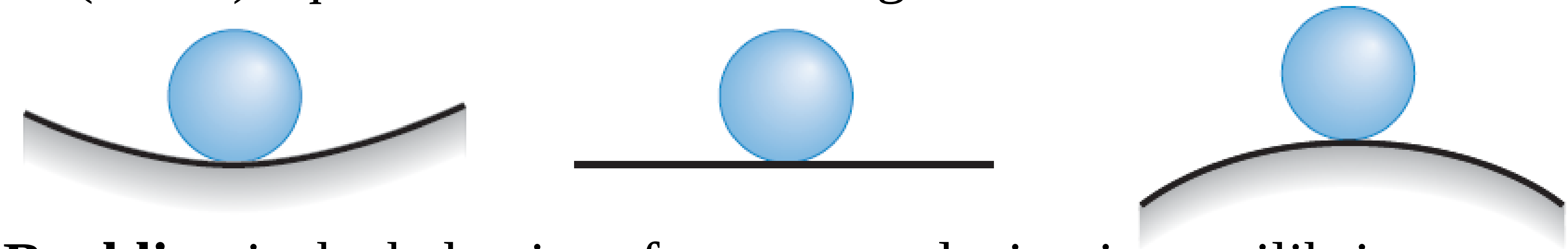
Column Buckling

mi@seu.edu.cn

Contents

Stability and Buckling

- **Stability** is characterized as the ability of a structure to maintain its (stable) equilibrium under working conditions.

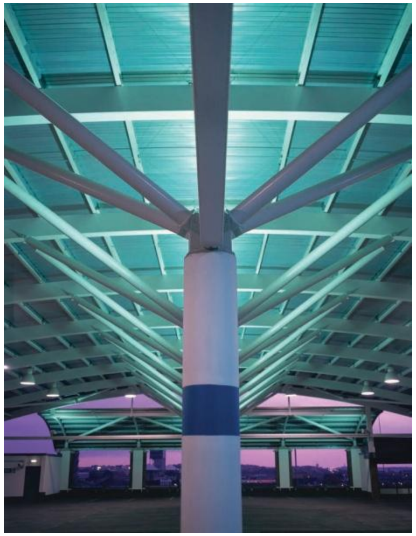


- **Buckling** is the behavior of a structure losing its equilibrium under working conditions. This is another type of failure criterion, in addition to strength (fracture/yielding), stiffness (deformation) and fatigue criteria.

- Buckling occurs suddenly and results in catastrophic accident.

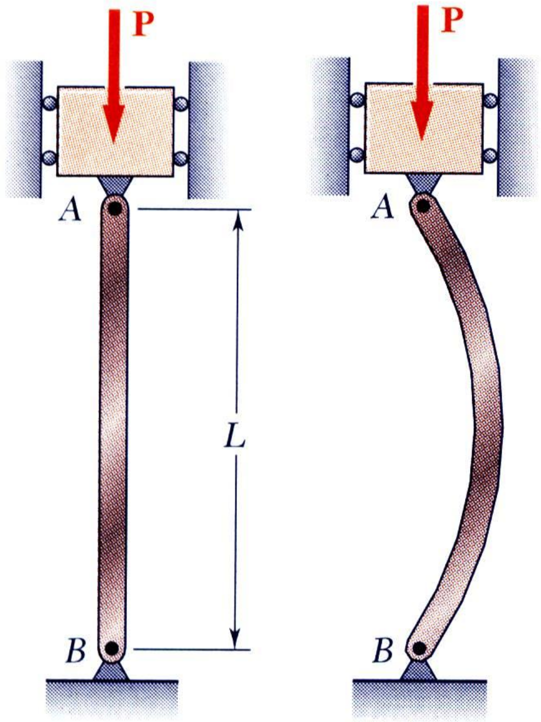
Examples of Columns







Conventional Design of Columns



- In the design of columns, cross-sectional area is selected such that
  - allowable stress is not exceeded

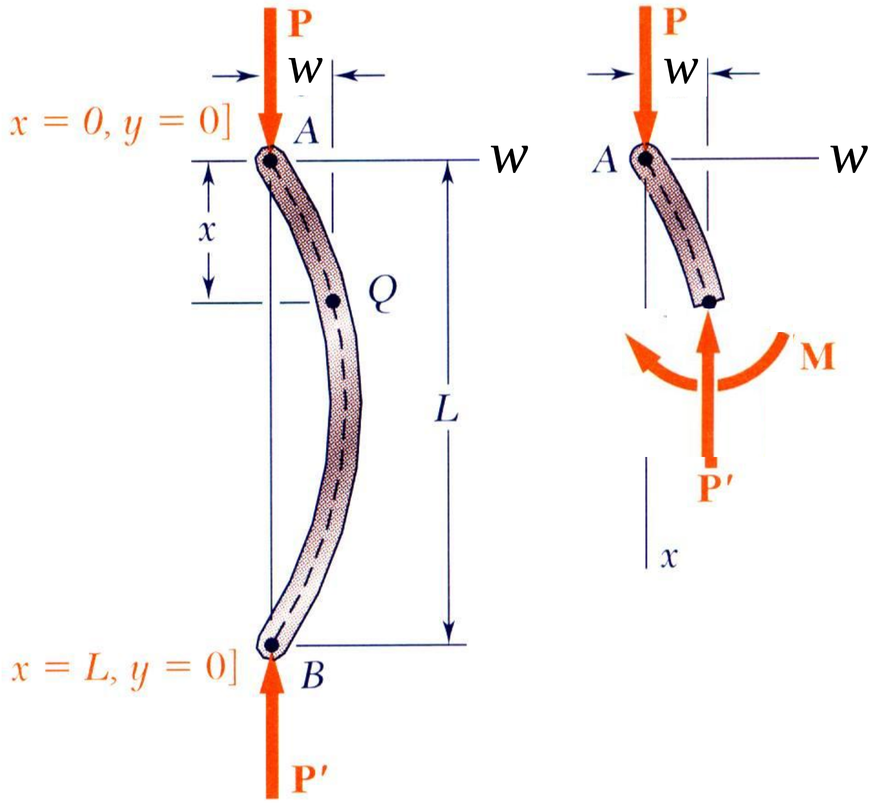
$$\sigma = \frac{P}{A} \leq [\sigma]$$

  - deformation falls within specifications

$$\varepsilon = \frac{\Delta L}{L} = \frac{P}{AE} \leq [\varepsilon]$$

- After these design calculations, many discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.

Euler's Formula for Pin-ended Columns



- Consider an axially loaded beam. After a small perturbation, the system reaches a neutral equilibrium configuration such that

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = -\frac{P_{cr}}{EI}w$$

$$\Rightarrow \frac{d^2w}{dx^2} + \frac{P_{cr}}{EI}w = 0$$

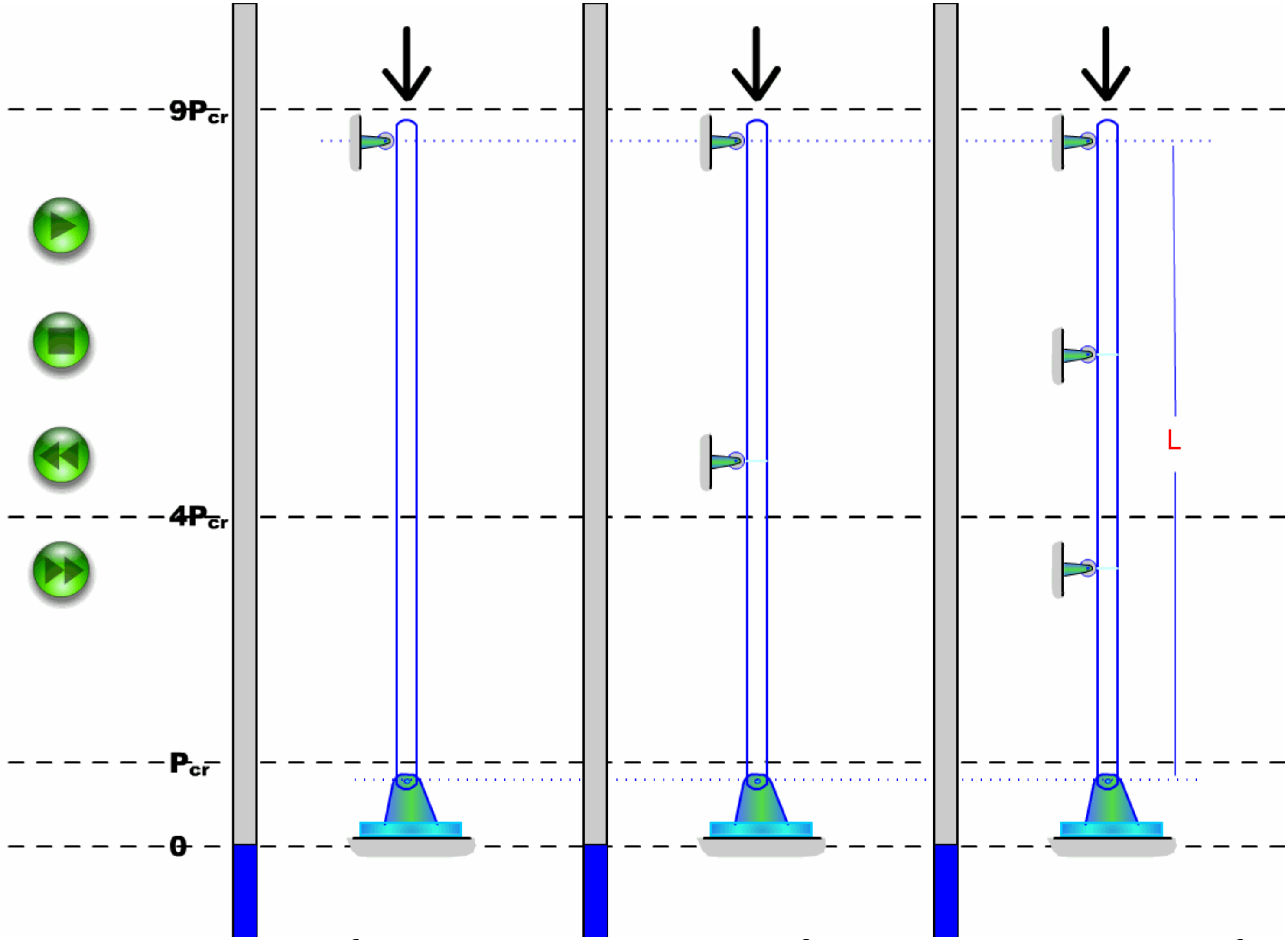
$$\Rightarrow w'' + k^2 w = 0, \quad k^2 = \frac{P_{cr}}{EI}$$

$$\Rightarrow w = A\sin kx + B\cos kx$$

$$0 = w(0) = w(L)$$

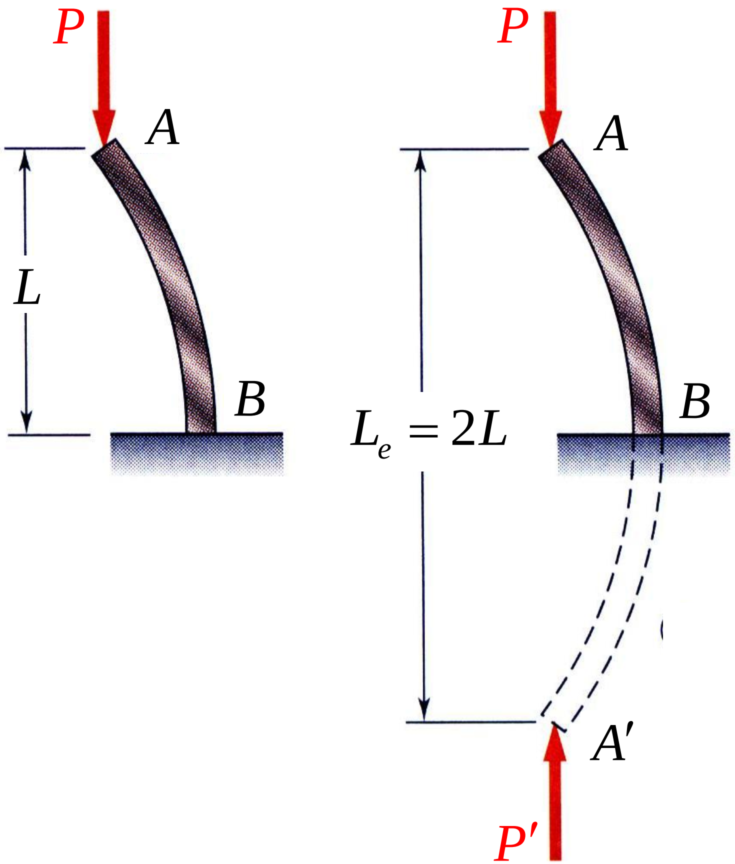
$$\Rightarrow \begin{cases} B = 0; \\ kL = n\pi \Rightarrow \boxed{P_{cr} = \dfrac{EIn^2\pi^2}{L^2}} \end{cases}$$

Buckling Modes



$$P_{cr} = \frac{EI\pi^2}{L^2}; \qquad P_{cr} = \frac{4EI\pi^2}{L^2}; \qquad P_{cr} = \frac{9EI\pi^2}{L^2}$$

Cantilevered Columns



- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.

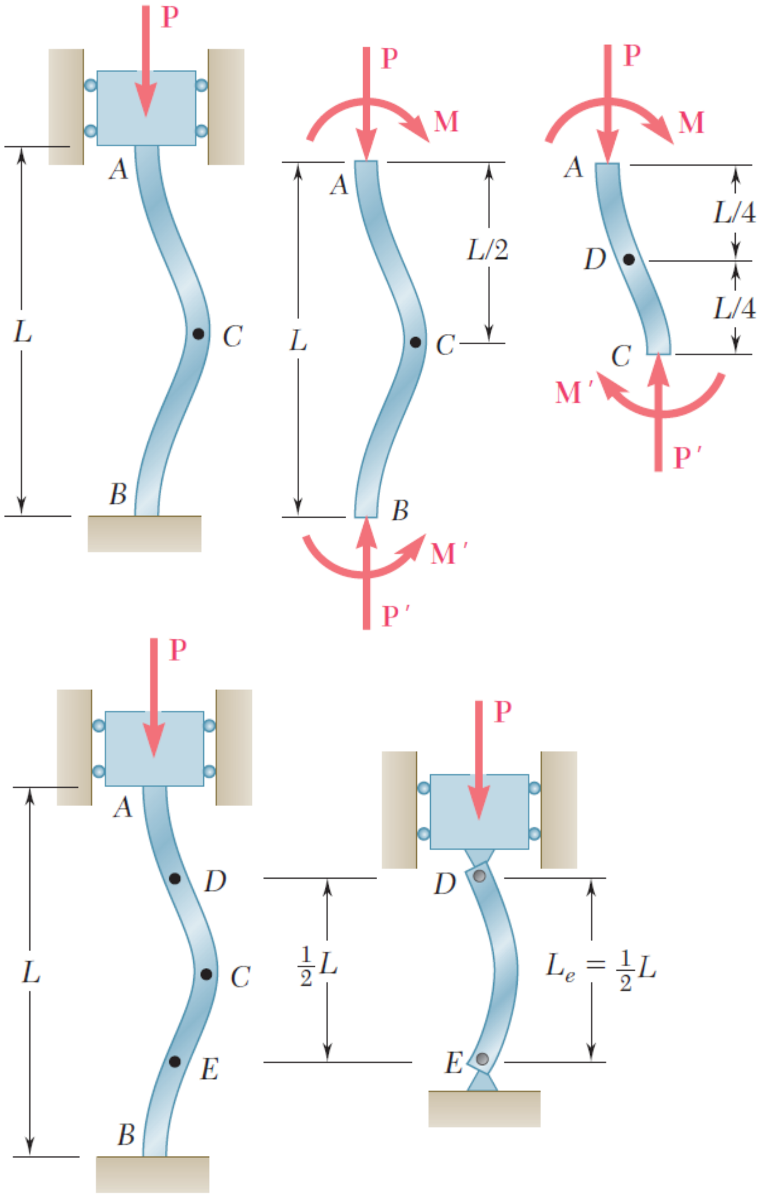
- The critical loading is calculated from Euler's formula,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{4L^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{\left(L_e/i_r\right)^2} = \frac{\pi^2 E}{4\left(L/i_r\right)^2}$$

$$L_e = 2L = \text{equivalent length}$$

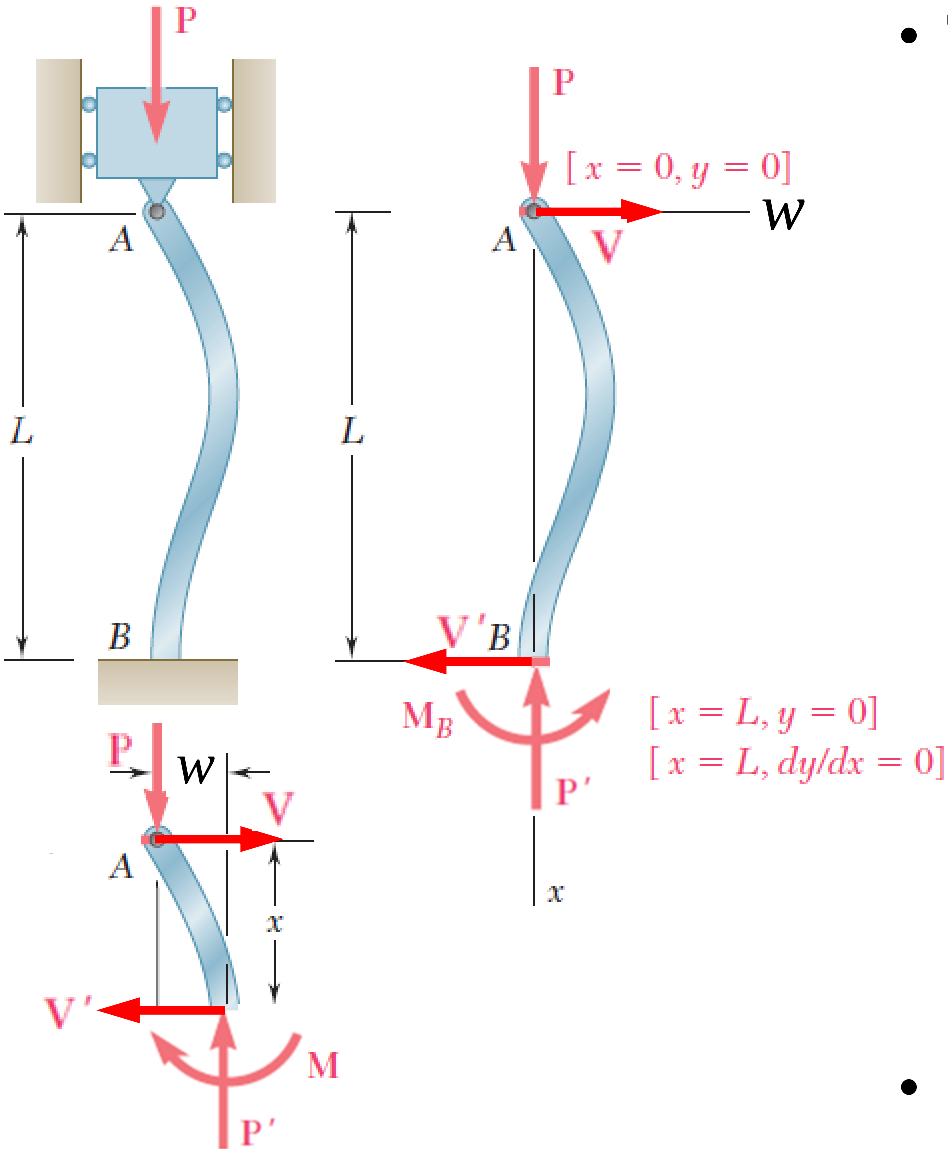
Columns with Two Fixed Ends



- The symmetry of the supports and of the loading requires that the shear at *C* and the horizontal reactions at both ends be zero.
- The equation of the deflection curve involves sine and cosine functions.
- Point *D* must be a point of inflection, where the bending moment is zero.
- It follows that the portion *DE* of the column must behave as a pin ended column.

$$L_e = 0.5L \qquad \Rightarrow P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{4\pi^2 EI}{L^2}$$

Columns with One Fixed End and One Free End



- The differential equation

$$\frac{d^2w}{dx^2} = -\frac{(Pw - Vx)}{EI}$$

$$\Rightarrow \frac{d^2w}{dx^2} + \frac{P}{EI}w = \frac{Vx}{EI}$$

$$\Rightarrow w = A\sin kx + B\cos kx + \frac{Vx}{P}, \quad k^2 = \frac{P}{EI}$$

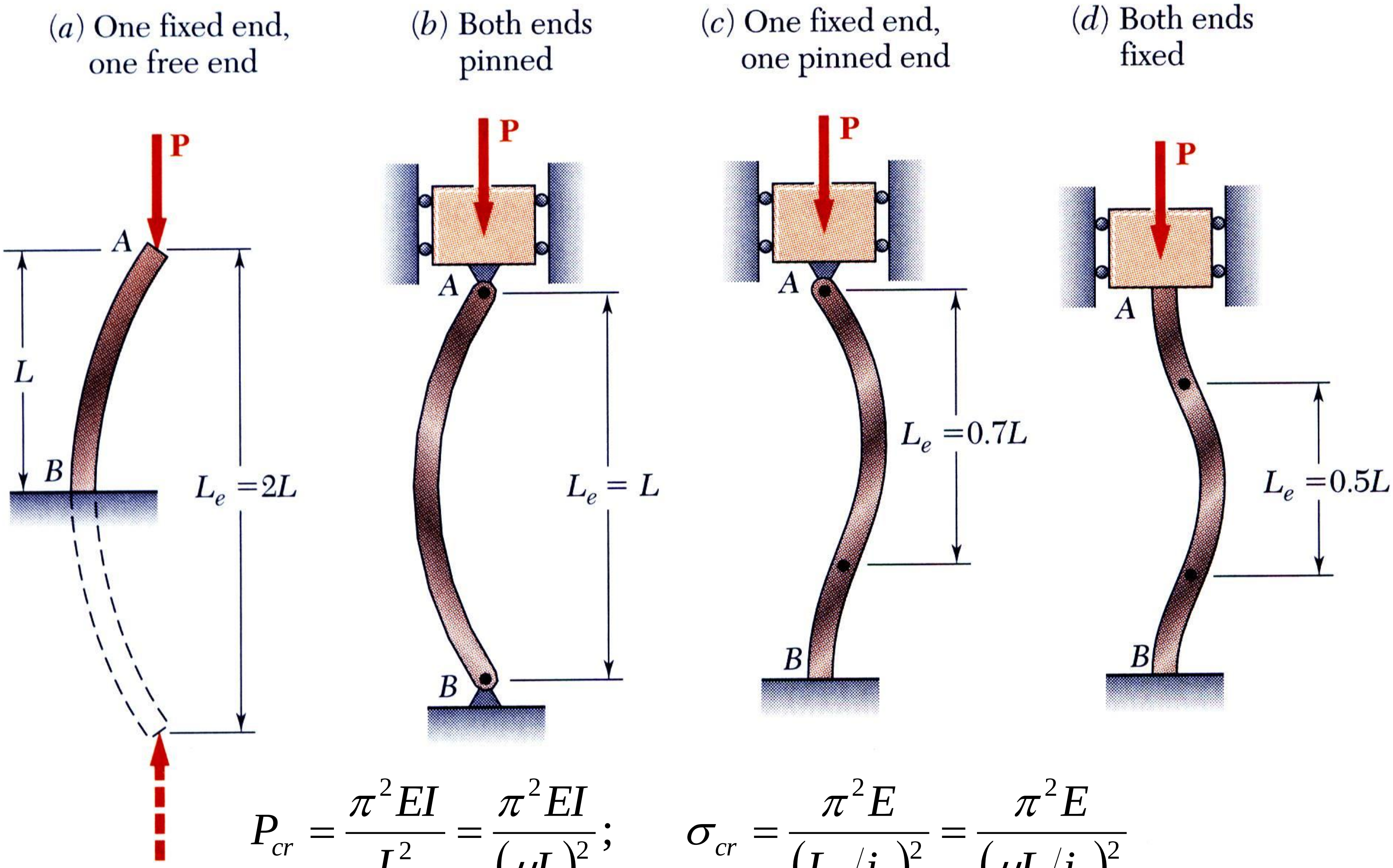
$$\begin{cases} 0 = w(0) = w(L) & \Rightarrow B = 0; A\sin kL = -\dfrac{VL}{P} \\ 0 = w'(L) & \Rightarrow Ak\cos kL = -\dfrac{V}{P} \end{cases}$$

$$\Rightarrow \tan kL = kL \qquad \Rightarrow k^2 = 20.19/L^2$$

$$\Rightarrow P_{cr} = EIk^2 = \frac{20.19EI}{L^2} \approx \frac{\pi^2 EI}{(0.699L)^2}$$

- Equivalent length: $L_e \approx 0.7L$

Extension of Euler's Formula

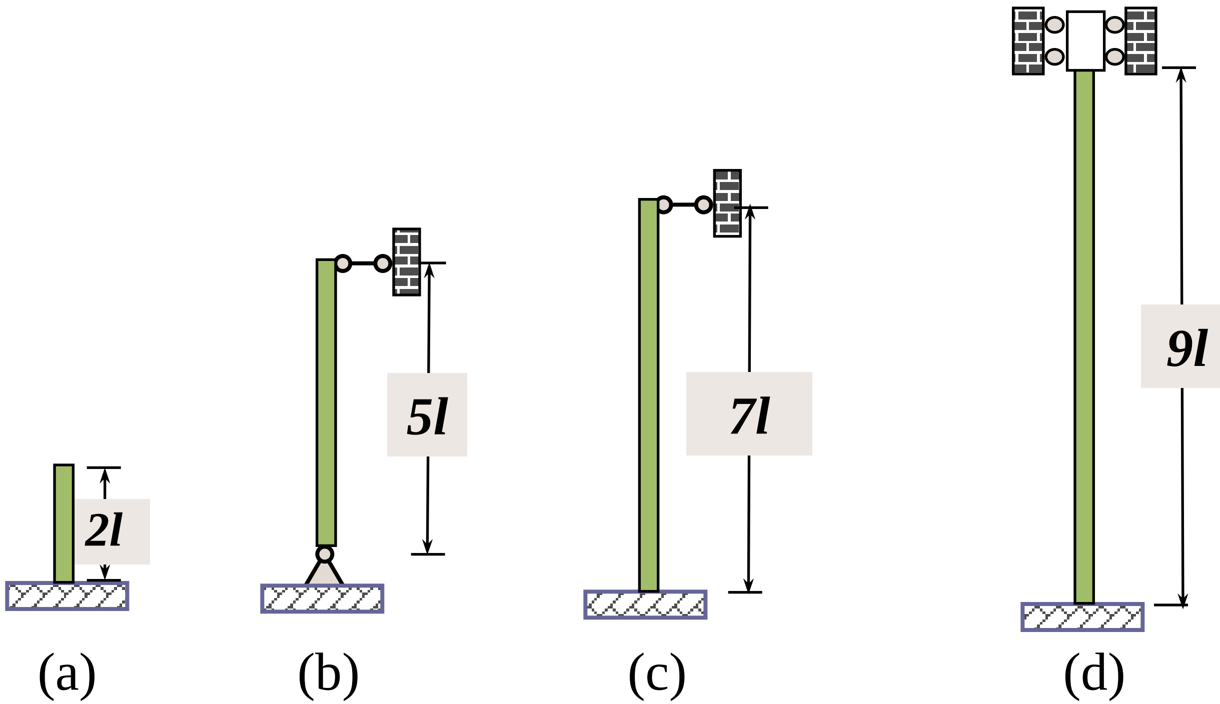


$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(\mu L)^2}; \qquad \sigma_{cr} = \frac{\pi^2 E}{(L_e/i_r)^2} = \frac{\pi^2 E}{(\mu L/i_r)^2}$$

L_e = equivalent length; μ = length coefficient

Sample Problem

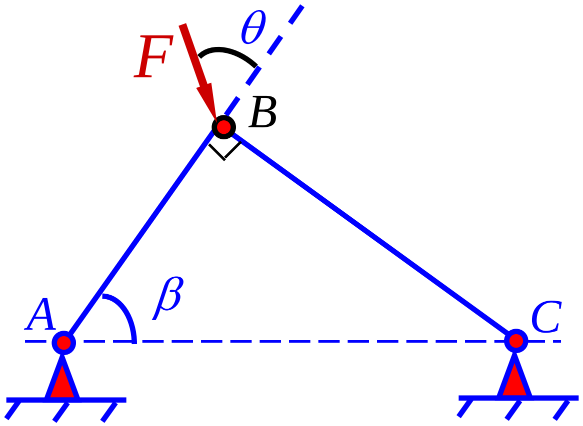
- Assuming the same material and cross-sectional dimension, which of the following four columns is most susceptible to buckling?



$$P_{cr} = \frac{\pi^2 EI}{(\mu l)^2} \quad (\text{Euler Formula})$$

Sample Problem

- For the truss shown, F, β, and L_{AC} are given. Find $0 < \theta < \pi/2$, under which column AB and BC reach critical stability simultaneously.
- Solution:



1. Euler's equation for column *AB*:

$$F_{AB} = F\cos\theta = \frac{\pi^2 EI}{l_{AB}^2} = \frac{\pi^2 EI}{l^2\cos^2\beta}$$

2. Euler's equation for column *BC*:

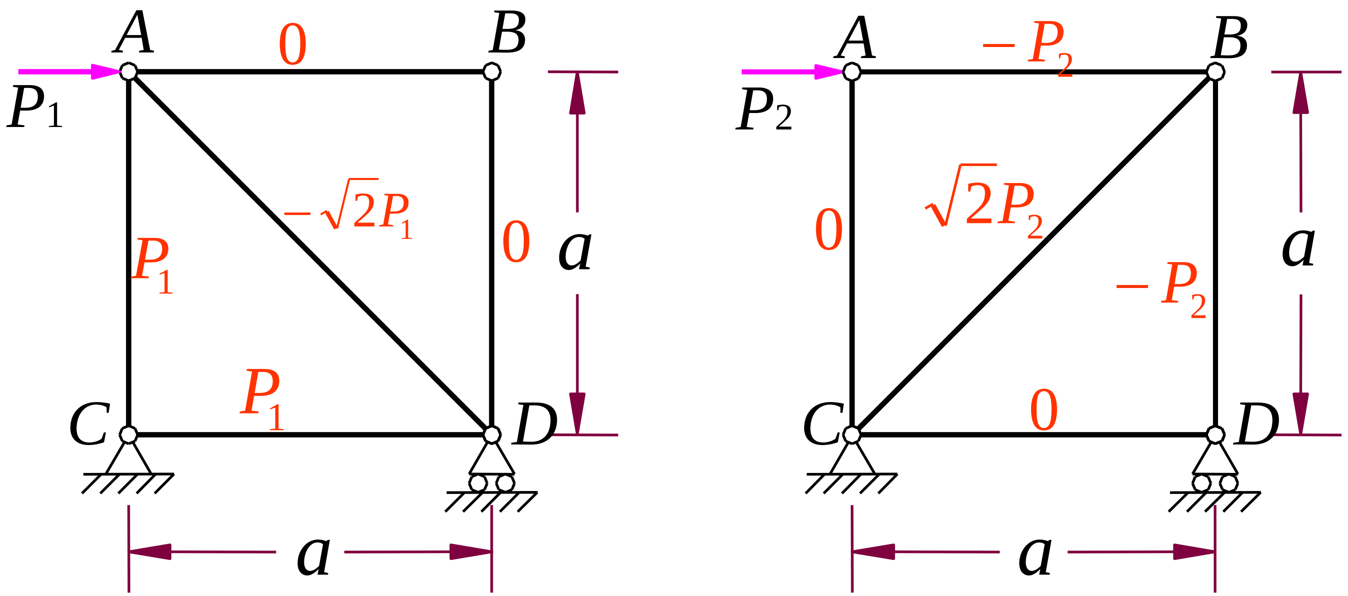
$$F_{BC} = F\sin\theta = \frac{\pi^2 EI}{l_{CB}^2} = \frac{\pi^2 EI}{l^2\sin^2\beta}$$

3. If column *AB* & *BC* reach critical stability simultaneously:

$$\left.\begin{aligned} F\cos\theta &= \frac{\pi^2 EI}{l^2\cos^2\beta} \\ F\sin\theta &= \frac{\pi^2 EI}{l^2\sin^2\beta} \end{aligned}\right\} \Rightarrow \cot\theta = \tan^2\beta$$

Sample Problem

- Which of the following best describes the relationship between critical P_1 and P_2? (a) $P_1 = P_2$; (b) $P_1 < P_2$; (c) $P_1 > P_2$; (d) not sure.

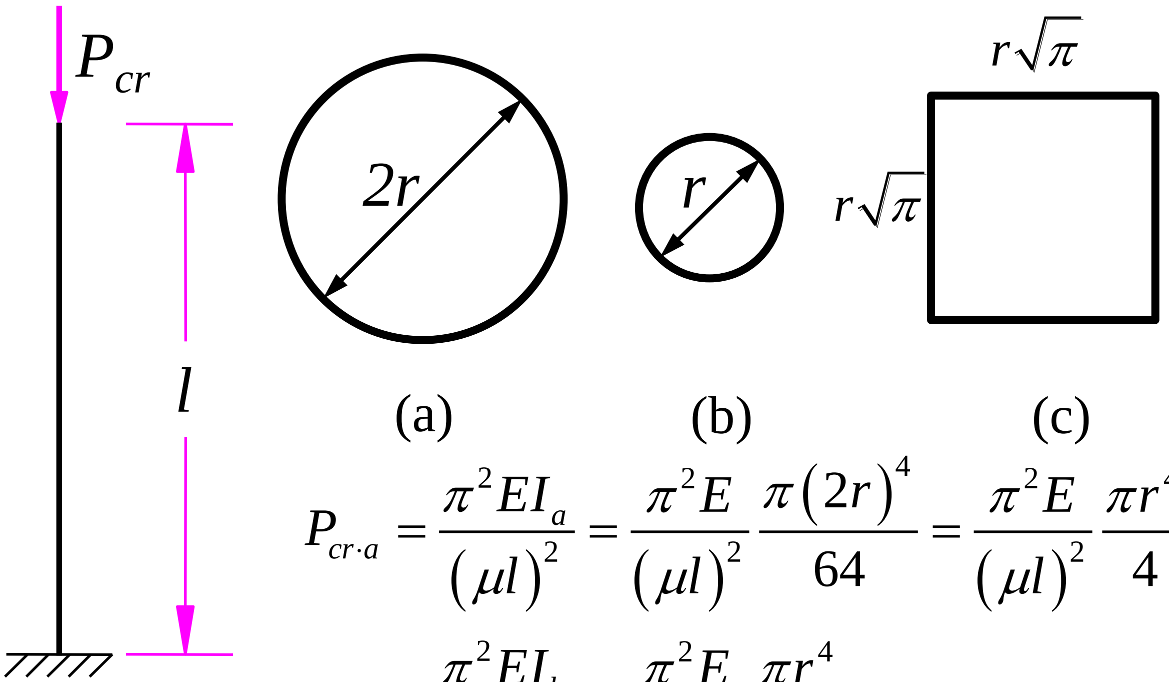


- Solution: $(a):\ F_{NAD} = \sqrt{2}P_1 = \dfrac{\pi^2 EI}{\left(\sqrt{2}a\right)^2} \Rightarrow P_1 = \dfrac{1}{2\sqrt{2}}\dfrac{\pi^2 EI}{a^2}$

$$(b):\ F_{NAB} = P_2 = \frac{\pi^2 EI}{a^2} \Rightarrow P_2 = \frac{\pi^2 EI}{a^2}$$

Sample Problem

- For the column shown, find the relationship between the critical loads corresponding to cross-sections described by (a), (b) and (c).



- Solution:

$$P_{cr\cdot a}=\frac{\pi^2 EI_a}{(\mu l)^2}=\frac{\pi^2 E}{(\mu l)^2}\frac{\pi(2r)^4}{64}=\frac{\pi^2 E}{(\mu l)^2}\frac{\pi r^4}{4}$$

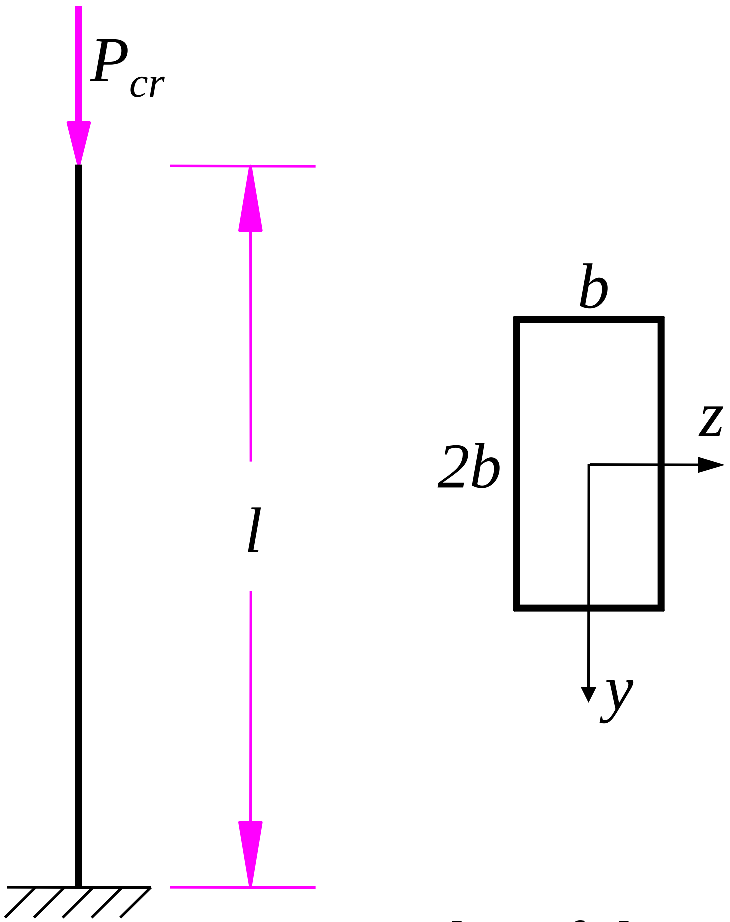
$$P_{cr\cdot b}=\frac{\pi^2 EI_b}{(\mu l)^2}=\frac{\pi^2 E}{(\mu l)^2}\frac{\pi r^4}{64}$$

$$P_{cr\cdot c}=\frac{\pi^2 EI_c}{(\mu l)^2}=\frac{\pi^2 E}{(\mu l)^2}\frac{\left(r\sqrt{\pi}\right)^4}{12}=\frac{\pi^2 E}{(\mu l)^2}\frac{\pi^2 r^4}{12}$$

$$\Rightarrow \begin{cases} P_{cr\cdot a} : P_{cr\cdot b} : P_{cr\cdot c} \\ =\frac{1}{4}:\frac{1}{64}:\frac{\pi}{12} \\ =1:1/16:\pi/3 \end{cases}$$

Buckling in Orthogonal Planes

- For the column shown, find the relationship between the critical loads corresponding to buckling in *x*-*y* and *x*-*z* plane respectively.



- Solution:

$$\left.\begin{aligned} P_{cr\cdot z} &= \frac{\pi^2 EI_z}{(\mu_z l)^2} \\ P_{cr\cdot y} &= \frac{\pi^2 EI_y}{(\mu_y l)^2} \end{aligned}\right\} \Rightarrow$$

$$\frac{P_{cr\cdot z}}{P_{cr\cdot y}} = \frac{I_z}{I_y} = \frac{b(2b)^3/12}{2bb^3/12} = 4$$

- What if the cross-section is increased to $2b \times 2b$?

Buckling in Orthogonal Planes

- Buckling in *x*-*z*:

$$\mu_y = 1 \qquad P_{cr \cdot y} = \frac{\pi^2 EI_y}{L^2}$$

- Buckling in *x*-*y*:

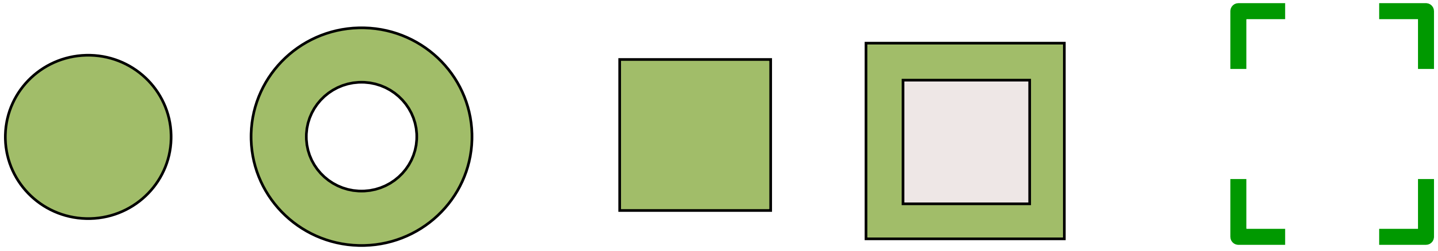
$$\mu_z = 0.5 \qquad P_{cr \cdot z} = \frac{\pi^2 EI_z}{(0.5L)^2}$$

Ways to Improve Column Stability

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(\mu L)^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(\mu L/i_r)^2} = \frac{\pi^2 E}{\lambda^2}$$

- Selection of materials.
- Decrease effective column length (μL).
- Increase moment of inertia for a given cross-sectional area.



- Coordination of end conditions and moment of inertia in orthogonal planes: $\mu_z / i_{r \cdot z} = \mu_y / i_{r \cdot y}$
- For a group of columns, make each one equally stable (avoid the last straw).

Applicability of Euler's Formula

- Critical buckling stress

$$\left.\begin{aligned} P_{cr} &= \frac{\pi^2 EI}{(\mu L)^2} \\ \sigma &= \frac{P}{A} \end{aligned}\right\} \Rightarrow \sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E\left(A i_r^2\right)}{(\mu L)^2 A} = \frac{\pi^2 E}{(\mu L / i_r)^2} = \frac{\pi^2 E}{\lambda^2}$$

$$\lambda = \mu L / i_r = \text{ slenderness ratio}$$

- Applicability of Euler's formula

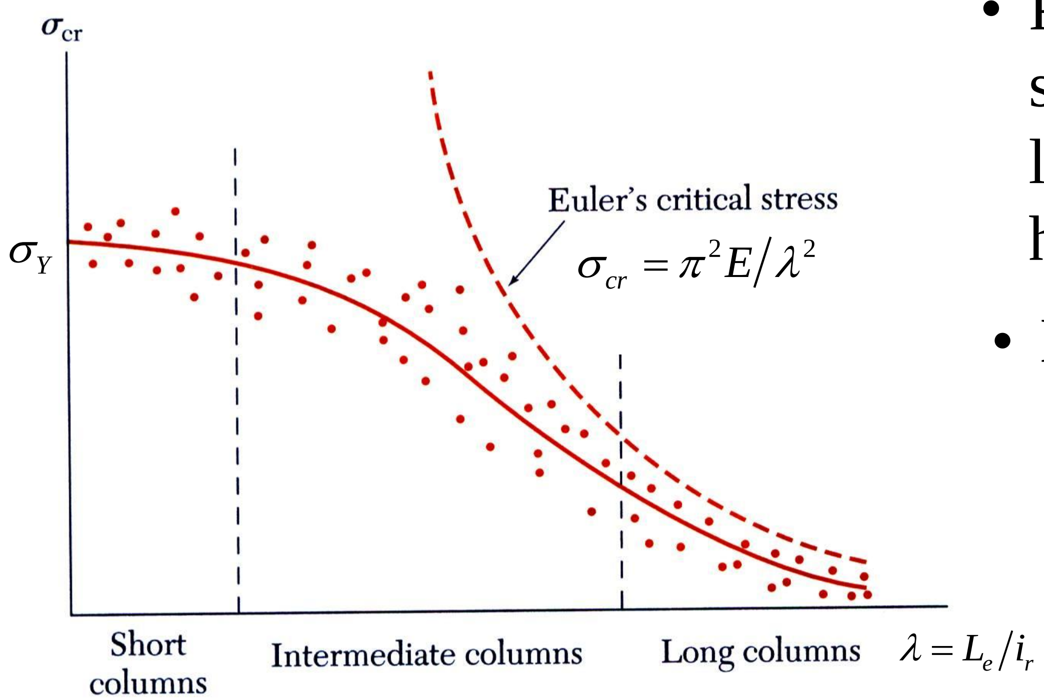
$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \le \sigma_P \qquad \Rightarrow \lambda \ge \lambda_P = \pi \sqrt{\frac{E}{\sigma_P}}$$

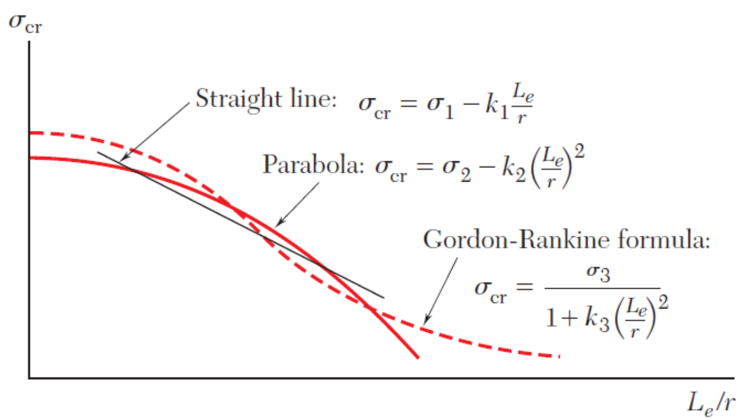
σ_P: proportion limit;
λ_p: critical slenderness ratio.

Q235: $$\lambda_P = \pi \sqrt{\frac{E}{\sigma_P}} = \pi \sqrt{\frac{206\ \text{GPa}}{200\ \text{MPa}}} = 100$$

$$\left(\sigma_Y = 235\ \text{MPa}\right)$$

Failure Diagram of Columns

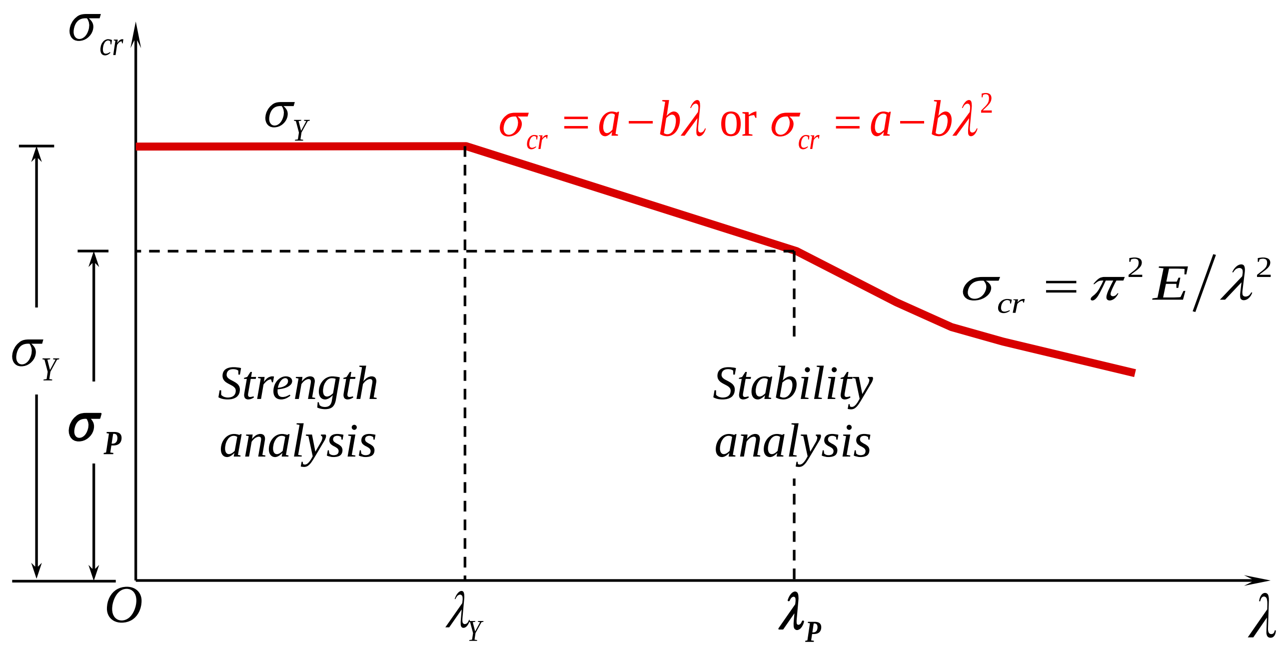




Empirical formulas for intermediate columns

- Previous analyses assumed stresses below the proportional limit and initially straight, homogeneous columns
- Experimental data demonstrate
  - for large L_e/i_r, σ_{cr} follows Euler's formula and depends upon E but not σ_Y.
  - for small L_e/i_r, σ_{cr} is determined by the yield strength σ_Y and not E.
  - for intermediate L_e/i_r, σ_{cr} depends on both σ_Y and E.

Critical Stress of Columns



1. Long columns: $\lambda \geq \lambda_P \left(\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \leq \sigma_p = \frac{\pi^2 E}{\lambda_P^2} \right)$

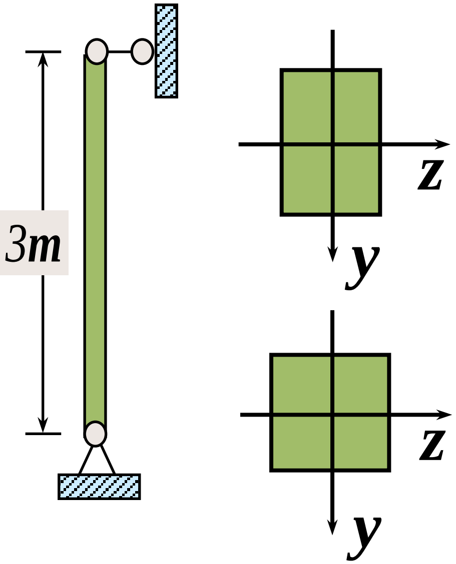
2. Intermediate columns:

$$\lambda_Y = \frac{a - \sigma_Y}{b} < \lambda < \lambda_P \left(\sigma_p < \sigma_{cr} = a - b\lambda < \sigma_Y \right)$$

3. Short columns: $\lambda < \lambda_Y \left(\sigma_{cr} > \sigma_Y \right)$

Sample Problem

- For the wood column shown, E = 10 GPa, σ_p = 9 MPa, σ_Y = 13 MPa, σ_{cr} = 28.9 − 0.19λ for intermediate columns. Find the critical loads for rectangular cross-sections: (1) h = 120 mm, b = 90 mm; (2) h = b = 104 mm
- Solution:



$$\lambda_p = \sqrt{\frac{\pi^2 E}{\sigma_p}} = 104.7$$

$$\lambda_Y = \frac{a - \sigma_Y}{b} = 82.6$$

$$1.\ \lambda = \frac{\mu l}{i_r} = 115.4$$

$$\because \lambda > \lambda_p \qquad \therefore \text{Slender column.}$$

$$P_{cr} = \frac{\pi^2 EI}{(\mu l)^2} = 79.9\ \text{kN}$$

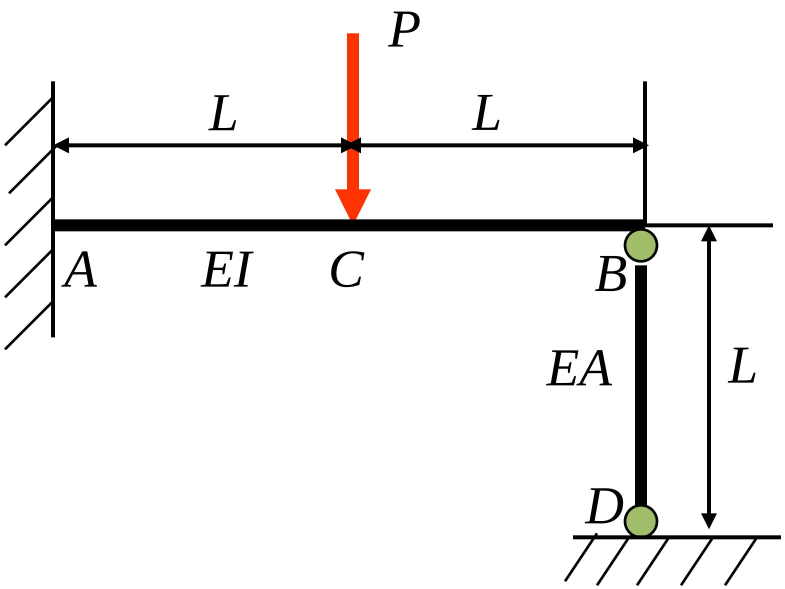
$$2.\ \lambda = 100$$

$$\therefore \text{Intermediate column.}$$

$$P_{cr} = \sigma_{cr} A = (a - b\lambda)A$$
$$= 107.08\ \text{kN}$$

Sample Problem

- For the composite beam and column structure shown, $E_{AB} = E_{BD} = E$, $d_{AB} = d_{BD} = d$, $L:d = 30$, $\lambda_{P.BD} = 100$. Find P, under which BD reaches the critical condition.



- Solution:

$$BD: \quad \lambda = \frac{\mu L}{i} = \frac{\mu L}{d/4} = 120 > \lambda_P$$

$$F_{cr \cdot BD} = \frac{\pi^2 EI}{L^2}$$

- Deformation compatibility:

$$\frac{5PL^3}{6EI} - \frac{8F_{BD}L^3}{3EI} = \frac{F_{BD}L}{EA}$$

$$\Rightarrow P = \frac{\pi^3 Ed^2}{18000}$$

Design of Columns

- Stability analysis of columns
  - Stability check
  - Cross-section design
  - Allowable load/stress

1. Method of safety factor

$$F \le \frac{F_{cr}}{n_{st}} = [F_{st}]$$

$$\sigma = \frac{F}{A} \le \frac{\sigma_{cr}}{n_{st}} = [\sigma_{st}]$$

n_{st}: safety factor

$[F_{st}]$: allowable load

$[\sigma_{st}]$: allowable stress

2. Method of discount factor

$$[\sigma_{st}] = \varphi[\sigma]$$

$$\sigma = \frac{F}{A} \le \varphi[\sigma]$$

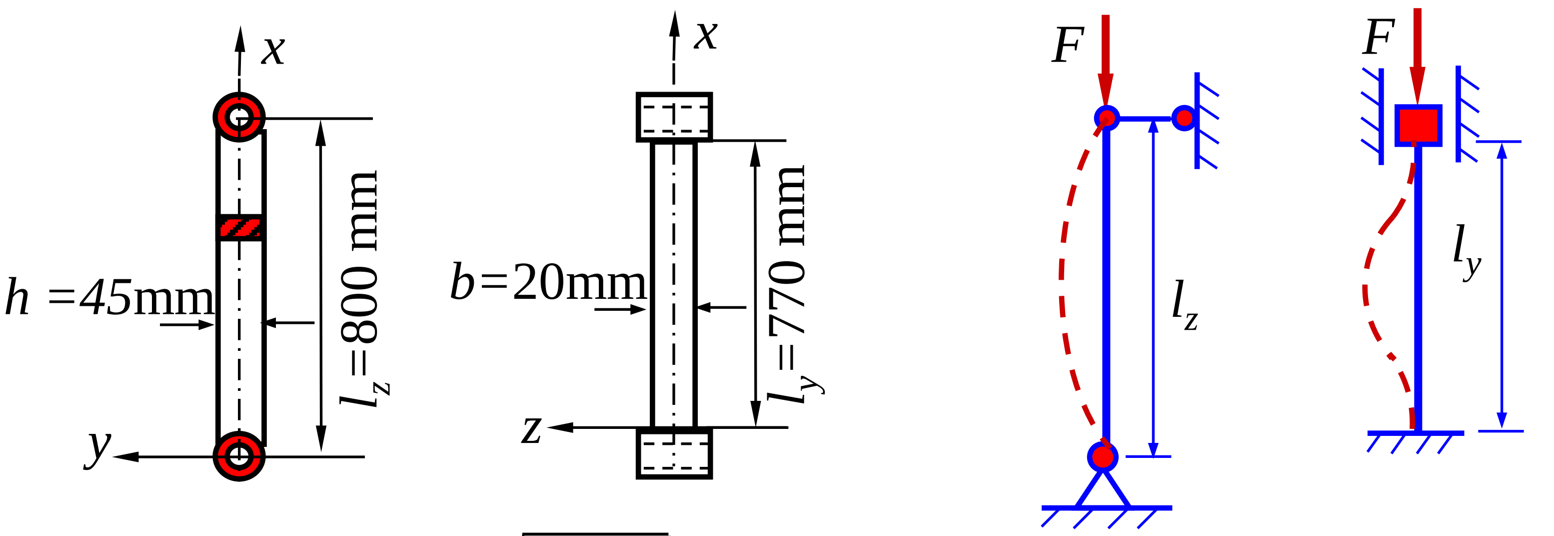
$\varphi = \varphi(\lambda)$: discount/stability factor

表 9－2　Q235 钢 a 类截面中心受压直杆的稳定因数 φ

λ	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
0	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.998	0.997	0.996
10	0.995	0.994	0.993	0.992	0.991	0.989	0.988	0.986	0.985	0.983
20	0.981	0.979	0.977	0.976	0.974	0.972	0.970	0.968	0.966	0.964
30	0.963	0.961	0.959	0.957	0.955	0.952	0.950	0.948	0.946	0.944
40	0.941	0.939	0.937	0.934	0.932	0.929	0.927	0.924	0.921	0.919
50	0.916	0.913	0.910	0.907	0.904	0.900	0.897	0.894	0.890	0.886
60	0.883	0.879	0.875	0.871	0.867	0.863	0.858	0.851	0.849	0.844
70	0.830	0.834	0.829	0.824	0.818	0.813	0.807	0.801	0.795	0.789
80	0.788	0.776	0.770	0.763	0.757	0.750	0.743	0.736	0.728	0.721
90	0.714	0.706	0.699	0.691	0.684	0.676	0.668	0.661	0.653	0.645
100	0.638	0.630	0.622	0.615	0.607	0.600	0.592	0.585	0.577	0.570
110	0.563	0.555	0.548	0.541	0.534	0.527	0.520	0.514	0.507	0.500
120	0.494	0.488	0.481	0.475	0.469	0.463	0.457	0.451	0.445	0.440
130	0.434	0.429	0.423	0.418	0.412	0.407	0.402	0.397	0.392	0.387
140	0.383	0.378	0.373	0.369	0.364	0.360	0.356	0.351	0.347	0.343
150	0.339	0.335	0.331	0.327	0.323	0.320	0.316	0.312	0.309	0.305
160	0.302	0.298	0.295	0.292	0.289	0.285	0.282	0.279	0.276	0.273
170	0.270	0.267	0.264	0.262	0.259	0.256	0.253	0.251	0.248	0.246
180	0.243	0.241	0.238	0.236	0.233	0.231	0.229	0.226	0.224	0.222
190	0.220	0.218	0.215	0.213	0.211	0.209	0.207	0.205	0.203	0.201
200	0.199	0.198	0.196	0.194	0.192	0.190	0.189	0.187	0.185	0.183
210	0.182	0.180	0.179	0.177	0.175	0.174	0.172	0.171	0.169	0.168
220	0.166	0.165	0.164	1.162	0.161	0.159	0.158	0.157	0.155	0.154
230	0.150	0.152	0.150	0.149	0.148	0.147	0.146	0.144	0.143	0.142
240	0.141	0.140	0.139	0.138	0.136	0.135	0.134	0.133	0.132	0.131

Sample Problem

- A Q275 steel column is cylindrically pinned at both ends. λ_p = 96, σ_Y = 275 MPa, σ_{cr} = 280 – 0.00872λ^2 for intermediate columns. Analyze the column stability for F = 60 kN and n_{st} = 3.5.



- Solution: $\lambda_p = 96, \quad \lambda_Y = \sqrt{\dfrac{a-\sigma_Y}{b}} \approx 24$

- In x-y plane, both ends are pinned.

$$i_z = \sqrt{\frac{I_z}{A}} = \frac{h}{2\sqrt{3}} = 12.99mm, \qquad \lambda_z = \frac{\mu_z l_z}{i_z} = \frac{1\times 800}{12.99} = 61.9$$

- In *x*-*z* plane, both ends are fixed.

$$i_y = \sqrt{\frac{I_y}{A}} = \frac{b}{2\sqrt{3}} = 5.77mm, \quad \lambda_y = \frac{\mu_y l_y}{i_y} = \frac{0.5\times 770}{5.77} = 66.7$$

$$\Rightarrow \left(\lambda_Y = 24\right) < \left(\lambda_z = 61.9\right) < \left(\lambda_y = 66.7\right) < \left(\lambda_p = 96\right)$$

- The column buckling first happens in the *x*-*z* plane.

$$\sigma_{cr} = 280 - 0.00872\lambda_y{}^2 = 241\,\text{MPa}$$

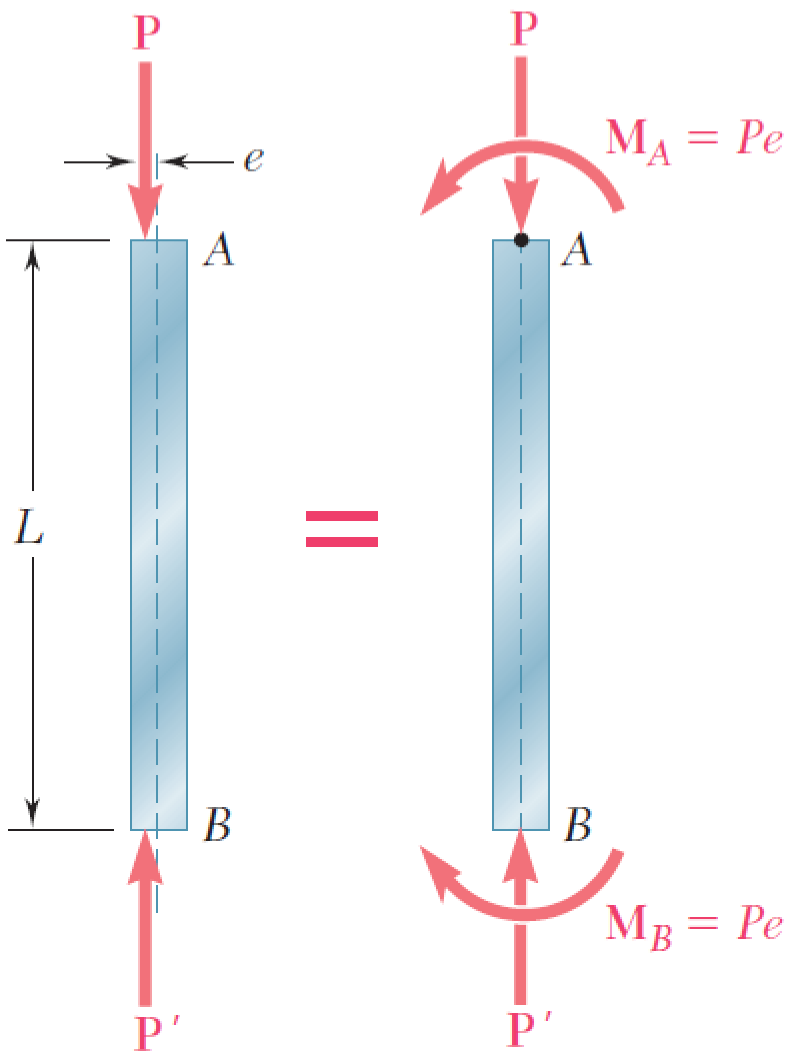
$$F_{cr} = \sigma_{cr} A = \left(241\times 10^6\right)\left(20\times 45\times 10^{-6}\right) = 217\text{ kN}$$

$$\left[F\right] = F_{cr}/n = 62\text{ kN}$$

$$\Rightarrow \left(F = 60\text{ kN}\right) < \left(\left[F\right] = 62\text{ kN}\right)$$

- The column is stable.

Eccentric Loading: The Secant Formula



- Eccentric loading is equivalent to a centric load and a moment.
- Bending occurs for any nonzero eccentricity. Question of buckling becomes whether the resulting deflection is excessive.
- The deflection become infinite when $P = P_{cr}$

$$\frac{d^2w}{dx^2} = \frac{-Pw - Pe}{EI} \quad \Rightarrow \frac{d^2w}{dx^2} + \frac{P}{EI}w = -\frac{Pe}{EI}$$

$$w = A\sin kx + B\cos kx - e, \quad k^2 = P/EI$$

$$0 = w(0) = w(L)$$

$$B = e; A\sin kL = e(1 - \cos kL) \quad \Rightarrow A = e\tan(kL/2)$$

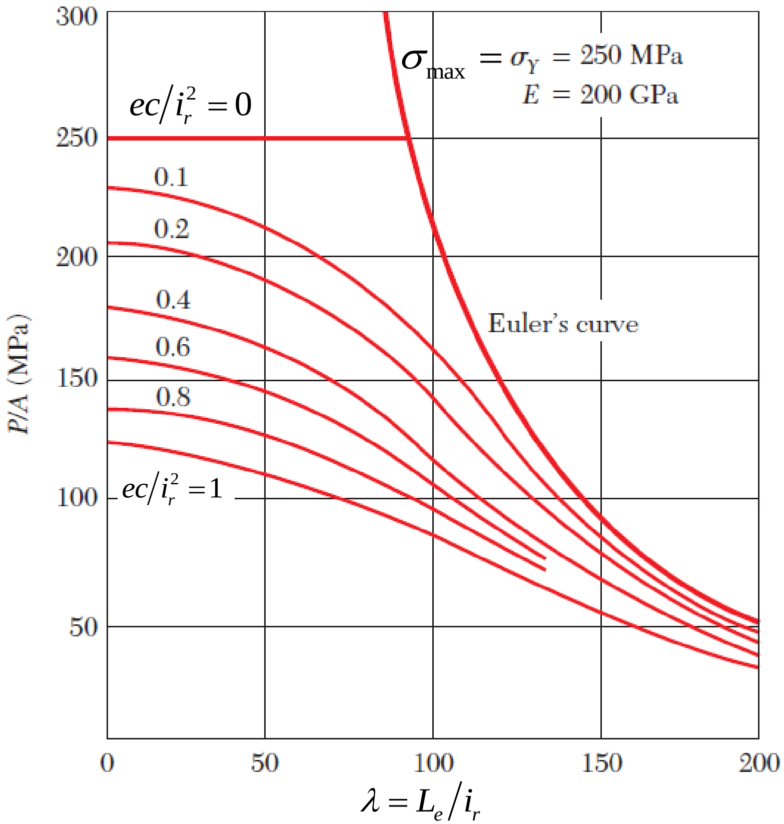
$$w = e(\tan(kL/2)\sin kx + \cos kx - 1)$$

$$w_{\max} = w(L/2) = e(\tan(kL/2)\sin(kL/2) + \cos(kL/2) - 1)$$

$$= e(\sec(kL/2) - 1) = e\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right]$$

$$= e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]. \quad \left(\frac{L}{2}\sqrt{\frac{P_{cr}}{EI}} = \frac{\pi}{2}, \quad P_{cr} = \frac{\pi^2 EI}{L_e^2}\right)$$

Eccentric Loading: The Secant Formula



- Maximum stress

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} = \frac{P}{A} + \frac{P(w_{max}+e)c}{I}$$

$$= \frac{P}{A}\left[1 + \frac{ec}{i_r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$= \frac{P}{A}\left[1 + \frac{ec}{i_r^2}\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)\right]$$

$$= \frac{P}{A}\left[1 + \frac{ec}{i_r^2}\sec\left(\frac{1}{2}\frac{L}{i_r}\sqrt{\frac{P}{EA}}\right)\right]$$

- Secant Formula

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{i_r^2}\sec\left(\frac{1}{2}\frac{L_e}{i_r}\sqrt{\frac{P}{EA}}\right)}$$

- For small slenderness ratio: $\frac{P}{A} \approx \sigma_{max} \Big/ \left(1 + \frac{ec}{i_r^2}\right)$.
- For large slenderness ratio, the curves get very close to Euler's curve, and thus that the effect of the eccentricity of the loading becomes negligible.
- The secant formula is chiefly useful for intermediate values of slenderness ratio.

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