Linear Elastic Materials

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Outline

- Introduction（引言）
- Linear elastic model（线弹性本构）
- Matrix representation（线性本构的矩阵表示）
- Symmetry of stiffness/compliance tensor（刚度/柔度张量的对称性）
- Linear elastic anisotropic models（线性各向异性模型）
- Linear elastic orthotropic model（线性正交对称模型）
- Linear elastic cubic model（线性立方对称模型）
- Linear elastic isotropic model（线性各向同性本构）
- Small stretches but large rotations（小应变大转动）

Introduction

- Relations that characterize the mechanical behavior of materials
- Perhaps one of the most challenging fields in mechanics, due to the endless variety of materials and loadings
- The mechanical behavior of solids is normally defined by constitutive stress-strain relations

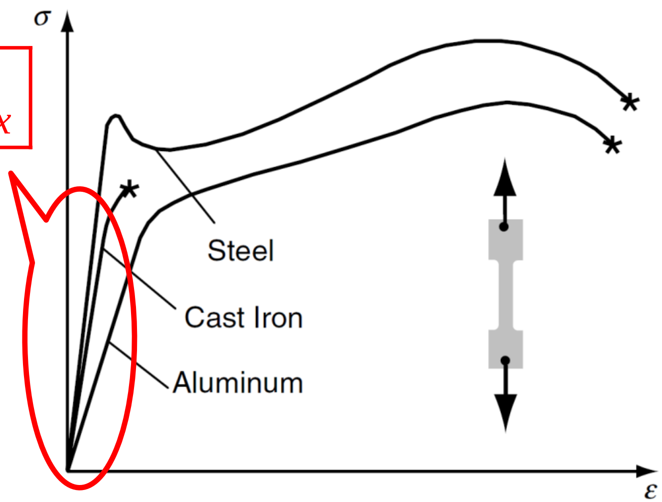
$$\boldsymbol{\sigma} = f\left(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, t, \mathbf{T}, \cdots\right)$$

- Linear elastic model (Hooke’s law)
- Elastic-plastic model
- Visco-elastic model
- Visco-plastic model

Introduction

- Neglect strain rate, time and loading history dependency
- Set aside thermal, electrical, pore-pressure, and other loads
- Include only mechanical loads
- Assume linear stress-strain relationship
- Defined as materials that recover original configuration when mechanical loads are removed
- Agree well with experimental tests of metals

$$\sigma_x = E\varepsilon_x$$



Linear Elastic Model

- Hooke's law in 1D:

$$\sigma = E\varepsilon$$

- In 3D, one might generalize this in tensor form as

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}; \qquad \boldsymbol{\varepsilon} = \mathbf{S} : \boldsymbol{\sigma} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

- Most generally, **C** has 81 independent components.
- Thanks to the (minor) symmetry properties

$$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl}, \qquad \varepsilon_{kl} = \varepsilon_{lk} \Rightarrow C_{ijkl} = C_{ijlk}$$

- The number of independent components is reduced to 36.

Matrix Representation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$= C_{ij11}\varepsilon_{11} + C_{ij22}\varepsilon_{22} + C_{ij33}\varepsilon_{33} + 2C_{ij12}\varepsilon_{12} + 2C_{ij13}\varepsilon_{13} + 2C_{ij23}\varepsilon_{23}$$

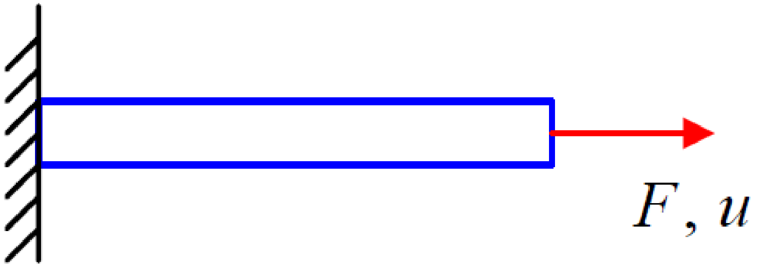
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

- 36 elastic constants

Major Symmetry of Stiffness/Compliance Tensor

- Assume an increment of **u**

  $u \to u + \delta u$



- The work increment done by **F**

$$\delta W = F\delta u = \sigma A \delta\left(l\varepsilon\right) = V\sigma\delta\varepsilon = \delta U$$

- Strain energy density:

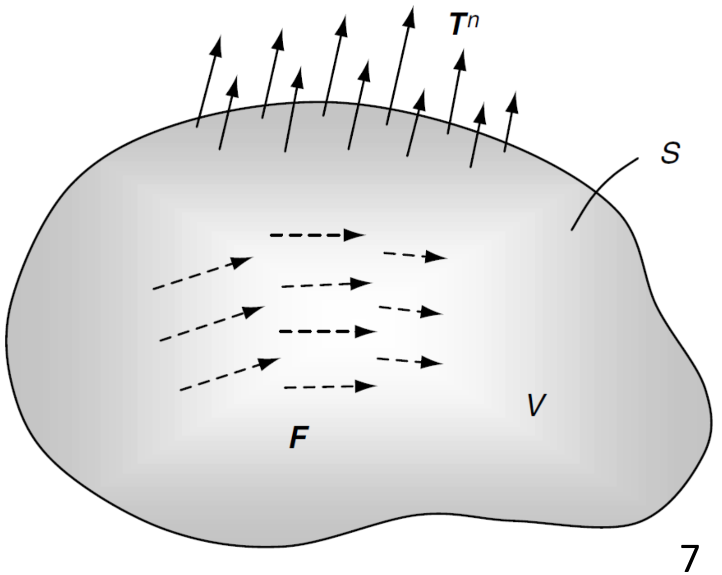
$$\delta U_0 = \delta U/V = \delta W/V = \sigma\delta\varepsilon; \quad U_0 = U_0\left(\varepsilon\right), \sigma = \frac{\partial U_0}{\partial \varepsilon}, U_0\left(\varepsilon\right) = \int_0^{\varepsilon} \sigma\delta\varepsilon$$

- Generalize to 3D

$$\delta W = \iiint_V \mathbf{F}\cdot\delta\mathbf{u}dV + \iint_{S_t} \mathbf{T}\cdot\delta\mathbf{u}dS$$

$$= \iiint_V F_i\delta u_i dV + \iint_{S_t} T_i\delta u_i dS$$

$$= \iiint_V F_i\delta u_i dV + \iint_{S_t} n_j\sigma_{ji}\delta u_i dS$$



Major Symmetry of Stiffness/Compliance Tensor

- Applying the divergence theorem on the surface integral:

$$\delta W = \iiint_V \left[F_i \delta u_i + \frac{\partial}{\partial x_j}\left(\sigma_{ji}\delta u_i\right) \right] dV = \iiint_V \left[F_i \delta u_i + \frac{\partial \sigma_{ji}}{\partial x_j}\delta u_i + \sigma_{ji}\frac{\partial \delta u_i}{\partial x_j} \right] dV$$

$$= \iiint_V \left[\left(F_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right)\delta u_i + \sigma_{ij}\left(\delta\varepsilon_{ij} + \delta\omega_{ij}\right) \right] dV$$

$$= \iiint_V \left[\left(F_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right)\delta u_i + \sigma_{ij}\delta\varepsilon_{ij} \right] dV = \iiint_V \sigma_{ij}\delta\varepsilon_{ij} dV = \delta U$$

- Major symmetry property

$$\delta U_0 = \sigma_{ij}\delta\varepsilon_{ij} \quad \Rightarrow U_0 = \int \sigma_{ij}\delta\varepsilon_{ij} = \int C_{ijkl}\varepsilon_{kl}\delta\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

$$\Rightarrow C_{ijkl} = \frac{\partial^2 U_0}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} \qquad \Rightarrow C_{ijkl} = C_{klij}$$

Matrix Representation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

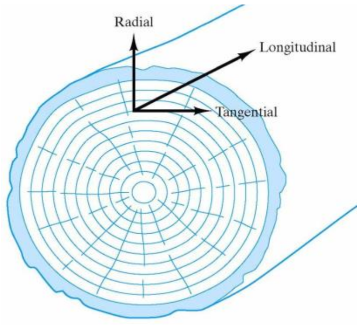
$$= C_{ij11}\varepsilon_{11} + C_{ij22}\varepsilon_{22} + C_{ij33}\varepsilon_{33} + 2C_{ij12}\varepsilon_{12} + 2C_{ij13}\varepsilon_{13} + 2C_{ij23}\varepsilon_{23}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ & & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ & & & C_{1212} & C_{1213} & C_{1223} \\ & \text{Symm.} & & & C_{1313} & C_{1323} \\ & & & & & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

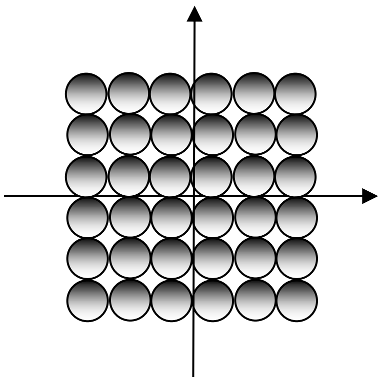
- 21 elastic constants

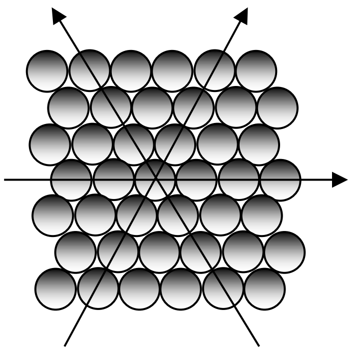
Linear Elastic Anisotropic Models

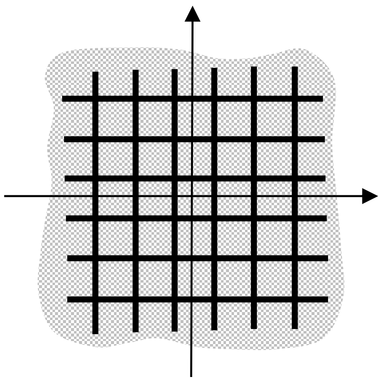
- Differences in material properties along different directions.
- Materials like wood, crystalline minerals, fiber-reinforced composites have such behavior.



Typical Wood Structure


Body-Centered Cubic Crystal

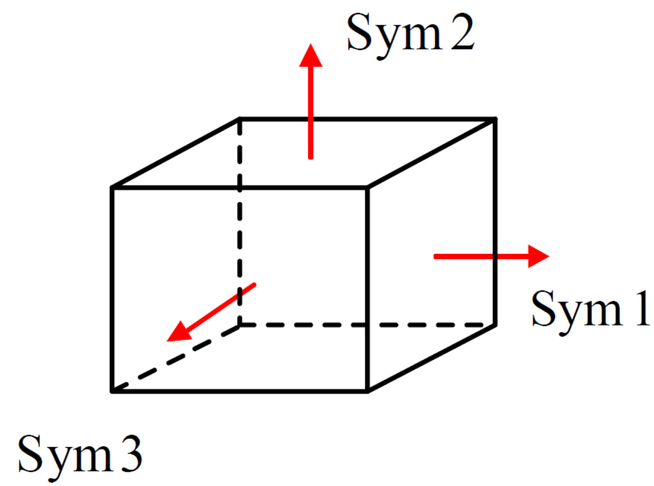

Hexagonal Crystal


Fiber Reinforced Composite

- Note particular material symmetries indicated by arrows.

Linear Elastic Orthotropic Model

- Wood is an example of an orthotropic material. Material properties in three perpendicular directions (axial, radial, and circumferential) are different.

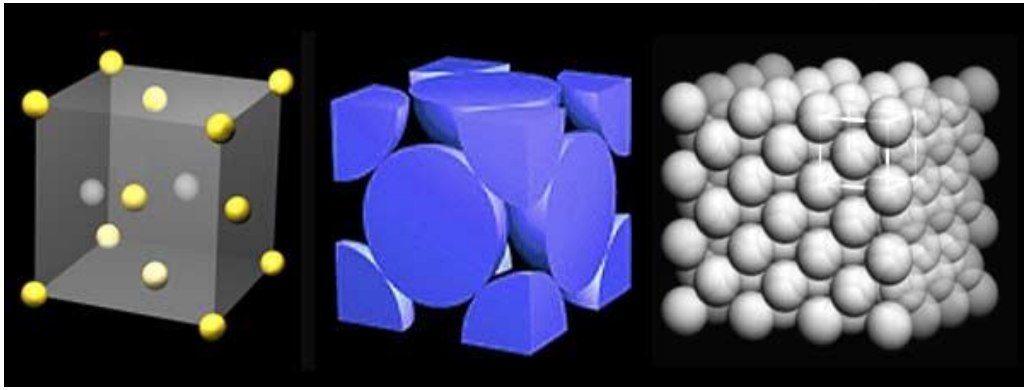


$$\mathbf{C} : \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix}$$

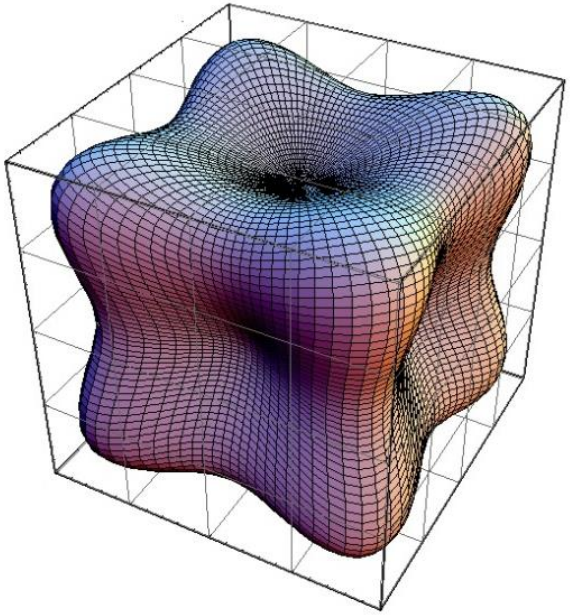
- 9 elastic constants

Linear Elastic Cubic Model

- Single crystal metals: FCC (Cu, Al, Ag, etc.); BCC (Fe, etc.)



$$\mathbf{C}: \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & Sym & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix}$$



- 3 elastic constants

Linear Elastic Isotropic Model

- Isotropic tensors are defined as those tensors whose components remain the same under all transformations
- The most general fourth order isotropic tensor

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

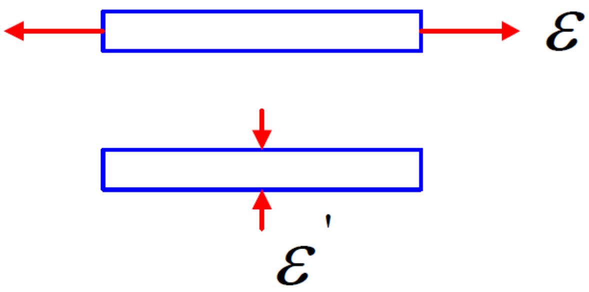
- Imposing the symmetry conditions

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad S_{ijkl} = d_1\delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ & \lambda+2\mu & \lambda & & & \\ & & \lambda+2\mu & & & \\ & & & \mu & & \\ & \text{Symm.} & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

- The constants are to be determined experimentally.

Linear Elastic Isotropic Model



- Since there are only 2 independent elastic constants, it suffices to conduct experiments in 1D: $E = \sigma/\varepsilon, \ \nu = -\varepsilon'/\varepsilon$
- Specify the general stress-strain relation:

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} = \left\{d_1\delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\}\sigma_{kl} = d_1\sigma_{kk}\delta_{ij} + 2d_2\sigma_{ij}$$

$$\sigma_{11} \neq 0, \sigma_{22} = \sigma_{33} = 0 \Rightarrow \begin{cases} \varepsilon_{11} = (d_1 + 2d_2)\sigma_{11} = \sigma_{11}/E \\ \varepsilon_{22} = d_1\sigma_{11} = -\nu\,\sigma_{11}/E \end{cases} \Rightarrow \begin{cases} d_1 = -\nu/E \\ d_2 = (1+\nu)/2E \end{cases}$$

$$\Rightarrow \begin{cases} S_{ijkl} = -\dfrac{\nu}{E}\delta_{ij}\delta_{kl} + \dfrac{1+\nu}{2E}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \\ \boxed{\varepsilon_{ij} = S_{ijkl}\sigma_{kl} = \dfrac{1+\nu}{E}\sigma_{ij} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij}} \end{cases}$$

Linear Elastic Isotropic Model

- Stresses in terms of strains

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\Rightarrow \varepsilon_{mm} = \frac{1+\nu}{E}\sigma_{mm} - \frac{3\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{mm} \quad \Rightarrow \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk}$$

$$\Rightarrow \frac{1+\nu}{E}\sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \varepsilon_{ij} + \frac{\nu}{E}\frac{E}{1-2\nu}\varepsilon_{kk}\delta_{ij}$$

$$\Rightarrow \boxed{\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = \left\{\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\}\varepsilon_{kl} \quad \Rightarrow \boxed{\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}}$$

$$\Rightarrow \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$$

Linear Elastic Isotropic Model

- A plot of shear stress vs. shear strain is similar to the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains

$$\tau_{xy} = G\gamma_{xy}$$

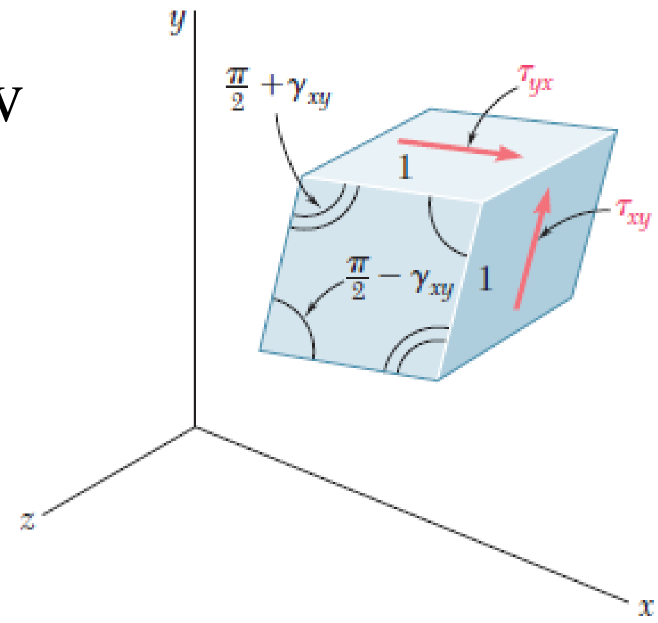
where G is the modulus of rigidity or shear modulus.

- In terms of the general Hooke's law

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\Rightarrow \sigma_{12} = \tau = 2\mu\varepsilon_{12} = \mu\gamma = G\gamma$$

$$\Rightarrow \boxed{G = \mu = \frac{E}{2(1+\nu)}}$$



Linear Elastic Isotropic Model

$$\left(E,\nu\right): \lambda = \frac{E\nu}{\left(1-2\nu\right)\left(1+\nu\right)}, G = \frac{E}{2\left(1+\nu\right)}, K = \frac{E}{3\left(1-2\nu\right)}$$

$$\left(\lambda,G\right): E = \frac{G\left(3\lambda+2G\right)}{\lambda+G}, \nu = \frac{\lambda}{2\left(\lambda+G\right)}, K = \frac{3\lambda+2G}{3}$$

$$\left(G,\nu\right): E = 2G\left(1+\nu\right), \lambda = \frac{2G\nu}{1-2\nu}, K = \frac{2G\left(1+\nu\right)}{3\left(1-2\nu\right)}$$

$$\left(\lambda,\nu\right): E = \frac{\lambda\left(1+\nu\right)\left(1-2\nu\right)}{\nu}, G = \frac{\lambda\left(1-2\nu\right)}{2\nu}, K = \frac{\lambda\left(1+\nu\right)}{3\nu}$$

$$\left(E,G\right): \nu = \frac{E-2G}{2G}, K = \frac{EG}{3\left(3G-E\right)}, \lambda = \frac{G\left(E-2G\right)}{3G-E}; \quad \left(E,K\right): \nu = \frac{3K-E}{6K}, G = \frac{3KE}{9K-E}, \lambda = \frac{3K\left(3K-E\right)}{9K-E}$$

$$\left(E,\lambda\right): \nu = \frac{2\lambda}{E+\lambda+R}, G = \frac{E-3\lambda+R}{4}, K = \frac{E+3\lambda+R}{6}; \quad \left(K,\nu\right): E = 3K\left(1-2\nu\right), G = \frac{3K\left(1-2\nu\right)}{2\left(1+\nu\right)}, \lambda = \frac{3K\nu}{1+\nu}$$

$$\left(G,K\right): E = \frac{9KG}{3K+G}, \nu = \frac{3K-2G}{6K+2G}, \lambda = \frac{3K-2G}{3}; \quad \left(K,\lambda\right): E = \frac{9K\left(K-\lambda\right)}{3K-\lambda}, \nu = \frac{\lambda}{3K-\lambda}, G = \frac{3}{2}\left(K-\lambda\right)$$

$$R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$$

Linear Elastic Isotropic Model

- Alternative derivation of the generalized Hooke's law

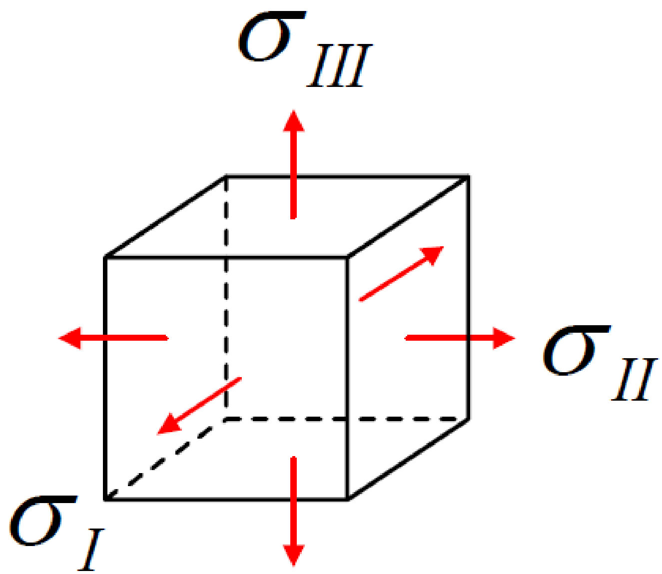
$$\varepsilon_I = \frac{\sigma_I}{E} - \nu\frac{\sigma_{II}}{E} - \nu\frac{\sigma_{III}}{E} = \frac{1+\nu}{E}\sigma_I - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$

$$\varepsilon_{II} = \frac{\sigma_{II}}{E} - \nu\frac{\sigma_I}{E} - \nu\frac{\sigma_{III}}{E} = \frac{1+\nu}{E}\sigma_{II} - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$

$$\varepsilon_{III} = \frac{\sigma_{III}}{E} - \nu\frac{\sigma_I}{E} - \nu\frac{\sigma_{II}}{E} = \frac{1+\nu}{E}\sigma_{III} - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\sigma_{kk}\mathbf{I}$$



Linear Elastic Isotropic Model

- Alternative forms in terms of deviatoric stresses/strains:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}, \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$$

$$\Rightarrow \varepsilon'_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1+\nu}{E}\left(\sigma'_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right) - \frac{\nu}{E}\sigma_{kk}\delta_{ij} = \frac{1+\nu}{E}\sigma'_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij}$$

$$\Rightarrow \boxed{\varepsilon'_{ij} = \frac{1+\nu}{E}\sigma'_{ij} = \frac{\sigma'_{ij}}{2G}, \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk} = \frac{\sigma_{kk}}{3K}, \varepsilon_{ij} = \varepsilon'_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{\sigma'_{ij}}{2G} + \frac{\sigma_{kk}}{9K}\delta_{ij}}$$

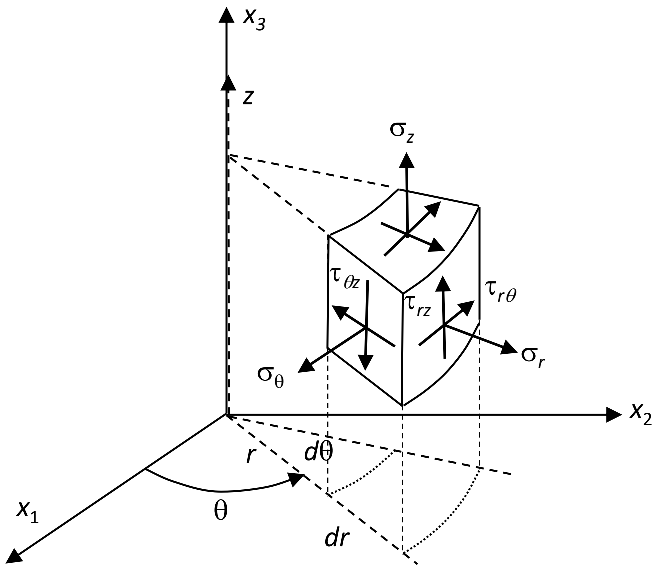
- Strain energy

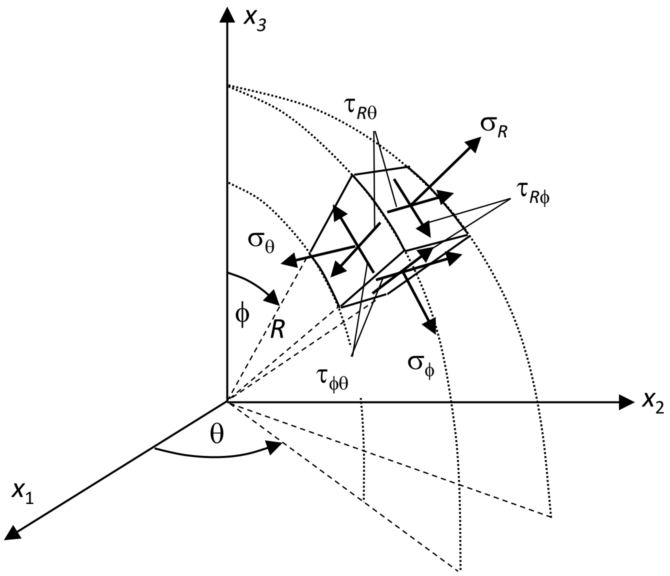
$$U = \int \sigma_{ij} d\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma'_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\varepsilon'_{ij} + \frac{1}{3}\varepsilon_{mm}\delta_{ij}\right)$$

$$= \frac{1}{2}\sigma'_{ij}\varepsilon'_{ij} + \frac{1}{6}\sigma_{kk}\varepsilon_{mm} = G\varepsilon'_{ij}\varepsilon'_{ij} + \frac{1}{2}K\varepsilon_{kk}\varepsilon_{mm} = \frac{1}{4G}\sigma'_{ij}\sigma'_{ij} + \frac{1}{18K}\sigma_{kk}\sigma_{mm}$$

Linear Elastic Isotropic Model

- Recall that, **the elastic stiffness tensor *C* is a fourth order isotropic tensor**.
- Its components remain unchanged under any orthogonal coordinate systems.
- The isotropic Hooke's law stays the same.





Small Stretches but Large Rotations

- Stress-strain law for an anisotropic, linear elastic material

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

- This stress-strain relation can only be used if the material is subjected to small deformations and small rotations.
- Compute the linear strain due to a rigid-body rotation:
- The error of linear strain is in the order of θ^2.
- The definition of linear strain must be discarded for deformations involving large rotation.

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} - \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} \varepsilon_1 = \dfrac{\partial u_1}{\partial x_1} = \cos\theta - 1, \quad \varepsilon_2 = \dfrac{\partial u_2}{\partial x_2} = \cos\theta - 1, \\ 2\varepsilon_{12} = \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} = 0. \end{cases}$$

$$\Rightarrow \varepsilon_1 = \varepsilon_2 = \cos\theta - 1 = 1 - \frac{\theta^2}{2} + O\left(\theta^4\right) - 1 \approx -\frac{\theta^2}{2}$$

Small Stretches but Large Rotations

- A nonlinear strain measure, together with its appropriate work-conjugate stress measure, needs to adopted.
- Since both are defined w.r.t. the undeformed configuration, we prefer to use the material stress (PK2)–Lagrange strain relation

$$\Sigma_{ij} = C_{ijkl}E_{kl}, \quad E_{kl} = \frac{1}{2}\left(F_{jk}F_{jl} - \delta_{kl}\right), \quad dP_j^{(\mathbf{n}0)} = dA_0 n_i^0 \Sigma_{ij}$$

- Cauchy stress-Eulerian strain relation

$$\mathbf{E} = \mathbf{F}^T \cdot \mathbf{E}^* \cdot \mathbf{F}, \quad E_{kl} = F_{mk}E_{mn}^* F_{nl}, \qquad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\boldsymbol{\Sigma}\cdot\mathbf{F}^T, \quad \sigma_{ij} = \frac{1}{J}F_{ip}\Sigma_{pq}F_{jq}$$

$$\Rightarrow \sigma_{ij} = \frac{1}{J}F_{ip}\Sigma_{pq}F_{jq} = \frac{1}{J}F_{ip}\left(C_{pqkl}E_{kl}\right)F_{jq} = \frac{1}{J}F_{ip}C_{pqkl}\left(F_{mk}E_{mn}^*F_{nl}\right)F_{jq}$$

$$\Rightarrow \boxed{\sigma_{ij} = C_{ijmn}^* E_{mn}^*, \quad C_{ijmn}^* = \frac{1}{J}F_{ip}F_{jq}F_{mk}F_{nl}C_{pqkl}}$$

Small Stretches but Large Rotations

- For isotropic materials

$$\Sigma_{ij} = C_{ijkl} E_{kl} = \frac{E}{(1+\nu)} E_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} E_{kk} \delta_{ij};$$

$$\sigma_{ij} = C^*_{ijmn} E^*_{mn} = \frac{E}{(1+\nu)} E^*_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} E^*_{kk} \delta_{ij}.$$

- C_{ijkl} and C_{ijkl}* are tensors of elastic moduli with orientations of the undeformed and deformed configurations, respectively.

- For isotropic materials and infinitesimal deformations

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} = \frac{E}{(1+\nu)} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij}$$