Energy Method and Variational Principle

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- Castigliano's Second Theorem
- Stress Variation
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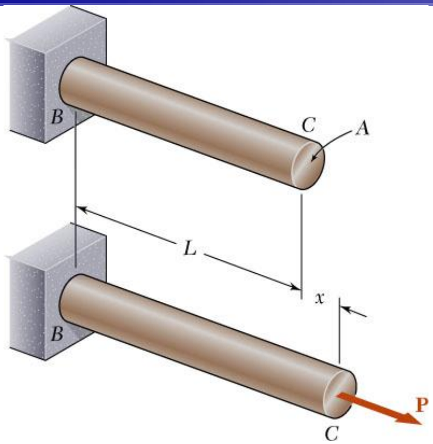
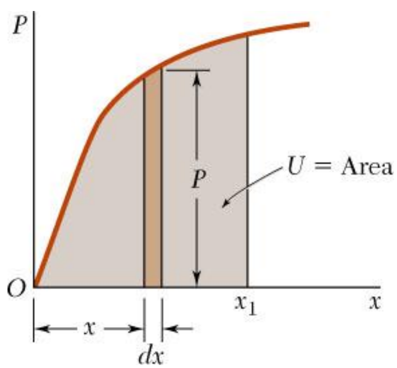
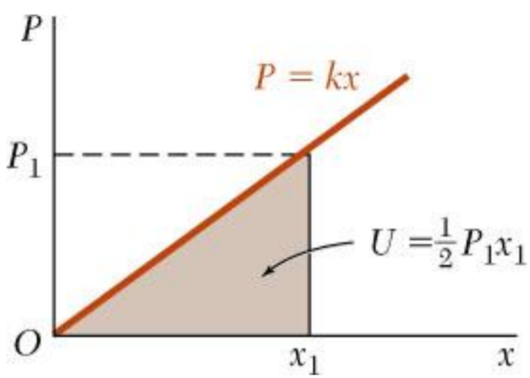
- Stress Variation: Application to Torsion of Cylinders

Work Done by External Load







A uniform rod is subjected to a slowly increasing load.

The *elementary work* done by the load *P* as the rod elongates by a small d*x* is

$$dU = P\,\mathrm{d}x = \text{elementary work}$$

which is equal to the area of width d*x* under the load-deformation diagram.

The *total work* done by the load for a deformation x_1,

$$U = \int_0^{x_1} P\,\mathrm{d}x = \text{total work} = \text{strain energy}$$

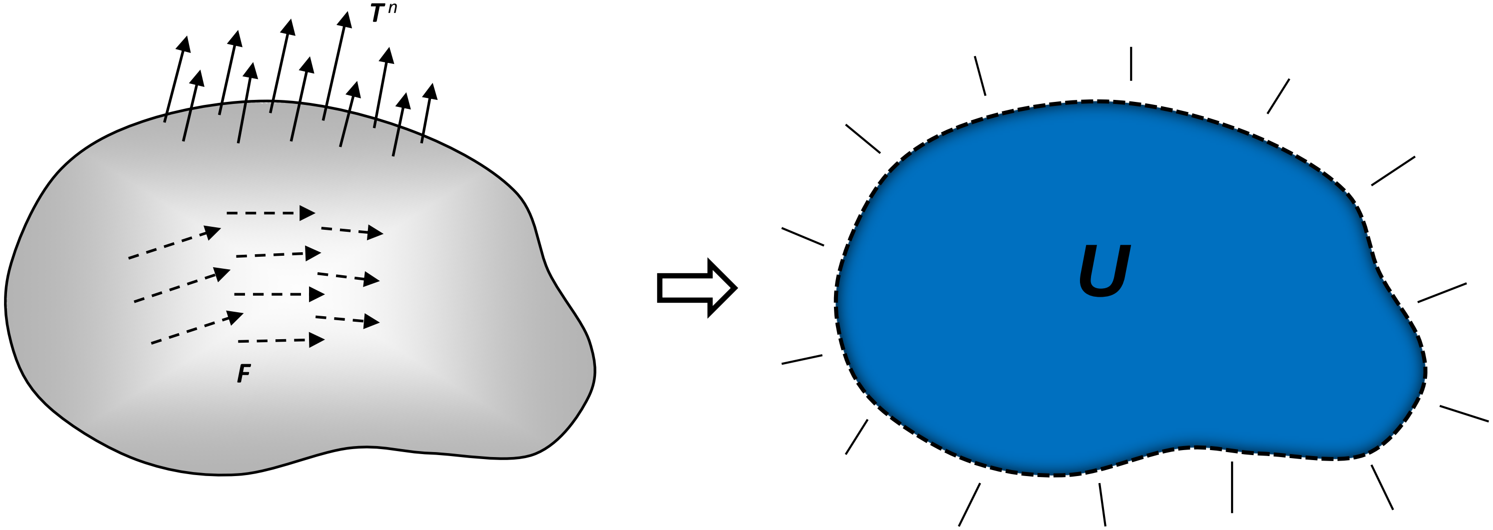
which results in an increase of *strain energy* in the rod.

In the case of a linear elastic deformation,

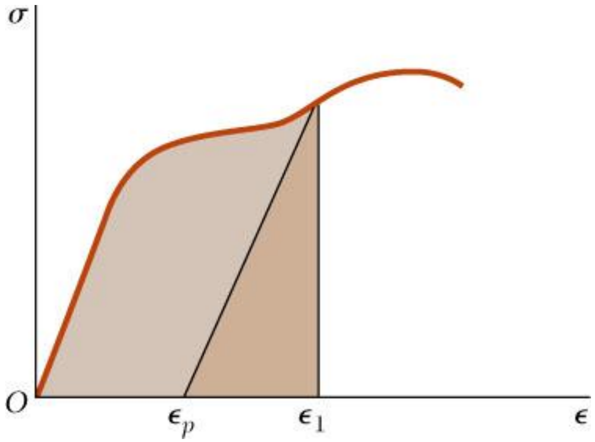
$$U = \int_0^{x_1} kx\,\mathrm{d}x = \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} P_1 x_1$$

Energy Conversion

- Work done by surface and body forces on elastic solids is stored inside the body in the form of strain energy.



Strain Energy Density



To eliminate the effects of size, evaluate the strain-energy per unit volume,

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

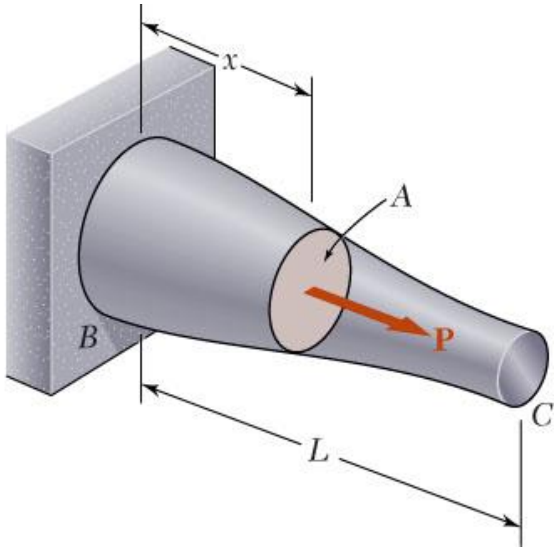
$$U_0 = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon_x = \textit{strain energy density}$$

The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

Remainder of the energy spent in deforming the material is dissipated as heat.

Strain Energy for Normal Stress



In an element with a nonuniform stress distribution,

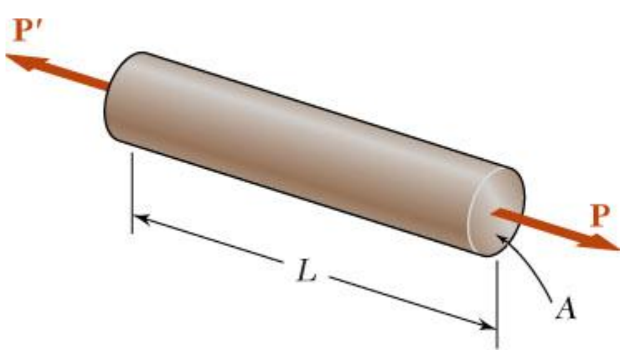
$$U_0 = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{\mathrm{d}U}{\mathrm{d}V} \qquad U = \int U_0 \,\mathrm{d}V = \text{ total strain energy}$$

For values of $U_0 < U_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} \mathrm{d}V = \text{elastic strain energy} \qquad \Rightarrow E > 0$$

Under axial loading, $\sigma_x = P/A \qquad \mathrm{d}V = A\,\mathrm{d}x$

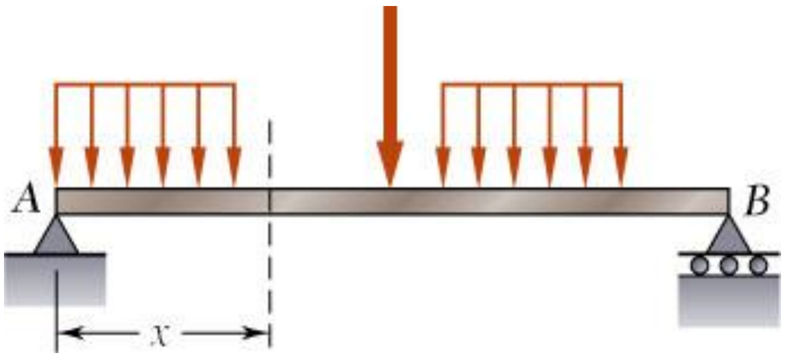
$$U = \int_0^L \frac{P^2}{2AE} \mathrm{d}x = \frac{1}{2}\int_0^L EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x$$



For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Strain Energy for Normal Stress



For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} \mathrm{d}V = \int \frac{M^2 y^2}{2EI^2} \mathrm{d}V$$

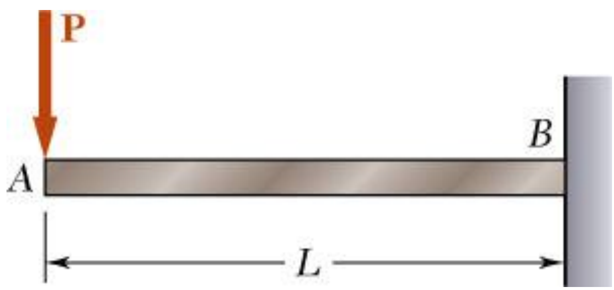
Setting $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}x$,

$$\sigma_x = \frac{M\,y}{I}$$

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} \mathrm{d}A\,\mathrm{d}x = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 \mathrm{d}A \right) \mathrm{d}x$$

$$= \int_0^L \frac{M^2}{2EI} \mathrm{d}x = \frac{1}{2} \int_0^L EI \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

$$\Rightarrow E > 0$$

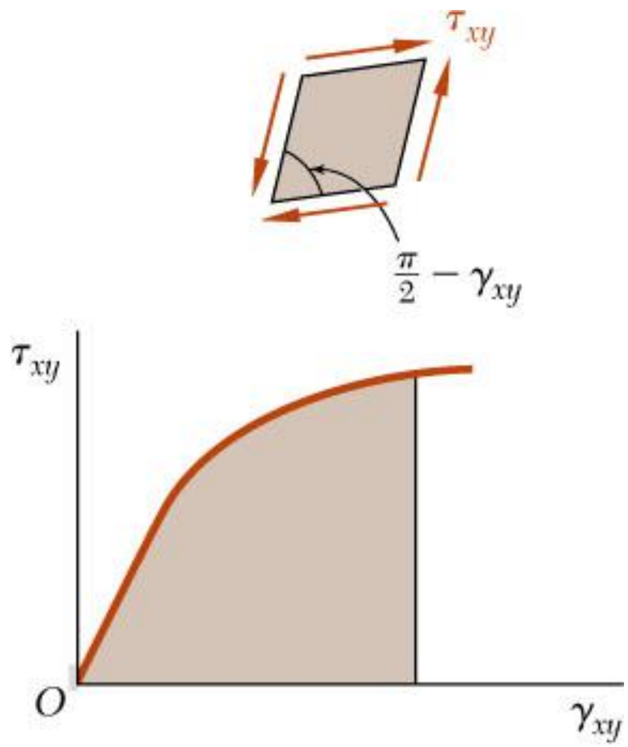


For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} \mathrm{d}x = \frac{P^2 L^3}{6EI}$$

Strain Energy for Shear Stress



For a material subjected to plane shearing stresses,

$$U_0 = \int_0^{\gamma_{xy}} \tau_{xy} \, \mathrm{d}\gamma_{xy}$$

For values of τ_{xy} within the proportional limit,

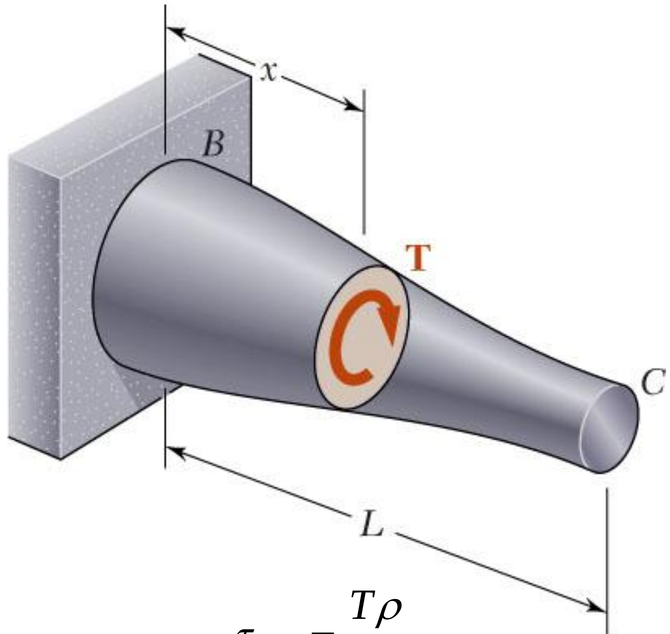
$$U_0 = \tfrac{1}{2} G \gamma_{xy}^2 = \tfrac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

The total strain energy is found from

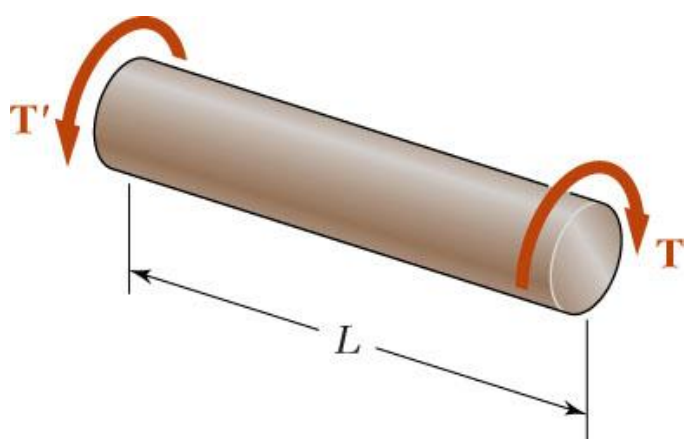
$$U = \int U_0 \mathrm{d}V = \int \frac{\tau_{xy}^2}{2G} \mathrm{d}V = \int \frac{(1+\nu)}{E} \tau_{xy}^2 \mathrm{d}V$$

$$\Rightarrow G > 0; \quad \nu > -1$$

Strain Energy for Shear Stress



$$\tau_{xy} = \frac{T\rho}{J}$$



For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{xy}^2}{2G} \mathrm{d}V = \int \frac{T^2 \rho^2}{2GJ^2} \mathrm{d}V$$

Setting $\mathrm{d}V = \mathrm{d}A\, \mathrm{d}x$,

$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} \mathrm{d}A\, \mathrm{d}x = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 \mathrm{d}A \right) \mathrm{d}x$$

$$= \int_0^L \frac{T^2}{2GJ} \mathrm{d}x = \frac{1}{2} \int_0^L GJ \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} \right)^2 \mathrm{d}x$$

In the case of a uniform shaft,

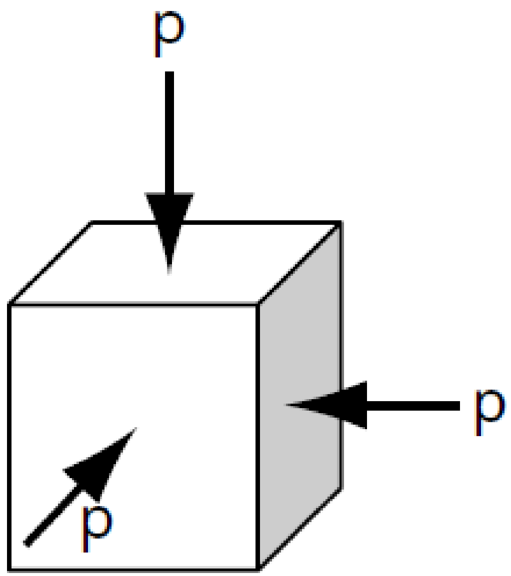
$$U = \frac{T^2 L}{2GJ}$$

Strain Energy for Hydrostatic Stress

$$\varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -p\frac{3(1-2\nu)}{E} = \frac{-3p}{3\lambda + 2G}$$

$$K = \frac{-p}{\Delta V} = \frac{E}{3(1-2\nu)} = \frac{3\lambda + 2G}{3}$$

$$U_0 = \frac{1}{2}(-p)\varepsilon_{kk} = \frac{1}{2}\sigma_m \varepsilon_{kk} = \frac{1}{2K}\sigma_m^2 = \frac{3(1-2\nu)}{2E}\sigma_m^2$$

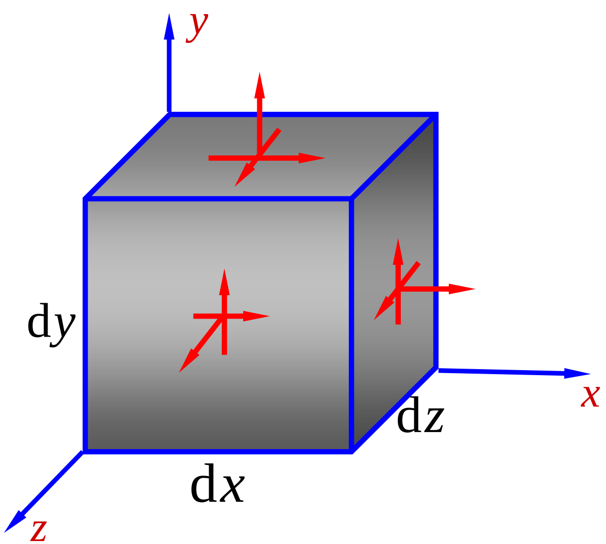


$$\Rightarrow K > 0; \quad \nu < 0.5$$

Strain Energy Density for a General Stress State

- Strain energy density of non-linearly elastic material under generalized 3-D stress states

$$dU_0 = \sigma_{ij} d\varepsilon_{ij}$$
$$= \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z$$
$$+ \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx}$$



- Strain energy density of linearly elastic material under generalized 3-D stress states

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\begin{bmatrix} \sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z \\ +\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} \end{bmatrix}$$

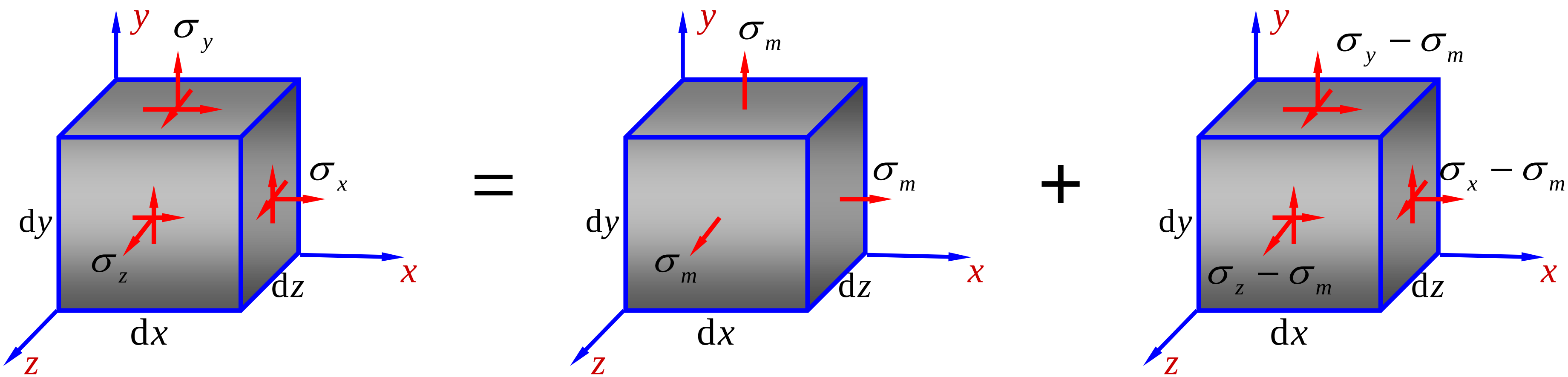
- In Terms of Strain

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}\right)\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon_{kk}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ij}$$
$$= \frac{1}{2}\lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)^2 + G\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}\gamma_{xy}^2 + \frac{1}{2}\gamma_{yz}^2 + \frac{1}{2}\gamma_{zx}^2\right)$$

- In Terms of Stress

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right) = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{kk}\sigma_{jj}$$
$$= \frac{1+\nu}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2\right) - \frac{\nu}{2E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2$$

Decomposition of Strain Energy Density



$$\boxed{U_0 = U_V + U_D}$$

(a) Spherical stress tensor

(b) Deviatoric stress tensor

- Volumetric energy density: $U_V = \dfrac{3(1-2\nu)}{2E}\sigma_m^{\ 2} = \dfrac{(1-2\nu)}{6E}(\sigma_x + \sigma_y + \sigma_z)^2$
- Distortion energy density:

$$U_D = U_0 - U_V = \frac{1+\nu}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2\right) - \frac{\nu}{2E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2 - \frac{(1-2\nu)}{6E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2$$

$$= \frac{1+\nu}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2\tau_{xy}^2 + 2\tau_{yz}^2 + 2\tau_{zx}^2\right) - \frac{(1+\nu)}{6E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2$$

$$= \frac{1+\nu}{6E}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2\right] + \frac{1+\nu}{E}\left(\tau_{xy}^{\ 2} + \tau_{yz}^{\ 2} + \tau_{zx}^{\ 2}\right)$$

Strain Energy Density in terms of Displacement

$$U_0 = \frac{1}{2}\lambda\varepsilon_{kk}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda u_{k,k}u_{j,j} + G\varepsilon_{ij}\varepsilon_{ij}$$

$$= \frac{1}{2}\lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)^2 + G\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2}\gamma_{xy}^2 + \frac{1}{2}\gamma_{yz}^2 + \frac{1}{2}\gamma_{zx}^2\right)$$

$$= \frac{1}{2}\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + G\left[\begin{array}{l}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\end{array}\right]$$

$$\boxed{\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)}}$$

Strain Energy Density for Plane Elasticity

$$U_0 = \frac{1}{2}\sigma_{\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1}{2}2G\left[\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right]\varepsilon_{\alpha\beta} = G\left[\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} - \frac{3-\kappa}{2(1-\kappa)}\varepsilon_{\gamma\gamma}\varepsilon_{\beta\beta}\right]$$

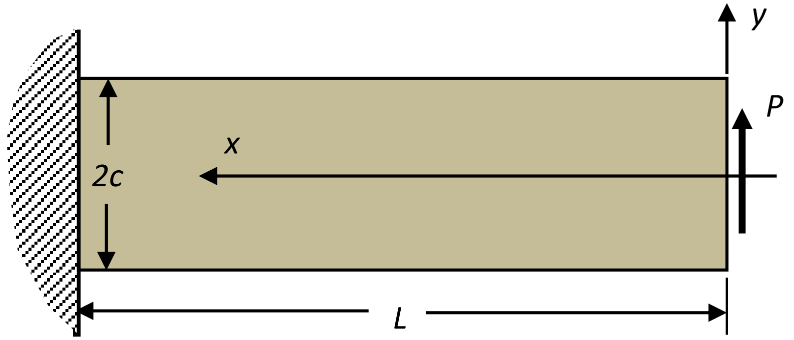
$$= G\left[(\varepsilon_x)^2 + (\varepsilon_y)^2 + 2(\varepsilon_{xy})^2 - \frac{3-\kappa}{2(1-\kappa)}(\varepsilon_x + \varepsilon_y)^2\right]$$

$$= G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \frac{3-\kappa}{2(1-\kappa)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right]$$

For plane strain: $\kappa = 3 - 4\nu : U_0 = G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right]$

For plane stress: $\kappa = \frac{3-\nu}{1+\nu} : U_0 = G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{\nu}{1-\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right]$

Strain Energy Density for a General Stress State



$$\sigma_x = -\frac{3P}{2c^3}xy\,,\tau_{xy} = -\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right),\ \sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = 0$$

$$U_0 = \frac{1+\nu}{2E}\left(\sigma_x^2 + 2\tau_{xy}^2\right) - \frac{\nu}{2E}\sigma_x^2 = \frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2$$

$$U = \iiint U_0 \mathrm{d}V = \int_0^1\int_{-c}^{c}\int_0^L\left(\frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_{-c}^{c}\int_0^L\left(\frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{2E}\int_{-c}^{c}\int_0^L\frac{9P^2}{4c^6}x^2y^2\mathrm{d}x\mathrm{d}y + \frac{1+\nu}{E}\int_{-c}^{c}\int_0^L\frac{9P^2}{16c^2}\left(1-\frac{y^2}{c^2}\right)^2\mathrm{d}x\mathrm{d}y$$

$$= \frac{P^2L^2}{4Ec^3} + \frac{9P^2L(1+\nu)}{Ec}$$

The Variation Operator

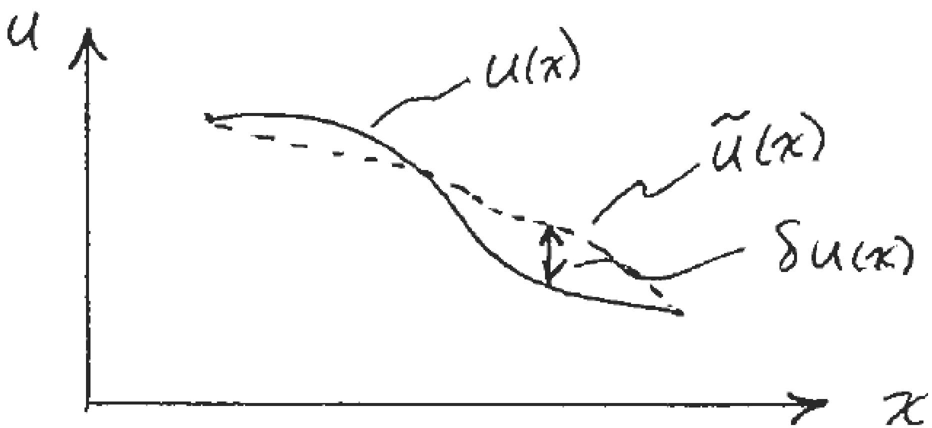
- Assuming $u(x)$ is the minimizing path for a functional:

$$I(u) = \int_a^b F(x, u, u')\,\mathrm{d}x$$

- Introducing a family of varied functions: $\tilde{u}(x) = u(x) + \varepsilon\eta(x)$
- We call $\varepsilon\eta(x)$ the variation of $u(x)$ and write

$$\varepsilon\eta(x) = \delta u(x) = \delta u = \tilde{u} - u, \qquad \varepsilon \to 0,\ \eta(a) = \eta(b) = 0$$

- The delta operator (δ) represents a small arbitrary change in the dependent variable u for a fixed value of the independent variable x, i.e. we do not associate a δx with a δu.



The difference between δu and a differential du

- A differential du has a dx associated with it.
- Consider the variation for the derivative:

$$\delta\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)=\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x}-\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\tilde{u}-u\right)=\frac{\mathrm{d}}{\mathrm{d}x}\delta u$$

- In a similar manner: $\delta\int u\left(x\right)\mathrm{d}x=\int\tilde{u}\left(x\right)\mathrm{d}x-\int u\left(x\right)\mathrm{d}x=\int\delta u\left(x\right)\mathrm{d}x$
- Consider a functional: $F=F\left(u_1\left(x\right),u_2\left(x\right),u_3\left(x\right),x\right)$
- Its variation:

$$\delta F=\frac{\partial F}{\partial u_1}\delta u_1+\frac{\partial F}{\partial u_2}\delta u_2+\frac{\partial F}{\partial u_3}\delta u_3$$

- In contrast, the differential is

$$\mathrm{d}F=\frac{\partial F}{\partial u_1}\mathrm{d}u_1+\frac{\partial F}{\partial u_2}\mathrm{d}u_2+\frac{\partial F}{\partial u_3}\mathrm{d}u_3+\frac{\partial F}{\partial x}\mathrm{d}x$$

Minimization of a Functional

- Consider the problem of minimizing $I(u) = \int_a^b F(x,u,u')\,\mathrm{d}x$
- For a varied path, the integrand may be written as

$$F(x, u+\delta u, u'+\delta u')$$

- Expanding the above in a Taylor series yields

$$F(x, u+\delta u, u'+\delta u') = F(x,u,u') + \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) + \mathrm{O}(\delta^2)$$

- The first variation of the functional I is defined by

$$\delta I = \int_a^b \delta F \mathrm{d}x \approx \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) \mathrm{d}x$$

$$= \int_a^b \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} \right) \delta u \mathrm{d}x + \left. \frac{\partial F}{\partial u'}\delta u \right|_a^b$$

- The minimizing process leads to Euler-Lagrange equation.
- Essential vs. natural BCs…

Principle of Virtual Work

- **A kinematically admissible displacement field** is one possessing continuous first partial derivatives in the interior of a domain *B* and satisfying all displacement boundary conditions on S_u.
- **A kinematically admissible displacement variation** $\boldsymbol{\delta u}$ **(virtual displacement)** is one possessing continuous first partial derivatives in the interior of a domain *B* and zero on S_u.
- **A statically admissible stress field** is one that satisfies the equilibrium equation over the interior of a domain *B* and all stress boundary conditions over S_t.

Principle of Virtual Work

- Now consider a body with statically admissible stress field and subjected to kinematically admissible virtual displacements.
- The work done by the external loads against the virtual displacements is

$$\delta W_E = \iiint_V \boldsymbol{F} \cdot \delta \boldsymbol{u} \mathrm{d}V + \iint_{S_t} \boldsymbol{T} \cdot \delta \boldsymbol{u} \mathrm{d}S$$

- In indicial notation

$$\delta W_E = \iiint_V \boldsymbol{F} \cdot \delta \boldsymbol{u} \mathrm{d}V + \iint_{S_t} \boldsymbol{T} \cdot \delta \boldsymbol{u} \mathrm{d}S = \iiint_V F_i \delta u_i \mathrm{d}V + \iint_{S_t} T_i \delta u_i \mathrm{d}S$$

$$= \iiint_V F_i \delta u_i \mathrm{d}V + \iint_{S_t} n_j \sigma_{ji} \delta u_i \mathrm{d}S = \iiint_V F_i \delta u_i \mathrm{d}V + \iint_S n_j \sigma_{ji} \delta u_i \mathrm{d}S$$

- **Recall that, $\delta \boldsymbol{u} = 0$ on S_u.**

Principle of Virtual Work

- Applying the divergence theorem on the surface integral:

$$\delta W_E = \iiint_V \left[F_i \delta u_i + \frac{\partial}{\partial x_j}\left(\sigma_{ji}\delta u_i\right)\right] \mathrm{d}V = \iiint_V \left[F_i \delta u_i + \frac{\partial \sigma_{ji}}{\partial x_j}\delta u_i + \sigma_{ji}\frac{\partial \delta u_i}{\partial x_j}\right] \mathrm{d}V$$

$$= \iiint_V \left[\left(F_i + \frac{\partial \sigma_{ji}}{\partial x_j}\right)\delta u_i + \sigma_{ij}\left(\delta\varepsilon_{ij} + \delta\omega_{ij}\right)\right] \mathrm{d}V$$

$$= \iiint_V \left[\left(F_i + \frac{\partial \sigma_{ji}}{\partial x_j}\right)\delta u_i + \sigma_{ij}\delta\varepsilon_{ij}\right] \mathrm{d}V$$

$$= \iiint_V \left(F_i + \frac{\partial \sigma_{ji}}{\partial x_j}\right)\delta u_i \mathrm{d}V + \delta W_I$$

- Balance between the external and internal virtual work is an alternative statement of equilibrium condition.

Principle of Virtual Work

- Principle of Virtual Work:

$$\delta W_E = \iiint_V \boldsymbol{F} \cdot \delta \boldsymbol{u} \mathrm{d}V + \iint_{S_t} \boldsymbol{T} \cdot \delta \boldsymbol{u} \mathrm{d}S = \iiint_V \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \mathrm{d}V = \delta W_I$$

$$\delta W_E = \iiint_V F_i \delta u_i \mathrm{d}V + \iint_{S_t} T_i \delta u_i \mathrm{d}S = \iiint_V \sigma_{ij} \delta \varepsilon_{ij} \mathrm{d}V = \delta W_I$$

- All forces and stresses are constant and need not to be actual forces and stresses.
- The stresses are independent of the virtual deformations.
- This principle is independent of any constitutive law.
- This principle is NOT about energy conservation, i.e. it is valid when energy is not conserved (plasticity, e.g.).
- This principle is applicable to simplified one- and two-dimensional theories as well, i.e. $\delta W_E = F_i \delta u_i$.

Principle of Minimum Total Potential Energy

- For an elastic solid

$$\delta W_I = \iiint_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \iiint_V \frac{\partial U_0}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij}\mathrm{d}V = \iiint_V \delta U_0 \mathrm{d}V = \delta U$$

where U is the strain energy.

- If we define the potential energy of applied loads as

$$V = -\iiint_V \boldsymbol{F}\cdot\boldsymbol{u}\mathrm{d}V - \iint_{S_t} \boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S = -\iiint_V F_i u_i \mathrm{d}V - \iint_{S_t} T_i u_i \mathrm{d}S$$

- For prescribed (constant) body and surface forces

$$\delta V = -\iiint_V \boldsymbol{F}\cdot\delta\boldsymbol{u}\mathrm{d}V - \iint_{S_t} \boldsymbol{T}\cdot\delta\boldsymbol{u}\mathrm{d}S = -\iiint_V F_i \delta u_i \mathrm{d}V - \iint_{S_t} T_i \delta u_i \mathrm{d}S$$

- Principle of Minimum Total Potential Energy

$$\delta\left(U + V\right) = \delta\Pi = 0.$$

- Restricted to elastic solids, both linear and nonlinear.

Principle of Minimum Total Potential Energy

- Elastic strain energy due to a strain variation

$$\delta U = \iiint_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V = \iiint_V \sigma_{ij}\delta\left(\varepsilon_{ij}+\omega_{ij}\right)\mathrm{d}V = \iiint_V \sigma_{ij}\delta\frac{\partial u_i}{\partial x_j}\mathrm{d}V = \iiint_V \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}\mathrm{d}V$$

$$= \iiint_V \left[\frac{\partial}{\partial x_j}\left(\sigma_{ij}\delta u_i\right) - \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i\right]\mathrm{d}V = -\iiint_V \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i \mathrm{d}V + \iint_{S_t} n_j\sigma_{ij}\delta u_i \mathrm{d}S$$

- The corresponding potential energy variation

$$\delta V = -\iiint_V F_i\delta u_i\mathrm{d}V - \iint_{S_t} T_i\delta u_i \mathrm{d}S$$

- Principle of Minimum Total Potential Energy

$$0 = \delta\Pi = \delta\left(U+V\right) = -\iiint_V\left(\frac{\partial \sigma_{ij}}{\partial x_j}+F_i\right)\delta u_i\mathrm{d}V + \iint_{S_t}\left(n_j\sigma_{ij}-T_i\right)\delta u_i\mathrm{d}S$$

- For an arbitrary displacement variation, the principle of minimum total potential energy yields the equilibrium equation and traction BCs.

Castigliano's First Theorem

- Consider an elastic system subjected to a set of generalized loads F_i (forces & moments) with corresponding generalized displacements u_i (deflection, rotation, angle of twist & extension/contraction). **Subsequently,**
- Express the variation of strain energy in terms of virtual displacements δu_i, i.e. $\delta U = \delta U(\delta u_i)$.
- The total potential energy variation may be expressed as

$$\delta\Pi = \delta U + \delta V = \delta U - \sum_{k=1}^{n} F_k \delta u_k = \delta\left(U - \sum_{k=1}^{n} F_k u_k \right)$$

- For equilibrium, we must require

$$\delta\Pi = \frac{\partial \Pi}{\partial u_i}\delta u_i = \frac{\partial}{\partial u_i}\left(U - \sum_{k=1}^{n} F_k u_k \right)\delta u_i = \left(\frac{\partial U}{\partial u_i} - F_k \delta_{ik} \right)\delta u_i = \left(\frac{\partial U}{\partial u_i} - F_i \right)\delta u_i = 0$$

- For arbitrary displacement variations: $\boxed{F_i = \dfrac{\partial U}{\partial u_i}}$

Castigliano’s First Theorem

$$F_i = \frac{\partial U}{\partial u_i}$$

- This theorem is simply an application of the minimum total potential energy.
- This theorem is valid for both linear and nonlinear elastic solids. The specific material behavior only affects the way how elastic strain energy is calculated.
- This theorem requires one to write the elastic strain energy in terms of generalized displacements, , i.e. $U = U(u_i)$.

Approximate Methods

- The Principle of Minimum Total Potential Energy states

$$0 = \delta\Pi = \delta\left(U + V\right) = -\iiint_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + F_i\right)\delta u_i \mathrm{d}V + \iint_{S_t} \left(n_j \sigma_{ij} - T_i\right)\delta u_i \mathrm{d}S$$

- Minimizing the total potential energy is equivalent to satisfying the equilibrium condition and traction BCs

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0, \quad \text{inside } V; \qquad T_i = n_j \sigma_{ij} \quad \text{on } S_t.$$

- In many instances, the solution to the above is untenable.
- Approximate methods need to be developed.
- The first will be to approximate the total potential energy.
- The second will be to approximate the d.e.
- Both are precursors to the Finite Element Method.

Ritz Method

- Based on approximating the displacement field as a linear combination of trial functions

$$u = u_0 + \sum A_m u_m ;\ v = v_0 + \sum B_m v_m ;\ w = w_0 + \sum C_m w_m ,$$

- where u_0, u_m, v_0, v_m, w_0, w_m are known functions and A_m, B_m, C_m, represent undetermined coefficients.
- u_0, v_0, w_0 must satisfy the displacement BCs on S_u.
- u_m, v_m, w_m must be differentiable inside V, zero on S_u, linearly independent and complete (trig or poly functions).
- The displacement variation is thus

$$\delta u = \sum \frac{\partial u}{\partial A_m} \delta A_m = \sum u_m \delta A_m ;\quad \delta v = \sum v_m \delta B_m ;\quad \delta w = \sum w_m \delta C_m ,$$

Ritz Method

- We now have reduced $\Pi(u, v, w)$ to $\Pi(A_m, B_m, C_m)$. The standard variation procedure yields

$$0 = \delta\Pi \;\Rightarrow\; \sum \left(\frac{\partial \Pi}{\partial A_m}\delta A_m + \frac{\partial \Pi}{\partial B_m}\delta B_m + \frac{\partial \Pi}{\partial C_m}\delta C_m \right) = 0$$

- For arbitrary variation of the coefficients A_m, B_m, C_m

$$\frac{\partial \Pi}{\partial A_m} = 0; \quad \frac{\partial \Pi}{\partial B_m} = 0; \quad \frac{\partial \Pi}{\partial C_m} = 0$$

- Given the total potential energy

$$\Pi = U - \iiint_V \left(F_x u + F_y v + F_z w \right) \mathrm{d}V - \iint_{S_t} \left(T_x u + T_y v + T_z w \right) \mathrm{d}S$$

$$\Rightarrow \begin{array}{l} \dfrac{\partial U}{\partial A_m} - \iiint_V F_x u_m \mathrm{d}V - \iint_{S_t} T_x u_m \mathrm{d}S = 0; \quad \dfrac{\partial U}{\partial B_m} - \iiint_V F_y v_m \mathrm{d}V - \iint_{S_t} T_y v_m \mathrm{d}S = 0 \\ \dfrac{\partial U}{\partial C_m} - \iiint_V F_z w_m \mathrm{d}V - \iint_{S_t} T_z w_m \mathrm{d}S = 0 \end{array}$$

A_m, B_m, C_m are determined from these equations.

Galerkin Method

- The Galerkin method for finding an approximate solution of a d.e. involves the direct use of the d.e. itself.
- No variational statement is required and hence the method has broader range of application.
- Recall the principle of minimum total potential Energy

$$0 = \delta\Pi = \delta\left(U + V\right) = -\iiint_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + F_i\right)\delta u_i \mathrm{d}V + \iint_{S_t}\left(n_j \sigma_{ij} - T_i\right)\delta u_i \mathrm{d}S$$

- We still assume an approximate solution for displacements

$$u = u_0 + \sum A_m u_m ; v = v_0 + \sum B_m v_m ; w = w_0 + \sum C_m w_m$$

$$\Rightarrow \delta u = \sum \frac{\partial u}{\partial A_m}\delta A_m = \sum u_m \delta A_m ; \quad \delta v = \sum v_m \delta B_m ; \quad \delta w = \sum w_m \delta C_m ,$$

Galerkin Method

- Substitute the displacement variation into the principle

$$-\sum\iiint\left[\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+F_x\right]u_m\delta A_m\mathrm{d}V+\sum\iint_{S_t}\left(n_x\sigma_x+n_y\tau_{xy}+n_z\tau_{xz}-T_x\right)u_m\delta A_m\mathrm{d}S=0$$

$$-\sum\iiint\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+F_y\right]v_m\delta B_m\mathrm{d}V+\sum\iint_{S_t}\left(n_x\tau_{xy}+n_y\sigma_y+n_z\tau_{yz}-T_y\right)v_m\delta B_m\mathrm{d}S=0$$

$$-\sum\iiint\left[\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+F_z\right]w_m\delta C_m\mathrm{d}V+\sum\iint_{S_t}\left(n_x\tau_{xz}+n_y\tau_{yz}+n_z\sigma_z-T_z\right)w_m\delta C_m\mathrm{d}S=0$$

- If the proposed displacements satisfy not only the displacement BCs on S_u, but also the traction BCs on S_t, i.e.

$$\boxed{\begin{aligned}n_x\sigma_x+n_y\tau_{xy}+n_z\tau_{xz}-T_x&=0\\ n_x\tau_{xy}+n_y\sigma_y+n_z\tau_{yz}-T_y&=0\\ n_x\tau_{xz}+n_y\tau_{yz}+n_z\sigma_z-T_z&=0\end{aligned}}$$

Galerkin Method

- Then, for arbitrary A_m, B_m, C_m

$$\iiint \left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x \right] u_m \mathrm{d}V = 0$$

$$\iiint \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y \right] v_m \mathrm{d}V = 0$$

$$\iiint \left[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z \right] w_m \mathrm{d}V = 0$$

- A_m, B_m, C_m are determined from these equations.

$$u = u_0 + \sum A_m u_m ; \; v = v_0 + \sum B_m v_m ; \; w = w_0 + \sum C_m w_m$$

Galerkin Method

- In terms of displacements

$$\iiint \left[G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_x \right] u_m \mathrm{d}V = 0$$

$$\iiint \left[G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_y \right] v_m \mathrm{d}V = 0$$

$$\iiint \left[G\nabla^2 w + (\lambda + G)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + F_z \right] w_m \mathrm{d}V = 0$$

- A_m, B_m, C_m are determined from these equations.

$$u = u_0 + \sum A_m u_m;\ v = v_0 + \sum B_m v_m;\ w = w_0 + \sum C_m w_m$$

Ritz Method: Application to Plane Elasticity

$$u = u_0 + \sum A_m u_m ; \; v = v_0 + \sum B_m v_m$$

$$\frac{\partial U}{\partial A_m} - \iint_A F_x u_m \mathrm{d}A - \int_{S_t} T_x u_m \mathrm{d}S = 0; \qquad \frac{\partial U}{\partial B_m} - \iint_A F_y v_m \mathrm{d}A - \int_{S_t} T_y v_m \mathrm{d}S = 0$$

- A_m, B_m are determined from these equations.

$$U = G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \frac{3-\kappa}{2(1-\kappa)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right]$$

$$\frac{\partial U}{\partial A_m} = \iint_A 2G\left[\frac{\partial u}{\partial x}\frac{\partial u_m}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial u_m}{\partial y} - \frac{3-\kappa}{2(1-\kappa)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial u_m}{\partial x}\right]\mathrm{d}A$$

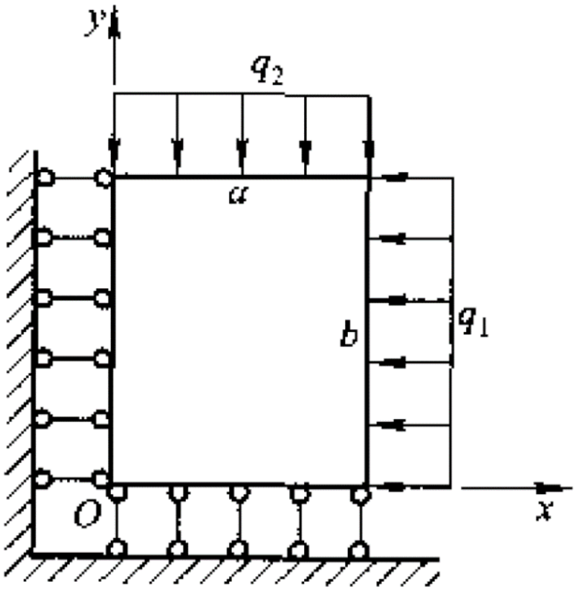
$$\frac{\partial U}{\partial B_m} = \iint_A 2G\left[\frac{\partial v}{\partial y}\frac{\partial v_m}{\partial y} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial v_m}{\partial x} - \frac{3-\kappa}{2(1-\kappa)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial v_m}{\partial y}\right]\mathrm{d}A$$

For plane strain: $\kappa = 3 - 4\nu \;\Rightarrow\; -\frac{3-\kappa}{2(1-\kappa)} = \frac{\nu}{1-2\nu}$

For plane stress: $\kappa = \frac{3-\nu}{1+\nu} \;\Rightarrow\; -\frac{3-\kappa}{2(1-\kappa)} = \frac{\nu}{1-\nu}$

Ritz Method: Application to Plane Elasticity

- The thin-plate is rolling-supported at the left and bottom edge.
- Propose an approximate displacement solution based on Ritz method and solve the plane stress problem. Neglect body forces.



$$u = x\left[A_1 + A_2 x + A_3 y + \cdots\right]$$

$$v = y\left[B_1 + B_2 x + B_3 y + \cdots\right]$$

- Note how the displacement BCs are satisfied.

Ritz Method: Application to Plane Elasticity

- If take only one term, i.e., $u = A_1 x, \quad v = B_1 y \quad \Rightarrow \quad u_1 = x, \quad v_1 = y$
- Substitute back into the principle

$$\Rightarrow \begin{cases} \int_0^b \int_0^a 2G\left[\frac{\partial u}{\partial x}\frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial u_1}{\partial y} + \frac{\nu}{1-\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial u_1}{\partial x}\right]\mathrm{d}x\mathrm{d}y - \int_0^b (-q_1) u_1(a)\,\mathrm{d}y = 0 \\ \int_0^b \int_0^a 2G\left[\frac{\partial v}{\partial y}\frac{\partial v_1}{\partial y} + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial v_1}{\partial x} + \frac{\nu}{1-\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\frac{\partial v_1}{\partial y}\right]\mathrm{d}x\mathrm{d}y - \int_0^a (-q_2) v_1(b)\,\mathrm{d}x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \int_0^b \int_0^a 2G\left[A_1(1) + \frac{1}{2}(0+0)(0) + \frac{\nu}{1-\nu}(A_1 + B_1)(1)\right]\mathrm{d}x\mathrm{d}y - \int_0^b (-q_1)\,a\,\mathrm{d}y = 0 \\ \int_0^b \int_0^a 2G\left[B_1(1) + \frac{1}{2}(0+0)(0) + \frac{\nu}{1-\nu}(A_1 + B_1)(1)\right]\mathrm{d}x\mathrm{d}y - \int_0^a (-q_2)\,b\,\mathrm{d}x = 0 \end{cases}$$

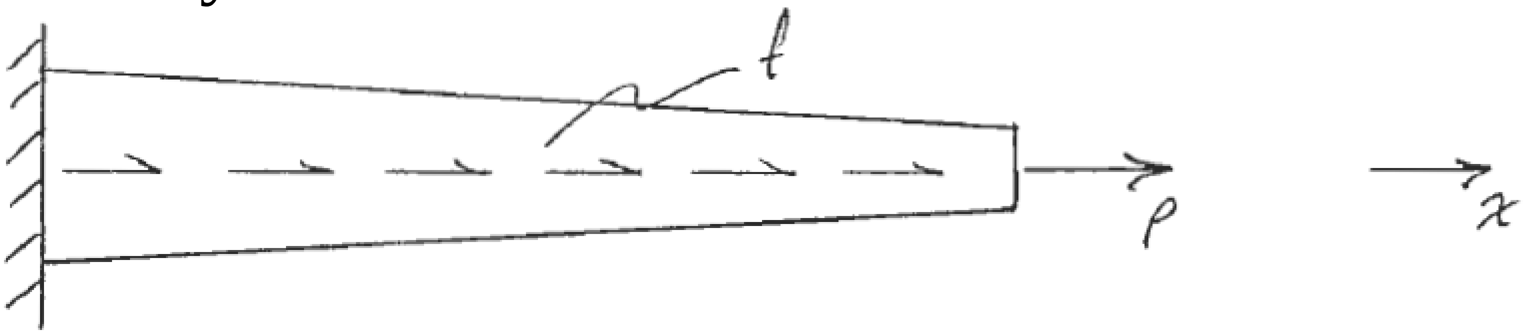
$$\Rightarrow \boxed{\frac{Eab}{1-\nu^2}\left[A_1 + \nu B_1\right] + q_1 ab = 0, \qquad \frac{Eab}{1-\nu^2}\left[B_1 + \nu A_1\right] + q_2 ab = 0}$$

$$\Rightarrow \boxed{A_1 = -\frac{q_1 - \nu q_2}{E}, \quad B_1 = -\frac{q_2 - \nu q_1}{E}}$$

- For the present case, A_1 and B_1 yield the exact solution. Just a special case!

Ritz Method: Application to Axial Loading

- Consider a variable cross-section rod subjected to a uniformly distributed load and a concentrated load.



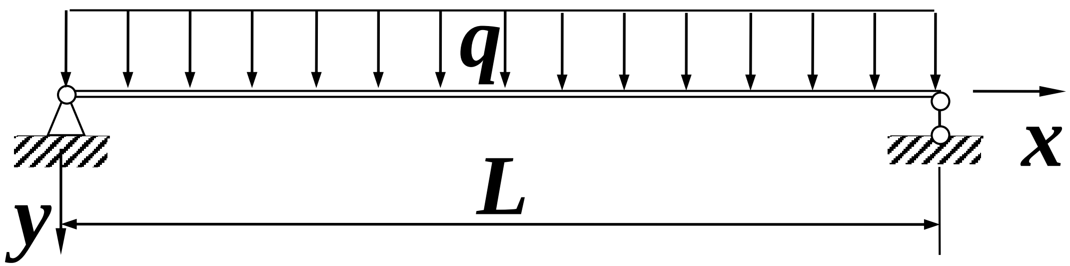
$$\Pi = U + V = \int_0^L \frac{F_N^2 \mathrm{d}x}{2EA} - \int_0^L fu\mathrm{d}x - Pu_{x=L} = \frac{1}{2}\int_0^L EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x - \int_0^L fu\mathrm{d}x - Pu_{x=L}$$

- Assume: $u = u_0 + \sum A_m u_m = 0 + A_1 x + A_2 x^2$
- Note how the displacement BCs are satisfied.
- The standard variation procedure yields: $\dfrac{\partial \Pi}{\partial A_1} = 0; \quad \dfrac{\partial \Pi}{\partial A_2} = 0.$
- Solving the above two equations for A_1 and A_2, an approximate solution are constructed.

Ritz Method: Application to Beam Theory

- Consider a beam subjected to a uniformly distributed load
- Assume:

$$v = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{L}$$



- Note how the displacement BCs are satisfied.

$$\Pi = U + V = \int_0^L \frac{M^2 \mathrm{d}x}{2EI} - \int_0^L qv\mathrm{d}x = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L qv\mathrm{d}x$$

$$= \frac{1}{2}\int_0^L EI\left[-\sum_{m=1}^{\infty} B_m \left(\frac{m\pi}{L}\right)^2 \sin\frac{m\pi x}{L}\right]\left[-\sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\right]\mathrm{d}x - \int_0^L q\sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{L}\mathrm{d}x$$

- Note the orthogonality of trigonometric functions

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\mathrm{d}x = \begin{cases} L/2 & m = n \\ 0 & m \neq n \end{cases}$$

Ritz Method: Application to Beam Theory

- Upon evaluating the integrals

$$\Pi = \frac{EI\pi^4}{4L^3}\sum_{m=1}^{\infty} m^4 B_m^2 - \frac{2qL}{\pi}\sum_{m=1,3,5,\cdots}^{\infty} \frac{B_m}{m}$$

- The standard variation procedure yields:

$$\frac{\partial \Pi}{\partial B_m} = 0 \quad \Rightarrow \quad B_m = \begin{cases} \dfrac{4qL^4}{EIm^5\pi^5} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

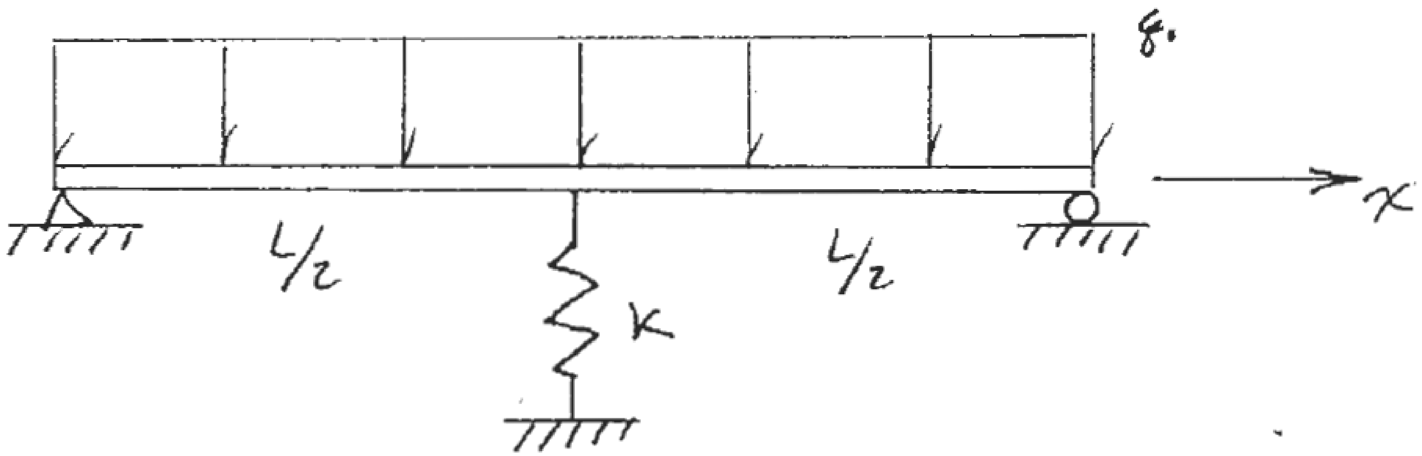
- The approximate solution is found

$$v = \frac{4qL^4}{EI\pi^5}\sum_{m=1,3,5,\cdots}^{\infty} \frac{1}{m^5}\sin\frac{m\pi x}{L}$$

- Symmetry requires all even terms vanish.

Ritz Method: Application to Beam Theory

- Consider a simply-supported beam enhanced by an elastic column as shown.



- We may still assume: $v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$

$$\Rightarrow \Pi = U + V = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L qv\mathrm{d}x - \frac{1}{2}kv^2\left(L/2\right)$$

- The rest is left as an exercise!

Galerkin Method: Application to Plane Elasticity

- In terms of stresses

$$\iint\left[\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x\right]u_m\mathrm{d}A=0,\qquad \iint\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y\right]v_m\mathrm{d}A=0$$

- In terms of displacements

$$\iint\left[G\nabla^2u-\frac{2G}{1-\kappa}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x\right]u_m\mathrm{d}A=0$$

$$\iint\left[G\nabla^2v-\frac{2G}{1-\kappa}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y\right]v_m\mathrm{d}A=0$$

Plane strain: $\kappa = 3-4\nu$

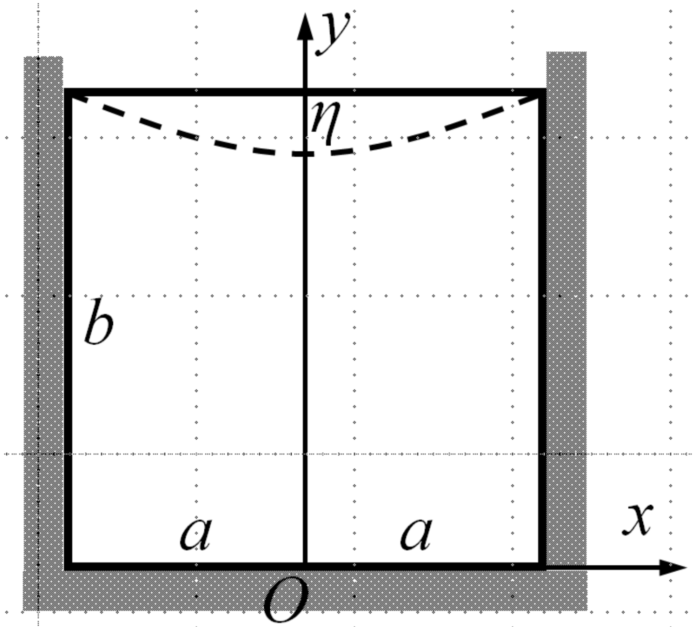
Plane stress: $\kappa = \dfrac{3-\nu}{1+\nu}$

- A_m, B_m are determined from these equations.

$$u=u_0+\sum A_m u_m;\ v=v_0+\sum B_m v_m$$

Exercise

- For the thin plate shown, the displacements along the top edge are confined to $u = 0; \quad v = -\eta\left(1 - x^2/a^2\right)$.
- Propose an approximate displacement solution based on Galerkin method and solve the plane stress problem. Neglect body forces.



$$u = \left(1 - \frac{x^2}{a^2}\right)\frac{x}{a}\frac{y}{b}\left(1 - \frac{y}{b}\right)\left[A_1 + A_2 y + A_3 x^2 + A_4 y^2 + \cdots\right]$$

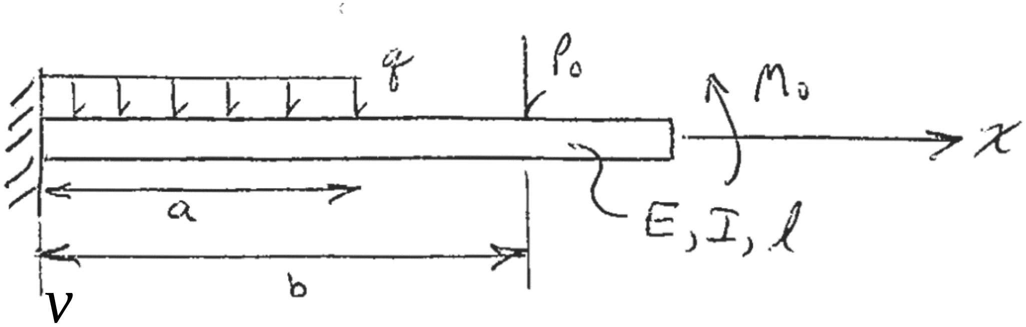
$$v = -\eta\left(1 - \frac{x^2}{a^2}\right)\frac{y}{b} + \left(1 - \frac{x^2}{a^2}\right)\frac{y}{b}\left(1 - \frac{y}{b}\right)\left[B_1 + B_2 y + B_3 x^2 + B_4 y^2 + \cdots\right]$$

- Note the symmetry property of the proposed displacements.

$$\iint\left[\nabla^2 u + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]u_m \mathrm{d}A = 0, \quad \iint\left[\nabla^2 v + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]v_m \mathrm{d}A = 0$$

Galerkin Method: Application to Beam Theory

- $u = w = 0,\ v = v_0 + \sum B_m v_m$
- v must also satisfy the force BCs.



- The second equation of the Galerkin method yields

$$\iiint \left[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y \right] v_m \mathrm{d}V = 0$$

$$\Rightarrow \int_0^L \left[\iint_A \frac{\partial \tau_{xy}}{\partial x} v_m \mathrm{d}A \right] \mathrm{d}x + \int_0^a q v_m \mathrm{d}x + P_0 v_m \big|_{x=b} - M_0 v_m' \big|_{x=L} = 0$$

$$\Rightarrow \int_0^L \frac{\partial V}{\partial x} v_m \mathrm{d}x + \int_0^a q v_m \mathrm{d}x + P_0 v_m \big|_{x=b} - M_0 v_m' \big|_{x=L} = 0$$

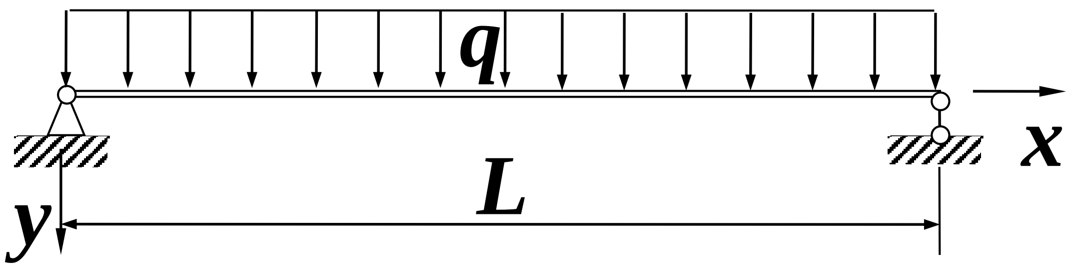
$$\Rightarrow \int_0^L \left(-EI \frac{d^4 v}{dx^4} \right) v_m \mathrm{d}x + \int_0^a q v_m \mathrm{d}x + P_0 v_m \big|_{x=b} - M_0 v_m' \big|_{x=L} = 0$$

- **Note the sign conventions of deflection, slope and moments.**

Sample Problem

- Let us revisit the beam problem
- We may still assume:

$$v=\sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{L}$$



- The displacement BCs are satisfied: $v(0)=v(L)=0$
- The traction BCs are also satisfied, i.e.

$$M(0)=EIv''(0)=0,\quad M(L)=EIv''(L)=0$$

- Galerkin method yields

$$\int_0^L\left(-EI\frac{d^4v}{dx^4}\right)v_m\mathrm{d}x+\int_0^L qv_m\mathrm{d}x=0$$

Sample Problem

- Plug in the proposed deflection

$$\Rightarrow \int_0^L \left(-EI \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L} \right)^4 \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} \mathrm{d}x + \int_0^L q \sin \frac{m\pi x}{L} \mathrm{d}x = 0$$

- Note the orthogonality of trigonometric functions

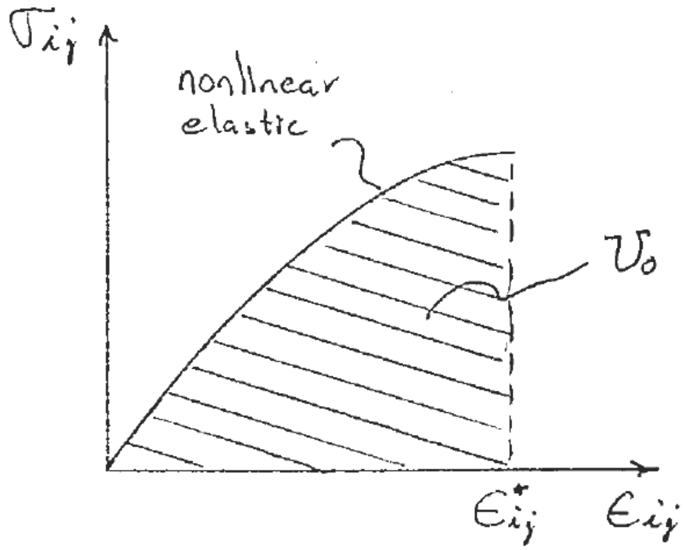
$$\Rightarrow -EIB_m \left(\frac{m\pi}{L} \right)^4 \frac{L}{2} - q \frac{L}{m\pi} (\cos m\pi - 1) = 0$$

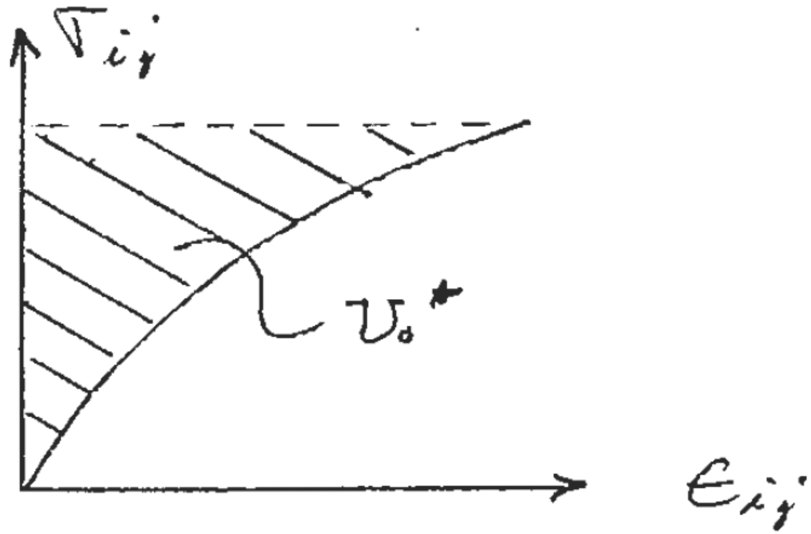
$$\Rightarrow B_m = -\frac{2qL^4 (\cos m\pi - 1)}{EIm^5 \pi^5} = \begin{cases} \dfrac{4qL^4}{EIm^5 \pi^5} & m = \text{odd} \\ 0 & m = \text{even} \end{cases}$$

- The same solution as that of Ritz method.

Complementary Strain Energy Density

- Recall that the strain energy density is defined as $dU_0 = \sigma_{ij} d\varepsilon_{ij}$
- Similarly, we define the complementary strain energy density $dU_0^* = \varepsilon_{ij} d\sigma_{ij}$
- It is the area "to the left" of the stress-strain curve.
- For a linear elastic solid, $U_0 = U_0^*$.
- U_0 is often expressed in terms of displacements or strains.
- U_0^* is often expressed in terms of forces or stresses.





Principle of Complementary Virtual Work

- Thus far we have focused on varying the displacement field while keeping the stress field fixed.
- Here we consider varying the stresses while holding displacements fixed.
- **A statically admissible stress field** is one that satisfies the equilibrium equation over the interior of a domain *B* and all stress boundary conditions over S_t.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0; \qquad n_j \sigma_{ij} = T_i \quad \text{on } S_t$$

- Consider a statically admissible variation in stresses

$$\left.\begin{aligned} &\sigma'_{ij} = \sigma_{ij} + \delta\sigma_{ij} \\ &\frac{\partial \sigma'_{ij}}{\partial x_j} + F_i = 0; \qquad n_j \sigma'_{ij} = T_i \quad \text{on } S_t \end{aligned}\right\} \Rightarrow \frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0; \qquad \delta\sigma_{ij} = 0 \quad \text{on } S_t$$

Principle of Complementary Virtual Work

- On S_u, a variation in surface traction is induced

$$\delta T_i = n_j \delta\sigma_{ij} \quad \text{on } S_u$$

- **The internal complementary virtual work** done by the virtual stresses against strains

$$\delta W_I^* = \iiint_V \varepsilon_{ij}\delta\sigma_{ij}\mathrm{d}V = \iiint_V \left(\varepsilon_{ij} + \omega_{ij}\right)\delta\sigma_{ij}\mathrm{d}V = \iiint_V \frac{\partial u_i}{\partial x_j}\delta\sigma_{ij}\mathrm{d}V$$

$$= \iiint_V \left[\frac{\partial}{\partial x_j}\left(u_i\delta\sigma_{ij}\right) - u_i \frac{\partial \delta\sigma_{ij}}{\partial x_j}\right]\mathrm{d}V = \iiint_V \frac{\partial}{\partial x_j}\left(u_i\delta\sigma_{ij}\right)\mathrm{d}V$$

$$= \iint_S n_j u_i \delta\sigma_{ij}\mathrm{d}S = \iint_{S_t} n_j u_i \delta\sigma_{ij}\mathrm{d}S + \iint_{S_u} n_j u_i \delta\sigma_{ij}\mathrm{d}S$$

$$= \iint_{S_u} u_i \delta T_i \mathrm{d}S = \delta W_E^*$$

- **is equal to the external complementary virtual work** done by the virtual tractions against displacements on S_u.

Principle of Complementary Virtual Work

- All displacements and strains are constant and need not to be actual displacements and strains.
- The strain and displacement fields are independent of the virtual stresses.
- This principle is independent of any constitutive law.
- This principle is applicable to simplified one- and two-dimensional theories as well, i.e. $\delta W_E^* = u_i \delta T_i$.

$$\delta W_I^* = \iiint_V \varepsilon_{ij} \delta \sigma_{ij} \mathrm{d}V = \iint_{S_u} u_i \delta T_i \mathrm{d}S = \delta W_E^*$$

Principle of Total Complementary Energy

- For an elastic solid

$$\delta W_I^* = \iiint_V \varepsilon_{ij}\delta\sigma_{ij}\mathrm{d}V = \iiint_V \frac{\partial U_0^*}{\partial \sigma_{ij}}\delta\sigma_{ij}\mathrm{d}V = \iiint_V \delta U_0^*\mathrm{d}V = \delta U^*$$

where U^* is the complementary strain energy.

- The complementary potential energy of applied loads

$$V = -\iiint_V \boldsymbol{u}\cdot\boldsymbol{F}\mathrm{d}V - \iint_S \boldsymbol{u}\cdot\boldsymbol{T}\mathrm{d}S = -\iiint_V u_i F_i \mathrm{d}V - \iint_S u_i T_i \mathrm{d}S$$

- For prescribed (constant) displacements

$$\delta V^* = -\iiint_V \boldsymbol{u}\cdot\delta\boldsymbol{F}\mathrm{d}V - \iint_S \boldsymbol{u}\cdot\delta\boldsymbol{T}\mathrm{d}S = -\iiint_V u_i\delta F_i\mathrm{d}V - \iint_S u_i\delta T_i\mathrm{d}S$$

$$= -\iiint_V u_i\left(-\frac{\partial\delta\sigma_{ij}}{\partial x_j}\right)\mathrm{d}V - \iint_{S_t} u_i n_j\,\delta\sigma_{ij}\,\mathrm{d}S - \iint_{S_u} u_i\delta T_i\mathrm{d}S$$

$$= -\iint_{S_u} u_i\delta T_i\mathrm{d}S = -\delta W_E^*$$

$$\delta W_I^* = \delta W_E^*$$

$$\Rightarrow \delta\left(U^* + V^*\right) = \delta\Pi^* = 0$$

Principle of Total Complementary Energy

- Of all stress fields that satisfy the equations of equilibrium and stress BCs on S_t, the actual one is distinguished by a minimum value of the complementary energy.
- Since the actual stress must satisfy the compatibility condition, this principle is an alternative statement to stress compatibility.
- Restricted to elastic bodies, both linear and nonlinear.
- This principle implies that the stress variation must satisfy the equilibrium equation with zero body forces inside V and traction BCs on S_t.

Castigliano's Second Theorem

- Consider an elastic system subjected to a set of generalized loads F_i (forces & moments) with corresponding generalized displacements u_i (deflection, rotation, angle of twist & extension/contraction). **Subsequently,**
- Express the variation of complimentary energy in terms of virtual loads δF_i, i.e. $\delta U^* = \delta U^*(\delta F_i)$.
- The total complementary energy variation Π^* is

$$\delta\Pi^* = \delta U^* + \delta V^* = \delta U^* - \sum_{k=1}^{n} u_k \delta F_k = \delta\left(U^* - \sum_{k=1}^{n} u_k F_k \right)$$

- For equilibrium, we must require

$$\delta\Pi^* = \frac{\partial \Pi^*}{\partial F_i}\delta F_i = \frac{\partial}{\partial F_i}\left(U^* - \sum_{k=1}^{n} u_k F_k \right)\delta F_i = \left(\frac{\partial U^*}{\partial F_i} - u_k \delta_{ik} \right)\delta F_i = \left(\frac{\partial U^*}{\partial F_i} - u_i \right)\delta F_i = 0$$

- For arbitrary force variations: $\boxed{u_i = \dfrac{\partial U^*}{\partial F_i}}$

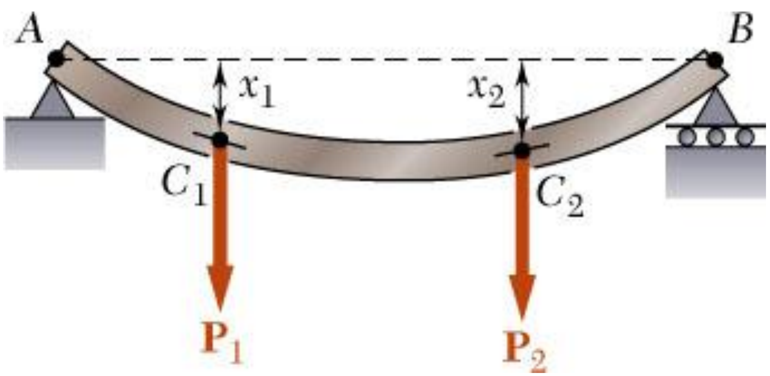
Castigliano's Second Theorem

$$u_i = \frac{\partial U^*}{\partial F_i}$$

- This theorem is simply an application of the complementary total potential energy.
- This theorem is valid for both linear and nonlinear elastic solids. The specific material behavior only affects the way how complementary energy is calculated.
- This theorem requires one to write the complementary energy in terms of generalized forces, , i.e. $U^* = U^*(F_i)$.

Application to Beams and Trusses

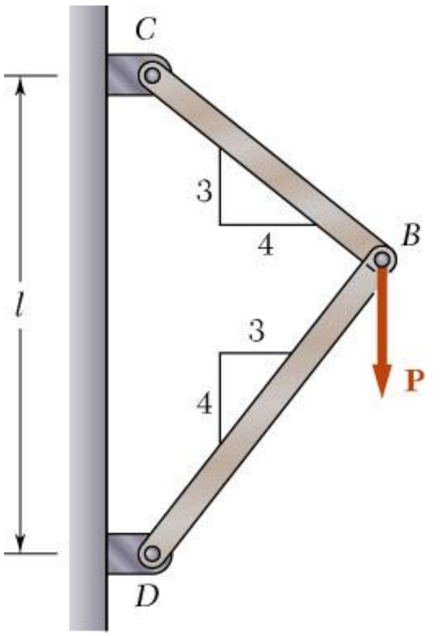
- For linearly elastic bodies: $U^* = U$.
- In the case of a beam:



$$U^* = U = \int_0^L \frac{M^2}{2EI}\,\mathrm{d}x$$

$$\Rightarrow \quad u_i = \frac{\partial U}{\partial P_i} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_i}\,\mathrm{d}x$$

- In the case of a truss:



$$U^* = U = \sum_{k=1}^{n} \frac{F_k^2 L_k}{2E_k A_k}$$

$$\Rightarrow \quad u_i = \frac{\partial U}{\partial P_i} = \sum_{k=1}^{n} \frac{F_k L_k}{E_k A_k}\frac{\partial F_k}{\partial P_i}$$

Approximate Solution

- The Principle of Total Complementary Energy states

$$0 = \delta\Pi^* = \delta\left(U^* + V^*\right) = \iiint_V u_i \frac{\partial \delta\sigma_{ij}}{\partial x_j} \mathrm{d}V - \iint_{S_t} u_i n_j \delta\sigma_{ij} \mathrm{d}S$$

- Minimizing the total complementary energy requires

$$\frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0, \quad \text{inside } V; \qquad \delta T_i = n_j \delta\sigma_{ij} = 0 \quad \text{on } S_t.$$

- In many instances, the solution to the above is untenable.
- Approximate methods need to be developed.
- We aim to find an approximate stress solution that satisfies the equilibrium condition inside V and the traction BCs on S_t.

Approximate Solution of Virtual Stresses

- Based on approximating the stress field as a linear combination of trial functions: $\sigma_{ij} = \sigma_{ij}^0 + \sum_m A_m \sigma_{ij}^m$
- where $\sigma_{ij}{}^0$ and $\sigma_{ij}{}^m$ are known functions and A_m represent undetermined coefficients.
- A_m stay the same for all six stress components, since altogether six stresses must satisfy compatibility.
- $\sigma_{ij}{}^0$ must satisfy the equilibrium condition inside V and the traction BCs on S_t.
- $\sigma_{ij}{}^m$ represent linearly independent functions, preferably form a complete base, and must satisfy

$$\frac{\partial \sigma_{ij}^m}{\partial x_j} = 0, \quad \text{inside } V; \qquad\qquad n_j \sigma_{ij}^m = 0 \quad \text{on } S_t.$$

Approximate Solution of Virtual Stresses

- The stress variation is thus: $\delta\sigma_{ij} = \sum_m \frac{\partial \sigma_{ij}}{\partial A_m}\delta A_m = \sum_m \sigma_{ij}^m \delta A_m$
- We now have reduced $\Pi^*(\sigma_{ij})$ to $\Pi^*(A_m)$. The standard variation procedure yields

$$0 = \delta\Pi^* = \delta U^* + \delta V^* = \delta U^* - \iint_{S_u} u_i \delta T_i \mathrm{d}S = \delta U^* - \iint_{S_u} u_i n_j \delta\sigma_{ij} \mathrm{d}S$$

$$= \delta U^* - \iint_{S_u} u_i n_j \sum_m \sigma_{ij}^m \delta A_m \mathrm{d}S = \sum \frac{\partial U^*}{\partial A_m}\delta A_m - \sum_m \left(\iint_{S_u} u_i n_j \sigma_{ij}^m \mathrm{d}S\right)\delta A_m$$

- For arbitrary variation of the coefficient A_m

$$\frac{\partial \Pi^*}{\partial A_m} = 0 \quad \Rightarrow \quad \frac{\partial U^*}{\partial A_m} = \iint_{S_u} u_i n_j \sigma_{ij}^m \mathrm{d}S$$

- If $u_i = 0$ on S_u or no S_u at all, the solution can further be simplified to $\partial U^* / \partial A_m = 0.$

Application to Plane Elasticity with S_t Only

- Recall that for a conservative body force field, the in-plane stress components of a plane problem are

$$\sigma_x = \frac{\partial^2\psi}{\partial y^2} + V, \sigma_y = \frac{\partial^2\psi}{\partial x^2} + V, \tau_{xy} = -\frac{\partial^2\psi}{\partial x\partial y} \qquad F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}.$$

- Instead of dealing with all three stresses, we choose to approximate the single Airy stress function as a linear combination of trial functions

$$\psi = \psi_0 + \sum_m A_m \psi_m$$

- where ψ_0 and ψ_m are known functions and A_m represent undetermined coefficients.

Application to Plane Elasticity with S_t Only

- The stress field resulted from ψ_0 must satisfy the in-plane equilibrium condition and the traction BCs on S_t.

$$\frac{\partial \sigma_x^0}{\partial x} + \frac{\partial \tau_{xy}^0}{\partial y} + F_x = 0, \quad \frac{\partial \tau_{xy}^0}{\partial x} + \frac{\partial \sigma_y^0}{\partial y} + F_y = 0$$

$$n_x \sigma_x^0 + n_y \tau_{xy}^0 = T_x, \quad n_x \tau_{xy}^0 + n_y \sigma_y^0 = T_y \qquad \text{on } S_t$$

- ψ_m represents m linearly independent functions, preferably forms a complete base, and results in stresses that satisfy

$$\frac{\partial \sigma_x^m}{\partial x} + \frac{\partial \tau_{xy}^m}{\partial y} = 0, \quad \frac{\partial \tau_{xy}^m}{\partial x} + \frac{\partial \sigma_y^m}{\partial y} = 0$$

$$n_x \sigma_x^m + n_y \tau_{xy}^m = 0, \quad n_x \tau_{xy}^m + n_y \sigma_y^m = 0 \qquad \text{on } S_t$$

- With the help Airy stress function, the equilibrium conditions are automatically satisfied.

Application to Plane Elasticity with S_t Only

- The principle of total complementary energy states

$$0=\delta\Pi^*=\delta U^*+\delta V^*=\delta U^*-\iint_{S_u} u_i\delta T_i\mathrm{d}S$$

- If $u_i = 0$ on S_u or no S_u at all: $\delta V^* = 0$.

$$0=\delta U^*=\iiint_V \delta U_0^*\mathrm{d}V=\iiint_V \frac{\partial U^*}{\partial\sigma_{ij}}\delta\sigma_{ij}\mathrm{d}V=\iiint_V \varepsilon_{ij}\delta\sigma_{ij}\mathrm{d}V$$

- For plane elasticity, the principle is reduced to

$$0=\delta U^*=\iint_A\left(\varepsilon_x\delta\sigma_x+\varepsilon_y\delta\sigma_y+2\varepsilon_{xy}\delta\tau_{xy}\right)\mathrm{d}A$$

- For plane strain problem (linear elasticity)

$$\varepsilon_x=\frac{1-\nu^2}{E}\left(\sigma_x-\frac{\nu}{1-\nu}\sigma_y\right),\quad \varepsilon_y=\frac{1-\nu^2}{E}\left(\sigma_y-\frac{\nu}{1-\nu}\sigma_x\right),\quad \varepsilon_{xy}=\frac{1+\nu}{E}\sigma_{xy}$$

- For plane stress (linear elasticity)

$$\varepsilon_x=\frac{1}{E}\left(\sigma_x-\nu\sigma_y\right),\quad \varepsilon_{xy}=\frac{1+\nu}{E}\sigma_{xy},\quad \varepsilon_y=\frac{1}{E}\left(\sigma_y-\nu\sigma_x\right)$$

Application to Plane Elasticity with S_t Only

- **If the plane domain is simply-connected, *V* is harmonic, and there is S_t only**

✓ The governing Airy function equation is biharmonic.

✓ The stress field is independent of elastic constants.

✓ The stress field is identical for plane strain and plane stress.

- **The principle can thus be reduced by setting $\nu = 0$:**

$$0 = \delta U^* = \iint_A \left(\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + 2\varepsilon_{xy}\delta\tau_{xy}\right)\mathrm{d}A = \frac{1}{E}\iint_A \left(\sigma_x \delta\sigma_x + \sigma_y \delta\sigma_y + 2\sigma_{xy}\delta\tau_{xy}\right)\mathrm{d}A$$

$$\psi = \psi_0 + \sum_m A_m \psi_m \quad \Rightarrow \quad \delta\psi = \sum \frac{\partial\psi}{\partial A_m}\delta A_m = \sum \psi_m \delta A_m$$

$$\Rightarrow \boxed{\begin{aligned} &\delta\sigma_x = \frac{\partial^2 \delta\psi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\sum \psi_m \delta A_m = \sum \frac{\partial^2 \psi_m}{\partial y^2}\delta A_m, \quad \delta\sigma_y = \frac{\partial^2 \delta\psi}{\partial x^2} = \sum \frac{\partial^2 \psi_m}{\partial x^2}\delta A_m, \\ &\delta\tau_{xy} = -\frac{\partial^2 \delta\psi}{\partial x \partial y} = -\sum \frac{\partial^2 \psi_m}{\partial x \partial y}\delta A_m \end{aligned}}$$

Application to Plane Elasticity with S_t Only

- Plug in the expressions of stresses and stress variations

$$0 = \iint_A \left[\sigma_x \delta\sigma_x + \sigma_y \delta\sigma_y + 2\sigma_{xy}\delta\tau_{xy} \right] \mathrm{d}A$$

$$= \iint_A \left[\left(\frac{\partial^2 \psi}{\partial y^2} + V \right) \sum \frac{\partial^2 \psi_m}{\partial y^2} \delta A_m + \left(\frac{\partial^2 \psi}{\partial x^2} + V \right) \sum \frac{\partial^2 \psi_m}{\partial x^2} \delta A_m + 2 \frac{\partial^2 \psi}{\partial x \partial y} \sum \frac{\partial^2 \psi_m}{\partial x \partial y} \delta A_m \right] \mathrm{d}A$$

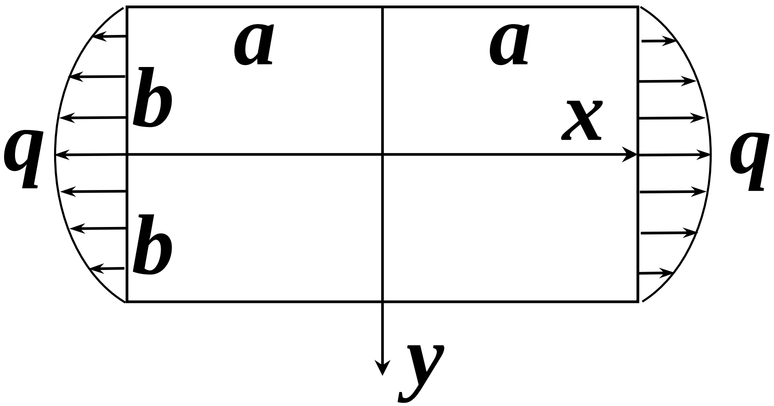
$$= \sum \delta A_m \iint_A \left[\left(\frac{\partial^2 \psi}{\partial y^2} + V \right) \frac{\partial^2 \psi_m}{\partial y^2} + \left(\frac{\partial^2 \psi}{\partial x^2} + V \right) \frac{\partial^2 \psi_m}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi_m}{\partial x \partial y} \right] \mathrm{d}A$$

- For arbitrary variation of the coefficient A_m

$$\iint_A \left[\left(\frac{\partial^2 \psi}{\partial y^2} + V \right) \frac{\partial^2 \psi_m}{\partial y^2} + \left(\frac{\partial^2 \psi}{\partial x^2} + V \right) \frac{\partial^2 \psi_m}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi_m}{\partial x \partial y} \right] \mathrm{d}A = 0.$$

Sample Problem

- Determine the stress field in the rectangular thin plate. ***F* = 0**.



$$(\sigma_x)_{x=\pm a} = q\left(1-\frac{y^2}{b^2}\right), \quad (\tau_{xy})_{x=\pm a} = 0;$$

$$(\sigma_y)_{y=\pm b} = 0, \quad (\tau_{xy})_{y=\pm b} = 0$$

- Solution: approximate the Airy stress function as

$$\psi = \psi_0 + \sum_m A_m \psi_m = \psi_0 + \sum_m A_m \psi_m$$

$$= \frac{1}{2} q y^2 \left(1-\frac{y^2}{6b^2}\right) + \left(x^2-a^2\right)^2 \left(y^2-b^2\right)^2 \left(A_1 + A_2 x^2 + A_3 y^2 + \cdots\right)$$

- ψ_0 satisfies the tractions BCs and ψ_m satisfies the zero-traction BCs, as required.

Sample Problem

- Include A_1 only and substitute into the principle

$$\iint_A \left[\frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi_m}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi_m}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi_m}{\partial x \partial y} \right] \mathrm{d}A = 0$$

$$\Rightarrow A_1 \left(\frac{64}{7} + \frac{256}{49} \cdot \frac{b^2}{a^2} + \frac{64}{7} \cdot \frac{b^4}{a^4} \right) = \frac{q}{a^4 b^2}$$

- For square plate:

$$A_1 = 0.0425 \frac{q}{a^6} \quad \Rightarrow$$

$$\sigma_x = q\left(1 - \frac{y^2}{a^2}\right) - 0.170q\left(1 - \frac{x^2}{a^2}\right)^2 \left(1 - \frac{3y^2}{a^2}\right)$$

$$\sigma_y = -0.170q\left(1 - \frac{3x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right)^2$$

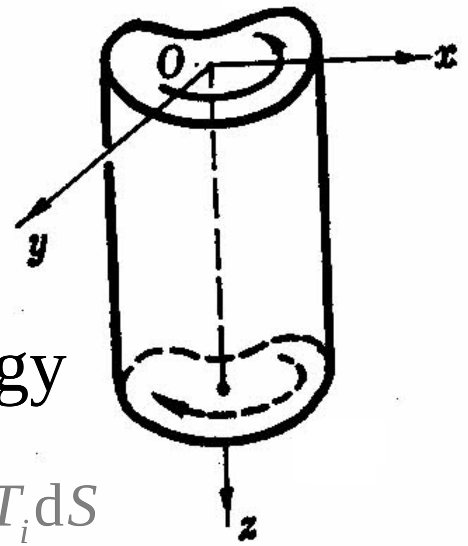
$$\tau_{xy} = -0.680q\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right)\frac{xy}{a^2}$$

- Higher accuracy can be achieved by including more terms.

Application to Torsion of Cylinders

- Two non-trivial stresses in terms of Prandtl Stress Function $\psi = \psi(x,y)$

$$\tau_{xz} = \frac{\partial \psi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \psi}{\partial x}$$



- The principle of total complementary energy

$$0 = \delta\Pi^* = \delta U^* + \delta V^* = \iiint_V \varepsilon_{ij}\delta\sigma_{ij}\mathrm{d}V - \iint_{S_u} u_i\delta T_i\mathrm{d}S$$

$$= \iiint_V \left[2\varepsilon_{xz}\delta\tau_{xz} + 2\varepsilon_{yz}\delta\tau_{yz} \right]\mathrm{d}V - \iint_{S_u} u_i\delta T_i\mathrm{d}S$$

$$= \frac{1}{G}\iiint_V \left[\tau_{xz}\delta\tau_{xz} + \tau_{yz}\delta\tau_{yz} \right]\mathrm{d}V - \iint_{S_u} u_i\delta T_i\mathrm{d}S$$

$$= \frac{1}{G}\iiint_V \left[\frac{\partial \psi}{\partial y}\frac{\partial \delta\psi}{\partial y} + \frac{\partial \psi}{\partial x}\frac{\partial \delta\psi}{\partial x} \right]\mathrm{d}V - \iint_{S_u} u_i\delta T_i\mathrm{d}S$$

Application to Torsion of Cylinders

- $\psi = \psi(x,y)$
- Relative angle of twist between ends: αL
- Variation of Torque at ends: δT

$$0 = \delta U^* + \delta V^* = \frac{L}{G}\iint_A \left[\frac{\partial \psi}{\partial y}\frac{\partial \delta\psi}{\partial y} + \frac{\partial \psi}{\partial x}\frac{\partial \delta\psi}{\partial x} \right] \mathrm{d}A - (\alpha L)\,\delta T$$

$$= \frac{L}{G}\iint_A \left[\frac{\partial \psi}{\partial y}\frac{\partial \delta\psi}{\partial y} + \frac{\partial \psi}{\partial x}\frac{\partial \delta\psi}{\partial x} \right] \mathrm{d}A - 2\alpha L \iint_A \delta\psi \,\mathrm{d}x\mathrm{d}y$$

- The total complementary energy results in

$$\iint_A \left[\frac{\partial \psi}{\partial x}\frac{\partial \delta\psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \delta\psi}{\partial y} - 2G\alpha\delta\psi \right] \mathrm{d}A = 0$$

- $\psi = \psi(x,y)$: Prandtl Stress Function for torsion.

Application to Torsion of Cylinders

- Propose an approximate solution of the form: $\psi = \sum_m A_m \psi_m$.
- where A_m are undetermined coefficients and ψ_m are known functions that satisfy $\psi_m = 0$ on lateral boundaries.
- The total complementary energy results in

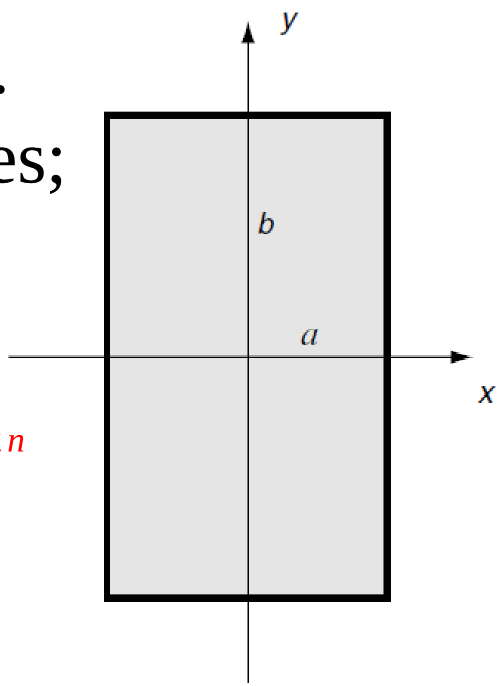
$$\Rightarrow \iint_A \left[\frac{\partial \psi}{\partial x}\left(\frac{\partial}{\partial x} \sum \frac{\partial \psi}{\partial A_m} \delta A_m \right) + \frac{\partial \psi}{\partial y}\left(\frac{\partial}{\partial y} \sum \frac{\partial \psi}{\partial A_m} \delta A_m \right) - 2G\alpha \left(\sum \frac{\partial \psi}{\partial A_m} \delta A_m \right) \right] \mathrm{d}A = 0$$

$$= \sum \delta A_m \iint_A \left[\frac{\partial \psi}{\partial x}\left(\frac{\partial}{\partial x} \frac{\partial \psi}{\partial A_m} \right) + \frac{\partial \psi}{\partial y}\left(\frac{\partial}{\partial y} \frac{\partial \psi}{\partial A_m} \right) - 2G\alpha \frac{\partial \psi}{\partial A_m} \right] \mathrm{d}A = 0$$

$$= \sum \delta A_m \iint_A \left[\frac{\partial \psi}{\partial x} \frac{\partial \psi_m}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi_m}{\partial y} - 2G\alpha \psi_m \right] \mathrm{d}A = 0$$

- For arbitrary δA_m: $\boxed{\iint_A \left[\frac{\partial \psi}{\partial x} \frac{\partial \psi_m}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi_m}{\partial y} - 2G\alpha \psi_m \right] \mathrm{d}A = 0}$

Sample Problem: Torsion of Rectangular Cylinder



- Boundary equation scheme does not work.
- Membrane analogy: $\psi = 0$ at the boundaries; symmetric about x & y.
- Propose an approximate solution

$$\psi = \sum A_{mn}\psi_{mn} = \left(x^2 - a^2\right)\left(y^2 - b^2\right)\sum A_{mn} x^{2m} y^{2n}$$

- $m \times n$ equations for $m \times n$ coefficients A_{mn}

$$\iint_A \left[\frac{\partial \psi}{\partial x}\frac{\partial \psi_{mn}}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial \psi_{mn}}{\partial y} - 2G\alpha\psi_{mn} \right] \mathrm{d}A = 0$$

- If we take three terms only:

$$\psi = \sum \left[A_{00}\psi_{00} + A_{10}\psi_{10} + A_{01}\psi_{01} \right] = \left(x^2 - a^2\right)\left(y^2 - b^2\right)\left(A_{00} + A_{10}x^2 + A_{01}y^2\right).$$

- The principle results in three equations

Sample Problem: Torsion of Rectangular Cylinder

$$\int_{-a}^{a}\int_{-b}^{b}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi_{00}}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\psi_{00}}{\partial y}-2G\alpha\psi_{00}\right]\mathrm{d}x\mathrm{d}y=0$$

$$\int_{-a}^{a}\int_{-b}^{b}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi_{10}}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\psi_{10}}{\partial y}-2G\alpha\psi_{10}\right]\mathrm{d}x\mathrm{d}y=0$$

$$\int_{-a}^{a}\int_{-b}^{b}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi_{01}}{\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial\psi_{01}}{\partial y}-2G\alpha\psi_{01}\right]\mathrm{d}x\mathrm{d}y=0$$

- Substitute ψ and ψ_{mn} into the above and implement the calculation

$$A_{00}=\frac{35G\alpha}{8\Delta}\left(19a^4+13a^2b^2+9b^4\right),\ A_{10}=\frac{105G\alpha}{8\Delta}\left(9a^2+b^2\right),\ A_{01}=\frac{105G\alpha}{8\Delta}\left(a^2+9b^2\right)$$

- where $\Delta=45a^6+509a^4b^2+509a^2b^4+45b^6.$
- Higher accuracy is achieved by including more terms.

Outline

- Work Done by External Load
- Strain Energy
- The Delta Operator
- Principle of Virtual Work
- Principle of Minimum Potential Energy
- Castigliano's First Theorem
- Displacement Variation: Ritz Method
- Displacement Variation: Galerkin Method
- Complimentary Strain Energy
- Principle of Complimentary Virtual Work
- Principle of Minimum Complimentary Potential Energy
- Castigliano's Second Theorem
- Stress Variation
- Stress Variation: Application to Plane Elasticity
- Stress Variation: Application to Torsion of Cylinders