Torsion of Prismatic Bars

Outline

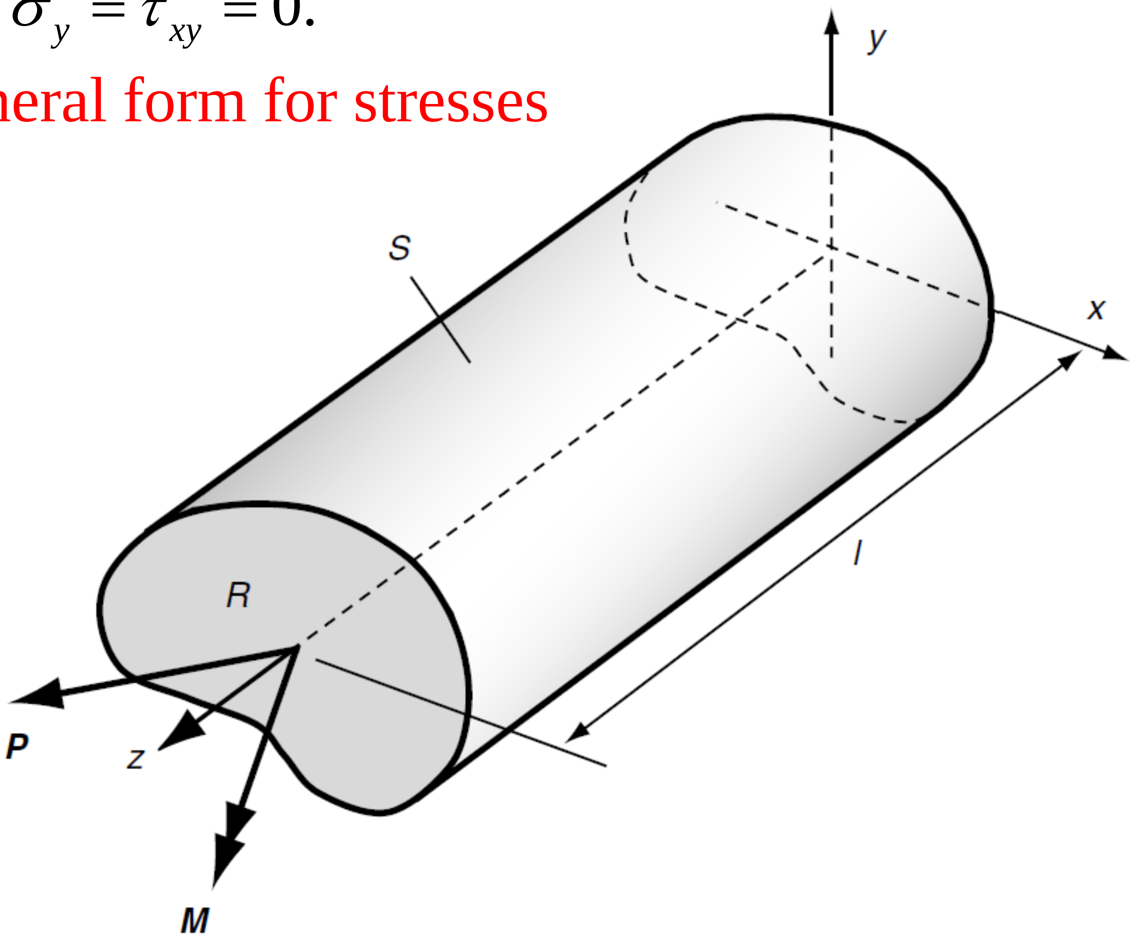
- Elastic Cylinders with End Loading
- Torsion of Cylinders: General formulation
- Stress-Function Formulation
- Displacement Formulation
- Membrane Analogy
- Solution: Boundary Equation Scheme
- Solution: Fourier Method – Rectangular Section
- Multiply Connected Cross-Sections
- Hollow Sections

Elastic Cylinders Subjected to End Loadings

- Semi-Inverse Method
- Zero lateral forces: $\sigma_x = \sigma_y = \tau_{xy} = 0.$
- Let us guess the most general form for stresses

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + F_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + F_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + F_z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \tau_{xz}}{\partial z} = 0 & \Rightarrow \tau_{xz} = \tau_{xz}(x, y) \\ \dfrac{\partial \tau_{yz}}{\partial z} = 0 & \Rightarrow \tau_{yz} = \tau_{yz}(x, y) \\ \dfrac{\partial^2 \sigma_z}{\partial z^2} = 0 \end{cases}$$



Elastic Cylinders Subjected to End Loadings

- Beltrami-Michell Equations

$$\begin{cases}\cancel{\nabla^2\sigma_x} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial x^2}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\dfrac{\nu}{1-\nu}\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}+\dfrac{\partial F_z}{\partial z}\right)-2\dfrac{\partial F_x}{\partial x}} & \Rightarrow \dfrac{\partial^2\sigma_z}{\partial x^2}=0 \\ \cancel{\nabla^2\sigma_y} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial y^2}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\dfrac{\nu}{1-\nu}\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}+\dfrac{\partial F_z}{\partial z}\right)-2\dfrac{\partial F_y}{\partial y}} & \Rightarrow \dfrac{\partial^2\sigma_z}{\partial y^2}=0 \\ \cancel{\nabla^2\tau_{xy}} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial x\partial y}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\left(\dfrac{\partial F_x}{\partial y}+\dfrac{\partial F_y}{\partial x}\right)}, & \Rightarrow \dfrac{\partial^2\sigma_z}{\partial x\partial y}=0 \\ \nabla^2\sigma_z + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial z^2}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\dfrac{\nu}{1-\nu}\left(\dfrac{\partial F_x}{\partial x}+\dfrac{\partial F_y}{\partial y}+\dfrac{\partial F_z}{\partial z}\right)-2\dfrac{\partial F_z}{\partial z}} & \text{satisfied}\end{cases}$$

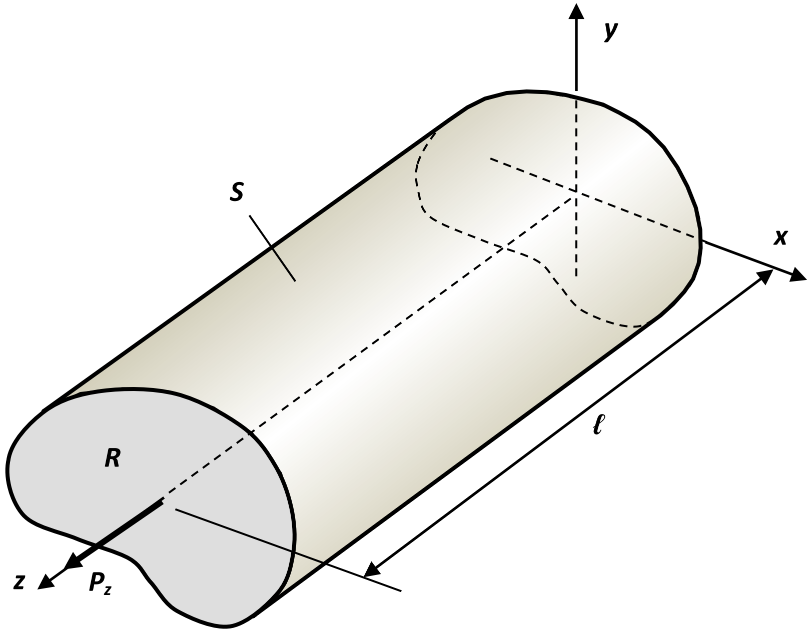
$$\frac{\partial^2\sigma_z}{\partial x^2}=\frac{\partial^2\sigma_z}{\partial y^2}=\frac{\partial^2\sigma_z}{\partial z^2}=\frac{\partial^2\sigma_z}{\partial x\partial y}=0 \Rightarrow \boxed{\sigma_z = C_1x+C_2y+C_3z+C_4xz+C_5yz+C_6}$$

$$\begin{cases}\nabla^2\tau_{xz} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial x\partial z}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\left(\dfrac{\partial F_x}{\partial z}+\dfrac{\partial F_z}{\partial x}\right)} & \Rightarrow \boxed{\dfrac{\partial^2\tau_{xz}}{\partial x^2}+\dfrac{\partial^2\tau_{xz}}{\partial y^2}=-\dfrac{C_4}{1+\nu}} \\ \nabla^2\tau_{yz} + \dfrac{1}{1+\nu}\dfrac{\partial^2}{\partial y\partial z}\left(\cancel{\sigma_x+\sigma_y}+\sigma_z\right) = \cancel{-\left(\dfrac{\partial F_y}{\partial z}+\dfrac{\partial F_z}{\partial y}\right)} & \Rightarrow \boxed{\dfrac{\partial^2\tau_{yz}}{\partial x^2}+\dfrac{\partial^2\tau_{yz}}{\partial y^2}=-\dfrac{C_5}{1+\nu}}\end{cases}$$

Extension of Cylinders

Assumptions

- Load P_z is applied at centroid of cross-section so no bending effects
- Using Saint-Venant Principle, exact end tractions are replaced by statically equivalent uniform loading
- Thus assume stress σ_z is uniform over any cross-section throughout the solid



$$\Rightarrow \ \sigma_z = \frac{P_z}{A}\ , \ \tau_{xz} = \tau_{yz} = 0$$

- Using stress results into Hooke's law and combining with the strain-displacement relations gives

$$\frac{\partial u}{\partial x} = -\frac{\nu P_z}{AE}\ , \ \frac{\partial v}{\partial y} = -\frac{\nu P_z}{AE}\ , \ \frac{\partial w}{\partial z} = \frac{P_z}{AE}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

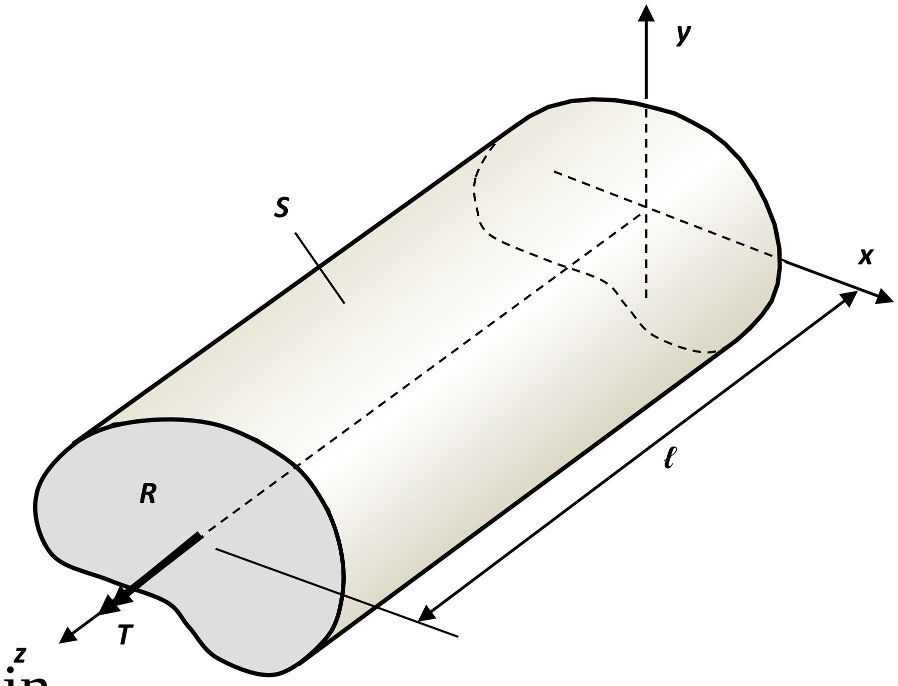
Integrating and dropping rigid-body motion terms such that displacements vanish at origin

$$\Rightarrow \quad u = -\frac{\nu P_z}{AE}x \qquad v = -\frac{\nu P_z}{AE}y \qquad w = \frac{P_z}{AE}z$$

Torsion of Cylinders

Guided by Observations from Strength of Materials:

- Projection of each section on *x*-*y* plane rotates as rigid-body about central axis
- Amount of projected section rotation is linear function of axial coordinate
- Plane cross-sections will not remain plane after deformation thus leading to a warping displacement



Torsional Deformation

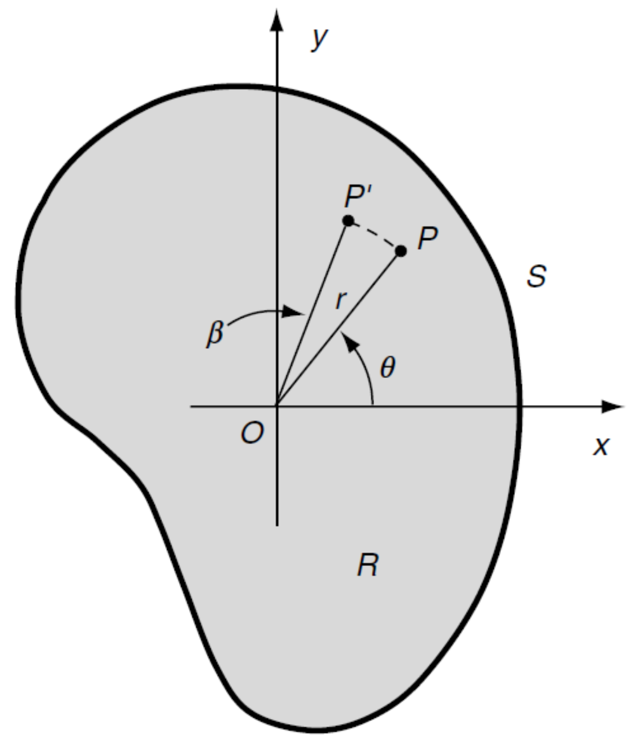
- In-plane / projected displacements

$$u = -r\beta \sin\theta = -\beta y, \quad v = r\beta \cos\theta = \beta x$$

- Angle of twist: $\beta = \alpha z.$
- The warping displacement is assumed to be a function of only the in-plane coordinates

$$\Rightarrow u = -\alpha yz, \quad v = \alpha xz, \quad w = w(x, y).$$

- Now must show assumed displacement form will satisfy all elasticity field equations



Stress Function Formulation

- Strain and stress field

$$u = -\alpha yz \quad v = \alpha xz \quad w = w(x, y)$$

$$\Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{xy} = 0, \quad \varepsilon_{xz} = \frac{1}{2}\left(-\alpha y + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\alpha x + \frac{\partial w}{\partial y}\right)$$

$$\Rightarrow \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \quad \tau_{xz} = G\left(-\alpha y + \frac{\partial w}{\partial x}\right), \quad \tau_{yz} = G\left(\alpha x + \frac{\partial w}{\partial y}\right)$$

- Equilibrium equations result in

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0$$

$$\Rightarrow \tau_{xz} = \frac{\partial \psi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \psi}{\partial x}$$

- $\psi = \psi(x,y)$
- **Prandtl Stress Function**

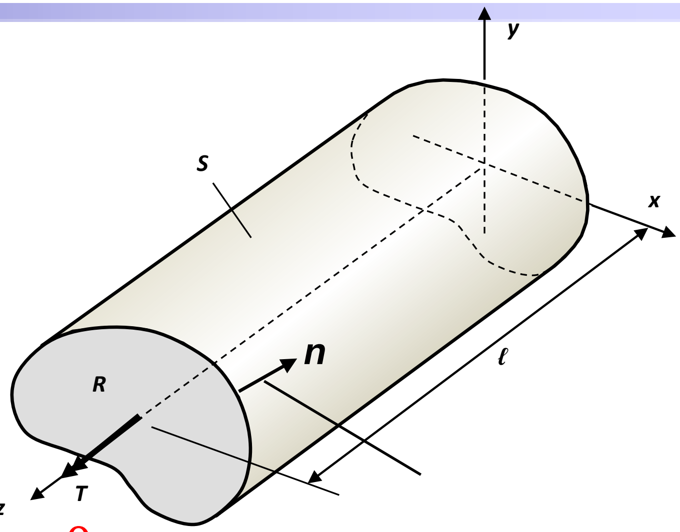
- Two stress components satisfy

$$\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -2G\alpha$$

$$\Rightarrow \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2G\alpha$$

- Stress compatibility is automatically satisfied.

Stress Function Formulation: BCs



- On lateral surface

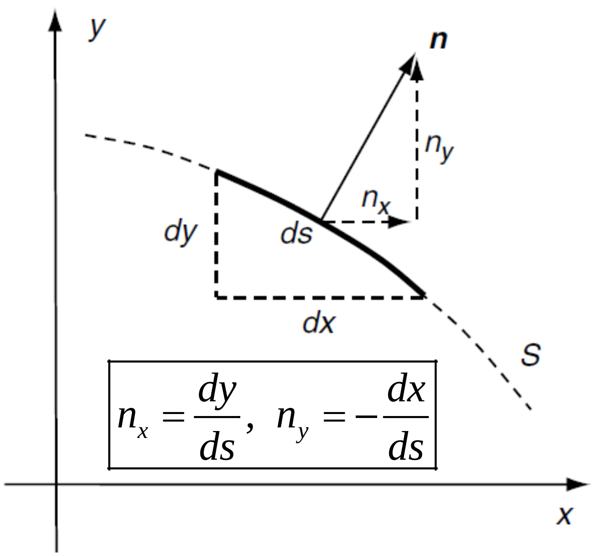
$$T_x^n = \cancel{\sigma_x} n_x + \cancel{\tau_{yx}} n_y + \tau_{zx} \cancel{n_z} = 0 \Rightarrow 0 = 0$$

$$T_y^n = \cancel{\tau_{xy}} n_x + \cancel{\sigma_y} n_y + \tau_{zy} \cancel{n_z} = 0 \Rightarrow 0 = 0$$

$$T_z^n = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z \cancel{n_z} = 0 \Rightarrow$$

$$\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \left(-\frac{\partial \psi}{\partial x}\right)\left(-\frac{dx}{ds}\right) = 0 \Rightarrow \frac{d\psi}{ds} = 0, \quad \text{i.e. set: } \psi = 0.$$

- On ends: $T_x^n = \pm\tau_{xz},\ T_y^n = \pm\tau_{yz},\ T_z^n = \sigma_z = 0$
- More interested in satisfying the resultant end-loadings



$$\boxed{1}: 0 = P_x = \iint_A \tau_{xz}\, dxdy, \quad \boxed{2}: 0 = P_y = \iint_A \tau_{yz}\, dxdy$$

$$\boxed{3}: 0 = P_z = \iint_A \cancel{\sigma_z}\, dxdy, \quad \boxed{4}: 0 = M_x = \iint_A y \cancel{\sigma_z}\, dxdy$$

$$\boxed{5}: 0 = M_y = \iint_A x \cancel{\sigma_z}\, dxdy, \quad \boxed{6}: T = M_z = \iint_A \left(x\tau_{yz} - y\tau_{xz}\right) dxdy$$

Stress Function Formulation: BCs

- On ends (with a big help from the Green's theorem)

$\boxed{3}-\boxed{5}$ are automatically satisfied.

$$\boxed{1}: \iint_A \tau_{xz}\,dxdy = \iint_A \frac{\partial\psi}{\partial y}\,dxdy = -\int_S \psi\,dx = \int_S \psi\,n_y\,ds = 0 \qquad \text{satisfied}$$

$$\boxed{2}: \iint_A \tau_{yz}\,dxdy = -\iint_A \frac{\partial\psi}{\partial x}\,dxdy = -\int_S \psi\,dy = -\int_S \psi\,n_x\,ds = 0 \qquad \text{satisfied}$$

$$\boxed{6}: T = \iint_A \left(x\tau_{yz} - y\tau_{xz}\right)dxdy = -\iint_A \left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right)dxdy = -\iint_A \left(\frac{\partial\left(x\psi\right)}{\partial x} + \frac{\partial\left(y\psi\right)}{\partial y} - 2\psi\right)dxdy$$

$$= \cancel{-\int_S\left(-y\psi\,dx + x\psi\,dy\right)} + 2\iint_A \psi\,dxdy \quad \Rightarrow \quad \boxed{T = 2\iint_A \psi\,dxdy}$$

- The assumed stress function yields a governing Poisson equation.
- The stress function vanishes on the lateral boundary.
- The overall torque is related to the integral of the stress Function.
- The remaining end conditions are automatically satisfied.

Displacement Formulation

- Expressing the equilibrium equation with warping displacement

$$G\cancel{\nabla^2 u}+(\lambda+G)\frac{\partial}{\partial x}\left(\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\cancel{\frac{\partial w}{\partial z}}\right)+\cancel{F_x}=0 \qquad \text{satisfied}$$

$$G\cancel{\nabla^2 v}+(\lambda+G)\frac{\partial}{\partial y}\left(\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\cancel{\frac{\partial w}{\partial z}}\right)+\cancel{F_y}=0 \qquad \text{satisfied}$$

$$G\nabla^2 w+(\lambda+G)\frac{\partial}{\partial z}\left(\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\cancel{\frac{\partial w}{\partial z}}\right)+\cancel{F_z}=0 \;\Rightarrow\; \boxed{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0}$$

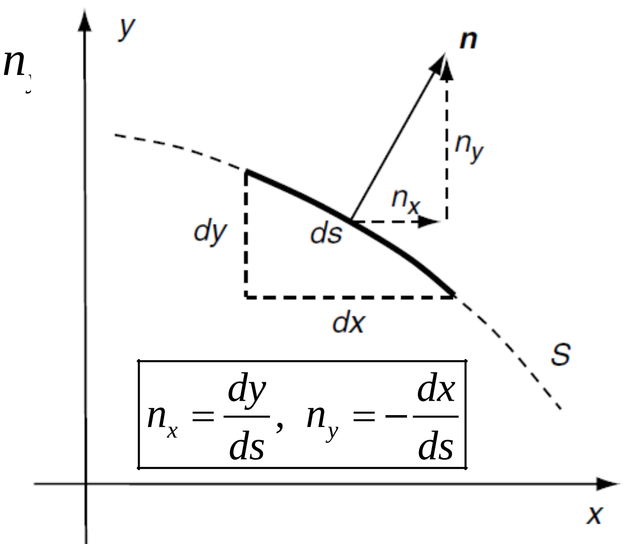
- BCs on the lateral surface:

$$0=T_z^n=\tau_{xz}n_x+\tau_{yz}n_y+\sigma_z\cancel{n_z}=G\left(-\alpha y+\frac{\partial w}{\partial x}\right)n_x+G\left(\alpha x+\frac{\partial w}{\partial y}\right)n_y$$

For arbitrary n: $\boxed{\dfrac{\partial w}{\partial x}=\alpha y, \qquad \dfrac{\partial w}{\partial y}=-\alpha x}$

or equivalently: $\dfrac{\partial w}{\partial x}n_x+\dfrac{\partial w}{\partial y}n_y=\alpha y n_x-\alpha x n_y=\alpha y\dfrac{dy}{ds}+\alpha x\dfrac{dx}{ds}$

$$\Rightarrow \boxed{\frac{dw}{dn}=\frac{\alpha}{2}\frac{d}{ds}\left(x^2+y^2\right)}$$



$$n_x=\frac{dy}{ds}, \quad n_y=-\frac{dx}{ds}$$

Displacement Formulation

- BCs on the ends

$$\boxed{1}: 0 = P_x = \iint_A \tau_{xz} dxdy, \ \boxed{2}: 0 = P_y = \iint_A \tau_{yz} dxdy$$

$$\boxed{3}: 0 = P_z = \iint_A \cancel{\sigma_z} dxdy, \ \boxed{4}: 0 = M_x = \iint_A y \cancel{\sigma_z} dxdy$$

$$\boxed{5}: 0 = M_y = \iint_A x \cancel{\sigma_z} dxdy, \ \boxed{6}: T = M_z = \iint_A \left(x\tau_{yz} - y\tau_{xz} \right) dxdy$$

$\boxed{1} - \boxed{5}$ are automatically satisfied.

$$\boxed{6}: T = M_z = \iint_A \left(x\tau_{yz} - y\tau_{xz} \right) dxdy = \iint_A \left[xG\left(\alpha x + \frac{\partial w}{\partial y} \right) - yG\left(-\alpha y + \frac{\partial w}{\partial x} \right) \right] dxdy$$

$$\Rightarrow T = G\iint_A \left(\alpha(x^2 + y^2) + x\frac{\partial w}{\partial y} - y\frac{\partial w}{\partial x} \right) dxdy = \alpha J$$

$$J = G\iint_A \left(x^2 + y^2 + \frac{x}{\alpha}\frac{\partial w}{\partial y} - \frac{y}{\alpha}\frac{\partial w}{\partial x} \right) dxdy \ldots \text{ Torsional Rigidity}$$

Formulation Comparison

- **Stress Function Formulation**

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2G\alpha \quad \in \quad R$$

$$\psi = 0 \quad \in \quad \text{Lateral Surface}$$

$$T = 2\iint_A \psi \, dxdy \quad \in \quad \text{Ends}$$

- Relatively Simple Governing Equation
- Relatively Simple Boundary Conditions

- **Displacement Formulation**

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad \in \quad R$$

$$\frac{dw}{dn} = \frac{1}{2}\alpha \frac{d}{ds}\left(x^2 + y^2\right)$$

$$\in \quad \text{Lateral Surface}$$

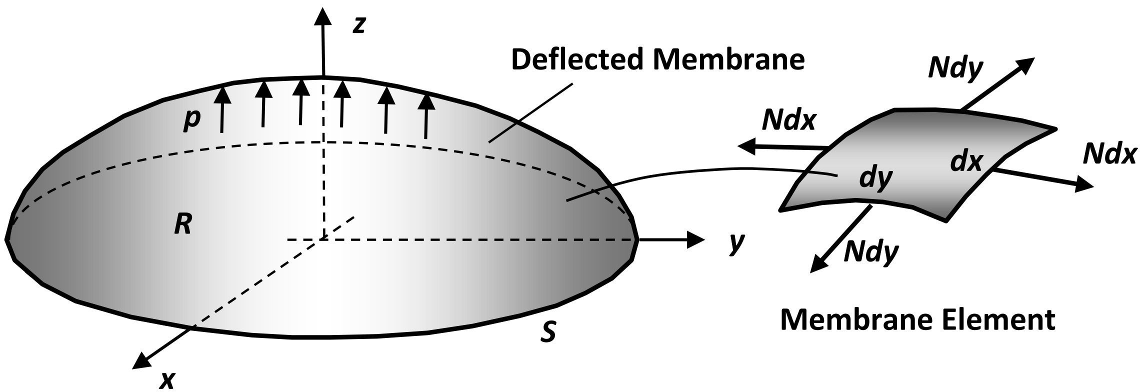
$$T = G\iint_A \left(\alpha(x^2 + y^2) + x\frac{\partial w}{\partial y} - y\frac{\partial w}{\partial x} \right) dxdy$$

$$\in \quad \text{Ends}$$

- Simple Governing Equation
- Complicated Boundary Condition

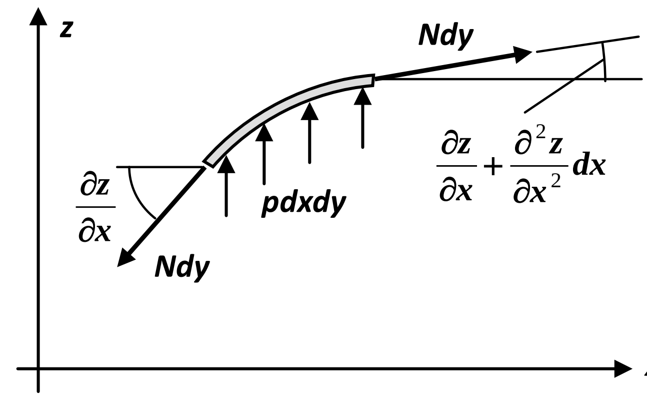
Membrane Analogy

- Consider a thin elastic membrane stretched with uniform tension over a closed frame and subjected to a uniform pressure.



Membrane Element

Static Deflection of a Stretched Membrane



Equilibrium of Membrane Element

- The equilibrium of a membrane element requires

$$0=\sum F_z = Ndy\left(\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial x^2}dx\right)-Ndy\left(\frac{\partial z}{\partial x}\right)+Ndx\left(\frac{\partial z}{\partial y}+\frac{\partial^2 z}{\partial y^2}dy\right)-Ndx\left(\frac{\partial z}{\partial y}\right)+pdxdy$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=-\frac{p}{N}}$$

Membrane Analogy

- The membrane is stretched over the boundary of the frame

$$z = 0 \text{ on } S$$

- The volume enclosed by the deflected membrane and the *x*-*y* plane

$$V = \iint_A z\,dx\,dy$$

- **Membrane Equations**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{N}$$

$$z = 0 \text{ on } S$$

$$V = \iint_A z\,dx\,dy$$

- **Torsion Equations**

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2G\alpha$$

$$\psi = 0 \text{ on } S$$

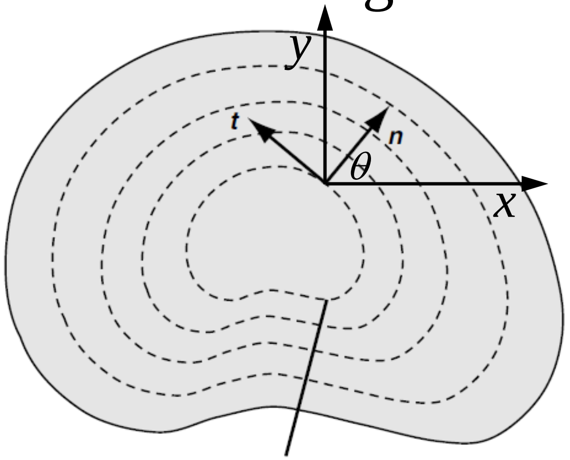
$$T = 2\iint_A \psi\,dx\,dy$$

- Equations are same with: $\psi = z$, $p/N = 2G\alpha$, $T = 2V$

Membrane Analogy

- Along any contour line, the resultant shear stress must be tangent

$$n_x = \frac{dy}{ds}, \quad n_y = -\frac{dx}{ds} \qquad \tau_{xz} = \frac{\partial \psi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \psi}{\partial x}$$



$$0 = \frac{dz}{ds} = \frac{d\psi}{ds} = \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} = \left(-\tau_{yz}\right)\left(-n_y\right) + \tau_{xz} n_x = \tau_{zn} = 0$$

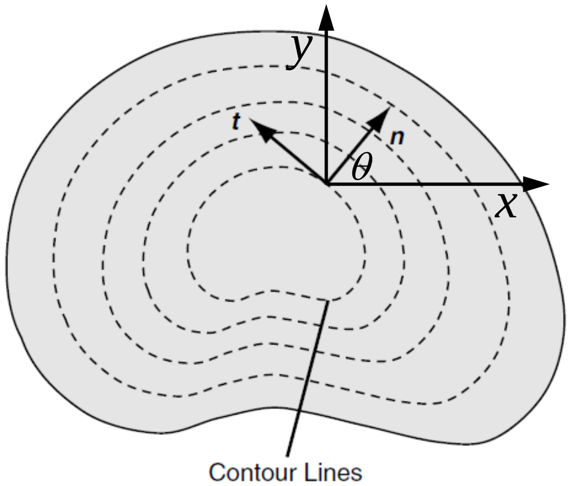
$$\Rightarrow \tau = \tau_{zt} = -\tau_{xz} n_y + \tau_{yz} n_x = -\frac{\partial \psi}{\partial y} n_y + \left(-\frac{\partial \psi}{\partial x}\right) n_x = -\frac{d\psi}{dn} = -\frac{dz}{dn}$$

$$\begin{bmatrix} \sigma_n & \tau_{nt} & \tau_{zn} \\ \tau_{nt} & \sigma_t & \tau_{zt} \\ \tau_{zn} & \tau_{zt} & \sigma_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cancel{\tau_{xz}\cos\theta + \tau_{yz}\sin\theta} \\ 0 & 0 & -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta \\ \cancel{\tau_{xz}\cos\theta + \tau_{yz}\sin\theta} & -\tau_{xz}\sin\theta + \tau_{yz}\cos\theta & 0 \end{bmatrix}$$

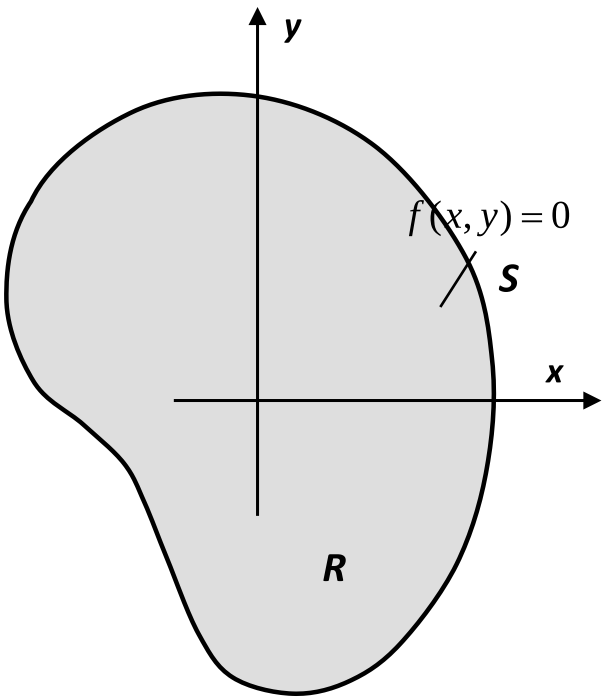
Membrane Analogy

- The shear stress at any point in the cross-section is given by the negative of the slope of the membrane in the direction normal to the contour line through the point.
- The maximum shear stress appears always to occur on the boundary where the largest slope of the membrane occurs.
- Using membrane visualizations, a useful qualitative picture of the stress function distribution can be determined and approximate solutions can be constructed.
- The torque is given as twice the volume under the membrane.



Solutions Derived from Boundary Equation

- If boundary is expressed by relation $f(x,y) = 0$, this suggests possible simple solution scheme of expressing stress function as $\psi = Kf(x,y)$ where K is an arbitrary constant.
- This form satisfies boundary condition on S, and for some simple geometric shapes it will also satisfy the governing equation with appropriate choice of K.
- Unfortunately this is not a general solution method and works only for special cross-sections of simple geometry.



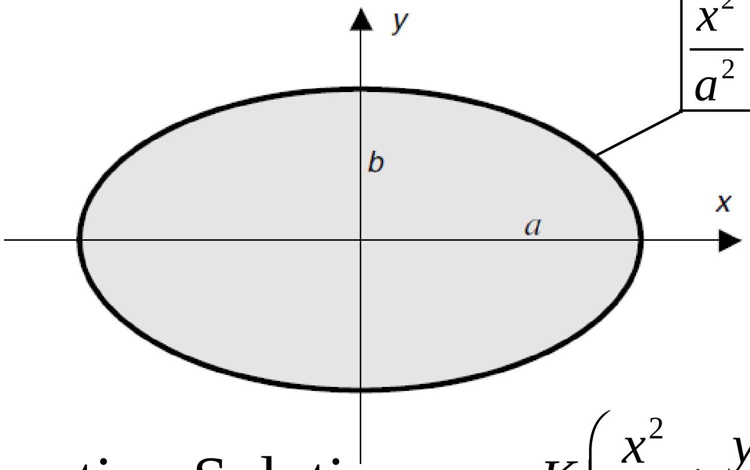
Boundary-Value Problem

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2G\alpha \quad \in \quad R$$

$$\psi = 0 \quad \in \quad S$$

$$T = 2\iint_A \psi\, dxdy \quad \in \quad \text{Ends}$$

Elliptical Section



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

- Look for Stress Function Solution: $\psi = K\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$
- ψ satisfies the BC and will satisfy the governing equation if

$$K=-\frac{a^2b^2G\alpha}{a^2+b^2} \quad \Rightarrow \quad \psi=-\frac{a^2b^2G\alpha}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

- **Since the governing equation and the BC are satisfied, we have found the solution.**

Elliptical Section

- Relation to the carrying torque

$$T = 2\iint_A \psi\, dxdy = -\frac{2a^2b^2G\alpha}{a^2+b^2}\left(\frac{1}{a^2}\iint_A x^2 dxdy + \frac{1}{b^2}\iint_A y^2 dxdy - \iint_A dxdy\right)$$

$$= -\frac{2a^2b^2G\alpha}{a^2+b^2}\left(\frac{1}{a^2}\frac{\pi a^3 b}{4} + \frac{1}{b^2}\frac{\pi a b^3}{4} - \pi ab\right) = \frac{\pi a^3b^3G\alpha}{a^2+b^2}$$

- Angle of twist per unit length

$$\alpha = \frac{a^2+b^2}{\pi a^3b^3G}T \qquad \Rightarrow \qquad \psi = -\frac{T}{\pi ab}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

- Stress field (two-fold symmetry)

$$\tau_{xz} = \frac{\partial\psi}{\partial y} = -\frac{2Ty}{\pi ab^3}, \qquad \tau_{yz} = -\frac{\partial\psi}{\partial x} = \frac{2Tx}{\pi a^3 b} \quad \Rightarrow \quad \tau = \sqrt{\tau_{xz}^2+\tau_{yz}^2} = \frac{2T}{\pi ab}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}}$$

- The maximum shear stress (Strength of Materials vs. Membrane analogy)

$$\tau_{max} = \tau(0,\pm b) = \frac{2T}{\pi ab^2}$$

Elliptical Section

- Warping displacement (two-fold symmetry)

$$\tau_{xz} = G\left(-\alpha y + \frac{\partial w}{\partial x}\right)$$

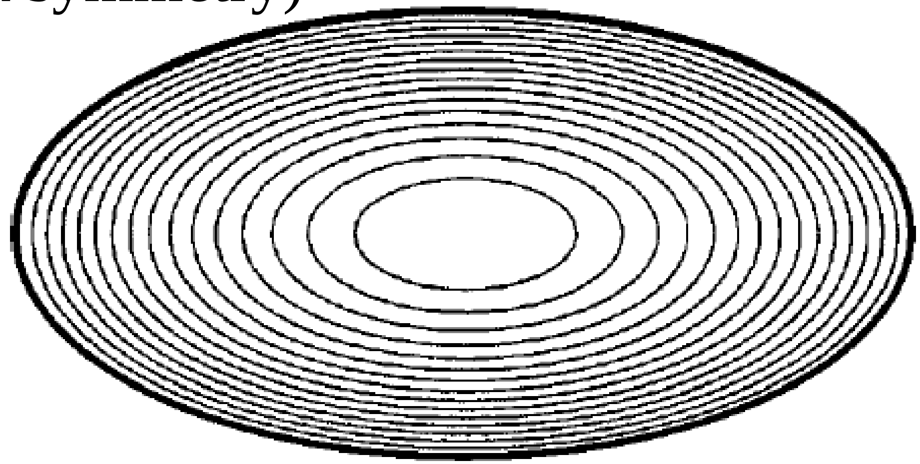
$$\Rightarrow$$

$$w = \int\left(\frac{\tau_{xz}}{G} + \alpha y\right)dx$$

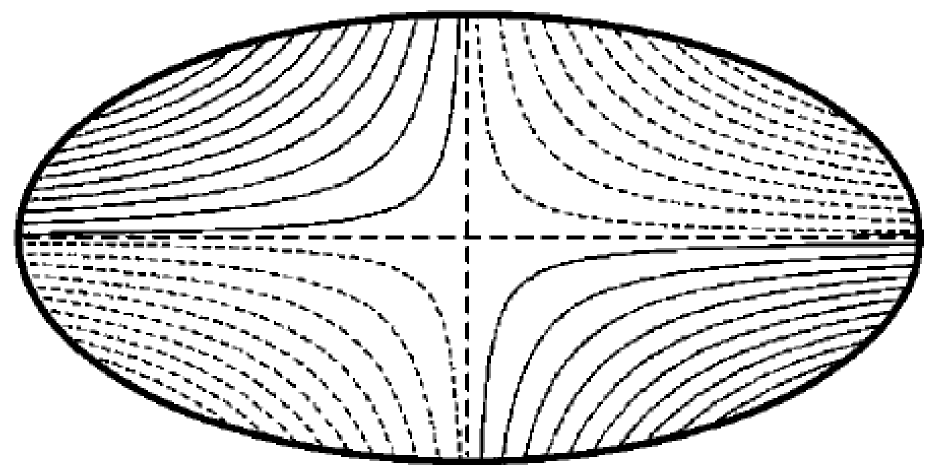
$$= \int\left(-\frac{2Ty}{\pi ab^3}\frac{1}{G} + \alpha y\right)dx$$

$$= \left(-\frac{2T}{\pi ab^3 G} + \alpha\right)xy$$

$$= \frac{T\left(b^2 - a^2\right)}{\pi a^3 b^3 G}xy$$



(Stress Function Contours)



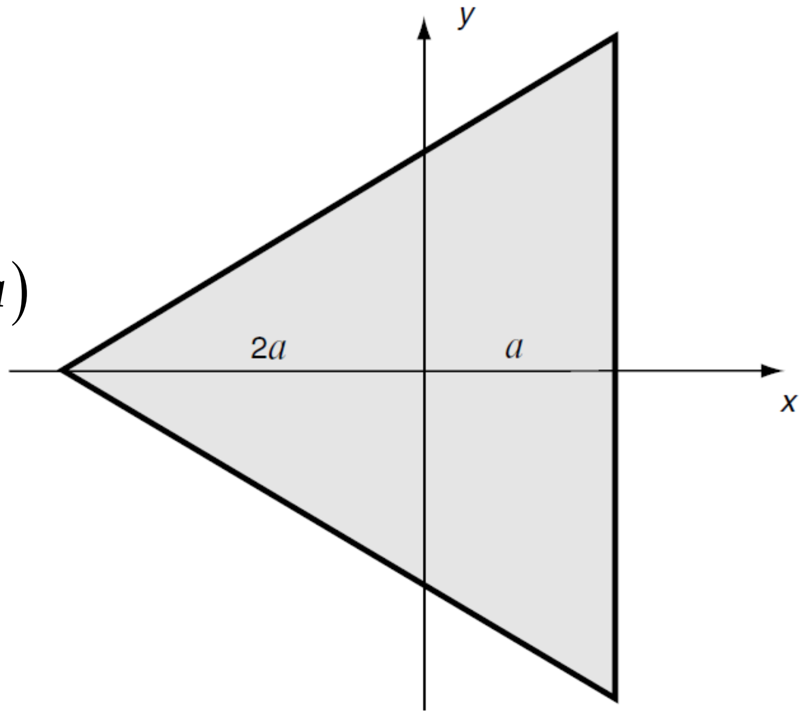
(Displacement Contours)

- If $a = b$, the results degenerate to torsion of circular shafts.

Equilateral Triangular Section

- For stress function try product form of each boundary line equation

$$\psi = K\left(x-\sqrt{3}y+2a\right)\left(x+\sqrt{3}y+2a\right)\left(x-a\right)$$



- ψ satisfies boundary condition and will satisfy governing equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-2G\alpha, \quad \text{if} \quad K=-\frac{G\alpha}{6a}.$$

- Since governing equation and boundary condition are satisfied, we have found the solution.

Equilateral Triangular Section

- Relation to the loading torque

$$T = 2\iint_A \psi\, dxdy = \frac{27}{5\sqrt{3}} G\alpha a^4 = \frac{3}{5} G\alpha I_p$$

$$\alpha = \frac{5\sqrt{3}T}{27Ga^4} = \frac{5T}{3GI_p}, \qquad I_p = \iint_A \left(x^2 + y^2\right) dxdy = 3\sqrt{3}a^4$$

- Stress field (three-fold symmetry)

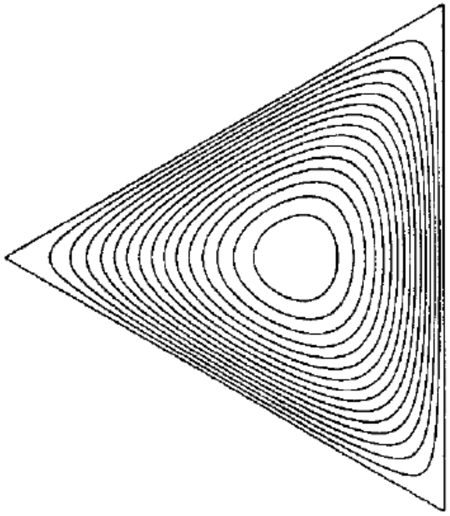
$$\tau_{xz} = \frac{\partial \psi}{\partial y} = \frac{G\alpha}{a}(x-a)y,$$

$$\tau_{yz} = -\frac{\partial \psi}{\partial x} = \frac{G\alpha}{2a}(x^2 + 2ax - y^2)$$

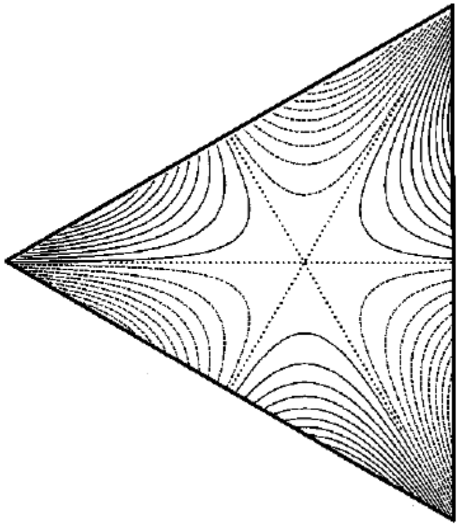
$$\Rightarrow \tau_{\max} = \tau_{yz}(a,0) = \frac{3}{2} G\alpha a = \frac{5\sqrt{3}T}{18a^3}$$

- Warping displacement (three-fold symmetry)

$$w = \int \left(\frac{\tau_{xz}}{G} + \alpha y \right) dx \ \Rightarrow \ w = \frac{\alpha}{6a} y\left(3x^2 - y^2\right)$$

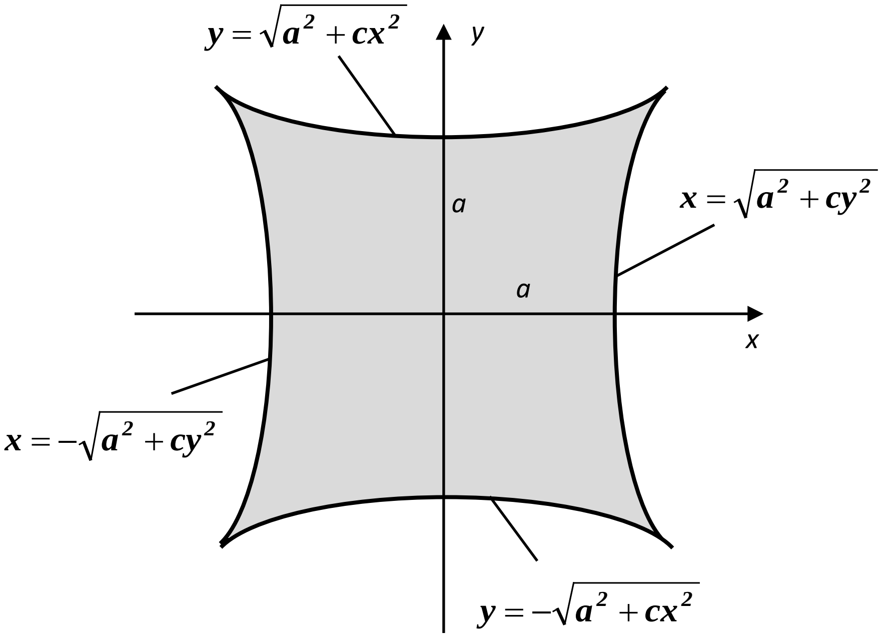


(Stress Function Contours)



(Displacement Contours)

Additional Examples Using Boundary Equation Scheme

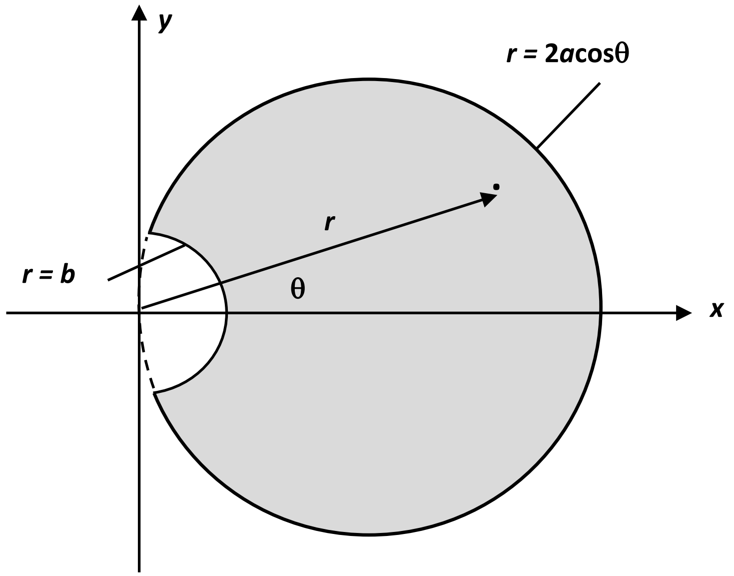


Section with Higher Order Polynomial Boundary

$$\psi = K(a^2 - x^2 + cy^2)(a^2 + cx^2 - y^2)$$

$$K = -\frac{G\alpha}{4a^2(1-\sqrt{2})}\ ,\ c = 3 - \sqrt{8}$$

$$\tau_{\max} = \tau(\pm a, 0) = \tau(0, \pm a) = \sqrt{2}G\alpha a$$



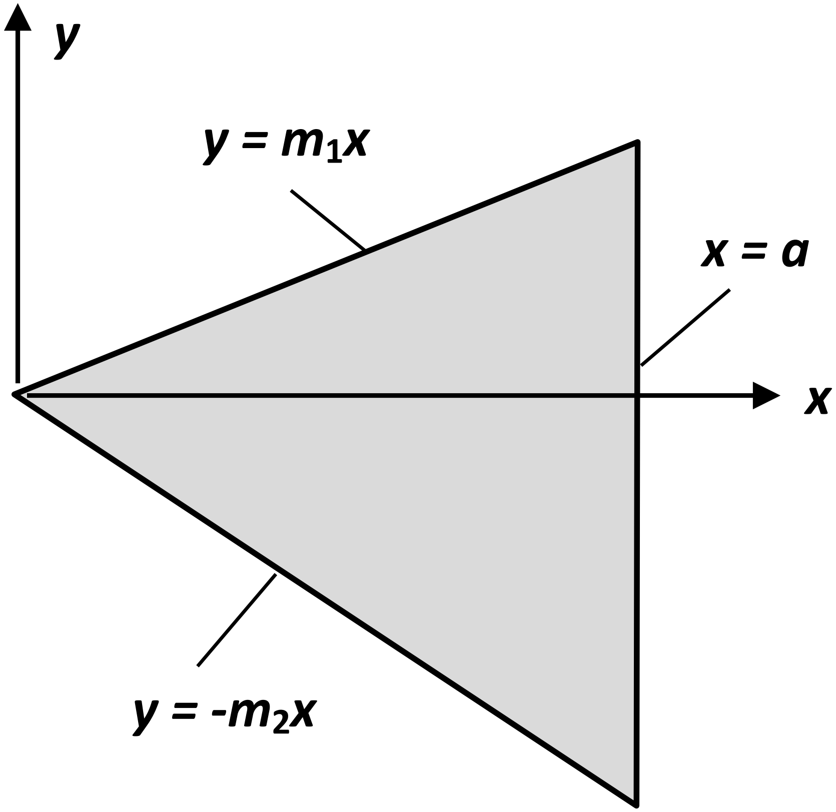
Circular Shaft with Circular Keyway

$$\psi = \frac{G\alpha}{2}\left(b^2 - r^2\right)\left(1 - \frac{2a\cos\theta}{r}\right)$$

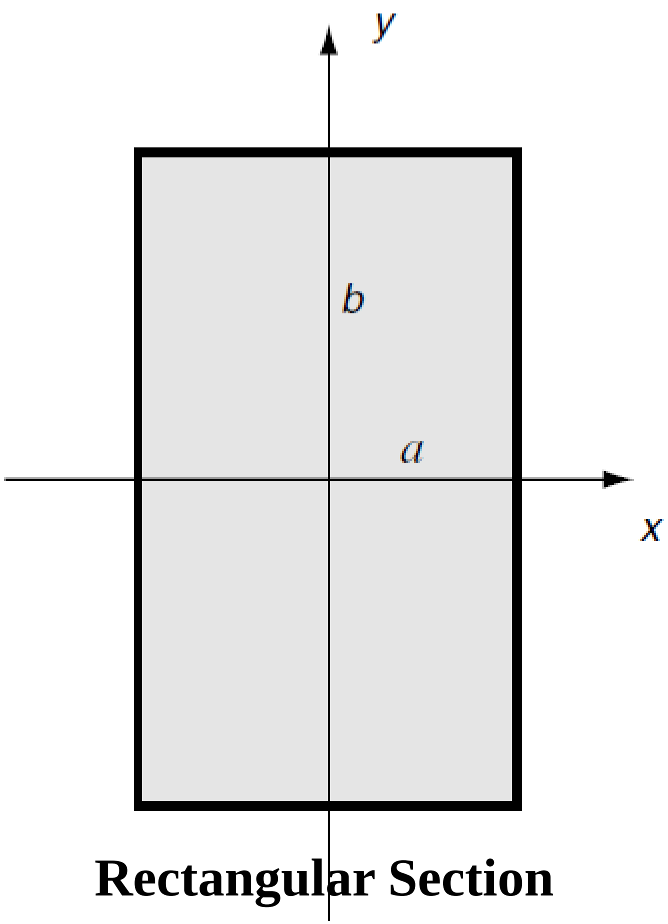
$$\text{As } b/a \to 0: \ \frac{(\tau_{\max})_{keyway}}{(\tau_{\max})_{shaft}} \to \frac{2G\alpha a}{G\alpha a} = 2$$

$\therefore$ Stress Concentration

When Boundary Equation Scheme Does Not Work?



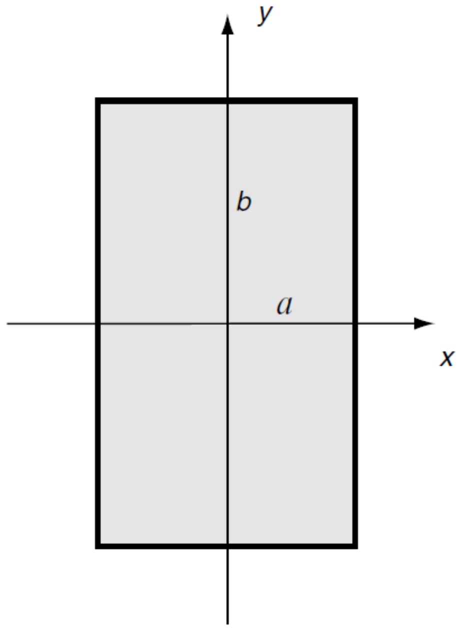
General Triangular Section



Rectangular Section

- **Trying the previous scheme of products of the boundary lines does not create a stress function that can satisfy the governing equation.**

Rectangular Section – Fourier Method Solution



- Previous boundary equation scheme will not creat a stress function that satisfies the governing equation. Thus we must use a more fundamental solution technique - Fourier method. Thus look for stress function solution of the standard form:

$$\psi = \psi_h + \psi_p \quad \text{with} \quad \psi_p(x,y) = G\alpha(a^2 - x^2)$$

- The homogeneous solution must then satisfy

$$\nabla^2 \psi_h = 0 \ , \ \ \psi_h(\pm a, y) = 0 \ , \ \ \psi_h(x, \pm b) = -G\alpha(a^2 - x^2)$$

- Separation of Variables Method

$$\psi_h(x,y) = X(x)Y(y): \ \ \nabla^2\psi_h = 0 \Rightarrow X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

$$\Rightarrow \begin{cases} X'' + \lambda^2 X = 0 \\ Y'' - \lambda^2 Y = 0 \end{cases} \Rightarrow \begin{cases} X = \cancel{A\sin\lambda x} + B\cos\lambda x \\ Y = \cancel{C\sinh\lambda y} + D\cosh\lambda y \end{cases}$$

$$\psi_h(\pm a, y) = 0 \Rightarrow \lambda a = \frac{1}{2}n\pi, \quad n = 1,3,5,\cdots$$

Rectangular Section – Fourier Method Solution

- Homogeneous solution

$$\psi_h(x,y) = BD\cos\lambda x \cosh\lambda y = \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a}, \quad n = 1, 3, 5, \cdots$$

$$\Rightarrow \psi_h(x,\pm b) = -G\alpha(a^2 - x^2) = \sum_{n=1}^{\infty}\left(B_n \cosh\frac{n\pi b}{2a}\right)\cos\frac{n\pi x}{2a}, \quad n = 1, 3, 5, \cdots$$

- By Fourier cosine series

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \quad a_n = \frac{2}{l}\int_0^l f(\xi)\cos\frac{n\pi\xi}{l}d\xi, \quad n = 0, 1, 2, \cdots}$$

$$B_n \cosh\frac{n\pi b}{2a} = \frac{2}{a}\int_0^a \left(-G\alpha(a^2 - \xi^2)\right)\cos\frac{n\pi\xi}{2a}d\xi \quad \Rightarrow \quad B_n = -\frac{32G\alpha a^2(-1)^{(n-1)/2}}{n^3\pi^3\cosh\frac{n\pi b}{2a}}$$

$$\Rightarrow \psi = \psi_h + \psi_p = G\alpha(a^2 - x^2) - \frac{32G\alpha a^2}{\pi^3}\sum_{n=1,3,5\cdots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^3\cosh\frac{n\pi b}{2a}}\cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a}$$

Rectangular Section – Fourier Method Solution

- Stress field

$$\tau_{xz} = \frac{\partial \psi}{\partial y} = -\frac{16G\alpha a}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2 \cosh \frac{n\pi b}{2a}} \cos \frac{n\pi x}{2a} \sinh \frac{n\pi y}{2a}$$

$$\tau_{yz} = -\frac{\partial \psi}{\partial x} = 2G\alpha x - \frac{16G\alpha a}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2 \cosh \frac{n\pi b}{2a}} \sin \frac{n\pi x}{2a} \cosh \frac{n\pi y}{2a}$$

$$\tau_{\max} = \tau_{yz}(a,0) = 2G\alpha a - \frac{16G\alpha a}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2 \cosh \frac{n\pi b}{2a}}$$

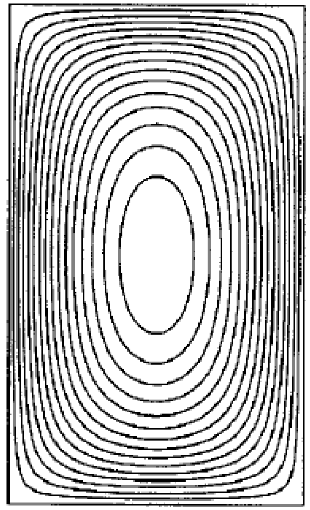
- Relation to the loading torque

$$T = 2\iint_A \psi \, dxdy = \frac{16G\alpha a^3 b}{3} - \frac{1024G\alpha a^4}{\pi^5} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi b}{2a} \qquad \Rightarrow \alpha = \cdots$$

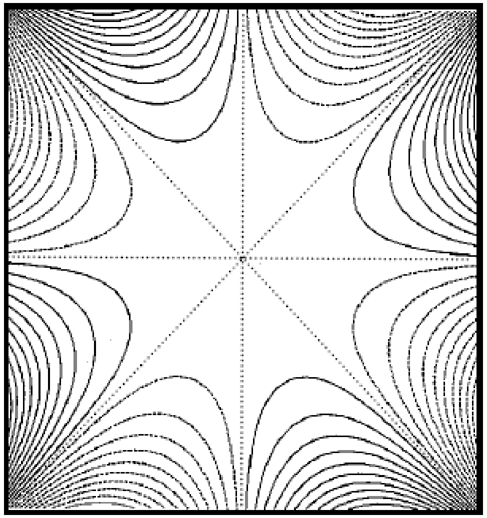
- Warping displacement

$$w = \int \left(\frac{\tau_{xz}}{G} + \alpha y \right) dx \quad \Rightarrow \quad w = \alpha xy - \frac{32\alpha a^2}{\pi^3} \sum_{n=1,3,5\cdots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^3 \cosh \frac{n\pi b}{2a}} \sin \frac{n\pi x}{2a} \sinh \frac{n\pi y}{2a}$$

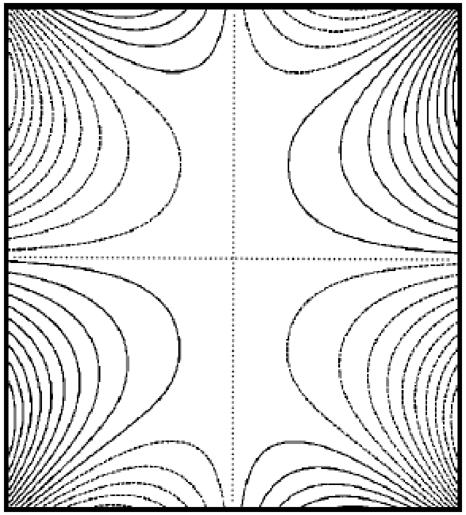
Rectangular Section – Fourier Method Solution



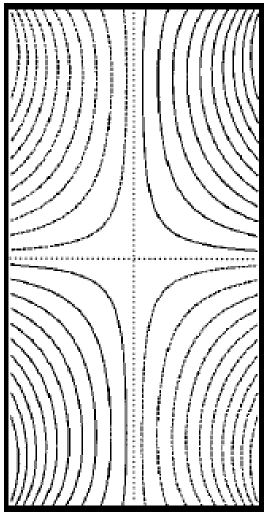
(Stress Function Contours)



(Displacement Contours, a/b = 1.0)

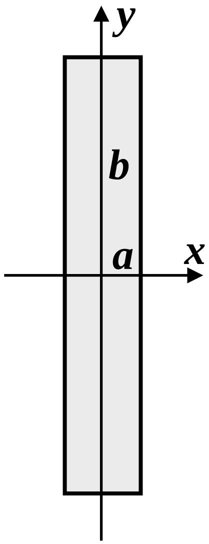


(Displacement Contours, a/b = 0.9)



(Displacement Contours, a/b = 0.5)

Thin Rectangles (a<<b): Open Thin-Walled Tubes



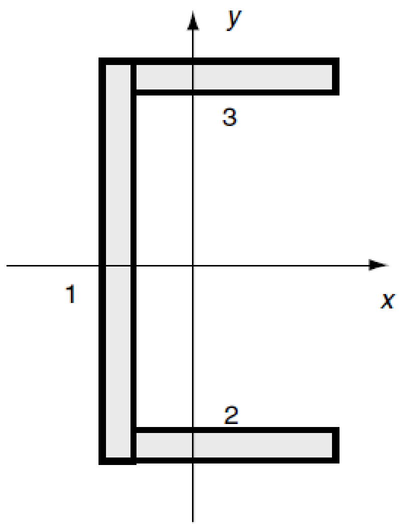
- Investigate results for special case of a *very thin rectangle* with $a << b$. Under conditions of $b/a >> 1$

$$\cosh\frac{n\pi b}{2a}\to\infty,\quad \tanh\frac{n\pi b}{2a}\to 1,\quad 2a\equiv t \qquad \Rightarrow T=\frac{16}{3}G\alpha a^3 b=\frac{1}{3}G\alpha A t^2,\quad \alpha=\frac{3T}{GAt^2}$$

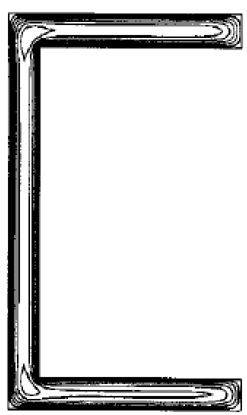
$$\psi=G\alpha\left(\frac{t^2}{4}-x^2\right)=\frac{3T}{At^2}\left(\frac{t^2}{4}-x^2\right),\qquad \Rightarrow \tau_{max}=\left(-\frac{d\psi}{dx}\right)_{x=t/2}=G\alpha t=\frac{3T}{At}$$

- **Torsion of sections composed of thin rectangles:** Neglecting local regions where rectangles are joined, we can use thin rectangular solution over each section. Stress function contours shown justify these assumptions. Thus load carrying torque for such composite section will be given by: $\alpha=3T\Big/G\sum_{i=1}^{N}A_i t_i^2$

- The maximum shear stress can be estimated for the narrowest rectangle.

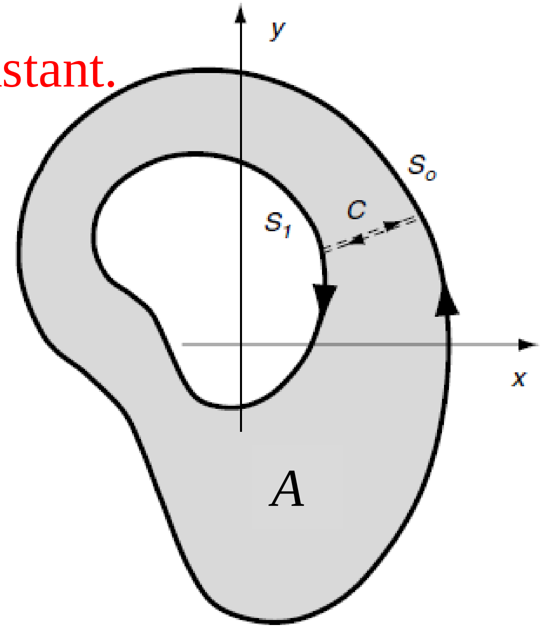


(Composite Section)



(Stress Function Contours)

Multiply Connected Cross-Sections



- On all lateral boundaries: $\frac{d\psi}{ds} = 0 \Rightarrow \psi_i = \text{Constant.}$
- Value of ψ_i may be arbitrarily chosen only on one boundary, commonly taken as zero on S_o .
- The single-valuedness of warping displacement requires:

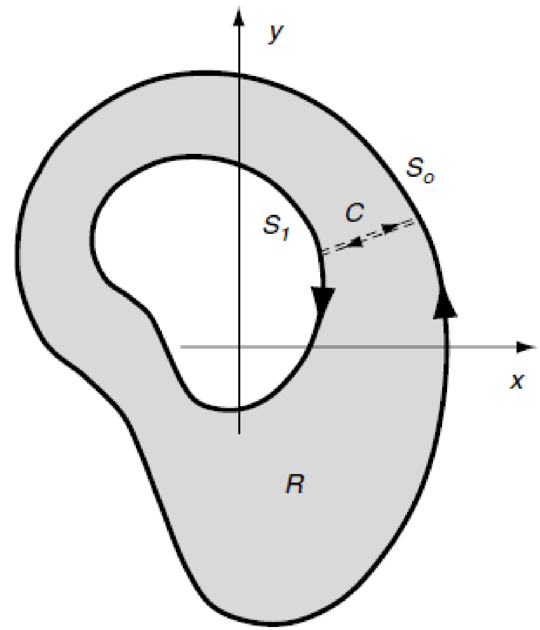
$$\tau_{xz} = G\left(-\alpha y + \frac{\partial w}{\partial x}\right), \quad \tau_{yz} = G\left(\alpha x + \frac{\partial w}{\partial y}\right)$$

$$0 = \oint_S dw = \oint_S \left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy\right) = \frac{1}{G}\oint_S \left(\tau_{xz}dx + \tau_{yz}dy\right) + \alpha \oint_S \left(ydx - xdy\right)$$

$$= \frac{1}{G}\oint_S \tau(s)\,ds + \alpha\left(-2A_i\right) \quad \Rightarrow \quad \boxed{\oint_S \tau(s)\,ds = 2G\alpha A},$$

- where A is the area enclosed by an arbitrary closed-path S that is encircling an inner boundary S_i.
- The value of ψ_i on the inner boundary S_i must be chosen so that the above relation is satisfied.

Multiply Connected Cross-Sections



- The BCs on cylinder ends require

$$\boxed{1}: 0 = P_x = \iint_A T_x^n dxdy, \quad \boxed{2}: 0 = P_y = \iint_A T_y^n dxdy$$

$$\boxed{3}: 0 = P_z = \iint_A \cancel{T_z^n} dxdy, \quad \boxed{4}: 0 = M_x = \iint_A y \cancel{T_z^n} dxdy$$

$$\boxed{5}: 0 = M_y = \iint_A x \cancel{T_z^n} dxdy, \quad \boxed{6}: T = M_z = \iint_A (xT_y^n - yT_x^n) dxdy$$

$\boxed{1} - \boxed{5}$ are automatically satisfied.

$$\boxed{6}: T = \iint_A \left(xT_y^n - yT_x^n\right) dxdy = -\iint_A \left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} \right) dxdy$$

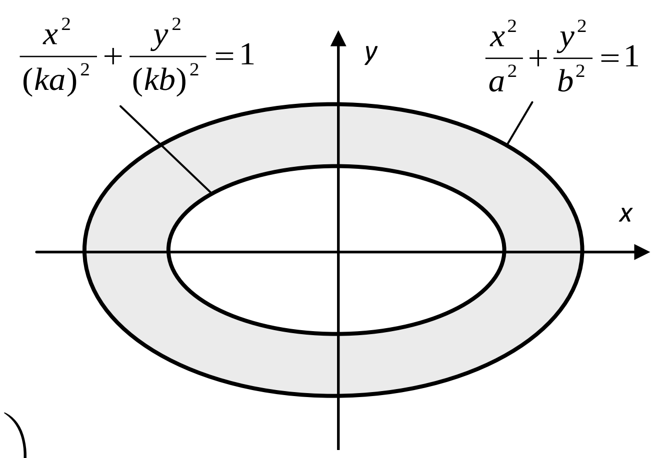
$$= -\iint_A \left(\frac{\partial (x\psi)}{\partial x} + \frac{\partial (y\psi)}{\partial y} - 2\psi \right) dxdy = \int_{S_o, C, S_1} \left(y\psi dx - x\psi dy \right) + 2\iint_A \psi \, dxdy$$

$$= -\int_{S_1} \left(y\psi_1 dx - x\psi_1 dy \right) + 2\iint_A \psi \, dxdy = 2\psi_1 A_1 + 2\iint_A \psi \, dxdy$$

$$\Rightarrow \quad \boxed{T = 2\iint_A \psi dxdy + 2\psi_1 A_1}$$

- For multiple holes: $\boxed{T = 2\iint_A \psi dxdy + 2\sum_i \psi_i A_i}$

Hollow Elliptical Section



- For this case lines of constant shear stress coincide with both inner and outer boundaries, and so no stress will act on these lateral surfaces. Therefore, hollow section solution is found by simply removing inner core from solid solution. This gives same stress function and stress distribution in remaining material.

$$\psi=\frac{a^2b^2G\alpha}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

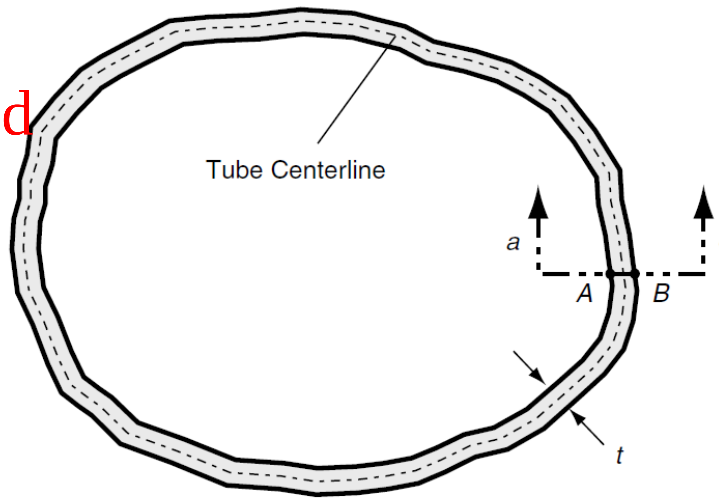
- Constant value of stress function on inner boundary is: $\psi_i=\frac{a^2b^2G\alpha}{a^2+b^2}\left(k^2-1\right)$.
- Load carrying capacity is determined by subtracting load carried by the removed inner cylinder from the torque relation for solid section

$$T=\frac{\pi a^3b^3G\alpha}{a^2+b^2}-\frac{\pi(ka)^3(kb)^3G\alpha}{(ka)^2+(kb)^2}=\frac{\pi G\alpha}{a^2+b^2}a^3b^3(1-k^4)$$

- Maximum stress still occurs at $x = 0$ and $y = \pm b$: $\tau_{max}=\frac{2T}{\pi ab^2}\frac{1}{1-k^4}$.

Closed Thin-Walled Tubes

- Thin wall thickness implies that the membrane slope *BC* can be approximated by a straight line.
- Since the membrane slope equals resultant shear stress: $\tau(s) = \psi_1 / t(s)$.
- Determine the value of ψ_1:



$$\oint_{S_c} \tau(s)\,ds = 2G\alpha A_c \quad \Rightarrow \quad \psi_1 = \frac{2G\alpha A_c}{\oint_{S_c} \frac{1}{t(s)} ds},$$ where S_c = length of tube centerline, A_c = area enclosed by tube centerline.

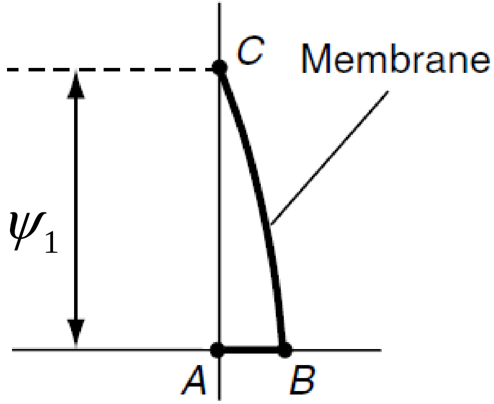
- Load carrying relation:

$$T = 2\iint_A \psi\, dxdy + 2\psi_1 A_1 = 2\left(A\frac{\psi_1}{2}\right) + 2\psi_1 A_1 = 2\psi_1 A_c.$$

where A = section area, A_1 = area enclosed by S_1.

- Combining these relations:

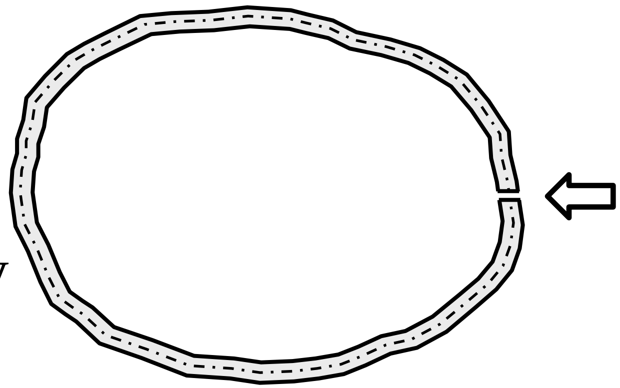
$$\boxed{\psi_1 = \frac{T}{2A_c}} \quad \boxed{\tau(s) = \frac{T}{2A_c t(s)}} \quad \boxed{\alpha = \frac{T}{4GA_c^2}\oint_{S_c} \frac{1}{t(s)} ds}$$



(Section *aa*)

- τ_{max} occurs across the narrowest wall.

Open vs. Closed Thin-Walled Tubes



- Cut creates an open tube and produces significant changes to stress function, stress field and load carrying capacity.
- Open tube solution can be approximately determined using results from thin rectangular solution.
- Stresses for open and closed tubes can be compared and for identical applied torques, the following relation can be established

$$\frac{\tau_{\text{Open Tube}}}{\tau_{\text{Closed Tube}}} \approx \frac{\dfrac{3T}{At}}{\dfrac{T}{2A_c t}} = 6\frac{A_c}{A}, \text{ but since } A_c >> A \Rightarrow \frac{\tau_{\text{Open Tube}}}{\tau_{\text{Closed Tube}}} >> 1$$

$$\Rightarrow \tau_{\text{Open Tube}} >> \tau_{\text{Closed Tube}}$$

$\therefore$ Stresses are higher in open tube and thus closed tube is stronger.

Outline

- Elastic Cylinders with End Loading
- Torsion of Cylinders: General formulation
- Stress-Function Formulation
- Displacement Formulation
- Membrane Analogy
- Solution: Boundary Equation Scheme
- Solution: Fourier Method – Rectangular Section
- Multiply Connected Cross-Sections
- Hollow Sections