q. \tag{A.10.6}$$

Using the dyadic representation of tensor (A.9.12) we can write the following dyadic representation for $\hat{Q}^2$

$$\hat{Q}^2 = \sum_{s=1}^{3} \lambda_s \overset{s\,s}{\mathbf{ee}} \cdot \sum_{k=1}^{3} \lambda_k \overset{k\,k}{\mathbf{ee}} = \sum_{s=1}^{3}\sum_{k=1}^{3} \lambda_s\lambda_k \overset{s\,k}{\mathbf{ee}} \, \delta_{sk} = \lambda_1^2 \overset{1\,1}{\mathbf{ee}} + \lambda_2^2 \overset{2\,2}{\mathbf{ee}} + \lambda_3^2 \overset{3\,3}{\mathbf{ee}}$$

and in general

$$\hat{Q}^n = \lambda_1^n \overset{1\,1}{\mathbf{ee}} + \lambda_2^n \overset{2\,2}{\mathbf{ee}} + \lambda_3^n \overset{3\,3}{\mathbf{ee}} . \tag{A.10.7}$$

This formula remains valid for both integer negative and for non-integer n. For instance

$$\hat{Q}^{-1} = \frac{1}{\lambda_1} \overset{11}{\mathbf{ee}} + \frac{1}{\lambda_2} \overset{22}{\mathbf{ee}} + \frac{1}{\lambda_3} \overset{33}{\mathbf{ee}} \tag{A.10.8}$$

and for $\lambda_s > 0$

$$\hat{Q}^{1/2} = \sqrt{\lambda_1} \overset{11}{\mathbf{ee}} + \sqrt{\lambda_2} \overset{22}{\mathbf{ee}} + \sqrt{\lambda_3} \overset{33}{\mathbf{ee}}, \tag{A.10.9}$$

since for this definition

$$\hat{Q}^{-1} \cdot \hat{Q} = \overset{11}{\mathbf{ee}} + \overset{22}{\mathbf{ee}} + \overset{33}{\mathbf{ee}} = \hat{E}, \quad \hat{Q}^{1/2} \cdot \hat{Q}^{1/2} = \lambda_1 \overset{11}{\mathbf{ee}} + \lambda_2 \overset{22}{\mathbf{ee}} + \lambda_3 \overset{33}{\mathbf{ee}} = \hat{Q},$$

which is required.

By eq. (A.6.10) we have $I_1\left(\hat{Q}^2\right) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ which allows $I_2\left(\hat{Q}\right)$ to be represented in the form

$$I_2\left(\hat{Q}\right) = \frac{1}{2}\left[I_1^2\left(\hat{Q}\right) - I_1\left(\hat{Q}^2\right)\right]. \tag{A.10.10}$$

Using eq. (A.10.3) and (A.10.7) we have

$$\hat{Q}^3 = \sum_{s=1}^{3} \overset{ss}{\mathbf{ee}} \left[-P_3(\lambda_s) + I_1\left(\hat{Q}\right)\lambda_s^2 - I_2\left(\hat{Q}\right)\lambda_s + I_3\left(\hat{Q}\right)\right] = \sum_{s=1}^{3} \lambda_s^3 \overset{ss}{\mathbf{ee}}$$

and since $P_3(\lambda_s) = 0$ we obtain

$$-\hat{Q}^3 + I_1\left(\hat{Q}\right)\hat{Q}^2 - I_2\left(\hat{Q}\right)\hat{Q} + I_3\left(\hat{Q}\right)\hat{E} = 0, \tag{A.10.11}$$

that is, the tensor satisfies the same characteristic equation as its principal invariants. This is the Cayley-Hamilton theorem which allows any integer power of tensor $\hat{Q}$ to be expressed in terms of $\hat{Q}^2, \hat{Q}, \hat{E}$ and its invariants of any power in terms of three principal invariants. The "negative powers" are also expressed in terms of $\hat{Q}^2, \hat{Q}, \hat{E}$. Indeed, by eq. (A.10.11)

$$\hat{Q}^{-1} = \frac{1}{I_3}\left(\hat{Q}^2 - I_1\hat{Q} + I_2\hat{E}\right), \tag{A.10.12}$$

$$\begin{aligned} \hat{Q}^{-2} &= \frac{1}{I_3}\left(\hat{Q} - I_1\hat{E} + I_2\hat{Q}^{-1}\right) \\ &= \frac{1}{I_3^2}\left[\left(I_2^2 - I_1 I_3\right)\hat{E} + \left(I_3 - I_1 I_2\right)\hat{Q} + I_2\hat{Q}^2\right], \end{aligned} \tag{A.10.13}$$

where for brevity $I_k = I_k\left(\hat{Q}\right)$.

Based on these equalities we construct expressions for the invariants $I_1\left(\hat{Q}^{-1}\right)$. Referring to eq. (A.10.10) we have

$$I_1\left(\hat{Q}^{-1}\right) = \frac{1}{I_3\left(\hat{Q}\right)}\left[I_1\left(\hat{Q}^2\right) - I_1^2\left(\hat{Q}\right) + 3I_2\left(\hat{Q}\right)\right] = \frac{I_2\left(\hat{Q}\right)}{I_3\left(\hat{Q}\right)}. \tag{A.10.14}$$

Further

$$\begin{aligned} I_2\left(\hat{Q}^{-1}\right) &= \frac{1}{2}\left[I_1\left(\hat{Q}^{-1}\right) - I_1\left(\hat{Q}^{-2}\right)\right] \\ &= \frac{1}{2I_3^2}\left[I_2^2 - 3\left(I_2^2 - I_1 I_3\right) - \left(I_3 - I_1 I_2\right) I_1 - I_2\left(I_1^2 - 2I_2\right)\right], \end{aligned}$$

so that

$$I_2\left(\hat{Q}^{-1}\right) = \frac{I_1\left(\hat{Q}\right)}{I_3\left(\hat{Q}\right)}. \tag{A.10.15}$$

Recalling now the well-known property of the determinant of the inverse matrix, we have

$$I_3\left(\hat{Q}^{-1}\right) = I_3^{-1}\left(\hat{Q}\right). \tag{A.10.16}$$

The Cayley-Hamilton theorem proved here for the symmetric tensor of second rank is valid for any (symmetric or nonsymmetric) matrix. Thus, the matrix satisfied its characteristic equation.

The symmetric tensor of second ranks $\hat{Q}$ is called positive if the quadratic form of the components of any vector $\mathbf{a}$ produced by means of this tensor is positive definite

$$\mathbf{a}\cdot\hat{Q}\cdot\mathbf{a} = q_{st}a_s a_t \geq 0,$$

an equality sign holding only for $a_1 = a_2 = a_3 = 0$. All principal invariants λ_s of the positive tensor are positive since

$$\mathbf{a}\cdot\hat{Q}\cdot\mathbf{a} = \mathbf{a}\cdot\sum_{s=1}^{3}\lambda_s \overset{s}{\mathbf{e}}\overset{s}{\mathbf{e}}\cdot\mathbf{a} = \lambda_1 a_1^2 + \lambda_2 a_2^2 + \lambda_3 a_3^2 \geq 0, \quad \lambda_s > 0,$$

where $a_k = \mathbf{a}\cdot\overset{k}{\mathbf{e}}$ denotes projections $\mathbf{a}$ onto the principal axes. Definition of tensor $\hat{Q}^{1/2}$ makes sense only for a positive tensor $\hat{Q}$. An example of a positive tensor is $\hat{P}\cdot\hat{P}^*$ if $\hat{P}$ is a non-singular tensor, i.e. ($|p_{st}| \neq 0$). Indeed

$$\mathbf{a}\cdot\hat{P}\cdot\hat{P}^*\cdot\mathbf{a} = \mathbf{a}\cdot p_{st}p_{qt}\mathbf{i}_s\mathbf{i}_q\cdot\mathbf{a} = p_{st}a_s p_{qt}a_q = \sum_t\left(p_{st}a_s\right)^2 \geq 0,$$

and the equality sign is possible only if

$$p_{st}a_s = 0 \quad (t = 1, 2, 3).$$

However this system of three equations only has the trivial solution ($a_s = 0$) as $|p_{st}| \neq 0$.

Now we can prove that the non-singular tensor $\hat{P}$ can be represented as the following products

$$\hat{P} = \hat{A} \cdot \hat{H} \quad \text{or} \quad \hat{P} = \hat{K} \cdot \hat{B}.$$

Here $\hat{A}$ and $\hat{B}$ are rotation tensors, while $\hat{H}$ and $\hat{K}$ are symmetric positive tensors. Indeed

$$\hat{P}^* = \hat{H} \cdot \hat{A}^* \quad \text{or} \quad \hat{P}^* = \hat{B}^* \cdot \hat{K},$$

and thus

$$\hat{P}^* \cdot \hat{P} = \hat{H} \cdot \hat{A}^* \cdot \hat{A} \cdot \hat{H} = \hat{H}^2 \quad \text{or} \quad \hat{P} \cdot \hat{P}^* = \hat{K} \cdot \hat{B} \cdot \hat{B}^* \cdot \hat{K} = \hat{K}^2.$$

According to the above-said $\hat{P}^* \cdot \hat{P}$ and $\hat{P} \cdot \hat{P}^*$ are positive tensors. Hence

$$\hat{H} = \left(\hat{P}^* \cdot \hat{P}\right)^{1/2}, \quad \hat{K} = \left(\hat{P} \cdot \hat{P}^*\right)^{1/2}.$$

We obtained

$$\hat{P} = \hat{A} \cdot \left(\hat{P}^* \cdot \hat{P}\right)^{1/2}, \quad \hat{P} = \left(\hat{P} \cdot \hat{P}^*\right)^{1/2} \cdot \hat{B} \tag{A.10.17}$$

which yields that

$$\hat{A} = \hat{P} \cdot \left(\hat{P}^* \cdot \hat{P}\right)^{-1/2}, \quad \hat{B} = \left(\hat{P} \cdot \hat{P}^*\right)^{-1/2} \cdot \hat{P} \tag{A.10.18}$$

are the rotation tensors. Indeed, tensors $\left(\hat{P}^* \cdot \hat{P}\right)^{-1/2}$ and $\left(\hat{P} \cdot \hat{P}^*\right)^{-1/2}$ are symmetric and thus

$$\hat{A}^* = \left(\hat{P}^* \cdot \hat{P}\right)^{-1/2} \cdot \hat{P}^*, \quad \hat{B}^* = \hat{P}^* \cdot \left(\hat{P} \cdot \hat{P}^*\right)^{-1/2},$$

$$\hat{A}^* \cdot \hat{A} = \left(\hat{P}^* \cdot \hat{P}\right)^{-1/2} \cdot \hat{P}^* \cdot \hat{P} \cdot \left(\hat{P}^* \cdot \hat{P}\right)^{-1/2} = \hat{E}, \quad \hat{A}^* = \hat{A}^{-1},$$

$$\hat{B} \cdot \hat{B}^* = \left(\hat{P} \cdot \hat{P}^*\right)^{-1/2} \cdot \hat{P} \cdot \hat{P}^* \cdot \left(\hat{P} \cdot \hat{P}^*\right)^{-1/2} = \hat{E}, \quad \hat{B}^{-1} = \hat{B}^*,$$

which is required.

A.10.1 Principal axes and principal directions of non-symmetric tensors

Similar to Section A.9 we restrict our consideration to the case of a tensor of second rank. We can introduce two systems ("right" $\mathbf{e}$ and "left" $\tilde{\mathbf{e}}$) of the principal directions

$$\hat{Q} \cdot \mathbf{e} = \lambda \mathbf{e} \quad \text{or} \quad \tilde{\mathbf{e}} \cdot \hat{Q} = \tilde{\lambda} \tilde{\mathbf{e}}. \tag{A.10.19}$$

The scalar multipliers λ and $\tilde{\lambda}$ are equal to each other since they are determined from the same cubic equation

$$\left|\hat{Q} - \lambda \hat{E}\right| = |q_{st} - \lambda \delta_{st}| = 0, \quad \left|\hat{Q} - \tilde{\lambda} \hat{E}\right| = \left|q_{ts} - \tilde{\lambda} \delta_{ts}\right| = 0.$$

An extended form of this equation is as follows

$$-\lambda^3 + I_1\left(\hat{Q}\right)\lambda^2 - I_2\left(\hat{Q}\right)\lambda + I_3\left(\hat{Q}\right) = 0,$$

where coefficients $I_k\left(\hat{Q}\right)$ are the principal invariants of $\hat{Q}$ and are expressed in terms of the components q_{st} by formulae similar to eqs. (A.10.4)-(A.10.6)

$$I_1\left(\hat{Q}\right) = \lambda_1 + \lambda_2 + \lambda_3 = q_{11} + q_{22} + q_{33}, \tag{A.10.20}$$

$$I_2\left(\hat{Q}\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = (q_{11}q_{22} - q_{12}q_{21}) + (q_{22}q_{33} - q_{23}q_{32}) + (q_{33}q_{11} - q_{31}q_{13}) = q\left(q^{11} + q^{22} + q^{33}\right) = qI_1\left(\hat{Q}^{-1}\right), \tag{A.10.21}$$

$$I_2\left(\hat{Q}\right) = \lambda_1\lambda_2\lambda_3 = |q_{ts}| = q. \tag{A.10.22}$$

Let us denote the real-valued root of this equation and the corresponding right principal value as λ_3 and $\mathbf{e}_3$ respectively. The remaining roots λ_1 and λ_2 can be either real-valued or complex-conjugate, i.e. $\lambda_1 = \mu_1 + i\mu_2, \lambda_2 = \mu_1 - i\mu_2$. Let us study first the latter assumption and introduce the corresponding complex-conjugate (right) vectors $\mathbf{e}_2 + i\mathbf{e}_1, \mathbf{e}_2 - i\mathbf{e}_1$. By definition

$$\hat{Q} \cdot (\mathbf{e}_2 \pm i\mathbf{e}_1) = (\mu_1 \pm i\mu_2)(\mathbf{e}_2 \pm i\mathbf{e}_1),$$

so that

$$\hat{Q} \cdot \mathbf{e}_2 = \mu_1\mathbf{e}_2 - \mu_2\mathbf{e}_1, \quad \hat{Q} \cdot \mathbf{e}_1 = \mu_1\mathbf{e}_1 - \mu_2\mathbf{e}_2, \quad \hat{Q} \cdot \mathbf{e}_3 = \lambda_3\mathbf{e}_3. \tag{A.10.23}$$

Now we introduce the (non-orthogonal vectorial) cobasis

$$\overset{1}{\mathbf{e}} = \frac{1}{g}\mathbf{e}_2 \times \mathbf{e}_3, \quad \overset{2}{\mathbf{e}} = \frac{1}{g}\mathbf{e}_3 \times \mathbf{e}_1, \quad \overset{3}{\mathbf{e}} = \frac{1}{g}\mathbf{e}_1 \times \mathbf{e}_2 \quad (g = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)),$$

such that

$$\overset{k}{\mathbf{e}} \cdot \mathbf{e}_s = \delta_{ks}. \tag{A.10.24}$$

This allows one to write the dyadic (right) representation of $\hat{Q}$

$$\hat{Q} = \mu_1 \left(\mathbf{e}_1 \overset{1}{\mathbf{e}} + \mathbf{e}_2 \overset{2}{\mathbf{e}} \right) + \mu_2 \left(\mathbf{e}_2 \overset{1}{\mathbf{e}} - \mathbf{e}_1 \overset{2}{\mathbf{e}} \right) + \lambda_3 \mathbf{e}_3 \overset{3}{\mathbf{e}}. \tag{A.10.25}$$

Indeed, due to eq. (A.10.24) the relationships (A.10.25) hold.

If all roots λ_s are real-valued and the corresponding vectors of the principal directions are denoted as $\mathbf{e}_s$ then defining the cobasis by the above relationships we obtain

$$\hat{Q} = \lambda_1 \mathbf{e}_1 \overset{1}{\mathbf{e}} + \lambda_2 \mathbf{e}_2 \overset{2}{\mathbf{e}} + \lambda_3 \mathbf{e}_3 \overset{3}{\mathbf{e}}. \tag{A.10.26}$$

When $\hat{Q}$ is a symmetric tensor then $\mathbf{e}_s$ is the orthonormalised vector basis coinciding with the cobasis $\overset{s}{\mathbf{e}}$ and we arrive at representation (A.9.12).

The left basis $\tilde{\mathbf{e}}_s$ and the corresponding cobasis $\overset{k}{\tilde{\mathbf{e}}}$ of the principal directions are introduced by analogy.

Example 1. The skew-symmetric tensor $\hat{\Omega}$ is presented in terms of the accompanying vector $\boldsymbol{\omega}$

$$\hat{\Omega} = \omega_{st} \mathbf{i}_s \mathbf{i}_t = \hat{E} \times \boldsymbol{\omega}. \tag{A.10.27}$$

Hence

$$\hat{\Omega} \cdot \mathbf{e} = \left(\hat{E} \times \boldsymbol{\omega} \right) \cdot \mathbf{e} = \boldsymbol{\omega} \times \mathbf{e} = \lambda \mathbf{e},$$

and this relationship can be satisfied by assuming

$$\lambda_3 = 0, \quad \mathbf{e}_3 = \frac{\boldsymbol{\omega}}{\omega} \quad (|\mathbf{e}_3| = 1).$$

The characteristic equation for tensor $\hat{\Omega}$

$$\begin{vmatrix} -\lambda & -\omega_3 & \omega_2 \\ \omega_3 & -\lambda & -\omega_1 \\ -\omega_2 & \omega_1 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda\omega^2 = 0 \quad \left(\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \right)$$

has two imaginary roots $\pm i\omega$ except for a zero one. Let $\mathbf{e}_2 \pm i\mathbf{e}_1$ denote the vectors $\mathbf{e}$ corresponding to these two roots, such that

$$\boldsymbol{\omega} \times (\mathbf{e}_2 \pm i\mathbf{e}_1) = \omega \mathbf{e}_3 \times (\mathbf{e}_2 \pm i\mathbf{e}_1) = \pm i\omega (\mathbf{e}_2 \pm i\mathbf{e}_1),$$
$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3, \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1.$$

Vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form an orthogonal basis and the unit vectors $\mathbf{e}_1, \mathbf{e}_2$ are determined up to a rotation about axis $\mathbf{e}_3$. In such an orthonormalised basis tensor $\hat{\Omega}$ is given by the dyadic representation

$$\hat{\Omega} = \omega \left(\mathbf{e}_2 \mathbf{e}_1 - \mathbf{e}_1 \mathbf{e}_2 \right). \tag{A.10.28}$$

Indeed,

$$\hat{\Omega} \cdot \left(\mathbf{e}_2 \pm i \mathbf{e}_1 \right) = \omega \left(\mathbf{e}_2 \mathbf{e}_1 - \mathbf{e}_1 \mathbf{e}_2 \right) \cdot \left(\mathbf{e}_2 \pm i \mathbf{e}_1 \right) = \pm i \omega \left(\mathbf{e}_2 \pm i \mathbf{e}_1 \right),$$

which is required.

Example 2. The rotation tensor is $\hat{A} = \alpha_{st} \mathbf{i}_s \mathbf{i}_t$. Taking into account the well-known relationships

$$|\alpha_{st}| = 1, \quad \alpha_{11} = \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} \quad \text{etc.}$$

we can set the characteristic equation in the form

$$-\lambda^3 + \lambda \left(\lambda - 1 \right) \left(\alpha_{11} + a_{22} + \alpha_{33} \right) + 1 = 0.$$

One of the roots $\lambda_3 = 1$ whilst the two remaining roots can be represented as $\lambda_1 = e^{-i\omega}, \lambda_2 = e^{i\omega}$ where

$$2 \cos \omega = 1 + \alpha_{11} + \alpha_{22} + \alpha_{33} = 1 + I_1 \left(\hat{A} \right). \tag{A.10.29}$$

Two left principal directions of tensor $\hat{A}$ (the right directions of tensor $\hat{A}^*$) are sought. Root $\lambda_3 = 1$ describes the direction of the unit vector $\mathbf{e}_3$ remaining unchanged under rotation (since it describes the rotation axis)

$$\mathbf{e}_3 \cdot \hat{A} = \grave{\mathbf{e}}_3 = \lambda_3 \mathbf{e}_3 = \mathbf{e}_3.$$

Vectors $\mathbf{e}_1 \pm i \mathbf{e}_2$ corresponding to roots λ_1, λ_2 are given by

$$\left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) \cdot \hat{A} = e^{\mp i\omega} \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right), \quad \hat{A}^* \cdot \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) = e^{\mp i\omega} \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right),$$

so that

$$\left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) \cdot \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) \left(1 - e^{\mp 2 i\omega} \right) = 0$$

because $\hat{A} \cdot \hat{A}^* = \hat{E}$. By analogy

$$\mathbf{e}_3 \cdot \hat{A} \cdot \hat{A}^* \cdot \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) = \mathbf{e}_3 \cdot \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right) = \mathbf{e}_3 \cdot e^{\mp i\omega} \left(\mathbf{e}_1 \pm i \mathbf{e}_2 \right)$$

such that, excluding the case of $e^{\mp i\omega} = 1$ we obtain

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \quad \mathbf{e}_2 \cdot \mathbf{e}_3 = 0,$$

where $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2$. Let the square of the absolute value of vectors $\mathbf{e}_1 \pm i \mathbf{e}_2$ be equal to 2

$$\left(\mathbf{e}_1 + i \mathbf{e}_2 \right) \cdot \left(\mathbf{e}_1 - i \mathbf{e}_2 \right) = \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_2 \cdot \mathbf{e}_2 = 2 \mathbf{e}_1 \cdot \mathbf{e}_1 = 2 \mathbf{e}_2 \cdot \mathbf{e}_2 = 2.$$

Then vectors $\mathbf{e}_1, \mathbf{e}_2$ are the unit vectors and vector $\mathbf{e}_1 + i\mathbf{e}_2$ is defined up to the multiplier $e^{i\lambda}$. Thus, vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form an orthonormalised trihedron in which vectors $\mathbf{e}_1, \mathbf{e}_2$ are determined up to a rotation about $\mathbf{e}_3$.

The rotation tensor is now presented in the form

$$\hat{A} = \mathbf{e}_3\mathbf{e}_3 + \frac{1}{2}\left[(\mathbf{e}_1 - i\mathbf{e}_2)(\mathbf{e}_1 + i\mathbf{e}_2)e^{-i\omega} + (\mathbf{e}_1 + i\mathbf{e}_2)(\mathbf{e}_1 - i\mathbf{e}_2)e^{i\omega}\right]. \tag{A.10.30}$$

Indeed, the required relationships

$$\mathbf{e}_3 \cdot \hat{A} = \mathbf{e}_3, \quad (\mathbf{e}_1 \pm i\mathbf{e}_2) \cdot \hat{A} = e^{\mp i\omega}(\mathbf{e}_1 \pm i\mathbf{e}_2)$$

hold. Another form of formula (A.10.26) repeating eq. (A.8.8) has the form

$$\begin{aligned}\hat{A} &= \mathbf{e}_3\mathbf{e}_3 + (\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2)\cos\omega + (\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1)\sin\omega \\ &= \hat{E}\cos\omega + \mathbf{e}_3\mathbf{e}_3(1-\cos\omega) - \mathbf{e}_3 \times \hat{E}\sin\omega.\end{aligned} \tag{A.10.31}$$

This invariant form for the rotation tensor $\hat{A}$ shows that the axis of rotation is given by vector $\mathbf{e}_3$ and ω denotes the angle of rotation about this axis. One can easily convince oneself of this by noticing that

$$\mathbf{e}_1^{\backslash} = \mathbf{e}_1 \cdot \hat{A} = \mathbf{e}_1\cos\omega + \mathbf{e}_2\sin\omega, \quad \mathbf{e}_2^{\backslash} = \mathbf{e}_2 \cdot \hat{A} = -\mathbf{e}_1\sin\omega + \mathbf{e}_2\cos\omega,$$

i.e. the rotated vectors $\mathbf{e}_1^{\backslash}, \mathbf{e}_2^{\backslash}$ are obtained by rotating vectors $\mathbf{e}_1, \mathbf{e}_2$ about $\overset{3}{\mathbf{e}}$ through angle ω in the positive direction.

The representations for the dyadics of the principal directions of the symmetric tensor follows from the relations

$$\begin{aligned}\overset{1\,1}{\mathbf{ee}} + \overset{2\,2}{\mathbf{ee}} + \overset{3\,3}{\mathbf{ee}} &= \hat{E}, \\ q_1\overset{1\,1}{\mathbf{ee}} + q_2\overset{2\,2}{\mathbf{ee}} + q_3\overset{3\,3}{\mathbf{ee}} &= \hat{Q}, \\ q_1^2\overset{1\,1}{\mathbf{ee}} + q_2^2\overset{2\,2}{\mathbf{ee}} + q_3^2\overset{3\,3}{\mathbf{ee}} &= \hat{Q}^2,\end{aligned}$$

where q_s denotes the principal values of $\hat{Q}$. We obtain

$$\begin{aligned}\overset{1\,1}{\mathbf{ee}} &= \frac{\hat{E}q_2q_3 - \hat{Q}(q_2+q_3) + \hat{Q}^2}{(q_1-q_2)(q_1-q_3)} = \frac{\left(\hat{Q} - \hat{E}q_2\right)\cdot\left(\hat{Q} - \hat{E}q_3\right)}{(q_1-q_2)(q_1-q_3)}, \\ \overset{2\,2}{\mathbf{ee}} &= \frac{\hat{E}q_3q_1 - \hat{Q}(q_3+q_1) + \hat{Q}^2}{(q_2-q_3)(q_2-q_1)} = \frac{\left(\hat{Q} - \hat{E}q_3\right)\cdot\left(\hat{Q} - \hat{E}q_1\right)}{(q_2-q_3)(q_2-q_1)}, \\ \overset{3\,3}{\mathbf{ee}} &= \frac{\hat{E}q_1q_2 - \hat{Q}(q_1+q_2) + \hat{Q}^2}{(q_3-q_1)(q_3-q_2)} = \frac{\left(\hat{Q} - \hat{E}q_1\right)\cdot\left(\hat{Q} - \hat{E}q_2\right)}{(q_3-q_1)(q_3-q_2)}.\end{aligned} \tag{A.10.32}$$

Now it is easy to obtain expressions for the square and the products of cosines of the angles $\overset{s}{e}_k = \overset{s}{\mathbf{e}} \cdot \mathbf{i}_k$ between the principal directions and the coordinate axes. For example,

$$\left(\overset{1}{e}_s\right)^2 = \frac{q_2 q_3 - q_{ss}(q_2 + q_3) + q_{st} q_{ts}}{(q_1 - q_2)(q_1 - q_3)} \quad (s = 1, 2, 3),$$

$$\overset{1}{e}_s \overset{1}{e}_k = \frac{q_{st} q_{tk} - q_{sk}(q_2 + q_3)}{(q_1 - q_2)(q_1 - q_3)}.$$

A.11 Splitting the symmetric tensor of second rank in deviatoric and spherical tensors

The isotropic tensor $\frac{1}{3} I_1 \left(\hat{Q}\right) \hat{E}$ is referred to as the spherical part of tensor $\hat{Q}$. The remainder of tensor $\hat{Q}$ is called the deviator and denoted as $\operatorname{Dev} \hat{Q}$

$$\hat{Q} = \frac{1}{3} I_1 \left(\hat{Q}\right) \hat{E} + \operatorname{Dev} \hat{Q}, \quad \operatorname{Dev} \hat{Q} = \hat{Q} - \frac{1}{3} I_1 \left(\hat{Q}\right) \hat{E}. \tag{A.11.1}$$

By eq. (A.9.5) the characteristic equation of deviator is set in the form

$$\left| q_{st} - \delta_{st} \left[\varkappa + \frac{1}{3} (\lambda_1 + \lambda_2 + \lambda_3) \right] \right| = 0, \tag{A.11.2}$$

that is, the principal values of the deviator are equal to

$$\varkappa_s = \lambda_s - \frac{1}{3} I_1 \left(\hat{Q}\right),$$

or

$$\varkappa_1 = \frac{1}{3}(2\lambda_1 - \lambda_2 - \lambda_3), \ \varkappa_2 = \frac{1}{3}(2\lambda_2 - \lambda_3 - \lambda_1), \ \varkappa_3 = \frac{1}{3}(2\lambda_3 - \lambda_1 - \lambda_2). \tag{A.11.3}$$

Its principal directions coincide with the principal directions of the tensor. Indeed, from the equality

$$\hat{Q} \cdot \mathbf{e} = \lambda \mathbf{e} = \frac{1}{3} I_1 \left(\hat{Q}\right) \mathbf{e} + \left(\operatorname{Dev} \hat{Q}\right) \cdot \mathbf{e}$$

we obtain

$$\left(\operatorname{Dev} \hat{Q}\right) \cdot \mathbf{e} = \left[\lambda - \frac{1}{3} I_1 \left(\hat{Q}\right)\right] \mathbf{e} = \varkappa \mathbf{e}, \tag{A.11.4}$$

that is, vector $\mathbf{e}$ determined by eq. (A.9.1) also satisfies eq. (A.11.4).

A.11.1 Invariants of deviator

It follows from formulae (A.11.3), (A.10.4) and (A.10.5) that

$$I_1\left(\operatorname{Dev}\hat{Q}\right) = \varkappa_1 + \varkappa_2 + \varkappa_3 = 0, \tag{A.11.5}$$

$$I_2\left(\operatorname{Dev}\hat{Q}\right) = \varkappa_1\varkappa_2 + \varkappa_2\varkappa_3 + \varkappa_3\varkappa_1 = I_2\left(\hat{Q}\right) - \frac{1}{3}I_1^2\left(\hat{Q}\right), \tag{A.11.6}$$

$$I_3\left(\operatorname{Dev}\hat{Q}\right) = \varkappa_1\varkappa_2\varkappa_3 = I_3\left(\hat{Q}\right) - \frac{1}{3}I_1\left(\hat{Q}\right)I_2\left(\hat{Q}\right) + \frac{2}{27}I_1^3\left(\hat{Q}\right). \tag{A.11.7}$$

It is easy to prove by means of eqs. (A.11.5) and (A.11.6) that the second invariant of the deviator is set in the form

$$I_2\left(\operatorname{Dev}\hat{Q}\right) = -\frac{1}{6}\left[(\lambda_1-\lambda_2)^2 + (\lambda_2-\lambda_3)^2 + (\lambda_3-\lambda_1)^2\right]. \tag{A.11.8}$$

Another expression for the second invariant of the deviator is obtained by replacing $\hat{Q}$ by $\operatorname{Dev}\hat{Q}$ in eq. (A.10.10) and taking into account eq. (A.11.5)

$$I_2\left(\operatorname{Dev}\hat{Q}\right) = -\frac{1}{2}I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^2\right]. \tag{A.11.9}$$

Clearly, $I_2\left(\operatorname{Dev}\hat{Q}\right) < 0$. We also need the expressions for the first invariant of the powers of the deviator in terms of the second and third invariants. To obtain them, we use the Cayley-Hamilton theorem (A.10.11) for the deviator. We obtain

$$\left.\begin{aligned}\left(\operatorname{Dev}\hat{Q}\right)^3 &= -I_2\left(\operatorname{Dev}\hat{Q}\right)\operatorname{Dev}\hat{Q} + I_3\left(\operatorname{Dev}\hat{Q}\right)\hat{E},\\ \left(\operatorname{Dev}\hat{Q}\right)^4 &= -I_2\left(\operatorname{Dev}\hat{Q}\right)\left(\operatorname{Dev}\hat{Q}\right)^2 + I_3\left(\operatorname{Dev}\hat{Q}\right)\operatorname{Dev}\hat{Q}\end{aligned}\right\} \tag{A.11.10}$$

etc. Referring to eqs. (A.11.5) and (A.11.9) we find

$$I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^3\right] = 3I_3\left(\operatorname{Dev}\hat{Q}\right), \quad I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^4\right] = 2I_2^2\left(\operatorname{Dev}\hat{Q}\right). \tag{A.11.11}$$

Using eqs. (A.11.5) and (A.10.3) we can write the characteristic equation of the deviator in the following form

$$\varkappa^3 + I_2\left(\operatorname{Dev}\hat{Q}\right)\varkappa = I_3\left(\operatorname{Dev}\hat{Q}\right). \tag{A.11.12}$$

This cubic equations has no term with $\varkappa^2$, thus it has only real-valued roots. Its solution can be presented in trigonometric form by assuming

$$\varkappa = \frac{2}{\sqrt{3}}\sqrt{-I_2\left(\operatorname{Dev}\hat{Q}\right)}\sin\psi = \frac{2\Gamma}{\sqrt{3}}\sin\psi, \tag{A.11.13}$$

where for brevity

$$\Gamma^2 = -I_2\left(\operatorname{Dev}\hat{Q}\right). \tag{A.11.14}$$

Insertion into eq. (A.11.12) yields

$$\varkappa^3 + I_2\left(\operatorname{Dev}\hat{Q}\right)\varkappa = \frac{2\Gamma^3}{3\sqrt{3}}\left(4\sin^3\psi - 3\sin\psi\right) = -\frac{2\Gamma^3}{3\sqrt{3}}\sin 3\psi,$$

so that

$$I_3\left(\operatorname{Dev}\hat{Q}\right) = -\frac{2}{3\sqrt{3}}\Gamma^3\sin 3\psi. \tag{A.11.15}$$

This determines three values of $\varkappa$ and the corresponding principal values of the deviator

$$\left.\begin{array}{ll} \varkappa_1 = \dfrac{2\Gamma}{\sqrt{3}}\sin\psi, & \varkappa_2 = \dfrac{2\Gamma}{\sqrt{3}}\sin\left(\psi + \dfrac{2\pi}{3}\right), \\ \varkappa_3 = \dfrac{2\Gamma}{\sqrt{3}}\sin\left(\psi + \dfrac{4\pi}{3}\right) & \left(|\psi| < \dfrac{\pi}{6}\right). \end{array}\right\} \tag{A.11.16}$$

Values Γ and ψ are expressed in terms of the principal invariants of the deviator and thus they are also its invariants. When dealing with some problems, the invariants $I_1\left(\operatorname{Dev}\hat{Q}\right)$, Γ and ψ are preferable to the principal invariants of tensor $\hat{Q}$.

Formulae (A.11.9)-(A.11.11) are now rewritten in the form

$$I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^2\right] = 2\Gamma^2, \quad I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^3\right] = -\frac{2\Gamma^3}{\sqrt{3}}\sin 3\psi,$$
$$I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^4\right] = 2\Gamma^4. \tag{A.11.17}$$

A.12 Functions of tensors

A.12.1 Scalar

Function $f\left(\hat{Q}\right)$ satisfying the condition

$$f\left(\hat{Q}\right) = f\left(\hat{A}^* \cdot \hat{Q} \cdot \hat{A}\right) = f\left(\hat{Q}'\right) \tag{A.12.1}$$

is referred to as the invariant scalar. However this notion contains a tautology since the scalar is invariant by definition. Relation (A.12.1) indicates that both the numerical value and the form of dependence $f\left(\hat{Q}\right)$ on the components of tensor $\hat{Q}$ are the same in all coordinate systems related by the transformation of rotation. Any function of the invariants of tensor, in particular, its principal invariants

$$f\left(\hat{Q}\right) = f\left(I_1\left(\hat{Q}\right), I_2\left(\hat{Q}\right), I_3\left(\hat{Q}\right)\right). \tag{A.12.2}$$

is the invariant scalar. In what follows the notion "invariant scalar" is used in this (limited) sense.

A.12.2 Tensor function of tensor $\hat{Q}$

It is assumed that n^2 functions $p_{sk} = f_{sk}(q_{mn})$ are prescribed where q_{mn} denotes the components of tensor $\hat{Q}$. The elements of matrix $\|f_{sk}\|$ are assumed to obey the transformation law of the tensor components. It determines a tensor function $\hat{P}\left(\hat{Q}\right)$ on tensor $\hat{Q}$.

A tensor power series

$$\hat{P} = \sum_{k=0}^{\infty} \alpha_k \hat{Q}^k \quad \left(\hat{Q}^0 = \hat{E}\right), \tag{A.12.3}$$

in which α_k are the invariant scalars, is a tensor function. The Cayley-Hamilton theorem allows all powers of $\hat{Q}$ higher than two to be expressed in terms of $\hat{Q}^0, \hat{Q}, \hat{Q}^2$. Hence any tensor function which is represented by a power series can be described by the quadratic trinomial

$$\hat{P} = A\hat{E} + B\hat{Q} + C\hat{Q}^2, \tag{A.12.4}$$

where A, B, C are invariant scalars.

An example of the tensor function which can not be presented by a power series is $\hat{P} = \hat{Q}^*$.

Tensor $\hat{P}$ is an isotropic function of tensor $\hat{Q}$ if the following equality

$$\hat{P}' = \hat{F}\left(\hat{Q}'\right) \quad \text{or} \quad \hat{A}^* \cdot \hat{P} \cdot \hat{A} = \hat{F}\left(\hat{A}^* \cdot \hat{Q} \cdot \hat{A}\right) \tag{A.12.5}$$

holds. This means that in the system of axes related by the rotation transformation the form of dependence of $\hat{P}$ on $\hat{Q}$ is conserved, the numerical values of tensor $\hat{P}$ (in axes $\mathbf{i}_s$) and tensor $\hat{P}'$ (in axes $\mathbf{i}'_s$) are equal to each other. Do not confuse the terms "isotropic tensor function" and "isotropic tensor". The components of the latter are the same in all system of axes, see Section A.6.

One can prove the theorem: a symmetric tensor function $\left(\hat{P} = \hat{P}^*\right)$ on the symmetric tensor $\hat{Q}$ is isotropic if and only if it is presented in the form (A.12.4) where A, B, C are scalar invariants of type (A.12.2).

A.12.3 Gradient of scalar with respect to a tensor

Let $f(q_{11}, \ldots, q_{33}) = f(q_{st})$ be a function of the components of tensor $\hat{Q}$ prescribed in axes $\mathbf{i}_s$. When the components of $\hat{Q}$ are given in new axes, the elements of matrix $\dfrac{\partial f}{\partial q_{st}}$ are transformed by rule (A.3.7). Indeed

$$\frac{\partial f}{\partial q'_{mn}} = \frac{\partial f}{\partial q_{st}} \frac{\partial q_{st}}{\partial q'_{mn}} = \alpha_{ms}\alpha_{nt}\frac{\partial f}{\partial q_{st}},$$

which is required. Hence the quantity

$$\frac{\partial f}{\partial \hat{Q}} = \hat{F}\left(\hat{Q}\right) = \frac{\partial f}{\partial q_{st}}\mathbf{i}_s\mathbf{i}_t \tag{A.12.6}$$

presents a tensor of second rank and is called the gradient of function f with respect to $\hat{Q}$.

This definition yields the relationship

$$\frac{\partial f}{\partial \hat{Q}} \cdot\cdot \delta\hat{Q}^* = \frac{\partial f}{\partial q_{st}}\mathbf{i}_s\mathbf{i}_t \cdot\cdot \mathbf{i}_m\mathbf{i}_n \delta q_{nm} = \frac{\partial f}{\partial q_{st}}\delta q_{st} = \delta f. \tag{A.12.7}$$

Let f be an invariant scalar, then according to definition (A.12.1) $f\left(\hat{Q}\right) = f\left(\hat{Q}'\right)$, $q_{st} = q'_{st}$ and by eq. (A.12.3)

$$\hat{F}\left(\hat{Q}'\right) = \hat{F}\left(\hat{A}^* \cdot \hat{Q} \cdot \hat{A}\right) = \frac{\partial f}{\partial q'_{st}}\mathbf{i}'_s\mathbf{i}'_t = \hat{A}^* \cdot \mathbf{i}_s \frac{\partial f}{\partial q_{st}}\mathbf{i}_t \cdot \hat{A} = \hat{A}^* \cdot \hat{F}\left(\hat{Q}\right) \cdot \hat{A},$$

so that $\hat{F}\left(\hat{Q}\right)$ is an isotropic function of tensor $\hat{Q}$.

A.12.4 Derivatives of the principal invariants of a tensor with respect to the tensor

Referring to eqs. (A.7.9) and (A.12.6) we have

$$\frac{\partial I_3\left(\hat{Q}\right)}{\partial \hat{Q}} = I_3\left(\hat{Q}\right)\left(\hat{Q}^*\right)^{-1}. \tag{A.12.8}$$

On the other hand

$$I_3\left(\hat{Q} - \lambda\hat{E}\right) = \left|\hat{Q} - \lambda\hat{E}\right| = -\lambda^3 + \lambda^2 I_1\left(\hat{Q}\right) - \lambda I_2\left(\hat{Q}\right) + I_3\left(\hat{Q}\right) \tag{A.12.9}$$

and by eq. (A.12.8)

$$\frac{\partial}{\partial \hat{Q}} I_3\left(\hat{Q} - \lambda\hat{E}\right) = I_3\left(\hat{Q} - \lambda\hat{E}\right)\left[\left(\hat{Q} - \lambda\hat{E}\right)^*\right]^{-1}$$
$$= \lambda^2 \frac{\partial I_1\left(\hat{Q}\right)}{\partial \hat{Q}} - \lambda \frac{\partial I_2\left(\hat{Q}\right)}{\partial \hat{Q}} + I_3\left(\hat{Q}\right)\left(\hat{Q}^*\right)^{-1}.$$

Hence

$$\hat{E}I_3\left(\hat{Q}-\lambda\hat{E}\right)=\left(\hat{Q}-\lambda\hat{E}\right)^*\cdot\left[\lambda^2\frac{\partial I_1\left(\hat{Q}\right)}{\partial\hat{Q}}-\right.$$
$$\left.\lambda\frac{\partial I_2\left(\hat{Q}\right)}{\partial\hat{Q}}+I_3\left(\hat{Q}\right)\left(\hat{Q}^*\right)^{-1}\right].$$

Replacing the left hand side according to eq. (A.12.9) and equating the coefficients in front of the same power of λ we obtain the expression for the tensor functions

$$\frac{\partial I_1\left(\hat{Q}\right)}{\partial\hat{Q}}=\hat{E},\quad\frac{\partial I_2\left(\hat{Q}\right)}{\partial\hat{Q}}=\hat{E}I_1\left(\hat{Q}\right)-\hat{Q}^*\tag{A.12.10}$$

and the relationship

$$\hat{Q}^*\cdot\frac{\partial I_2\left(\hat{Q}\right)}{\partial\hat{Q}}+I_3\left(\hat{Q}\right)\left(\hat{Q}^*\right)^{-1}=\hat{E}I_2\left(\hat{Q}\right).$$

By eq. (A.12.10) the latter is transformed to the form

$$-\hat{Q}^{*^2}+\hat{Q}^*I_1\left(\hat{Q}\right)-\hat{E}I_2\left(\hat{Q}\right)+I_3\left(\hat{Q}\right)\left(\hat{Q}^*\right)^{-1}=0,\tag{A.12.11}$$

which is one of the forms of the Cayley-Hamilton theorem (A.10.11). Indeed, bearing in mind that $I_1\left(\hat{Q}^*\right)=I_1\left(\hat{Q}\right)$ and multiplying eq. (A.12.11) by $\hat{Q}^*$ we can represent this equation in form (A.10.11).

A.12.5 Gradient of an invariant scalar

The expression for this quantity follows directly from formulae (A.12.8) and (A.12.20)

$$\frac{\partial f}{\partial\hat{Q}}=\left(\frac{\partial f}{\partial I_1}+I_1\left(\hat{Q}\right)\frac{\partial f}{\partial I_2}\right)\hat{E}-\frac{\partial f}{\partial I_2}\hat{Q}+\frac{\partial f}{\partial I_3}I_3\left(\hat{Q}\right)\hat{Q}^{-1}\quad\left(\hat{Q}=\hat{Q}^*\right).\tag{A.12.12}$$

Replacing here $\hat{Q}^{-1}$ by eq. (A.10.12) we arrive at another form

$$\frac{\partial f}{\partial\hat{Q}}=\left[\frac{\partial f}{\partial I_1}+I_1\left(\hat{Q}\right)\frac{\partial f}{\partial I_2}+I_2\left(\hat{Q}\right)\frac{\partial f}{\partial I_3}\right]\hat{E}-$$
$$\left(\frac{\partial f}{\partial I_2}+I_1\left(\hat{Q}\right)\frac{\partial f}{\partial I_3}\right)\hat{Q}+\frac{\partial f}{\partial I_3}\hat{Q}^2.\tag{A.12.13}$$

The isotropic tensor function obtained with the help of invariant scalar $f\left(\hat{Q}\right)$ is presented in the form of a quadratic trinomial (or in the equivalent form (A.12.12)) on tensor $\hat{Q} = \hat{Q}^*$ with the coefficients which are equivalent scalars. This is a particular case of representation (A.12.4) for the isotropic tensor function $\hat{P} = f\left(\hat{Q}\right)$.

Remark. Relationship (A.12.4) does not exhaust the diversity of relationships between two symmetric tensors. The expression for the invariant relation between them

$$\hat{P} = \hat{F}\left(\hat{Q}, I_s\left(\hat{Q}\right), I_k\left(\hat{P}\right), \hat{R}, \hat{S}, \ldots\right)$$

may contain some additional tensors $\hat{R}, \hat{S}$. An example is the following relation

$$\hat{P} = {}^{(4)}\hat{R} \cdot\cdot \hat{Q}, \quad p_{st} = r_{stmn} q_{mn}, \tag{A.12.14}$$

in which tensor $\hat{P}$ is obtained by double contraction of the tensor of sixth rank ${}^{(4)}\hat{R}\hat{Q}$. The components of tensor ${}^{(4)}\hat{R}$ must be symmetric about indices in each pair st, mn which reduces their number from 81 to 36. In particular, eq. (A.12.14) describes the relation between the stress tensor and strain tensor in an elastic anisotropic solid (the number of independent components of ${}^{(4)}\hat{R}$ reduces to 21).

A.13 Extracting spherical and deviatoric parts

Returning to the general representation of isotropic tensor function (A.12.4) we extract the spherical and deviatoric parts of tensor $\hat{P}$

$$I_1\left(\hat{P}\right) = 3A + BI_1\left(\hat{Q}\right) + CI_1\left(\hat{Q}^2\right), \tag{A.13.1}$$

$$\operatorname{Dev}\hat{P} = \hat{P} - \frac{1}{3}\hat{E}I_1\left(\hat{P}\right) = B\operatorname{Dev}\hat{Q} + C\operatorname{Dev}\hat{Q}^2. \tag{A.13.2}$$

By eqs. (A.10.10) and (A.11.6)

$$I_1\left(\hat{Q}^2\right) = I_1^2\left(\hat{Q}\right) - 2I_2\left(\hat{Q}\right) = \frac{1}{3}I_1^2\left(\hat{Q}\right) - 2I_2\left(\operatorname{Dev}\hat{Q}\right),$$

$$\hat{Q}^2 = \left[\operatorname{Dev}\hat{Q} + \frac{1}{3}\hat{E}I_1\left(\hat{Q}\right)\right]^2 = \left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{2}{3}I_1\left(\hat{Q}\right)\operatorname{Dev}\hat{Q} + \frac{1}{9}\hat{E}I_1^2\left(\hat{Q}\right),$$

so that

$$\begin{aligned}\operatorname{Dev}\hat{Q}^2 &= \hat{Q}^2 - \frac{1}{3}\hat{E}I_1\left(\hat{Q}^2\right) \\ &= \left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{2}{3}I_1\left(\hat{Q}\right)\operatorname{Dev}\hat{Q} + \frac{2}{3}I_2\left(\operatorname{Dev}\hat{Q}\right)\hat{E},\end{aligned}$$

and substitution into eq. (A.13.1) leads to the relationships

$$I_1\left(\hat{P}\right) = 3A + BI_1\left(\hat{Q}\right) + C\left[\frac{1}{3}I_1^2\left(\hat{Q}\right) + 2\Gamma^2\right], \tag{A.13.3}$$

$$\operatorname{Dev}\hat{P} = \left[B + \frac{2}{3}CI_1\left(\hat{Q}\right)\right]\operatorname{Dev}\hat{Q} + C\left[\left(\operatorname{Dev}\hat{Q}\right)^2 - \frac{2}{3}\hat{E}\Gamma^2\right]. \tag{A.13.4}$$

Introducing the new denotation

$$I_1\left(\hat{P}\right) = 3kI_1\left(\hat{Q}\right), \quad \alpha = B + \frac{2}{3}CI_1\left(\hat{Q}\right), \quad \beta = C, \tag{A.13.5}$$

we can present the general form of the dependence between two coaxial tensors in the following form

$$\hat{P} = kI_1\left(\hat{Q}\right)\hat{E} + \alpha\operatorname{Dev}\hat{Q} + \beta\left[\left(\operatorname{Dev}\hat{Q}\right)^2 - \frac{2}{3}\hat{E}\Gamma^2\right], \tag{A.13.6}$$

where

$$\operatorname{Dev}\hat{P} = \alpha\operatorname{Dev}\hat{Q} + \beta\left[\left(\operatorname{Dev}\hat{Q}\right)^2 - \frac{2}{3}\hat{E}\Gamma^2\right]. \tag{A.13.7}$$

Constants α and β can be obtained in terms of the invariants of $\operatorname{Dev}\hat{P}$. For obtaining the second invariant we use the first formula (A.11.7) in the form

$$2\tau^2 = I_1\left[\left(\operatorname{Dev}\hat{P}\right)^2\right],$$

where

$$\tau^2 = -I_2\left(\operatorname{Dev}\hat{P}\right). \tag{A.13.8}$$

Then we have

$$\left(\operatorname{Dev}\hat{P}\right)^2 = \alpha^2\left(\operatorname{Dev}\hat{Q}\right)^2 = 2\alpha\beta\left[\left(\operatorname{Dev}\hat{Q}\right)^3 - \frac{2}{3}\Gamma^2\operatorname{Dev}\hat{Q}\right] + \beta^2\left[\left(\operatorname{Dev}\hat{Q}\right)^4 - \frac{4}{3}\Gamma^2\left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{4}{9}\hat{E}\Gamma^4\right]$$

and referring again to eq. (A.11.17) we find after a simple calculation that

$$\begin{aligned}\tau^2 = \frac{1}{2}I_1\left[\left(\operatorname{Dev}\hat{P}\right)^2\right] &= \Gamma^2\left(\alpha^2 - \frac{2}{\sqrt{3}}\alpha\beta\Gamma\sin 3\psi + \Gamma^2\frac{\beta^2}{3}\right)\\ &= \Gamma^2\left[\left(\alpha - \frac{\Gamma}{\sqrt{3}}\beta\sin 3\psi\right)^2 + \left(\frac{\Gamma}{\sqrt{3}}\beta\cos 3\psi\right)^2\right].\end{aligned}$$

Denoting

$$\mu = \frac{\tau}{\Gamma} = \sqrt{\frac{I_2\left(\operatorname{Dev}\hat{P}\right)}{I_2\left(\operatorname{Dev}\hat{Q}\right)}}, \tag{A.13.9}$$

we obtain

$$\mu^2 = \left(\alpha - \frac{\Gamma}{\sqrt{3}}\beta\sin 3\psi\right)^2 + \left(\frac{\Gamma}{\sqrt{3}}\beta\cos 3\psi\right)^2.$$

One can satisfy this relationship by assuming

$$\mu\cos\omega = \alpha - \frac{\Gamma}{\sqrt{3}}\beta\sin 3\psi, \quad \mu\sin\omega = -\frac{\Gamma}{\sqrt{3}}\beta\cos 3\psi,$$

or

$$\alpha = \mu\frac{\cos(\omega + 3\psi)}{\cos 3\psi}, \quad \beta = -\frac{\sqrt{3}}{\Gamma}\mu\frac{\sin\omega}{\cos 3\psi}. \tag{A.13.10}$$

Formula (A.13.7) now takes the form

$$\operatorname{Dev}\hat{P} = \frac{\mu}{\cos 3\psi}\left\{\cos(\omega + 3\psi)\operatorname{Dev}\hat{Q} - \frac{\sqrt{3}}{\Gamma}\sin\omega\left[\left(\operatorname{Dev}\hat{Q}\right)^2 - \frac{2}{3}\hat{E}\Gamma^2\right]\right\}, \tag{A.13.11}$$

where the auxiliary angle ω can be expressed in terms of the principal value of $\operatorname{Dev}\hat{P}$. Following eq. (A.11.16) we present them in the form

$$\left.\begin{array}{ll} \nu_1 = \dfrac{2\tau}{\sqrt{3}}\sin\chi, & \nu_2 = \dfrac{2\tau}{\sqrt{3}}\sin\left(\chi + \dfrac{2\pi}{3}\right), \\ \nu_3 = \dfrac{2\tau}{\sqrt{3}}\sin\left(\chi + \dfrac{4\pi}{3}\right) & \left(|\chi| < \dfrac{\pi}{6}\right) \end{array}\right\} \tag{A.13.12}$$

Using eq. (A.3.11) we can write the equality relating the principal values of tensor $\operatorname{Dev}\hat{P}$ and $\operatorname{Dev}\hat{Q}$ as follows

$$\nu = \frac{\mu}{\cos 3\psi}\left[\cos(\omega + 3\psi)\varkappa - \frac{\sqrt{3}}{\Gamma}\sin\omega\left(\varkappa^2 - \frac{2}{3}\Gamma^2\right)\right].$$

Replacing ν and $\varkappa$ by means of eqs. (A.13.12), (A.11.16) and taking into account eq. (A.13.9) we obtain

$$\begin{aligned}\sin\chi &= \frac{1}{\cos 3\psi}\left[\cos(\omega + 3\psi)\sin\psi - \sin\omega\left(2\sin^2\psi - 1\right)\right] \\ &= \cos\omega\sin\psi + \frac{\sin\omega}{\cos 3\psi}\left(\cos 2\psi - \sin 3\psi\sin\psi\right) = \cos\omega\sin\psi + \sin\omega\cos\psi.\end{aligned}$$

Hence the value of ω is given by the equality

$$\sin\chi = \sin(\omega + \psi), \quad \omega = \chi - \psi. \tag{A.13.13}$$

For $\omega = 0, \chi = \psi$ the ratio of the principal values of deviators of tensors $\hat{P}$ and $\hat{Q}$ are equal to each other

$$\frac{\nu_1}{\varkappa_1} = \frac{\nu_2}{\varkappa_2} = \frac{\nu_3}{\varkappa_3} = \frac{\tau}{\Gamma} = \mu \tag{A.13.14}$$

and such tensors are called similar. This gives grounds to refer to ω as the angle of similarity of the deviators. The coaxial tensors are similar at $\omega = 0$ and the degree of their "dissimilarity" is determined by the value of ω.

Hence formula (A.13.6) relating two coaxial tensors is reduced to the form

$$\hat{P} = kI_1\left(\hat{Q}\right)\hat{E} + \frac{\mu}{\cos 3\psi}\left\{\cos(\omega + 3\psi)\operatorname{Dev}\hat{Q} - \frac{\sqrt{3}}{\Gamma}\sin\omega\left[\left(\operatorname{Dev}\hat{Q}\right)^2 - \frac{2}{3}\hat{E}\Gamma^2\right]\right\}. \tag{A.13.15}$$

Three values are seen when prescribing tensor $\hat{P}$, namely moduli k and μ given by formulae (A.13.5) and (A.13.9) and the angle of similarity of deviators ω. They should be considered as being functions of three invariant characteristics of tensor $\hat{Q}$, which are the first invariant $I_1\left(\hat{Q}\right)$ as well as Γ and ψ obtained in terms of the second and third invariants of $\operatorname{Dev}\hat{Q}$ by means of eqs. (A.11.14) and (A.11.15).

It is evident that under permutation of $\hat{P}$ and $\hat{Q}$ it is necessary to replace Γ and ψ by τ and χ respectively and k and μ by

$$k_1 = \frac{1}{3}\frac{I_1\left(\hat{Q}\right)}{I_1\left(\hat{P}\right)} = \frac{1}{9k}, \quad \mu_1 = \sqrt{\frac{I_2\left(\operatorname{Dev}\hat{Q}\right)}{I_2\left(\operatorname{Dev}\hat{P}\right)}} = \frac{1}{\mu} \tag{A.13.16}$$

respectively. This solves the problem of inversion of relation (A.13.15)

$$\hat{Q} = \frac{1}{9k}I_1\left(\hat{P}\right)\hat{E} + \frac{1}{\mu\cos 3\chi}\left\{\cos(3\chi - \omega)\operatorname{Dev}\hat{P} + \frac{\sqrt{3}}{\tau}\sin\omega\left[\left(\operatorname{Dev}\hat{P}\right)^2 - \frac{2}{3}\hat{E}\tau^2\right]\right\}.$$

V.V. Novozhilov suggested to reduce the initial relationship (A.12.4) between two coaxial tensors to the form (A.13.15) which allows expressing the initial coefficients A, B, C via the values which can be interpreted in terms of the mechanics of solids. The solution of the inverse problem then becomes transparently simple. Of course, one can suggest a direct solution, to this end it is sufficient to express tensor $\hat{Q}$ in terms of tensor $\hat{P}$

$$\hat{Q} = A_1 \hat{E} + B_1 \hat{P} + C_1 \hat{P}^2, \tag{A.13.17}$$

substitute expression (A.12.4) for $\hat{P}$ and replace $\hat{Q}^3, \hat{Q}^4$ in terms of $\hat{E}, \hat{Q}, \hat{Q}^2$ with the help of the Cayley-Hamilton theorem. We arrive at three linear equations enabling expression of the unknowns A_1, B_1, C_1 via A, B, C and the principal invariants $I_k\left(\hat{Q}\right)$. The relationships turn out to be cumbersome and they can be simplified if one extracts the spherical part from $\hat{P}$, presents $\operatorname{Dev}\hat{P}$ in the form of eq. (A.13.7) and seeks the solution of the inverse problem in the following form

$$\operatorname{Dev}\hat{Q} = \alpha_1 \operatorname{Dev}\hat{P} + \beta_1 \left[\left(\operatorname{Dev}\hat{P}\right)^2 + \frac{2}{3}\hat{E} I_2\left(\operatorname{Dev}\hat{P}\right)\right]. \tag{A.13.18}$$

Here by eqs. (A.13.1) and (A.13.5)

$$\left.\begin{aligned} I_1\left(\hat{Q}\right) &= 3A_1 + B_1 I_1\left(\hat{P}\right) + C_1\left[\frac{1}{3} I_1^2\left(\hat{P}\right) - 2I_2\left(\operatorname{Dev}\hat{P}\right)\right], \\ \alpha_1 &= B_1 + \frac{2}{3} C_1 I_1\left(\hat{P}\right), \quad \beta_1 = C_1. \end{aligned}\right\} \tag{A.13.19}$$

In order to obtain α and β we substitute expression (A.13.7) into the right hand side of eq. (A.13.18). Using the Cayley-Hamilton theorem yields the following result

$$\begin{aligned} \operatorname{Dev}\hat{Q} = {}& \alpha_1 \left\{\alpha \operatorname{Dev}\hat{Q} + \beta\left[\left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{2}{3}\hat{E} I_2\left(\operatorname{Dev}\hat{Q}\right)\right]\right\} + \\ & \beta_1 \left\{\alpha^2\left[\left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{2}{3}\hat{E} I_2\left(\operatorname{Dev}\hat{Q}\right)\right] - \frac{2}{3}\alpha\beta \operatorname{Dev}\hat{Q} I_2\left(\operatorname{Dev}\hat{Q}\right) + \right. \\ & \left. \beta^2\left[\operatorname{Dev}\hat{Q} I_3\left(\operatorname{Dev}\hat{Q}\right) + \frac{1}{3} I_2\left(\operatorname{Dev}\hat{Q}\right)\left(\operatorname{Dev}\hat{Q}\right)^2 + \frac{2}{9}\hat{E} I_2^2\left(\operatorname{Dev}\hat{Q}\right)\right]\right\} + \\ & \beta_1 \hat{E}\left[\frac{2}{3} I_2\left(\operatorname{Dev}\hat{P}\right) - \frac{2}{3}\alpha^2 I_2\left(\operatorname{Dev}\hat{Q}\right) + 2\alpha\beta I_3\left(\operatorname{Dev}\hat{Q}\right) + \frac{2}{9}\beta^2 I_2^2\left(\operatorname{Dev}\hat{Q}\right)\right] \end{aligned}$$

Noticing that

$$I_1\left(\operatorname{Dev}\hat{Q}\right) = 0, \quad I_1\left[\left(\operatorname{Dev}\hat{Q}\right)^2\right] = -2I_2\left(\operatorname{Dev}\hat{Q}\right)$$

and using the condition of vanishing $I_1\left(\operatorname{Dev}\hat{Q}\right)$ we arrive at the relationship

$$I_2\left(\operatorname{Dev}\hat{P}\right) = \alpha^2 I_2\left(\operatorname{Dev}\hat{Q}\right) - 3\alpha\beta I_3\left(\operatorname{Dev}\hat{Q}\right) - \frac{1}{3}\beta^2 I_2^2\left(\operatorname{Dev}\hat{Q}\right), \tag{A.13.20}$$

presenting another form of the relationship between τ and Γ. For determining α_1 and β_1 we obtain two equations

$$\alpha_1\alpha + \beta_1 \left[-\frac{2}{3}\alpha\beta I_2\left(\operatorname{Dev}\hat{Q}\right) + \beta^2 I_3\left(\operatorname{Dev}\hat{Q}\right)\right] = 1,$$
$$\alpha_1\beta + \beta_1\left[\alpha^2 + \frac{1}{3}\beta^2 I_2\left(\operatorname{Dev}\hat{Q}\right)\right] = 0,$$

from which we have

$$\left.\begin{array}{l} \alpha_1 = \dfrac{1}{H}\left[\alpha^2 + \dfrac{1}{3}\beta^2 I_2\left(\operatorname{Dev}\hat{Q}\right)\right], \quad \beta_1 = -\dfrac{\beta}{H}, \\ H = \alpha^3 + I_2\left(\operatorname{Dev}\hat{Q}\right)\alpha\beta^2 - I_3\left(\operatorname{Dev}\hat{Q}\right)\beta^2. \end{array}\right\} \tag{A.13.21}$$

A.14 Linear relationship between tensors

In this case the relations become very simple. We have

$$\hat{P} = A\hat{E} + B\hat{Q}, \tag{A.14.1}$$

so that

$$I_1\left(\hat{P}\right) = 3A + BI_1\left(\hat{Q}\right), \quad \operatorname{Dev}\hat{P} = B\operatorname{Dev}\hat{Q} \tag{A.14.2}$$

and

$$I_2\left(\operatorname{Dev}\hat{P}\right) = B^2 I_2\left(\operatorname{Dev}\hat{Q}\right). \tag{A.14.3}$$

We obtain

$$\hat{P} = \frac{1}{3}I_1\left(\hat{Q}\right)\frac{I_1\left(\hat{P}\right)}{I_1\left(\hat{Q}\right)}\hat{E} + \sqrt{\frac{I_2\left(\operatorname{Dev}\hat{P}\right)}{I_2\left(\operatorname{Dev}\hat{Q}\right)}}\operatorname{Dev}\hat{Q}$$

or under denotation (A.13.5) and (A.13.9)

$$\hat{P} = kI_1\left(\hat{Q}\right)\hat{E} + \mu\operatorname{Dev}\hat{Q}. \tag{A.14.4}$$

Referring to eq. (A.11.6) it is straightforward to obtain the relationship

$$I_2\left(\hat{P}\right) = 3A^2 + 2ABI_1\left(\hat{Q}\right) + B^2 I_2\left(\hat{Q}\right). \tag{A.14.5}$$

Appendix B
Main operations of tensor analysis

B.1 Nabla-operator

Given a scalar field $\varphi(x_1, x_2, x_3)$, one can define a vector $\operatorname{grad}\varphi$, referred to as the gradient, whose projections on the axes of an orthogonal Cartesian coordinate system are equal to the partial derivatives of scalar φ with respect to x_s

$$\operatorname{grad}\varphi = \frac{\partial\varphi}{\partial x_s}\mathbf{i}_s. \tag{B.1.1}$$

The scalar product of $\operatorname{grad}\varphi$ and $d\mathbf{r}$ describing the mutual position of two infinitely close points M and M' with the corresponding position radii is determined by the following scalar

$$\operatorname{grad}\varphi \cdot d\mathbf{r} = \frac{\partial\varphi}{\partial x_s}\mathbf{i}_s \cdot \mathbf{i}_k dx_k = \frac{\partial\varphi}{\partial x_s}dx_s = d\varphi, \tag{B.1.2}$$

which proves that $\operatorname{grad}\varphi$ is a vector, see Section A.1.

The operation of calculating gradient can be written with the help of the symbolic vector which is the Hamilton nabla-operator

$$\nabla = \mathbf{i}_s\frac{\partial}{\partial x_s}, \tag{B.1.3}$$

so that

$$\operatorname{grad}\varphi = \nabla\varphi.$$

The projections of the nabla-operator which are the differentiation operators $\partial/\partial x_s$ are subjected to the law of transformation of vector projections. Indeed, according to formulae (A.1.6) in the case of rotation of the coordinate system we have

$$\frac{\partial \varphi}{\partial x_s} = \frac{\partial \varphi}{\partial x'_k}\frac{\partial x'_k}{\partial x_s} = \alpha_{ks}\frac{\partial \varphi}{\partial x'_k}, \quad \frac{\partial}{\partial x_s} = \alpha_{ks}\frac{\partial}{\partial x'_k}.$$

The well-known rule of differentiation of the product also holds for the nabla-operator

$$\nabla\varphi\psi = \psi\nabla\varphi + \varphi\nabla\psi. \tag{B.1.4}$$

B.2 Differential operations on a vector field

It is known (see Section A.6) that the operations on two vectors reduce to the scalar invariant $\mathbf{a}\cdot\mathbf{b}$, vector $\mathbf{a}\times\mathbf{b}$ and dyadic $\mathbf{ab}$. Hence the simplest differential operation in the vector field is the scalar product of the nabla-operator and a vector

$$\nabla\cdot\mathbf{a} = \mathbf{i}_s\frac{\partial}{\partial x_s}\cdot a_k\mathbf{i}_k = \frac{\partial a_s}{\partial x_s} = \operatorname{div}\mathbf{a}. \tag{B.2.1}$$

This scalar is termed the divergence of the vector.

The next step is forming the vector

$$\nabla\times\mathbf{a} = \mathbf{i}_s\frac{\partial}{\partial x_s}\times a_k\mathbf{i}_k = e_{rsk}\frac{\partial a_k}{\partial x_s}\mathbf{i}_r = \operatorname{rot}\mathbf{a}, \tag{B.2.2}$$

referred to as the rotor (or curl) of the vector.

Finally, forming a dyadic leads to the tensor of second rank

$$\nabla\mathbf{a} = \mathbf{i}_s\frac{\partial}{\partial x_s}\mathbf{i}_k a_k = \mathbf{i}_s\mathbf{i}_k\frac{\partial a_k}{\partial x_s} = \operatorname{grad}\mathbf{a} \tag{B.2.3}$$

which is the gradient of the vector. The transposed tensor

$$(\nabla\mathbf{a})^* = \mathbf{i}_k\mathbf{i}_s\frac{\partial a_k}{\partial x_s} \tag{B.2.4}$$

has the following matrix of components

$$\begin{Vmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{Vmatrix}. \tag{B.2.5}$$

Postmultiplying this tensor by vector $d\mathbf{r}$ leads to vector $d\mathbf{a}$

$$(\nabla \mathbf{a})^* \cdot d\mathbf{r} = \mathbf{i}_k \mathbf{i}_s \frac{\partial a_k}{\partial x_s} \cdot \mathbf{i}_q dx_q = \mathbf{i}_k \frac{\partial a_k}{\partial x_s} dx_s = \mathbf{i}_k da_k = d\mathbf{a}.$$

Based upon this equality it is natural to refer to tensor $(\nabla \mathbf{a})^*$ as the derivative of vector $\mathbf{a}$ with respect to the position vector $d\mathbf{r}$ and to adopt the notation

$$(\nabla \mathbf{a})^* = \frac{d\mathbf{a}}{d\mathbf{r}}. \tag{B.2.6}$$

Extracting the symmetric part of tensor $(\nabla \mathbf{a})^*$ we obtain the tensor

$$\frac{1}{2}\left[(\nabla \mathbf{a})^* + \nabla \mathbf{a}\right] = \operatorname{def} \mathbf{a} = \frac{1}{2} \mathbf{i}_k \mathbf{i}_s \left(\frac{\partial a_k}{\partial x_s} + \frac{\partial a_s}{\partial x_k} \right), \tag{B.2.7}$$

called the deformation of vector $\mathbf{a}$. By eq. (A.4.9) vector $\boldsymbol{\omega}$ accompanying tensor $(\nabla \mathbf{a})^*$ is given by the equality

$$\boldsymbol{\omega} = \frac{1}{2} e_{rst} \frac{\partial a_t}{\partial x_s} \mathbf{i}_r = \frac{1}{2} \mathbf{i}_s \times \mathbf{i}_t \frac{\partial a_t}{\partial x_s} = \frac{1}{2} \mathbf{i}_s \frac{\partial}{\partial x_s} \times \mathbf{i}_t a_t = \frac{1}{2} \nabla \times \mathbf{a} = \frac{1}{2} \operatorname{rot} \mathbf{a}. \tag{B.2.8}$$

Denoting the skew-symmetric part of $(\nabla \mathbf{a})^*$ by $\hat{\Omega}$ we have

$$(\nabla \mathbf{a})^* = \operatorname{def} \mathbf{a} + \hat{\Omega}, \quad \nabla \mathbf{a} = \operatorname{def} \mathbf{a} - \hat{\Omega}, \quad (\nabla \mathbf{a})^* = \nabla \mathbf{a} + 2\hat{\Omega}, \tag{B.2.9}$$

where

$$\hat{\Omega} = \frac{1}{2} \mathbf{i}_k \mathbf{i}_s \left(\frac{\partial a_k}{\partial x_s} - \frac{\partial a_s}{\partial x_k} \right), \quad \Omega_{ks} = e_{rsk} \omega_r. \tag{B.2.10}$$

Then we obtain

$$d\mathbf{a} = (\nabla \mathbf{a})^* \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla \mathbf{a} = \operatorname{def} \mathbf{a} \cdot d\mathbf{r} + \hat{\Omega} \cdot d\mathbf{r} = \operatorname{def} \mathbf{a} \cdot d\mathbf{r} + \boldsymbol{\omega} \times d\mathbf{r}. \tag{B.2.11}$$

Application of the nabla-operator to the composition of two vectors is demonstrated in the following examples:

1. *Gradient of the scalar product*

$$\begin{aligned}\nabla \mathbf{a} \cdot \mathbf{b} &= \mathbf{i}_s \frac{\partial}{\partial x_s} a_k b_k = \mathbf{i}_s \frac{\partial a_k}{\partial x_s} b_k + a_k \mathbf{i}_s \frac{\partial b_k}{\partial x_s} = \mathbf{b} \cdot \mathbf{i}_k \mathbf{i}_s \frac{\partial a_k}{\partial x_s} + \mathbf{a} \cdot \mathbf{i}_k \mathbf{i}_s \frac{\partial b_k}{\partial x_s} \\ &= \mathbf{b} \cdot (\nabla \mathbf{a})^* + \mathbf{a} \cdot (\nabla \mathbf{b})^* = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a}. \end{aligned} \tag{B.2.12}$$

This rule of differentiation is quite analogous to that in eq. (B.1.4). It can be written in another form by introducing the vector called the derivative of $\mathbf{b}$ with respect to direction $\mathbf{a}$

$$\mathbf{a} \cdot \nabla \mathbf{b} = \mathbf{a} \cdot \left[(\nabla \mathbf{b})^* - 2\Omega\right] = \mathbf{a} \cdot (\nabla \mathbf{b})^* + 2\boldsymbol{\omega} \times \mathbf{a} = \mathbf{a} \cdot (\nabla \mathbf{b})^* - \mathbf{a} \times \operatorname{rot} \mathbf{b},$$

so that

$$\mathbf{a}\cdot(\nabla\mathbf{b})^* = \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{a}\times\operatorname{rot}\mathbf{b},$$
$$\operatorname{grad}\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{a} + \mathbf{a}\times\operatorname{rot}\mathbf{b} + \times\mathbf{b}\operatorname{rot}\mathbf{a}. \tag{B.2.13}$$

2. *Divergence of the vector product*

$$\operatorname{div}\mathbf{a}\times\mathbf{b} = \nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{i}_s\cdot\left(\mathbf{i}_q\frac{\partial a_q}{\partial x_s}\times\mathbf{b}\right) + \mathbf{i}_s\cdot\left(\mathbf{a}\times\mathbf{i}_q\frac{\partial b_q}{\partial x_s}\right)$$
$$= \left(\mathbf{i}_s\times\mathbf{i}_q\frac{\partial a_q}{\partial x_s}\right)\cdot\mathbf{b} - \mathbf{a}\cdot\left(\mathbf{i}_s\times\mathbf{i}_q\frac{\partial b_q}{\partial x_s}\right) = \mathbf{b}\cdot\operatorname{rot}\mathbf{a} - \mathbf{a}\operatorname{rot}\mathbf{b}. \tag{B.2.14}$$

3. *Rotor of the vector product*

$$\operatorname{rot}\mathbf{a}\times\mathbf{b} = \nabla\times(\mathbf{a}\times\mathbf{b}) = [\mathbf{i}_s\times(\mathbf{i}_q\times\mathbf{i}_r)]\frac{\partial}{\partial x_s}a_q b_r$$
$$= (\mathbf{i}_q\delta_{rs} - \mathbf{i}_r\delta_{sq})\left(\frac{\partial a_q}{\partial x_s}b_r + a_q\frac{\partial b_r}{\partial x_s}\right),$$

which yields

$$\operatorname{rot}\mathbf{a}\times\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{a} + \mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{a}\cdot\nabla\mathbf{b} + \mathbf{b}\operatorname{div}\mathbf{a}. \tag{B.2.15}$$

B.3 Differential operations on tensors

The content of Section B.2 is generalised to the tensor fields of any rank. The rank of tensor decreases by one under premultiplying it by the nabla-operator, i.e. under constructing divergence of the tensor

$$\nabla\cdot{}^{(n)}\hat{Q} = \mathbf{i}_s\frac{\partial}{\partial x_s}\cdot q_{s_1s_2\ldots s_n}\mathbf{i}_{s_1}\mathbf{i}_{s_2}\ldots\mathbf{i}_{s_n}$$
$$= \mathbf{i}_{s_2}\ldots\mathbf{i}_{s_n}\frac{\partial}{\partial x_s}q_{ss_2\ldots s_n} = \operatorname{div}^{(n)}\hat{Q}. \tag{B.3.1}$$

The vector product results in a tensor of the same rank called the rotor of the tensor

$$\nabla\times{}^{(n)}\hat{Q} = e_{ss_1t}\mathbf{i}_t\mathbf{i}_{s_2}\ldots\mathbf{i}_{s_n}\frac{\partial}{\partial x_s}q_{s_1\ldots s_n}\operatorname{rot}^{(n)}\hat{Q}. \tag{B.3.2}$$

Finally the gradient of a tensor is as follows

$$\nabla^{(n)}\hat{Q} = \mathbf{i}_{s_1}\mathbf{i}_{s_2}\ldots\mathbf{i}_{s_n}\frac{\partial}{\partial x_s}q_{s_1s_2\ldots s_n} = \operatorname{grad}^{(n)}\hat{Q}.$$

In particular, the divergence of a tensor of second rank is a vector

$$\operatorname{div}\hat{Q} = \nabla\cdot\hat{Q} = \mathbf{i}_s\frac{\partial}{\partial x_s}\cdot q_{rt}\mathbf{i}_r\mathbf{i}_t = \frac{\partial q_{st}}{\partial x_s}\mathbf{i}_t, \tag{B.3.3}$$

and for the skew-symmetric tensor we have

$$\operatorname{div}\hat{\Omega} = \mathbf{i}_t \frac{\partial \Omega_{rt}}{\partial x_r} = \mathbf{i}_t e_{str} \frac{\partial \omega_s}{\partial x_r} = \mathbf{i}_r \times \mathbf{i}_s \frac{\partial \omega_s}{\partial x_r} = \nabla \times \boldsymbol{\omega} = \operatorname{rot} \boldsymbol{\omega}. \tag{B.3.4}$$

Hence extracting the skew-symmetric part we obtain

$$\operatorname{div}\hat{Q} = \operatorname{div}\hat{S} + \operatorname{rot}\boldsymbol{\omega}. \tag{B.3.5}$$

The divergence of a dyadic is expressed in terms of the differential operations on its vectors

$$\operatorname{div}\mathbf{ab} = \mathbf{b}\operatorname{div}\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{b}. \tag{B.3.6}$$

The rotor of a tensor of second rank is again a tensor of second rank

$$\operatorname{rot}\hat{Q} = \nabla \times \hat{Q} = \mathbf{i}_s \times \mathbf{i}_r \mathbf{i}_t \frac{\partial q_{rt}}{\partial x_s} = e_{srq}\mathbf{i}_q\mathbf{i}_t \frac{\partial q_{rt}}{\partial x_s}. \tag{B.3.7}$$

The extended matrix of its components is given in table (A.5.9) in which a_s needs to be replaced by $\partial/\partial x_s$ Referring to eq. (A.4.9) we obtain the trace of this tensor

$$\operatorname{tr}\operatorname{rot}\hat{Q} = e_{srq}\delta_{qt}\frac{\partial q_{rt}}{\partial x_s} = e_{srt}\frac{\partial q_{rt}}{\partial x_s} = -2\frac{\partial \omega_s}{\partial x_s} = -2\operatorname{div}\boldsymbol{\omega}, \tag{B.3.8}$$

where $\boldsymbol{\omega}$ denotes the vector accompanying tensor $\hat{Q}$.

In the case of a skew-symmetric tensor we have

$$\begin{aligned}\operatorname{rot}\hat{\Omega} &= e_{srq}\mathbf{i}_q\mathbf{i}_t\frac{\partial \omega_{rt}}{\partial x_s} = e_{srq}e_{trm}\frac{\partial \omega_m}{\partial x_s}\mathbf{i}_q\mathbf{i}_t = \left(\delta_{st}\delta_{qm} - \delta_{sm}\delta_{qt}\right)\frac{\partial \omega_m}{\partial x_s}\mathbf{i}_q\mathbf{i}_t \\ &= \mathbf{i}_q\mathbf{i}_t\frac{\partial \omega_q}{\partial x_s} - \mathbf{i}_q\mathbf{i}_q\frac{\partial \omega_s}{\partial x_s} = (\nabla\boldsymbol{\omega})^* - \hat{E}\operatorname{div}\boldsymbol{\omega}. \end{aligned} \tag{B.3.9}$$

The formula for the divergence of vector $\hat{Q}\cdot\mathbf{a}$ is repeatedly used in the mechanics of solids

$$\operatorname{div}\hat{Q}\cdot\mathbf{a} = \mathbf{i}_s\frac{\partial}{\partial x_s}\cdot q_{rt}\mathbf{i}_r a_t = \frac{\partial q_{st}}{\partial x_s}a_t + q_{st}\frac{\partial a_t}{\partial x_s} = \mathbf{a}\cdot\operatorname{div}\hat{Q} + \hat{Q}\cdot\cdot(\nabla\mathbf{a})^*, \tag{B.3.10}$$

since, by eqs. (A.6.7) and (B.2.4),

$$q_{st}\frac{\partial a_t}{\partial x_s} = \hat{Q}\cdot\cdot(\nabla\mathbf{a})^* = \hat{Q}^*\cdot\cdot\nabla\mathbf{a}. \tag{B.3.11}$$

B.4 Double differentiation

Given vector $\nabla\varphi$, one can introduce the symmetric tensor of second rank

$$\nabla\nabla\varphi = \mathbf{i}_s\mathbf{i}_k \frac{\partial^2\varphi}{\partial x_s \partial x_k} = \mathbf{i}_k\mathbf{i}_s \frac{\partial^2\varphi}{\partial x_s \partial x_k}. \tag{B.4.1}$$

Its trace $\nabla \cdot \nabla\varphi = \nabla^2\varphi$ is the Laplace operator of scalar φ

$$\nabla^2\varphi = \mathbf{i}_k \cdot \mathbf{i}_s \frac{\partial^2\varphi}{\partial x_s \partial x_k} = \frac{\partial^2\varphi}{\partial x_s \partial x_s} = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}. \tag{B.4.2}$$

The vector accompanying the symmetric tensor is equal to zero. Applying this property to tensor $\nabla\nabla\varphi$ results in the well-known property of the gradient of a scalar

$$\nabla \times \nabla\varphi = \operatorname{rot}\operatorname{grad}\varphi = 0.$$

A tensor of third rank $\nabla\nabla\mathbf{a}$ admits the following contractions reducing the rank by two:

i) vectorial Laplace operator

$$\nabla \cdot \nabla\mathbf{a} = \nabla^2\mathbf{a} = \operatorname{div}\operatorname{grad}\mathbf{a} = \mathbf{i}_k \frac{\partial^2 a_k}{\partial x_s \partial x_s}; \tag{B.4.3}$$

ii) vectorial gradient of the divergence of $\mathbf{a}$

$$\nabla\nabla \cdot \mathbf{a} = \operatorname{grad}\operatorname{div}\mathbf{a} = \mathbf{i}_s \frac{\partial^2 a_k}{\partial x_s \partial x_k}; \tag{B.4.4}$$

iii) vectorial rotor of the rotor of $\mathbf{a}$

$$\nabla \times (\nabla \times \mathbf{a}) = \mathbf{i}_s \frac{\partial}{\partial x_s} \times \left(\mathbf{i}_t \frac{\partial}{\partial x_t} \times \mathbf{i}_q a_q\right) = (\mathbf{i}_t \delta_{sq} - \mathbf{i}_q \delta_{st}) \frac{\partial^2 a_q}{\partial x_s \partial x_t},$$

so that

$$\operatorname{rot}\operatorname{rot}\mathbf{a} = \operatorname{grad}\operatorname{div}\mathbf{a} - \nabla^2\mathbf{a}. \tag{B.4.5}$$

The tensors of second rank are obtained by means of the rotor of the gradient of vector $\mathbf{a}$

$$\begin{aligned} \operatorname{rot}\operatorname{grad}\mathbf{a} = \nabla \times \nabla\mathbf{a} &= \frac{\partial}{\partial x_s}\mathbf{i}_s \times \mathbf{i}_t\mathbf{i}_q \frac{\partial a_q}{\partial x_t} = e_{str}\mathbf{i}_r\mathbf{i}_q \frac{\partial^2 a_q}{\partial x_s \partial x_t} \\ &= e_{tsr}\mathbf{i}_r\mathbf{i}_q \frac{\partial^2 a_q}{\partial x_t \partial x_s} = -e_{str}\mathbf{i}_r\mathbf{i}_q \frac{\partial^2 a_q}{\partial x_s \partial x_t} = 0 \end{aligned} \tag{B.4.6}$$

and the gradient of the rotor of $\mathbf{a}$

$$\operatorname{grad}\operatorname{rot}\mathbf{a} = \nabla(\nabla \times \mathbf{a}) = \mathbf{i}_s\mathbf{i}_r \times \mathbf{i}_t \frac{\partial^2 a_t}{\partial x_s \partial x_r} = e_{rtq}\mathbf{i}_s\mathbf{i}_q \frac{\partial^2 a_t}{\partial x_s \partial x_r}. \tag{B.4.7}$$

The trace of this tensor is equal to zero

$$\nabla \cdot (\nabla \times \mathbf{a}) = \operatorname{div}\operatorname{rot}\mathbf{a} = e_{rts}\frac{\partial^2 a_t}{\partial x_r \partial x_s} = e_{str}\frac{\partial^2 a_t}{\partial x_s \partial x_r} = -e_{rts}\frac{\partial^2 a_t}{\partial x_r \partial x_s} = 0. \tag{B.4.8}$$

Let us proceed to the tensor of fourth rank $\nabla\nabla\hat{Q}$

$$\nabla\nabla\hat{Q} = \mathbf{i}_s\mathbf{i}_t\mathbf{i}_k\mathbf{i}_m\frac{\partial^2 q_{km}}{\partial x_s \partial x_t}.$$

Among all possible contractions with respect to two indices we mention the following one

$$\operatorname{div}\operatorname{div}\hat{Q} = \nabla\cdot\nabla\cdot\hat{Q} = \delta_{kt}\delta_{sm}\frac{\partial^2 q_{km}}{\partial x_s \partial x_t} = \frac{\partial^2 q_{ts}}{\partial x_s \partial x_t}. \tag{B.4.9}$$

The contractions with respect to one index yields tensors of second rank

$$\nabla\cdot\nabla\hat{Q} = \operatorname{div}\operatorname{grad}\hat{Q} = \nabla^2\hat{Q},\ \ \nabla\nabla\cdot\hat{Q} = \operatorname{grad}\operatorname{div}\hat{Q},\ \ \nabla\nabla\operatorname{tr}\hat{Q}. \tag{B.4.10}$$

One can also construct tensors of third rank

$$\nabla\times\left(\nabla\hat{Q}\right) = \operatorname{rot}\operatorname{grad}\hat{Q}, \qquad \nabla\left(\nabla\times\hat{Q}\right) = \operatorname{grad}\operatorname{rot}\hat{Q} \tag{B.4.11}$$

and the tensor of second rank

$$\nabla\times\left(\nabla\times\hat{Q}\right) = \operatorname{rot}\operatorname{rot}\hat{Q} = \nabla\nabla\cdot\hat{Q} - \nabla^2\hat{Q}. \tag{B.4.12}$$

Of fundamental importance in the mechanics of solids is the tensor of second rank which is the rotor of the transposed rotor of the tensor of second rank

$$\hat{M} = \operatorname{rot}\left(\operatorname{rot}\hat{Q}\right)^* = \nabla\times\left(\nabla\times\hat{Q}\right)^* = e_{tmq}e_{srk}\mathbf{i}_k\mathbf{i}_q\frac{\partial^2 q_{mr}}{\partial x_s \partial x_t}. \tag{B.4.13}$$

This tensor is symmetric if $\hat{Q}$ is a symmetric tensor. Indeed,

$$m_{kq} = e_{tmq}e_{srk}\frac{\partial^2 q_{mr}}{\partial x_s \partial x_t} = e_{srq}e_{tmk}\frac{\partial^2 q_{rm}}{\partial x_t \partial x_s} = e_{tmk}e_{srq}\frac{\partial^2 q_{mr}}{\partial x_s \partial x_t} = m_{qk}.$$

This tensor is denoted as $\operatorname{inc}\hat{Q}$ (denoting incompatibility, see Subsection 2.2.1)

$$\operatorname{inc}\hat{Q} = \operatorname{rot}\left(\operatorname{rot}\hat{Q}\right)^* = \nabla\times\left(\nabla\times\hat{Q}\right)^*. \tag{B.4.14}$$

The components of $\operatorname{inc}\hat{Q}$ for the symmetric tensor $\hat{Q}$ are given by

$$\left.\begin{aligned}
m_{11} &= \frac{\partial^2 q_{33}}{\partial x_2^2} + \frac{\partial^2 q_{22}}{\partial x_3^2} - 2\frac{\partial^2 q_{23}}{\partial x_2 \partial x_3},\\
m_{22} &= \frac{\partial^2 q_{11}}{\partial x_3^2} + \frac{\partial^2 q_{33}}{\partial x_1^2} - 2\frac{\partial^2 q_{31}}{\partial x_3 \partial x_1},\\
m_{33} &= \frac{\partial^2 q_{22}}{\partial x_1^2} + \frac{\partial^2 q_{11}}{\partial x_2^2} - 2\frac{\partial^2 q_{12}}{\partial x_1 \partial x_2},\\
m_{12} = m_{21} &= \frac{\partial}{\partial x_3}\left(\frac{\partial q_{23}}{\partial x_1} + \frac{\partial q_{31}}{\partial x_2} - \frac{\partial q_{12}}{\partial x_3}\right) - \frac{\partial^2 q_{33}}{\partial x_1 \partial x_2},\\
m_{23} = m_{32} &= \frac{\partial}{\partial x_1}\left(\frac{\partial q_{31}}{\partial x_2} + \frac{\partial q_{12}}{\partial x_3} - \frac{\partial q_{23}}{\partial x_1}\right) - \frac{\partial^2 q_{11}}{\partial x_2 \partial x_3},\\
m_{31} = m_{13} &= \frac{\partial}{\partial x_2}\left(\frac{\partial q_{12}}{\partial x_3} + \frac{\partial q_{23}}{\partial x_1} - \frac{\partial q_{12}}{\partial x_3}\right) - \frac{\partial^2 q_{22}}{\partial x_3 \partial x_1},
\end{aligned}\right\} \tag{B.4.15}$$

Under contraction of the pair of indices tensors of third rank (B.4.11) produce the following vectors

$$\left.\begin{aligned}
&\operatorname{rot}\operatorname{grad}\operatorname{tr}\hat{Q} = 0,\\
&\operatorname{rot}\operatorname{div}\hat{Q} = \nabla\times\left(\nabla\cdot\hat{Q}\right) = e_{sqt}\mathbf{i}_t\frac{\partial^2 q_{rq}}{\partial x_s \partial x_r},\\
&\operatorname{div}\operatorname{rot}\hat{Q} = \nabla\cdot\left(\nabla\times\hat{Q}\right) = e_{srq}\mathbf{i}_t\frac{\partial^2 q_{rt}}{\partial x_s \partial x_q} = 0.
\end{aligned}\right\} \tag{B.4.16}$$

Some other differential operations on the products of vector and scalar are as follows

$$\nabla\psi\mathbf{a} = (\nabla\psi)\,\mathbf{a} + \psi\nabla\mathbf{a}, \tag{B.4.17}$$

$$\operatorname{div}\psi\mathbf{a} = \psi\operatorname{div}\mathbf{a} + \mathbf{a}\cdot\operatorname{div}\psi, \quad \operatorname{rot}\psi\mathbf{a} = \psi\operatorname{rot}\mathbf{a} + \operatorname{grad}\psi\times\mathbf{a}, \tag{B.4.18}$$

$$\nabla^2\psi\mathbf{a} = \psi\nabla^2\mathbf{a} + \mathbf{a}\nabla^2\psi + 2\nabla\psi\cdot\nabla\mathbf{a}. \tag{B.4.19}$$

The Laplace operator of the product of two scalars is given by the relationship

$$\nabla^2\varphi\psi = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\nabla\varphi\cdot\nabla\psi. \tag{B.4.20}$$

B.5 Transformation of a volume integral into a surface integral

The Gauss-Ostrogradsky formula

$$\iiint\limits_V \frac{\partial\varphi}{\partial x_s}d\tau = \iint\limits_O n_s\varphi do \tag{B.5.1}$$

is assumed to be known. Here $d\tau$ and do denote respectively the element of volume V and surface O bounding this volume and n_s is the projection of the unit vector of the outward normal $\mathbf{n}$ to this surface on axis x_s. Function φ is continuous together with its partial derivatives of the first order in the closed volume $V + O$. The simplest generalisation of formula (B.5.1) is the following equality

$$\iiint\limits_V \nabla \cdot \mathbf{a} d\tau = \iiint\limits_V \frac{\partial a_s}{\partial x_s} d\tau = \iint\limits_O n_s a_s do = \iint\limits_O \mathbf{n} \cdot \mathbf{a} do, \qquad \text{(B.5.2)}$$

relating the integral of the divergence over the closed volume with the vector flux through the surface bounding this volume.

The rule of replacing the nabla-operator in the volume integral by vector $\mathbf{n}$ in the surface integral can be generalised to more complex relationships since the Gauss-Ostrogradsky formula (B.5.1) is eventually used. For example

$$\iiint\limits_V \nabla \times \mathbf{a} d\tau = \iiint\limits_V \operatorname{rot} \mathbf{a} d\tau = \iint\limits_O \mathbf{n} \times \mathbf{a} do, \qquad \text{(B.5.3)}$$

since

$$\iiint\limits_V \nabla \times \mathbf{a} d\tau = \iiint\limits_V \mathbf{i}_s \times \mathbf{i}_k \frac{\partial a_k}{\partial x_s} d\tau = \iint\limits_O \mathbf{i}_s n_s \times \mathbf{i}_k a_k do = \iint\limits_O \mathbf{n} \times \mathbf{a} do.$$

Other examples which are straightforward to prove are

$$\iiint\limits_V \nabla \mathbf{a} d\tau = \iint\limits_O \mathbf{n}\mathbf{a} do, \quad \iiint\limits_V (\nabla \mathbf{a})^* d\tau = \iint\limits_O (\mathbf{n}\mathbf{a})^* do = \iint\limits_O \mathbf{a}\mathbf{n} do. \qquad \text{(B.5.4)}$$

Applying this to the tensor of second rank yields

$$\iiint\limits_V \nabla \cdot \hat{Q} d\tau = \iint\limits_O \mathbf{n} \cdot \hat{Q} do, \quad \iiint\limits_V \nabla \times \hat{Q} d\tau = \iint\limits_O \mathbf{n} \times \hat{Q} do. \qquad \text{(B.5.5)}$$

Another example is the transformation

$$\begin{aligned}
\iint\limits_O \mathbf{r} \times \left(\mathbf{n} \cdot \hat{Q}\right) do &= -\iint\limits_O \left(\mathbf{n} \cdot \hat{Q}\right) \times \mathbf{r} do \\
&= -\iiint\limits_V \left(\nabla \cdot \hat{Q}\right) \times \mathbf{r} d\tau - \iiint\limits_V \left(\mathbf{i}_s \cdot \hat{Q}\right) \times \frac{\partial \mathbf{r}}{\partial x_s} d\tau \\
&= \iiint\limits_V \mathbf{r} \times \nabla \cdot \hat{Q} d\tau + \iiint\limits_V \mathbf{i}_s \times \left(\mathbf{i}_s \cdot \hat{Q}\right) d\tau.
\end{aligned}$$

However

$$\mathbf{i}_s \times \left(\mathbf{i}_s \cdot \hat{Q}\right) = \mathbf{i}_s \times \mathbf{i}_r q_{sr} = e_{srt} q_{sr} \mathbf{i}_t = -2\boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ denotes the vector accompanying tensor $\hat{Q}$. Hence

$$\iint\limits_O \mathbf{r} \times \left(\mathbf{n} \cdot \hat{Q}\right) do = \iiint\limits_V \left(\mathbf{r} \times \operatorname{div} \hat{Q} - 2\boldsymbol{\omega}\right) d\tau. \tag{B.5.6}$$

B.6 Stokes's transformation

It is known that the linear integral (circulation) of the vector along arc C

$$\int\limits_{M_0}^{M} \mathbf{a} \cdot d\mathbf{r} \tag{B.6.1}$$

does not depend upon the choice of C and is determined only by the coordinates of the initial M_0 and final M points provided that

$$\operatorname{rot} \mathbf{a} = 0, \quad \mathbf{a} = \operatorname{grad} \varphi = \nabla \varphi. \tag{B.6.2}$$

Then

$$\int\limits_{M_0}^{M} \mathbf{a} \cdot d\mathbf{r} = \int\limits_{M_0}^{M} \nabla \varphi \cdot d\mathbf{r} = \int\limits_{M_0}^{M} d\varphi = \varphi_M - \varphi_{M_0}. \tag{B.6.3}$$

The analogous equality for the linear integral of the tensor

$$\int\limits_{M_0}^{M} \hat{Q} \cdot d\mathbf{r} = \mathbf{a}_M - \mathbf{a}_{M_0} \tag{B.6.4}$$

holds if

$$\hat{Q} = \frac{d\mathbf{a}}{d\mathbf{r}} = (\nabla \mathbf{a})^*, \quad \hat{Q}^* = \nabla \mathbf{a}, \quad \operatorname{rot} \hat{Q}^* = 0. \tag{B.6.5}$$

Indeed, in this case

$$\int\limits_{M_0}^{M} \hat{Q} \cdot d\mathbf{r} = \int\limits_{M_0}^{M} \frac{d\mathbf{a}}{d\mathbf{r}} \cdot d\mathbf{r} = \int\limits_{M_0}^{M} d\mathbf{a} = \mathbf{a}_M - \mathbf{a}_{M_0}.$$

By analogy one can prove that the condition for the integral

$$\int\limits_{M_0}^{M} d\mathbf{r} \cdot \hat{Q} = \int\limits_{M_0}^{M} \hat{Q}^* \cdot d\mathbf{r} \tag{B.6.6}$$

to be independent of the integration path is given by the equality

$$\operatorname{rot} \hat{Q} = 0, \quad \hat{Q} = \nabla \mathbf{a}. \tag{B.6.7}$$

According to Stokes's theorem

$$\oint \mathbf{a} \cdot d\mathbf{r} = \iint\limits_{S} \mathbf{n} \cdot \operatorname{rot} \mathbf{a} do,$$

i.e. the circulation of the vector along the closed contour is equal to the flux of the rotor of this vector through any arbitrary surface supported on this contour. The surface within this volume can be supported by a closed contour which is reduced to a point by means of a continuous transformation, the surface boundary not being intersected. Any closed contour in a simply-connected volume satisfies this condition. However in the double-connected volume, for instance in the torus or in the space between two coaxial cylinders there exist contours (let us denote them $K-$contours) which are reduced to each other by a continuous transformation and not reduced to a point. These are the axial line in the torus, a circle lying in the plane perpendicular to the cylinder axis and embracing the inner cylinder. A double-connected domain can be transformed into a simply-connected domain by means of a barrier.

Let the integrability condition (B.6.2) be satisfied. Since Stokes's theorem can not be applied to $K-$contour the circulation along this contour can differ from zero and in this case scalar φ is a multiple-valued function of the coordinates

$$\oint \mathbf{a} \cdot d\mathbf{r} = \oint d\varphi = \chi, \tag{B.6.8}$$

the cyclic constant χ being the same for all $K-$contours. The proof consists of considering the integral over the closed contour $N_1 M_1 M_2 N_2 M_2 M_1 N_3 N_1$, Fig. B.1, which is reduced to a point by a continuous transformation. Stokes's theorem is applicable and gives

$$\int\limits_{N_1}^{M_1} \mathbf{a} \cdot d\mathbf{r} + \int\limits_{M_1}^{M_2} \mathbf{a} \cdot d\mathbf{r} + \oint\limits_{K_2} \mathbf{a} \cdot d\mathbf{r} + \int\limits_{M_2}^{M1} \mathbf{a} \cdot d\mathbf{r} + \oint\limits_{K_1} \mathbf{a} \cdot d\mathbf{r} + \int\limits_{M_1}^{N_1} \mathbf{a} \cdot d\mathbf{r} = 0,$$

and since

$$\int\limits_{M_1}^{M_2} \mathbf{a} \cdot d\mathbf{r} + \int\limits_{M_2}^{M1} \mathbf{a} \cdot d\mathbf{r} = 0, \quad \int\limits_{M_1}^{N_1} \mathbf{a} \cdot d\mathbf{r} + \int\limits_{N_1}^{M_1} \mathbf{a} \cdot d\mathbf{r} = 0,$$

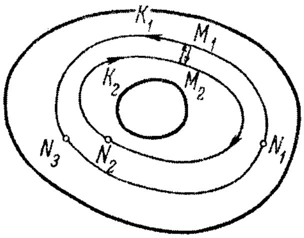

FIGURE B.1.

we have

$$\oint_{K_1} \mathbf{a} \cdot d\mathbf{r} + \oint_{K_2} \mathbf{a} \cdot d\mathbf{r} = 0, \quad \oint_{K_1} \mathbf{a} \cdot d\mathbf{r} = \oint_{K_2} \mathbf{a} \cdot d\mathbf{r},$$

which is required.

The result is valid for integrals of the type (B.6.4). If the integrability condition (B.6.5) holds we obtain

$$\oint_K \hat{Q} \cdot d\mathbf{r} = \oint_K d\mathbf{a} = \mathbf{c} \tag{B.6.9}$$

where $\mathbf{c}$ is a constant cyclic vector which is the same for all K–contours.

Appendix C
Orthogonal curvilinear coordinates

C.1 Definitions

Three numbers prescribing the position of a point in the space and denoted by q^1, q^2, q^3 are referred to as its curvilinear coordinates. The Cartesian coordinates are related to the curvilinear ones by three equalities

$$x_s = x_s\left(q^1, q^2, q^3\right) \tag{C.1.1}$$

or in the vector form

$$\mathbf{R} = \mathbf{i}_s x_s = \mathbf{R}\left(q^1, q^2, q^3\right), \tag{C.1.2}$$

where $\mathbf{R}$ denotes the position vector. In the domain of definition, the functions in eq. (C.1.1) are assumed to be continuous, single-valued and having continuous partial derivatives up to and including the third order. They should be uniquely resolved for q^1, q^2, q^3 which is equivalent to the requirement of nonvanishing Jacobian

$$J = \left|\frac{\partial x_s}{\partial q^k}\right|. \tag{C.1.3}$$

The coordinate numbering is assumed to be chosen such that the Jacobian is positive.

Transformation (C.1.1) determines three families of surface $q^r = q_0^r$ and the coordinate lines are the curves of intersection of the coordinate surfaces. Along the coordinate line denoted as $[q^s]$ the coordinate q^s varies. The coordinate lines of the same family do not intersect if condition (C.1.3) holds.

Well-known examples are the cylindrical and spherical coordinates. For cylindrical coordinates, $q^1 = r, q^2 = \varphi, q^3 = z$ are the radius, azimuthal angle and height respectively. Formulae (C.1.1) take the form

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{C.1.4}$$

The domain of definition is given by the inequalities

$$0 < r < \infty, \quad 0 \le \varphi \le 2\pi, \quad -\infty < z < \infty.$$

The coordinate surfaces are the circular cylinders $r = r_0$ with axis Ox_3, half-planes $\varphi = \varphi_0$ passing through this axis and planes $z = z_0$ perpendicular to this axis. The coordinate lines are straight lines $[z]$ parallel to axis Ox_3, radially directed half-lines $[r]$ and circles $[\varphi]$. The Jacobian $J = r$ vanishes on axis Ox_3 where the planes $\varphi = \varphi_0$ are intersecting. This axis is not included in the domain of definition of this coordinate system.

For spherical coordinates, $q^1 = R, q^2 = \vartheta, q^3 = \lambda$ are respectively the radius, the angle measured along the meridian from the north pole and the longitude. We have

$$x_1 = R\sin\vartheta\cos\lambda, \quad x_2 = R\sin\vartheta\sin\lambda, \quad x_3 = R\cos\vartheta. \tag{C.1.5}$$

The domain of the definition is given by the inequalities

$$0 < R < \infty, \quad 0 < \vartheta < \pi, \quad 0 \le \lambda \le 2\pi.$$

The coordinate surfaces are spheres $R = R_0$ with centre at the coordinate origin O, circular cones $\vartheta = \vartheta_0$ passing through axis Ox_3 and having the vertex at the origin O and half-planes $\lambda = \lambda_0$. The coordinate lines are the parallel circles $[\lambda]$, the half-lines $[R]$ from centre O and meridians $[\vartheta]$. The Jacobian $J = R^2\sin\vartheta$ vanishes at centre O and at the poles of the spheres.

C.2 Square of a linear element

We introduce into consideration the triple of vectors

$$\mathbf{R}_k = \frac{\partial \mathbf{R}}{\partial q^k}, \tag{C.2.1}$$

having the directions of the tangents to the coordinate lines $\left[q^k\right]$. In the vicinity of point $M\left(q^1, q^2, q^3\right)$ they define an infinitesimally small vector

$$d\mathbf{R} = \mathbf{R}_k dq^k. \tag{C.2.2}$$

The square of its length is the square of the linear element expressed in terms of the curvilinear coordinates and is given by the equality

$$ds^2 = d\mathbf{R} \cdot d\mathbf{R} = \mathbf{R}_s dq^s \cdot \mathbf{R}_k dq^k = g_{sk} dq^s dq^k. \tag{C.2.3}$$

The following six values

$$g_{sk} = g_{ks} = \mathbf{R}_s \cdot \mathbf{R}_k \tag{C.2.4}$$

determine the metric of the coordinate system under consideration. They are referred to as the covariant components of the metric tensor, see Section E.1.

The second derivatives of the position vector $\mathbf{R}$ are denoted by

$$\mathbf{R}_{st} = \frac{\partial^2 \mathbf{R}}{\partial q^s \partial q^t} = \mathbf{R}_{ts}. \tag{C.2.5}$$

In what follows we need the scalar products $\mathbf{R}_{st} \cdot \mathbf{R}_k$. They are expressed in terms of the derivatives of the covariant components of the metric tensor. We have

$$\frac{\partial g_{st}}{\partial q^k} = \frac{\partial}{\partial q^k} \mathbf{R}_s \cdot \mathbf{R}_t = \mathbf{R}_{sk} \cdot \mathbf{R}_t + \mathbf{R}_s \cdot \mathbf{R}_{tk}$$

and two similar equations obtained from the latter by the circular permutation of indices. We arrive at the relationship

$$\left.\begin{aligned} \frac{\partial g_{st}}{\partial q^k} &= \mathbf{R}_{sk} \cdot \mathbf{R}_t + \mathbf{R}_s \cdot \mathbf{R}_{tk}, \\ \frac{\partial g_{tk}}{\partial q^s} &= \mathbf{R}_{ts} \cdot \mathbf{R}_k + \mathbf{R}_t \cdot \mathbf{R}_{ks}, \\ \frac{\partial g_{ks}}{\partial q^t} &= \mathbf{R}_{kt} \cdot \mathbf{R}_s + \mathbf{R}_k \cdot \mathbf{R}_{st}. \end{aligned}\right\} \tag{C.2.6}$$

The sought expression for $\mathbf{R}_{st} \cdot \mathbf{R}_k$ is obtained by subtracting the first equation from the sum of the second and third

$$\mathbf{R}_{st} \cdot \mathbf{R}_k = \frac{1}{2}\left(\frac{\partial g_{sk}}{\partial q^t} + \frac{\partial g_{kt}}{\partial q^s} - \frac{\partial g_{st}}{\partial q^k}\right) = [st, k] = [ts, k]. \tag{C.2.7}$$

They are referred to as Christoffel's symbols of the first kind or Christoffel's square brackets.

C.3 Orthogonal curvilinear coordinate system, base vectors

For the orthogonal system of curvilinear coordinates the following equalities

$$\mathbf{R}_s \cdot \mathbf{R}_k = g_{sk} = \begin{cases} 0, & s \neq k, \\ H_s^2, & s = k \end{cases} \tag{C.3.1}$$

hold. Here H_s is referred to as Lame's coefficients. They are equal to the magnitudes of vectors $\mathbf{R}_s$

$$|\mathbf{R}_s| = H_s = \sqrt{\left(\frac{\partial x_1}{\partial q^s}\right)^2 + \left(\frac{\partial x_2}{\partial q^s}\right)^2 + \left(\frac{\partial x_3}{\partial q^s}\right)^2}. \tag{C.3.2}$$

The orthogonal trihedron of the unit vectors of the tangents to the coordinate lines $[q^s]$ in the direction of increasing q^s

$$\mathbf{e}_s = \frac{\mathbf{R}_s}{H_s}, \quad \mathbf{e}_s \cdot \mathbf{e}_k = \delta_{sk} \quad (\not\Sigma_s) \tag{C.3.3}$$

forms the vector basis in the considered system of orthogonal curvilinear coordinates, symbol $\not\Sigma_s$ implying no summation over s. Vectors $\mathbf{e}_s$ also have the directions of the normals to the coordinate surfaces q_0^s.

Vectors and tensors are given by their representations in vector basis $\mathbf{e}_s$

$$\mathbf{a} = a_s \mathbf{e}_s, \quad \hat{Q} = q_{sk} \mathbf{e}_s \mathbf{e}_k. \tag{C.3.4}$$

However, in contrast to base vectors $\mathbf{i}_k$ of the Cartesian orthogonal system vectors $\mathbf{e}_s$ do not conserve fixed directions. For instance, components a_s of a constant vector $\mathbf{a}$ are changing from point to point and, on the contrary, $a_s = \text{const}$ does not means that $\mathbf{a}$ is a constant vector.

A consequence of eq. (C.3.3) is the following formulae

$$\mathbf{e}_s = \mathbf{i}_k \frac{1}{H_s} \frac{\partial x_k}{\partial q^s}, \quad \mathbf{i}_k = \mathbf{e}_s \frac{1}{H_s} \frac{\partial x_k}{\partial q^s}, \quad \frac{\partial q^s}{\partial x_k} = \frac{1}{H_s^2} \frac{\partial x_k}{\partial q^s}, \tag{C.3.5}$$

where the last formula is proved in the following way

$$\frac{\partial q^s}{\partial q^t} = \delta_{st} = \frac{\partial q^s}{\partial x_r} \frac{\partial x_r}{\partial q^t} = \frac{1}{H_s^2} \frac{\partial x_r}{\partial q^s} \frac{\partial x_r}{\partial q^t} = \frac{1}{H_s^2} \mathbf{R}_s \cdot \mathbf{R}_t = \begin{cases} 0, & s \neq t, \\ 1, & s = t. \end{cases}$$

By eqs. (C.2.3) and (C.3.1) the square of the linear element in the orthogonal curvilinear coordinates is given by the expression

$$ds^2 = H_1^2 dq^{1^2} + H_2^2 dq^{2^2} + H_3^2 dq^{3^2}. \tag{C.3.6}$$

Carrying out the operations of vector and tensor analysis in curvilinear coordinates is completely related to the values of g_{sk} and in the case of the orthogonal curvilinear coordinates they rely on Lame's coefficients H_s. In order to calculate the latter one can avoid using formulae (C.1.1). One can consider the element $d_k s$ of the arc of the coordinate line $\left[q^k\right]$

$$d_k s = H_k dq^k \quad (\not\Sigma_k). \tag{C.3.7}$$

In the following, symbol $\not\Sigma_k$ is omitted when the index appears on both the left and right hand sides of the formula.

Similar to Section B.1 the nabla-operator ∇ is introduced with the help of the definition of the gradient of a scalar field

$$d\varphi = \frac{\partial \varphi}{\partial q^s} dq^s = d\mathbf{R} \cdot \nabla \varphi = \mathbf{R}_s dq^s \cdot \nabla \varphi = H_s \mathbf{e}_s \cdot \operatorname{grad} \varphi dq^s,$$

so that by eq. (C.3.4)

$$\mathbf{e}_s \cdot \operatorname{grad} \varphi = \frac{1}{H_s} \frac{\partial \varphi}{\partial q^s}, \quad \operatorname{grad} \varphi = \nabla \varphi = \frac{\mathbf{e}_s}{H_s} \frac{\partial \varphi}{\partial q^s} \tag{C.3.8}$$

and the nabla-operator is defined by the equality

$$\nabla = \frac{1}{H_s} \mathbf{e}_s \frac{\partial}{\partial q^s}. \tag{C.3.9}$$

The element of the volume in the orthogonal curvilinear coordinates is given by the evident relation

$$\begin{aligned} d\tau &= \mathbf{R}_1 dq^1 \cdot \left(\mathbf{R}_2 dq^2 \times \mathbf{R}_3 dq^3\right) = H_1 H_2 H_3 \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)\, dq^1 dq^2 dq^3 \\ &= H_1 H_2 H_3 dq^1 dq^2 dq^3 \end{aligned}$$

or

$$d\tau = \sqrt{g} dq^1 dq^2 dq^3, \quad \sqrt{g} = H_1 H_2 H_3 = \mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3) = J, \tag{C.3.10}$$

where J is the Jacobian (C.1.3).

C.4 Differentiation of base vectors

Performing the operations of vector and tensor analysis in curvilinear coordinates is made difficult by the fact that the vector basis $\mathbf{e}_s$ has changing direction. Therefore, it is necessary to derive the equations for the derivatives of these vectors with respect to q^k.

Presenting formulae (C.3.1) in the form

$$g_{sk} = H_s^2 \delta_{sk} = H_k^2 \delta_{sk} = H_s H_k \delta_{sk},$$

we have by eq. (C.2.7)

$$\mathbf{R}_{st} \cdot \mathbf{R}_k = H_s \frac{\partial H_k}{\partial q^t} \delta_{sk} + H_t \frac{\partial H_k}{\partial q^s} \delta_{kt} - H_t \frac{\partial H_s}{\partial q^k} \delta_{st}.$$

Taking into account that the left hand side of this relationship can be set in the form

$$\frac{\partial}{\partial q^t} (H_s \mathbf{e}_s) \cdot H_k \mathbf{e}_k = \frac{\partial H_s}{\partial q^t} H_k \delta_{sk} + H_s H_k \frac{\partial \mathbf{e}_s}{\partial q^t} \cdot \mathbf{e}_k,$$

we arrive at the equality

$$\frac{\partial \mathbf{e}_s}{\partial q^t} \cdot \mathbf{e}_k = \frac{H_t}{H_s H_k} \left(\frac{\partial H_k}{\partial q^s} \delta_{kt} - \frac{\partial H_s}{\partial q^k} \delta_{st} \right) = \frac{\partial H_k}{H_s \partial q^s} \delta_{kt} - \frac{\partial H_s}{H_k \partial q^k} \delta_{st}. \tag{C.4.1}$$

The right hand side is equal to zero for $s = k$ and changes sign under permutation of s and k. It is not surprising since

$$\frac{\partial}{\partial q^t} \mathbf{e}_s \cdot \mathbf{e}_k = \frac{\partial \mathbf{e}_s}{\partial q^t} \cdot \mathbf{e}_k + \mathbf{e}_s \cdot \frac{\partial \mathbf{e}_k}{\partial q^t} = 0. \tag{C.4.2}$$

Hence, the matrix of scalar products

$$\|a_{ks}\| = \left\| \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_s}{\partial q^t} \right\| \tag{C.4.3}$$

(for fixed t) is skew-symmetric and the following three numbers

$$\overset{t}{o}_r = \frac{1}{2} e_{rsk} \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_s}{\partial q^t} = \frac{1}{2} \left(\frac{\partial H_t}{H_s \partial q^s} e_{rst} + \frac{\partial H_t}{H_k \partial q^k} e_{rkt} \right) = \frac{\partial H_t}{H_m \partial q^m} e_{rmt} \tag{C.4.4}$$

(summation over m) are sufficient to prescribe matrix (C.4.3). Introducing the vectors

$$\overset{t}{\mathbf{o}} = \mathbf{e}_r \overset{t}{o}_r,$$

we obtain the expression for vector $\overset{t}{\mathbf{o}}$ accompanying matrix (C.4.3)

$$\overset{t}{\mathbf{o}} = \frac{\partial H_t}{H_m \partial q^m} e_{rmt} \mathbf{e}_r = \frac{\partial H_t}{H_m \partial q^m} \mathbf{e}_m \times \mathbf{e}_t = \operatorname{grad} H_t \times \mathbf{e}_t. \tag{C.4.5}$$

Using eqs. (C.4.4) and (A.4.6) we can write the matrix components in the following form

$$\frac{\partial \mathbf{e}_s}{\partial q^t} \cdot \mathbf{e}_k = e_{rsk} \overset{t}{o}_r, \tag{C.4.6}$$

$$\frac{\partial \mathbf{e}_s}{\partial q^t} = e_{rsk} \overset{t}{o}_r \mathbf{e}_k = \mathbf{e}_r \times \mathbf{e}_s \overset{t}{o}_r = \overset{t}{\mathbf{o}} \times \mathbf{e}_s. \tag{C.4.7}$$

These are the sought formulae for differentiating base vectors. By means of eq. (C.4.5) they can be written in another form

$$\frac{\partial \mathbf{e}_s}{\partial q^t} = \mathbf{e}_s \times (\mathbf{e}_t \times \operatorname{grad} H_t) = \mathbf{e}_t \frac{\partial H_t}{H_s \partial q^s} - \delta_{st} \operatorname{grad} H_t. \tag{C.4.8}$$

Formulae (C.4.7) admits a kinematic interpretation known as "the method of moving trihedron". Let vertex M of the trihedron of base vectors $\mathbf{e}_s$

move with unit speed $v = 1$ along the coordinate line $[q^m]$ so that $d_m s = H_m dq^m = dt$ (here t denotes time). In any instantaneous position of the trihedron, vectors $\mathbf{e}_s$ must have the direction of the tangents to the coordinate lines at point M. Hence the motion of the point is accompanied by rotation of the trihedron about the coordinate line. Let us denote the angular velocity vector of this rotation by $\overset{m}{\boldsymbol{\omega}}$. The velocity of the ends of the unit vectors $\mathbf{e}_s$ about the trihedron vertex are equal to

$$\frac{d\mathbf{e}_s}{dt} = \frac{\partial \mathbf{e}_s}{H_m \partial q^m} = \overset{m}{\boldsymbol{\omega}} \times \mathbf{e}_s,$$

and comparison with eq. (C.4.7) yields

$$\overset{m}{\boldsymbol{\omega}} = \frac{\overset{m}{\mathbf{o}}}{H_m}. \tag{C.4.9}$$

Vector $\overset{m}{\boldsymbol{\omega}}$ can often be found without calculation by using this kinematic interpretation.

C.5 Differential operations in orthogonal curvilinear coordinates

The calculations are based on the definition of the nabla-operator (C.3.9) and formula (C.4.8).

1. *Gradient of a vector.* We have

$$\begin{aligned}\nabla \mathbf{a} &= \frac{\mathbf{e}_t}{H_t}\frac{\partial}{\partial q^t} a_s \mathbf{e}_s = \mathbf{e}_t \mathbf{e}_s \frac{\partial a_s}{H_t \partial q^t} + \mathbf{e}_t \frac{a_s}{H_t} \overset{t}{\mathbf{o}} \times \mathbf{e}_s \\ &= \mathbf{e}_t \mathbf{e}_s \frac{\partial a_s}{H_t \partial q^t} + \frac{a_s}{H_t} \overset{t}{o}_r \, e_{rsk} \mathbf{e}_t \mathbf{e}_k = \mathbf{e}_t \mathbf{e}_s \left(\frac{\partial a_s}{H_t \partial q^t} + \frac{a_k}{H_t} \overset{t}{o}_r \, e_{rks} \right).\end{aligned}$$

By eq. (C.4.4)

$$\begin{aligned}\overset{t}{o}_r \, e_{rks} &= \frac{\partial H_t}{H_m \partial q^m} e_{rmt} e_{rks} = \frac{\partial H_t}{H_m \partial q^m} \left(\delta_{mk}\delta_{ts} - \delta_{ms}\delta_{tk}\right) \\ &= \frac{\partial H_t}{H_k \partial q^k}\delta_{ts} - \frac{\partial H_t}{H_s \partial q^s}\delta_{tk} \quad \left(\not{\Sigma}_{kst}\right),\end{aligned}$$

and the expression for $\nabla \mathbf{a}$ reduces to the form

$$\nabla \mathbf{a} = \mathbf{e}_t \mathbf{e}_s \left(\frac{\partial a_s}{H_t \partial q^t} - \frac{a_t}{H_t H_s}\frac{\partial H_t}{\partial q^s} + \delta_{ts} \frac{a_k}{H_t H_k}\frac{\partial H_t}{\partial q^k} \right). \tag{C.5.1}$$

2. *Divergence of a vector.* Using eq. (C.3.10) we write down the expression

$$\frac{\partial \ln \sqrt{g}}{\partial q^s} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial q^s} = \frac{\partial \ln H_1}{\partial q^s} + \frac{\partial \ln H_2}{\partial q^s} + \frac{\partial \ln H_3}{\partial q^s}. \tag{C.5.2}$$

The trace of $\nabla\mathbf{a}$ is as follows

$$\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a} = \frac{\partial a_s}{H_s\partial q^s} - \frac{a_s}{H_s^2}\frac{\partial H_s}{\partial q^s} + \frac{a_k}{H_k}\frac{\partial\sqrt{g}}{\sqrt{g}\partial q^k} = \frac{\partial}{\partial q^s}\frac{a_s}{H_s} + \frac{a_s}{H_s}\frac{\partial\sqrt{g}}{\sqrt{g}\partial q^s}$$

and finally

$$\begin{aligned}\operatorname{div}\mathbf{a} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^s}\left(\frac{\sqrt{g}}{H_s}a_s\right)\\ &= \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial q^1}H_2H_3a_1 + \frac{\partial}{\partial q^2}H_3H_1a_2 + \frac{\partial}{\partial q^3}H_1H_2a_3\right).\end{aligned} \tag{C.5.3}$$

3. *Laplace operator of a scalar.* If in particular

$$\mathbf{a} = \operatorname{grad}\psi, \quad a_s = \frac{1}{H_s}\frac{\partial\psi}{\partial q^s}, \tag{C.5.4}$$

then

$$\begin{aligned}\nabla^2\psi = \operatorname{div}\operatorname{grad}\psi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^s}\frac{\sqrt{g}}{H_s^2}\frac{\partial\psi}{\partial q^s} \qquad (C.5.5)\\ &= \frac{1}{H_1H_2H_3}\left(\frac{\partial}{\partial q^1}\frac{H_2H_3}{H_1}\frac{\partial\psi}{\partial q^1} + \frac{\partial}{\partial q^2}\frac{H_3H_1}{H_2}\frac{\partial\psi}{\partial q^2} + \frac{\partial}{\partial q^3}\frac{H_1H_2}{H_3}\frac{\partial\psi}{\partial q^3}\right).\end{aligned}$$

4. *Rotor of a vector.* In eq. (C.5.1) the dyadics $\mathbf{e}_t\mathbf{e}_s$ should be replaced by the vector products

$$\mathbf{e}_t\times\mathbf{e}_s = \frac{1}{2}\mathbf{e}_t\times\mathbf{e}_s - \frac{1}{2}\mathbf{e}_s\times\mathbf{e}_t.$$

Then we arrive at the expression

$$\begin{aligned}\nabla\times\mathbf{a} = \operatorname{rot}\mathbf{a} = \frac{1}{2}\left(\frac{\partial a_s}{H_t\partial q^t} - \frac{a_t}{H_tH_s}\frac{\partial H_t}{\partial q^s}\right)\mathbf{e}_t\times\mathbf{e}_s &-\\ \frac{1}{2}\left(\frac{\partial a_t}{H_s\partial q^s} - \frac{a_s}{H_sH_t}\frac{\partial H_s}{\partial q^t}\right)\mathbf{e}_t\times\mathbf{e}_s&\end{aligned}$$

or

$$\operatorname{rot}\mathbf{a} = \frac{1}{2H_sH_t}\left(\frac{\partial}{\partial q^t}H_sa_s - \frac{\partial}{\partial q^s}H_ta_t\right)\mathbf{e}_t\times\mathbf{e}_s. \tag{C.5.6}$$

The projections of this vector are equal to

$$\mathbf{e}_q\cdot\operatorname{rot}\mathbf{a} = \frac{e_{qts}}{2H_tH_s}\left(\frac{\partial}{\partial q^t}H_sa_s - \frac{\partial}{\partial q^s}H_ta_t\right). \tag{C.5.7}$$

5. *Tensor* def **a**. This is defined by formula (B.2.7). Referring to eq. (C.5.1) we obtain

$$\hat{\varepsilon} = \operatorname{def} \mathbf{a} = \frac{1}{2}\mathbf{e}_t\mathbf{e}_s\left(\frac{\partial a_s}{H_t\partial q^t} + \frac{\partial a_t}{H_s\partial q^s} - \frac{a_t}{H_sH_t}\frac{\partial H_t}{\partial q^s} - \frac{a_s}{H_sH_t}\frac{\partial H_s}{\partial q^t} + 2\delta_{ts}\frac{a_k}{H_tH_k}\frac{\partial H_t}{\partial q^k}\right). \quad \text{(C.5.8)}$$

The expressions for the components of this tensor are set in the form

$$\left.\begin{aligned} \varepsilon_{11} &= \frac{\partial a_1}{H_1\partial q^1} + \frac{a_2}{H_1H_2}\frac{\partial H_1}{\partial q^2} + \frac{a_3}{H_1H_3}\frac{\partial H_1}{\partial q^3}, \\ \varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\left(\frac{\partial a_1}{H_2\partial q^2} + \frac{\partial a_2}{H_1\partial q^1} - \frac{a_1}{H_1H_2}\frac{\partial H_1}{\partial q^2} - \frac{a_2}{H_1H_2}\frac{\partial H_2}{\partial q^1}\right). \end{aligned}\right\} \quad \text{(C.5.9)}$$

6. *Divergence of a tensor of second rank.*

$$\begin{aligned} \operatorname{div}\hat{T} = \nabla\cdot\hat{T} &= \frac{\mathbf{e}_r}{H_r}\cdot\frac{\partial}{\partial q^r}t_{st}\mathbf{e}_s\mathbf{e}_t \\ &= \frac{1}{H_s}\frac{\partial t_{st}}{\partial q^s}\mathbf{e}_t + \frac{t_{st}}{H_r}\left[\mathbf{e}_r\cdot\left(\overset{r}{\mathbf{o}}\times\mathbf{e}_s\right)\mathbf{e}_t + \delta_{rs}\frac{\partial \mathbf{e}_t}{\partial q^r}\right]. \end{aligned}$$

By eq. (C.4.5)

$$\begin{aligned} \mathbf{e}_r\cdot\left(\overset{r}{\mathbf{o}}\times\mathbf{e}_s\right) &= \mathbf{e}_r\cdot\left[\left(\operatorname{grad}H_r\times\mathbf{e}_r\right)\times\mathbf{e}_s\right] \\ &= \frac{\partial H_r}{H_s\partial q^s} - \delta_{rs}\frac{\partial H_r}{H_r\partial q^r} = \frac{\partial H_r}{H_s\partial q^s} - \frac{\partial H_s}{H_s\partial q^s}, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{div}\hat{T} &= \left(\frac{1}{H_s}\frac{\partial t_{st}}{\partial q^s} + \frac{t_{st}}{H_rH_s}\frac{\partial H_r}{\partial q^s} - \frac{t_{st}}{H_s^2}\frac{\partial H_s}{\partial q^s}\right)\mathbf{e}_t + \frac{t_{st}}{H_s}\frac{\partial \mathbf{e}_t}{\partial q^s} \\ &= \left(\frac{\partial}{\partial q^s}\frac{t_{st}}{H_s} + \frac{t_{st}}{H_s}\frac{\partial\sqrt{g}}{\sqrt{g}\partial q^s}\right)\mathbf{e}_t + \frac{t_{st}}{H_s}\frac{\partial \mathbf{e}_t}{\partial q^s} \end{aligned}$$

and in the final form

$$\begin{aligned} \operatorname{div}\hat{T} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^s}\frac{\sqrt{g}}{H_s}t_{st}\mathbf{e}_t \qquad \text{(C.5.10)} \\ &= \frac{1}{H_1H_2H_3}\left(\frac{\partial}{\partial q^1}H_2H_3t_{1t}\mathbf{e}_t + \frac{\partial}{\partial q^2}H_3H_1t_{2t}\mathbf{e}_t + \frac{\partial}{\partial q^3}H_1H_2t_{3t}\mathbf{e}_t\right). \end{aligned}$$

7. *Tensor* $\nabla\nabla\psi$. The calculation yields

$$\begin{aligned} \nabla\nabla\psi &= \frac{\mathbf{e}_s}{H_s}\frac{\partial}{\partial q^s}\frac{\mathbf{e}_t}{H_t}\frac{\partial\psi}{\partial q^t} \qquad \text{(C.5.11)} \\ &= \frac{\mathbf{e}_s\mathbf{e}_t}{H_sH_t}\left(\frac{\partial^2\psi}{\partial q^s\partial q^t} - \frac{\partial\ln H_t}{\partial q^s}\frac{\partial\psi}{\partial q^t} - \frac{\partial\ln H_s}{\partial q^t}\frac{\partial\psi}{\partial q^s}\right) + \frac{\mathbf{e}_s\mathbf{e}_t}{H_r^2}\frac{\partial\ln H_s}{\partial q^r}\frac{\partial\psi}{\partial q^r}. \end{aligned}$$

This allows the following formula for the Laplace operator to be obtained

$$\nabla \cdot \nabla \psi = \nabla^2 \psi = \frac{1}{H_s^2}\left(\frac{\partial^2 \psi}{\partial q^{s2}} - 2\frac{\partial \ln H_s}{\partial q^s}\frac{\partial \psi}{\partial q^s}\right) + \frac{1}{H_r^2 H_s}\frac{\partial H_s}{\partial q^r}\frac{\partial \psi}{\partial q^r}, \tag{C.5.12}$$

which can be easily transformed into that in eq. (C.5.5).

C.6 Lame's dependences

Using formulae (C.4.7) we can present the relationship

$$\frac{\partial^2 \mathbf{e}_s}{\partial q^r \partial q^t} = \frac{\partial^2 \mathbf{e}_s}{\partial q^t \partial q^r} \tag{C.6.1}$$

in the following form

$$\frac{\partial}{\partial q^r}\left(\overset{t}{\mathbf{o}} \times \mathbf{e}_s\right) = \frac{\partial \overset{t}{\mathbf{o}}}{\partial q^r} \times \mathbf{e}_s + \overset{t}{\mathbf{o}} \times \left(\overset{r}{\mathbf{o}} \times \mathbf{e}_s\right) = \frac{\partial \overset{r}{\mathbf{o}}}{\partial q^t} \times \mathbf{e}_s + \overset{r}{\mathbf{o}} \times \left(\overset{t}{\mathbf{o}} \times \mathbf{e}_s\right)$$

or

$$\left(\frac{\partial \overset{t}{\mathbf{o}}}{\partial q^r} - \frac{\partial \overset{r}{\mathbf{o}}}{\partial q^t} + \overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}}\right) \times \mathbf{e}_s = 0 \quad (s = 1, 2, 3).$$

The obtained differential relationship between vectors $\overset{t}{\mathbf{o}}$ and $\overset{r}{\mathbf{o}}$

$$\frac{\partial \overset{t}{\mathbf{o}}}{\partial q^r} - \frac{\partial \overset{r}{\mathbf{o}}}{\partial q^t} + \overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}} = 0 \tag{C.6.2}$$

are transformed into equalities relating Lame's coefficients. To this aim, we calculate the projections of the vectors in eq. (C.6.2) onto the axes $\mathbf{e}_s$ of the trihedron. By eq. (C.4.7) we have

$$\begin{aligned}
\mathbf{e}_s \cdot &\left(\frac{\partial \overset{t}{\mathbf{o}}}{\partial q^r} - \frac{\partial \overset{r}{\mathbf{o}}}{\partial q^t} + \overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}}\right) = \\
&= \frac{\partial \mathbf{e}_s \cdot \overset{t}{\mathbf{o}}}{\partial q^r} - \frac{\partial \mathbf{e}_s \cdot \overset{r}{\mathbf{o}}}{\partial q^t} - \overset{t}{\mathbf{o}} \cdot \left(\overset{r}{\mathbf{o}} \times \mathbf{e}_s\right) + \overset{r}{\mathbf{o}} \cdot \left(\overset{t}{\mathbf{o}} \times \mathbf{e}_s\right) + \left(\overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}}\right) \cdot \mathbf{e}_s \\
&= \frac{\partial \overset{t}{\mathbf{o}} \cdot \mathbf{e}_s}{\partial q^r} - \frac{\partial \overset{r}{\mathbf{o}} \cdot \mathbf{e}_s}{\partial q^t} - \left(\overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}}\right) \cdot \mathbf{e}_s = 0.
\end{aligned} \tag{C.6.3}$$

Further,

$$\overset{t}{\mathbf{o}} \cdot \mathbf{e}_s = \operatorname{grad} H_t \cdot (\mathbf{e}_t \times \mathbf{e}_s) = \frac{1}{H_m}\frac{\partial H_t}{\partial q^m} e_{tsm},$$

$$\left(\overset{t}{\mathbf{o}} \times \overset{r}{\mathbf{o}}\right) \cdot \mathbf{e}_s = \frac{\partial H_t}{H_m \partial q^m}\frac{\partial H_r}{H_s \partial q^s} e_{trm} - (\operatorname{grad} H_r \times \operatorname{grad} H_t) \cdot \mathbf{e}_t \delta_{rs},$$

and substitution into eq. (C.6.3) yields the equality

$$e_{tsm}\frac{\partial}{\partial q^r}\frac{1}{H_m}\frac{\partial H_t}{\partial q^m} - e_{rsm}\frac{\partial}{\partial q^t}\frac{1}{H_m}\frac{\partial H_r}{\partial q^m} + \frac{\partial H_t}{H_m \partial q^m}\frac{\partial H_r}{H_s \partial q^s}e_{rtm} + \\ (\operatorname{grad} H_r \times \operatorname{grad} H_t)\cdot \mathbf{e}_t \delta_{rs} = 0. \quad \text{(C.6.4)}$$

Clearly, this equality is satisfied for $t = r$. Hence it is necessary to consider the cases i) $s \neq t \neq r$ and ii) $s = t \neq r$.

i) $e_{tsm} = e_{tsr}, e_{rsm} = e_{rst}, e_{rtm} = e_{rts}, \delta_{rs} = 0$. We obtain

$$\frac{\partial}{\partial q^r}\frac{1}{H_r}\frac{\partial H_t}{\partial q^r} + \frac{\partial}{\partial q^t}\frac{1}{H_t}\frac{\partial H_r}{\partial q^t} + \frac{1}{H_s^2}\frac{\partial H_r}{\partial q^s}\frac{\partial H_t}{\partial q^s} = 0, \quad \text{(C.6.5)}$$

and since it is symmetric in r and t, and $s \neq t \neq r$ we have only three different relationships

$$\left.\begin{aligned}
\frac{\partial}{\partial q^1}\frac{1}{H_1}\frac{\partial H_2}{\partial q^1} + \frac{\partial}{\partial q^2}\frac{1}{H_2}\frac{\partial H_1}{\partial q^2} + \frac{1}{H_3^2}\frac{\partial H_1}{\partial q^3}\frac{\partial H_2}{\partial q^3} = 0,\\
\frac{\partial}{\partial q^2}\frac{1}{H_2}\frac{\partial H_3}{\partial q^2} + \frac{\partial}{\partial q^3}\frac{1}{H_3}\frac{\partial H_2}{\partial q^3} + \frac{1}{H_1^2}\frac{\partial H_2}{\partial q^1}\frac{\partial H_3}{\partial q^1} = 0,\\
\frac{\partial}{\partial q^3}\frac{1}{H_3}\frac{\partial H_1}{\partial q^3} + \frac{\partial}{\partial q^1}\frac{1}{H_1}\frac{\partial H_3}{\partial q^1} + \frac{1}{H_2^2}\frac{\partial H_3}{\partial q^2}\frac{\partial H_1}{\partial q^2} = 0.
\end{aligned}\right\} \quad \text{(C.6.6)}$$

ii) In the case $s = t \neq r$

$$\frac{\partial^2 H_r}{\partial q^t \partial q^m} = \frac{1}{H_m}\frac{\partial H_m}{\partial q^t}\frac{\partial H_r}{\partial q^m} + \frac{1}{H_t}\frac{\partial H_r}{\partial q^t}\frac{\partial H_t}{\partial q^m}, \quad \text{(C.6.7)}$$

where m, t, r are each different. We obtain another three relationships

$$\left.\begin{aligned}
\frac{\partial^2 H_1}{\partial q^2 \partial q^3} = \frac{1}{H_3}\frac{\partial H_3}{\partial q^2}\frac{\partial H_1}{\partial q^3} + \frac{1}{H_2}\frac{\partial H_1}{\partial q^2}\frac{\partial H_2}{\partial q^3},\\
\frac{\partial^2 H_2}{\partial q^3 \partial q^1} = \frac{1}{H_1}\frac{\partial H_1}{\partial q^3}\frac{\partial H_2}{\partial q^1} + \frac{1}{H_3}\frac{\partial H_2}{\partial q^3}\frac{\partial H_3}{\partial q^1},\\
\frac{\partial^2 H_3}{\partial q^1 \partial q^2} = \frac{1}{H_2}\frac{\partial H_2}{\partial q^1}\frac{\partial H_3}{\partial q^2} + \frac{1}{H_1}\frac{\partial H_3}{\partial q^1}\frac{\partial H_1}{\partial q^2}.
\end{aligned}\right\} \quad \text{(C.6.8)}$$

Six Lame's dependences (C.6.6), (C.6.8) hold identically if Lame's coefficients are obtained by means of the point transformation (C.1.1) and formulae (C.3.2). Inversely, when these dependences are satisfied then three prescribed functions $H\left(q^1, q^2, q^3\right)$ are Lame's coefficients for some transformation determined by the system of differential equations (C.3.2) and Lame's dependences form the integrability condition for this system.

C.7 Cylindrical coordinates

The base vectors $\mathbf{e}_1 = \mathbf{e}_r, \mathbf{e}_2 = \mathbf{e}_\varphi, \mathbf{e}_3 = \mathbf{k}$ have the directions of the radius of circles, the tangents to circles and the axis of the concentric cylinders respectively. Lame's coefficients are equal to

$$H_1 = H_r = 1, \quad H_2 = H_\varphi = r, \quad H_3 = H_z = 1. \tag{C.8.1}$$

Vectors $\overset{3}{\mathbf{o}}, \overset{1}{\mathbf{o}}$ are equal to zero because of the translatory motion of the vertex of the basis trihedron along the coordinate lines $[z], [r]$. The angular velocity of the trihedron under the motion of its vertex along circle $[\varphi]$ is given by the vector

$$\overset{\varphi}{\boldsymbol{\omega}} = \frac{1}{r}\mathbf{k}.$$

Hence

$$\overset{z}{\mathbf{o}} = 0, \quad \overset{r}{\mathbf{o}} = 0, \quad \overset{\varphi}{\mathbf{o}} = \mathbf{k} \tag{C.7.2}$$

and the nonvanishing derivatives are as follows

$$\frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{k} \times \mathbf{e}_r = \mathbf{e}_\varphi, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = \mathbf{k} \times \mathbf{e}_\varphi = -\mathbf{e}_r. \tag{C.7.3}$$

By definition (C.3.9) for the nabla-operator we have

$$\nabla \mathbf{e}_r = \frac{\mathbf{e}_\varphi}{r} \frac{\partial \mathbf{e}_r}{\partial \varphi} = \frac{1}{r}\mathbf{e}_\varphi \mathbf{e}_\varphi, \quad \nabla \mathbf{e}_\varphi = \frac{\mathbf{e}_\varphi}{r} \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\frac{1}{r}\mathbf{e}_\varphi \mathbf{e}_r \tag{C.7.4}$$

and

$$\nabla^2 \mathbf{e}_r = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\varphi}{r} \frac{\partial}{\partial \varphi}\right) \cdot \frac{1}{r}\mathbf{e}_\varphi \mathbf{e}_\varphi = -\frac{1}{r^2}\mathbf{e}_r, \quad \nabla^2 \mathbf{e}_\varphi = -\frac{1}{r^2}\mathbf{e}_\varphi. \tag{C.7.5}$$

By eq. (C.5.5) the Laplace operator in cylindrical coordinates is written in the form

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi^2}. \tag{C.7.6}$$

C.8 Spherical coordinates

The base vectors $\mathbf{e}_1 = \mathbf{e}_R, \mathbf{e}_2 = \mathbf{e}_\vartheta, \mathbf{e}_3 = \mathbf{e}_\lambda = \mathbf{e}_R \times \mathbf{e}_\vartheta$ have the directions of the radius, the tangent to the meridian (southern direction) and perpendicular to the meridional plane (eastern direction) respectively. Lame's coefficients are determined by eq. (C.3.7)

$$H_1 = H_R = 1, \quad H_2 = H_\vartheta = R, \quad H_3 = H_\lambda = R \sin\vartheta. \tag{C.8.1}$$

Further $\overset{R}{\mathbf{o}} = 0$ and the angular velocities of the trihedron under the motion of its vertex along the meridian $[\vartheta]$ and the parallel circle $[\lambda]$ are given respectively by vectors $\overset{\vartheta}{\boldsymbol{\omega}}= \frac{1}{R}\mathbf{e}_\lambda$ and $\overset{\lambda}{\boldsymbol{\omega}}= \frac{1}{R\sin\vartheta}\mathbf{k}$ where $\mathbf{k}$ denotes the unit vector directed along axis Ox_3 to the north pole.

Hence

$$\overset{R}{\mathbf{o}}= 0, \quad \overset{\vartheta}{\mathbf{o}}= \mathbf{e}_\lambda, \quad \overset{\lambda}{\mathbf{o}}= \mathbf{k} = \mathbf{e}_R\cos\vartheta - \mathbf{e}_\vartheta \sin\vartheta \tag{C.8.2}$$

and by derivation formulae (C.4.7) we have

$$\frac{\partial \mathbf{e}_s}{\partial R} = 0, \quad \frac{\partial \mathbf{e}_s}{\partial \vartheta} = \mathbf{e}_\lambda \times \mathbf{e}_s, \quad \frac{\partial \mathbf{e}_s}{\partial \lambda} = \mathbf{e}_R \times \mathbf{e}_s \cos\vartheta - \mathbf{e}_\vartheta \times \mathbf{e}_s \sin\vartheta, \tag{C.8.3}$$

so that

$$\left.\begin{array}{lll} \dfrac{\partial \mathbf{e}_R}{\partial \vartheta} = \mathbf{e}_\vartheta, & \dfrac{\partial \mathbf{e}_\vartheta}{\partial \vartheta} = -\mathbf{e}_R, & \dfrac{\partial \mathbf{e}_\lambda}{\partial \vartheta} = 0, \\ \dfrac{\partial \mathbf{e}_R}{\partial \lambda} = \mathbf{e}_\lambda \sin\vartheta, & \dfrac{\partial \mathbf{e}_\vartheta}{\partial \lambda} = \mathbf{e}_\lambda \cos\vartheta, & \dfrac{\partial \mathbf{e}_\lambda}{\partial \lambda} = -\left(\mathbf{e}_R \sin\vartheta + \mathbf{e}_\vartheta \cos\vartheta\right). \end{array}\right\} \tag{C.8.4}$$

The Laplace operator in spherical coordinates is written in the form

$$\nabla^2\psi = \frac{1}{R^2}\left(\frac{\partial}{\partial R}R^2\frac{\partial\psi}{\partial R} + \frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\lambda^2}\right), \tag{C.8.5}$$

where the new variable

$$\mu = \cos\vartheta$$

has been introduced.

C.9 Bodies of revolution

Formulae (C.1.1) expressing Cartesian coordinates in terms of the curvilinear coordinates $q^1, q^2, q^3 = \varphi$ are given in the form

$$x_1 = r\left(q^1, q^2\right)\cos\varphi, \quad x_2 = r\left(q^1, q^2\right)\sin\varphi, \quad x_3 = z\left(q^1, q^2\right), \tag{C.9.1}$$

where φ is the azimuthal angle of the cylindrical coordinate system. The coordinate surface q_0^1 and q_0^2 are the surfaces of revolution about axis Ox_3 whose meridional cross-sections (by planes φ^0) are the orthogonal families of the curves

$$\left.\begin{array}{lll} q^1 = q_0^1: & r = r\left(q_0^1, q^2\right), & z = z\left(q_0^1, q^2\right), \\ q^2 = q_0^2: & r = r\left(q^1, q_0^2\right), & z = z\left(q^1, q_0^2\right). \end{array}\right\} \tag{C.9.2}$$

The position vector $\mathbf{R} = r\left(q^1, q^2, \varphi\right)$ of the point is as follows

$$\mathbf{R} = r\left(q^1, q^2\right)\mathbf{e}_r + z\left(q^1, q^2\right)\mathbf{k}. \tag{C.9.3}$$

Then we have

$$\mathbf{R}_s = \frac{\partial r}{\partial q^s}\mathbf{e}_r + \frac{\partial z}{\partial q^s}\mathbf{k} \quad (s = 1, 2), \qquad \mathbf{R}_3 = r\left(q^1, q^2\right)\mathbf{e}_\varphi, \tag{C.9.4}$$

so that

$$H_s^2 = \left(\frac{\partial r}{\partial q^s}\right)^2 + \left(\frac{\partial z}{\partial q^s}\right)^2, \quad H_3 = r, \tag{C.9.5}$$

and the condition of orthogonality of the family of curves (C.9.2) is written in the form

$$\mathbf{R}_1 \cdot \mathbf{R}_2 = \frac{\partial r}{\partial q^1}\frac{\partial r}{\partial q^2} + \frac{\partial z}{\partial q^1}\frac{\partial z}{\partial q^2} = 0. \tag{C.9.6}$$

The unit vectors of the orthogonal trihedron of the tangents to the coordinate lines $[q^s]$ are given by

$$\mathbf{e}_s = \frac{1}{H_s}\mathbf{R}_s \quad (s = 1, 2), \qquad \mathbf{e}_3 = \mathbf{e}_\varphi. \tag{C.9.7}$$

They are related to the unit vectors $\mathbf{k}, \mathbf{e}_r$ of the cylindrical system of axes by the following equalities

$$\left.\begin{aligned}
\mathbf{e}_1 &= \frac{1}{H_1}\frac{\partial z}{\partial q^1}\mathbf{k} + \frac{1}{H_1}\frac{\partial r}{\partial q^1}\mathbf{e}_r = \frac{1}{H_2}\frac{\partial r}{\partial q^2}\mathbf{k} - \frac{1}{H_2}\frac{\partial z}{\partial q^2}\mathbf{e}_r, \\
\mathbf{e}_2 &= \frac{1}{H_2}\frac{\partial z}{\partial q^2}\mathbf{k} + \frac{1}{H_2}\frac{\partial r}{\partial q^2}\mathbf{e}_r = -\frac{1}{H_1}\frac{\partial r}{\partial q^1}\mathbf{k} + \frac{1}{H_1}\frac{\partial z}{\partial q^1}\mathbf{e}_r.
\end{aligned}\right\} \tag{C.9.8}$$

The third group of formulae in eq. (C.5.5) take the form

$$\frac{\partial q^s}{\partial z} = \frac{1}{H_s^2}\frac{\partial z}{\partial q^s}, \qquad \frac{\partial q^s}{\partial r} = \frac{1}{H_s^2}\frac{\partial r}{\partial q^s}. \tag{C.9.9}$$

According to eq. (C.4.5) the vectors $\overset{s}{\mathbf{o}}$ are written as follows

$$\left.\begin{aligned}
\overset{s}{\mathbf{o}} &= \operatorname{grad} H_s \times \mathbf{e}_s \quad (s = 1, 2), \\
\overset{3}{\mathbf{o}} &= -\frac{1}{H_1}\frac{\partial r}{\partial q^1}\mathbf{e}_2 + \frac{1}{H_2}\frac{\partial r}{\partial q^2}\mathbf{e}_1 = \frac{1}{H_2}\frac{\partial z}{\partial q^2}\mathbf{e}_2 + \frac{1}{H_1}\frac{\partial z}{\partial q^1}\mathbf{e}_1,
\end{aligned}\right\} \tag{C.9.10}$$

and the derivation formulae take the form

$$\left.\begin{aligned}
&\frac{\partial \mathbf{e}_s}{\partial q^t} = \mathbf{e}_t\frac{\partial H_t}{H_s\partial q^s} - \delta_{st}\operatorname{grad} H_t, \quad \frac{\partial \mathbf{e}_s}{\partial \varphi} = \mathbf{e}_\varphi\frac{\partial r}{H_s\partial q^s} \quad (s, t = 1, 2), \\
&\frac{\partial \mathbf{e}_\varphi}{\partial q^t} = 0, \quad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\left(\frac{\mathbf{e}_1}{H_1}\frac{\partial r}{\partial q^1} + \frac{\mathbf{e}_2}{H_2}\frac{\partial r}{\partial q^2}\right).
\end{aligned}\right\} \tag{C.9.11}$$

Lame's dependences are simplified since two of them are identically satisfied whereas the remaining four are written as follows

$$
\left.
\begin{array}{c}
\dfrac{\partial}{\partial q^1}\dfrac{1}{H_1}\dfrac{\partial r}{\partial q^1} + \dfrac{1}{H_2^2}\dfrac{\partial r}{\partial q^2}\dfrac{\partial H_1}{\partial q^2} = 0, \\
\dfrac{\partial}{\partial q^2}\dfrac{1}{H_2}\dfrac{\partial r}{\partial q^2} + \dfrac{1}{H_1^2}\dfrac{\partial r}{\partial q^1}\dfrac{\partial H_2}{\partial q^1} = 0, \\
\dfrac{\partial^2 r}{\partial q^1 \partial q^2} = \dfrac{1}{H_1}\dfrac{\partial r}{\partial q^1}\dfrac{\partial H_1}{\partial q^2} + \dfrac{1}{H_2}\dfrac{\partial r}{\partial q^2}\dfrac{\partial H_2}{\partial q^1} = 0, \\
\dfrac{\partial}{\partial q^1}\dfrac{1}{H_1}\dfrac{\partial H_2}{\partial q^1} + \dfrac{\partial}{\partial q^2}\dfrac{1}{H_2}\dfrac{\partial H_1}{\partial q^2} = 0.
\end{array}
\right\} \tag{C.9.12}
$$

C.10 Degenerated elliptic coordinates

One of the families of coordinate surfaces are the ellipsoids of revolution about axis Ox_3. We will separately consider two cases: the first when the axis of revolution is its smaller axis (oblate ellipsoid, spheroid) and the second when this axis is the larger.

The cylindrical are related to the curvilinear coordinates as follows

$$q^1 = s, \quad q^2 = \mu, \quad q^3 = \varphi.$$

In the first case

$$r = a\sqrt{1+s^2}\sqrt{1-\mu^2}, \quad z = as\mu. \tag{C.10.1}$$

The coordinate surfaces $s =$ const are the oblate ellipsoids of revolution and $\mu =$ const describe one-sheet hyperboloids of revolution about axis z. Two mutually orthogonal families of curves in the meridional cross-section $\varphi =$ const are the ellipses

$$\frac{r^2}{1+s^2} + \frac{z^2}{s^2} - a^2 = 0 \tag{C.10.2}$$

and confocal hyperbolas

$$\frac{r^2}{1-\mu^2} - \frac{z^2}{\mu^2} - a^2 = 0. \tag{C.10.3}$$

The "ellipsoid" $s = 0$ is degenerated into a circular plate of radius a

$$r = a\sqrt{1-\mu^2}, \quad z = 0, \tag{C.10.4}$$

on the "upper" and "lower" sides of which $\mu > 0$ and $\mu < 0$ respectively.

The "hyperboloid" $\mu = 0$ presents a part of plane $z = 0$ outside of the circle of the radius

$$r = a\sqrt{1+s^2}, \quad z = 0. \tag{C.10.5}$$

The circle $r = a$ in the plane $z = 0$ is a locus of the foci of surfaces (C.10.2) and (C.10.3) (the so-called focal circle) in which $\mu = 0, s = 0$.

The domain of definition of parameters s, μ for the ellipsoids is given by the inequalities

$$0 \le s < \infty, \quad |\mu| \le 1, \tag{C.10.6}$$

and for the hyperboloids it is as follows

$$-\infty < s < \infty, \quad 0 \le \mu \le 1. \tag{C.10.7}$$

Lame's coefficients are calculated by means of formulae (C.9.5)

$$\left.\begin{aligned} H_1 &= H_s = a\sqrt{\frac{s^2+\mu^2}{1+s^2}}, \\ H_2 &= H_\mu = a\sqrt{\frac{s^2+\mu^2}{1-\mu^2}}, \\ H_3 &= H_\varphi = a\sqrt{(1+s^2)(1-\mu^2)}, \end{aligned}\right\} \tag{C.10.8}$$

such that the Jacobian of the transformation is equal to

$$\sqrt{g} = a^3\left(s^2+\mu^2\right)$$

and the focal circle is a singular line of the transformation.

The unit vectors of the trihedron of the tangents to the coordinate lines $[q^s]$ or (which is the same in the case of the orthogonal system) the normals to surfaces $q^s = \text{const}$ are expressed in terms of the unit vectors of the cylindrical systems in the following way

$$\left.\begin{aligned} \mathbf{e}_1 &= \frac{1}{\sqrt{s^2+\mu^2}}\left(\mathbf{k}\mu\sqrt{1+s^2} + \mathbf{e}_r s\sqrt{1-\mu^2}\right), \\ \mathbf{e}_2 &= \frac{1}{\sqrt{s^2+\mu^2}}\left(\mathbf{k}s\sqrt{1-\mu^2} - \mathbf{e}_r\mu\sqrt{1+s^2}\right), \\ \mathbf{e}_3 &= \mathbf{e}_\varphi. \end{aligned}\right\} \tag{C.10.9}$$

By eq. (C.5.5) the Laplace operator of the scalar is given by

$$\begin{aligned} \nabla^2\psi = \frac{1}{a^2\left(s^2+\mu^2\right)}&\left[\frac{\partial}{\partial s}\left(1+s^2\right)\frac{\partial\psi}{\partial s} + \right. \\ &\left.\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu} + \left(\frac{s^2}{1+s^2} + \frac{\mu^2}{1-\mu^2}\right)\frac{\partial^2\psi}{\partial\varphi^2}\right]. \end{aligned} \tag{C.10.10}$$

In the second case, i.e. in the case of an oblong ellipsoid we have

$$r = a\sqrt{(1-\mu^2)(s^2-1)}, \quad z = as\mu \tag{C.10.11}$$

and the orthogonal families of curves in the meridional plane $\varphi = \text{const}$ are the ellipses

$$\frac{r^2}{s^2-1} + \frac{z^2}{s^2} = a^2 \tag{C.10.12}$$

and the confocal two-sheeted hyperboloids

$$\frac{z^2}{\mu^2} - \frac{r^2}{1-\mu^2} - a^2 = 0 \tag{C.10.13}$$

with the common foci at points $\pm a$ on axis z. The domain of definition for variables s, μ is

$$1 \le s \le \infty, \quad -1 \le \mu \le 1. \tag{C.10.14}$$

The "ellipsoid" $s = 1$ is degenerated into a cut $|z| \le a$ of axis z whereas the "hyperboloids" $\mu = \pm 1$ is degenerated into half-lines $a \le z \le \infty$ and $-\infty \le z \le -a$ of this axis. Lame's coefficients are equal to

$$\begin{gathered} H_1 = H_s = a\sqrt{\frac{s^2-\mu^2}{s^2-1}}, \quad H_2 = H_\mu = a\sqrt{\frac{s^2-\mu^2}{1-\mu^2}}, \\ H_3 = H_\varphi = a\sqrt{\left(s^2-1\right)\left(1-\mu^2\right)}. \end{gathered} \tag{C.10.15}$$

The Laplace operator has the form

$$\begin{aligned} \nabla^2\psi = \frac{1}{a^2\left(s^2-\mu^2\right)} & \left[\frac{\partial}{\partial s}\left(s^2-1\right)\frac{\partial\psi}{\partial s} + \right. \\ & \left. \frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu} + \left(\frac{s^2}{s^2-1} + \frac{\mu^2}{1-\mu^2}\right)\frac{\partial^2\psi}{\partial\varphi^2}\right]. \end{aligned} \tag{C.10.16}$$

C.11 Elliptic coordinates (general case)

We consider the ellipsoid

$$\frac{x_1^2}{\rho_0^2} + \frac{x_2^2}{\rho_0^2-e^2} + \frac{x_3^2}{\rho_0^2-1} - a^2 = 0 \quad (\rho_0 > 1 > e) \tag{C.11.1}$$

and the family of confocal surfaces

$$\frac{x_1^2}{\sigma} + \frac{x_2^2}{\sigma-e^2} + \frac{x_3^2}{\sigma-1} - a^2 = 0, \tag{C.11.2}$$

where σ is a variable parameter. Considering x_1, x_2, x_3 as being prescribed, let us consider (C.11.2) as a cubic equation for σ

$$Q(\sigma) = \frac{P(\sigma)}{f(\sigma)} = 0, \tag{C.11.3}$$

in which

$$P(\sigma) = x_1^2 \left(\sigma - e^2\right)(\sigma - 1) + x_2^2 (\sigma - 1)\sigma + x_3^2 \left(\sigma - e^2\right)\sigma - a^2 f(\sigma), \tag{C.11.4}$$

$$f(\sigma) = \sigma\left(\sigma - e^2\right)(\sigma - 1). \tag{C.11.5}$$

For some positive values of σ we have the following value of $P(\sigma)$

$$P(0) > 0, \quad P\left(e^2\right) < 0, \quad P(1) > 0, \quad P(\infty) < 0,$$

hence the roots of polynomial $P(\sigma)$ lie in the intervals $\left(0, e^2\right), \left(e^2, 1\right)$ and $(1, \infty)$. Let us denote these roots as ν^2, μ^2 and ρ^2 respectively

$$0 \le \nu^2 \le e^2, \quad e^2 \le \mu^2 \le 1, \quad 1 \le \rho^2 < \infty. \tag{C.11.6}$$

This determines three families of confocal surfaces of second order, namely the two-sheeted hyperboloids $\nu = \text{const}$

$$\frac{x_1^2}{\nu^2} - \frac{x_2^2}{e^2 - \nu^2} - \frac{x_3^2}{1 - \nu^2} - a^2 = 0, \tag{C.11.7}$$

the one-sheet hyperboloids

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - e^2} - \frac{x_3^2}{1 - \mu^2} - a^2 = 0 \tag{C.11.8}$$

and the ellipsoids

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - e^2} + \frac{x_3^2}{\rho^2 - 1} - a^2 = 0. \tag{C.11.9}$$

The polynomial $P(\sigma)$ can now be set in the form

$$P(\sigma) = -a^2\left(\sigma - \nu^2\right)\left(\sigma - \mu^2\right)\left(\sigma - \rho^2\right), \tag{C.11.10}$$

and by means of eqs. (C.11.2) and (C.11.3) we arrive at the basic identity

$$\frac{x_1^2}{\sigma} + \frac{x_2^2}{\sigma - e^2} + \frac{x_3^2}{\sigma - 1} - a^2 = -a^2 \frac{\left(\sigma - \nu^2\right)\left(\sigma - \mu^2\right)\left(\sigma - \rho^2\right)}{f(\sigma)}. \tag{C.11.11}$$

From this identity one can express x_1^2, x_2^2, x_3^2 in terms of ν, μ, ρ. For example, multiplying both sides of this identity by σ and setting $\sigma = 0$ we obtain x_1^2 whereas multiplying by $\sigma - e^2$ and $\sigma - 1$ and inserting respectively $\sigma = e^2$ and $\sigma = 1$ we obtain x_2^2 and x_3^2. The result is as follows

$$\left.\begin{aligned} &e^2 x_1^2 = a^2 \rho^2 \mu^2 \nu^2, \\ &\left(1 - e^2\right) e^2 x_2^2 = a^2\left(\rho^2 - e^2\right)\left(\mu^2 - e^2\right)\left(e^2 - \nu^2\right), \\ &\left(1 - e^2\right) x_3^2 = a^2\left(\rho^2 - 1\right)\left(1 - \mu^2\right)\left(1 - \nu^2\right). \end{aligned}\right\} \tag{C.11.12}$$

Thus, for given values of coordinates x_1, x_2, x_3 of point M one uniquely determines three coordinate surfaces (C.11.7)-(C.11.9) passing through this point. Inversely, prescribing ν, μ, ρ we can determine (up to a sign) the Cartesian coordinates of points of intersection of these surfaces in each of eight octants of the coordinate system Ox_1, x_2, x_3.

The numbers

$$q^1 = \rho, \quad q^2 = \mu, \quad q^3 = \nu \tag{C.11.13}$$

determine the curvilinear system of the elliptic coordinates. The position of a point of the surface of ellipsoid $\rho = 1$ is given by parameters μ, ν and any function on this surface can be expressed in terms of these parameters. In particular for $\rho = 1$

$$e^2 x_1^2 = a^2 \mu^2 \nu^2, \quad e^2 x_2^2 = a^2 \left(\mu^2 - e^2\right)\left(e^2 - \nu^2\right), \quad x_3 = 0. \tag{C.11.14}$$

The curves $\mu = 1$ describe ellipses E with the semi-axes $a\mu$ and $a\sqrt{\mu^2 - e^2}$

$$(E) \quad \frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - e^2} - a^2 = 0, \tag{C.11.15}$$

lying between $(-ae, ae)$ of axis x_1 and ellipse E_0 with the semi-axes a and $a\sqrt{1 - e^2}$

$$(E_0) \quad x_1^2 + \frac{x_2^2}{1 - e^2} - a^2 = 0. \tag{C.11.16}$$

Hence, the "ellipsoid" $\rho = 1$ is an elliptic plate in the plane $x_3 = 0$ bounded by ellipse E_0.

At $\mu = 1$ the one-sheet hyperboloid degenerates into a part of plane $x_3 = 0$ outside of ellipse E_0 where

$$e^2 x_1^2 = a^2 \rho^2 \nu^2, \quad e^2 x_2^2 = a^2 \left(\rho^2 - e^2\right)\left(e^2 - \nu^2\right). \tag{C.11.17}$$

Ellipse E_0 is a line of intersection of surfaces $\rho = 1$ and $\mu = 1$. It is the locus of the foci of the system of coordinate surfaces.

Let us now proceed to calculating Lame's coefficients. By eq. (C.11.12)

$$\left.\begin{aligned}
\frac{dx_1}{x_1} &= \frac{d\rho}{\rho} + \frac{d\mu}{\mu} + \frac{d\nu}{\nu}, \\
\frac{dx_2}{x_2} &= \frac{\rho d\rho}{\rho^2 - e^2} + \frac{\mu d\mu}{\mu^2 - e^2} - \frac{\nu d\nu}{e^2 - \nu^2}, \\
\frac{dx_3}{x_3} &= \frac{\rho d\rho}{\rho^2 - 1} - \frac{\mu d\mu}{1 - \mu^2} - \frac{\nu d\nu}{1 - \nu^2}.
\end{aligned}\right\} \tag{C.11.18}$$

This yields the table of derivatives

$$\left.\begin{aligned}
&\frac{\partial x_1}{\partial \rho} = \frac{x_1}{\rho}, && \frac{\partial x_2}{\partial \rho} = \frac{x_2 \rho}{\rho^2 - e^2}, && \frac{\partial x_3}{\partial \rho} = \frac{x_3 \rho}{\rho^2 - 1}, \\
&\frac{\partial x_1}{\partial \mu} = \frac{x_1}{\mu}, && \frac{\partial x_2}{\partial \mu} = \frac{x_2 \mu}{\mu^2 - e^2}, && \frac{\partial x_3}{\partial \mu} = -\frac{x_3 \mu}{1 - \mu^2}, \\
&\frac{\partial x_1}{\partial \nu} = \frac{x_1}{\nu}, && \frac{\partial x_2}{\partial \nu} = -\frac{x_2 \nu}{e^2 - \nu^2}, && \frac{\partial x_3}{\partial \nu} = -\frac{x_3 \nu}{1 - \nu^2}
\end{aligned}\right\} \tag{C.11.19}$$

and by eq. (C.2.4) we find

$$\begin{aligned}
g_{11} = g_{\rho\rho} &= \left(\frac{\partial x_1}{\partial \rho}\right)^2 + \left(\frac{\partial x_2}{\partial \rho}\right)^2 + \left(\frac{\partial x_3}{\partial \rho}\right)^2 \\
&= \rho^2 \left[\frac{x_1^2}{\rho^4} + \frac{x_2^2}{\left(\rho^2 - e^2\right)^2} + \frac{x_3^2}{\left(\rho^2 - 1\right)^2}\right], \\
g_{12} = g_{\rho\mu} &= \frac{\partial x_1}{\partial \rho}\frac{\partial x_1}{\partial \mu} + \frac{\partial x_2}{\partial \rho}\frac{\partial x_2}{\partial \mu} + \frac{\partial x_3}{\partial \rho}\frac{\partial x_3}{\partial \mu} \\
&= \rho\mu \left[\frac{x_1^2}{\rho^2\mu^2} + \frac{x_2^2}{\left(\rho^2 - e^2\right)\left(\mu^2 - e^2\right)} + \frac{x_3^2}{\left(\rho^2 - 1\right)\left(\mu^2 - 1\right)}\right] \quad \text{etc.}
\end{aligned}$$

For $s \neq k$ we replace the Cartesian coordinates in the expressions for g_{sk} according to eq. (C.11.12), to obtain

$$g_{12} = g_{23} = g_{31} = 0, \tag{C.11.20}$$

which establishes the orthogonality of the system of elliptic coordinates. Further we have

$$\left.\begin{aligned}
g_{\rho\rho} = H_\rho^2 = \rho^2 \left[\frac{x_1^2}{\rho^4} + \frac{x_2^2}{\left(\rho^2 - e^2\right)^2} + \frac{x_3^2}{\left(\rho^2 - 1\right)^2}\right] &= \rho^2 D_\rho^2, \\
g_{\mu\mu} = H_\mu^2 = \mu^2 \left[\frac{x_1^2}{\mu^4} + \frac{x_2^2}{\left(\mu^2 - e^2\right)^2} + \frac{x_3^2}{\left(1 - \mu^2\right)^2}\right] &= \mu^2 D_\mu^2, \\
g_{\nu\nu} = H_\nu^2 = \nu^2 \left[\frac{x_1^2}{\nu^4} + \frac{x_2^2}{\left(e^2 - \nu^2\right)^2} + \frac{x_3^2}{\left(1 - \nu^2\right)^2}\right] &= \nu^2 D_\nu^2.
\end{aligned}\right\} \tag{C.11.21}$$

The Cartesian coordinates are excluded from these expressions by differentiating the basic identity (C.11.11) with respect to σ

$$\begin{aligned}
\frac{x_1^2}{\sigma^2} + \frac{x_2^2}{\left(\sigma^2 - e^2\right)^2} + \frac{x_3^2}{\left(\sigma^2 - 1\right)^2} &= \frac{a^2}{f(\sigma)} \left[\frac{f'(\sigma)}{f(\sigma)} \left(\sigma - \nu^2\right)\left(\sigma - \mu^2\right)\left(\sigma - \rho^2\right) + \right. \\
&\left. \left(\sigma - \mu^2\right)\left(\sigma - \rho^2\right) + \left(\sigma - \rho^2\right)\left(\sigma - \nu^2\right) + \left(\sigma - \nu^2\right)\left(\sigma - \mu^2\right)\right]
\end{aligned}$$

and consequently setting $\sigma = \rho^2, \sigma = \mu^2, \sigma = \nu^2$. The result is as follows

$$H_\rho^2 = \rho^2 \frac{a^2}{f\left(\rho^2\right)} \left(\rho^2 - \nu^2\right)\left(\rho^2 - \mu^2\right) = a^2 \frac{\left(\rho^2 - \nu^2\right)\left(\rho^2 - \mu^2\right)}{\left(\rho^2 - e^2\right)\left(\rho^2 - 1\right)} \quad \text{etc.}$$

We arrive at the equalities

$$\left.\begin{aligned}
H_\rho &= a\sqrt{\frac{\left(\rho^2 - \nu^2\right)\left(\rho^2 - \mu^2\right)}{\left(\rho^2 - e^2\right)\left(\rho^2 - 1\right)}}, \\
H_\mu &= a\sqrt{\frac{\left(\rho^2 - \mu^2\right)\left(\mu^2 - \nu^2\right)}{\left(\mu^2 - e^2\right)\left(1 - \mu^2\right)}}, \\
H_\nu &= a\sqrt{\frac{\left(\rho^2 - \nu^2\right)\left(\mu^2 - \nu^2\right)}{\left(e^2 - \nu^2\right)\left(1 - \nu^2\right)}}.
\end{aligned}\right\} \tag{C.11.22}$$

Under the denotation

$$\left.\begin{aligned} \Delta(\rho) &= \sqrt{(\rho^2 - e^2)(\rho^2 - 1)}, \\ \Delta_1(\mu) &= \sqrt{(\mu^2 - e^2)(1 - \mu^2)}, \\ \Delta(\nu) &= \sqrt{(e^2 - \nu^2)(1 - \nu^2)} \end{aligned}\right\} \tag{C.11.23}$$

we write the Jacobian in the form

$$\sqrt{g} = H_\rho H_\mu H_\nu = a^3 \frac{(\rho^2 - \mu^2)(\rho^2 - \nu^2)(\mu^2 - \nu^2)}{\Delta(\rho)\Delta(\mu)\Delta(\nu)}, \tag{C.11.24}$$

and using eq. (C.5.5) we lead the expression for the Laplace operator to the form

$$\nabla^2\psi = \frac{1}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)(\mu^2 - \nu^2)} \left[(\mu^2 - \nu^2)\Delta(\rho)\frac{\partial}{\partial\rho}\Delta(\rho)\frac{\partial\psi}{\partial\rho} + \right.$$
$$\left.(\rho^2 - \nu^2)\Delta_1(\mu)\frac{\partial}{\partial\mu}\Delta_1(\mu)\frac{\partial\psi}{\partial\mu} + (\rho^2 - \mu^2)\Delta(\nu)\frac{\partial}{\partial\nu}\Delta(\nu)\frac{\partial\psi}{\partial\nu}\right]. \tag{C.11.25}$$

Referring to eqs. (C.3.5) and (C.11.19) we obtain the following formulae

$$\left.\begin{aligned} \frac{\partial\rho}{\partial x_1} &= \frac{1}{H_\rho^2}\frac{x_1}{\rho} = \frac{x_1}{\rho^3 D_\rho^2}, \\ \frac{\partial\rho}{\partial x_2} &= \frac{1}{H_\rho^2}\frac{x_2\rho}{\rho^2 - e^2} = \frac{x_2}{\rho(\rho^2 - e^2) D_\rho^2}, \\ \frac{\partial\rho}{\partial x_3} &= \frac{1}{H_\rho^2}\frac{x_3\rho}{\rho^2 - 1} = \frac{x_2}{\rho(\rho^2 - 1) D_\rho^2}. \end{aligned}\right\} \tag{C.11.26}$$

The passage to the limiting case of the oblate ellipsoid (spheroid) having the coordinates s, q, φ is carried out by assuming $e = 0, \nu \to 0$ (however v/e remains finite) and putting

$$\rho^2 = 1 + s^2, \quad \mu^2 = 1 - q^2, \quad \nu = e\cos\varphi. \tag{C.11.27}$$

Then by eq. (C.11.12) we obtain

$$x_1 = \sqrt{1 + s^2}\sqrt{1 - q^2}\cos\varphi, \quad x_2 = \sqrt{1 + s^2}\sqrt{1 - q^2}\sin\varphi, \quad x_3 = asq, \tag{C.11.28}$$

which is required (see eq. (C.10.1)). Here q denotes the coordinate denoted as μ is Section C.10.

For transforming to the coordinates of an oblong spheroid we assume

$$\rho = s, \quad e \to 1, \quad \frac{\nu}{e} \to q, \quad \sqrt{\frac{1 - \mu^2}{1 - e^2}} \to \sin\varphi, \quad \sqrt{\frac{\mu^2 - e^2}{1 - e^2}} \to \cos\varphi.$$

Then by eq. (C.11.12)

$$x_1 = asq, \quad x_2 = a\sqrt{s^2-1}\sqrt{1-q^2}\cos\varphi, \quad x_3 = a\sqrt{s^2-1}\sqrt{1-q^2}\sin\varphi \tag{C.11.29}$$

and the coordinate surfaces $s = \text{const}$ are the ellipsoids of revolution about axis x_1

$$\frac{x_1^2}{a^2 s^2} + \frac{x_2^2 + x_3^2}{a^2\left(s^2-1\right)} - 1 = 0. \tag{C.11.30}$$

Appendix D

Tensor algebra in curvilinear basis

D.1 Main basis and cobasis

Three noncomplanar vectors denoted by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are introduced into consideration. They are not unit vectors and not mutually orthogonal ones. The volume of the parallelepiped spanned by these vectors is equal to

$$v = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) . \tag{D.1.1}$$

Arranging the numbering of the vectors we can obtain $v > 0$. Vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form the main basis. The cobasis is introduced by the vectors

$$\mathbf{e}^1 = \frac{1}{v}\mathbf{e}_2 \times \mathbf{e}_3, \quad \mathbf{e}^2 = \frac{1}{v}\mathbf{e}_3 \times \mathbf{e}_1, \quad \mathbf{e}^3 = \frac{1}{v}\mathbf{e}_1 \times \mathbf{e}_2, \tag{D.1.2}$$

so that

$$\mathbf{e}^s \cdot \mathbf{e}_k = g_k^s = \begin{cases} 0, & s \neq k, \\ 1, & s = k. \end{cases} \tag{D.1.3}$$

The scalar products of the vectors of the main basis and cobasis are also introduced into consideration

$$\mathbf{e}_k \cdot \mathbf{e}_s = g_{ks} = g_{sk}, \quad \mathbf{e}^k \cdot \mathbf{e}^s = g^{ks} = g^{sk}. \tag{D.1.4}$$

It is straightforward to prove that the basis which is the reciprocal to the cobasis is the main basis. Indeed

$$v^* = \mathbf{e}^1 \cdot \left(\mathbf{e}^2 \times \mathbf{e}^3\right) = \frac{1}{v^2}\mathbf{e}^1 \cdot \left[(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2)\right] = \frac{1}{v}\mathbf{e}^1 \cdot \mathbf{e}_1 = \frac{1}{v}, \tag{D.1.5}$$

and thus

$$\frac{1}{v^2}\mathbf{e}^2 \times \mathbf{e}^3 = \frac{1}{v}(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2) = \mathbf{e}_1,$$

which is required.

D.2 Vectors in an oblique basis

Vector $\mathbf{a}$ can be presented by the expansion in both the main basis and the cobasis

$$\mathbf{a} = a^s \mathbf{e}_s = a_s \mathbf{e}^s, \tag{D.2.1}$$

where s is a dummy index. In the earlier denotation when the orthogonal Cartesian coordinates were used, there was no need to distinguish between the upper and lower indices. In the general tensor analysis the dummy indices always have a superscript and a subscript whereas the free indices have the same position in both sides of the formula. No summation is carried out over two superscripts or two subscripts. For example, $g_3^3 = 3$ has three terms while g_{ss} means a single term (the value of g_{st} at $s = t$).

Using eqs. (D.2.1) and (D.1.3) we obtain

$$a^s = \mathbf{a} \cdot \mathbf{e}^s, \quad a_s = \mathbf{a} \cdot \mathbf{e}_s. \tag{D.2.2}$$

The quantities a^s and a_s are referred to as the contravariant and covariant components of vector $\mathbf{a}$ respectively. They are equal to the projections of vector $\mathbf{a}$ onto the vectors of the main basis and the cobasis multiplied by the absolute value of the corresponding vectors

$$|\mathbf{e}^s| = \sqrt{g^{ss}}, \quad |\mathbf{e}_s| = \sqrt{g_{ss}}. \tag{D.2.3}$$

Another interpretation relies on representation (D.2.1). Each term in sums $a^s\mathbf{e}_s$ and $a_s\mathbf{e}^s$ is an edge of the skew-angled parallelepiped built on the vectors of the main basis and cobasis, $\mathbf{a}$ being the diagonal of this parallelepiped. This is shown in Fig. D.1 in which vector $\mathbf{e}_3$ is perpendicular to the plane of vectors $\mathbf{e}_1, \mathbf{e}_2$ and vector $\mathbf{a}$ lies in this plane. An explanation of the terms "covariant" and "contravariant" is given later, see Section D.6.

The formulae relating the covariant and contravariant components follow from eqs. (D.1.3) and (D.1.4)

$$a^s = g^{sk} a_k, \quad a^s = g_{sk} a^k. \tag{D.2.4}$$

The quantities

$$\frac{a^s}{\sqrt{g_{ss}}} = a_{(s)}, \quad \frac{a^s}{\sqrt{g^{ss}}} = a^{(s)} \tag{D.2.5}$$

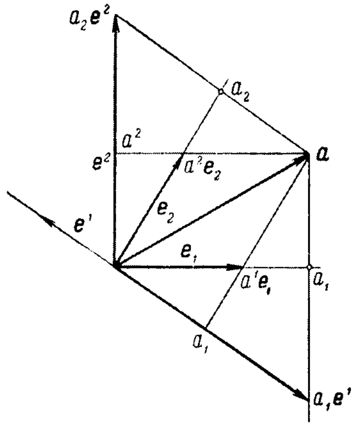

FIGURE D.1.

are called the physical components of $\mathbf{a}$ and they are equal to the projection of this vector onto $\mathbf{e}_s$ and $\mathbf{e}^s$. The square of the magnitude of the vector can be presented in the following form

$$\left.\begin{aligned} a^2 &= \mathbf{a}\cdot\mathbf{a} = a^s\mathbf{e}_s \cdot a^k\mathbf{e}_k = g_{sk}a^s a^k, \\ a^2 &= \mathbf{a}\cdot\mathbf{a} = a_s\mathbf{e}^s \cdot a_k\mathbf{e}^k = g^{sk}a_s a_k, \\ a^2 &= \mathbf{a}\cdot\mathbf{a} = a_s\mathbf{e}^s \cdot a^k\mathbf{e}_k = g_s^k a_s a^k = a_s a^s. \end{aligned}\right\} \tag{D.2.6}$$

D.3 Metric tensor

It follows from formulae (D.2.6) that the values g_{sk} (or g^{sk}) are the coefficients of the invariant quadratic form of variables a^s (or a_s). Hence according to Section A.4 one concludes that these values determine a symmetric tensor of second rank denoted as $\hat{g}$. Its co- and contravariant components are g_{sk} and g^{sk} respectively and the mixed components g_k^s are coefficients of the bilinear form of variables a_s and a^k. In the taken basis tensor $\hat{g}$ determines the square of the length which explains why it is called a metric tensor. The dyadic representation of tensor $\hat{g}$ is set in three following forms

$$\hat{g} = g^{sk}\mathbf{e}_s\mathbf{e}_k = g_{sk}\mathbf{e}^s\mathbf{e}^k = g_k^s\mathbf{e}^k\mathbf{e}_s = \mathbf{e}^s\mathbf{e}_s = \mathbf{e}_s\mathbf{e}^s. \tag{D.3.1}$$

They yield

$$g^{sk}\mathbf{e}^m \cdot \mathbf{e}_s\mathbf{e}_k \cdot \mathbf{e}_q = \mathbf{e}^m \cdot \mathbf{e}_r\mathbf{e}^r \cdot \mathbf{e}_q; \quad g^{sk}g_s^m g_{kq} = g_r^m g_q^r,$$

or

$$g^{mk}g_{kq} = g_q^m. \tag{D.3.2}$$

The latter formula determines the rule of contraction over the dummy index using the metric tensor whilst formulae (D.2.4) explain the transformation from the covariant to contravariant components (and back) by multiplying by g^{sk} (or g_{sk}) with a further contraction with respect to the dummy index.

The metric tensor in the oblique basis plays the role of the unit tensor. It follows from the fact that postmultiplying and premultiplying it by vector $\mathbf{a}$ yields the same vector

$$\left.\begin{aligned} \hat{g}\cdot\mathbf{a} = g_{sk}\mathbf{e}^s\mathbf{e}^k\cdot\mathbf{a} = \mathbf{e}^s g_{sk}a^k = \mathbf{e}^s a_s = \mathbf{a}, \\ \mathbf{a}\cdot\hat{g} = \mathbf{a}\cdot g^{sk}\mathbf{e}_s\mathbf{e}_k = a_s g^{sk}\mathbf{e}_k = a^k\mathbf{e}_k = \mathbf{a}. \end{aligned}\right\} \tag{D.3.3}$$

It is much easier to prove it by utilising the bilinear representation

$$\mathbf{a}\cdot\hat{g} = \mathbf{a}\cdot\mathbf{e}^s\mathbf{e}_s = a^s\mathbf{e}_s = \mathbf{a}, \quad \hat{g}\cdot\mathbf{a} = \mathbf{e}^s\mathbf{e}_s\cdot\mathbf{a} = \mathbf{e}^s a_s = \mathbf{a}.$$

Let us denote, for the time being, tensor $\hat{g}$ in the contravariant representation by $\hat{g}^*$. Then

$$\begin{aligned} \hat{g}\cdot\hat{g}^* &= g_{sk}\mathbf{e}^s\mathbf{e}^k\cdot g^{rt}\mathbf{e}_r\mathbf{e}_t = g_{sk}g^{rt}g^k_r\mathbf{e}^s\mathbf{e}_t = g_{st}g^{kt}\mathbf{e}^s\mathbf{e}_t \\ &= g^t_s\mathbf{e}^s\mathbf{e}_t = \mathbf{e}^s\mathbf{e}_t = \hat{g} = \hat{E}, \end{aligned}$$

so that

$$\hat{g}^* = \hat{g}^{-1}. \tag{D.3.4}$$

This should be expected since the unit tensor is equal to its inverse. This property holds in any coordinate basis and $\hat{g}^*$ is just another denotation for tensor $\hat{g}$.

It follows from eq. (D.3.4) that matrices $\|g_{sk}\|$ and $\|g^{sk}\|$ are inverse matrices. Hence denoting the algebraic adjunct of element g_{sk} of the first matrix by $A^{sk} = A^{ks}$ we have

$$g^{sk} = \frac{A^{sk}}{g}, \tag{D.3.5}$$

where $g = |g_{sk}|$ denotes the determinant of the matrix of the covariant components of the metric tensor. It is easy to obtain this result by considering the product of the determinants g and g^* of matrices $\|g_{st}\|$ and $\|g^{st}\|$ respectively.

The area $\overset{1}{o}$ of the parallelogram spanned by the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ can be presented in the following two ways

$$\begin{aligned} &\overset{1}{o} = |\mathbf{e}_2\times\mathbf{e}_3| = v\left|\mathbf{e}^1\right| = v\sqrt{g^{11}}, \\ &\overset{1}{o} = \sqrt{\mathbf{e}_2\cdot\mathbf{e}_2\mathbf{e}_3\cdot\mathbf{e}_3 - (\mathbf{e}_2\cdot\mathbf{e}_3)^2} = \sqrt{g_{22}g_{33} - g_{23}^2} = \sqrt{A^{11}} = \sqrt{g^{11}g}. \end{aligned}$$

Hence

$$v = \sqrt{g}, \quad \overset{s}{o} = \sqrt{g^{ss}g}. \tag{D.3.6}$$

D.4 The Levi-Civita tensor

The components of this tensor in the oblique main basis and cobasis are given, by analogy to eq. (A.2.1) by the formulae

$$\epsilon_{skt} = \mathbf{e}_s \cdot (\mathbf{e}_k \times \mathbf{e}_t), \quad \epsilon^{skt} = \mathbf{e}^s \cdot \left(\mathbf{e}^k \times \mathbf{e}^t\right). \tag{D.4.1}$$

They are equal to zero if there are coincident indices among skt. They are equal to $\sqrt{g}$ and $1/\sqrt{g}$ in the first and second definition (D.4.1) if the indices have the order of 123, 231, 312. When this order is violated they are equal to$-\sqrt{g}$ and $-1/\sqrt{g}$ respectively. The above-said can be written down as follows

$$\epsilon_{skt} = \sqrt{g} e_{skt}, \quad \epsilon^{skt} = \frac{1}{\sqrt{g}} e^{skt}, \tag{D.4.2}$$

where symbols e_{skt}, e^{skt} are equal to zero or ± 1 according to the above rules.

Using these definitions we can construct expressions for the covariant and contravariant components of the vector product

$$\begin{aligned} \mathbf{c} = \mathbf{a} \times \mathbf{b} = a^k b^t \mathbf{e}_k \times \mathbf{e}_t, \quad c_s = \mathbf{c} \cdot \mathbf{e}_s = a^k b^t \epsilon_{kts}, \\ \mathbf{c} = \mathbf{a} \times \mathbf{b} = a_k b_t \mathbf{e}^k \times \mathbf{e}^t, \quad c^s = \mathbf{c} \cdot \mathbf{e}^s = a_k b_t \epsilon^{kts}, \end{aligned}$$

so that

$$c_s = \epsilon_{skt} a^k b^t = \sqrt{g} e_{skt} a^k b^t, \quad c^s = \epsilon^{skt} a_k b_t = \frac{1}{\sqrt{g}} e^{skt} a_k b_t. \tag{D.4.3}$$

In particular

$$\mathbf{e}_k \times \mathbf{e}_s = \sqrt{g} e_{kst} \mathbf{e}_t, \quad \mathbf{e}^k \times \mathbf{e}^s = \frac{1}{\sqrt{g}} e^{skt} \mathbf{e}_t, \tag{D.4.4}$$

and the inverse formulae take the form

$$\mathbf{e}^t = \frac{1}{2\sqrt{g}} e^{kst} \mathbf{e}_k \times \mathbf{e}_s, \quad \mathbf{e}_t = \frac{1}{2} \sqrt{g} e_{kst} \mathbf{e}^k \times \mathbf{e}^s. \tag{D.4.5}$$

D.5 Tensors in an oblique basis

With the help of the vectors of the main basis and the cobasis, one constructs four types of dyadics

$$\mathbf{e}_s \mathbf{e}_k, \quad \mathbf{e}^s \mathbf{e}^k, \quad \mathbf{e}_s \mathbf{e}^k, \quad \mathbf{e}^s \mathbf{e}_k. \tag{D.5.1}$$

The corresponding expressions are as follows

$$p^{sk} \mathbf{e}_s \mathbf{e}_k, \quad p_{sk} \mathbf{e}^s \mathbf{e}^k, \quad p^s_{\cdot k} \mathbf{e}_s \mathbf{e}^k, \quad p_s^{\cdot k} \mathbf{e}^s \mathbf{e}_k. \tag{D.5.2}$$

If postmultiplication of these four expressions by vector $\mathbf{a}$ leads to the same vector $\mathbf{b}$ then they are nothing more than the different expressions for the same invariant quantity termed the tensor of second rank $\hat{P}$

$$\mathbf{b} = \hat{P} \cdot \mathbf{a} = p^{sk}\mathbf{e}_s a_k = p_{sk}\mathbf{e}^s a^k = p^{s}_{\cdot k}\mathbf{e}_s a^k = p_s^{\cdot k}\mathbf{e}^s a_k. \tag{D.5.3}$$

Here

$$p^{sk}, \quad p_{sk}, \quad p^{s}_{\cdot k}, \quad p_s^{\cdot k}$$

determines the contravariant, covariant, contracovariant and cocontravariant components of this tensor. The relations between them are easy to establish. Indeed, by eqs. (D.5.3) and (D.2.4)

$$b^s = p^{sk}a_k = g^{sr}b_r = g^{sr}p_{rm}a^m = p_{rm}g^{sr}g^{km}a_k,$$

so that

$$p^{sk} = g^{sr}g^{km}p_{rm}$$

and further by eq. (D.3.2)

$$p^{sk}g_{sq}g_{kt} = g^{sr}g_{sq}g^{km}g_{kt}p_{rm} = g_q^r g_t^m p_{rm} = p_{qt}.$$

We arrive at the relationships

$$p^{sk} = g^{sr}g^{km}p_{rm}, \quad p_{sk} = g_{sr}g_{km}p^{rm}, \quad p^{s}_{\cdot k} = g^{sr}p_{rk}, \quad p_s^{\cdot k} = g^{kr}p_{st} \tag{D.5.4}$$

etc. confirming the above-obtained rules of operations on the indices.

For the symmetric tensor $p^{sk} = p^{ks}, p_{sk} = p_{ks}$ there is no need to indicate the position of the index of the mixed components $\left(p^{s}_{\cdot k} = p_s^{\cdot k} = p_s^k\right)$. The property of the tensor to be symmetric is invariant with respect to the choice of the basis. The tensor which is symmetric in the orthogonal basis of the unit vectors $\mathbf{i}_s$ remains symmetric in the oblique basis. Indeed, denoting the components of $\hat{P}$ in basis $\mathbf{i}_s$ by $p_{(st)}$ we have

$$\begin{aligned} \mathbf{e}_i \cdot \hat{P} \cdot \mathbf{e}_k = p^{ik} &= p_{(st)}\mathbf{e}_i \cdot \mathbf{i}_s\mathbf{i}_t \cdot \mathbf{e}_k = p_{(ts)}\mathbf{e}_i \cdot \mathbf{i}_t\mathbf{i}_s \cdot \mathbf{e}_k \\ &= p_{(st)}\mathbf{e}_i \cdot \mathbf{i}_t\mathbf{i}_s \cdot \mathbf{e}_k = p^{ki}. \end{aligned} \tag{D.5.5}$$

One can prove that $p_{ik} = p_{ki}$ by analogy.

D.6 Transformation of basis

The vectors of the new basis are denoted as $\mathbf{e}'_s$. They are related by the linear relationships to the vectors of the original basis

$$\mathbf{e}'_s = c_s^{\prime r}\mathbf{e}_r, \quad c_s^{\prime r} = \mathbf{e}'_s \cdot \mathbf{e}^r \tag{D.6.1}$$

with the nonvanishing determinant $\left|c_s^{\prime r}\right|$. The inverse relations are as follows

$$\mathbf{e}_s = c_s^r \mathbf{e}'_r, \quad c_s^r = \mathbf{e}_s \cdot \mathbf{e}'^r, \tag{D.6.2}$$

so that

$$\mathbf{e}'_s = c_s^{\prime r} c_r^k \mathbf{e}'_k, \quad c_s^{\prime r} c_r^k = \delta_s^k = \begin{cases} 0, & s \neq k, \\ 1, & s = k. \end{cases} \tag{D.6.3}$$

The formulae determining vectors $\mathbf{e}'^q$ of the cobasis are obtained by referring to eqs. (D.3.3) and (D.6.2)

$$\left.\begin{aligned} \mathbf{e}'^q &= \hat{g} \cdot \mathbf{e}'^q = \mathbf{e}^r \mathbf{e}_r \cdot \mathbf{e}'^q = \mathbf{e}^r c_r^q, \\ \mathbf{e}^q &= \hat{g}' \cdot \mathbf{e}^q = \mathbf{e}'^r \mathbf{e}'_r \cdot \mathbf{e}^q = \mathbf{e}'^r c_r^{\prime q}. \end{aligned}\right\} \tag{D.6.4}$$

Now we can present vector $\mathbf{a}$ in terms of the covariant components, such that

$$\mathbf{a} = a_q \mathbf{e}^q = a_q \mathbf{e}'^r c_r^{\prime q} = a'_r \mathbf{e}'^r, \quad a'_r = a_q c_r^{\prime q}, \tag{D.6.5}$$

and comparison with eq. (D.6.1) shows that these components are transformed as the base vectors which explains the origin of "covariant". The contravariant components are transformed as vectors of the cobasis

$$\mathbf{a} = a^s \mathbf{e}_s = a^s c_s^r \mathbf{e}'_r = a'^r \mathbf{e}'_r, \quad a'^r = a^s c_s^r. \tag{D.6.6}$$

By analogy we obtain the formulae for transformation of the tensor components

$$p'_{st} = c_s^{\prime r} c_t^{\prime q} p_{rq}, \quad p'^{st} = c_r^s c_q^t p^{rq}, \quad p'^{s}_{\cdot t} = c_r^s c_t^{\prime q} p^r_{\cdot q}. \tag{D.6.7}$$

D.7 Principal axes and principal invariants of symmetric tensor

Based upon the invariant definition of Section A.9 of the principal directions of the tensor

$$\hat{P} \cdot \mathbf{n} = \lambda \mathbf{n},$$

where $\mathbf{n}$ denotes the unit vector we have

$$p_s^t \mathbf{e}^s \mathbf{e}_t \cdot \mathbf{n} = p_s^t \mathbf{e}^s n_t = \lambda \mathbf{e}^t n_t = \lambda g_s^t \mathbf{e}^s n_t. \tag{D.7.1}$$

We arrive at the system of three equations

$$\left(p_s^t - \lambda g_s^t\right) n_t = 0, \tag{D.7.2}$$

in which the unknowns n_t are related by the additional equality

$$\mathbf{n} \cdot \mathbf{n} = g^{rt} n_r n_t = 1. \tag{D.7.3}$$

The characteristic equation for the tensor

$$P_3(\lambda) = \left| p_s^{\cdot t} - \lambda g_s^{t} \right| = 0 \tag{D.7.4}$$

differs from eq. (A.9.5) only in that the role of the components in the orthogonal system is now played by the mixed components. For this reason it is sufficient to modify the expressions for the principal components (A.10.4), (A.10.6) and use the formulae of transformation (D.5.4). The result is as follows

$$I_1\left(\hat{P}\right) = p_s^{\cdot s} = g^{sk} p_{ks} = g_{sk} p^{ks}, \tag{D.7.5}$$

$$I_3\left(\hat{P}\right) = \left|p_t^{\cdot r}\right| = \left|g^{rk} p_{kt}\right| = \left|g^{rk}\right| \left|p_{kt}\right| = \frac{1}{g} \left|p_{kt}\right| = \left|g_{tk} p^{kr}\right| = g \left|p^{kr}\right|. \tag{D.7.6}$$

The second invariant is obtained by using eq. (A.10.14)

$$I_2\left(\hat{P}\right) = I_1\left(\hat{P}^{-1}\right) I_3\left(\hat{P}\right). \tag{D.7.7}$$

Here tensor $\hat{P}^{-1}$, by eq. (A.7.8), is given by the expression

$$\hat{P}^{-1} = \tilde{p}^{st} \mathbf{e}_s \mathbf{e}_t, \tag{D.7.8}$$

where

$$\tilde{p}^{st} p_{tq} = \delta_q^s. \tag{D.7.9}$$

Then

$$\hat{P}^{-1} \cdot \hat{P} = \tilde{p}^{sr} \mathbf{e}_s \mathbf{e}_r \cdot p_{tq} \mathbf{e}^t \mathbf{e}^q = \tilde{p}^{sr} p_{rq} \mathbf{e}_s \mathbf{e}^q = \delta_q^s \mathbf{e}_s \mathbf{e}^q = \mathbf{e}_s \mathbf{e}^s = \hat{g},$$

which is required since $\hat{g}$ is the unit tensor. It is a erroneous to identify $\tilde{p}^{ts}$ with the contravariant components of $\hat{P}$. The latter are obtained in terms of its covariant components with the help of eq. (D.5.4) whereas determining $\tilde{p}^{ts}$ requires construction of the matrix inverse to $\|p_{st}\|$.

By eqs. (D.7.5) and (D.7.7) we obtain

$$I_2\left(\hat{P}\right) = \frac{p}{g} g_{st} \tilde{p}^{ts}, \quad p = \left|p_{st}\right|. \tag{D.7.10}$$

Another form for the second invariant is obtained by means of eq. (A.10.10). We have

$$\hat{P}^2 = p_{st} p_{qr} \mathbf{e}^s \mathbf{e}^r g^{tq}, \quad I_1\left(\hat{P}^2\right) = g^{sr} g^{tq} p_{st} p_{qr}$$

and thus

$$I_2\left(\hat{P}\right) = \frac{1}{2}\left|I_1^2\left(\hat{P}\right) - I_1\left(\hat{P}^2\right)\right| = \frac{1}{2}\left(g^{st}g^{qr} - g^{sr}g^{tq}\right)p_{st}p_{qr}. \quad \text{(D.7.11)}$$

The value in the parentheses can also be written as follows

$$\begin{aligned} \mathbf{e}^s \cdot \left(\mathbf{e}^t\mathbf{e}^q \cdot \mathbf{e}^r - \mathbf{e}^r\mathbf{e}^t \cdot \mathbf{e}^q\right) &= \mathbf{e}^s \cdot \left[\mathbf{e}^q \times \left(\mathbf{e}^t \times \mathbf{e}^r\right)\right] \\ &= \left(\mathbf{e}^s \times \mathbf{e}^q\right) \cdot \left(\mathbf{e}^t \times \mathbf{e}^r\right) = \epsilon^{sqm}\epsilon^{trn}\mathbf{e}_m \cdot \mathbf{e}_n, \end{aligned}$$

so that by eq. (D.4.2)

$$I_2\left(\hat{P}\right) = \frac{1}{2g}g_{mn}e^{sqm}e^{trn}p_{st}p_{qr}. \quad \text{(D.7.12)}$$

It is evident that this formula can be obtained directly from eq. (D.7.10) by utilising definition (A.7.11) for the components of the inverse tensor.

Appendix E

Operations of tensor analysis in curvilinear coordinates

E.1 Introducing the basis

In contrast to the previous denotation, in this Appendix we denote the Cartesian coordinates and the position vector of a point as a_1, a_2, a_3 and $\mathbf{r}$

$$\mathbf{r} = \mathbf{i}_s a_s \tag{E.1.1}$$

respectively. The curvilinear coordinates are denoted as q^1, q^2, q^3 such that

$$a_s = a_s\left(q^1, q^2, q^3\right), \quad \mathbf{r} = \mathbf{r}\left(q^1, q^2, q^3\right), \tag{E.1.2}$$

where in the domain of the definition, the Jacobian

$$J = \left|\frac{\partial a_s}{\partial q^k}\right| \tag{E.1.3}$$

differs from zero and is positive.

The main vector basis is described by the triple of vectors

$$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial q^s}, \tag{E.1.4}$$

and relationships (D.1.2) and (D.4.5) introduce the cobasis

$$\mathbf{r}^k = \frac{1}{2}\epsilon^{kst}\mathbf{r}_s \times \mathbf{r}_t = \frac{1}{2\sqrt{g}}e^{kst}\mathbf{r}_s \times \mathbf{r}_t. \tag{E.1.5}$$

The metric tensor is introduced as follows

$$\hat{g} = g_{sk}\mathbf{r}^s\mathbf{r}^k = g^{sk}\mathbf{r}_s\mathbf{r}_k = g_k^s\mathbf{r}_s\mathbf{r}^k = \mathbf{r}_s\mathbf{r}^s, \tag{E.1.6}$$

where the values

$$g_{sk} = \mathbf{r}_s \cdot \mathbf{r}_k, \quad g^{sk} = \mathbf{r}^s \cdot \mathbf{r}^k, \quad g_k^s = \mathbf{r}^s \cdot \mathbf{r}_k \tag{E.1.7}$$

represent its covariant, contravariant and mixed components respectively.

The infinitesimally small vector $d\mathbf{r}$ is determined by the evident equation

$$d\mathbf{r} = \mathbf{r}_s dq^s, \tag{E.1.8}$$

whereas the square of its length, i.e. the square of the linear element, is expressed in terms of the covariant components of the metric tensor

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \mathbf{r}_s dq^s \cdot \mathbf{r}_k dq^k = g_{sk} dq^s dq^k. \tag{E.1.9}$$

All results of Appendix D remain valid provided that $\mathbf{e}_s$ is replaced by $\mathbf{r}_s$.

We show only the expressions for the element of volume

$$d\tau = \mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)\, dq^1 dq^2 dq^3 = \sqrt{g} dq^1 dq^2 dq^3 \tag{E.1.10}$$

and the elements of surfaces

$$d\overset{1}{o} = \sqrt{gg^{11}} dq^2 dq^3, \quad d\overset{2}{o} = \sqrt{gg^{22}} dq^3 dq^1, \quad d\overset{3}{o} = \sqrt{gg^{33}} dq^1 dq^2, \tag{E.1.11}$$

where

$$g = |g_{st}|, \quad \sqrt{g} = \frac{1}{6} e^{stq} \mathbf{r}_s \cdot (\mathbf{r}_t \times \mathbf{r}_q) = J. \tag{E.1.12}$$

Formulae (E.1.11) can be presented in the unified form

$$d\overset{s}{o} = \sqrt{g^{ss}} \frac{d\tau}{dq^s}. \tag{E.1.13}$$

E.2 Derivatives of base vectors

The vectors

$$\mathbf{r}_{st} = \frac{\partial \mathbf{r}_s}{\partial q^t} = \frac{\partial^2 \mathbf{r}}{\partial q^t \partial q^s} = \mathbf{r}_{ts} \tag{E.2.1}$$

can be represented by expansions in terms of the base vectors

$$\mathbf{r}_{st} = \mathbf{r}_{ts} = \left\{ {k \atop st} \right\} \mathbf{r}_k. \tag{E.2.2}$$

The expressions in the braces are named Christoffel's symbols of second kind and are denoted as follows

$$\Gamma_{st}^{k} = \left\{ \begin{matrix} k \\ st \end{matrix} \right\}. \tag{E.2.3}$$

One can see that in formula (E.2.2) the subscript is the free index. The symmetry with respect to the subscripts follows from definition (E.2.2)

$$\left\{ \begin{matrix} k \\ st \end{matrix} \right\} = \left\{ \begin{matrix} k \\ ts \end{matrix} \right\}, \tag{E.2.4}$$

so that the total number of symbols is equal to 18. By eq. (E.2.2) we have

$$\mathbf{r}_{st} \cdot \mathbf{r}_q = g_{qk} \left\{ \begin{matrix} k \\ st \end{matrix} \right\}. \tag{E.2.5}$$

The scalar products on the left hand side are introduced in Section C.3 and termed Christoffel's symbols of first kind. They are given by eq. (C.2.7)

$$\mathbf{r}_{st} \cdot \mathbf{r}_q = [st, q] = [ts, q] = \frac{1}{2} \left(\frac{\partial g_{sq}}{\partial q^t} + \frac{\partial g_{tq}}{\partial q^s} - \frac{\partial g_{st}}{\partial q^q} \right). \tag{E.2.6}$$

Returning to eq. (E.2.5) we have

$$[st, q] = g_{kq} \left\{ \begin{matrix} k \\ st \end{matrix} \right\}, \quad \left\{ \begin{matrix} r \\ st \end{matrix} \right\} = g^{rq} [st, q], \tag{E.2.7}$$

where the second set of formulae is obtained from the first one with the help of eq. (D.3.2).

The derivatives of the vectors of the main basis are now obtained. In order to derive the derivatives of the vectors of the cobasis we begin with the relationship

$$\frac{\partial}{\partial q^t} g_k^s = \frac{\partial}{\partial q^t} \mathbf{r}^s \cdot \mathbf{r}_k = \frac{\partial \mathbf{r}^s}{\partial q^t} \cdot \mathbf{r}_k + \mathbf{r}^s \cdot \left\{ \begin{matrix} m \\ tk \end{matrix} \right\} \mathbf{r}_m = 0.$$

It yields

$$\frac{\partial \mathbf{r}^s}{\partial q^t} \cdot \mathbf{r}_k = - \left\{ \begin{matrix} s \\ tk \end{matrix} \right\} \tag{E.2.8}$$

and referring to eqs. (D.2.2) and (D.2.1) we find

$$\frac{\partial \mathbf{r}^s}{\partial q^t} = - \left\{ \begin{matrix} s \\ tq \end{matrix} \right\} \mathbf{r}^q. \tag{E.2.9}$$

E.3 Covariant differentiation

Carrying out operations on the vector and tensor quantities requires the coordinate basis to be introduced and the covariant, contravariant and mixed components to be considered. Changes in the invariant quantity (scalar, vector, tensor) due to changes in the position of the point are caused only by properties of the invariant. A different situation arises with the components of this invariant since their change depends also on the values and directions of the base vectors. For example, let the contravariant components a^k of vector $\mathbf{a}$ be independent of coordinates q^k. Then the derivatives with respect to these variables are equal to zero. However it is erroneous to think that vector $\mathbf{a}$ does not change. The inverse statement is also true: the components a_s and a^s of a constant vector $\mathbf{a}$ do not remain constant. The goal of further analysis is to introduce such characteristics of vectors and tensors which take into account change in both the quantities and the coordinate basis. This is achieved by means of the operation of covariant (or absolute) differentiation.

Let us consider the derivative of $\mathbf{a}$ with respect to q^s and begin with the case in which $\mathbf{a}$ is given by the contravariant components. Then we have

$$\frac{\partial \mathbf{a}}{\partial q^s} = \frac{\partial}{\partial q^s} a^k \mathbf{r}_k = \frac{\partial a^k}{\partial q^s}\mathbf{r}_k + a^k \left\{ \begin{matrix} t \\ sk \end{matrix} \right\} \mathbf{r}_t$$

or after replacing the dummy indices

$$\frac{\partial \mathbf{a}}{\partial q^s} = \left(\frac{\partial a^k}{\partial q^s} + \left\{ \begin{matrix} k \\ st \end{matrix} \right\} a^t \right) \mathbf{r}_k. \tag{E.3.1}$$

Referring to eq. (E.2.9) we obtain by analogy

$$\frac{\partial \mathbf{a}}{\partial q^s} = \left(\frac{\partial a_k}{\partial q^s} - \left\{ \begin{matrix} t \\ sk \end{matrix} \right\} a_t \right) \mathbf{r}^k. \tag{E.3.2}$$

The expressions

$$\nabla_s a^k = \frac{\partial a^k}{\partial q^s} + \left\{ \begin{matrix} k \\ st \end{matrix} \right\} a^t, \quad \nabla_s a_k = \frac{\partial a_k}{\partial q^s} - \left\{ \begin{matrix} t \\ sk \end{matrix} \right\} a_t \tag{E.3.3}$$

are referred to as the covariant (absolute) derivatives of the contravariant and covariant components of vector $\mathbf{a}$. Under the above denotation we have

$$\frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{r}_k \nabla_s a^k = \mathbf{r}^k \nabla_s a_k. \tag{E.3.4}$$

The quantities $\nabla_s a^k$ and $\nabla_s a_k$ present respectively the contravariant and covariant components of vector $\partial \mathbf{a}/\partial q^s$.

This is generalised to the tensors of any rank. For example, if we describe the tensor of second rank by the contravariant components then we have

$$\frac{\partial \hat{P}}{\partial q^r} = \frac{\partial}{\partial q^r} p^{sk}\mathbf{r}_s\mathbf{r}_k = \frac{\partial p^{sk}}{\partial q^r}\mathbf{r}_s\mathbf{r}_k + p^{sk}\left(\left\{ {t \atop sr} \right\}\mathbf{r}_t\mathbf{r}_k + \left\{ {t \atop kr} \right\}\mathbf{r}_s\mathbf{r}_t\right)$$
$$= \left(\frac{\partial p^{sk}}{\partial q^r} + \left\{ {s \atop qr} \right\} p^{qk} + \left\{ {k \atop qr} \right\} p^{sq}\right)\mathbf{r}_s\mathbf{r}_k,$$

or

$$\frac{\partial \hat{P}}{\partial q^r} = \mathbf{r}_s\mathbf{r}_k\nabla_r p^{sk}, \quad \nabla_r p^{sk} = \frac{\partial p^{sk}}{\partial q^r} + \left\{ {s \atop rq} \right\} p^{qk} + \left\{ {k \atop rq} \right\} p^{sq}. \tag{E.3.5}$$

By analogy we obtain

$$\frac{\partial \hat{P}}{\partial q^r} = \mathbf{r}^s\mathbf{r}^k\nabla_r p_{sk}, \quad \nabla_r p_{sk} = \frac{\partial p_{sk}}{\partial q^r} - \left\{ {q \atop rs} \right\} p_{qk} - \left\{ {q \atop rk} \right\} p_{sq}, \tag{E.3.6}$$

$$\frac{\partial \hat{P}}{\partial q^r} = \mathbf{r}^s\mathbf{r}_k\nabla_r p_s^{\cdot k}, \quad \nabla_r p_s^{\cdot k} = \frac{\partial p_s^{\cdot k}}{\partial q^r} - \left\{ {q \atop rs} \right\} p_q^{\cdot k} + \left\{ {k \atop rq} \right\} p_s^{\cdot q}. \tag{E.3.7}$$

Tensor $\partial\hat{P}/\partial q^r$ is presented here by the contravariant, covariant and mixed components.

Of frequent use is Ricci's theorem: the covariant derivative of components of the metric tensor is equal to zero. This follows from the relationship

$$\frac{\partial}{\partial q^t}\hat{g} = \frac{\partial}{\partial q^t}\mathbf{e}^s\mathbf{e}_s = -\left\{ {s \atop tr} \right\}\mathbf{e}^r\mathbf{e}_s + \left\{ {r \atop ts} \right\}\mathbf{e}^s\mathbf{e}_r$$
$$= -\left\{ {s \atop tr} \right\}\mathbf{e}^r\mathbf{e}_s + \left\{ {s \atop tr} \right\}\mathbf{e}^r\mathbf{e}_s = 0$$

or

$$\frac{\partial}{\partial q^t}\hat{g} = \mathbf{r}^s\mathbf{r}^k\nabla_t g_{sk} = \mathbf{r}_s\mathbf{r}_k\nabla_t g^{sk} = \mathbf{r}^s\mathbf{r}_k\nabla_t g_s^k = 0.$$

Hence,

$$\nabla_t g_{sk} = 0, \quad \nabla_t g^{sk} = 0, \quad \nabla_t g_s^k = 0, \tag{E.3.8}$$

which is required. It should be expected as the metric tensor plays the role of the unit tensor.

In covariant differentiation the components of the metric tensor play the role of the constants, i.e. they can be placed behind the symbol ∇_s (however not $\partial/\partial q^t$). For instance,

$$\frac{\partial \hat{P}}{\partial q^t} = \mathbf{r}^s\mathbf{r}^k\nabla_t p_{sk} = \mathbf{r}^s\mathbf{r}^k\nabla_t g_{ms}g_{qk}p^{mq}$$
$$= g_{ms}\mathbf{r}^s g_{qk}\mathbf{r}^k\nabla_t p^{mq} = \mathbf{r}_m\mathbf{r}_q\nabla_t p^{mq} = \frac{\partial \hat{P}}{\partial q^t},$$

which is to be expected since the derivative of the invariant quantity (tensor $\hat{P}$) is an invariant which is independent of the way it is described (in terms of the covariant $\nabla_t p_{sk}$ or contravariant $\nabla_t p^{mq}$ components).

In covariant differentiation, the rule of differentiation of the product holds, i.e.

$$\nabla_r a_s b_t = (\nabla_r a_s)\, b_t + a_s \nabla_r b_t. \tag{E.3.9}$$

The covariant derivative of the Levi-Civita tensor vanishes. For example, considering the covariant components we have

$$\nabla_r \epsilon_{stq} = 0. \tag{E.3.10}$$

In the extended form this equality has the form

$$\frac{\partial}{\partial q^r}\epsilon_{stq} - \left(\left\{ {m \atop sr} \right\} \epsilon_{mtq} + \left\{ {m \atop tr} \right\} \epsilon_{smq} + \left\{ {m \atop qr} \right\} \epsilon_{stm} \right) = 0.$$

As stq denotes a triple of different indices, then only one of three triples

$$mtq, \quad smq, \quad stm$$

has no repeating indices, namely at $m = s, m = t$ and $m = q$ in the first, second and third triples respectively. Let the first triple be such a triple. Noticing that

$$\frac{\partial}{\partial q^r}\epsilon_{stq} = \frac{\partial \sqrt{g}}{\partial q^r} e_{stq} = \frac{\partial \ln \sqrt{g}}{\partial q^r}\epsilon_{stq},$$

we have

$$\nabla_r \epsilon_{stq} = \left(\frac{\partial \ln \sqrt{g}}{\partial q^r} - \left\{ {s \atop sr} \right\} \right) \epsilon_{stq},$$

and the above-said follows from the relationship

$$\left\{ {s \atop sr} \right\} = \frac{\partial \ln \sqrt{g}}{\partial q^r} = \frac{1}{2g}\frac{\partial g}{\partial q^r}. \tag{E.3.11}$$

The derivation is based on the definitions (E.2.7) and (E.2.6). We have

$$\left\{ {s \atop sr} \right\} = g^{st}\,[sr, t] = \frac{1}{2} g^{st} \left(\frac{\partial g_{st}}{\partial q^r} + \frac{\partial g_{rt}}{\partial q^s} - \frac{\partial g_{sr}}{\partial q^t} \right).$$

The terms

$$g^{st}\frac{\partial g_{rt}}{\partial q^s}, \quad -g^{st}\frac{\partial g_{sr}}{\partial q^t}$$

cancel, hence using formula (A.7.9) for differentiation of a determinant we find

$$\left\{ {s \atop sr} \right\} = \frac{1}{2} g^{st}\frac{\partial g_{st}}{\partial q^r} = \frac{1}{2} g^{st}\frac{\partial g_{st}}{\partial q^r} = \frac{1}{2g}\frac{\partial g}{\partial g_{st}}\frac{\partial g_{st}}{\partial q^r} = \frac{1}{2g}\frac{\partial g}{\partial q^r},$$

which is required.

E.4 Differential operations in curvilinear coordinates

The total differential of scalar $\varphi\left(q^1, q^2, q^3\right)$ is presented in two forms

$$d\varphi = \frac{\partial \varphi}{\partial q^s} dq^s = \nabla\varphi \cdot d\mathbf{r} = \nabla\varphi \cdot \mathbf{r}_s dq^s,$$

which means that the derivatives of φ with respect to q^s are the covariant components of vector $\nabla\varphi$ in the vector basis $\mathbf{r}^s$ and by eq. (D.2.1)

$$\nabla\varphi = \operatorname{grad}\varphi = \mathbf{r}^s \frac{\partial \varphi}{\partial q^s}. \tag{E.4.1}$$

The total differential of the vector is obtained by analogy. By eq. (B.2.11) we have

$$d\mathbf{a} = \frac{\partial \mathbf{a}}{\partial q^s} dq^s = \frac{\partial \mathbf{a}}{\partial \mathbf{r}} \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla\mathbf{a} = dq^s \mathbf{r}_s \cdot \nabla\mathbf{a},$$

and tensor $\nabla\mathbf{a}$ is the sum of the following dyadics

$$\nabla\mathbf{a} = \mathbf{r}^s \frac{\partial \mathbf{a}}{\partial q^s}. \tag{E.4.2}$$

Equations (E.4.1) and (E.4.2) suggest the following representation of the nabla-operator

$$\nabla = \mathbf{r}^s \frac{\partial}{\partial q^s}. \tag{E.4.3}$$

1. *Divergence of a vector*

$$\operatorname{div}\mathbf{a} = \nabla \cdot \mathbf{a} = \mathbf{r}^s \cdot \frac{\partial}{\partial q^s}\mathbf{a} = \mathbf{r}^s \cdot \mathbf{r}_k \nabla_s a^k = \nabla_k a^k = \frac{\partial a^k}{\partial q^k} + \left\{ {k \atop kr} \right\} a^r,$$

and referring to eq. (E.3.11) we obtain

$$\operatorname{div}\mathbf{a} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} a^r}{\partial q^r}. \tag{E.4.4}$$

2. *Rotor of a vector*

$$\operatorname{rot}\mathbf{a} = \nabla \times \mathbf{a} = \mathbf{r}^s \times \mathbf{r}^k \nabla_s a_k = \epsilon^{skt}\mathbf{r}_t \left(\frac{\partial a_k}{\partial q^s} - \left\{ {m \atop sk} \right\} a_m \right).$$

However

$$\left\{ {m \atop sk} \right\} = \left\{ {m \atop ks} \right\}, \quad \epsilon^{skt} \left\{ {m \atop sk} \right\} = \epsilon^{kst} \left\{ {m \atop ks} \right\} = -\epsilon^{skt} \left\{ {m \atop sk} \right\} = 0,$$

so that

$$\operatorname{rot}\mathbf{a} = \frac{1}{\sqrt{g}} e^{skt} \mathbf{r}_t \frac{\partial a_k}{\partial q^s}.$$

3. *Gradient of a vector*

$$\nabla \mathbf{a} = \mathbf{r}^s \frac{\partial \mathbf{a}}{\partial q^s} = \mathbf{r}^s \mathbf{r}^k \nabla_s a_k = \mathbf{r}^s \mathbf{r}^k \left(\frac{\partial a_k}{\partial q^s} - \left\{ {r \atop sk} \right\} a_r \right)$$
$$= \mathbf{r}^s \mathbf{r}_k \left(\frac{\partial a^k}{\partial q^s} + \left\{ {k \atop sr} \right\} a^r \right). \quad \text{(E.4.5)}$$

4. *Tensor of deformation*

$$\operatorname{def}\mathbf{a} = \left[\nabla \mathbf{a} + (\nabla \mathbf{a})^* \right] = \mathbf{r}^s \mathbf{r}^k \left[\frac{1}{2} \left(\frac{\partial a_k}{\partial q^s} + \frac{\partial a_s}{\partial q^k} \right) - \left\{ {r \atop sk} \right\} a_r \right]. \quad \text{(E.4.6)}$$

The quantities in the brackets are the covariant components of this tensor.

5. *Divergence of a tensor of second rank*

$$\operatorname{div}\hat{P} = \nabla \cdot \hat{P} = \mathbf{r}^s \cdot \mathbf{r}_k \mathbf{r}_t \nabla_s p^{kt} = \mathbf{r}_t \nabla_s p^{st}$$
$$= \mathbf{r}_t \left(\frac{\partial p^{st}}{\partial q^s} + \left\{ {s \atop sr} \right\} p^{rt} + \left\{ {t \atop sr} \right\} p^{sr} \right).$$

Referring to eq. (E.3.11) and (E.2.2) we have

$$\operatorname{div}\hat{P} = \mathbf{r}_t \left(\frac{\partial p^{st}}{\partial q^s} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^r} p^{rt} \right) + p^{sr} \frac{\partial \mathbf{r}_s}{\partial q^r}.$$

Therefore

$$\nabla \cdot \hat{P} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^s} \left(\sqrt{g} p^{st} \mathbf{r}_t \right). \quad \text{(E.4.7)}$$

6. *Laplace operator of a scalar*

$$\nabla \cdot \nabla \psi = \nabla^2 \psi = \mathbf{r}^s \frac{\partial}{\partial q^s} \cdot \mathbf{r}^k \frac{\partial \psi}{\partial q^k} = g^{sk} \frac{\partial^2 \psi}{\partial q^s \partial q^k} - \frac{\partial \psi}{\partial q^k} \left\{ {k \atop sr} \right\} g^{sr},$$

or

$$\nabla^2 \psi = g^{sk} \left(\frac{\partial^2 \psi}{\partial q^s \partial q^k} - \left\{ {r \atop sk} \right\} \frac{\partial \psi}{\partial q^r} \right). \quad \text{(E.4.8)}$$

The expressions for the differential operations of second order on vectors and tensors are very cumbersome. As an example we show the Laplace operator of a vector

$$\nabla^2 \mathbf{a} = \nabla \cdot \nabla \mathbf{a} = \mathbf{r}^t \frac{\partial}{\partial q^t} \cdot \mathbf{r}^s \mathbf{r}^k \nabla_s a_k = \mathbf{r}^t \cdot \frac{\partial}{\partial q^t} \mathbf{r}^s \mathbf{r}^k \left(\frac{\partial a_k}{\partial q^s} - \left\{ {r \atop sk} \right\} a_r \right)$$
$$= g^{ts} \mathbf{r}^k \left[\frac{\partial^2 a_k}{\partial q^t \partial q^s} - \left\{ {r \atop kt} \right\} \frac{\partial a_r}{\partial q^s} - \left\{ {r \atop st} \right\} \frac{\partial a_k}{\partial q^r} - \left\{ {r \atop sk} \right\} \frac{\partial a_r}{\partial q^t} - \right.$$
$$\left. a_r \left(\frac{\partial}{\partial q^t} \left\{ {r \atop sk} \right\} - \left\{ {r \atop qk} \right\} \left\{ {q \atop ts} \right\} - \left\{ {r \atop qs} \right\} \left\{ {q \atop tk} \right\} \right) \right], \qquad \text{(E.4.9)}$$

see also Section E.7.

E.5 Transition to orthogonal curvilinear coordinates

In the case of an orthogonal trihedron of base vectors, the metric tensor is diagonal and its covariant components are equal to

$$g_{sk} = \begin{cases} 0, & s \neq k, \\ H_s^2, & s = k, \end{cases} \qquad \text{(E.5.1)}$$

where H_s is Lame's coefficient. Further

$$g = |g_{sk}| = H_1^2 H_2^2 H_3^2 \qquad \text{(E.5.2)}$$

and using eq. (D.1.2) for determining the vectors of the cobasis we obtain

$$\left.\begin{aligned} \mathbf{r}^1 &= \frac{1}{\sqrt{g}} \mathbf{r}_2 \times \mathbf{r}_3 = \frac{1}{\sqrt{g}} |\mathbf{r}_2 \times \mathbf{r}_3| \frac{\mathbf{r}_1}{H_1} = \frac{\mathbf{r}_1}{H_1^2}, \\ \mathbf{r}^2 &= \frac{\mathbf{r}_2}{H_2^2}, \quad \mathbf{r}^3 = \frac{\mathbf{r}_3}{H_3^2}. \end{aligned}\right\} \qquad \text{(E.5.3)}$$

The directions of these vectors coincide with those of the original basis. The corresponding unit vectors are the tangents to the coordinate lines $[q^s]$ and are denoted as $\boldsymbol{\tau}_s$

$$\boldsymbol{\tau}_s = \frac{\mathbf{r}_s}{H_s} = H_s \mathbf{r}^s. \qquad \text{(E.5.4)}$$

The representation of vector

$$\mathbf{a} = a_s \mathbf{r}^s = a^s \mathbf{r}_s = \sum_{s=1}^{3} a_{(s)} \boldsymbol{\tau}_s$$

yields the following expressions for the contravariant and covariant components in terms of the "physical" components $a_{(s)}$

$$a^s = \frac{a_{(s)}}{H_s}, \quad a_s = H_s a_{(s)}. \qquad \text{(E.5.5)}$$

The analogous expressions for the tensor of second rank are set in the form

$$p^{st} = \frac{p_{(st)}}{H_s H_t}, \quad p_{st} H_s H_t p_{(st)}, \quad p^{s}_{\cdot t} = \frac{H_t}{H_s} p_{(st)}, \quad p_s^{\cdot t} = \frac{H_s}{H_t} p_{(st)}. \tag{E.5.6}$$

Christoffel's symbols of first and second kind are obtained by means of eqs. (E.2.6) and (E.2.7) respectively. Taking into account that the contravariant components of the metric tensor are equal to

$$g^{sk} = \mathbf{r}^s \cdot \mathbf{r}^k = \begin{cases} 0, & s \neq k, \\ \dfrac{1}{H_s^2}, & s = k \end{cases} \tag{E.5.7}$$

we find

$$\left.\begin{array}{lll} s \neq k \neq t: & [sk, t] = 0, & \left\{ \begin{matrix} t \\ sk \end{matrix} \right\} = 0, \\ s \neq t: & [ss, t] = -H_s \dfrac{\partial H_s}{\partial q^t}, & \left\{ \begin{matrix} t \\ ss \end{matrix} \right\} = -\dfrac{H_s}{H_t^2} \dfrac{\partial H_s}{\partial q^t}, \\ & [st, s] = H_s \dfrac{\partial H_s}{\partial q^t}, & \left\{ \begin{matrix} s \\ st \end{matrix} \right\} = \dfrac{1}{H_s} \dfrac{\partial H_s}{\partial q^t} \quad (\not{\Sigma}_s). \end{array}\right\} \tag{E.5.8}$$

The expressions for the derivatives of the base vectors are constructed by means of formulae (E.2.2). These expressions can be used for deriving formulae for differentiation of the unit vectors, see Section C.4.

The brevity and symmetry of the formulae of general tensor analysis is lost when the orthogonal curvilinear coordinates and the physical components of tensors are used. The formulae become cumbersome and for this reason it is preferable to perform the operations in curvilinear coordinates by using the approaches of Appendix C.

E.6 The Riemann-Christoffel tensor

The square of the linear element in the Euclidean space E_3 is given by the sum of squares of differentials of the Cartesian coordinates

$$ds^2 = da_1^2 + da_2^2 + da_3^2. \tag{E.6.1}$$

When the curvilinear coordinates are introduced by means of transformation (E.1.2) this expression takes a quadratic form of the differentials

$$ds^2 = g_{st} dq^s dq^t, \tag{E.6.2}$$

whose coefficients are the covariant components of the metric tensor and are calculated by the formula

$$g_{st} = \mathbf{r}_s \cdot \mathbf{r}_t = \frac{\partial a_k}{\partial q^s} \frac{\partial a_k}{\partial q^t} \tag{E.6.3}$$

provided that transformation (E.1.2) is given. Let us state the problem in another way. It is assumed that the quadratic form (E.6.2) is prescribed by the coefficients g_{st} and that it is positive definite. Then this form is said to define the metric in the Riemannian space R_3. Now the very transformation (E.1.2) is unknown and seeking it reduces to integrating the system of six equations (E.6.3) with three unknown functions a_1, a_2, a_3. This transformation exists only if the conditions for the system integrability are satisfied. If these conditions hold then the Riemannian space degenerates to Euclidean space and the position of a point can be determined in the Cartesian coordinate system and the square of the linear element can be presented in the Euclidean form (E.6.1).

By means of a linear transformation of the variables the positive definite form (E.6.2) can be lead to the sum of three squares

$$ds^2 = b_{11}\left(z^k\right) dz^{1^2} + b_{22}\left(z^k\right) dz^{2^2} + b_{33}\left(z^k\right) dz^{3^2}, \tag{E.6.4}$$

where z^1, z^2, z^3 are the new variables. Strictly speaking $\sqrt{b_{ss}\left(z^k\right)}dz^s$ (do not sum!) is not a differential of some quantity and one can put $da_s = \sqrt{b_{ss}\left(z^k\right)}dz^s$ only by fixing z^k which defines a local Cartesian system of axes a_s in the vicinity of the considered point in R_3. This proves the possibility of a local metric E_3 in R_3 while the sought conditions of integrability must guarantee the existence of the metric in the whole domain.

Provided that these conditions are satisfied, then there exist three functions $a_s\left(q^1, q^2, q^3\right)$. Equivalently there is a possibility of describing the position of any point by the position vector

$$\mathbf{r} = \mathbf{r}\left(q^1, q^2, q^3\right) \tag{E.6.5}$$

and a possibility of constructing the coordinate basis with the vectors $\mathbf{r}_k$ being equal to the derivatives of $\mathbf{r}$ with respect to coordinates q^k. Then

$$d\mathbf{r}_s = \mathbf{r}_{sk}dq^k = \left\{ {t \atop sk} \right\} \mathbf{r}_t dq^k, \quad \left\{ {t \atop sk} \right\} = \left\{ {t \atop ks} \right\}, \tag{E.6.6}$$

and the conditions for the integrability of these relationships are written as follows

$$\frac{\partial}{\partial q^r} \left\{ {t \atop sk} \right\} \mathbf{r}_t - \frac{\partial}{\partial q^k} \left\{ {t \atop sr} \right\} \mathbf{r}_t = 0. \tag{E.6.7}$$

When these conditions are satisfied the following expression

$$d\mathbf{r} = \mathbf{r}_s dq^s$$

is integrable since the conditions for its integrability

$$\frac{\partial \mathbf{r}_s}{\partial q^k} = \frac{\partial \mathbf{r}_k}{\partial q^s}$$

hold for the adopted definitions (E.2.6), (E.2.7) of Christoffel's symbols and the symmetry of these quantities due to the symmetry of the components of the metric tensor (the coefficients of quadratic form (E.6.2)).

An extended form of the integrability conditions (E.6.7) is as follows

$$\mathbf{r}_t \left[\frac{\partial}{\partial q^r}\left\{ {t \atop sk} \right\} - \frac{\partial}{\partial q^k}\left\{ {t \atop sr} \right\} + \left\{ {m \atop sk} \right\}\left\{ {t \atop rm} \right\} - \left\{ {m \atop sr} \right\}\left\{ {t \atop km} \right\}\right] = R^t_{krs\cdot}\mathbf{r}_t. \tag{E.6.8}$$

An expression of the same structure is obtained if we consider the difference

$$\nabla_r\nabla_s a^t - \nabla_s\nabla_r a^t.$$

Indeed, $\nabla_s a^t$ is a mixed component of tensor $\nabla\mathbf{a}$. Hence

$$\nabla_r\nabla_s a^t = \frac{\partial}{\partial q^r}\nabla_s a^t - \left\{ {q \atop rs} \right\}\nabla_q a^t + \left\{ {t \atop rq} \right\}\nabla_s a^q$$

and further

$$\nabla_r\nabla_s a^t - \nabla_s\nabla_r a^t = \frac{\partial}{\partial q^r}\nabla_s a^t - \frac{\partial}{\partial q^s}\nabla_r a^t + \left\{ {t \atop rq} \right\}\nabla_s a^q - \left\{ {t \atop sq} \right\}\nabla_r a^q =$$
$$= \left[\frac{\partial}{\partial q^r}\left\{ {t \atop sq} \right\} - \frac{\partial}{\partial q^s}\left\{ {t \atop rq} \right\} + \left\{ {t \atop rm} \right\}\left\{ {m \atop sq} \right\} - \left\{ {t \atop sm} \right\}\left\{ {m \atop rq} \right\}\right] a^q$$

or by eq. (E.6.8)

$$\nabla_r\nabla_s a^t - \nabla_s\nabla_r a^t = R^t_{srq\cdot}a^q. \tag{E.6.9}$$

The structure of this expression shows that quantities $R^t_{srq\cdot}$ represent the components of a tensor of fourth rank which are three times covariant with respect to indices srq and contravariant with respect to index t. This tensor is the Riemann-Christoffel tensor of curvature and its components are calculated in terms of the components of the metric tensor. If the latter are given such that the Riemann-Christoffel tensor is equal to zero then equations (E.6.6) are integrable and the space with the linear element (E.6.2) is the Euclidean one (E_3).

Referring to Ricci's theorem of Section E.3 we can rewrite condition (E.6.9) in the form

$$\nabla_r\nabla_s a_t - \nabla_s\nabla_r a_t = g_{mt}R^m_{srq\cdot}a^q = R_{srqt}a^q. \tag{E.6.10}$$

Here the four times covariant components of the Riemann-Christoffel tensor are introduced. They are expressed in terms of Christoffel's symbols of first kind and can be obtained with relative ease. Indeed,

$$R_{srqt} = g_{mt}\frac{\partial}{\partial q^r}g^{ml}[sq,l] - g_{mt}\frac{\partial}{\partial q^s}g^{ml}[rq,l] +$$
$$g^{lp}\left([sq,p][rl,t] - [rq,p][sl,t]\right).$$

Taking into account that

$$g_{mt}\frac{\partial g^{ml}}{\partial q^r} = -g^{ml}\frac{\partial g_{mt}}{\partial q^r},$$

and making the replacements

$$\frac{\partial g_{mt}}{\partial q^r} = [mr,t] + [rt,m], \quad \frac{\partial g_{mt}}{\partial q^s} = [ms,t] + [st,m],$$

we arrive at the following expressions for the covariant components of the Riemann-Christoffel tensor

$$R_{srqt} = \frac{1}{2}\left(\frac{\partial^2 g_{st}}{\partial q^r \partial q^q} - \frac{\partial^2 g_{sq}}{\partial q^r \partial q^t} + \frac{\partial^2 g_{rq}}{\partial q^s \partial q^t} - \frac{\partial^2 g_{rt}}{\partial q^s \partial q^q}\right) + \\ g^{ml}\left([rq,m]\,[st,l] - [sq,m]\,[rt,l]\right). \quad \text{(E.6.11)}$$

It follows from the latter equation that there is:

i) a symmetry with respect to pairs of indices sr and qt

$$R_{srqt} = R_{qtsr};$$

ii) a skew-symmetry with respect to pairs of indices s and r, q and t

$$R_{srqt} = -R_{rsqt} = -R_{srtq};$$

iii) Ricci's identities

$$R_{srqt} + R_{rqst} + R_{qsrt} = 0.$$

Taking into account these properties it can be proved that among 81 components there are only six independent ones

$$R_{2323}, \quad R_{2331}, \quad R_{2312}, \quad R_{3131}, \quad R_{3112}, \quad R_{1212}.$$

They can be presented in terms of the symmetric tensor of second rank named Ricci's tensor

$$A^{mn} = \frac{1}{4}\epsilon^{msr}\epsilon^{nqt}R_{srqt}. \quad \text{(E.6.12)}$$

Indeed,

$$\left.\begin{array}{lll} A^{11} = \frac{1}{g}R_{2323}, & A^{12} = \frac{1}{g}R_{2331}, & A^{13} = \frac{1}{g}R_{2312}, \\ & A^{22} = \frac{1}{g}R_{3131}, & A^{23} = \frac{1}{g}R_{3112}, \\ & & A^{33} = \frac{1}{g}R_{1212}, \end{array}\right\} \quad \text{(E.6.13)}$$

and in the Euclidean space

$$A^{mn} = 0. \quad \text{(E.6.14)}$$

In the orthogonal curvilinear coordinates, Lame's dependences, see Section C.6, correspond to these equations.

E.7 Tensor inc $\hat{P}$

The definition of this tensor is given in Section B.4 by formula (B.4.13)

$$\operatorname{inc}\hat{P} = \operatorname{rot}\left(\operatorname{rot}\hat{P}\right)^{*}.$$

Tensor $\hat{P}$ is assumed to be symmetric, then

$$\operatorname{rot}\hat{P} = \nabla\times\hat{P} = \mathbf{r}^{l}\times\mathbf{r}^{s}\mathbf{r}^{t}\nabla_{l}p_{st} = \epsilon^{lsq}\mathbf{r}_{q}\mathbf{r}^{t}\nabla_{l}p_{st},$$

so that

$$\left(\operatorname{rot}\hat{P}\right)^{*} = \mathbf{r}^{t}\mathbf{r}_{q}\epsilon^{lsq}\nabla_{l}p_{st}$$

and further

$$\operatorname{inc}\hat{P} = \mathbf{r}^{k}\times\frac{\partial}{\partial q^{k}}\left(\mathbf{r}^{t}\mathbf{r}_{q}\epsilon^{lsq}\nabla_{l}p_{st}\right).$$

The quantity in the parentheses is the tensor of second rank presented by the contravariant components. Taking into account eq. (E.3.10) we have

$$\operatorname{inc}\hat{P} = \mathbf{r}^{k}\times\mathbf{r}^{t}\mathbf{r}_{q}\nabla_{k}\epsilon^{lsq}\nabla_{l}p_{st} = \mathbf{r}_{p}\mathbf{r}_{q}\epsilon^{ktp}\epsilon^{lsq}\nabla_{k}\nabla_{l}p_{st}, \tag{E.7.1}$$

and it remains to prove the permutation of the operations of the covariant differentiation which is analogous to that in eq. (E.6.10)

$$\nabla_{k}\nabla_{l}p_{st} - \nabla_{l}\nabla_{k}p_{st} = R_{klmt}p_{s}^{\cdot m} + R_{lkms}p_{\cdot t}^{m} = 0. \tag{E.7.2}$$

This enables one to prove the symmetry of the considered tensor

$$\begin{aligned}\hat{M} &= \operatorname{inc}\hat{P} = \mathbf{r}_{p}\mathbf{r}_{q}\epsilon^{ktp}\epsilon^{lsq}\nabla_{k}\nabla_{l}p_{st}\\ &= \mathbf{r}_{p}\mathbf{r}_{q}\epsilon^{lsp}\epsilon^{ktq}\nabla_{l}\nabla_{k}p_{ts} = \mathbf{r}_{q}\mathbf{r}_{p}\epsilon^{lsq}\epsilon^{ktp}\nabla_{k}\nabla_{l}p_{st},\end{aligned}$$

which is required. We arrive at the relationships

$$\left.\begin{aligned}M^{11} &= \frac{1}{g}\left(\nabla_{2}^{2}p_{33} + \nabla_{3}^{2}p_{22} - 2\nabla_{2}\nabla_{3}p_{23}\right),\\ M^{12} &= \frac{1}{g}\left[-\nabla_{1}\nabla_{2}p_{33} + \nabla_{3}\left(\nabla_{1}p_{23} + \nabla_{2}p_{31} - \nabla_{3}p_{12}\right)\right]\end{aligned}\right\} \tag{E.7.3}$$

and others obtained from them by a circular permutation of the indices. This is a natural generalisation of formulae (B.4.15) in which differentiation is replaced by covariant differentiation. The extended expressions for operations $\nabla_{k}\nabla_{l}p_{st}$ are very cumbersome.

E.8 Transformation of the surface integral into a volume one

Let us consider the integral

$$\iint\limits_O n_s \varphi\left(q^1, q^2, q^3\right) do = \iint\limits_O \mathbf{n} \cdot \mathbf{r}_s \varphi do,$$

in which n_s denotes the covariant components of the vector of the external normal $\mathbf{n} = n_s \mathbf{r}^s$ to the closed surface O. Then by eqs. (B.5.2), (D.4.2), (E.2.2) and (E.3.11)

$$\begin{aligned}
\iint\limits_O \mathbf{n} \cdot \mathbf{r}_s \varphi do &= \iiint\limits_V \nabla \cdot \mathbf{r}_s \varphi d\tau = \iiint\limits_V \mathbf{r}^k \frac{\partial}{\partial q^k} \cdot \mathbf{r}_s \varphi d\tau \\
&= \iiint\limits_V \frac{\partial \varphi}{\partial q^s} d\tau + \iiint\limits_V \mathbf{r}^k \cdot \left\{ \begin{matrix} t \\ ks \end{matrix} \right\} \mathbf{r}_t \varphi d\tau \\
&= \iiint\limits_V \left(\frac{\partial \varphi}{\partial q^s} + \left\{ \begin{matrix} k \\ ks \end{matrix} \right\} \varphi \right) d\tau = \iiint \frac{d\tau}{\sqrt{g}} \frac{\partial}{\partial q^s} \left(\sqrt{g}\varphi\right).
\end{aligned}$$

Hence

$$\iint\limits_O n_s \varphi do = \iiint\limits_V \frac{\partial}{\partial q^s} \left(\sqrt{g}\varphi\right) \frac{d\tau}{\sqrt{g}}. \tag{E.8.1}$$

Applying it to vector $\mathbf{a}$ we have

$$\iint\limits_O \mathbf{n} \cdot \mathbf{a} do = \iint\limits_O n_s a^s do = \iiint\limits_V \frac{\partial}{\partial q^s} \left(\sqrt{g} a^s\right) \frac{d\tau}{\sqrt{g}} = \iiint\limits_V \operatorname{div} a d\tau, \tag{E.8.2}$$

and we arrive at expression (E.4.4).

Appendix F

Some information on spherical and ellipsoidal functions

F.1 Separating variables in Laplace's equation

In the case of spherical coordinates, the solution of Laplace's equations (C.8.5)

$$\nabla^2 \psi = \frac{\partial}{\partial R} R^2 \frac{\partial \psi}{\partial R} + \frac{\partial}{\partial \mu} \left(1 - \mu^2\right) \frac{\partial \psi}{\partial \mu} + \frac{1}{1 - \mu^2} \frac{\partial^2 \psi}{\partial \lambda^2} = 0, \tag{F.1.1}$$

in which, instead of ϑ, a new independent variable

$$\mu = \cos \vartheta \quad (-1 \leq \mu \leq 1) \tag{F.1.2}$$

is introduced, is sought in the form of the following product

$$\psi = f(R) M(\mu) \begin{matrix} \cos \\ \sin \end{matrix} m\lambda. \tag{F.1.3}$$

Let us denote the separation constant as $n(n+1)$. This constant does not change its value when replacing n by $-(n+1)$. We arrive at two differential equations for the sought functions $f(R)$ and $M(\mu)$

$$\left[R^2 f'(R)\right]' - n(n+1) f(R) = 0, \tag{F.1.4}$$

$$\left[\left(1 - \mu^2\right) M'(\mu)\right]' + \left[n(n+1) - \frac{m^2}{1 - \mu^2}\right] M(\mu) = 0. \tag{F.1.5}$$

In the following it is sufficient to assume that n is an integer. Equation (F.1.4) has two particular solutions

$$f_1(R) = R^n, \quad f_2(R) = R^{-(n+1)}. \tag{F.1.6}$$

The first is used for solving the internal boundary-value problem for the sphere $(0 \le R \le R_0)$, whilst the second is needed for solving the external boundary-value problem $(R_0 \le R < \infty)$. Solving boundary-value problems for the hollow sphere requires both solutions.

Using eq. (C.10.10) we have in the spheroidal coordinates (coordinates of the oblate ellipsoid)

$$\nabla^2 \psi = \frac{\partial}{\partial s}\left(1+s^2\right)\frac{\partial \psi}{\partial s} + \frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial \psi}{\partial \mu} + \left(\frac{s^2}{1+s^2} + \frac{\mu^2}{1-\mu^2}\right)\frac{\partial^2 \psi}{\partial \varphi^2} = 0 \tag{F.1.7}$$

and assuming

$$\psi = S(s) M(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\varphi, \tag{F.1.8}$$

we arrive, after separation of variables s, μ to the differential equations

$$\left[\left(1+s^2\right) S'(s)\right]' - \left(\lambda + \frac{m^2 s^2}{1+s^2}\right) S(s) = 0,$$
$$\left[\left(1-\mu^2\right) M'(\mu)\right]' + \left(\lambda - \frac{m^2 \mu^2}{1-\mu^2}\right) M(\mu) = 0.$$

Introducing the new denotation $n(n+1) = \lambda + \mu^2$ for the separation constant we have

$$\left[\left(1+s^2\right) S'(s)\right]' - \left[n(n+1) - \frac{s^2}{1+s^2}\right] S(s) = 0, \tag{F.1.9}$$

$$\left[\left(1-\mu^2\right) M'(\mu)\right]' + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right] M(\mu) = 0. \tag{F.1.10}$$

Equation (F.1.10) reduces to eq. (F.1.9) by replacing μ by is. It is also worth mentioning that equations (F.1.10) and (F.1.5) are identical.

Let us proceed to the case of the elliptic coordinates ρ, μ, ν defined in Section C.11. Looking for the solution of Laplace's equation (C.11.25) in the form of "Lame's product"

$$\psi = R(\rho) M(\mu) N(\nu) \tag{F.1.11}$$

we arrive at the form

$$\left(\mu^2 - \nu^2\right)\frac{\Delta(\rho)}{R(\rho)}\left[\Delta(\rho) R'(\rho)\right]' - \left(\nu^2 - \rho^2\right)\frac{\Delta_1(\mu)}{M(\mu)}\left[\Delta_1(\mu) M'(\mu)\right]' +$$
$$\left(\rho^2 - \mu^2\right)\frac{\Delta(\nu)}{N(\nu)}\left[\Delta(\nu) N'(\nu)\right]' = 0.$$

On the other hand we have the evident identity

$$\left(H\rho^2+h\right)\left(\mu^2-\nu^2\right)+\left(H\mu^2+h\right)\left(\nu^2-\rho^2\right)+\left(H\nu^2+h\right)\left(\rho^2-\mu^2\right)=0,$$

in which H and h are arbitrary constants. Taking this into account we obtain

$$\begin{aligned}\left(\mu^2-\nu^2\right)&\left\{\frac{\Delta(\rho)}{R(\rho)}\left[\Delta(\rho)R'(\rho)\right]'-\left(H\rho^2+h\right)\right\}+\\ &\left(\nu^2-\rho^2\right)\left\{-\frac{\Delta_1(\mu)}{M(\mu)}\left[\Delta_1(\mu)M'(\mu)\right]'-\left(H\mu^2+h\right)\right\}+\\ &\qquad\left(\rho^2-\mu^2\right)\left\{\frac{\Delta(\nu)}{N(\nu)}\left[\Delta(\nu)N'(\nu)\right]'-\left(H\nu^2+h\right)\right\}=0.\end{aligned}$$

Making use of the arbitrariness of H and h we can equate the expressions in the braces to zero

$$\Delta(\rho)\left[\Delta(\rho)R'(\rho)\right]'=\left[n(n+1)\rho^2+h\right]R(\rho), \tag{F.1.12}$$

$$-\Delta_1(\mu)\left[\Delta_1(\mu)M'(\mu)\right]'=\left[n(n+1)\mu^2+h\right]M(\mu), \tag{F.1.13}$$

$$\Delta(\nu)\left[\Delta(\nu)N'(\nu)\right]'=\left[n(n+1)\nu^2+h\right]N(\nu). \tag{F.1.14}$$

Here H is denoted as $n(n+1)$.

F.2 Laplace's spherical functions

Let $\varphi_n(x,y,z)$ denote a homogeneous polynomial of degree n. Altogether there are $\frac{1}{2}(n+1)(n+2)$ linearly independent polynomials. According to the definition of homogeneity

$$\varphi_n(kx,ky,kz)=k^n\varphi_n(x,y,z), \tag{F.2.1}$$

and the result of this functional relationship is Euler's theorem

$$x\frac{\partial\varphi_n}{\partial x}+y\frac{\partial\varphi_n}{\partial y}+z\frac{\partial\varphi_n}{\partial z}=\mathbf{R}\cdot\nabla\varphi_n=n\varphi_n. \tag{F.2.2}$$

These homogeneous harmonic polynomials satisfy Laplace's equation and are denoted as $P_n(x,y,z)$ in what follows. The expression for the polynomial $\nabla^2 P_n$ of degree $(n-2)$ has $\frac{1}{2}n(n-1)$ arbitrary coefficients. Thus, the requirement that this polynomial is equal to zero yields $\frac{1}{2}n(n-1)$ equalities relating $\frac{1}{2}(n+1)(n+2)$ coefficients of P_n. It can be proved that these

$\frac{1}{2}n(n-1)$ equalities are linearly independent. Hence there exist

$$\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1) = 2n+1$$

linearly independent harmonic polynomials of $n-th$ degree. For example for $n = 0, 1, 2$ they are as follows

$$1; \quad x, y, z; \quad x^2 - y^2, y^2 - z^2, xy, yz, zx. \tag{F.2.3}$$

By eq. (F.2.1) the harmonic polynomial can be presented in terms of the spherical coordinates in the following form

$$\begin{aligned} P_n(x, y, z) &= P_n(R\sin\vartheta\cos\lambda, R\sin\vartheta\sin\lambda, R\cos\vartheta) \\ &= R^n P_n(\sin\vartheta\cos\lambda, \sin\vartheta\sin\lambda, \cos\vartheta). \end{aligned} \tag{F.2.4}$$

The coefficient associated with R^n denoted as $Y_n(\vartheta, \lambda)$

$$Y_n(\vartheta, \lambda) = P_n(\sin\vartheta\cos\lambda, \sin\vartheta\sin\lambda, \cos\vartheta), \tag{F.2.5}$$

is referred to as Laplace's spherical function. It describes the value of the polynomial of $n-th$ degree on the sphere of unit radius. It is evident that Laplace's spherical function can be presented in the form of a trigonometric polynomial with respect to argument λ

$$Y_n(\vartheta, \lambda) = c_0(\vartheta) + \sum_{m=1}^{n} \left[c_n^m(\vartheta)\cos m\lambda + c_n'^m(\vartheta)\sin m\lambda\right]. \tag{F.2.6}$$

The sum of degrees $q_1 + q_2 + q_3$ of each term $x^{q_1}y^{q_2}z^{q_3}$ of the harmonic polynomial $P_n(x, y, z)$ is equal to n. Hence the terms in the expression for Laplace's spherical function are as follows

$$\begin{aligned} \sin^{q_1}\lambda\cos^{q_2}\lambda\sin^{q_1+q_2}\vartheta\cos^{q_3}\vartheta &= \sin^{q_1}\lambda\cos^{q_2}\lambda\sin^m\vartheta\cos^{n-m}\vartheta, \\ \sin^{q_2}\lambda\cos^{q_1}\lambda\sin^{q_1+q_2}\vartheta\cos^{q_3}\vartheta &= \sin^{q_2}\lambda\cos^{q_1}\lambda\sin^m\vartheta\cos^{n-m}\vartheta, \end{aligned}$$

where $m = q_1 + q_2$. Replacing now the trigonometric functions of λ by the following representations

$$\cos\lambda = \frac{1}{2}\left(e^{i\lambda} + e^{-i\lambda}\right), \quad \sin\lambda = \frac{1}{2i}\left(e^{i\lambda} - e^{-i\lambda}\right),$$

one can prove that the exponent of power of $\sin\vartheta$ in the coefficients $c_n^m(\vartheta)$, $c_n'^m(\vartheta)$ at $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\lambda$ in eq. (F.2.6) has the same evenness as m. On the other hand, $\sin^m\vartheta$ appears as a multiplier in the coefficients $c_n^m(\vartheta)$ and $c_n'^m(\vartheta)$. Hence, the latter can be set in the form

$$c_n^m(\vartheta) = a_{nm}P_n^m(\mu), \quad c_n'^m(\vartheta) = b_{nm}P_n^m(\mu), \tag{F.2.7}$$

where, as above, $\mu = \cos\vartheta$, a_{nm}, b_{nm} are constants and $P_n^m(\mu)$ denotes the product of $\left(1-\mu^2\right)^{m/2}$ and the polynomial of μ of degree $n-m$.

Thus we arrive at the following representation for the independent harmonic polynomials $P_n(x,y,z)$

$$R^n P_n(\mu); \quad R^n P_n^m(\mu)\cos m\lambda, \quad R^n P_n^m(\mu)\sin m\lambda, \tag{F.2.8}$$

where $P_n^0(\mu) = P_n(\mu)$.

Referring to eq. (F.1.3) we can conclude that the differential equations (F.1.5) and in turn (F.1.10) have the following particular solutions

$$M(\mu) = P_n^m(\mu), \quad m = 0, 1, \dots, n, \tag{F.2.9}$$

which are the products of $\left(1-\mu^2\right)^{m/2}$ and the polynomial of degree $(n-m)$ in μ. On the other hand the general representation of Laplace's spherical function is written in the form

$$Y_n(\mu,\vartheta) = a_0 P_n(\mu) + \sum_{m=1}^{n}\left(a_{nm}\cos m\lambda + b_{nm}\sin m\lambda\right) P_n^m(\vartheta). \tag{F.2.10}$$

The polynomial solution of equation (F.1.5) for $m=0$ is well known. It is Legendre's polynomial

$$\begin{aligned} P_n(\mu) &= \frac{1}{2^n n!}\frac{d^n}{d\mu^n}\left(\mu^2-1\right)^n \\ &= \frac{1\cdot 3\dots(2n-3)(2n-1)}{n!}\left[\mu^n - \frac{n(n-1)}{2\cdot(2n-1)}\mu^{n-2} + \right. \\ &\qquad \left. \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}\mu^{n-4} - \dots\right], \end{aligned} \tag{F.2.11}$$

where the series ends with a term dependent on μ for odd n and a term independent of μ for even n. In particular

$$\left.\begin{aligned} &P_0(\mu) = 1,\ P_1(\mu) = \mu,\ P_2(\mu) = \frac{1}{2}\left(3\mu^2-1\right), \\ &P_3(\mu) = \frac{1}{2}\left(5\mu^3-3\mu\right),\ P_4(\mu) = \frac{1}{8}\left(35\mu^4-30\mu^2+3\right) \quad \text{etc.} \end{aligned}\right\} \tag{F.2.12}$$

The real-valued polynomial solutions of eq. (F.1.9) can be set in the form

$$p_n(s) = \frac{1}{2^n n!}\frac{d^n}{ds^n}\left(s^2+1\right)^n \tag{F.2.13}$$

and in particular

$$\left.\begin{aligned} &p_0(s) = 1,\ p_1(s) = s,\ p_2(s) = \frac{1}{2}\left(3s^2+1\right), \\ &p_3(s) = \frac{1}{2}\left(5s^3+3s\right),\ p_4(s) = \frac{1}{8}\left(35s^4+30s^2+3\right) \quad \text{etc.} \end{aligned}\right\} \tag{F.2.14}$$

Returning to eq. (F.1.5) for $m \neq 0$ we can convince ourselves by a direct substitution that function $P_n^m(\mu)$ is related to the $n-th$ Legendre polynomial by the equation

$$P_n^m(\mu) = \left(1-\mu^2\right)^{m/2} \frac{d^m P_n(\mu)}{d\mu^n}, \tag{F.2.15}$$

that is, the above mentioned polynomial of $(n-m)-th$ degree is the $m-th$ derivative of $P_n(\mu)$ with respect to μ. The solutions $P_n^m(\mu)$ are referred to as the adjoint to $P_n(\mu)$. They are defined for $m \leq n$.

In particular, we have

$$\left.\begin{array}{l} P_1^1(\mu) = \sqrt{1-\mu^2},\ P_2^1(\mu) = 3\sqrt{1-\mu^2}\mu, \\ P_2^2(\mu) = 3\left(1-\mu^2\right),\ P_3^1(\mu) = \sqrt{1-\mu^2}\dfrac{3}{2}\left(5\mu^2-1\right), \\ P_3^2(\mu) = 15\left(1-\mu^2\right)\mu,\ P_3^3(\mu) = 15\left(1-\mu^2\right)^{3/2} \quad \text{etc.} \end{array}\right\} \tag{F.2.16}$$

The corresponding harmonic polynomials are presented in the form

$$\left.\begin{array}{l} RP_0(\mu) = 1,\ RP_1(\mu) = z,\ RP_1^1(\mu)\cos\lambda = x, \\ RP_1^1(\mu)\sin\lambda = y,\ R^2P_2(\mu) = \dfrac{1}{2}\left(2z^2-x^2-y^2\right), \\ R^2P_2^1(\mu)\cos\lambda = 3zx,\ R^2P_2^1(\mu)\sin\lambda = 3yz, \\ R^2P_2^2(\mu)\cos 2\lambda = 3\left(x^2-y^2\right),\ R^2P_2^2(\mu)\sin 2\lambda = 6xy \quad \text{etc.} \end{array}\right\} \tag{F.2.17}$$

The solutions of differential equation (F.1.9) for $m \neq 0$ are obtained by analogy

$$p_n^m(s) = \left(1+s^2\right)^{m/2} \frac{d^m\left(1+s^2\right)^n}{ds^m}. \tag{F.2.18}$$

In particular

$$\left.\begin{array}{l} p_1^1(s) = \sqrt{1+s^2},\ p_2^1(s) = 3\sqrt{1+s^2}s, p_2^2(s) = 3\left(1+s^2\right), \\ p_3^1(s) = \dfrac{3}{2}\sqrt{1+s^2}\left(5s^2-1\right),\ p_3^2(s) = 15\left(1+s^2\right)s, \\ p_3^3(s) = 15\left(1+s^2\right)^{3/2} \quad \text{etc.} \end{array}\right\} \tag{F.2.19}$$

F.3 Solutions $Q_n(\mu)$ and $q_n(s)$

It is known that if a particular solution of the differential equation of second order is obtained, then the second particular solution is obtained by quadrature. Designating the linear independent solutions of Legendre's equation with non-zero Wronskian W by M_1 and M_2 we have

$$W' = \left(M_1M_2' - M_2M_1'\right)' = \frac{2\mu}{1-\mu^2}\left(M_1M_2' - M_2M_1'\right) = \frac{2\mu}{1-\mu^2}W$$

and further

$$W = M_1 M_2' - M_2 M_1' = M_1^2 \left(\frac{M_2}{M_1}\right)' = \frac{C}{1-\mu^2},$$
$$M_2 = M_1 \left[C \int \frac{d\sigma}{(1-\sigma^2) M_1^2(\sigma)} + C_1\right]. \tag{F.3.1}$$

Choosing the constants C_1 and C_2 in a proper way and taking $M_1 = P_n(\mu)$ we obtain the second solution of Legendre's equation which is the Legendre function of second kind $(m = 0)$

$$Q_n(\mu) = P_n(\mu) \int_{\infty}^{\mu} \frac{d\sigma}{(1-\sigma^2)\left[P_n(\sigma)\right]^2}. \tag{F.3.2}$$

This solution is equal to zero as $\mu \to \infty$. The points $\mu = \pm 1$ are the logarithmic singularities, so that $P_n(\mu)$ has the single regular solution at $\mu = \pm 1$ (i.e. at the sphere poles $\vartheta = 0, \vartheta = \pi$). The integration path in expression (F.3.2) is assumed to coincide with the real axis and function $Q_n(\mu)$ is real-valued for $|\mu| > 1$.

For $n = 0, 1, 2$ we obtain

$$\left.\begin{aligned} Q_0(\mu) &= \frac{1}{2} \ln \frac{\mu+1}{\mu-1}, \quad Q_1(\mu) = \frac{1}{2} P_1(\mu) \ln \frac{\mu+1}{\mu-1} - 1, \\ Q_2(\mu) &= \frac{1}{2} P_2(\mu) \ln \frac{\mu+1}{\mu-1} - \frac{3}{2}\mu \end{aligned}\right\} \tag{F.3.3}$$

and, as shown in the theory of spherical functions, the general representation is given by

$$Q_n(\mu) = \frac{1}{2} P_n(\mu) \ln \frac{\mu+1}{\mu-1} + R_{n-1}(\mu), \tag{F.3.4}$$

where the polynomial $R_{n-1}(\mu)$ of $(n-1)-th$ degree is as follows

$$R_{n-1}(\mu) = -2\left[\frac{2n-1}{1 \cdot 2n} P_{n-1}(\mu) + \frac{2n-5}{3(2n-2)} P_{n-3}(\mu) + \frac{2n-9}{5(2n-4)} P_{n-5}(\mu) + \ldots\right]. \tag{F.3.5}$$

The second solution of Legendre's equation (F.1.9) at $m = 0$ is obtained by analogy, to give

$$q_n(s) = p_n(s) \int_{s}^{\infty} \frac{d\sigma}{(\sigma^2+1)\left[p_n(\sigma)\right]^2}. \tag{F.3.6}$$

It is real-valued on the entire real axis and vanishes when $s \to \infty$. At $n = 0, 1, 2$ the second solution is as follows

$$q_0(s) = \arctan s, \; q_1(s) = p_1(s) \arctan s - 1, \; q_2(s) = p_s(s) \arctan s - \frac{3}{2}s, \tag{F.3.7}$$

and the general representation is given by

$$\left.\begin{aligned} q_n(s) &= p_n(s) \arctan s - r_{n-1}(s), \\ r_{n-1}(s) &= 2\left[\frac{2n-1}{2n}p_{n-1}(s) - \frac{2n-5}{3(2n-2)}p_{n-3}(s) + \right. \\ &\qquad \left. \frac{2n-9}{5(2n-4)}p_{n-5}(s) - \dots\right]. \end{aligned}\right\} \tag{F.3.8}$$

Starting from the representation

$$\frac{1}{2}\ln\frac{\mu+1}{\mu-1} = \frac{1}{\mu} + \frac{1}{3\mu^3} + \frac{1}{5\mu^5} + \dots, \qquad \arctan s = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \dots,$$

one can prove that at $\mu \to \infty, s \to \infty$

$$Q_n(\mu) \approx \frac{n!}{1 \cdot 3 \cdot 5 \dots (2n+1)}\mu^{-(n+1)}, \; q_n(s) \approx \frac{n!}{1 \cdot 3 \cdot 5 \dots (2n+1)}s^{-(n+1)}. \tag{F.3.9}$$

The Wronskians are obtained directly from eqs. (F.3.2) and (F.3.6)

$$\left.\begin{aligned} P_n'(\mu) Q_n(\mu) - P_n(\mu) Q_n'(\mu) &= \frac{1}{\mu^2-1}, \\ p_n'(s) q_n(s) - p_n(s) q_n'(s) &= \frac{1}{s^2+1}. \end{aligned}\right\} \tag{F.3.10}$$

For $m \neq 0$ the solutions of equations (F.1.5) and (F.1.9), that is, the functions adjoint to $Q_n(\mu)$ and $q_n(s)$ are given by

$$Q_n^m(\mu) = \left(1-\mu^2\right)^{m/2} \frac{d^m Q_n(\mu)}{d\mu^m}, \quad q_n^m(s) = \left(1+s^2\right)^{m/2} \frac{d^m q_n(\mu)}{ds^m}, \tag{F.3.11}$$

where m can take any integer value. In particular

$$Q_0^1(\mu) = \frac{1}{\sqrt{1-\mu^2}}, \quad Q_0^2(\mu) = \frac{2\mu}{1-\mu^2}, \quad Q_1^2(\mu) = \frac{2}{1-\mu^2} \quad \text{etc.} \tag{F.3.12}$$

$$q_0^1(s) = -\frac{1}{\sqrt{1+s^2}}, \quad q_0^2(s) = \frac{2s}{1+s^2}, \quad q_1^2(s) = -\frac{2}{1+s^2} \quad \text{etc.} \tag{F.3.13}$$

and one can prove that in the general case for $m > n$

$$Q_n^m(\mu) = \frac{L_{n+m-1}(\mu)}{(1-\mu^2)^{m/2}}, \quad q_n^m(s) = \frac{l_{n+m-1}(s)}{(1+s^2)^{m/2}}, \tag{F.3.14}$$

where $L_{n+m-1}(\mu)$ and $l_{n+m-1}(s)$ are the polynomials of degree not higher than $n+m-1$. Thus we have constructed one system of solutions of Legendre's equations for $m > n$. The second one can be determined by relationship (F.3.1). Let us denote these by $\tilde{P}_n^m(\mu)$ and $\tilde{p}_n^m(s)$

$$\left.\begin{aligned} \tilde{P}_n^m(\mu) &= Q_n^m(\mu)\left[C\int^{\mu}\frac{\left(1-\sigma^2\right)^{m-1}}{\left[L_{n+m-1}(\sigma)\right]^2} + C_1\right], \\ \tilde{p}_n^m(s) &= q_n^m(s)\left[c\int^{s}\frac{\left(1+\sigma^2\right)^{m+1}}{\left[l_{n+m-1}(\sigma)\right]^2} + c_1\right] \end{aligned}\right\} \tag{F.3.15}$$

with the appropriate choice of constants. In such a way we obtain the solutions

$$\tilde{P}_0^1(\mu) = \frac{\mu}{\sqrt{1-\mu^2}}, \quad \tilde{P}_0^2(\mu) = \frac{1+\mu^2}{1-\mu^2}, \quad \tilde{P}_1^2(\mu) = \frac{\mu - \dfrac{1}{3}\mu^3}{1-\mu^2} \quad \text{etc.} \tag{F.3.16}$$

$$\tilde{p}_0^1(s) = \frac{s}{\sqrt{1+s^2}}, \quad \tilde{p}_0^2(s) = \frac{1-s^2}{1+s^2}, \quad \tilde{p}_1^2(s) = \frac{s + \dfrac{1}{3}s^3}{1+s^2} \quad \text{etc.} \tag{F.3.17}$$

F.4 Solution of the external and internal problems for a sphere

It is assumed that the function prescribed on the sphere surface $R = R_0$ can be presented in series in terms of Laplace's spherical functions

$$f(\mu,\lambda) = \sum_{n=0}^{\infty} Y_n(\mu,\lambda). \tag{F.4.1}$$

The following function

$$\Phi = \begin{cases} \displaystyle\sum_{n=0}^{\infty}\frac{R^n}{R_0^n}Y_n(\mu,\lambda) = \Phi_i, & R < R_0, \\ \displaystyle\sum_{n=0}^{\infty}\frac{R^{n+1}}{R_0^{n+1}}Y_n(\mu,\lambda) = \Phi_e, & R > R_0. \end{cases} \tag{F.4.2}$$

is harmonic in the sphere ($R < R_0$) and outside of it ($R > R_0$). In addition to this, it takes the value (F.4.1) on this sphere and, in the external problem, tends to zero not slower than R^{-1} as $R \to \infty$. This function is continuous over the entire space and describes the potential of the simple layer distributed over the sphere surface $R = R_0$ with density $\rho(\mu, \lambda)$

$$\rho(\mu, \lambda) = \frac{1}{4\pi} \mathbf{n} \cdot \nabla (\Phi_i - \Phi_e) = \frac{1}{4\pi} \left(\frac{\partial \Phi_i}{\partial n} - \frac{\partial \Phi_e}{\partial n} \right), \tag{F.4.3}$$

where $\mathbf{n}$ denotes the unit vector of the external normal to the surface. In the considered case we have

$$\rho(\mu, \lambda) = \frac{1}{4\pi R_0} \sum_{n=0}^{\infty} (2n+1) Y_n(\mu, \lambda). \tag{F.4.4}$$

The terms of the series (F.4.1) are Laplace's spherical functions which are determined through function $f(\mu, \lambda)$ prescribed on the sphere

$$Y_n(\mu, \lambda) = \frac{2n+1}{4\pi} \int_0^{2\pi} d\lambda' \int_{-1}^{1} d\mu' f(\mu', \lambda') P_n(\cos\gamma), \tag{F.4.5}$$

where

$$\begin{aligned} \cos\gamma &= \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\lambda - \lambda') \\ &= \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)} \cos(\lambda - \lambda') \end{aligned} \tag{F.4.6}$$

and

$$P_n(\cos\gamma) = P_n(\mu) P_n(\mu') + 2 \sum_{k=1}^{n} \frac{(n-k)!}{(n+k)!} P_n^k(\mu) P_n^k(\mu') \cos k(\lambda - \lambda'). \tag{F.4.7}$$

Formula (F.4.5) simplifies in axially symmetric problems, i.e. when f does not depend on λ. By eqs. (F.4.5) and (F.4.1) we obtain

$$\left. \begin{aligned} Y_n(\mu) &= \frac{2n+1}{2} P_n(\mu) \int_{-1}^{1} f(\mu') P_n(\mu') d\mu', \\ f(\mu) &= \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(\mu) \int_{-1}^{1} f(\mu') P_n(\mu') d\mu'. \end{aligned} \right\} \tag{F.4.8}$$

This is the well-known expansion of function $f(\mu)$ in series in terms of Legendre polynomials.

F.5 External and internal Dirichlet's problems for an oblate ellipsoid

Comparison of the form of the particular solutions of Laplace's equation (F.1.3) and (F.1.8) for the sphere and spheroid shows that the sought harmonic function on the surface of ellipsoid $s = s_0$ can be thought in the form of series (F.4.1). However in this case it is necessary to present each terms of the series in the form

$$Y_n(\mu) = a_{n0} P_n(\mu) + \sum_{m=1}^{n} (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi) P_n^m(\mu). \quad \text{(F.5.1)}$$

On the other hand

$$P_n^m(\mu) \begin{matrix} \cos \\ \sin \end{matrix} m\varphi$$

presents the value of the following functions

$$\frac{p_n^m(s)}{p_n^m(s_0)} P_n^m(\mu) \begin{matrix} \cos \\ \sin \end{matrix} m\varphi, \quad \frac{q_n^m(s)}{q_n^m(s_0)} P_n^m(\mu) \begin{matrix} \cos \\ \sin \end{matrix} m\varphi, \quad \text{(F.5.2)}$$

at $s = s_0$, the first and second ones being harmonic in $(s < s_0)$ and outside $(s > s_0)$ the spheroid respectively. The second representation in the latter equation satisfies the condition of vanishing at infinity, i.e. for $s \to \infty$.

We arrive at the solution

$$s < s_0: \quad \Phi(s, \mu, \varphi) = \sum_{n=0}^{\infty} \left[a_{n0} P_n(\mu) \frac{p_n(s)}{p_n(s_0)} + \right.$$
$$\left. \sum_{m=1}^{n} (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi) P_n^m(\mu) \frac{p_n^m(s)}{p_n^m(s_0)} \right], \quad \text{(F.5.3)}$$

$$s > s_0: \quad \Phi(s, \mu, \varphi) = \sum_{n=0}^{\infty} \left[a_{n0} P_n(\mu) \frac{q_n(s)}{q_n(s_0)} + \right.$$
$$\left. + \sum_{m=1}^{n} (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi) P_n^m(\mu) \frac{q_n^m(s)}{q_n^m(s_0)} \right]. \quad \text{(F.5.4)}$$

By eq. (C.3.8) the length of the normal δn to the ellipsoid surface $s = s_0$ is determined by the equality

$$\delta n = H_s \delta s = a \sqrt{\frac{s^2 + \mu^2}{1 + s^2}} \delta s.$$

Hence,

$$\left[\frac{\partial}{\partial n}\left(\frac{p_n^m(s)}{p_n^m(s_0)} - \frac{q_n^m(s)}{q_n^m(s_0)}\right)\right]_{s=s_0} = \tag{F.5.5}$$

$$= \frac{1}{2}\sqrt{\frac{1+s_0^2}{s_0^2+\mu^2}}\left[p_n^m(s_0)\, q_n^m(s_0)\right]^{-1}\left\{\left[p_n^m(s_0)\right]' q_n^m(s_0) - p_n^m(s_0)\left[q_n^m(s_0)\right]'\right\}$$

so that referring to formula (F.3.10) for the Wronskian of solutions $p_n^m(s)$, $q_n^m(s)$ we have by eq. (F.4.3)

$$\rho(\mu,\varphi) = \frac{1}{4\pi a\sqrt{(s_0^2+\mu^2)(s_0^2+1)}}\sum_{n=0}^{\infty}\left[\frac{a_{n0}P_n(\mu)}{p_n(s_0)\, q_n(s_0)} + \sum_{m=1}^{n}\frac{P_n^m(\mu)}{p_n^m(s_0)\, q_n^m(s_0)}\left(a_{nm}\cos m\varphi + b_{nm}\sin m\varphi\right)\right]. \tag{F.5.6}$$

This is the expression for the density of the simple layer potential of the spheroid surface $s = s_0$ describing the harmonic function given by eq. (F.5.3) in the spheroid and by eq. (F.5.4) outside it.

F.6 Representation of harmonic polynomials by means of Lame's products

It is proved in the theory of Lame's functions that for integer n and an appropriate choice of h in the differential equations (F.1.12)-(F.1.14) the Lame products (F.1.11) are presented by the harmonic polynomials of $n-th$ degree

$$F_n(x,y,z) = R(\rho)\, M(\mu)\, N(\nu). \tag{F.6.1}$$

For the present book it is sufficient to consider the cases $n = 0, 1, 2$.

1. $n = 0$. Then

$$R_0(\rho) = 1, \quad M_0(\mu) = 1, \quad N_0(\nu) = 1 \tag{F.6.2}$$

and the mentioned equations are satisfied at $h = 0$.

2. $n = 1$. We have three harmonic polynomials of the first degree which are the Cartesian coordinates presented by means of eq. (C.11.12) in the form

$$\left.\begin{aligned} F_1^{(1)} &= x = \frac{a}{e}\rho\mu\nu, \\ F_1^{(2)} &= y = \frac{a}{e\sqrt{1-e^2}}\sqrt{(\rho^2-e^2)(\mu^2-e^2)(\nu^2-e^2)}, \\ F_1^{(3)} &= z = \frac{a}{\sqrt{1-e^2}}\sqrt{(\rho^2-1)(1-\mu^2)(1-\nu^2)}, \end{aligned}\right\} \tag{F.6.3}$$

and one should take

$$R_1^{(1)}(\rho) = \rho, \quad R_1^{(2)} = \sqrt{(\rho^2 - e^2)}, \quad R_1^{(3)} = \sqrt{\rho^2 - 1}, \tag{F.6.4}$$

and the constant h should take the following values respectively

$$h_1^{(1)} = -\left(1 + e^2\right), \quad h_1^{(2)} = -1, \quad h_1^{(3)} = -e^2.$$

3. $n = 2$. Three of five harmonic polynomials are easy to guess. They are

$$F_2^{(1)} = yz, \quad F_2^{(2)} = zx, \quad F_2^{(3)} = xy, \tag{F.6.5}$$

for which

$$\begin{aligned} &R_2^{(1)}(\rho) = \sqrt{(\rho^2 - e^2)(\rho^2 - 1)}, \quad R_2^{(2)}(\rho) = \sqrt{\rho^2 - 1}\rho, \\ &R_2^{(3)}(\rho) = \rho\sqrt{\rho^2 - e^2}, \end{aligned} \tag{F.6.6}$$

which follows immediately from eq. (F.6.3). In this case

$$h_2^{(1)} = -\left(1 + e^2\right), \quad h_2^{(2)} = -\left(1 + 4e^2\right), \quad h_2^{(3)} = -\left(4 + e^2\right). \tag{F.6.7}$$

In order to construct the remaining two harmonic polynomials of second degree we require that the left hand side of the main identity (C.11.11)

$$\frac{x^2}{\sigma} + \frac{y^2}{\sigma - e^2} + \frac{z^2}{\sigma - 1} - a^2 = -a^2 \frac{(\sigma - \nu^2)(\sigma - \mu^2)(\sigma - \rho^2)}{f(\sigma)}$$

satisfies Laplace's equation

$$\begin{aligned} \nabla^2 \left(\frac{x^2}{\sigma} + \frac{y^2}{\sigma - e^2} + \frac{z^2}{\sigma - 1} - a^2\right) &= 2\left(\frac{1}{\sigma} + \frac{1}{\sigma - e^2} + \frac{1}{\sigma - 1}\right) \\ &= \frac{2}{f(\sigma)}\left[3\sigma^2 - 2\left(1 + e^2\right)\sigma + e^2\right]. \end{aligned} \tag{F.6.8}$$

This yields two values of constant σ

$$\sigma_1 = \frac{1}{3}\left(1 + e^2\right) + \beta, \ \sigma_2 = \frac{1}{3}\left(1 + e^2\right) - \beta, \ \beta = \frac{1}{3}\sqrt{1 - e^2 + e^4}, \tag{F.6.9}$$

and the corresponding harmonic polynomials

$$\left.\begin{aligned} F_2^{(4)} &= \frac{x^2}{a^2\sigma_1} + \frac{y^2}{a^2(\sigma_1 - e^2)} + \frac{z^2}{a^2(\sigma_1 - 1)} - 1, \\ F_2^{(5)} &= \frac{x^2}{a^2\sigma_2} + \frac{y^2}{a^2(\sigma_2 - e^2)} + \frac{z^2}{a^2(\sigma_2 - 1)} - 1. \end{aligned}\right\} \tag{F.6.10}$$

For them

$$R_2^{(4)}(\rho) = \sigma_1 - \rho^2, \quad R_2^{(5)} = \sigma_2 - \rho^2, \tag{F.6.11}$$

and the values of the constant in eqs. (F.1.12)-(F.1.14) are respectively equal to

$$h_2^{(4)} = -6\sigma_2, \quad h_2^{(5)} = -6\sigma_1. \tag{F.6.12}$$

F.7 Functions $S_i^{(k)}(\rho)$

To each of the solutions of differential equation (F.1.12)

$$R_i^{(k)}(\rho) \quad (i = 0, 1, 2; \quad k = 1, 2, \dots, 2i + 1)$$

constructed in the previous subsection we can find the second solution vanishing at infinity (as $\rho \to \infty$). Similar to eq. (F.3.2) it can be represented in the form

$$S(\rho) = R(\rho) \int_\rho^\infty \frac{d\lambda}{[R(\lambda)]^2 \Delta(\lambda)} = R(\rho)\,\omega(\rho). \tag{F.7.1}$$

This means that the Wronskian of solutions R and S is equal to

$$R'(\rho)\,S(\rho) - R(\rho)\,S'(\rho) = \frac{1}{\Delta(\rho)}. \tag{F.7.2}$$

We arrive at the solutions

$$1. \quad n = 0 \quad S_0(\rho) = \int_\rho^\infty \frac{d\lambda}{\Delta(\lambda)} = \omega_0(\rho); \tag{F.7.3}$$

$$2. \quad n = 1 \quad S_1^{(k)}(\rho) = R_1^{(k)}(\rho)\,\omega_1^{(k)}(\rho) \quad (k = 1, 2, 3), \tag{F.7.4}$$

where $\omega_1^{(k)}(\rho)$ is presented, similar to $\omega_0^{(k)}(\rho)$, by the elliptic integrals

$$\left.\begin{aligned} \omega_1^{(1)}(\rho) &= \int_\rho^\infty \frac{d\lambda}{\lambda^2 \Delta(\lambda)}, \\ \omega_1^{(2)}(\rho) &= \int_\rho^\infty \frac{d\lambda}{(\lambda^2 - e^2)\,\Delta(\lambda)}, \\ \omega_1^{(3)}(\rho) &= \int_\rho^\infty \frac{d\lambda}{(\lambda^2 - 1)\,\Delta(\lambda)}. \end{aligned}\right\} \tag{F.7.5}$$

$$3. \quad n = 2 \quad S_2(\rho) = R_2^{(k)} \omega_2^{(k)}(\rho) \quad (k = 1, 2, \dots, 5). \tag{F.7.6}$$

Here

$$\omega_2^{(1)}(\rho) = \int_\rho^\infty \frac{d\lambda}{(\lambda^2 - e^2)(\lambda^2 - 1)\,\Delta(\lambda)} = \frac{1}{1 - e^2}\left[\omega_1^{(3)}(\rho) - \omega_1^{(2)}(\rho)\right], \tag{F.7.7}$$

$$\omega_2^{(2)}(\rho) = \int_\rho^\infty \frac{d\lambda}{(\lambda^2 - 1)\lambda^2 \Delta(\lambda)} = -\left[\omega_1^{(1)}(\rho) - \omega_1^{(3)}(\rho)\right], \tag{F.7.8}$$

$$\omega_2^{(3)}(\rho) = \int_\rho^\infty \frac{d\lambda}{\lambda^2 (\lambda^2 - e^2)\Delta(\lambda)} = \frac{1}{e^2}\left[\omega_1^{(2)}(\rho) - \omega_1^{(1)}(\rho)\right] \tag{F.7.9}$$

and further

$$\omega_2^{(4)}(\rho) = \int_\rho^\infty \frac{d\lambda}{(\sigma_1 - \lambda^2)^2 \Delta(\lambda)}, \quad \omega_2^{(5)}(\rho) = \int_\rho^\infty \frac{d\lambda}{(\sigma_2 - \lambda^2)^2 \Delta(\lambda)}. \tag{F.7.10}$$

All these elliptic integrals reduce to Legendre's normal forms of first and second kind. The absence of an integral of third kind is the result of some properties of Lame's functions.

F.8 Simple layer potentials on an ellipsoid

Let us compare two solutions of Laplace's equation. The first one is presented by the harmonic polynomial

$$F_n(x, y, z) = R(\rho) M(\mu) N(\nu), \tag{F.8.1}$$

whereas the second is given by the formula

$$\frac{1}{\omega(\rho_0)} S(\rho) M(\mu) N(\nu) = \frac{\omega(\rho)}{\omega(\rho_0)} R(\rho) M(\mu) N(\nu) = \\ = \frac{\omega(\rho)}{\omega(\rho_0)} F_n(x, y, z), \tag{F.8.2}$$

see eq. (F.7.1). We arrive at the function

$$G(x, y, z) = \begin{cases} F_n(x, y, z), & \rho < \rho_0, \\ \dfrac{\omega(\rho)}{\omega(\rho_0)} F_n(x, y, z), & \rho > \rho_0, \end{cases} \tag{F.8.3}$$

which is harmonic inside $(\rho < \rho_0)$ and outside $(\rho > \rho_0)$ the ellipsoid

$$\frac{x^2}{a^2 \rho_0^2} + \frac{y^2}{a^2 (\rho_0^2 - e^2)} + \frac{z^2}{a^2 (\rho_0^2 - 1)} - 1 = 0. \tag{F.8.4}$$

This function vanishes at $\rho \to \infty$, is continuous in the whole space and is equal to

$$G(x,y,z)|_{\rho=\rho_0} = F_n(x,y,z)|_{\rho=\rho_0} \tag{F.8.5}$$

on the surface O of the ellipsoid $\rho = \rho_0$. Function $G(x,y,z)$ possesses all of the characteristic properties of a simple layer potential distributed over this surface and is presented in the form

$$G(x,y,z) = \iint_O \frac{q(\xi,\eta,\zeta)\,do}{\left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{1/2}}. \tag{F.8.6}$$

The density of this potential q is obtained by means of the well-known relationship

$$\left[\frac{\partial G(x,y,z)}{\partial n}\right]_{\rho\to\rho_0-0} - \left[\frac{\partial G(x,y,z)}{\partial n}\right]_{\rho\to\rho_0+0} = 4\pi q(x,y,z),$$

where $\mathbf{n}$ denotes the external normal to the ellipsoid $\rho = \rho_0$ and $\delta n = H_\rho \delta\rho$. Here Lame's coefficient is determined by eqs. (C.11.21) or (C.11.22). On the other hand

$$\begin{aligned}
\left.\frac{\partial G(x,y,z)}{\partial n}\right|_{\rho\to\rho_0-0} &= \frac{1}{H_\rho^0} R'(\rho_0) M(\mu) N(\nu) = \left.\frac{R'(\rho_0)}{H_\rho^0 R(\rho_0)} F_n(x,y,z)\right|_{\rho=\rho_0} \\
\left.\frac{\partial G(x,y,z)}{\partial n}\right|_{\rho\to\rho_0+0} &= \frac{1}{H_\rho^0 \omega(\rho_0)} S'(\rho_0) M(\mu) N(\nu) \\
&= \left.\frac{S'(\rho_0)}{H_\rho^0 S(\rho_0)} F_n(x,y,z)\right|_{\rho=\rho_0},
\end{aligned}$$

so that

$$4\pi q(x,y,z) = \frac{F_n(x,y,z)}{H_\rho^0 S(\rho_0) R(\rho_0)} \left[R'(\rho_0) S(\rho_0) - R(\rho_0) S'(\rho_0)\right]$$

and by eqs. (F.7.2) and (F.7.1)

$$\begin{aligned}
q(x,y,z) &= \frac{1}{4\pi a} \frac{[F_n(x,y,z)]_{\rho=\rho_0}}{\omega(\rho_0) R^2(\rho_0) \sqrt{(\rho_0^2 - \nu^2)(\rho_0^2 - \mu^2)}} \\
&= \frac{1}{4\pi} \frac{[F_n(x,y,z)]_{\rho=\rho_0}}{\omega(\rho_0) R^2(\rho_0) \rho_0 D_\rho^0 \Delta(\rho_0)}.
\end{aligned} \tag{F.8.7}$$

The solutions listed in Sections F.6 and F.7 have the corresponding potentials

$$1. \quad n = 0 \quad G_0(x,y,z) = \begin{cases} 1, & \rho < \rho_0, \\ \dfrac{\omega_0(\rho)}{\omega_0(\rho_0)}, & \rho > \rho_0. \end{cases} \tag{F.8.8}$$

This is the field of electrostatic potential caused by the conducting surface of ellipsoid $\rho = \rho_0$ with constant potential. The distribution of charge on this surface is due to formula (F.8.7) in which $F = 1, R = 1$ and $\omega_0(\rho_0)$ is given by the elliptic integral (F.7.3).

2. $n = 1$. The three solutions (F.7.4) have the corresponding potentials

$$G_1^{(s)}(x_1, x_2, x_3) = \begin{cases} x_s, & \rho < \rho_0, \\ x_s \dfrac{\omega_1^{(s)}(\rho)}{\omega_1^{(s)}(\rho_0)}, & \rho > \rho_0, \end{cases} \quad (s = 1, 2, 3). \tag{F.8.9}$$

3. For $n = 2$ we have three potentials equal to xy, yz, zx on the surface $\rho = \rho_0$. For instance

$$G_2^{(1)}(x, y, z) = \begin{cases} yz, & \rho < \rho_0, \\ yz \dfrac{\omega_2^{(1)}(\rho)}{\omega_2^{(1)}(\rho_0)}, & \rho > \rho_0. \end{cases} \tag{F.8.10}$$

Other two potentials, given by eq. (F.6.10) on the surface $\rho = \rho_0$, are constructed with the help of functions $\omega_2^{(4)}(\rho), \omega_2^{(5)}(\rho)$ obtained from eq. (F.7.10)

$$G_2^{(4,5)}(x, y, z) = \begin{cases} F_2^{(4,5)}(x, y, z), & \rho < \rho_0, \\ \dfrac{\omega_2^{(4,5)}(\rho)}{\omega_2^{(4,5)}(\rho_0)} F_2^{(4,5)}(x, y, z), & \rho > \rho_0. \end{cases} \tag{F.8.11}$$

At $\rho_0 = 1$ expression (F.8.3) for functions $F_n(x, y, z)$ which are even with respect to z determines the potential of the plate bounded by the focal ellipse E_0, see eq. (C.11.16). This potential on the plate surface has the value

$$G(x, y, 0) = [F_n(x, y, 0)]_{\rho=1} = R(1) M(\mu) N(\nu), \tag{F.8.12}$$

while outside of it

$$G(x, y, z) = \frac{\omega(\rho)}{\omega(1)} F_n(x, y, z). \tag{F.8.13}$$

The expression for the density is obtained by proceeding to the limit in eq. (F.8.7) with multiplication of the result by 2, which corresponds to layers on the "upper" and "lower" sides of the ellipsoid degenerated into the plate

$$q(x, y) = \frac{1}{2\pi a} \frac{[F_n(x, y, 0)]_{\rho=1}}{\omega(1) R^2(1) \sqrt{(1 - \nu^2)(1 - \mu^2)}}. \tag{F.8.14}$$

Referring to eq. (C.11.14) we can easily transform this expression to the form

$$q(x,y) = \frac{[F_n(x,y,0)]_{\rho=1}}{2\pi a\sqrt{1-e^2}\omega(1)R^2(1)}\left[1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right]^{-1/2}. \quad \text{(F.8.15)}$$

On the border of the plate, i.e. on the focal ellipse E_0, the density becomes unbounded. However it is zero on E_0 if $[F_n(x,y,0)]$ has the multiplier

$$1-\frac{x^2}{a^2}=\frac{y^2}{a^2(1-e^2)}.$$

This potential of the plate with the continuous density can be constructed as a linear combination of potentials $G_0, G_2^{(4)}, G_2^{(5)}$

$$G_*(x,y,z) = C_0G_0(x,y,z) + C_1G_2^{(4)}(x,y,z) + C_2G_2^{(5)}(x,y,z), \quad \text{(F.8.16)}$$

provided that, according to eq. (F.8.15), the constants C_0, C_1, C_2 are determined by the condition

$$\frac{C_0}{\omega_0(1)}+\frac{C_1}{\omega_2^{(4)}(1)(\sigma_1-1)^2}\left(\frac{x^2}{a^2\sigma_1}+\frac{y^2}{a^2(1-e^2)}-1\right)+$$
$$\frac{C_2}{\omega_2^{(5)}(1)(\sigma_2-1)^2}\left(\frac{x^2}{a^2}+\frac{y^2}{a^2(\sigma_2-e^2)}-1\right)=1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}, \quad \text{(F.8.17)}$$

see eqs. (F.6.10) and (F.6.11). The expression for the density corresponding to potential G_* is as follows

$$q(x,y) = \frac{1}{2\pi a\sqrt{1-e^2}}\left[1-\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}\right]^{1/2}. \quad \text{(F.8.18)}$$

Equation (F.8.17) yields three equations determining constants C_0, C_1, C_2 and in turn potential G_*. Omitting the intermediate manipulations, we show the final result

$$G_* = \frac{1}{2}\int_\rho^\infty \frac{d\lambda}{\Delta(\lambda)}\left[1-\frac{x^2}{a^2\lambda^2}-\frac{y^2}{a^2(\lambda^2-e^2)}-\frac{z^2}{a^2(\lambda^2-1)}\right]. \quad \text{(F.8.19)}$$

It is the volume (Newtonian) potential for the ellipsoid

$$\frac{x^2}{a^2\rho_0^2}+\frac{y^2}{a^2(\rho_0^2-e^2)}+\frac{z^2}{a^2(\lambda^2-1)}-1=0$$

of the constant density at the external point $(\rho > \rho_0)$. It also presents the potential of the layer on the elliptic plate bounded by ellipse E_0 with the density obeying law (F.8.18). The Newtonian potential at the internal point ($\rho < \rho_0$) is given by the quadratic function of coordinates x, y, z

$$G_* = \frac{1}{2}\int\limits_{\rho_0}^{\infty} \frac{d\lambda}{\Delta(\lambda)}\left[1 - \frac{x^2}{a^2\lambda^2} - \frac{y^2}{a^2\left(\lambda^2 - e^2\right)} - \frac{z^2}{a^2\left(\lambda^2 - 1\right)}\right]. \qquad \text{(F.8.20)}$$

Bibliographic References

Basic references

First of all it is worth referring to the fundamental treatise:

[1] Love A.E.H. A Treatise on the Mathematical Theory of Elasticity. Dower, New York, 1944.

The basic equations of the theory of elasticity, the plane problem, the problem of torsion and bending of a rod are exposed in:

[2] Muskhelishvili N.I. Some basic problems of the mathematical theory of elasticity. P. Noordhoff, Groningen, 1953. (English translation of the 3rd Russian edition of 1949).

The course by S.P. Timoshenko on the theory of elasticity contains a number of engineering applications and is more accessible from the mathematical perspective. The last American edition of this book appeared in conjunction with J. Goodier:

[3] Timoshenko S. P., Goodier J.N. Theory of Elasticity, McGraw-Hill, New York, 1951.

A modern exposition using tensor calculus and including material on the nonlinear theory of elasticity is contained in:

[4] Green A. E., Zerna W. Theoretical Elasticity, Clarendon Press, Oxford, 1954.

A brief exposition is found in the books:

[5] Sneddon J.N., Berry D.S. The classical theory of elasticity. Handbuch der Physik, Band VI, Springer-Verlag, Berlin, 1958.

[6] Trefftz E. Mathematische Elastizitätstheorie. Leipzig, Teubner, 1931.

A presentation of the basic equations of the nonlinear theory of elasticity and some problems of the linear theory is given in the book:

[7] Novozhilov V.V. Theory of elasticity (in Russian). Sudpromgiz, Leningrad, 1958.

Engineering aspects of the theory are the main subject of the books:

[8] Wang C.-T. Applied Elasticity. Mc Graw-Hill, New York, 1953.

[9] Southwell R.V. An introduction to the theory of elasticity for engineers and physicists - 2. ed., reprint. Oxford University Press, London, 1949.

The characteristic properties of the following books are the richness of the material and the pedagogical presentation

[10] Fung Y.C. Foundations of Solid Mechanics, Prentice-Hall, New York 1965.

[11] Sokolnikoff I.S. Mathematical Theory of Elasticity. MacGraw-Hill, New York, 1956.

The following textbook in Russian is the university course

[12] Leibenzon L.S. A course on the elasticity theory (in Russian). Gostekhizdat, Moscow, 1947.

whilst the following textbooks are intended for the universities of technology

[13] Kats A.M. Theory of elasticity (in Russian). Gostekhizdat, Moscow, 1956.

[14] Zhemochkin B.N. Theory of elasticity (in Russian). Stroivoenmorizdat, Moscow, 1948.

[15] Filonenko-Borodich M.M. Theory of elasticity (in Russian). Gostekhizdat, Moscow, 1947.

An original construction of theory and a richness of applications are the typical features of the work:

[16] Papkovich P.F. Theory of elasticity (in Russian). Oboronizdat, Moscow, 1939.

The classical treatment

[17] Clebsch A. Théorie de l'élasticité des corps solides, Paris 1883. Translation of the German course: Clebsch A. Theorie der Elastizität fester Korper, Leipzig 1862 with the comments by Saint-Venant.

is of importance nowadays.

The main stages in the development of the theory of elasticity are elucidated in the book:

[18] Timoshenko S.P. History of Strength of Materials. Dower 1956.

An outline of the development of the theory of elasticity is contained in the introduction to the book by Love [1].

During the preparation of the present book the material of the reviews:

[19] Grioli G. Mathematical Theory of Elastic Equilibrium, Springer, Berlin 1962.

[20] Truesdell C. The Mechanical Foundations of Elasticity and Fluid Dynamics. International Science Review Series, 8, Gordon and Breech, 1966 (reproduction from the Journal of Rational Mechanics and Analysis, 1952, vol. 1, pp. 125-300 and 1953, vol. 2, pp. 595-616).

was essentially used. The majority of the listed treatises [1, 2, 3, 11, 19, 20] contain extensive literature reviews.

Chapters 1 and 2

While presenting the basic definitions of the mechanics of solids the material of the following books was:

[21] Sedov L.I. Introduction to solid mechanics (in Russian). Fizmatgiz, Moscow, 1962.

[22] Eringen A.C. Nonlinear Theory of Continuous Media, McGraw-Hill, New York 1962,

as well as the references [19, 20].

Subsection 1.1.3

The fundamental work on moment elasticity is the following book:

[23] Cosserat E. et F. Théorie des corps déformables, Herrmann, Paris 1909.

Among the great number of works devoted to construction of the general theory and considering special problems we identify the following ones:

[24] Aero E.L., Kuvshinsky E.V. The main equations of theory of elasticity of media with rotational interaction of particles (in Russian). Fizika Tverdogo Tela, 1960, vol.2, pp. 1399-1409.

[25] Palmov V.A. The principal equations of non-symmetrical elasticity (in Russian). Prikladnaya Matematika i Mekhanika, 1964, vol. 28, No. 3, pp. 401-407.

Palmov V.A. Plain problem in the non-symmetrical theory of elasticity (in Russian). Prikladnaya Matematika i Mekhanika, 1964, vol. 28, No. 6, pp. 1117-1120.

[26] Mindlin R.D. Influence of the moment stresses on stress concentration (in Russian). In: Collection of translations "Mekhanika" 1964, vol. 4 (88), pp. 115-128.

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Mindlin R. D. Stress Functions for a Cosserat Continuum. International Journal of Solids and Structures, vol. 1, pp. 73-78, 1965.

[27] Koiter W.T. Couple-Stresses in the Theory of Elasticity. Kon. Nederl. Akad. v. Wetenschappen-Amsterdam, Proc. Ser. B, 67, N 1, pp. 17-44, 1964.

[28] Muki R., Sternberg E. The Influence of Couple-Stresses on Singular Stress Concentration in Elastic Solids. Zeitschrift für angewandte Mathematik und Physik, vol. 16, No. 5, pp. 611-618, 1965.

The theory of non-symmetric elasticity is presented in Chapter 9 of the book by Grioli [19].

Subsection 1.1.6

The stress function tensors of particular forms were considered by Maxwell (Scientific Papers, 2, 1870) and Morera (Rendi-conti d. Lincei, Ser. 5, 1, pp. 137-141, 1892). A combination of these solutions, suggested by Beltrami (ibid pp. 141-145), results in the tensor of general form introduced in the monograph:

[29] Krutkov Yu.A. Stress function tensor and general solutions in the statical theory of elasticity (in Russian). Publishers of the Academy of Sciences of USSR, 1949

and in the papers:

[30] Finzi B. Integrazione della equazione della Meccanica dei sistemi continui. Rendiconti d. Lincei, Ser. VI, 19, 1934.

[31] Blokh V.I. Stress functions in theory of elasticity (in Russian). Prikladnaya Matematika I Mekhanika, 1950, vol. 14, pp. 415-422.

Blokh V.I. Theory of elasticity (in Russian). Publishers of Kharkov University, Kharkov, 1950.

Subsection 1.3.4
The definition of the stress tensor is suggested in the papers:

[32] Trefftz E. Zur Theorie der Stabilität des elastischen Gleichgewichts. Zeitschrift für angewandte Mathematik und Mechanik, 12, pp. 160-165, 1933.

[33] Kappus R. Zur Elastizitätstheorie endlicher Verschiebungen. Zeitschrift für angewandte Mathematik und Mechanik, 19, pp. 271-285, 344-361, 1939,

and in the book:

[34] Hamel G. Theoretische Mechanik. Springer, Berlin 1949.

Subsections 1.4.1-1.4.12
In this subsection we present the content of Chapter 5 of Grioli's book [19] and the following paper:

[35] Signorini A. Sopra alcune questioni di statica dei sistemi continui. Ann. Scuola Norm. Sup. Pisa, Ser. II, 2. pp. 3-23, 1933.

Subsections 2.2.1 and 2.2.2
The derivation of the continuity conditions of Saint-Venant as being the conditions of integrability of the system of differential equations enabling determination of the displacements in terms of the given linear strain tensor is demonstrated in 22nd lecture of the classical work:

[36] Kirchhoff G.R. Mechanik. Vorlesungen über mathematische Physik. Teubner, Leipzig, 1874.

Kirchhoff only outlined the idea of the integration process, see also [6]. The formulae determining displacements in terms of strains were given by Cesaro in

[37] Cesaro E. Sulle formole del Volterra, fondamentali nella teoria delle distorsioni elastiche. Rendiconti d. Academia R. di Napoli, 12, pp. 311- 321, 1906.

Subsection 2.2.4
The concept of distortion is introduced in the fundamental memoir:

[38] Volterra V. Sur l'équilibre des corps élastiques multiplement connexes. Annales de l'Ec. Norm. Sup., 3-me serie, 24, pp. 401-507, 1907 (reproduced in "Opere matematiche", vol. 2).

Sections 2.3 and 2.4
Presentation of these subsections is based upon the review [20] by Truesdell, see also Chapter 9 of the book:

[39] Prager W. Introduction to Mechanics of Continua. Dower, 1963

The logarithmic strain measures suggested by Hencky in

[40] Hencky H. Über die Form des Elastizitätsgesetzes bei ideal-elastischen Stoffen. Zeitschrift für Technische Physik, 9, pp. 214-227, 1928,

present some advantages for interpreting the experimental results. The following paper

[41] Reiner M. Phenomenological microrheology (in Russian). In Collection of papers "Rheology", IL, Moscow, pp. 22-86, 1962,

is devoted to the possible strain measures.
Subsection 2.3.5
The geometric interpretation of tensor $\hat{G}^{\times^{-1}}$ suggested by formula (3.5.5) is proposed in:

[42] Truesdell C. Geometric Interpretation for the Reciprocal Deformation Tensors. Quart. of Applied Mathematics, vol. 15, No. 4, pp. 434-435, 1958.

Subsection 2.6.2
Construction of the rotation tensor for the plane displacement field is suggested by Signorini in the paper:

[43] Signorini A. Transformazioni termoelastiche finite. Ann. di Mat. pure e appl Ser. IV, 22, pp. 33-143, 1943.

Chapter 3

Construction of the linear theory of the relationship between the stress tensor and strain tensor adopted in this chapter relies on the articles:

[44] Hencky H. Zur Theorie plastischer Deformationen und der hierdurch im Material herforgerufenen Nachspannungen. Zeitschrift für angewandte Mathematik und Mechanik, 4, pp. 323-334, 1924

[45] Kachanov L.M. Elastic-plastic state of solids (in Russian). Prikladnaya Matematika i Mekhanika, 1941, vol. 5, No. 3, pp. 431-437.

In Subsection 3.2.1 we use the presentation of Chapter 8 of the book:

[46] Jeffreys H. Cartesian Tensors. Cambridge University Press, 1931.

The Table in Subsection 3.2.3 is taken from the book:

[47] Goldenblat I.I. Some topics of mechanics of deformable bodies (in Russian). Gostekhizdat, Moscow, 1955.

Subsections 3.3.4-3.3.6

[48] Nowacki W. Thermoelasticity. Reading, MA: Addison-Wesley, 1962, 628 p.

[49] Boley B.A., Weiner J.H. Theory of Thermal Stresses. Dover Publications, Inc., Mineola, NY, 1st edition, 1960.

[50] Kovalenko A.D. Introduction into thermoelasticity (in Russian). Naukova Dumka, Kiev, 1965.

Chapter 4

Subsection 4.1.4

The first publication by P.F. Papkovich in Proceedings of the Academy of Sciences of USSR (Series of Mathematical and Natural Sciences) appeared in 1932. A detailed presentation is given in his course [16]. The same form of the solution was obtained by H. Neuber in the paper:

[51] Neuber H. Ein neuer Ansatz zur Lösung raumlicher Probleme der Elastizitätstheorie. Zeitschrift für angewandte Mathematik und Mechanik, vol. 14, No. 4, 1934.

The content of this paper was repeated and completed by the formulae for the components of the stress tensor in orthogonal coordinates in the book:

[52] Neuber H. Kerbspannungslehre. Theorie der Spannungskonzentration. Genaue Berechnung der Festigkeit. Springer 2000.

The question of whether it is admissible to use three (rather than four) harmonic functions for solving the equilibrium equations in terms of displacements is considered in the papers:

[53] Slobodyansky M.G. General forms of solutions of the elasticity equations for simply connected and multiple-connected domains expressed in terms of the harmonic functions (in Russian). Prikladnaya Matematika i Mekhanika, 1954, vol. 18, pp. 55-74.

[54] Eubanks R.A., Sternberg E. On completeness of the Boussinesq-Papkovich stress functions (in Russian). In Collection of translations "Mekhanika" 1957, No. 6 (46), pp. 99-109.

Subsection 4.1.7
See [28] and [31] as well as the papers:

[55] Galerkin B.G. To the question of investigating the stresses and strains in an elastic isotropic body (in Russian). In: Galerkin B.G Collection of works, Publishers of the Academy of Sciences of USSR, 1953, vol. 1, pp. 318-321, first published in the Transactions of the Academy of Sciences of USSR, 1930.

[56] Mindlin R. D. Note on the Galerkin and Papkovich Stress Functions. Bulletin of American Mathematical Society, vol. 42, p. 373, 1936.

Section 4.2
The principles of the minimum strain energy and complementary work are presented in the majority of the above listed courses on elasticity theory. These principles and the various applications are the subject of the book:

[57] Leibenzon L.S. Variational principles of solving the problems of the theory of elasticity (in Russian). Gostekhizdat, Moscow, 1943, (see also the Complete Works by L.S. Leibenzon, vol. 1 (in Russian). Publishers of the Academy of Sciences of USSR, Moscow, 1951.

Subsection 4.2.5
The natural relation between the principle of minimum complementary work and the bounded problem of the calculus of variation was pointed out by Southwell (1936). The proof by Southwell is reproduced in [57] and [12].
Subsection 4.2.6
Presentation of the mixed principle of stationarity is based on the paper:

[58] Reissner E. On Some Variational Theorems in Elasticity. Problems of Continuum Mechanics. Contributions in Honour of N. I. Muskhelishvili, Philadelphia, 1961, pp. 370-381.

An excellent presentation of the variational principles of the elasticity theory is suggested in book [6].
Subsection 4.2.7
The variational principles with account of temperature terms are treated in the monograph:

[59] Maizel V.M. Thermal problem of the theory of elasticity (in Russian). Publishers of the Academy of Sciences of Ukrainian SSR, Kiev, 1951.

A complete review of the variational principles of elasticity theory is given in the paper:

[60] Tonti E. Variational Principles in Elastostatics. Meccanica, 1967, vol. 2, No 4, pp. 201-208.

A historical essay of the development of Saint-Venant's principle and the methods of its proof is given in the paper:

[61] Dzhanelidze G.Yu. Saint-Venant's principle (to centenary of the principle) (in Russian). Transactions of the Leningrad Polytechnic Institute, Dinamika i Prochnost Mashin, 1958, No. 192.

which also contains comprehensive references. See also [10].

Subsections 4.3.1-4.3.4

A treatment of the reciprocity theorem and some simple applications of it is contained in the courses [1, 3, 6, 10]. See [49, 50] for account of the thermal terms.

Subsection 4.3.5

The problem of the action of a concentrated force in an unbounded elastic medium (construction of the influence tensoı) was first considered by W. Thomson (Kelvin) in his memoir of 1848, see also

[62] Thomson W. Note on the Integration of the Equations of Equilibrium of an Elastic Solid. Mathematical and Physical Papers, 1, Cambridge, 1882.

Subsections 4.3.6 and 4.3.7

For the theory of potential see the book:

[63] Sretensky L.N. Theory of Newtonian potential (in Russian). Gostekhizdat, Moscow 1948.

Subsection 4.4.1

For the proof of the uniqueness theorem by Kirchhoff see [36] as well as courses [1, 2, 6] etc.

Statement of the uniqueness theorem in the case of concentrated forces is given in the paper:

[64] Sternberg E., Eubanks R.A. On the concept of concentrated forces and extension of the uniqueness theorem in the linear theory of elasticity (in Russian). In: Collection of translations "Mekhanika", 1956, No. 5 (39), pp. 56-84.

In presenting Subsections 4.4.2-4.4.8 (derivation of the integral equations for the first and second boundary-value problems and the proofs of existence of the solutions) the following book was used:

[65] Kupradze V.D. Methods of the potential in the theory of elasticity (in Russian).Fizmatgiz, Moscow 1963.

The work by Kupradze V.D., Gegeliya T.G., Basheleishvili M.O., Burchuladze T.V. Three-dimensional problems of the mathematical theory of elasticity (in Russian), Publishers of the Tbilisi University, Tbilisi, 1968, is devoted to investigations on the existence and uniqueness of solutions of the boundary-value problems of statics and stationary oscillations of the elastic body.

A very brief statement and solution of these problems are presented in the book:

[66] Mikhlin S.G. Multidimensional singular integrals and integral equations (in Russian). Fizmatgiz, Moscow 1962.

See also the review lecture:

[67] Kupradze V.D. Method of singular integral equations in the three-dimensional problem of elasticity (in Russian). Proceedings of the All-Union Congress on Theoretical and Applied Mechanics, Publishers of the Academy of Sciences of USSR, Moscow, 1962.

The question of numerical realization of the solution of integral equations is considered in the papers:

[68] Kupradze V.D. On one method of approximate solving of the limiting problems of mathematical physics (in Russian). Zhurnal Vychislitelnoy Matematiki i Matematicheskoi Fiziki vol. 4, No. 6, pp. 1118, 1964.

[69] Kupradze V.D. Methods of potential in elasticity theory. Applications of the theory of functions to mechanics of solids (in Russian). In: Proceedings of the International Symposium in Tbilisi, pp. 211-216, Nauka, Moscow, 1965.

The proofs by Lichtenstein and Korn of the existence of the solution of the boundary-value problems in the elasticity theory are presented in [6].

Treatment of Subsections 4.5.1-4.5.5 is entirely based on memoir [38].

Chapter 5

The content of this chapter differs essentially in the method of solution and considered material from the book of the present author:

[70] Lurie A.I. Three-dimensional problems of the theory of elasticity (in Russian). Gostekhizdat, Moscow, 1955.

Subsection 5.1.4

The papers by Boussinesq are collected in his classical treatise:

[71] Boussinesq J. Application des poténtieles a l'étude de l'équilibre et du mouvements de solides élastiques, Paris, 1885.

Subsection 5.1.5
The same results are shown by Nowacki [48].
Subsection 5.1.6
The content of this subsection is a revised presentation of the Section "Determination of the field of elastic stresses caused by the ellipsoidal inclusion and the related problems" in:

[72] Eshelby J.D. The continuum theory of lattice defects. In Solid State Physics, eds. Seith, F., Turnball, D., vol.3, Academic Press, New York, pp. 79-156, 1956.

Subsections 5.2.1-5.2.4
The solution of the problem of a concentrated force acting on the elastic half-space normal to its plane boundary was first given by Boussinesq [71]. A more general problem on loading the half-space by a system of normal and tangential surface forces was considered by Cerruti by means of Betti's integration method at the same time as Boussinesq in the memoir:

[73] Cerruti V. Ricerche intorno all'equilibrio de corpi elastici isotropi. Atti della R. Accademia dei Lincei, Memoriae della classe di scienze fisiche, matematiche e naturali, 13, pp. 81, 1881-1882.

See also [1, 70].
Subsection 5.2.5
The efficiency of the method of image in the problems of elasticity theory was first pointed out by Somigliana

[74] Somigliana C. Sul principo delle imagini di Lord Kelvin e le equazioni dell' Elasticita. Rendiconti d. Lincei, Ser. 5, 11, pp. 145, 1902.

The solution of the problem on the state of stress in the elastic half-space caused by a concentrated force was given by Mindlin in the paper:

[75] Mindlin R. D. Force at a point in the interior of a semi-infinite solid. Proc. First Midwestern Conf. Solid Mech., Univ. of Illinois, pp. 111, Urbana, 1953.

Subsection 5.2.6
The problem of thermal stresses in the elastic half-space is studied in detail in the paper:

[76] Sternberg E., McDowell E. L. On the Steady-State Thermoelastic Problem for the Half-Space. Quart. of Applied Mathematics, 1957, vol. 14, No. 4, pp. 381-398.

The absence of the stresses in the planes parallel to the boundary of the half-space is proved earlier in book [70]. See also the paper:

[77] Sneddon I.N., Tait R.I. On Lurie's Solution of the Equations of Thermoelastic Equilibrium. Problems of Continuum Mechanics. Contributions in Honor of N. I. Muskhelishvili, pp. 497-513, Philadelphia, 1961.

Subsections 5.2.10 and 5.2.11

Solutions of the boundary-value problems for the elastic-half-space are presented also in the paper:

[78] Michell I.H. The Transmission of Stress across a Plane of Discontinuity in an Isotropic Elastic Solid, and the Potential Solutions for a Plane Boundary. The Collected Mathematical Works of I.H.M. and A.G. Michell, pp. 189-195, Noordhoff, 1964 (first published in 1899).

Subsections 5.2.12-5.2.14

See [10, 61] as well as

[79] Mises R.V. On Saint-Venant's Principle. Bull. Amer. Math. Soc., vol. 51, pp. 555, 1945.

[80] Sternberg E. On Saint-Venant's Principle. Quart. of Appl. Math. vol. 11, No. 4, pp. 393-402, 1954 (January).

Section 5.3

The solution of the problem of equilibrium of an elastic sphere in spherical coordinates was first given in the classical treatise:

[81] Lamé G. Leçons sur les cordonnées curvilignes et leurs applications, Paris 1859.

An essential step forward was the paper:

[82] Thomson W. Dynamical Problems Regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid. Mathematical and Physical Papers, 3, pp. 351, 1892 (first published in 1863)

in which the solution was presented in the Cartesian coordinates in terms of three harmonic functions sought in the form of series in terms of the homogeneous harmonic polynomials. This solution is reproduced in the classical treatise:

[83] Thomson W., Tait P. G. Treatise on Natural Philosophy, vol. 1, part 2, 1883.

Thomson's solution of the first boundary-value problem for the solid sphere (Subsection 5.3.2) is reproduced in [1, 12, 6]. The way of solving the second boundary-value problem (Subsection 5.3.5) is also shown in papers [1, 6]. For the more difficult case of a hollow sphere Thomson gave the solution of the first boundary-value problem and suggested a way of solving the second problem.

The problem of equilibrium of the sphere is considered in the papers:

[84] Tedone O. Saggio di una teoria generate delle equazioni dell' equilibrio elastico per un corpo isotropo. Annali di Matematica pura et applicata, Ser. IIIa, 10, pp. 13, 1904.

[85] Somigliana C. Sopra l'equilibrio di un corpo elastico isotropo limitato da una o due superfici sferiche. Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matem., Ser. I, pp. 100, 1887.

[86] Cerruti V. Sulla deformazione di un involucre sferico isotropo per date forze agenti sulle due superfici limiti. Atti della Reale Acad. dei Lincei, Mem. della Classe di Sc. Fisiche, Matematice e Naturali, 1891.

[87] Lurie A.I. Equilibrium of elastic hollow sphere (in Russian). Prikladnaya Matematika i Mekhanika 1953, vol. 17, No. 3, p. 311.

Solutions of the boundary-value problems for solid and hollow spheres are given in Chapter 8 of book [70]. The case of the symmetrically loaded sphere is considered in Chapter 6 of the present book and in the following papers:

[88] Galerkin B.G. Equilibrium of an elastic spherical shell (in Russian). Prikladnaya Matematika i Mekhanika, 1942, vol. 6, p. 487.

[89] Lurie A.I. Equilibrium of an elastic symmetrically-loaded spherical shell (in Russian). Prikladnaya Matematika i Mekhanika, 1943, vol. 7, p. 393.

[90] Weber C. Kugel mit normalgerichteten Einzelkräften. Zeitschrift für angewandte Mathematik und Mechanik, 32, No. 6, pp. 186, 1952.

[91] Sternberg E., Rosental F. The Elastic Sphere under Concentrated Loads. Journal of Applied Mechanics, vol. 19, No. 4, pp. 413, 1952.

[92] Fichera G. Sur colcolo delle deformazioni, dotate di simmetria assiale, di uno strato sferico elastica. Atti dell' Accad. Nazionale dei Lincei, Classe di Sc. Fisiche, Ser. 8, 6, pp. 583. 1949.

Subsections 5.3.4 and 5.3.8.

The problem of the state of stress in the sphere under a transient thermal regime is considered in book [3] and is based on the result of the paper:

[93] Grünberg G. Über die in einer isotropen Kugel durch ungleichförmige Erwärmung erregten Spannungszustände. Zeitschrift für Physik, 35, pp. 548, 1925.

Subsections 5.3.9 and 5.3.10

The problems on the state of stress in the vicinity of the spherical cavity were considered by

[94] Southwell R. V. On the Concentration of Stress in the Neighborhood of a Small Spherical Flow. Phil. Mag., Ser. 7, 1, pp 71, 1926.

[95] Larmor J. The Influence of Flaws and Air-Cavities on the Strength of Materials. Phil. Mag, Ser. 5, 33, pp. 70, 1892.

In the context of the geophysical applications (theory of Earth's shape etc.) an extensive literature review appears in book [1] and in the paper:

[96] Jeffreys H. The Earth: Its Origin, History, and Physical Constitution, 6th ed. Cambridge, England: Cambridge University Press, 1976

which is devoted to the problems of Subsections 5.3.12 and 5.3.13.

Section 5.4

A detailed review of the diverse directions of investigations of three-dimensional problems of elasticity theory is given in the paper:

[97] Abramyan B.L., Alexandrov A.Ya. Axially symmetric problems of theory of elasticity (in Russian). Proceedings of the Second All-Union Congress on Theoretical and Applied Mechanics, Mechanics of Solids. Nauka, Moscow, 1966.

The paper has 241 references and considerable attention is paid to the method of solving the axially symmetric problems with the help of functions of complex variable (which is not considered in the present book).

See also the following review with many references:

[98] Sternberg E. Three-dimensional stress concentration in theory of elasticity (in Russian). In Collection of translations "Mekhanika", 1958, No. 6 (52), pp. 73-80.

The solutions of the problems on torsion, tension and bending of one-sheet hyperboloid of revolution considered in Subsections 5.4.1-5.4.4 were first given by H. Neuber. In his book [52] one finds numerous representations of the stress distribution, formulae and numerical tables.

The basic equations for the problem of torsion of the bodies of revolution (which is studied in the present book only for the case of cylinder, hyperboloid and the domain with a spherical cavity) were apparently first suggested by Michell in 1899 in the following paper:

[99] Michell I.H. The Uniform Torsion and Flexure of Incomplete Tores, with Application to Helical Springs. The Collected Mathematical Works, see [78].

This problem is the subject of the monograph:

[100] Solyanik-Krassa K.V. Torsion of shafts of variable cross-section (in Russian). Gostekhizdat, Moscow, 1949.

Considerable attention is paid to this problem in the book [131].

For the problems of equilibrium of the circular cone see [70] and the paper:

[101] Nuller B.M. To the solution of the problem of elasticity theory on truncated cone (in Russian). Mekhanika Tderdogo Tela, 1968, No. 5, pp. 102.

Section 5.5

The problem in Subsection 5.5.1-5.5.4 is considered in the paper:

[102] Lurie A.I. Elastostatic Robin's problem for a triaxial ellipsoid (in Russian). Mekhanika Tderdogo Tela, 1967, No. 1, pp. 80-83.

Subsections 5.5.6-5.5.8

The problem of the state of stress around an ellipsoidal cavity is considered in the paper:

[103] Sternberg E., Sadowsky M.A. Stress Concentration around a Triaxial Ellipsoidal Cavity. Journal of Applied Mechanics, 1949, vol. 16, No. 2. p. 149.

The solution is presented in terms of Jacobi's elliptic functions of the curvilinear elliptic oordinates. The solution obtained in the present book is expressed in terms of the Cartesian coordinate system and elliptic integrals. The error contained which appeared in book [70] is corrected in the book:

[104] Podilchuk Yu.N. State of stress in the vicinity of an ellipsoidal cavity under arbitrary constant forces at infinity (in Russian). Transactions of the Academy of Sciences of Ukrainian SSR, 1964, No. 9, pp. 1150-1154.

The problems of Subsection 5.5.9 were considered by H. Neuber in paper [52]. The problems of Subsections 5.5.10 and 5.5.11 are presented in the books:

[105] Sneddon I.N. Fourier Transforms. Dower, New York 1951

[106] Uflyand Ya.S. Integral transformations in theory of elasticity (in Russian).

and in the paper:

[107] Podilchuk Yu.N. Plane elliptic crack in arbitrary homogeneous field of stress (in Russian). Prikladnaya Mekhanika, 1968, vol.4, No. 8, pp. 93-100.

The solution of the problem on elliptic cracks is also presented in the book:

[107A] Panasyuk V.V. The limiting equilibrium of fragile bodies with cracks (in Russian). Naukova Dumka, Kiev, 1968, pp. 194-204.

Section 5.6

The first contact problem dates back to the classical memoir by H. Hertz:

[108] Hertz H. Über die Berührung fester elastischer Körper. Gesammelte Werke, vol. 1, p. 155, Leipzig 1895 (first published in Journal für reine und angewandte Mathematik (Crelle), vol. 92, p. 156, 1882).

The next sixty years were directed toward the experimental proof of the theory and development of engineering applications. Among the papers of this direction it is worth mentioning:

[109] Dinnik A.N. Impact and compression of elastic bodies (in Russian). Collection of papers, vol. 1, Publishers of the Academy of Sciences of Ukrainian SSR, 1952 (first published in 1909).

[110] Belyaev N.M. Local stresses under compression of elastic bodies (in Russian). In: Collection of papers "Inghenernye Sooruzheniya i Stroitelnaya Mekhanika, Put, Leningrad; 1924.

An incentive for the mathematical consideration of new contact problems of elasticity theory was a series of works by I.Ya. Shtaerman (the first one is dated 1939) unified in his monograph:

[111] Shtaerman I.Ya. Contact problem of the theory of elasticity (in Russian). Gostekhizdat, Moscow; 1949.

Solutions of the contact problems were further developed in studies by L.A. Galin, presented in the book:

[112] Galin L.A. Contact problems of theory of elasticity (in Russian). Gostekhizdat, Moscow; 1953.

More attention than in the present book is paid to contact problems in book [70]. The solutions of numerous problems are given in monograph [106]. A rather complete review of investigations of contact, three-dimensional and plane problems (134 references) is given in the paper:

[113] Popov G.Ya., Rostovtsev N.A. Contact (mixed) problems of the theory of elasticity (in Russian). Proceedings of the Second All-Union Congress on Theoretical and Applied Mechanics. Mechanics of Solids, Nauka, Moscow, 1966.

Subsections 5.6.2 and 5.6.5

See the paper:

[114] Mossakovsky V.I. The question of estimating displacement in three-dimensional contact problems (in Russian). Prikladnaya Matematika i Mekhanika, 1951, vol. 15, No. 5.

Section 5.7

The classical works related to the problem of the state of stress in circular cylinders (solid and hollow) are

[115] Filon L. On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Loads. Phil. Trans. of the Royal Soc. London, Ser. A, 198, 1902.

[116] Schiff P.A. Sur l'équilibre d'ue cylindre élastique. Journ. de math. pures et appliquées, Ser. 3, vol. .9, pp. 407, 1883.

Subsection 5.7.5

The problem of torsion by forces distributed on the end face was first considered in [116], see also [1].

Subsection 5.7.6

In the paper:

[117] Valov G.M. On axially symmetric deformation of a solid circular cylinder of finite length (in Russian). Prikladnaya Matematika i Mekhanika, 1962, vol. 26, No. 4, p. 650,

the solution is presented in terms of series whose coefficients are given by a finite (quite regular) system of equations.

Subsection 5.7.7

The problem of the state of stress in the cylinder loaded by the normal pressure on a part of lateral surface is considered in book [70]. The same problem for other types of loading is the subject of the papers:

[118] Livshits P.Z. State of stress in an elastic cylinder loaded by tangential forces on the lateral surface (in Russian). Inzhenernyi Sbornik, 1960, vol. 30, p. 47; Transaction of the Academy of Sciences of USSR, Mekhanika i Mashinostroenie, 1964, No. 4, p. 105.

Livshits P.Z. On the problem of bending the rod of circular cross-section (in Russian). Transactions of the Academy of Sciences of USSR, Mekhanika i Mashinostroenie, No. 1, pp. 76, 1963.

[119] Nikishin V.S. State of stress of a symmetrically loaded elastic cylinder (in Russian). Proceedings of Computational Centre of the Academy of Sciences of USSR, 1965.

Nikishin V.S. Thermal stresses in a composed cylinder under arbitrary temperature distribution over its height (in Russian). Ibid, 1964.

The tables of the influence functions simplifying calculations of stresses are presented in [119].

The solution in terms of trigonometric series is considered in the paper:

[120] Berton M. W. The Circular Cylinder with a Band of Uniform Pressure on a Finite Length of the Surface. Journ. of. Appl. Mech., 8, No. 3. pp. 97, 1941.

Using a Fourier integral this problem for a hollow cylinder is solved in the paper:

[121] Shapiro G.S. On compression of infinite hollow cylinder loaded on a part of its lateral surface (in Russian). Prikladnaya Matematika i Mekhanika, 1943, vol. 7, No. 5, p. 379.

The solution in the form of a series is given in the paper:

[122] Galerkin B.G. Elastic equilibrium of a hollow circular cylinder and a part of a cylinder (in Russian). Collection of Works, Publishers of the Academy of Sciences of USSR, vol. 1, 1953, p. 342 (first published in 1933).

The case of axially symmetric loading is studied in the papers:

[123] Prokopov V.K. Equilibrium of an elastic thick-walled axially-symmetric cylinder (in Russian). Prikladnaya Matematika i Mekhanika, 1949, vol. 12, No. 2, p. 135.

Prokopov V.K. Axially symmetric problem for isotropic cylinder (in Russian). Transactions of the Leningrad Polytechnic Institute, 1950, No. 2, p. 286.

Subsection 5.7.10

The property of generalized orthogonality of homogeneous solutions was first introduced by P.A. Schiff in [116]. This paper was undeservedly forgotten and the present author is obliged to B.M. Nuller for the indication. While presenting Subsection 5.7.10 the manuscript by B.M. Nuller was used and the main result by P.A. Schiff is presented in terms of the denotation of Section 5.7. See also [123].

Independently of P.A. Schiff, the property of orthogonality in the problem of bending of a rectangular plate was established by P.F. Papkovich in his book:

[124] Papkovich P.F. Structural mechanics of a ship (in Russian). Sudpromgiz, Leningrad, 1941, vol. 2, p. 634.

The feasibility of the simultaneous representation of two functions by series in terms of the homogeneous solutions (as being applied to Papkovich's problem) is the subject of the paper:

[125] Grinberg G.A. On the method suggested by P.F. Papkovich for solving the plane problem for a rectangular domain and bending of rectangular plate and some generalisations (in Russian). Prikladnaya Matematika i Mekhanika, 1953, vol. 17, No. 2, p. 211.

Studies of analogous questions for the cylinder are not known by the present author.

Approximate methods of satisfying the boundary conditions on the end faces of a cylinder are suggested by V.L. Biderman in the book:

[126] Biderman V.L., Likhachev K.K., Makushin V.M., Malinin N.N., Feodosiev V.I. Strength analysis in mechanical engineering (in Russian). Mashgiz, Moscow 1958, vol. 2, Chapter 5

and in the paper:

[127] Horvay G., Mirabal I. A. The End Problem of Cylinders. Journ of Appl. Mech. Trans. ASME, Paper N 58-A-24, pp. 1-10, 1958.

An estimate of the rate of decrease of stresses in the cylinder loaded on the end face by a statically equilibrated system of forces is given in the paper:

[127A] Knowles J. K., Morgan C.O. On the Exponential Decay of Stresses in Circular Elastic Cylinders Subjected to Axisymmetric Self-Equilibrated End Loads. International Journal of Solid and Structures, 1969, vol. 5, pp. 33-50.

Chapter 6

The term Saint-Venant's problem was introduced by Clebsch [17]. The celebrated works by Saint-Venant are

[128] Saint-Venant B. Memoir on the torsion of prisms.

Considerable attention is paid to Saint-Venant's problem in the courses [1, 3, 12, 16]. The method of functions of a complex variable is developed in detail in [2] for rods with cavities filled by the material with different elastic constants (composed rods).

Subsections 6.2.5 and 6.2.6

The problem of determining the coordinates of the centre of rigidity is considered in great detail in the paper:

[129] Dzhanelidze G.Yu. Determining the coordinates of the centre of rigidity in the torsion problem by means of various stress functions (in Russian). Transactions of the Leningrad Polytechnic Institute, Dinamika i Prochnost Mashin, 1963, No. 226, pp. 93-102,

containing a detailed references, see also [7]. The formulae for the multiply-connected domain are given in the paper:

[130] Prokopov V.K. On the centre of rigidity for a multiply-connected profile (in Russian). Scientific and Informational Bulletin of Leningrad Polytechnic Institute, 1960, No. 7, p. 91.

Section 6.3

Extensive literature references and the original solutions of numerous problems of rod torsion are given in the monograph:

[131] Arutyunyan N.Kh., Abramyan B.L. Torsion of elastic bodies (in Russian). Fizmatgiz, Moscow, 1963.

Application of functions of complex variable to the problem of torsion is developed in detail in the book:

[132] Weber C., Gunter W. Torsionstheorie. Akademie-Verlag, Berlin, 1958.

Subsections 6.3.5-6.3.7

The membrane analogy was suggested in the paper:

[133] Prandtl L. Eine neue Darstellung der Torsionsspannungen bei prismatischen Stäben von beliebigem Querschnitt, Jahresbericht Deutscher Math.-Vereins, 1904, vol. 13, p. 31.

For application of the analogy to experimentally solving the problem of torsion (the soap film method) see the references in [131].

A great number of isoperimetric problems related to the problem of torsion is considered in the monograph:

[134] Pólya G., Szegö G. Isoperimetric Inequalities in Mathematical Physics. Princeton University Press, 1951.

Inequality (3.7.7) is obtained in the paper:

[135] Nikolai E.L. On the problem of the elastic line of double curvature (in Russian). Works on Mechanics (Library of Russian science), Gostekhizdat, 1955 (first published in 1916).

Subsections 6.3.13-6.3.17

For the variational methods of solving the problem of torsion see [57]. The first publication on the variational method of Kantorovich is

[136] Kantorovich L.V. One direct method of approximately solving the problem of the minimum of a double integral (in Russian). Publishers of the Academy of Sciences of USSR, Division of Physical and Mathematical Sciences, No. 5, pp. 647-652, 1933.

Considerable attention is paid in Chapter 8 of monograph [131] to torsion of extended and thin-walled profiles. The problem of torsion of multiple-connected thin-walled profiles is considered in detail in [16].

Section 6.4

The problem of bending by a force is considered in courses [1, 2, 3, 12] and monograph [57] at great length. The relevant literature is not as extensive as for the problem of torsion. The graphic-analytical method of determining the true shear stress (rather than the mean one) is suggested in the paper mentioned in Subsection 6.4.1:

[137] Tricomi F. Sulla problema della trave soggetto a una sforza di traglio. Atti della Accad. Naz. del Lincei, Ser. VI, 1934, vol. 18, pp. 484-488.

The material of papers [57, 129] is used for presenting Subsections 6.4.5, 6.4.6 and 6.4.8. The results by Dunkan and Griffith mentioned in the text are contained in the papers:

[138] Griffith A, Taylor G. The Problem of Flexure and its Solution. Reports and Memoranda, N 399, 1917.

[139] Dunkan W. Torsion and Flexure of Cylinders and Tubes, Reports ani Memoranda, No. 1444, 1932.

Calculations of Subsection 6.4.7 are taken from the paper:

[140] Lurie A.I. Approximate solution of some problems of torsion and bending of a rod (in Russian). Transactions of the Leningrad Industrial Institute, 1939, No. 3, pp. 121-125.

Section 6.5

The theory was first suggested by Michell in the paper:

[141] Michell I.H. The Theory of Uniformly Loaded Beams. Quart. Journ. of Math., vol. 32, pp. 28-42, 1900 and published again in book [78].

Michell's problem is treated in [1] with unsubstantiated complexity. Presentation of Subsections 6.5.1-6.5.6 relies on the paper:

[142] Lurie A.I. Michell's problem (in Russian). In Stroitelnaya Mekhanika. Collection of papers celebrating 80th year of I.M. Rabinovich. Stroiizdat, Moscow, 1966.

For the centre of bending in Michell's problem see the paper:

[143] Khasis A.L. Michell's problem and the line of centre of bending (in Russian). Publishers of the Academy of Sciences of USSR, OTN, Mekhanika i Mashinostroenie, 1960, No. 5, pp. 58-65.

The calculation of Subsection 6.5.7 (bending of a heavy rod) is also carried out in [1]. In Subsection 6.5.8 there is a generalization of the problem considered in Subsection 6.4.1 to the case of a uniformly loaded beam. See also [19].

Subsection 6.5.9

The statement and way of solving the problem of a beam loaded on the lateral surface due to a polynomial law are given in the papers:

[144] Almansi E. Sopra la deformazione dei cylindri sollecitati lateralmente. Rendiconti delta Reale Accad. dei Lincei. Ser. 6, 10, pp. 333-338, 400-408, 1901.

A transparent form of solution of Almansi's problem is given in the paper:

[145] Dzhanelidze G.Yu. Almansi's problem (in Russian). Transactions of the Leningrad Polytechnic Institute, Dinamika i Prochnost Mashin, 1960, No. 210, pp. 25-38.

Chapter 7

The efficiency of applying the complex variable to the plane problem of elasticity was first pointed out by G.V. Kolosov in the monograph:

[146] Kolosov G.V. About one application of the theory of functions of complex variable to plane problem of mathematical problem of elasticity (in Russian). Yuriev, 1909.

The statement, proof of existence and the practical way of solving the boundary-value problems was given by N.I. Muskhelishvili in paper [2] (the first edition appeared in 1933) and the preceeding publications, among which the fundamental ones are:

[147] Muskhelishvili N.I. Sur l'intégration de 1'équation biharmonique, Transaction of Russian Academy of Sciences, 1919, pp. 663-686.

Muskhelishvili N.I. Applications des intégrales analogues a celles de Cauchy a quelques problèmes de la physique mathématiques, Tiflis, édition de l'Université, 1922.

Utilising complex variables in the plane problem of elasticity is the subject of the monographs:

[148] Babuška I., Rektorys K., Vyčichlo F. Mathematische Elastizitätstheorie der ebenen Probleme, Akademie-Verlag, Berlin, 1960.

[149] Milne-Thomson L. M. Plane Elastic Systems. Ergebnisse der angewandten Mathematik, No. 6, Springer-Verlag, Berlin - Gottingen, 1960.

[150] Belonosov S.M. Main plane static problems of the theory of elasticity (in Russian). Publishers of the Academy of Sciences of USSR, Siberian Division, Novosibirsk, 1962.

Many particular problems are considered in the two-volume monograph:

[151] Teodorescu P. P. Probleme plane in teoria elasticitatii, vol. 1, 995 p., 1960; vol. 2, 669 p., 1965.

In the present book the application of complex variable to the plane problem is reduced to examples of solving the simple (first and second) boundary-value problems. The mixed boundary-value problems requiring application of methods of linear adjunction and singular integral equations are presented in detail in the last editions of book [2] as well as in [149, 150]. In book [148] much space is given to application of the integral equations.

There is no possibility and need to mention an enormous number of papers devoted to the application of the methods of complex variables and integral equations to the plane problem. An extensive bibliography in given in [2] and the reviews:

[152] Vekua I.N., Muskhelishvili N.I. Methods of theory of analytical functions in the theory of elasticity (in Russian). Proceedings of the All-Union Congress on Theoretical and Applied Mechanics, Publishers of the Academy of Sciences of USSR, 1962.

Sherman D.I. Method of integral equations in plane and three-dimensional problems of the static theory of elasticity (in Russian). Proceedings of the All-Union Congress on Theoretical and Applied Mechanics, Publishers of the Academy of Sciences of USSR, 1962.

The solutions to a number of plane problems which are not based on the methods of the theory of functions are presented in the books [3, 16] and in the paper:

[153] Timpe A. Probleme der Spannungsverteilung in ebenen Systemen, einfach gelöst mit Hilfe der Airyschen Funktion. Zeitsch. für Math. u. Physik, 52, pp. 348-383, 1905.

Section 7.2

See book [3] for the bibliography related to the papers by A. Mesnager (1901), C. Ribiere, (1898), L. Filon, (1903) and Kh. Golovin. The solutions for the strip and a bar with a circular axis suggested in Subsections 7.2.3-7.2.10 are obtained by the methods developed in Chapters 3 and 4 of book [70] for the elastic layer and thick plate. Integral Fourier transforms were utilised in the problem of elastic strip (Subsection 7.2.8) in the papers:

[154] v. Karman Th. Über die Grundlagen der Balkentheorie. Abhandlungen aus dem Aerodynamischen Institut Aachen, pp. 3-10, 1927.

Seewald F. Die Spannungen und Formänderungen von Balken mit rechteckigem Querschnitt. Ibid., pp. 11-33.

The results by Seewald in the form of curves of corrections to the elementary theory of beam are reproduced in detail in book [3]. The homogeneous solutions of the problem on equilibrium of the elastic layer were first suggested in the paper:

[155] Fadle I. Die Selbstspannungs-Eigenwertfunktionen der quadratischen Scheibe. Ingenieur-Archiv, vol. 11, No. 4, pp. 125-149, 1940.

The Table of roots in Subsection 7.2.12 is taken from this paper. In addition to investigations [124, 125] using homogeneous solutions for taking into account the distortion of stresses due to the influence of the end faces it is worth mentioning

[156] Grinberg G.A., Lebedev N.N., Uflyand Ya.S. Method for solving the general biharmonic problem prescribed on the contour by the value and the normal derivative (in Russian). Prikladnaya Matematika i Mekhanika, 1953, vol. 17, No. 1, pp. 73-86.

A review of studies on homogeneous solutions is contained in the paper:

[157] Dzhanelidze G.Yu., Prokopov V.K. Method of homogeneous solutions in mathematical theory of elasticity (in Russian). Proceedings of the All-Union Mathematical Congress, Nauka, Moscow, 1964.

It was S. P. Timoshenko who initiated utilizing approximate solutions in the plane problem in the paper:

[158] Timoshenko S. P. The Approximate Solution of Two-Dimensional Problems in Elasticity, Phil. Mag., 47, pp. 1095-1104, 1924 (reproduced in Collected Papers of S.P. Timoshenko, McGraw-Hill, 1953).

These approaches together with the method of homogeneous solutions were next developed in the papers:

[159] Horvay G. The End Problem of Rectangular Strip. Journ. of Appl. Mech. Trans. ASME, No. 52-A-2, pp. 87-94, 1953.

[160] Horvay G., Born J. S. The Use of Self-Equilibrating Functions in Solution of Beam-Problems. Proc. of the 2nd V. S. Nat. Congr. Appl. Mech., Ann-Arbor, Mich., 1954.

Section 7.3

The solution to the problem of a concentrated force in the elastic plane (Subsection 7.3.1) is given by Michell in the paper:

[161] Michell I.H. Elementary Distributions of Plane Stress. Proc. London Math. Soc., 32, pp. 35-61, 1900 (reproduced in [78]).

This paper is also devoted to other plane problems on action of concentrated singularities.

Subsections 7.3.5-7.3.8

The theory of the plane contact problem is considered in books [2, 111], see also review [113]. Apparently, the simplest base of the plane die (Subsection 7.3.7) was first considered by M.A. Sadowsky in the paper:

[162] Sadowsky M.A. Zweidimensionale Probleme der Elastizitätstheorie. Zeitschrift für angewandte Mathematik und Mechanik, 8, pp. 515-518, 1928.

Section 7.4

The problem of Subsection 7.4.1 for a concentrated force at the wedge vertex was first solved in [161]. The Mellin integral transform in the problem on a wedge under arbitrary loading of its sides was first applied by V.M. Abramov in the paper:

[163] Abramov V.M. Distribution of stresses in a plane unbounded wedge under arbitrary load (in Russian). Proceedings of the Conference on Optical Method of Stress Studying, ONTI, 1937.

The case of loading the edge with a concentrated force is considered in the paper:

[164] Lurie A.I., Brachkovsky B.Z. Solution of the plane problem of elasticity for a wedge (in Russian). Transactions of the Leningrad Polytechnic Institute, 1941, No. 3, pp. 158-165.

An explanation of the Carothers paradox (Subsection 7.4.3) is given in book [150]. This question is also the subject of the thorough paper:

[165] Sternberg E., Koiter W. The Wedge under Concentrated Couple: A Paradox in the Two-Dimensional Theory of Elasticity. Journ. of Appl. Mech., 25, N 4, pp. 581-585, 1958.

Another treatment is given in the paper:

[166] Neuber H. Lösung des Carothers-Problems mittels Prinzipien der Kraftübertragung (Keil mit Moment an der Spitze). Zeitschrift für angewandte Mathematik und Mechanik, 43, No. 4-5, pp. 211-228, 1963.

The significance of the principle of "transmission of stresses through the surface of force transfer" is not limited by the explanation of the Carothers paradox.

The problem of the wedge is considered in book [106] at greater length, the Table of Subsection 7.4.4 being taken from this book. The case of loading the wedge is studied in greater detail in [105] than in Subsection 7.4.4.

Section 7.5

Clearly, the basic source for writing this section was book [2]. The material of book [149] was used as well.

A general representation for the stress function in the double-connected domain (Subsections 7.5.5-7.5.6) was given by Michell in the paper:

[167] Michell I.H. On the Direct Determination of Stress in an Elastic Solid with Application to the Theory of Plates. Proc. Lond. Math. Soc., vol. 31, pp. 100-124, 1899 (reproduced in [78]).

Section 7.6

The problem of Subsections 7.6.1 and 7.6.2 for a disc loaded by concentrated forces is considered in [2] in detail. Some specific examples of loading by forces on the circle which are in equilibrium with a concentrated force and a moment applied at the disc centre are considered in [167]. Graphical illustrations of the stress distribution in the disc are provided.

The classical Kirsch problem (1898) on tension of the plane weakened by a circular opening (Subsection 7.6.12) caused numerous investigations of local stresses in the vicinity of cavities in the plane field of stresses. They are presented in greater detail in the book:

[168] Savin G.N. Distribution of stresses near openings (in Russian). Naukova Dumka, Kiev, 1968 (first edition appeared in 1951). The book contains an extensive bibliography.

A great number of specific problems are considered in the paper:

[169] Naiman M.I. Stresses in a beam with a curvilinear opening (in Russian). Transactions of TsAGI, 1937, No. 313.

The method for solving the plane boundary-value problem presented in Subsections 7.6.13 and 7.6.14 is given in book [149].

Section 7.7

The problem of Subsections 7.7.2 and 7.7.3 on the annular ring is solved in a simpler way than that appears in [2]. The closed-form solution in terms of the elliptic functions is suggested in the book:

[170] Kolosov G.V. Application of complex variables to the theory of elasticity (in Russian). ONTI, 1935.

The solution by means of the analytic continuation suggested in paper [149] is not correct. The error is corrected here.

Section 7.8

While considering the plane problems for the domains which are transformed into a circle by a polynomial we limit ourselves to indicating the method of solution. The problem is considered in [2] in detail. Numerous examples with resulting formulae are presented in papers [168, 169]. The examples of Subsections 7.8.4 and 7.8.6 are treated in [149]. The results of solving the problem of the non-concentric ring are shown in [168]. This problem was considered in the paper:

[171] Chaplygin S.A., Arzhannikov N.S. On the question of deformation of a tube bounded by two eccentric cylinders and compressed by constant pressure (in Russian). Transactions of TsAGI, No. 123, 1933 (reproduced in Collection of Works by Chaplygin S.A., 1935, vol. 3, pp. 323-338).

Bipolar coordinates are applied to solving the plane problem for the domain between two non-concentric circles in the paper:

[172] Jeffery G. B. Plane Stress and Plane Strain in Bipolar Coordinates. Phil. Trans. Roy. Soc. London, Ser. A, pp. 265-293, 1921.

Chapter 8

Along with the above-mentioned books [4, 7, 19-22] the following books are general treatises:

[173] Green A.E., Adkins J. E. Large Elastic Deformations, 2nd ed. Oxford, England: Clarendon Press, 1970.

[174] Murnaghan F. D. Finite Deformation of an Elastic Solid, 1951.

[175] Varga O.H. Stress-Strain Behavior of Elastic Materials. Selected Problems of Large Deformations, Interscience Publichers, 1966.

[176] Truesdell C., Noll W. Nonlinear Field Theory of Mechanics, Handbuch der Physik, vol. 3, 1965.

The aim of this fundamental work is to construct a unified theory of the behaviour of solids based on a minimum numbers of basic assumptions (the principles of invariance, determinism, local action). A class of "simple materials" is proposed. The stress tensor of these materials depends on the time-history of change in the displacement gradient (and not on the gradients of higher order). Elastic and hyperelastic bodies belong to this class. An extensive review of solutions of particular problems is given and much attention is paid to establishing the acceptable forms of dependence of the specific strain energy of the hyperelastic body on the strain invariants. The book is provided with an extensive bibliography on the nonlinear theory of elasticity up to 1965.

[177] Brillouin L. Les Tenseurs en Mécanique et en Élasticité, Paris 1938.

Subsection 8.2.4

The constitutive law in the form equivalent to (2.4.6) is suggested in the paper:

[178] Finger J. Über die allgemeinsten Beziehungen zwischen Deformationen und den zugehörigen Spannungen in aelotropen und isotropen Substanzen. Sitzungsberichte der Akademie der Wissenschaften Wien, Ser. lla, 103, pp. 1073-1100, 1894.

Subsection 8.2.9

[179] Grioli G. On the Thermodynamic Potential for Continuums with Reversible Transformations - Some Possible Types, Meccanica, Journ. of the Italian Ass. of Theoretical and Applied Mechanics, 1, N 1-2, pp. 15-20, 1966.

The rotated stress tensor and its representation in terms of the Cauchy stress tensor are considered in book [174].

Subsection 8.3.4

Representation of the energy tensor with the help of moduli k, μ and the similarity phase is given in the paper:

[180] Novozhilov V.V. On the relationship between the stresses and strains in a nonlinear medium (in Russian). Prikladnaya Matematika i Mekhanika 1952, vol. 15, No. 2, pp. 183-194.

Subsections 8.4.1-8.4.2

The quadratic constitutive equation is formulated in Signorini's memoirs:

[181] Signorini A. Transformazioni termoelastiche finite. Ann. Mat. pur. appi, Ser. IV, 22, pp. 33-143. 1943; Ser. IV, 30, pp. 1-72, 1948.

Subsection 8.4.1

Remarks 1 and 2

[182] Zvolinsky N.V., Riz P.M. On some problems of the nonlinear theory of elasticity (in Russian). Prikladnaya Matematika i Mekhanika, 1939, vol. 2, No. 4, pp. 417-426.

[183] Seth B.R. Finite Strain in Elastic Problems, Phil. Trans. Roy. Soc. London. Ser. A, 234, pp. 231-264, 1935.

Subsections 8.4.6-8.4.9

See [173, 174, 175], as well as

[184] Mooney M. A Theory of Large Elastic Deformation. Journ. Appl. Phys. 11, pp. 582-592, 1940.

[185] Rivlin R. S., Saunders D. W. Large Elastic Deformations of Isotropic Materials. VII. Experiments on the Deformation of Rubber. Phil. Trans. Roy. Soc. London, Ser. A, 243, pp. 251-288, 1951.

[186] Bridgmann P.W. The Compession of 39 Substances to 100,000 kg/cm^2. Proc. Acd. Sci. Amsterdam, vol. 76, pp. 55-70, 1948.

The Table in Subsection 8.4.6 is taken from the book:

[187] Zarembo A.K., Krasilnikov B.A. Introduction into nonlinear acoustics (in Russian). Nauka, Moscow, 1966,

and the papers listed in the Subsection.

Subsection 8.4.10

Neuber suggested the constitutive law (4.10.10), (4.10.13) in the paper:

[188] Neuber H. Statische Stabilität nichtlinear elastischer Kontinua mit Anwendung auf Schalen. Zeitschrift für angewandte Mathematik und Mechanik, vol. 46, No. 3-4, pp. 211-220, 1966.

Section 8.5 and Subsections 8.5.3 and 8.5.4

The principle of stationarity of complementary work is considered in the paper:

[189] Levinson, A theorem on complementary energy in the nonlinear theory of elasticity, Journal of Applied Mechanics, Transactions ASME, 1965, No. 4.

Chapter 8 see also

[190] Tolokonnikov L.A. Equations of the nonlinear theory of elasticity in terms of displacements (in Russian). Prikladnaya Matematika i Mekhanika 1957, vol. 21, No. 6.

Tolokonnikov L.A. On the relations between stresses and strains in nonlinear theory of elasticity (in Russian). Prikladnaya Matematika i Mekhanika, 1956 , vol. 20, No. 3.

Chapter 9

Sections 9.1-9.3

see [20, 4, 176] and the works by Rivlin which are fundamental in the nonlinear theory of elasticity:

[191] Rivlin R. S. Large Elastic Deformation of Isotropic Materials, Further Developments of the General Theory. Part IV, Phil. Trans. Roy. Soc. London, Ser. A., 241, pp. 379-397, 1948;

Rivlin R. S. Large Elastic Deformation of Isotropic Materials. Part V. Problem of Flexure. Proc. Roy. Soc. London, Ser. A, 195, pp. 463-473, 1949;

Rivlin R. S. A Note on the Torsion of an Incompressible Highly-Elastic Cylinder. Proc. Cambridge Phil. Soc., 45, pp. 485-487, 1949;

Rivlin R. S. Large Elastic Deformation of Isotropic Materials, Part VI, Further Results in the Theory of Torsion, Shear and Flexure. Phil. Trans. Roy. Soc. London. Ser. A, 242, pp. 173-195. 1949.

The contents of **Subsection 9.2.2-9.2.4** is based upon the paper:

[192] Klingbeil W. W., Schield R.T. Large-Deformation Analyses of Bonded Elastic Mounts. Zeitsch. für angew. Math. u. Phys., vol. 17, No. 2, pp. 281-305, 1966.

Section 9.4 and Subsections 9.4.1-9.4.5

The suggested derivation of the equilibrium equations of initially loaded elastic bodies differs from that in [4]. See also [32, 33] and the papers:

[193] Biezeno C.B., Hencky H. On the General Theory of Elastic Stability, K. Akad. Wet. Amsterdam Proc., 31, pp. 569-592, 1929; 32, pp. 444-456, 1930.

[194] Pearson C.E. General Theory of Elastic Stability. Quart. of Appl. Math., 14, pp. 133-144, 1956.

[195] Lurie A.I. Bifurcation of equilibrium of ideally elastic body (in Russian). Prikladnaya Matematika i Mekhanika, 1966, vol. 30, No. 4, pp. 718-731.

Subsection 9.4.6

The problem of torsion of a tensioned rod was considered in [182] and in the paper:

[196] Green A.E., Shield R.T, Finite Extension and Torsion of Cylinders, Phil. Trans. Roy. Soc. London, Ser. A, 244, pp. 47-84, 1951.

The problem of equilibrium of an initially compressed rod is considered in [195].

Section 9.5

For the second order effects see [191] as well as

[197] Green A.E, Rivlin R.S., Shield R.T. General Theory of Small Elastic Deformations, Proc. Roy. Soc. London, Ser. A, 211, pp. 128-154, 1952.

[198] Rivlin R.S. The Solution of Problems in Second Order Elasticity Theory. Journ. of Rat. Mech. and Analysis, 2, pp. 53-81, 1953.

Subsection 9.5.3

[199] Toupin R.A., Rivlin R.S. Dimensional Changes in Crystals Caused by Dislocation. Journ. of Mathematical Physics, vol. 1, No. 1, pp. 8-15, 1960.

Section 9.6.
The theory of finite plane strain is developed in the papers:

[200] Adkins J.E., Green A.E., Shield R.T. Finite Plane Strain. Phil Trans, Ser. A, 246, pp. 181-213, 1953.

[201] Adkins J.E., Green A.E., Nicholas G.C. Two-dimensional Theory of Elasticity for Finite Deformations. Phil. Transactions of the Royal Society London, Ser. A, 247, pp. 279-306, 1954.

[202] Tolokonnikov L.A. Plane strain of the incompressible material (in Russian). Transaction of the Academy of Sciences of USSR, 1957, vol. 119, No. 6.

The plane problem for a wedge is considered in the paper:

[203] Klingbeil W.W., Shield R.T. On a Class of Solutions in Physical Finite Elasticity. Zeitschrift für Mathematik und Physik, vol. 17, No. 4, pp. 489-511, 1966.

A great number of papers are devoted to solutions by the method of successive approximation (Muskhelishvili's method). Let us mention the following ones:

[204] Carlson D.E., Shield R.T. Second and Higher Order Effects in a Class of Problems in Plane Finite Elasticity, Archive for Rational Mechanics and Analysis, 1965, vol. 19, No. 3, pp. 189-214.

[205] Koifman Yu.I. Solution to the plane problem of the nonlinear theory of elasticity with a curvilinear opening (in Russian). Izvestiya Vysshikh Uchebnykh Zavedenyi, Stroitelstvo I Arkhitektura, 1961, No. 1, pp. 44-51, Novosibirsk 1961.

Koifman Yu.I. Solutions of problems of nonlinear plane theory of elasticity (in Ukrainian). Transactions of Lvov State University, Division of Mechanics and Mathematics, 1962, No. 9.

Koifman Yu.I. Stress-strain state of tubes and annular discs of highly elastic nonlinear elastic material (in Russian). Dinamika i Prochnost Mashin, Publishers of Kharkov University, 1966, No. 3, pp. 75-81.

Savin G.N., Koifman Yu.I. Nonlinear effects in problems of stress concentration near openings with a stiffened edge (in Russian). Prikladnaya Mekhanika, 1965, vol. 1, No. 9, pp. 1-13.

Koifman Yu.I., Langleiben A.Sh. Large elastic deformations of a two-layered cylinder (in Russian). Prikladnaya Mekhanika, 2, No. 9, pp. 71-72, 1966.

[206] Gromov V.G. On the influence of physical nonlinearity on stress concentration around a circular opening under large strains (in Russian). Prikladnaya Mekhanika, 1965, vol. 1, No. 10.

Gromov V.G. Stress concentration around a circular cylindrical cavity in an infinite nonlinear elastic body (in Russian). Scientific Reports of Rostov University, Series of Exact and Natural Sciences, 1964, 67.

Gromov V.G., Tolokonnikov L.A. On calculation of approximations in the problem of finite deformations of incompressible material (in Russian). Transactions of the Academy of Sciences of USSR , Division of Technical Sciences, 1953, vol. 2.

The plane problem of the nonlinear theory of elasticity is considered in Chapter IX of book [168]. A comprehensive bibliography is provided.

Section 9.7

The constitutive law equivalent to eq. (7.1.1) is presented in the paper:

[207] Sensenig C.B. Instability of Thick Elastic Solid. Commun. Pure and Appl. Math., vol. 17, No. 4, pp. 451-491, 1964.

The statement of the boundary-value problems relevant to the case of plane strain is given in the paper:

[208] John P. Plane Strain Problems for a Perfectly Elastic Material of Harmonic Type. Commun. Pure and Appl. Math., vol. 13, No. 2, pp. 239-296, 1960.

Subsections 9.7.9 and 9.7.10

[209] Southwell R.V. On the General Theory of Elastic Stability. Phil. Trans. Roy. Soc. London, Ser. A, vol. 213, pp. 187-244, 1913.

[210] Biezeno C.B., Grammel R. Technische Dynamik. Springer, Berlin, 1953, vol. 1.

The problems of Subsections 9.7.11-9.7.13 and some analogous ones (hollow cylinder under external pressure, circular plane loaded on its edge) are considered in [207]. Another statement of the problem for the spherical shell is considered in the paper:

[211] Feodosiev V.I. On the forms of equilibrium of a rubber spherical shell under internal pressure (in Russian). Prikladnaya Matematika i Mekhanika, 1968, vol. 32, No. 2 pp. 339-344.

Section 9.7, see also

[212] Lurie A.I. Theory of elasticity for a semi-linear material (in Russian). Prikladnaya Matematika i Mekhanika, 1968, vol. 32, No. 6.

Appendices A-E

More details about tensor calculus can be found in the books:

[213] Kilchevsky N.A. Elements of tensor calculus and the applications to mechanics (in Russian). Gostekhizdat, Moscow 1954.

[214] MacConnell A.J. Application of Tensor Analysis. New York, Dower, 1957.

[215] Schouten J.A. Tensor analysis for physicists, New York, Dower, 1954.

While writing **Appendix F** book [63] and the following books were used:

[216] Hobson E.W. The Theory of Spherical and Ellipsoidal Harmonics. New York: Chelsea, 1955.

[217] Appell P. Traité de Méchanique Rationnelle. Figures d'équilibre d'une mase liquide homogène en rotation. Gauthier-Villars, Paris, 1932.

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