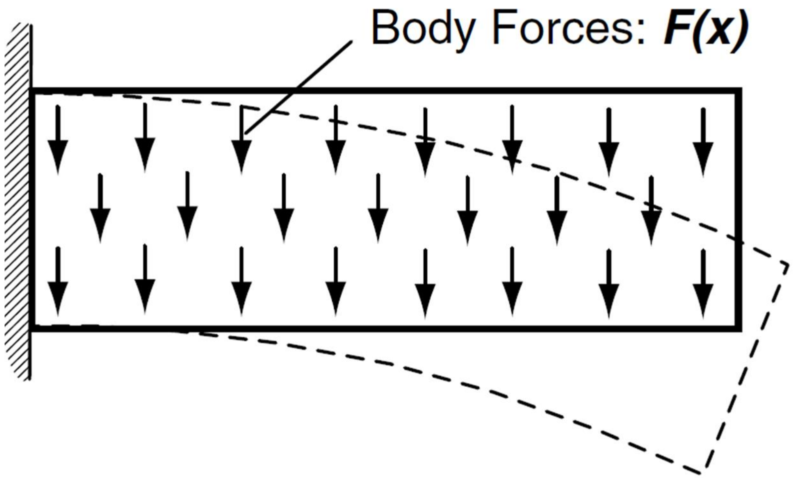
Stress and Equilibrium

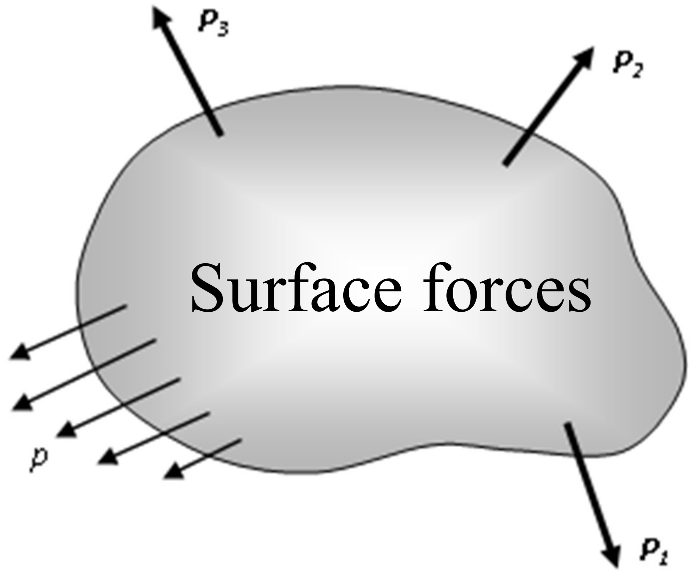
Outline

- Body and Surface Forces
- Traction/Stress Vector
- Stress Tensor
- Traction on Oblique Planes
- Principal Stresses and Directions
- Mohr’s Circles of Stresses
- Octahedral Stresses
- Spherical and Deviatoric Stresses
- Conservation of Linear Momentum
- Conservation of Angular Momentum
- Equilibrium Equations
- Equilibrium Equations in Curvilinear Coordinates

Body and Surface Forces

- External loads include body and surface forces.





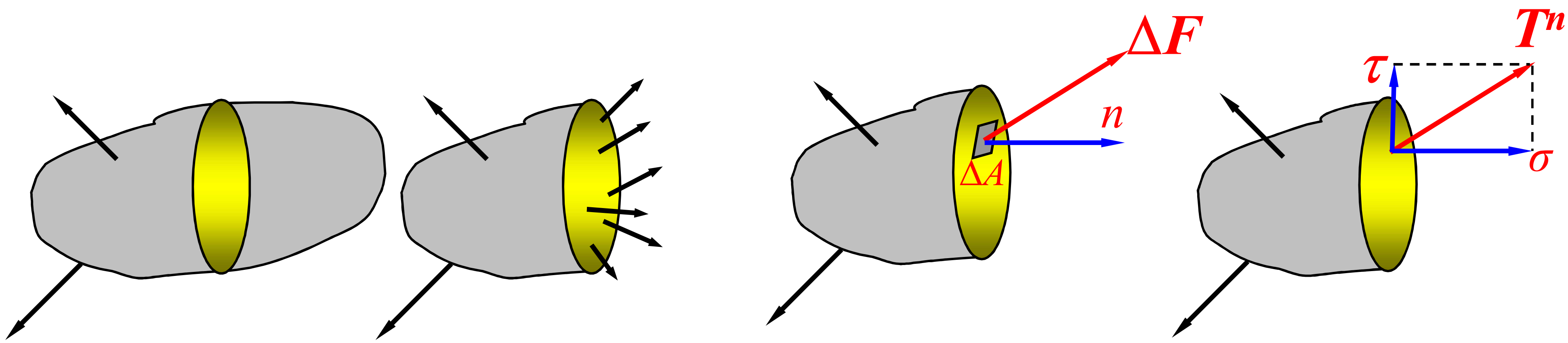
- Forces are vectors (unit: N)

$$\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{e}_3 = F_i\boldsymbol{e}_i$$

- Often interpreted in terms of density: body force density and surface force density

$$\boldsymbol{F}_b = \iiint_V \boldsymbol{F}(\boldsymbol{x})\mathrm{d}V \qquad \boldsymbol{F}_S = \iint_S \boldsymbol{T}^{\boldsymbol{n}}(\boldsymbol{x})\mathrm{d}S$$

Traction/Stress Vector



- Given $\Delta \boldsymbol{F}$ as the force transmitted across ΔA, a stress traction vector can be defined as

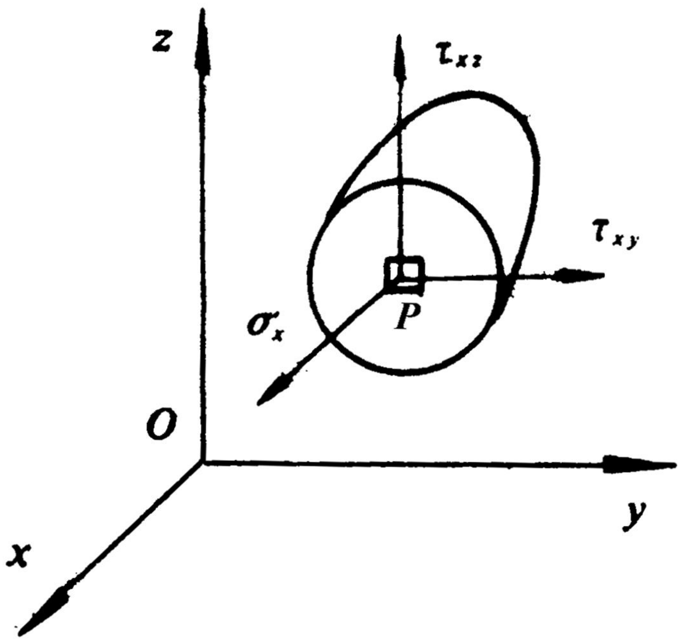
$$\boldsymbol{T}^n(\boldsymbol{x}) = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A}$$

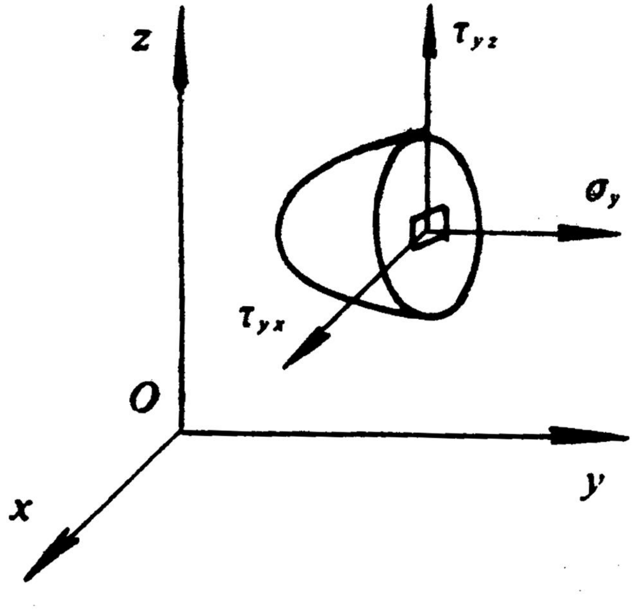
Units: Pa (N/m^2), 1 MPa = 10^6 Pa, 1 GPa = 10^9 Pa.

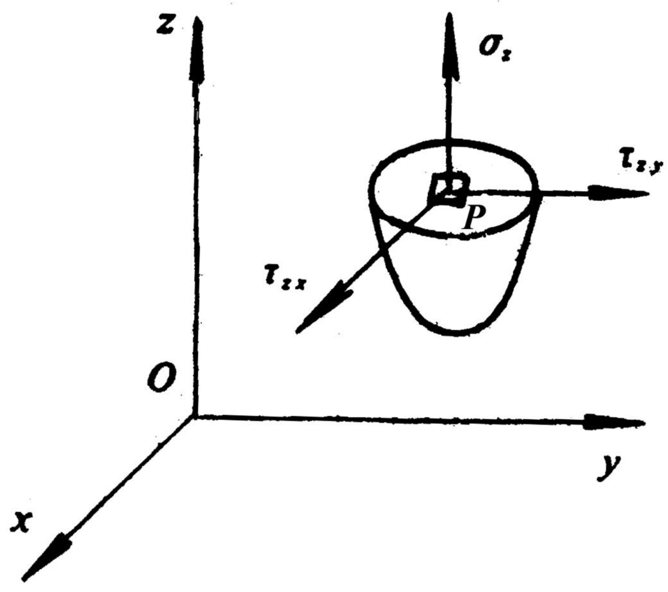
- Decomposition of the traction vector

$$\boldsymbol{T}^n(\boldsymbol{x}) = \sigma \boldsymbol{n} + \tau \boldsymbol{t} = \sigma \boldsymbol{n} + \tau' \boldsymbol{t}' + \tau'' \boldsymbol{t}''$$

Stress Tensor



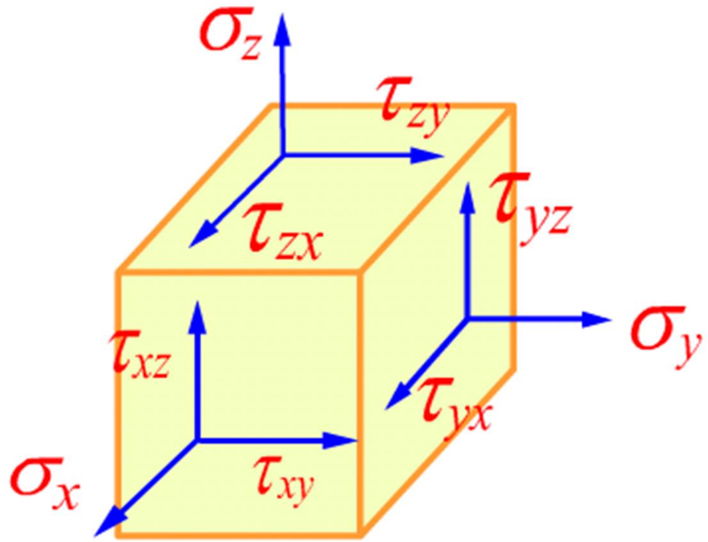


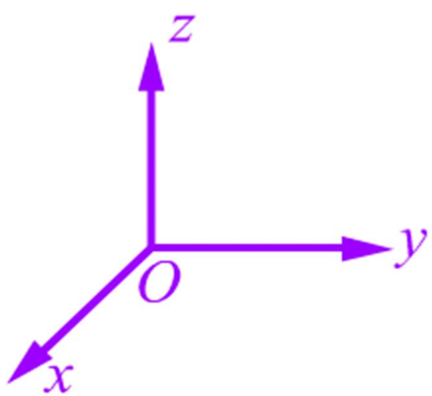


$$\boldsymbol{T}^n(\boldsymbol{x}) = \sigma_x \boldsymbol{e}_x + \tau_{xy}\boldsymbol{e}_y + \tau_{xz}\boldsymbol{e}_z$$

$$\boldsymbol{T}^n(\boldsymbol{x}) = \tau_{yx} \boldsymbol{e}_x + \sigma_{y}\boldsymbol{e}_y + \tau_{yz}\boldsymbol{e}_z$$

$$\boldsymbol{T}^n(\boldsymbol{x}) = \tau_{zx} \boldsymbol{e}_x + \tau_{zy}\boldsymbol{e}_y + \sigma_{z}\boldsymbol{e}_z$$



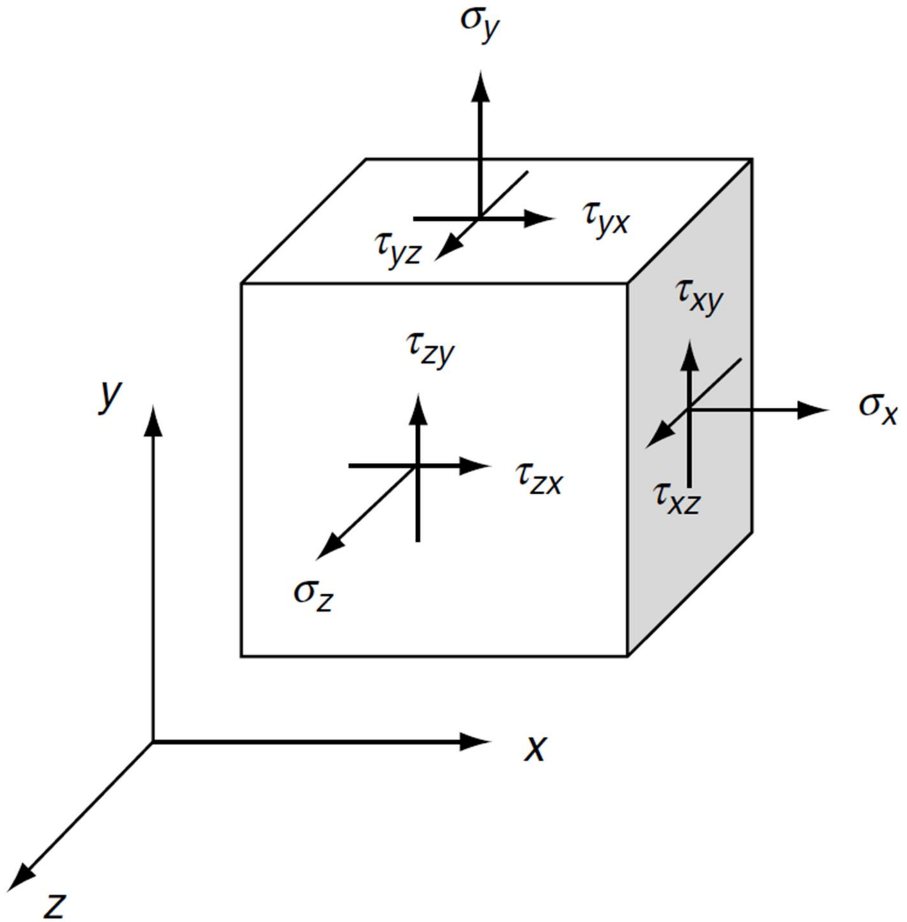


$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

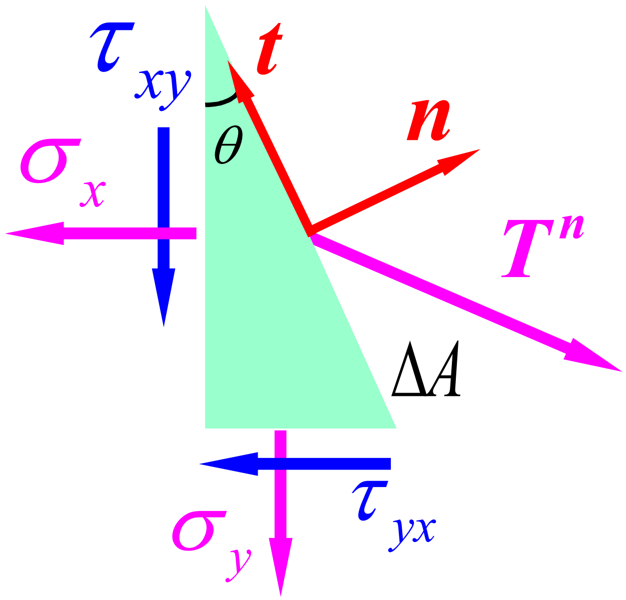
von Karman Notation

Sign Convention

- Normal stress: tension positive / compression negative
- Shear stress: product of the surface normal (the first subscript) and the stress direction (the second subscript)
- All stress components shown in the figure are positive.



Traction on an Oblique Plane - 2D



$$\begin{cases} 0 = \sum F_x \\ 0 = \sum F_y \end{cases}$$

$$\Rightarrow \begin{cases} T_x^n \Delta A = \sigma_x \Delta A \cos\theta + \tau_{yx} \Delta A \sin\theta \\ T_y^n \Delta A = \tau_{xy} \Delta A \cos\theta + \sigma_y \Delta A \sin\theta \end{cases}$$

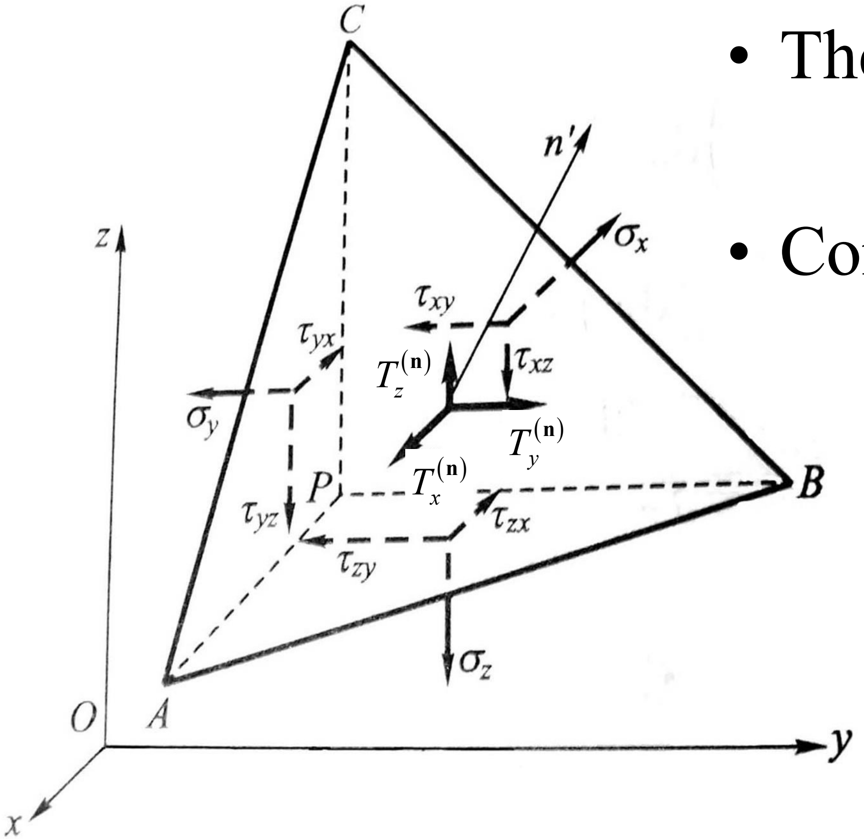
$$\Rightarrow \begin{cases} T_x^n = \sigma_x n_x + \tau_{yx} n_y \\ T_y^n = \tau_{xy} n_x + \sigma_y n_y \end{cases}$$

$$\Rightarrow \{T_x^n \quad T_y^n\} = \{n_x \quad n_y\} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{array}{l} T_\alpha^n = n_\beta \sigma_{\beta\alpha} \\ \boldsymbol{T}^n = \boldsymbol{n} \cdot \boldsymbol{\sigma} \end{array}}$$

2D Cauchy's relation

Traction on an Oblique Plane - 3D



- The state of stress at a point is defined by:

$$\sigma_x, \tau_{xy}, \tau_{xz}, \tau_{yx}, \sigma_y, \tau_{yz}, \tau_{zx}, \tau_{zy}, \sigma_z$$

- Consider the tetrahedron with unit normal $\boldsymbol{n}$

$$n_i = \frac{\boldsymbol{n} \cdot \boldsymbol{e}_i}{\|\boldsymbol{n}\| \, \|\boldsymbol{e}_i\|} = \cos\left(\boldsymbol{n}, \boldsymbol{e}_i\right)$$

$$0 = \sum F_x, \quad 0 = \sum F_y, \quad 0 = \sum F_z$$

$$\Rightarrow \begin{cases} T_x^{\boldsymbol{n}} \Delta A = \sigma_x \Delta A n_x + \tau_{yx} \Delta A n_y + \tau_{zx} \Delta A n_z \\ T_y^{\boldsymbol{n}} \Delta A = \tau_{xy} \Delta A n_x + \sigma_y \Delta A n_y + \tau_{zy} \Delta A n_z \\ T_z^{\boldsymbol{n}} \Delta A = \tau_{xz} \Delta A n_z + \tau_{yz} \Delta A n_y + \sigma_z \Delta A n_z \end{cases}$$

3D Cauchy's relation

$$\Rightarrow \boxed{\begin{array}{l} T_i^{\boldsymbol{n}} = n_j \sigma_{ji} \\ \boldsymbol{T}^{\boldsymbol{n}} = \boldsymbol{n} \cdot \boldsymbol{\sigma} \end{array}}$$

Principal Stresses and Directions

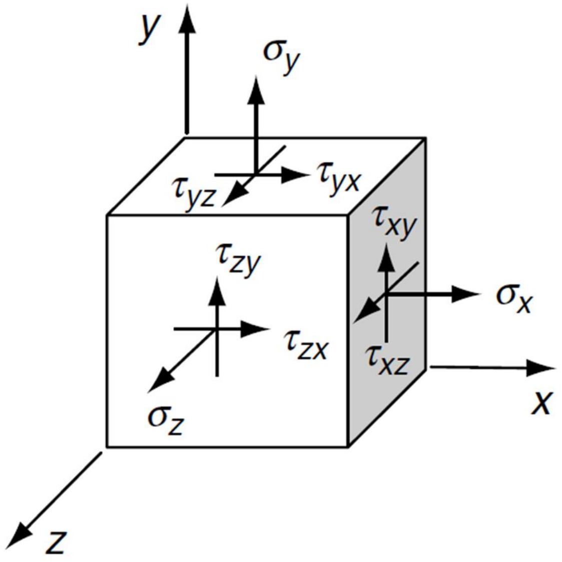
- Seeking the solution through an eigenvalue equation

$$\boldsymbol{T}^n = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\boldsymbol{n} = \sigma_n \boldsymbol{n} \Rightarrow \det\left[\sigma_{ij} - \sigma_n \delta_{ij}\right] = 0$$

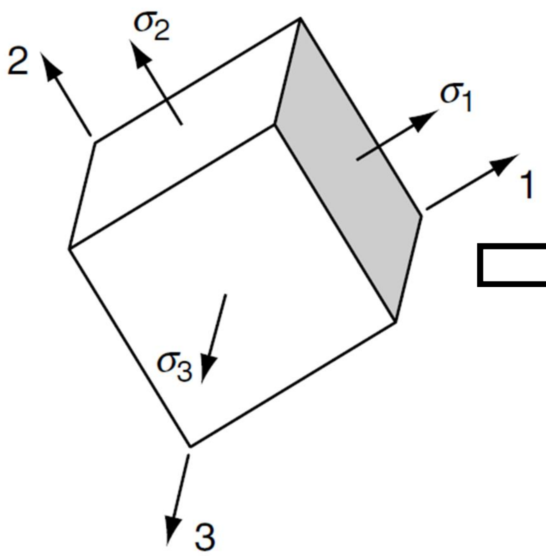
$$\Rightarrow \boxed{-\sigma_n^{\ 3} + I_1 \sigma_n^{\ 2} - I_2 \sigma_n + I_3 = 0}$$

- Three invariants of the stress tensor

$$I_1 = \sigma_{kk}, \quad I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right), \quad I_3 = \det\left[\sigma_{ij}\right].$$



(General Coordinate System)



(Principal Coordinate System)

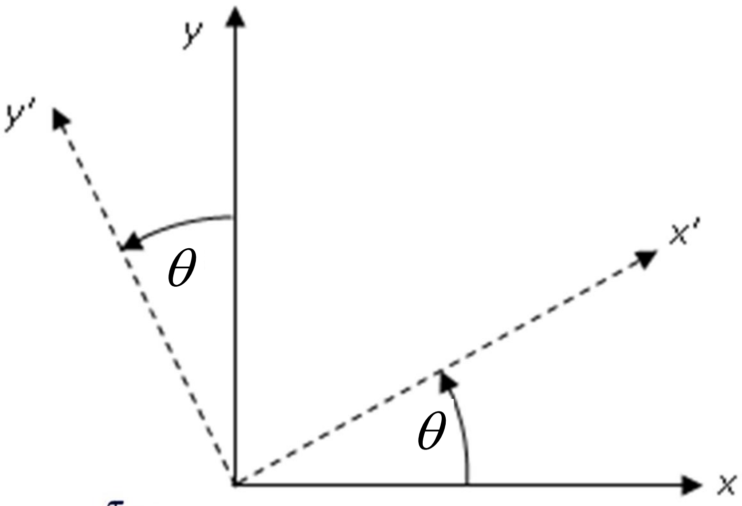
$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

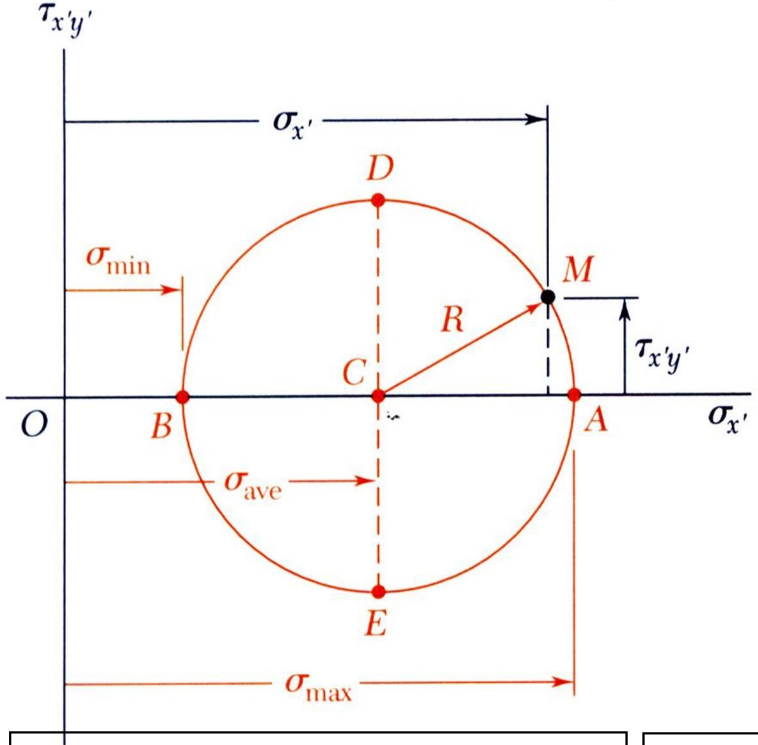
$$I_1 = \sigma_1 + \sigma_2 + \sigma_3,$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1,$$

$$I_3 = \sigma_1\sigma_2\sigma_3.$$

Stress Transformation - 2D





$$\sigma'_{\alpha\beta} = Q_{\alpha\gamma} Q_{\beta\delta} \sigma_{\gamma\delta}$$

$$\boldsymbol{\sigma}' = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$$

$$\boldsymbol{\sigma}' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$\Rightarrow \begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

$$\Rightarrow \begin{cases} \left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau_{x'y'}^2 = R^2 \\ \sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2};\ R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{cases}$$

Mohr's stress circle

Stress symmetry will be proved shortly.

Stress Transformation - 3D

- Take (the simplest) spherical transformation as an example

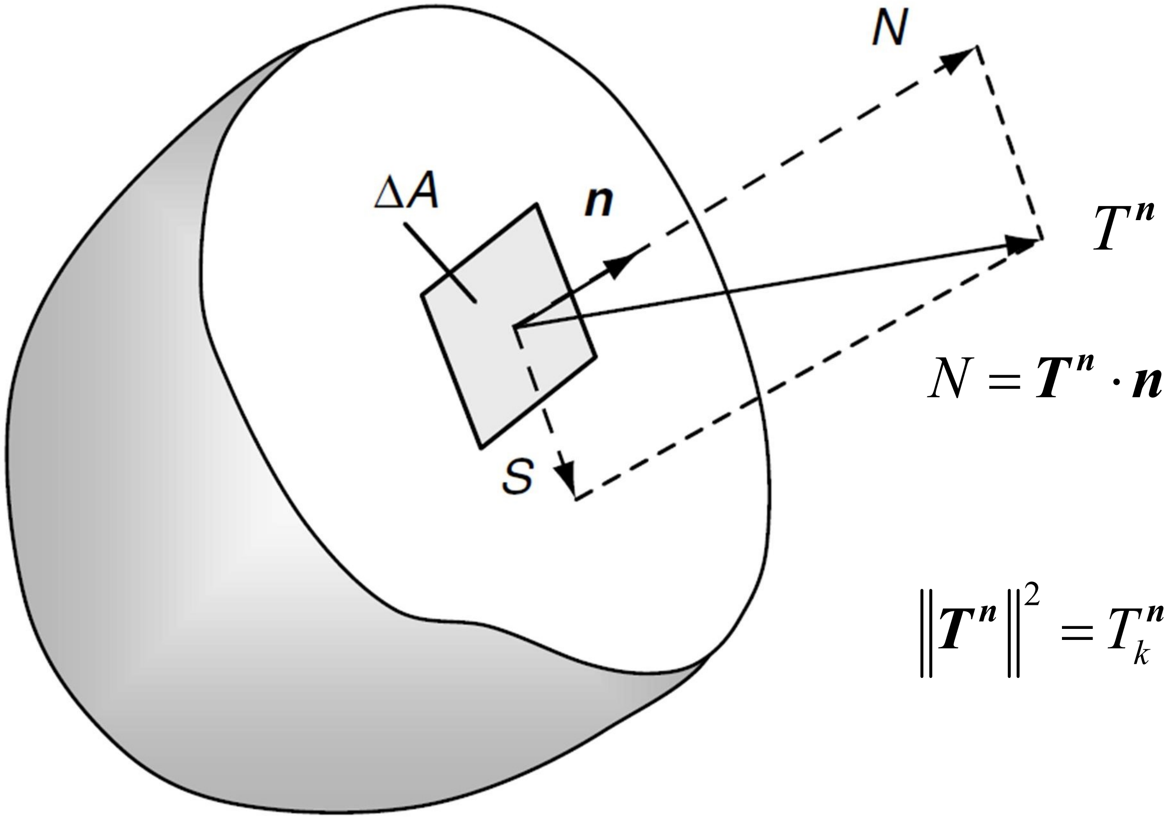
$$\begin{Bmatrix} \boldsymbol{e}_R \\ \boldsymbol{e}_\varphi \\ \boldsymbol{e}_\theta \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_z \\ \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{Bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \boldsymbol{e}_z \\ \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{Bmatrix}$$

$$\sigma'_{ij} = Q_{ik} Q_{jl} \sigma_{kl}; \quad \boldsymbol{\sigma}' = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$$

$$\begin{bmatrix} \sigma'_R & \tau'_{R\varphi} & \tau'_{R\theta} \\ \tau'_{R\varphi} & \sigma'_\varphi & \tau'_{\theta\varphi} \\ \tau'_{R\theta} & \tau'_{\theta\varphi} & \sigma'_\theta \end{bmatrix} = \begin{bmatrix} Q_{Rz} & Q_{Rx} & Q_{Ry} \\ Q_{\varphi z} & Q_{\varphi x} & Q_{\varphi y} \\ Q_{\theta z} & Q_{\theta x} & Q_{\theta y} \end{bmatrix} \begin{bmatrix} \sigma_z & \tau_{zx} & \tau_{zy} \\ \tau_{zx} & \sigma_x & \tau_{xy} \\ \tau_{zy} & \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} Q_{Rz} & Q_{Rx} & Q_{Ry} \\ Q_{\varphi z} & Q_{\varphi x} & Q_{\varphi y} \\ Q_{\theta z} & Q_{\theta x} & Q_{\theta y} \end{bmatrix}^T$$

- No *Mohr's sphere* exists.
- Not meaningful in the context of geometry.

Traction Vector Decomposition



$$N = \boldsymbol{T^n} \cdot \boldsymbol{n} = T_i^{\boldsymbol{n}} n_i = \sigma_{ji} n_j n_i = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

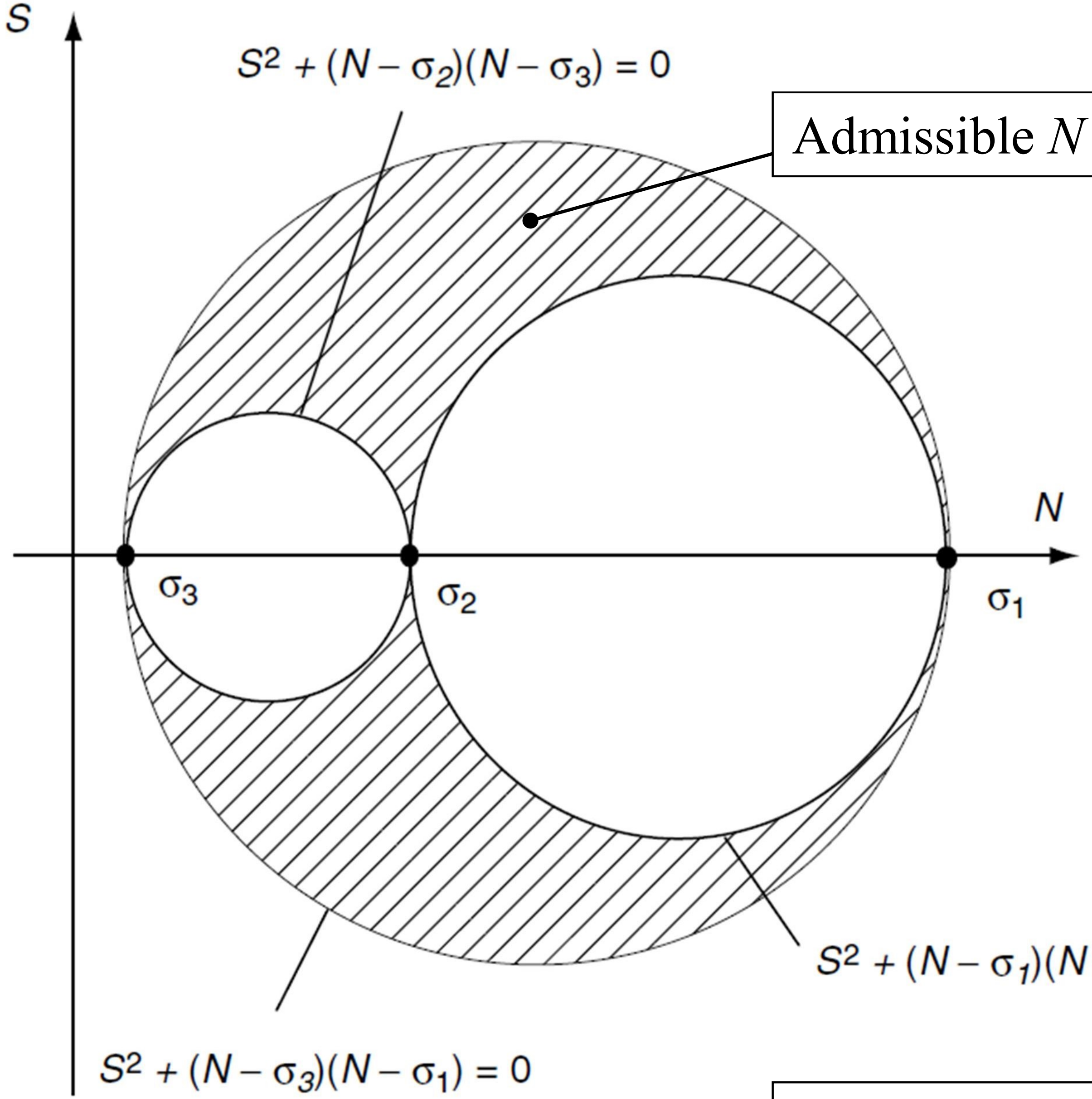
$$N^2 + S^2 = \left\| \boldsymbol{T^n} \right\|^2$$

$$\left\| \boldsymbol{T^n} \right\|^2 = T_k^{\boldsymbol{n}} T_k^{\boldsymbol{n}} = \sigma_{ik} n_i \sigma_{jk} n_j = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

$$\Rightarrow \begin{cases} \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = N \\ \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = N^2 + S^2 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{cases} \Rightarrow \begin{cases} n_1^2 = \dfrac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\ n_2^2 = \dfrac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\ n_3^2 = \dfrac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \end{cases}$$

The principal space is taken as the reference.

Mohr's Circles of Stress



Admissible N and S values lie in the shaded area.

For $\sigma_1 \geq \sigma_2 \geq \sigma_3$

$$\Rightarrow \begin{cases} S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0 \\ S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \\ S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0 \end{cases}$$

$$S_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

The principal space is taken as the reference.

Sample Problem

- For the following state of stress, determine the principal stresses and directions and find the traction vector on a plane with the given unit normal. Also, determine the normal and shear stresses on this plane.

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad \boldsymbol{n} = \left\{ 0 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \right\}.$$

- Solution:

$$I_1 = 3, \quad I_2 = -6, \quad I_3 = -8$$

$$\Rightarrow -\sigma_n{}^3 + 3\sigma_n{}^2 + 6\sigma_n - 8 = 0$$

$$\Rightarrow \sigma_1 = 4, \quad \sigma_2 = 1, \quad \sigma_3 = -2.$$

$$\Rightarrow \begin{cases} -n_1^{(1)} + n_2^{(1)} + n_3^{(1)} = 0 \\ n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} = 0 \\ n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} = 0 \end{cases}$$

$$\Rightarrow \boldsymbol{n}^{(1)} = \left(2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\right)/\sqrt{6}$$

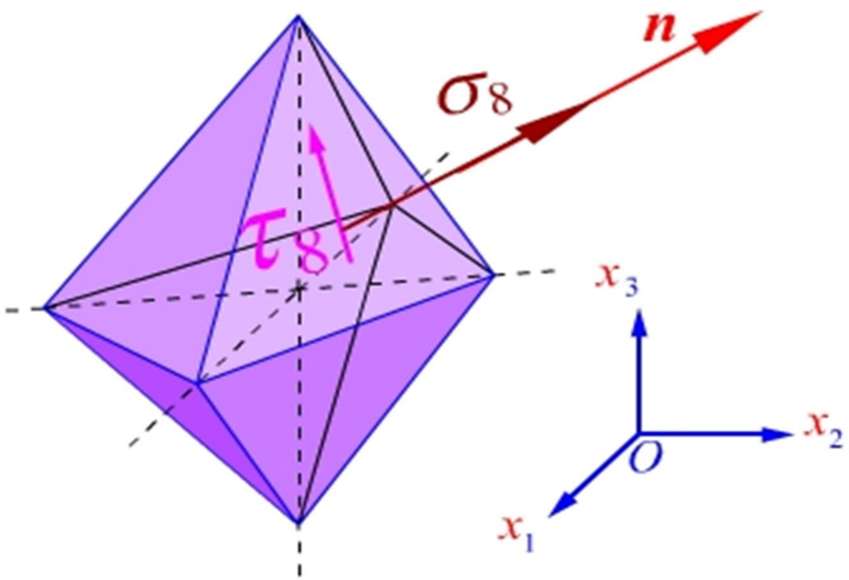
$$\Rightarrow \begin{cases} \boldsymbol{n}^{(2)} = \left(-\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\right)/\sqrt{3} \\ \boldsymbol{n}^{(3)} = \left(-\boldsymbol{e}_2 + \boldsymbol{e}_3\right)/\sqrt{2} \end{cases}$$

$$\boldsymbol{T}^n = \boldsymbol{n} \cdot \boldsymbol{\sigma} = \left\{ 0 \quad 1/\sqrt{2} \quad 1/\sqrt{2} \right\} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \sqrt{2}\left(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\right)$$

$$N = \boldsymbol{T}^n \cdot \boldsymbol{n} = \sqrt{2}\left(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\right) \cdot \left(\boldsymbol{e}_2 + \boldsymbol{e}_3\right)/\sqrt{2} = 2$$

$$S = \sqrt{\left\|\boldsymbol{T}^n\right\|^2 - N^2} = \sqrt{2}$$

Sample Problem − Octahedral Stress



$$\boldsymbol{n} = \left(\pm \boldsymbol{e}_1 \pm \boldsymbol{e}_2 \pm \boldsymbol{e}_3\right) / \sqrt{3}$$

$$\Rightarrow \begin{cases} \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = N \\ \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 = N^2 + S^2 \end{cases}$$

$$\Rightarrow \boxed{N = \sigma_8 = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{1}{3} I_1}$$

$$\Rightarrow \begin{aligned} S = \tau_8 &= \sqrt{\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - N^2} \\ &= \frac{1}{3}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2} \\ &= \frac{1}{3}\sqrt{\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 - 6\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)} \\ &= \frac{1}{3}\sqrt{2I_1^2 - 6I_2} \end{aligned}$$

- Octahedral shear stress is proportional to the equivalent stress of the maximum distortion energy yield criterion.
- Extremely significant for plastic deformation

Spherical and Deviatoric Stress

- Decomposition of the stress tensor

$$\sigma_{ij} = \tilde{\sigma}_{ij} + \hat{\sigma}_{ij}$$

- Spherical (mean) stress tensor: volume change + isotropic

$$\tilde{\sigma}_{ij} = \sigma_m \delta_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)\delta_{ij}$$

- Deviatoric (octahedral) stress tensor: shape change

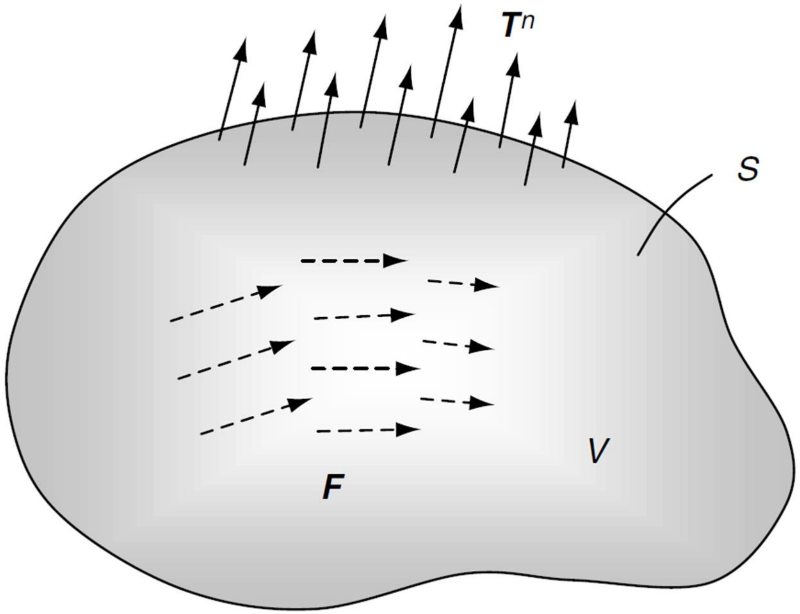
$$\hat{\sigma}_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij}$$

- Relationships among principal stresses and directions

$$\hat{\sigma}_{ij}\hat{n}_j = \hat{\sigma}_n \hat{n}_i \Rightarrow \left(\sigma_{ij} - \sigma_m \delta_{ij}\right)\hat{n}_j = \hat{\sigma}_n \hat{n}_i \Rightarrow \sigma_{ij}\hat{n}_j = \left(\hat{\sigma}_n + \sigma_m\right)\hat{n}_i$$

$$\sigma_{ij} n_j = \sigma_n n_i \Rightarrow \begin{cases} \hat{n}_i = n_i \\ \hat{\sigma}_n = \sigma_n - \sigma_m \end{cases}$$

Conservation of Linear Momentum

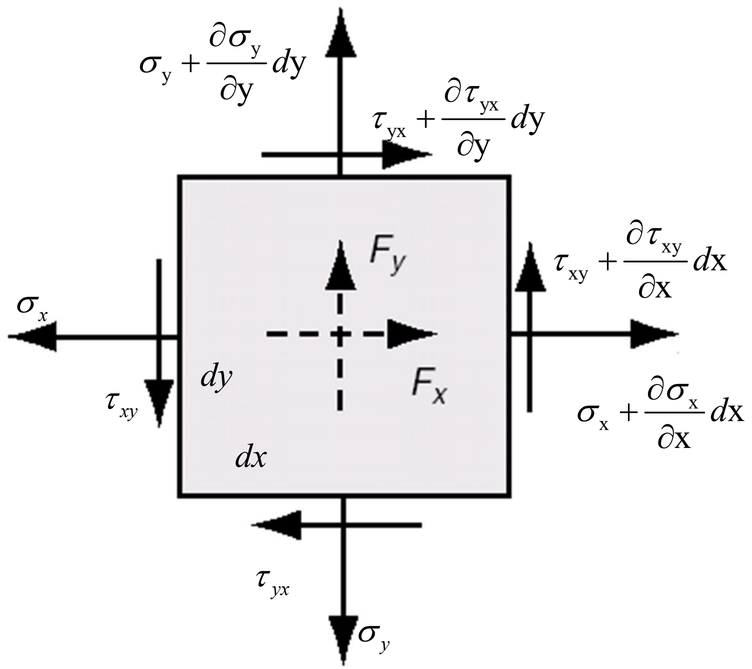


$$\sum F_i = \iint_S T_i^{\boldsymbol{n}} \mathrm{d}S + \iiint_V F_i \mathrm{d}V = \iint_S \sigma_{ji} n_j \mathrm{d}S + \iiint_V F_i \mathrm{d}V$$

$$= \iiint_V \left(\sigma_{ji,j} + F_i \right) \mathrm{d}V$$

3-D

$$\Rightarrow \boxed{\sigma_{ji,j} + F_i = 0} \Rightarrow \boxed{\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0 \end{aligned}}$$



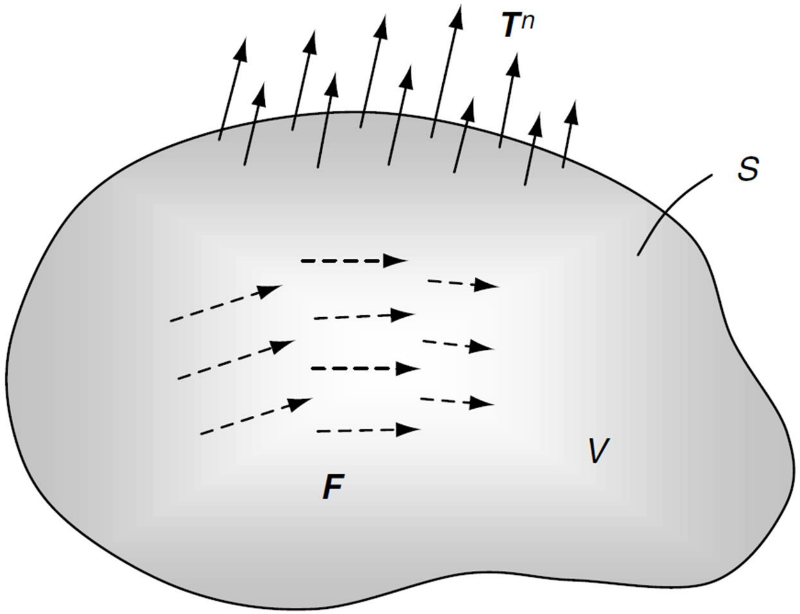
2-D

$$0 = \sum F_x = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \mathrm{d}x \right) \mathrm{d}y - \sigma_x \mathrm{d}y$$

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \mathrm{d}y \right) \mathrm{d}x - \tau_{yx} \mathrm{d}x + F_x \mathrm{d}x \mathrm{d}y$$

$$\Rightarrow \boxed{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0} \Rightarrow \boxed{\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0}$$

Conservation of Angular Momentum



3-D ⇒

$$\sum \boldsymbol{r}\times\boldsymbol{F} = \iint_S \boldsymbol{r}\times\boldsymbol{T}^{\boldsymbol{n}}\,\mathrm{d}S + \iiint_V \boldsymbol{r}\times\boldsymbol{F}\,\mathrm{d}V$$

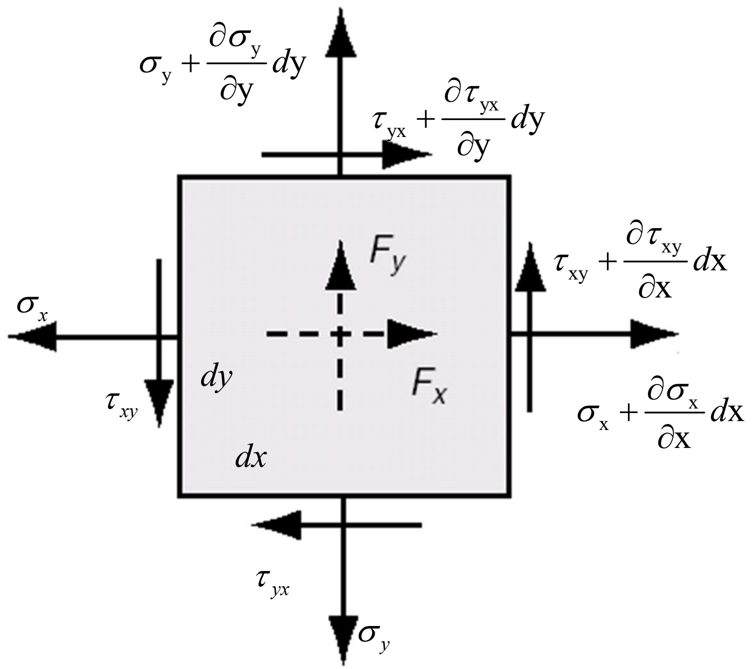
$$= \iint_S \varepsilon_{ijk} x_j \sigma_{lk} n_l\,\mathrm{d}S + \iiint_V \varepsilon_{ijk} x_j F_k\,\mathrm{d}V$$

$$= \iiint_V \left(\left(\varepsilon_{ijk} x_j \sigma_{lk} \right)_{,l} + \varepsilon_{ijk} x_j F_k \right) \mathrm{d}V$$

$$= \iiint_V \left(\varepsilon_{ijk} \delta_{jl} \sigma_{lk} + \xcancel{\varepsilon_{ijk} x_j \sigma_{lk,l} + \varepsilon_{ijk} x_j F_k} \right) \mathrm{d}V$$

$$= \iiint_V \varepsilon_{ijk} \sigma_{jk}\,\mathrm{d}V$$

$$\varepsilon_{ijk}\sigma_{jk} = 0_i \Rightarrow \sigma_{jk} = \sigma_{kj} \Rightarrow \begin{cases} \tau_{xy} = \tau_{yx} \\ \tau_{yz} = \tau_{zy} \\ \tau_{zx} = \tau_{xz} \end{cases}$$



2-D ⇒

$$0 = \sum M = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\mathrm{d}x \right) \mathrm{d}y \frac{1}{2}\mathrm{d}x + \tau_{xy}\mathrm{d}y\frac{1}{2}\mathrm{d}x$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\mathrm{d}y \right) \mathrm{d}x \frac{1}{2}\mathrm{d}y - \tau_{yx}\mathrm{d}x\frac{1}{2}\mathrm{d}y$$

$$\Rightarrow \boxed{\tau_{xy} = \tau_{yx}}$$

Equilibrium Equations

$$\boldsymbol{\sigma}\cdot\overleftarrow{\nabla}=\left(\sigma_{ij}\boldsymbol{e}_i\boldsymbol{e}_j\right)\cdot\left(\frac{\overleftarrow{\partial}}{\partial x_k}\boldsymbol{e}_k\right)=\frac{\partial\sigma_{ij}}{\partial x_k}\boldsymbol{e}_i\left(\boldsymbol{e}_j\cdot\boldsymbol{e}_k\right)=\sigma_{ij,j}\boldsymbol{e}_i$$

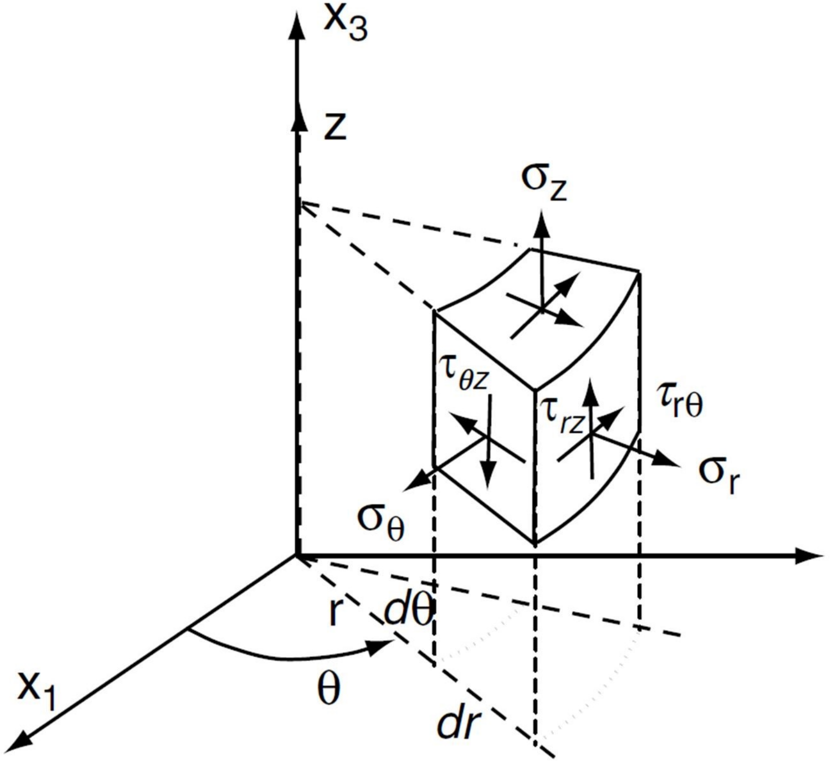
$$\nabla\cdot\boldsymbol{\sigma}=\left(\frac{\partial}{\partial x_k}\boldsymbol{e}_k\right)\cdot\left(\sigma_{ji}\boldsymbol{e}_j\boldsymbol{e}_i\right)=\frac{\partial\sigma_{ji}}{\partial x_k}\left(\boldsymbol{e}_k\cdot\boldsymbol{e}_j\right)\boldsymbol{e}_i=\sigma_{ji,j}\boldsymbol{e}_i$$

$$\boldsymbol{\sigma}\cdot\overleftarrow{\nabla}=\nabla\cdot\boldsymbol{\sigma}^T$$

$$\left\{\begin{matrix}\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{F}=\boldsymbol{\sigma}\cdot\overleftarrow{\nabla}+\boldsymbol{F}=\boldsymbol{0}\\ \sigma_{ji,j}+F_i=\sigma_{ij,j}+F_i=0_i\end{matrix}\right\},\quad \begin{array}{l}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}+\dfrac{\partial\tau_{xz}}{\partial z}+F_x=0\\ \dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{yz}}{\partial z}+F_y=0\\ \dfrac{\partial\tau_{xz}}{\partial x}+\dfrac{\partial\tau_{yz}}{\partial y}+\dfrac{\partial\sigma_z}{\partial z}+F_z=0\end{array},\quad \begin{bmatrix}\sigma_x&\tau_{xy}&\tau_{xz}\\ \tau_{xy}&\sigma_y&\tau_{yz}\\ \tau_{xz}&\tau_{yz}&\sigma_z\end{bmatrix}\left\{\begin{matrix}\dfrac{\overleftarrow{\partial}}{\partial x}\\ \dfrac{\overleftarrow{\partial}}{\partial y}\\ \dfrac{\overleftarrow{\partial}}{\partial z}\end{matrix}\right\}+\left\{\begin{matrix}F_x\\F_y\\F_z\end{matrix}\right\}=\left\{\begin{matrix}0\\0\\0\end{matrix}\right\}$$

$$\left\{\begin{matrix}\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\end{matrix}\right\}\begin{bmatrix}\sigma_x&\tau_{xy}&\tau_{xz}\\ \tau_{xy}&\sigma_y&\tau_{yz}\\ \tau_{xz}&\tau_{yz}&\sigma_z\end{bmatrix}+\left\{\begin{matrix}F_x&F_y&F_z\end{matrix}\right\}=\left\{\begin{matrix}0&0&0\end{matrix}\right\}$$

Cylindrical Equilibrium Equations



$\nabla \cdot \boldsymbol{\sigma}$ = contraction on the first and third index of $\boldsymbol{\sigma}\bar{\nabla}$

$$\nabla \cdot \boldsymbol{\sigma} = \left(\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} \right)\boldsymbol{e}_r + \left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r} \right)\boldsymbol{e}_\theta + \left(\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{z}}{\partial z} + \frac{\tau_{rz}}{r} \right)\boldsymbol{e}_z$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

$$\boldsymbol{T}^r = \sigma_r \boldsymbol{e}_r + \tau_{r\theta}\boldsymbol{e}_\theta + \tau_{rz}\boldsymbol{e}_z$$

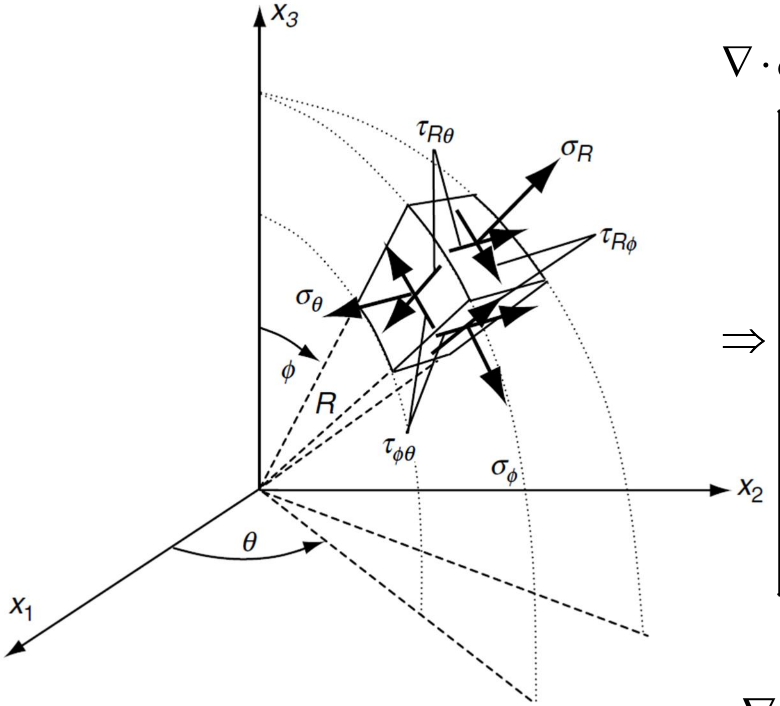
$$\boldsymbol{T}^\theta = \tau_{r\theta} \boldsymbol{e}_r + \sigma_{\theta}\boldsymbol{e}_\theta + \tau_{\theta z}\boldsymbol{e}_z$$

$$\boldsymbol{T}^z = \tau_{rz} \boldsymbol{e}_r + \tau_{\theta z}\boldsymbol{e}_\theta + \sigma_{z}\boldsymbol{e}_z$$

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F} = \boldsymbol{0}$$

$$\Rightarrow \begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0, \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0, \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{z}}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0. \end{aligned}$$

Spherical Equilibrium Equations



$\nabla\cdot\boldsymbol{\sigma}$ = contraction on the first and third index of $\boldsymbol{\sigma}\bar{\nabla}$

$$\nabla\cdot\boldsymbol{\sigma}=\left(\frac{\partial\sigma_R}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\varphi R}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta R}}{\partial\theta}+\frac{2\sigma_R-\sigma_\theta-\sigma_\varphi+\cot\varphi\,\tau_{\varphi R}}{R}\right)\boldsymbol{e}_R$$

$$\Rightarrow\quad+\left(\frac{\partial\tau_{R\varphi}}{\partial R}+\frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{2\tau_{R\varphi}+\tau_{\varphi R}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R}\right)\boldsymbol{e}_\varphi$$

$$+\left(\frac{\partial\tau_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{R\theta}+\tau_{\theta R}+\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R}\right)\boldsymbol{e}_\theta$$

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_R & \tau_{R\varphi} & \tau_{R\theta}\\ \tau_{R\varphi} & \sigma_\varphi & \tau_{\theta\varphi}\\ \tau_{R\theta} & \tau_{\theta\varphi} & \sigma_\theta\end{bmatrix}$$

$$\boldsymbol{T}^R=\sigma_R\boldsymbol{e}_R+\tau_{R\varphi}\boldsymbol{e}_\varphi+\tau_{R\theta}\boldsymbol{e}_\theta$$

$$\boldsymbol{T}^\varphi=\tau_{R\varphi}\boldsymbol{e}_R+\sigma_\varphi\boldsymbol{e}_\varphi+\tau_{\theta\varphi}\boldsymbol{e}_\theta$$

$$\boldsymbol{T}^\theta=\tau_{R\theta}\boldsymbol{e}_R+\tau_{\theta\varphi}\boldsymbol{e}_\varphi+\sigma_\theta\boldsymbol{e}_\theta$$

$\nabla\cdot\boldsymbol{\sigma}+\boldsymbol{F}=\boldsymbol{0}$

$$\frac{\partial\sigma_R}{\partial R}+\frac{1}{R}\frac{\partial\tau_{R\varphi}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{R\theta}}{\partial\theta}+\frac{2\sigma_R-\sigma_\theta-\sigma_\varphi+\cot\varphi\,\tau_{R\varphi}}{R}+F_R=0,$$

$$\frac{\partial\tau_{R\varphi}}{\partial R}+\frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{3\tau_{R\varphi}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R}+F_\varphi=0,$$

$$\frac{\partial\tau_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{3\tau_{R\theta}+2\tau_{\theta\varphi}\cot\varphi}{R}+F_\theta=0.$$

Outline

- Body and Surface Forces
- Traction/Stress Vector
- Stress Tensor
- Traction on Oblique Planes
- Principal Stresses and Directions
- Mohr's Circles of Stresses
- Octahedral Stresses
- Spherical and Deviatoric Stresses
- Conservation of Linear Momentum
- Conservation of Angular Momentum
- Equilibrium Equations
- Equilibrium Equations in Curvilinear Coordinates